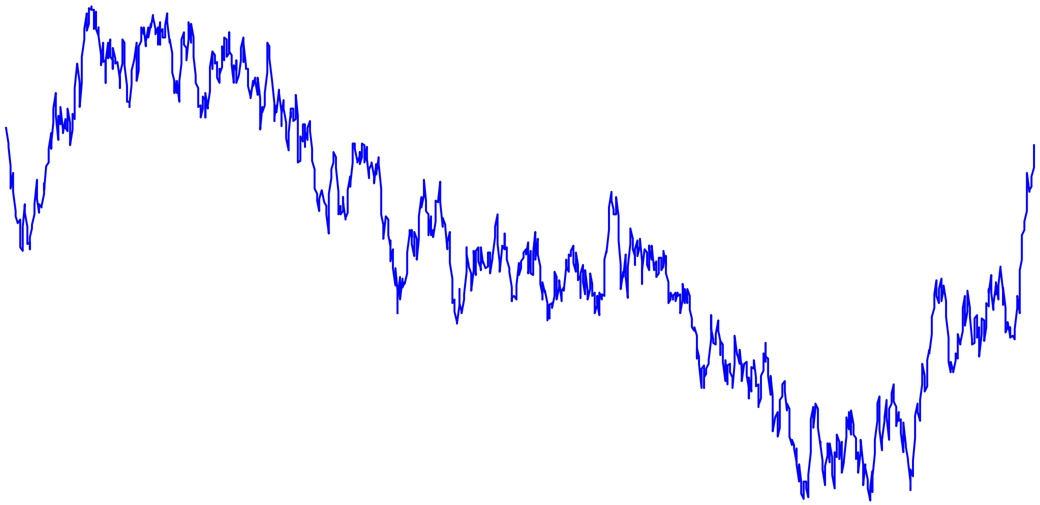
Distances, Approximations and Global Structural Features

Riccardo Guidotti

Syllabus

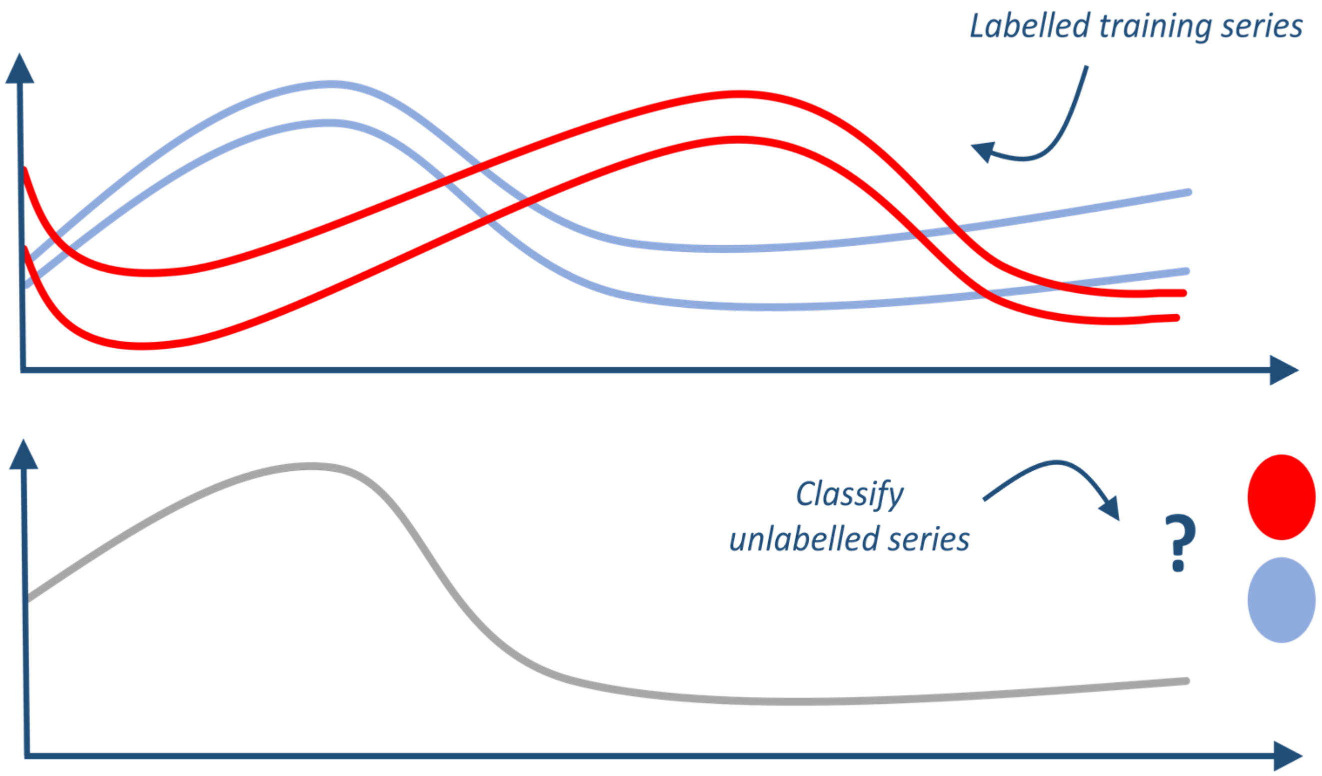
- Distances
  - Euclidean Distance
  - Dynamic Time Warping
- Approximations
  - Piecewise Aggregate Approximation
  - Symbolic Aggregate approXimation
  - Discrete Fourier Transformation
  - Symbolic Fourier Approximation
  - Singular Value Decomposition
  - Principal Component Analysis
- Global Structural Features
  - Simple Standard Features
  - Catch22 Features
  - TSFresh Features



A Simple ML Classifier

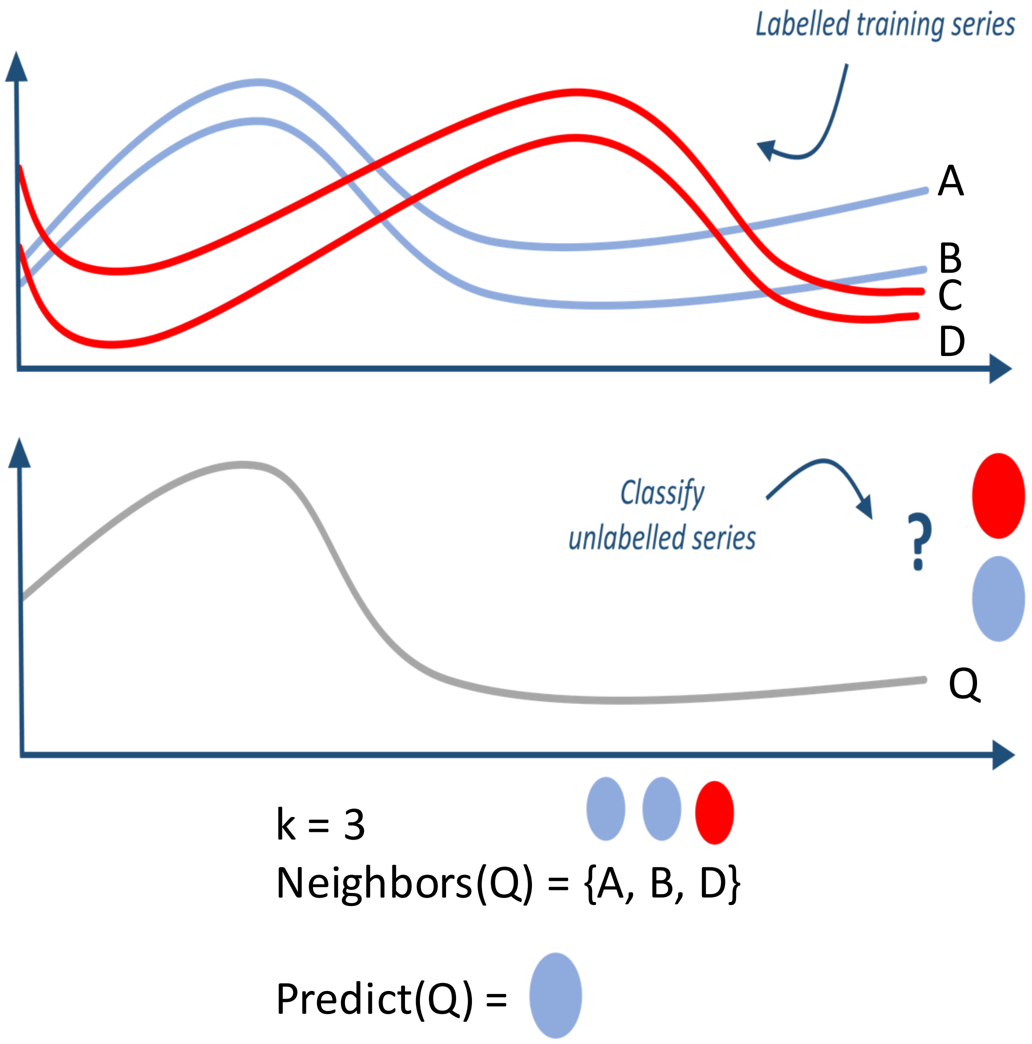
Time Series Classification - TSC

- Given a dataset $X = \{T_1, \ldots T_n\}$, TSC is the task of training a model f to predict an exogenous <u>categorical</u> output y for each time series T, i.e., $f(T) = y$.



Nearest-Neighbor Classifier (K-NN)

- Basic idea: If it walks like a duck, quacks like a duck, then it is probably a duck.
- Given a set of training records, and a test record:
    1. Compute the distances from the test to the training records.
    2. Identify the k "nearest" records.
    3. Use class labels of nearest neighbors to determine the class label of test record (e.g., by taking majority vote).



Performance Evaluation

- Let suppose we have a vector y of actual/real class labels:
- y = [0 0 0 1 1 1 0 1 0 1 0 1 1 1 0 0]

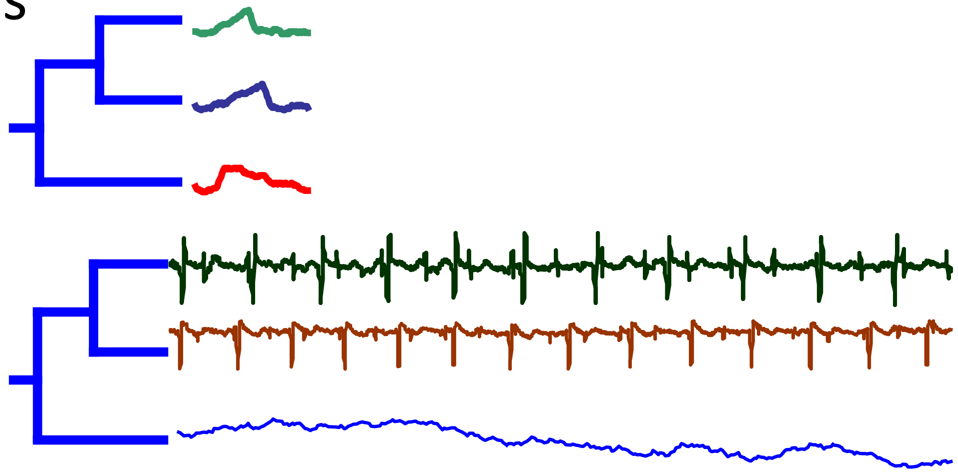
- Let name y' the vector returned by a kNN:
- y' = [0 0 1 1 1 0 0 1 0 1 1 1 0 0 0 0]

- **Accuracy** = # record correctly classified / # records classified
- Accuracy = 11 / 16 = 0.69

Time Series Distances

Distances and Similarities

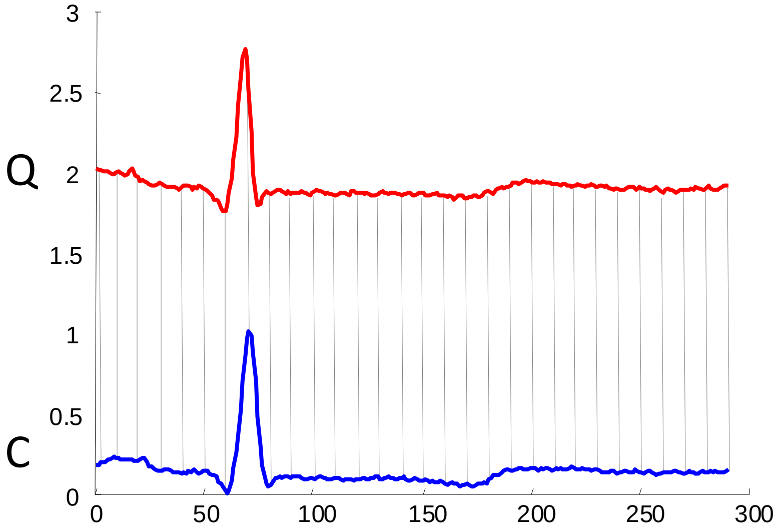
- Time series problems such as classification, forecasting, clustering, etc. require the usage of a notion of distance or similarity.
- What is similarity?
- It is the quality or state of being similar, likeness, resemblance, as a similarity of features.
- In TSA we recognize two types of similarity measures depending on the data representation considered:
  - **shape-based similarity**
  - **structural-based similarity**



Shape vs Structural Similarities

Shape-based Similarity

- The original values of the time series are compared taking time into account.
- Better for short time series.



Structural Similarity

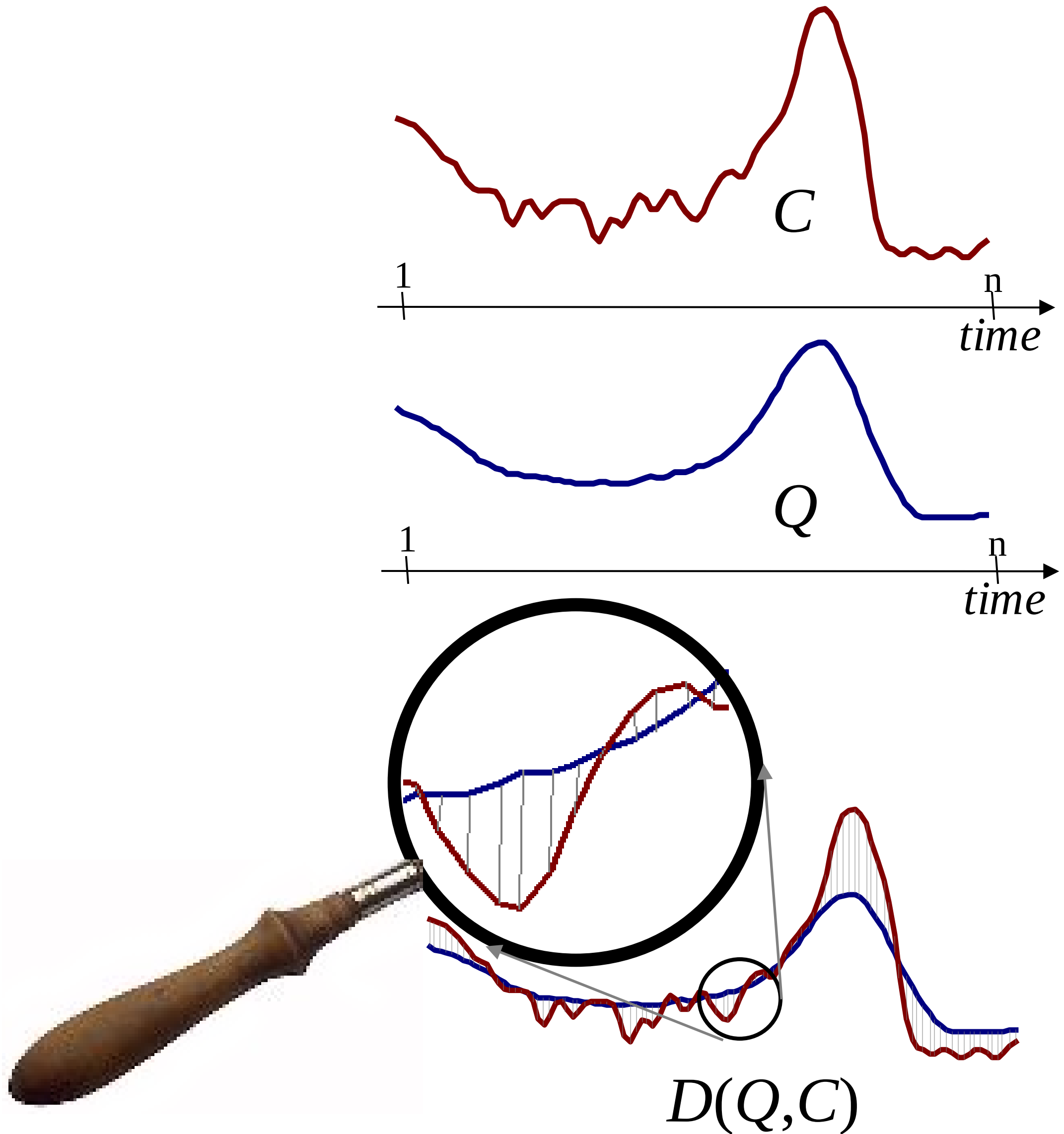
- Time series are transformed into an alternative representation where the novel features are time-independent.
- Better for long time series.

	min	max	mean	std
Q	1.8	2.9	2.0	1.3
C	0.0	1.0	0.2	1.2

Euclidean Distance

- Given two time series:
  - $Q = q_1 \ldots q_n$
  - $C = c_1 \ldots c_n$

$$D(Q,C) \equiv \sqrt{\sum_{i=1}^{n} (q_i - c_i)^2}$$

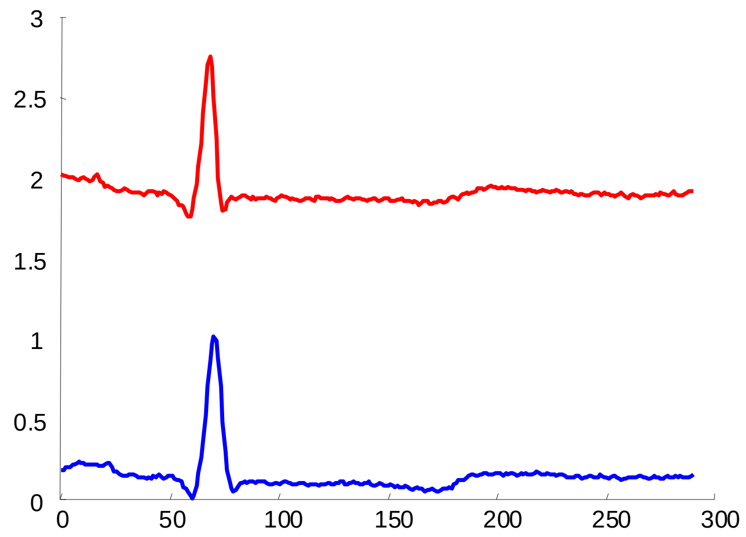


- *T1 = < 56, 176, 110, 95 >*
- *T2 = < 36, 126, 180, 80 >*

D(T1,T2) = sqrt [$(56-36)^2 + (176-126)^2 + (110-180)^2 + (95-80)^2$]

Problems with Euclidean Distance

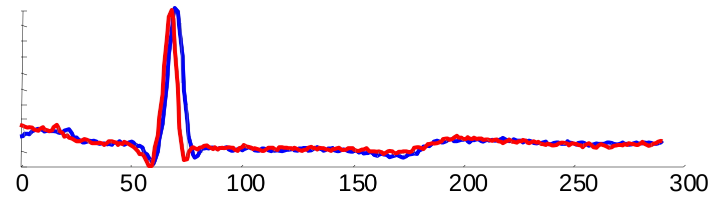
- Euclidean distance is very sensitive to "distortions" in the data.
- These distortions are dangerous and should be removed.
- Most common distortions:
    - Offset Translation
    - Amplitude Scaling
    - Linear Trend
    - Noise
- They can be removed by using the appropriate normalization.



Q = Q - mean(Q)

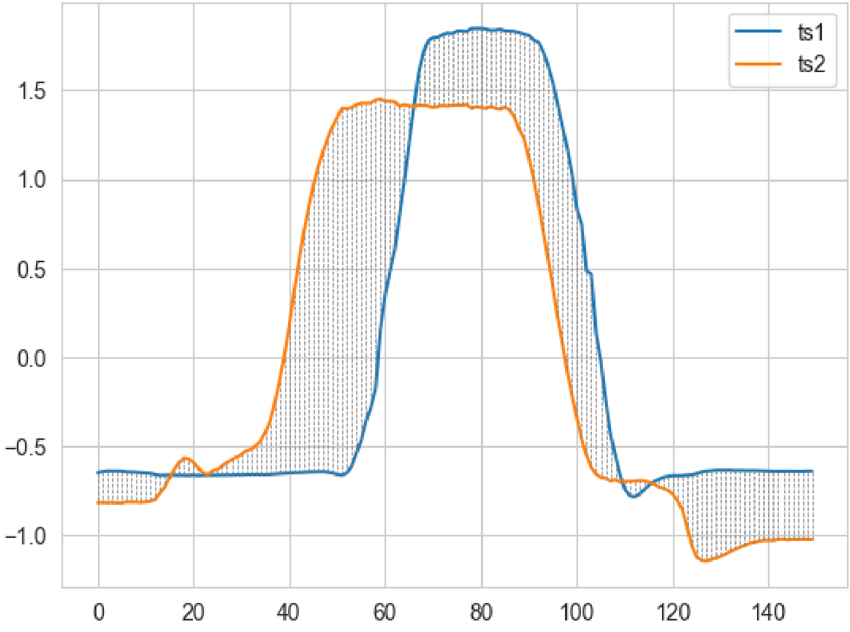
C = C - mean(C)

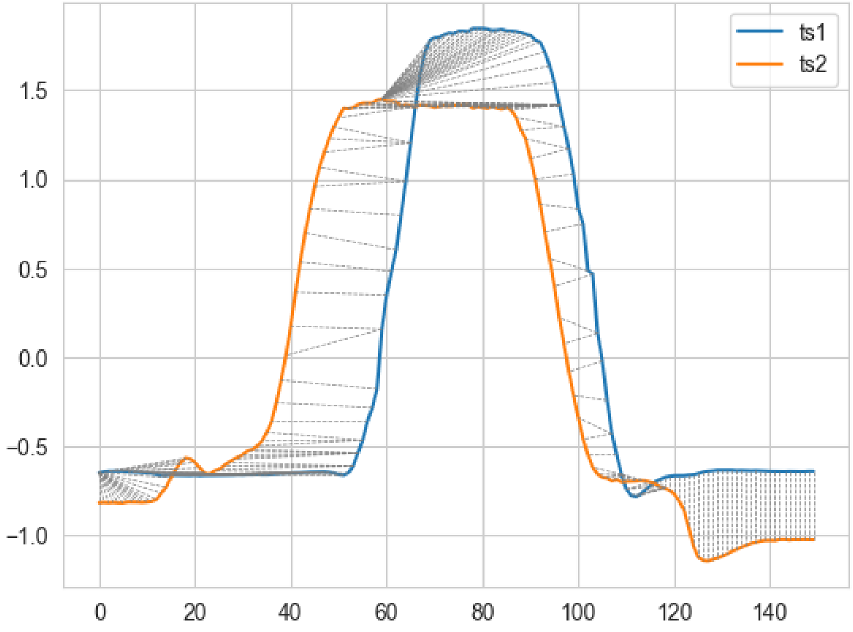
D(Q,C)



Further Problems with Euclidean Distance

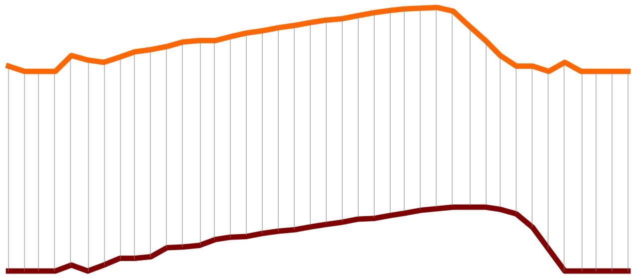
- Even after normalization, the Euclidean distance may still be unsuitable for some time series domains since it does not allow for acceleration and deceleration along the time axis.



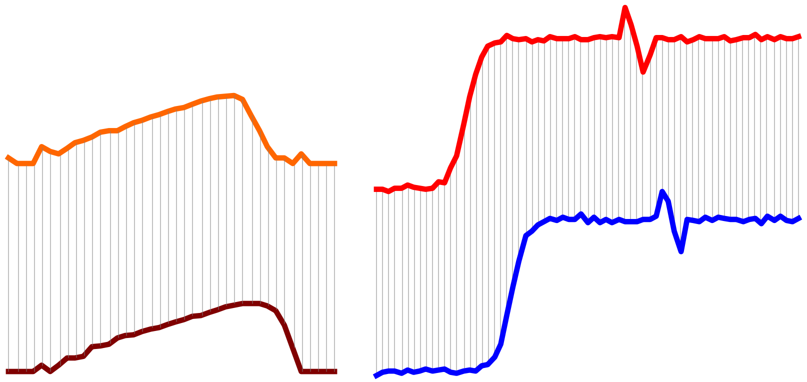


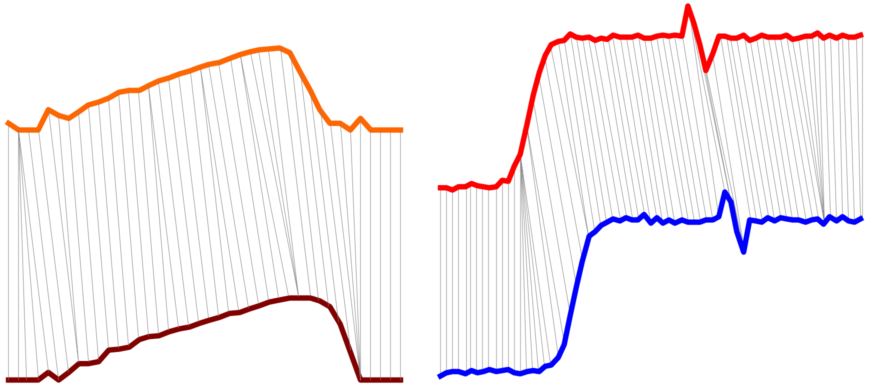
Dynamic Time Warping

- Sometimes two time series that are conceptually equivalent evolve at different speeds, at least in some moments.

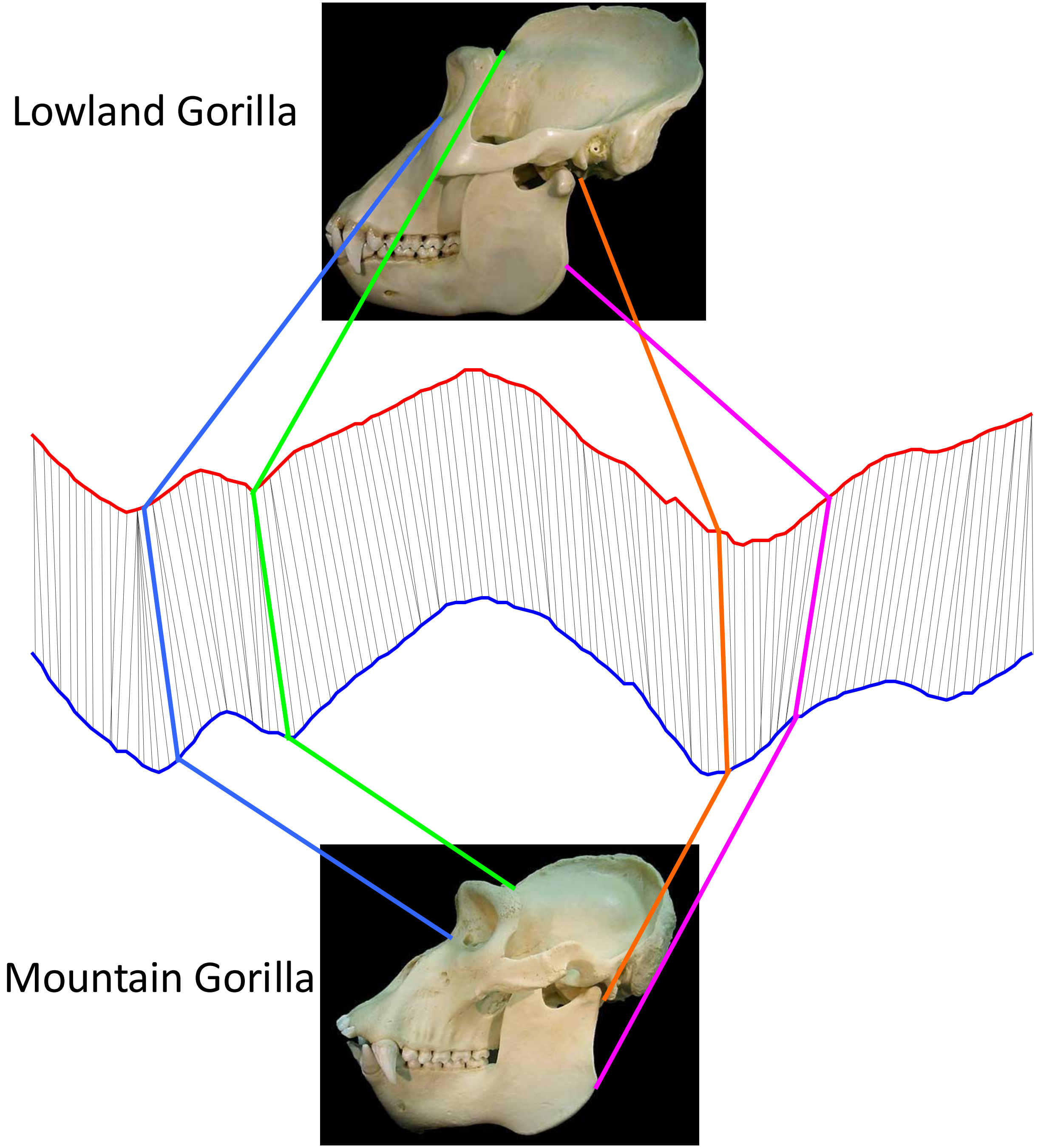


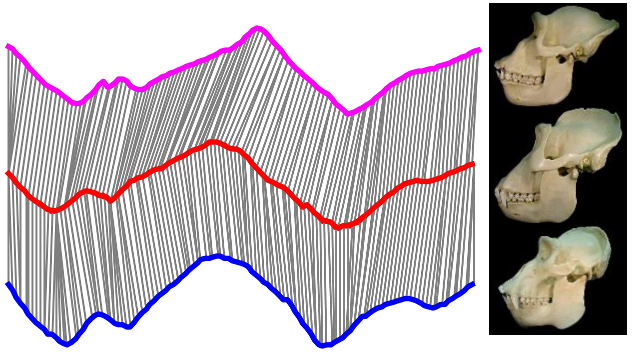
E.g. correspondence of peaks in two similar time series





- **Euclidean distance - Fixed Time Axis**: Sequences are aligned “one to one”. Greatly suffers from the misalignment in data.
- **Dynamic Time Warping - Warped Time Axis**: Nonlinear alignments are possible. Can correct misalignments in data.

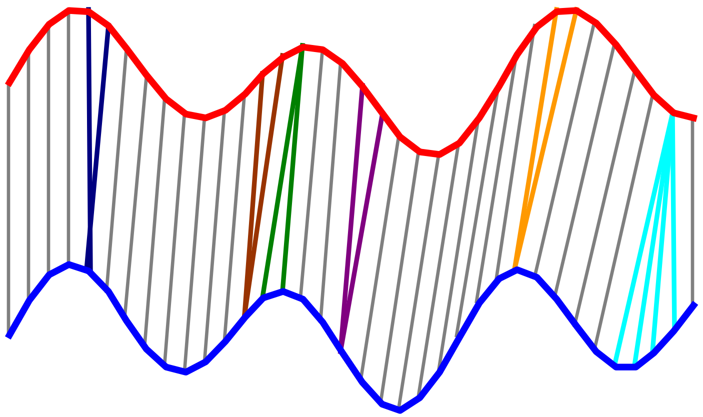


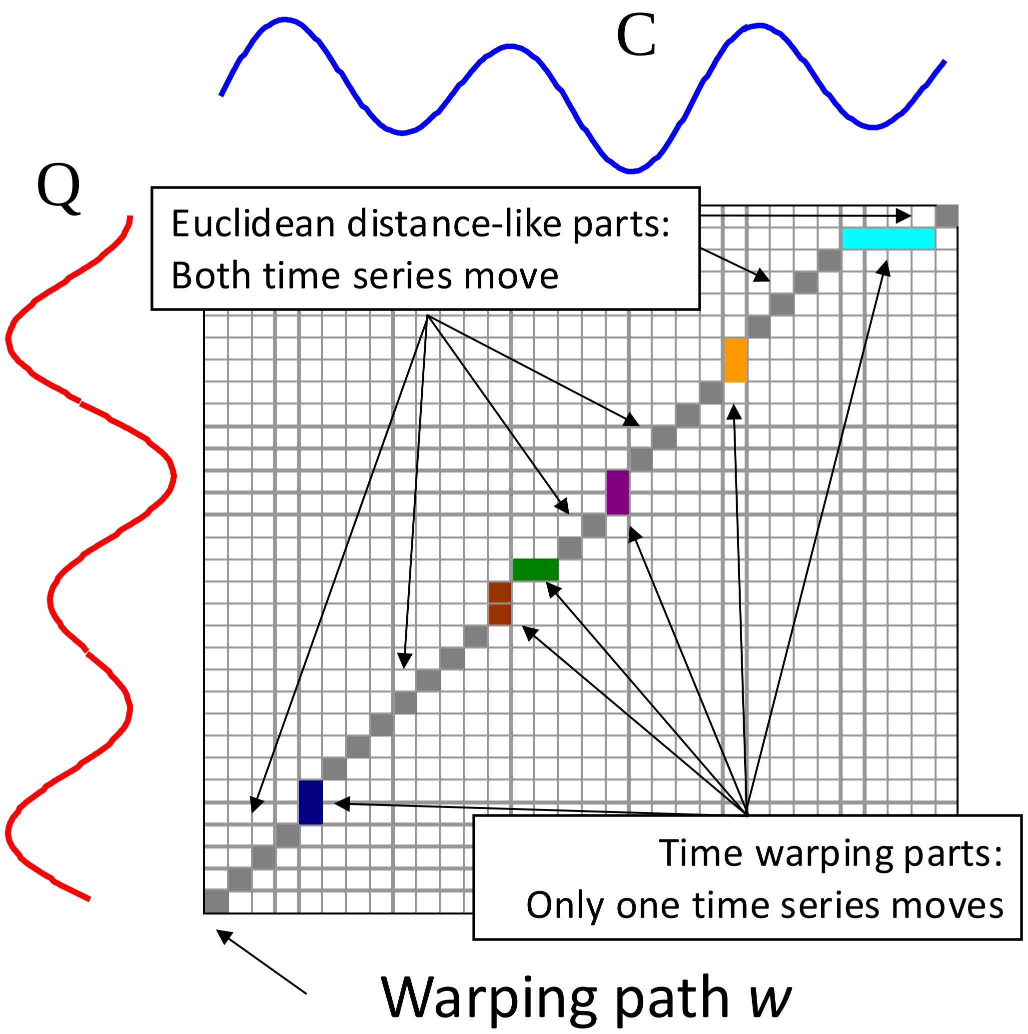


https://izbicki.me/blog/converting-images-into-time-series-for-data-mining.html

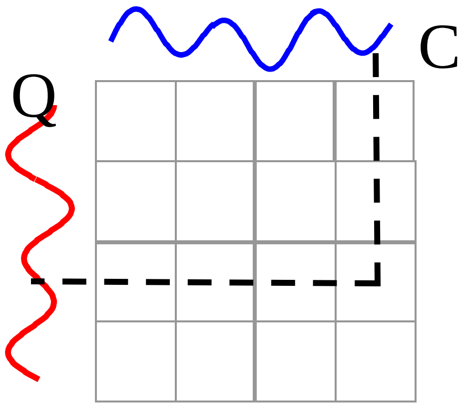
How is DTW Calculated?

- Every possible warping between two time series, is a path through the matrix.
- The constrained sequence of comparisons performed:
  - Start from pair of points *(0,0)*
  - After point *(i,j)*, either *i* or *j* increase by one, or both of them
  - End the process on *(n,m)*





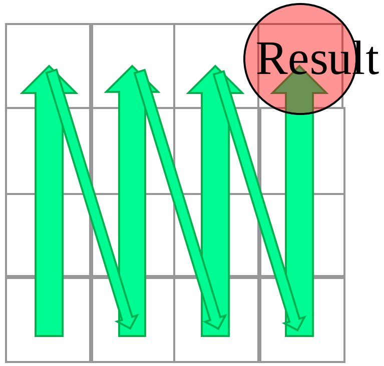
Dynamic Programming Approach



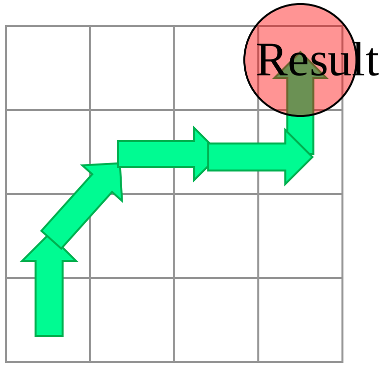
Step 1: Compute the matrix of all *point-to-point* distances $d(q_i,c_j) = | Q_i - C_j |$

Step 2: Compute the cumulative cost matrix as $\gamma(i,j) = d(q_i,c_j) + \min\{\gamma(i-1,j-1), \gamma(i-1,j), \gamma(i,j-1)\}$

- Start from cell *(1,1)*
- Compute *(2,1), (3,1), …, (n,1)*
- Repeat for columns *2, 3, …, n*
- Final distance value is in the last cell computed



Step 3: find the path with the lowest values, i.e., the best alignment between Q and C



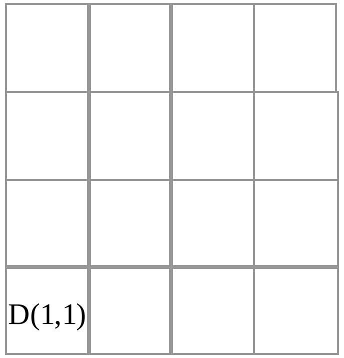
Dynamic Programming Approach

$\gamma(i,j) \quad = \quad d(q_i,c_j) + \min\{\ \gamma(i-1,j-1),\ \gamma(i-1,j\ ),\ \gamma(i,j-1)\ \}$

Step 2: compute the matrix of all path costs $\gamma(i,j)$

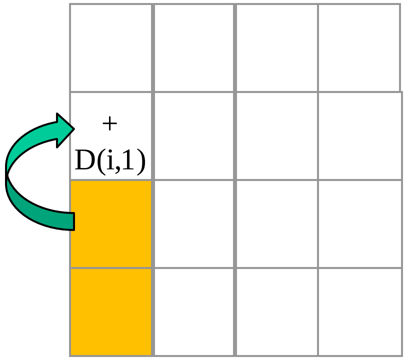
- Start from cell *(1,1)*

$$\begin{aligned}\gamma(1,1) &= d(q_1,c_1) + \min\{\ \gamma(\cancel{0,0}),\ \gamma(\cancel{0,1}),\ \gamma(\cancel{1,0})\} \\ &= d(q_1,c_1) \\ &= D(1,1)\end{aligned}$$

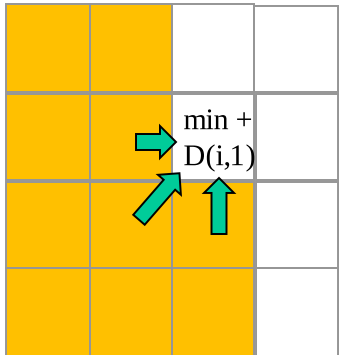


- Compute (2,1), (3,1), …, (n,1)

$$\begin{aligned}\gamma(i,1) &= d(q_i,c_1) + \min\{\ \gamma(\cancel{i-1,0}),\ \gamma(i-1,1),\ \gamma(\cancel{i,0})\ \} \\ &= d(q_i,c_1) + \gamma(i-1,1) \\ &= D(i,1) + \gamma(i-1,1)\end{aligned}$$

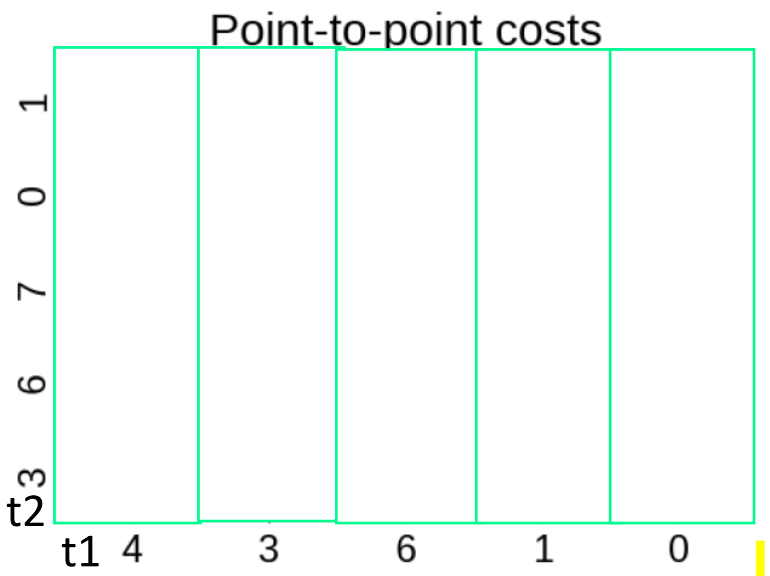


- Repeat for columns *2, 3, …, n*
  - The general formula applies



DTW – Example

t1	< 4, 3, 6, 1, 0 >
t2	< 3, 6, 7, 0, 1 >



Result: 4

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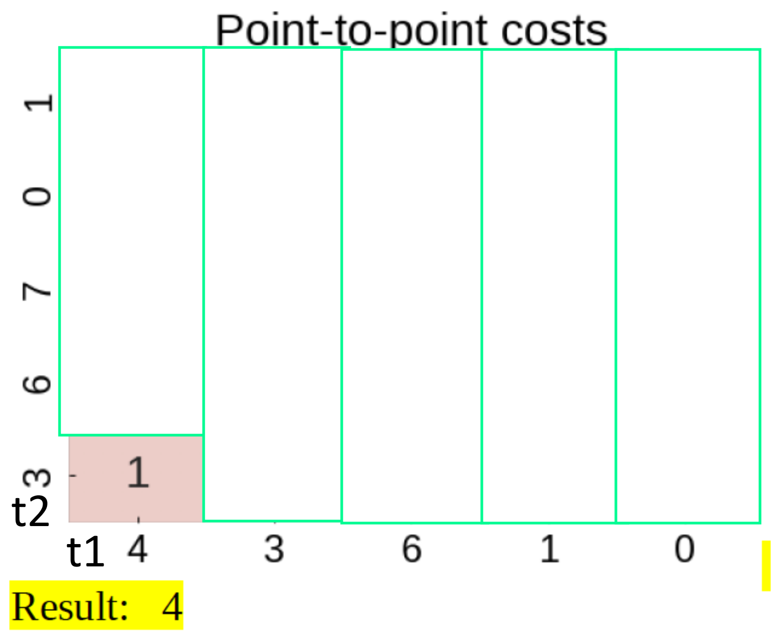

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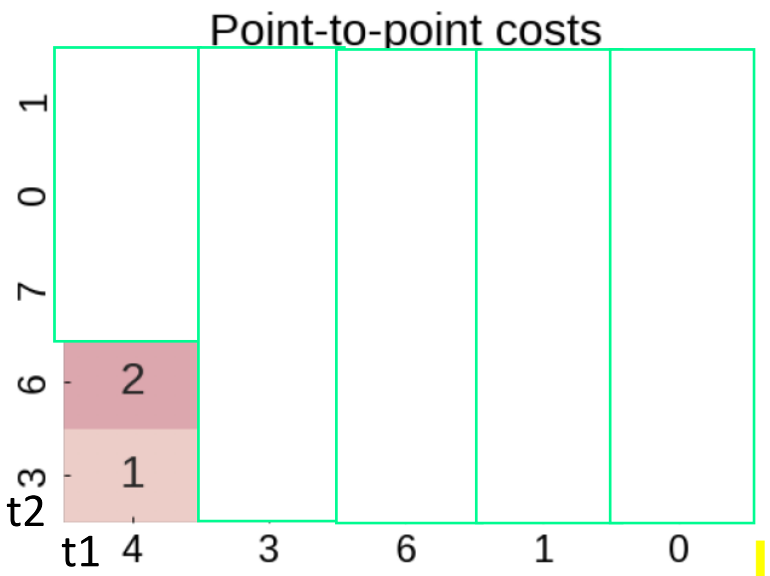

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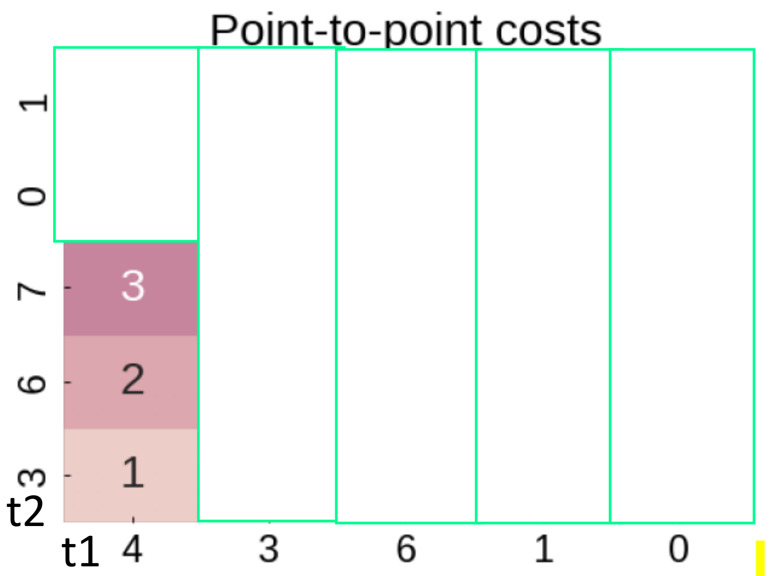

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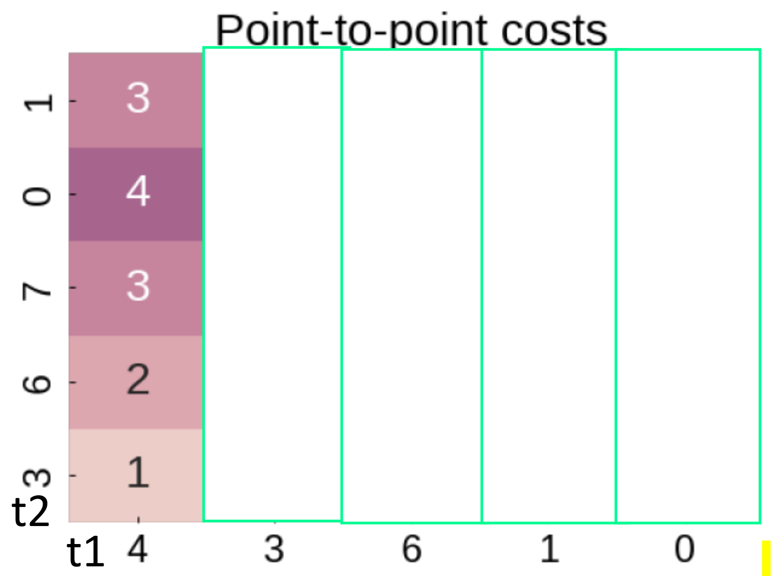

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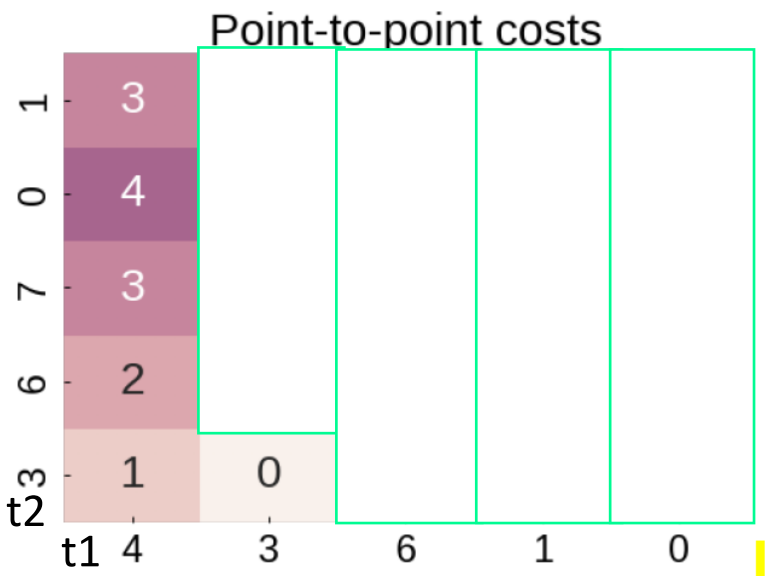

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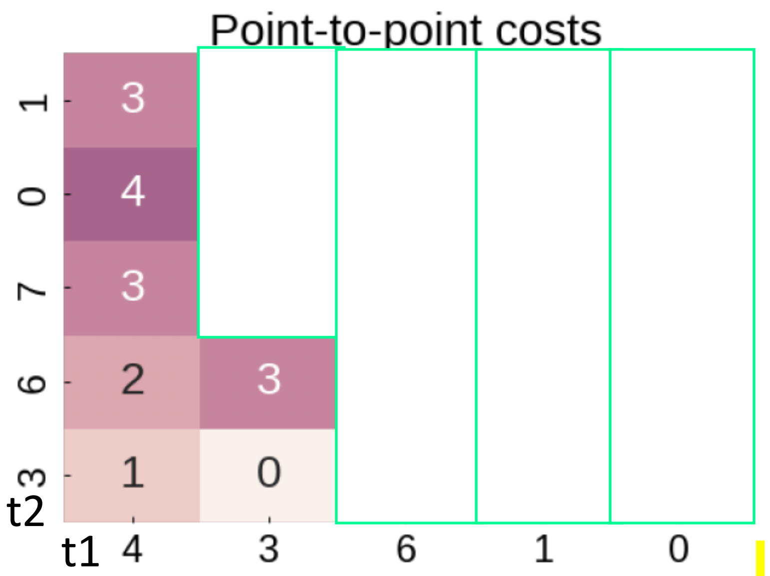

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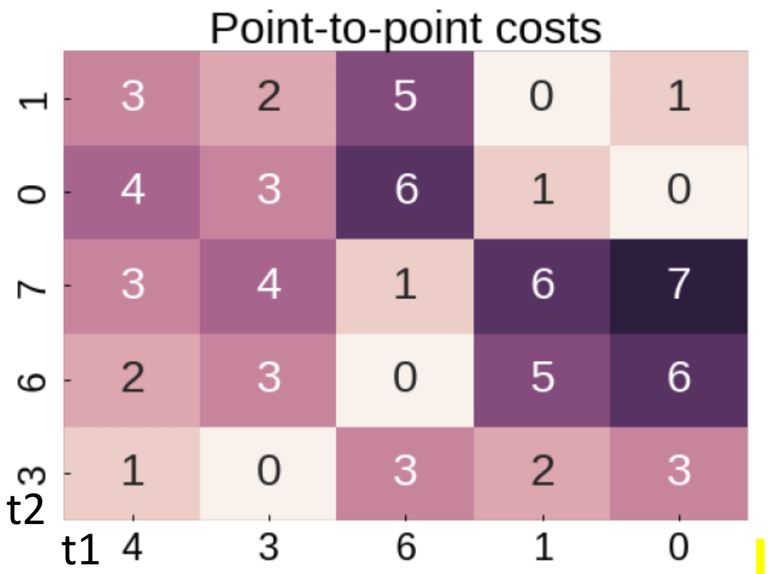

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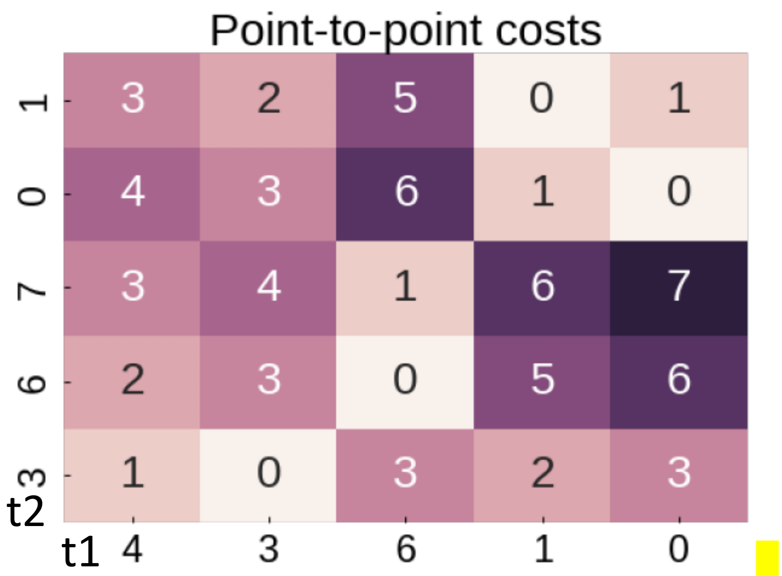

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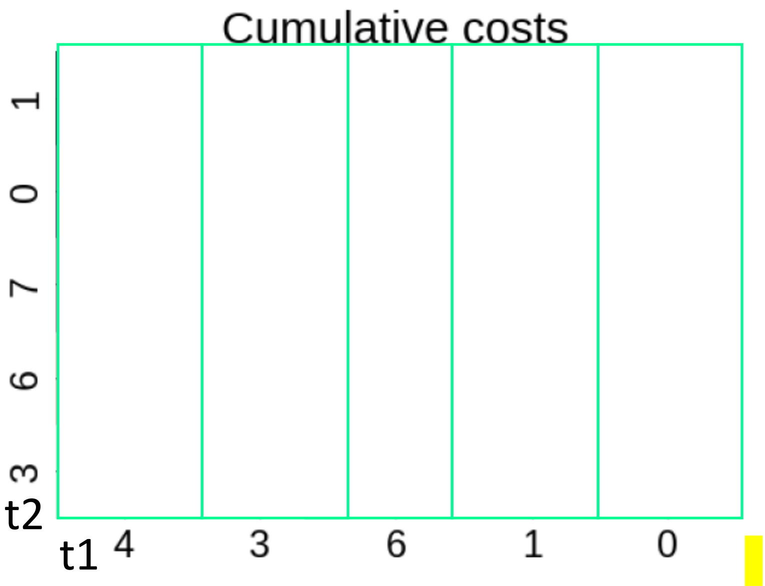

$$\gamma(i,j) \quad = \quad d(q_i,c_j) + \min\{\ \gamma(i\text{-}1,j\text{-}1),\ \gamma(i\text{-}1,j\ ),\ \gamma(i,j\text{-}1)\ \}$$

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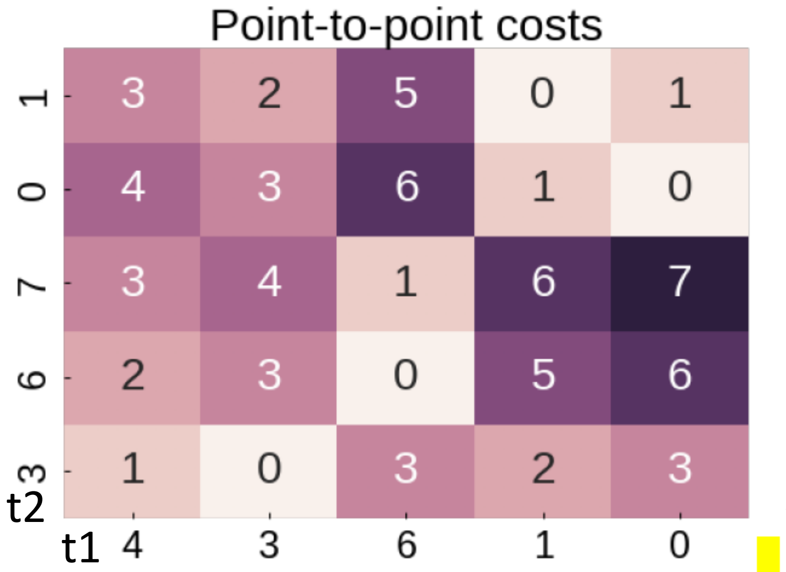

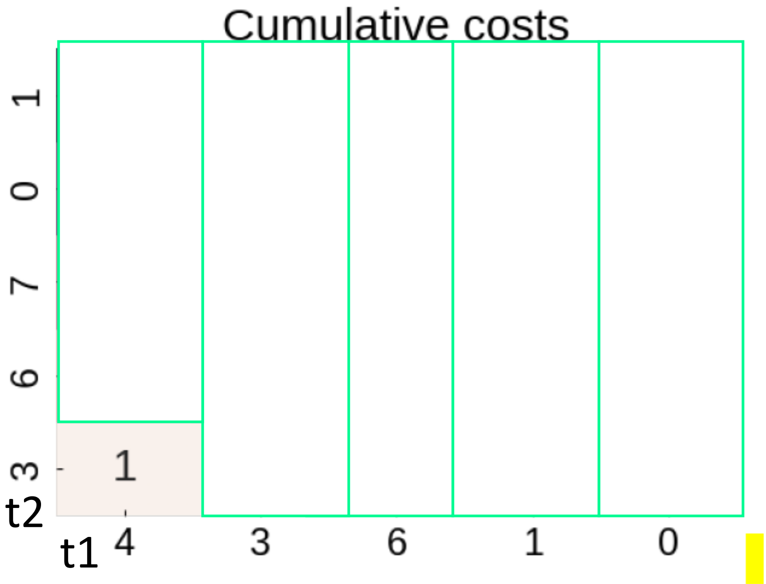

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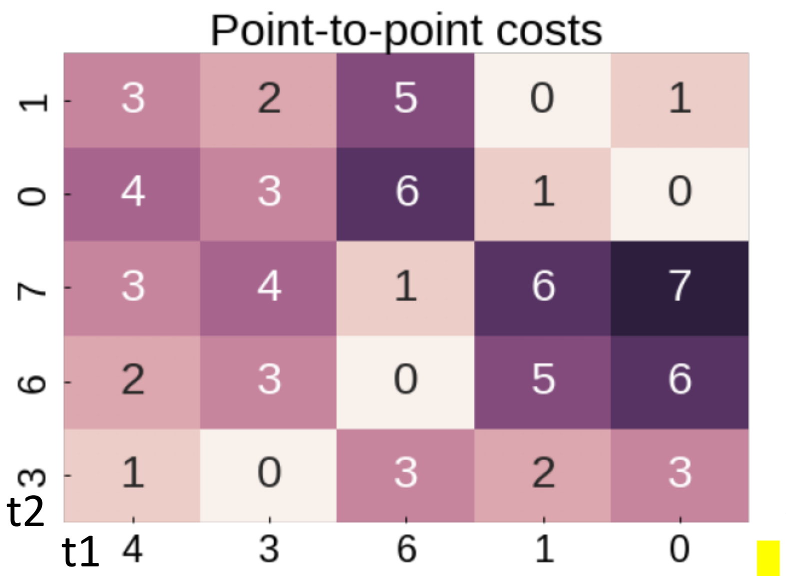

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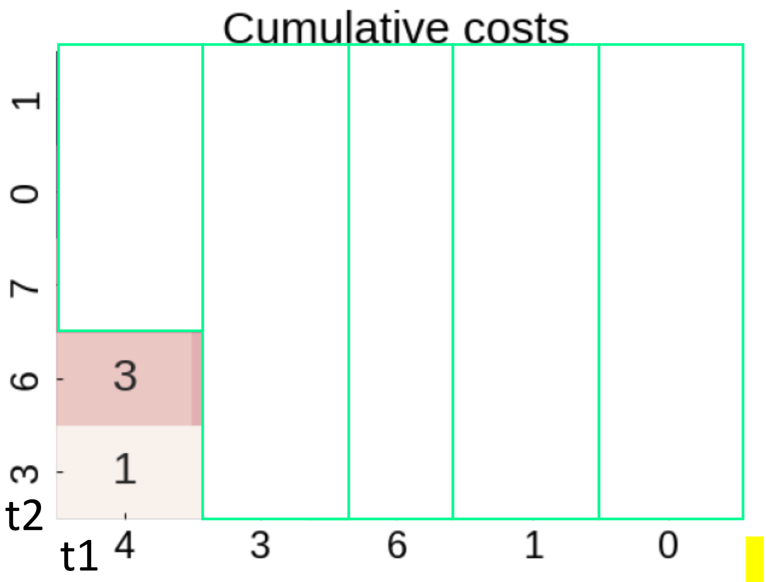

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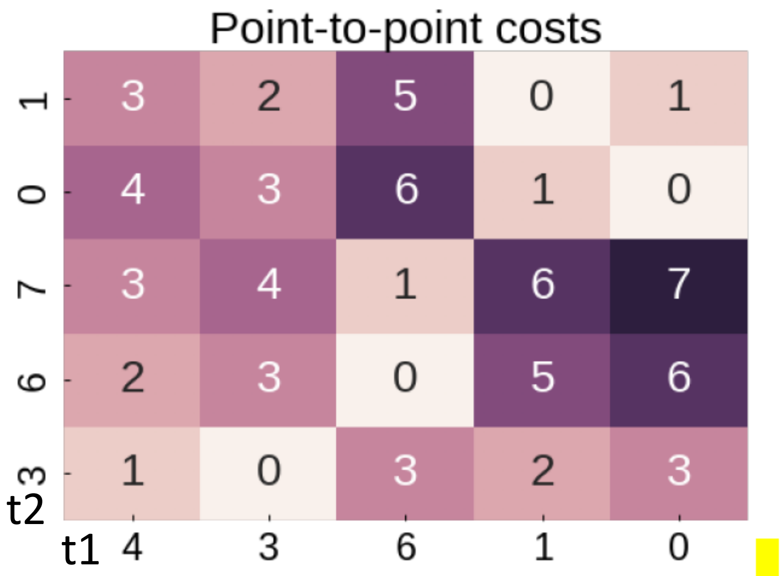

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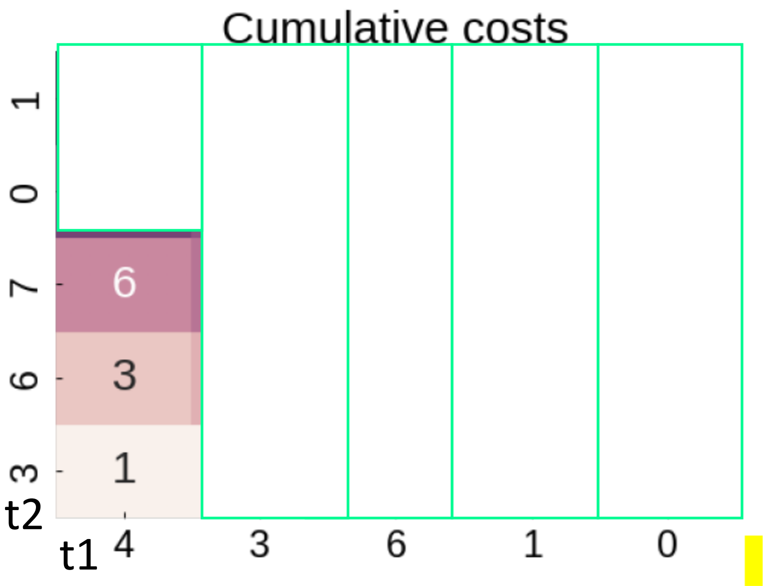

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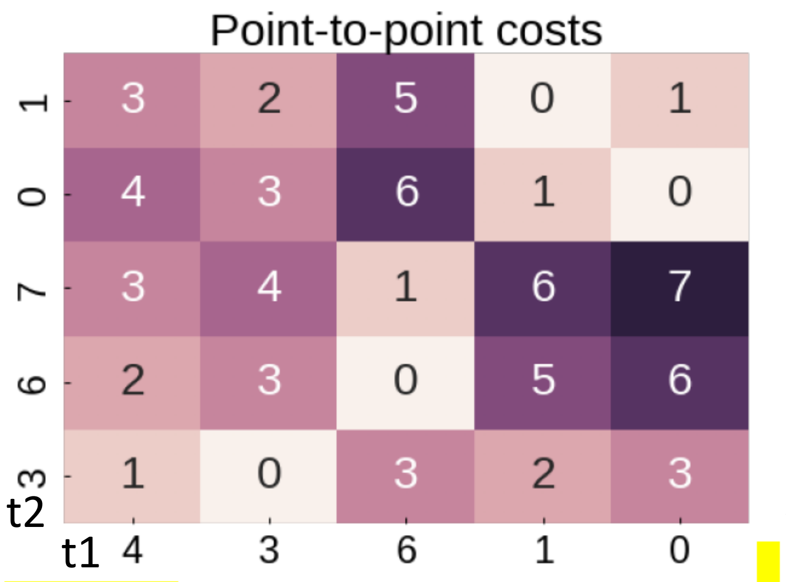

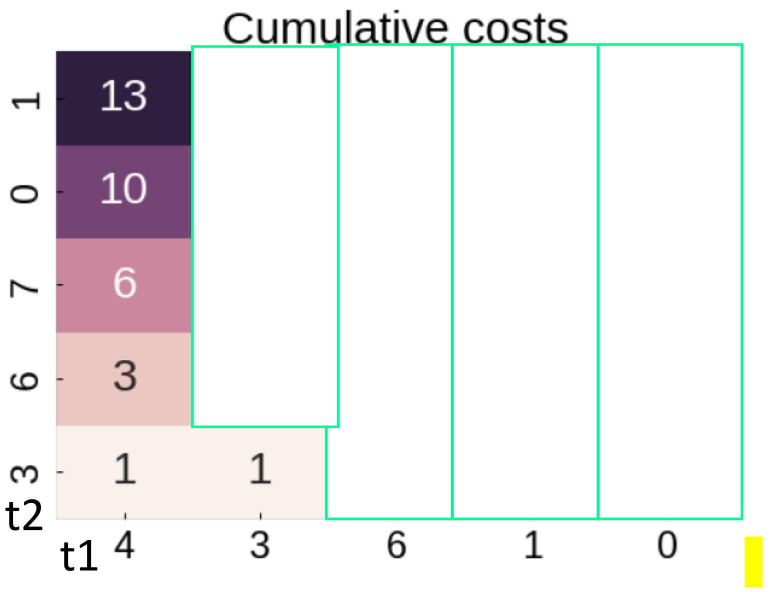

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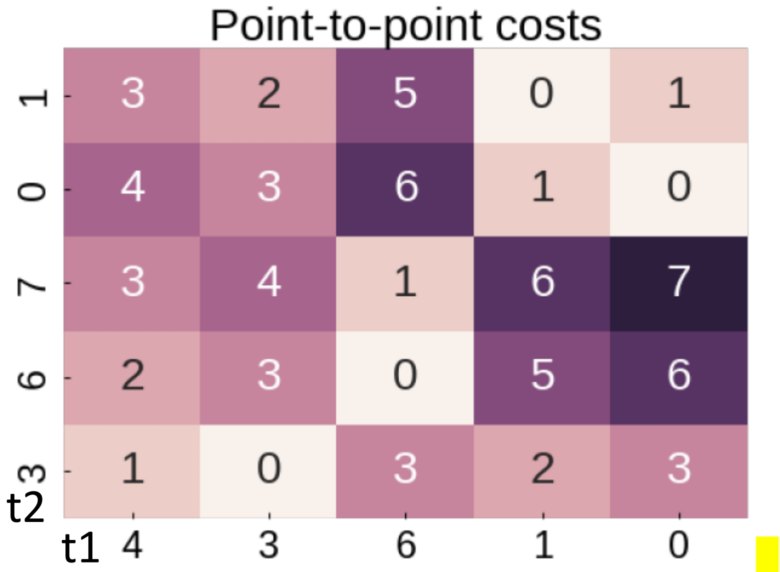

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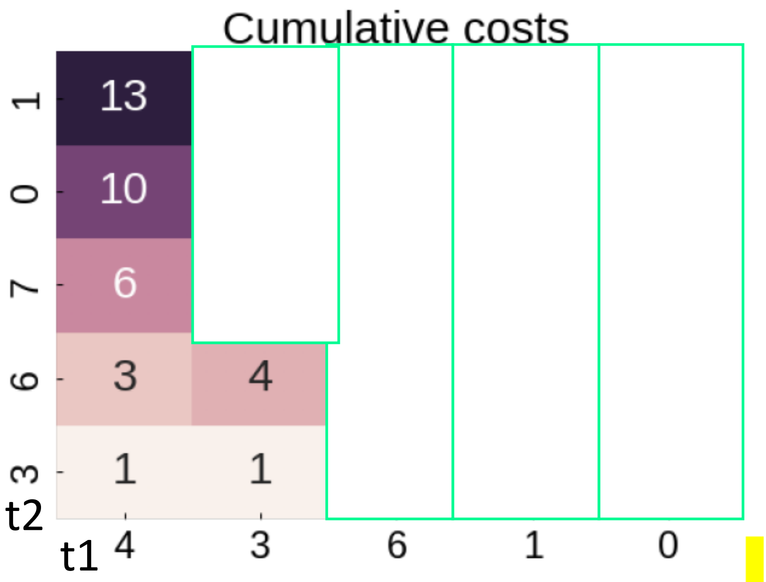

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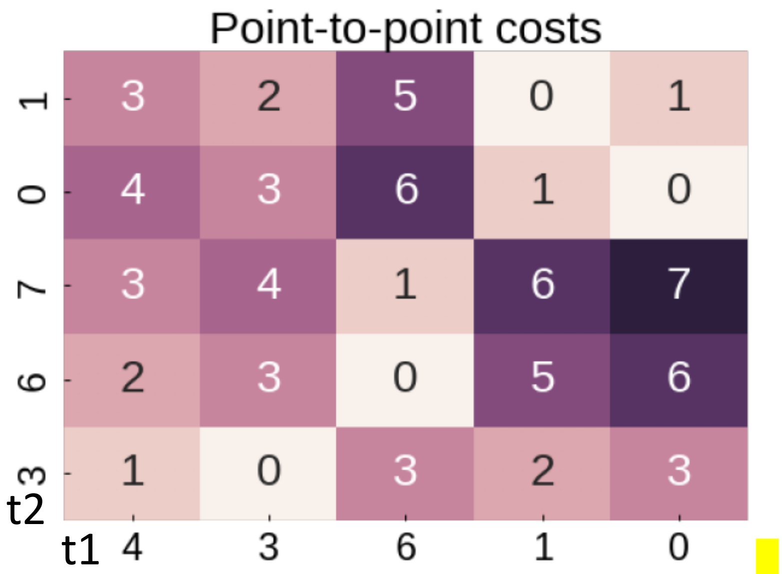

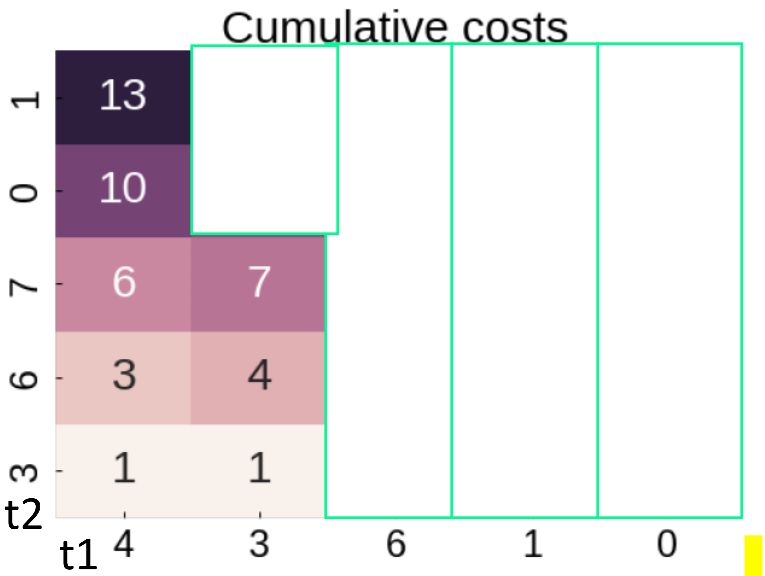

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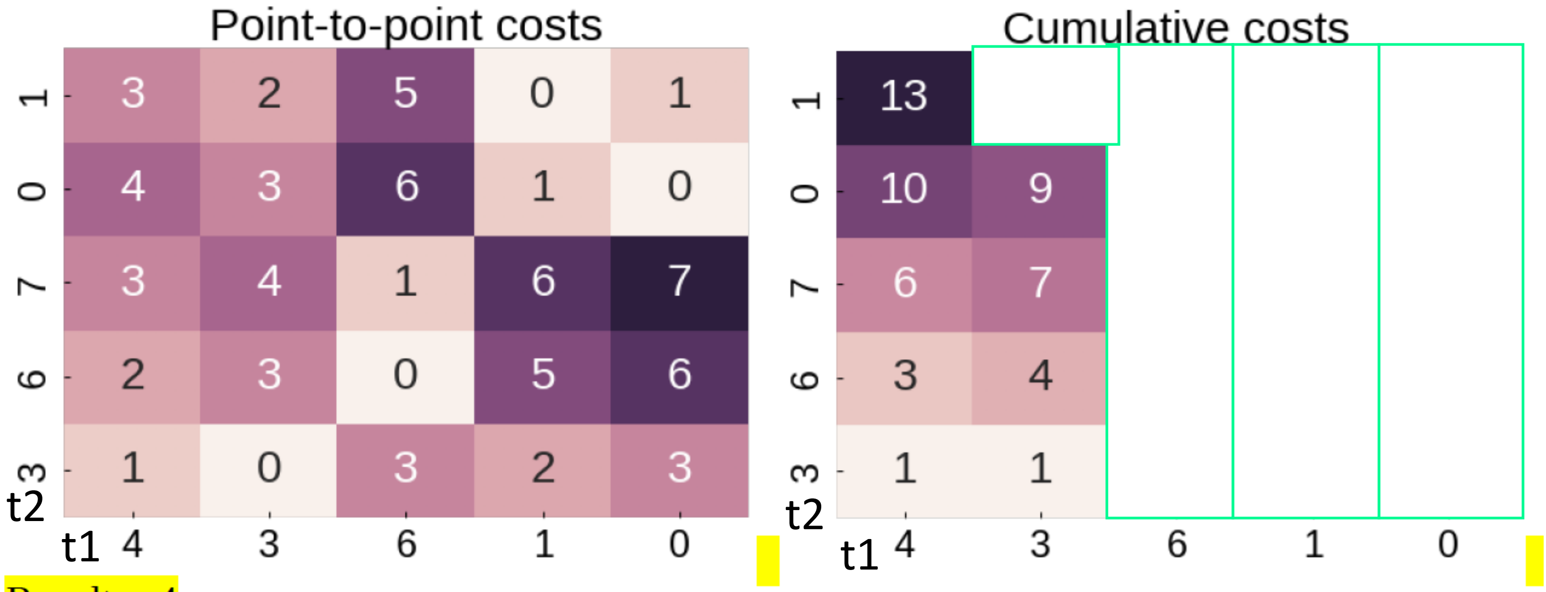

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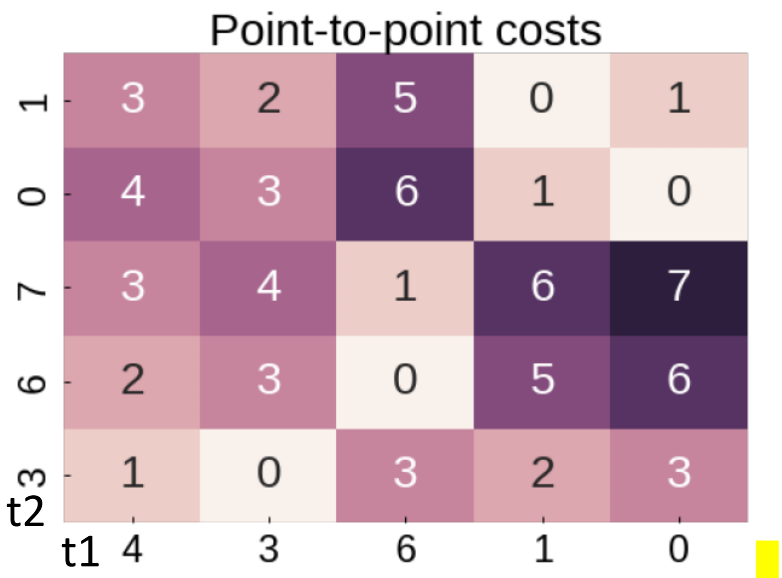
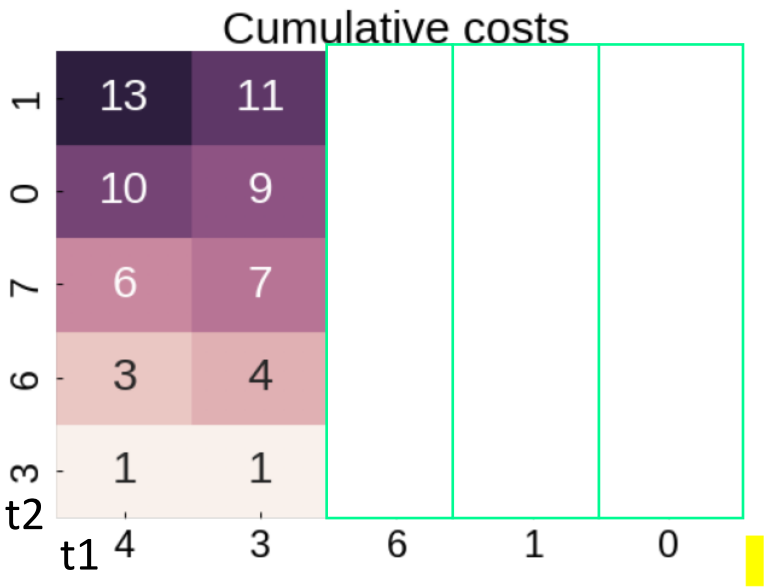


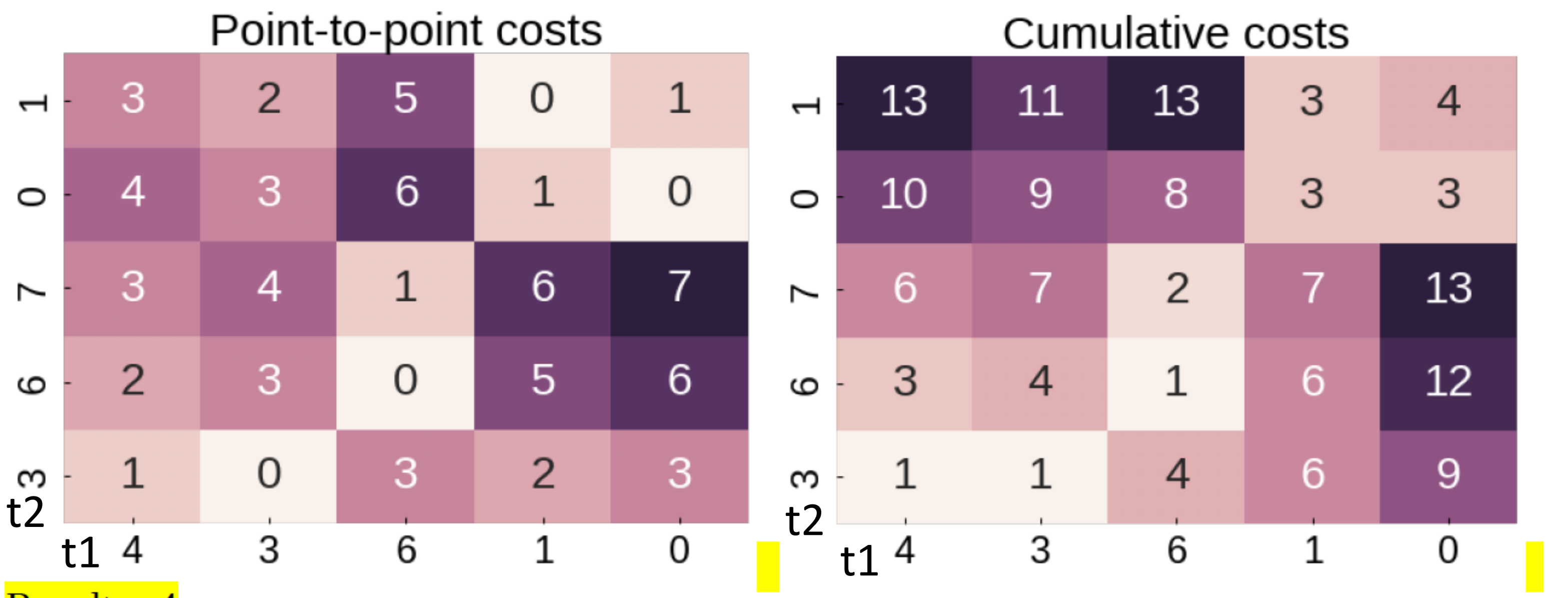


Result: 4

$$\gamma(i,j) \quad = \quad d(q_i,c_j) + \min\{\ \gamma(i\text{-}1,j\text{-}1),\ \gamma(i\text{-}1,j\ ),\ \gamma(i,j\text{-}1)\ \}$$

DTW – Example

t1	< 4, 3, 6, 1, 0 >
t2	< 3, 6, 7, 0, 1 >

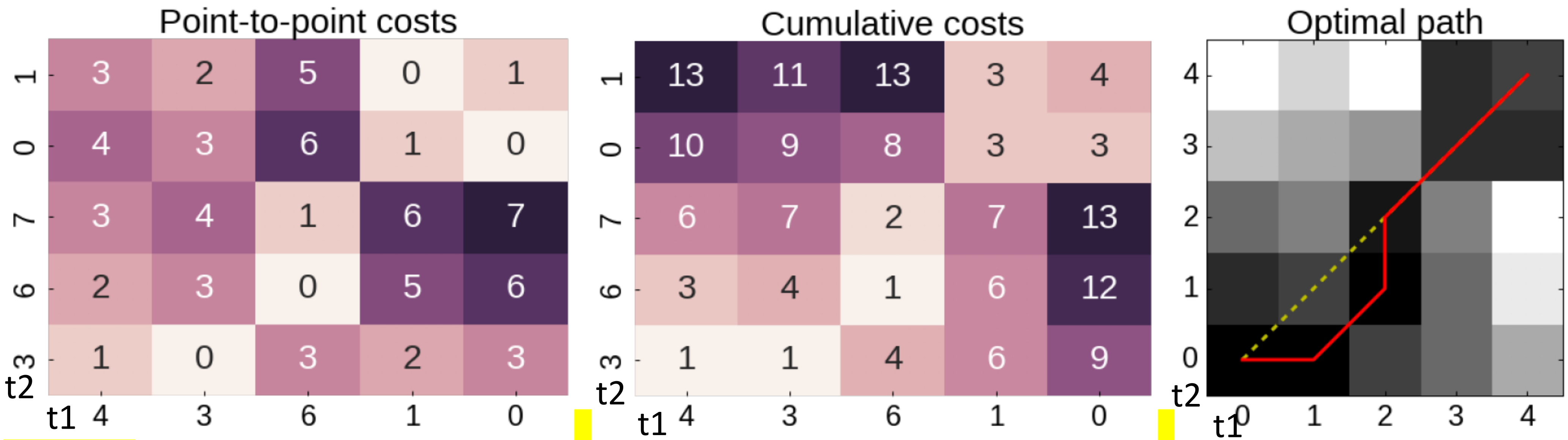


Result: 4

$$\gamma(i,j) \quad = \quad d(q_i,c_j) + \min\{\, \gamma(i-1,j-1),\ \gamma(i-1,j\,),\ \gamma(i,j-1)\,\}$$

DTW – Example

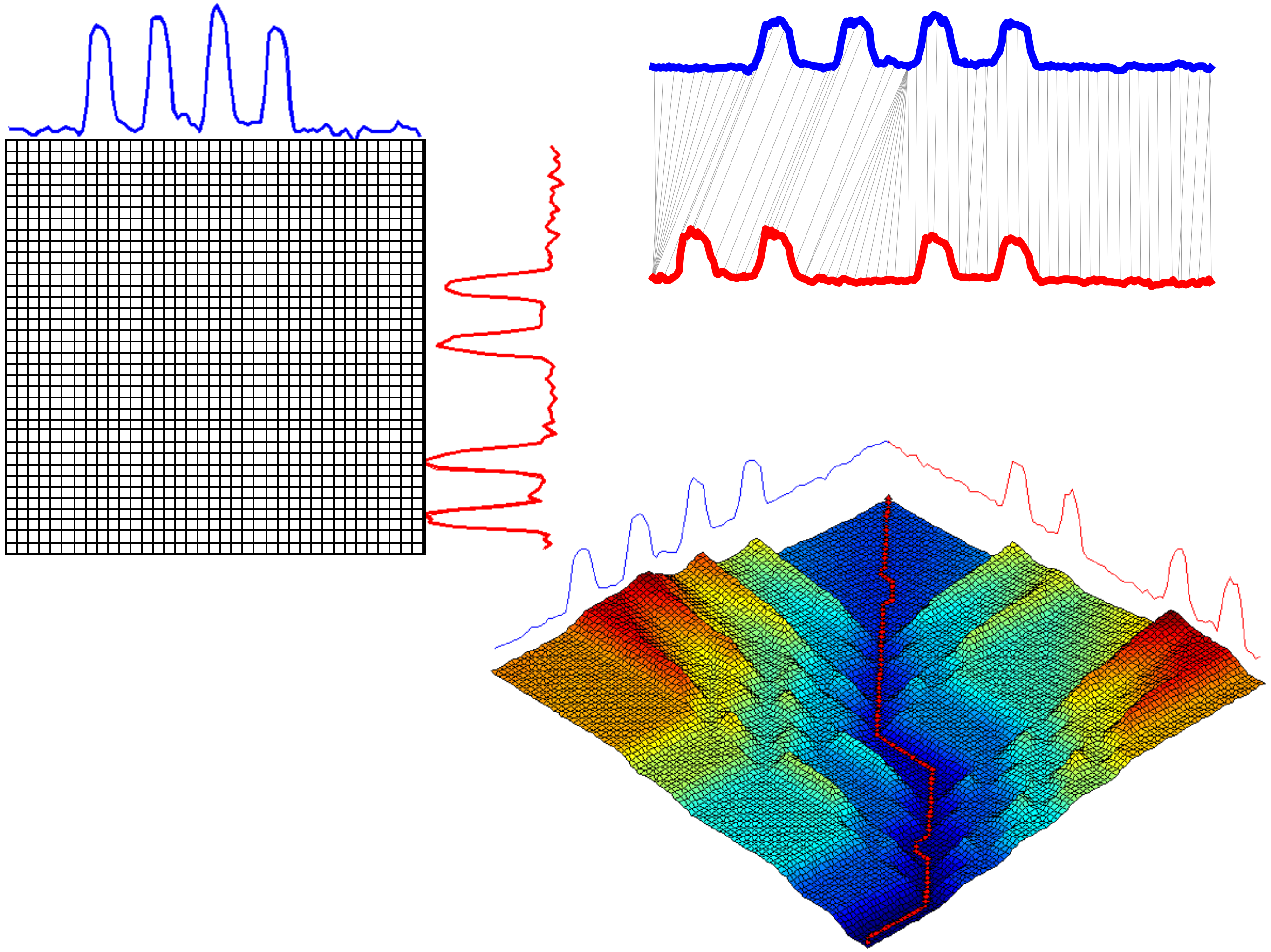
t1	< 4, 3, 6, 1, 0 >
t2	< 3, 6, 7, 0, 1 >



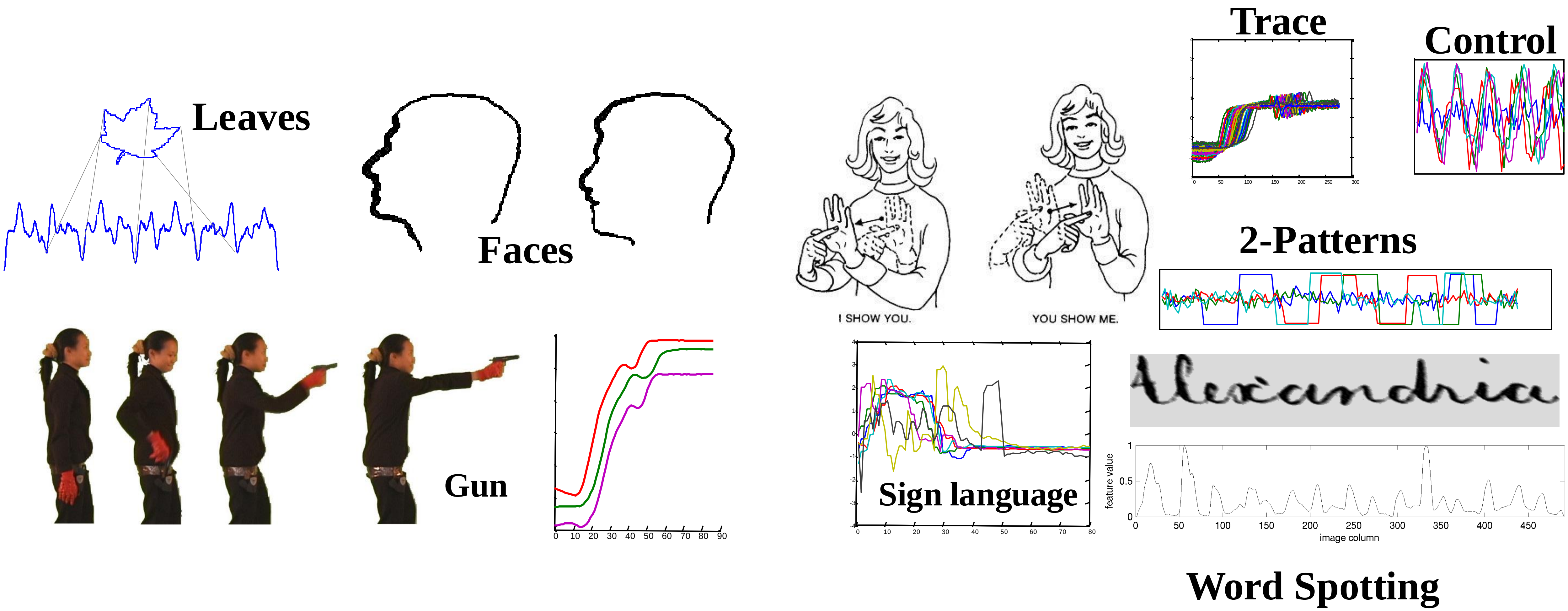
Result: 4

DTW – A Real Example

- This example shows 2 one-week periods from the power demand time series.
- Note that although they both describe 4-day work weeks, the blue sequence had Monday as a holiday, and the red sequence had Wednesday as a holiday.



Comparison of Euclidean Distance and DTW



Comparison of Euclidean Distance and DTW

- Classification using 1-NN
- Class(x) = class of most similar training object
- Leaving-one-out evaluation
- For each object: use it as test set, return overall average

Accuracy

Dataset	Euclidean	DTW
Word Spotting	0.95	0.99
Sign language	0.71	0.74
GUN	0.95	0.99
Nuclear Trace	0.89	1.00
Leaves#	0.67	0.96
(4) Faces	0.94	0.97
Control Chart*	0.93	1.00
2-Patterns	0.99	1.00

Comparison of Euclidean Distance and DTW

- Classification using 1-NN
- Class(x) = class of most similar training object
- Leaving-one-out evaluation
- For each object: use it as test set, return overall average
- DTW is two to three orders of magnitude slower than Euclidean distance.

Milliseconds

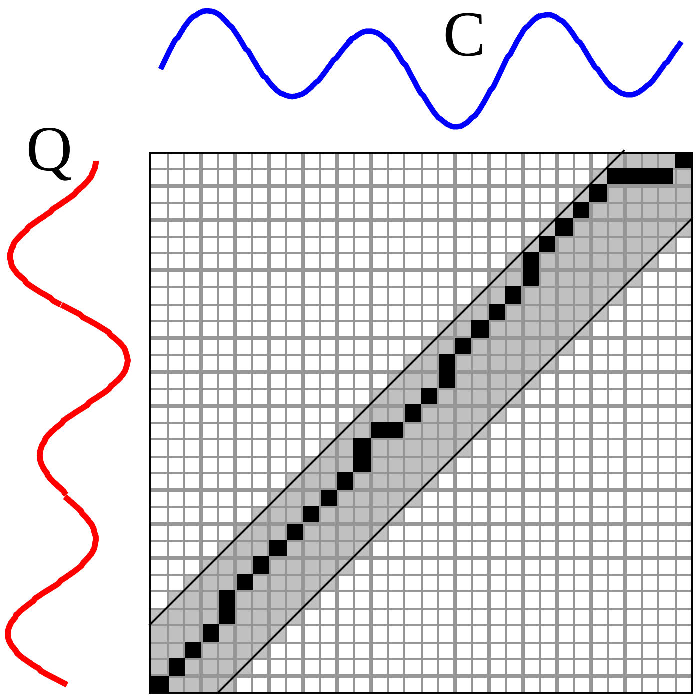
Dataset	Euclidean	DTW
Word Spotting	40	8,600
Sign language	10	1,110
GUN	60	11,820
Nuclear Trace	210	144,470
Leaves	150	51,830
(4) Faces	50	45,080
Control Chart	110	21,900
2-Patterns	16,890	545,123

Problems with Dynamic Time Warping

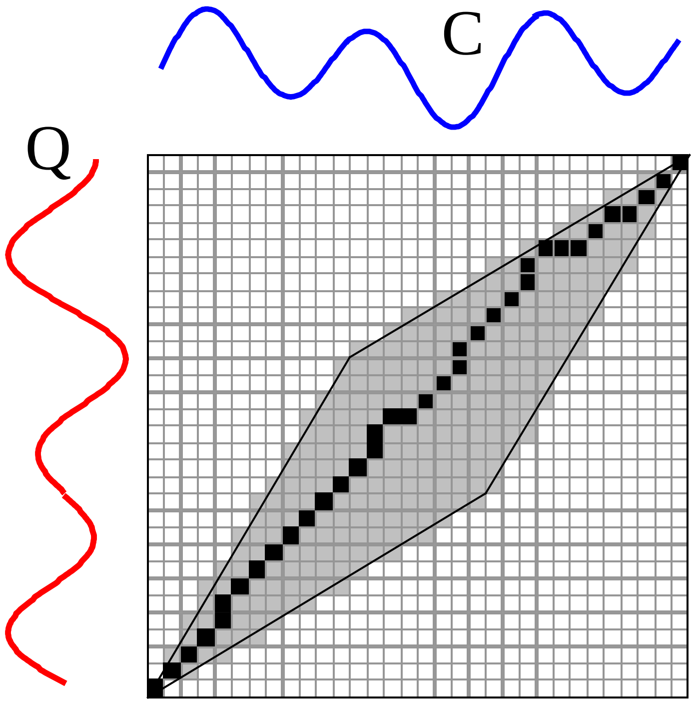
- Dynamic Time Warping gives much better results than Euclidean distance on many problems.
- Dynamic Time Warping is very very slow to calculate!
- Is there anything we can do to speed up similarity search under DTW?

Global Constraints

- Slightly speed up the calculations
- Prevent pathological warpings



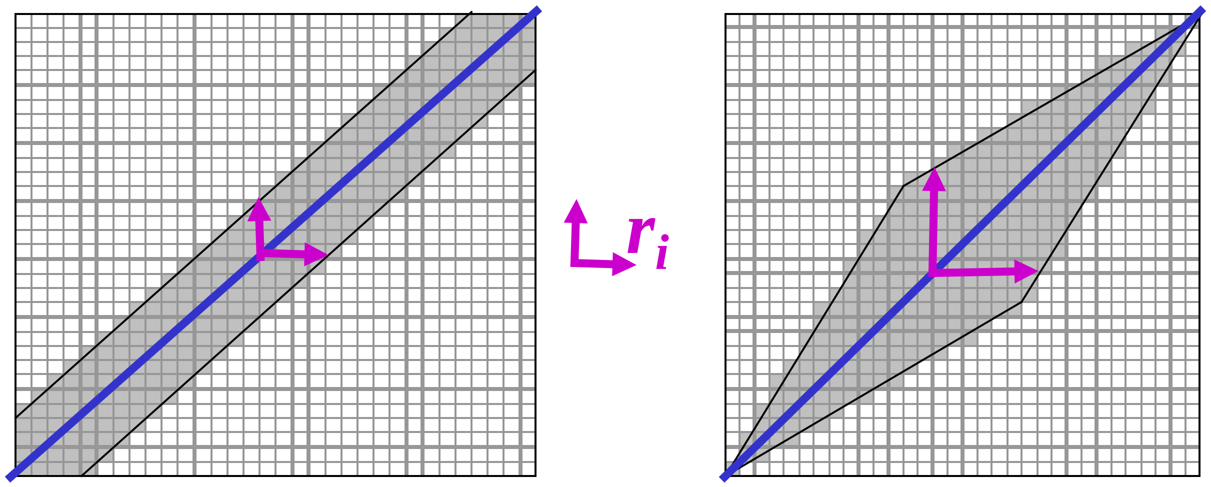
Sakoe-Chiba Band



Itakura Parallelogram

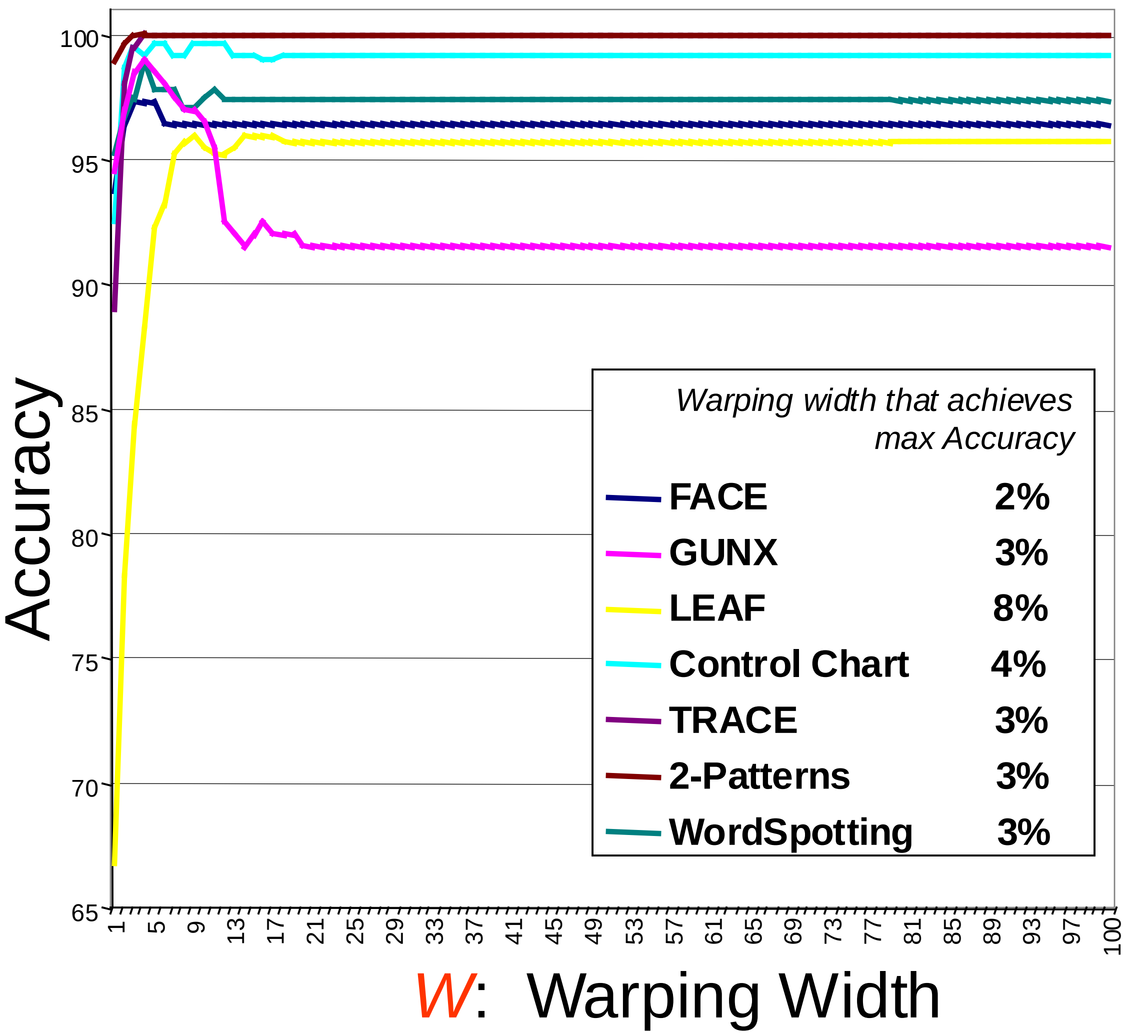
Global Constraints

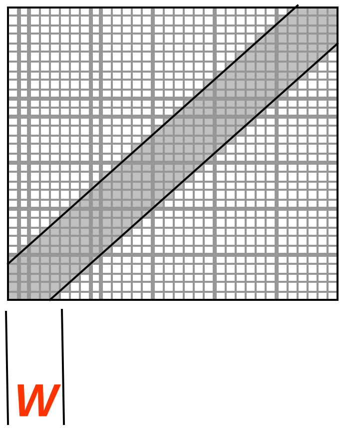
- A global constraint constrains the indices of the warping path $w_k = (i,j)_k$ such that $j\text{-}r \leq i \leq j\text{+}r$, *where* r is a term defining allowed range of warping for a given point in a sequence.
- r can be considered as a *window* that reduces the number of calculus.



Sakoe-Chiba Band Itakura Parallelogram

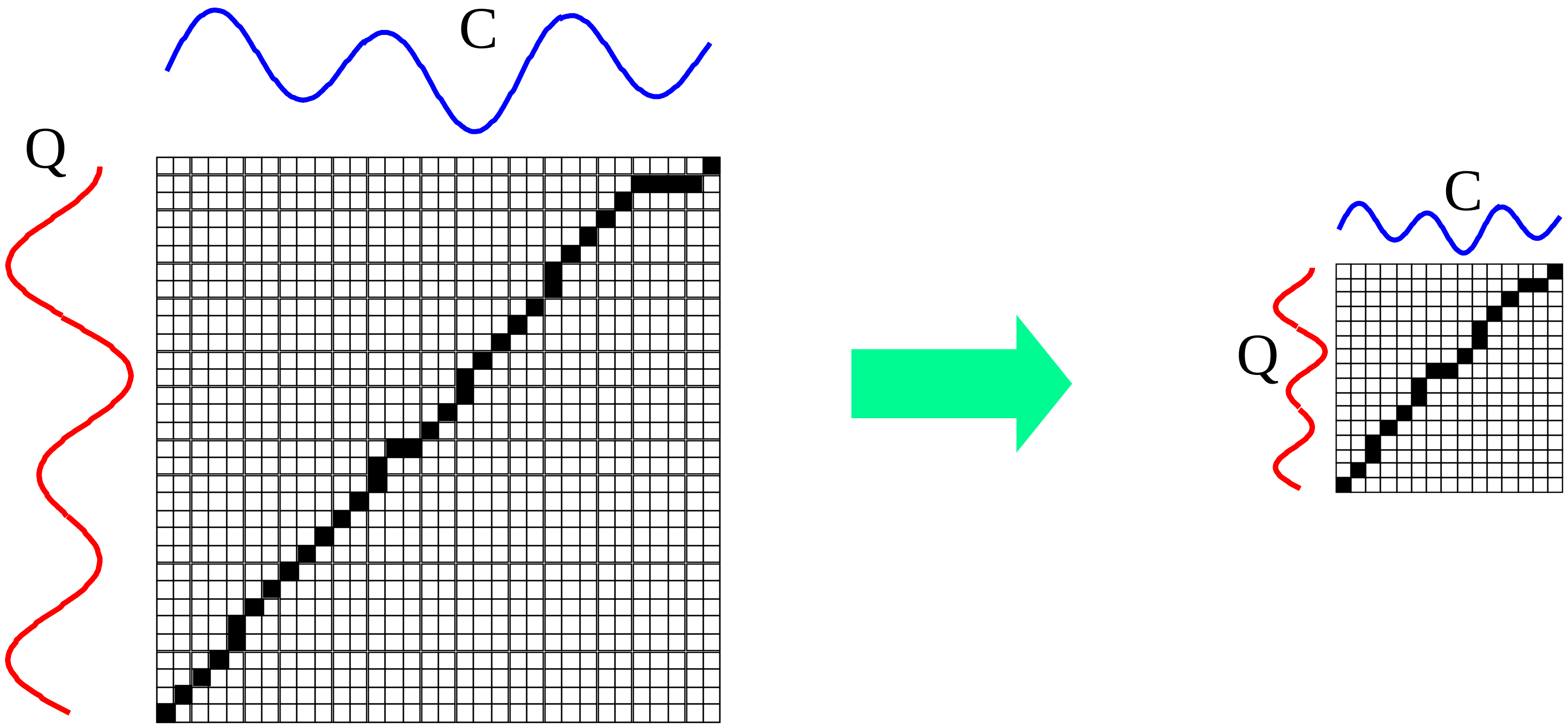
Accuracy vs. Width of Warping Window





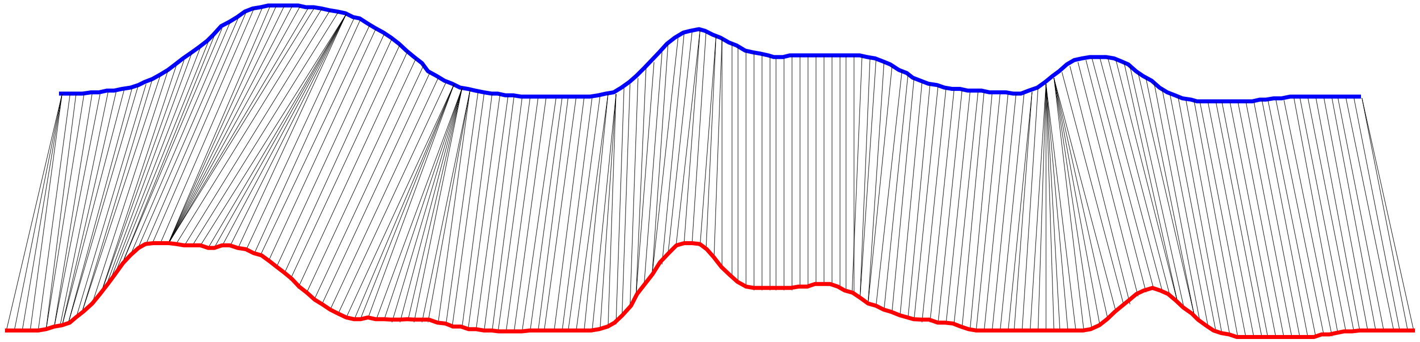
Fast Approximations to DTW

- Approximate the time series with some compressed or downsampled representation and do DTW on the new representation.

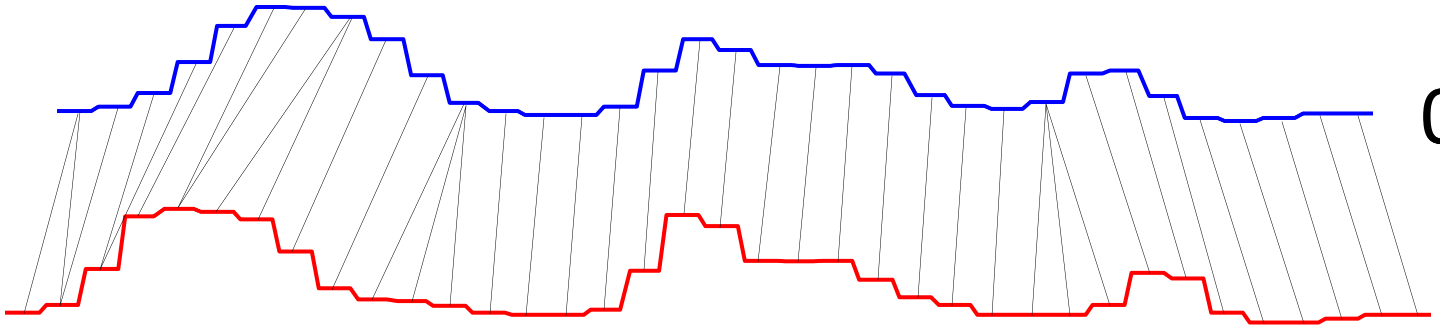


Fast Approximations to DTW

- There is strong visual evidence to suggests it works well
- In the literature there is good experimental evidence for the utility of the approach on clustering, classification, etc.



1.03 sec



0.07 sec

Distances and Normalizations

- If measuring a distance to account for a shape-based similarity it is important to consider the level then the level, i.e., the mean, should not be removed.
- This kind of reasoning applies also to other features of the TS.

Time Series Approximations

Dimensionality Reduction

- Dimensionality reduction is the process of reducing the number of variables under consideration by obtaining a subset of principal variables.
- Dimensionality Reduction approaches can be divided into:
  - **Feature Selection:** variables are selected among the existing ones
  - **Feature Projection:** new variables are created to compactly represent the existing ones.

X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
1.1	1.0	0.3	0.5	0.4	1.8	1.6	1.5	1.3	2.4
1.2	1.2	0.3	0.7	2.1	0.7	3.2	1.9	1.8	3.6
…	…	…	…	…	…	…	…	…	…

X_2	X_8
1.0	1.5
1.2	1.9
…	…

Selection

X_A	X_B
1.8	5.4
1.9	6.3
…	…

Projection

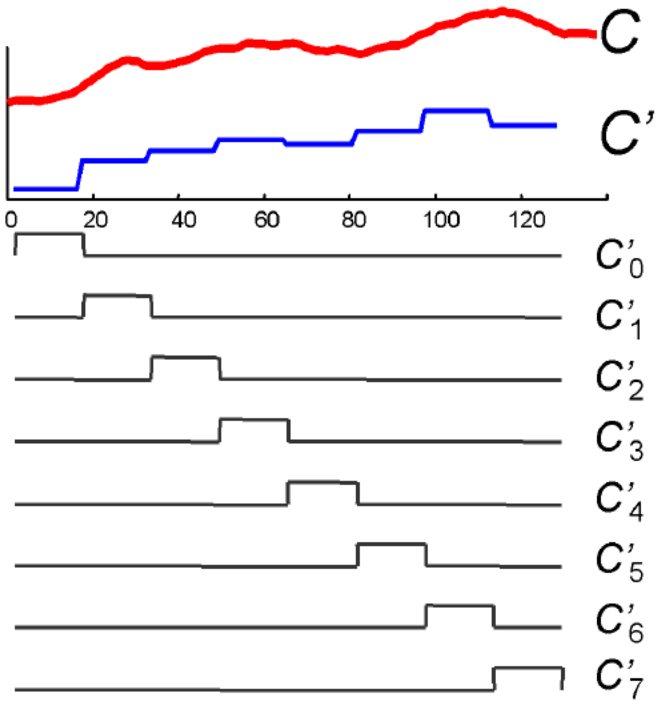
Time Series Approximation

- Time Series Approximation is a special form of Dimensionality Reduction specifically designed for TSs.
- Time Series Approximation consists in representing a TS into a smaller and simpler space that is later used for further calculus (e.g. DTW).
- Approximation vs Compression: the approximated space is always understandable, while the compressed space is not necessarily understandable.
- Approximated representations can be
  - **Time-Dependent**: the approximated values maintain a temporal ordering
  - **Time-Independent**: the approximated values loose the temporal ordering
  - **Instance-wise**: the approximation operation involves only the values of a single TS
  - **Dataset-wise**: the approximation operation involves the values of a dataset of TS

Piecewise Aggregate Approximation (PAA)

- PAA approximates a TS by dividing it into equal-length segments and using the mean value of the data points that fall within the segment as representation.
- PPA represent the TS as a sequence of box basis functions with each box of the same size.
- Given $T = \{x_1, \dots, x_n\}$, PAA reduces T from a vector with n dimensions to a vector $\bar{T} = \{\bar{x}_1, \dots, \bar{x}_N\}$ with N dimensions (with $N < n$) by dividing T into N equi-sized ``frames'' with length n/N where the *i-th* element is calculated as

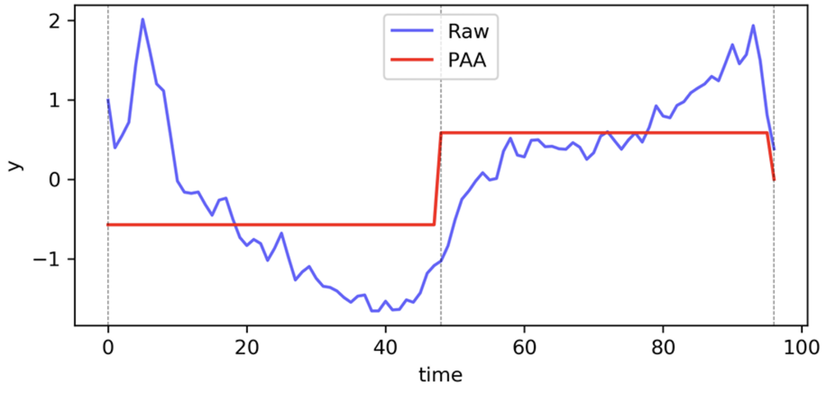
$$\bar{x}_i = \frac{N}{n} \sum_{j=\frac{n}{N}(i-1)+1}^{\frac{n}{N}i} x_j$$



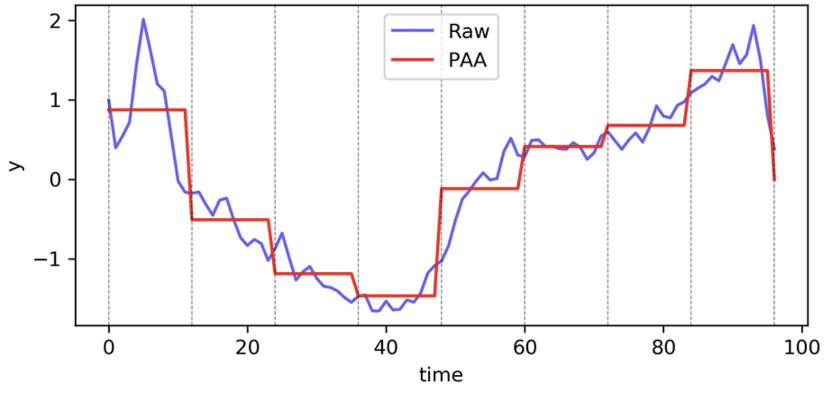
Pros

- Extremely fast to calculate
- Supports non-Euclidean measures
- Supports weighted Euclidean distance

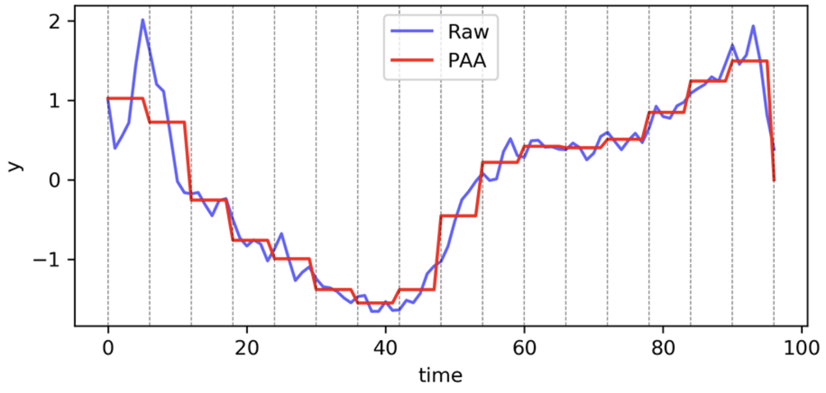
PAA - Example



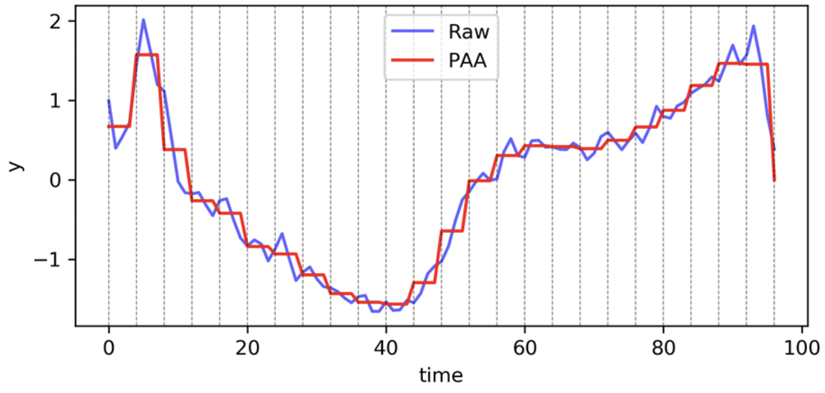
(a) segment $w = 2$.



(b) segment $w = 8$.

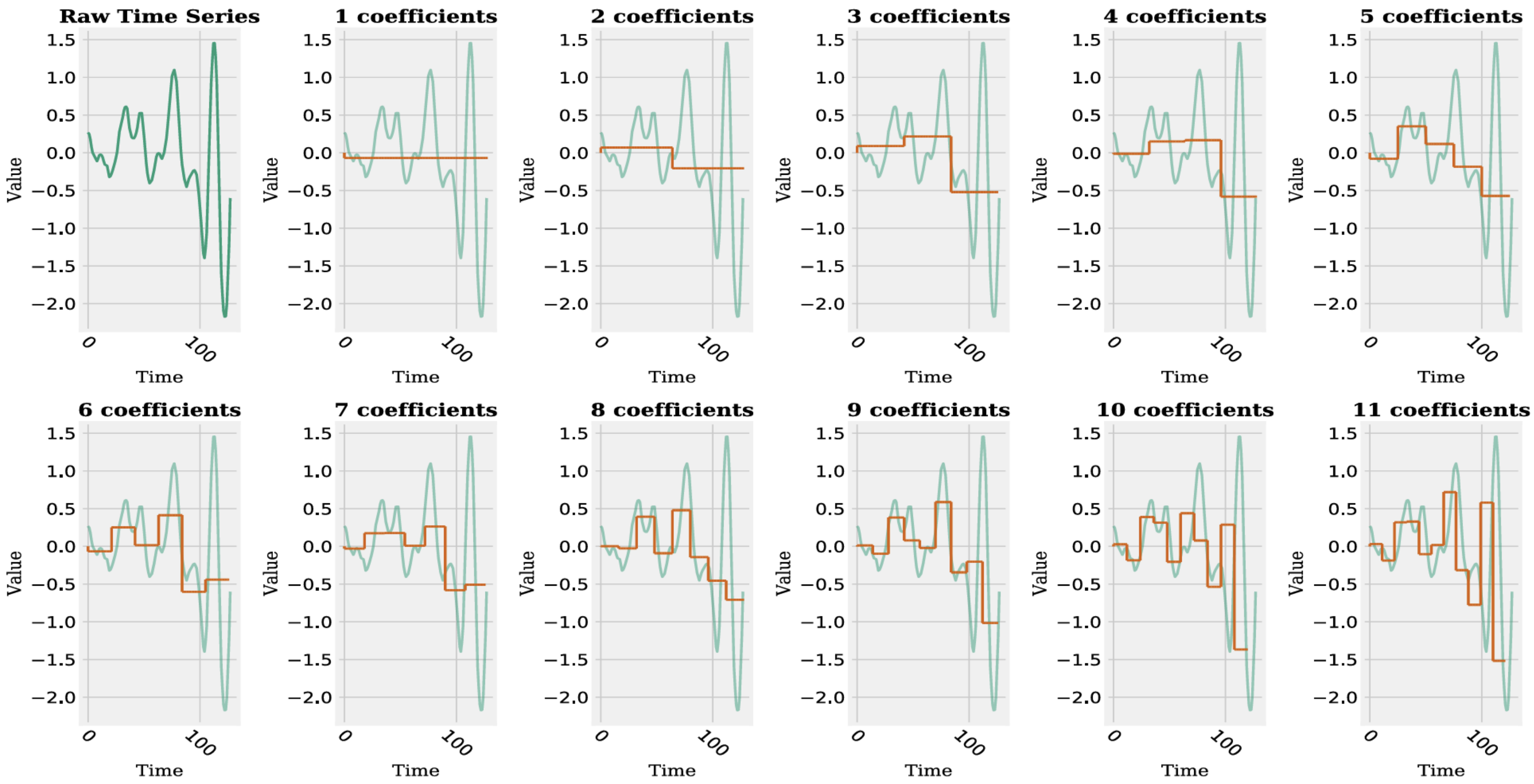


(c) segment $w = 16$.



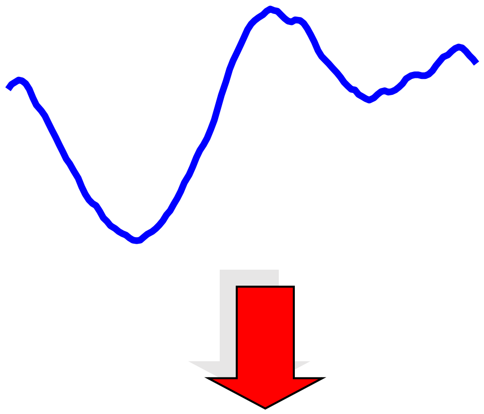
(d) segment $w = 24$.

PAA - Example


Raw Time Series
1 coefficients
2 coefficients
3 coefficients
4 coefficients
5 coefficients
6 coefficients
7 coefficients
8 coefficients
9 coefficients
10 coefficients
11 coefficients
Value
Time
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
0
100

Symbolic Aggregate Approximation (SAX)

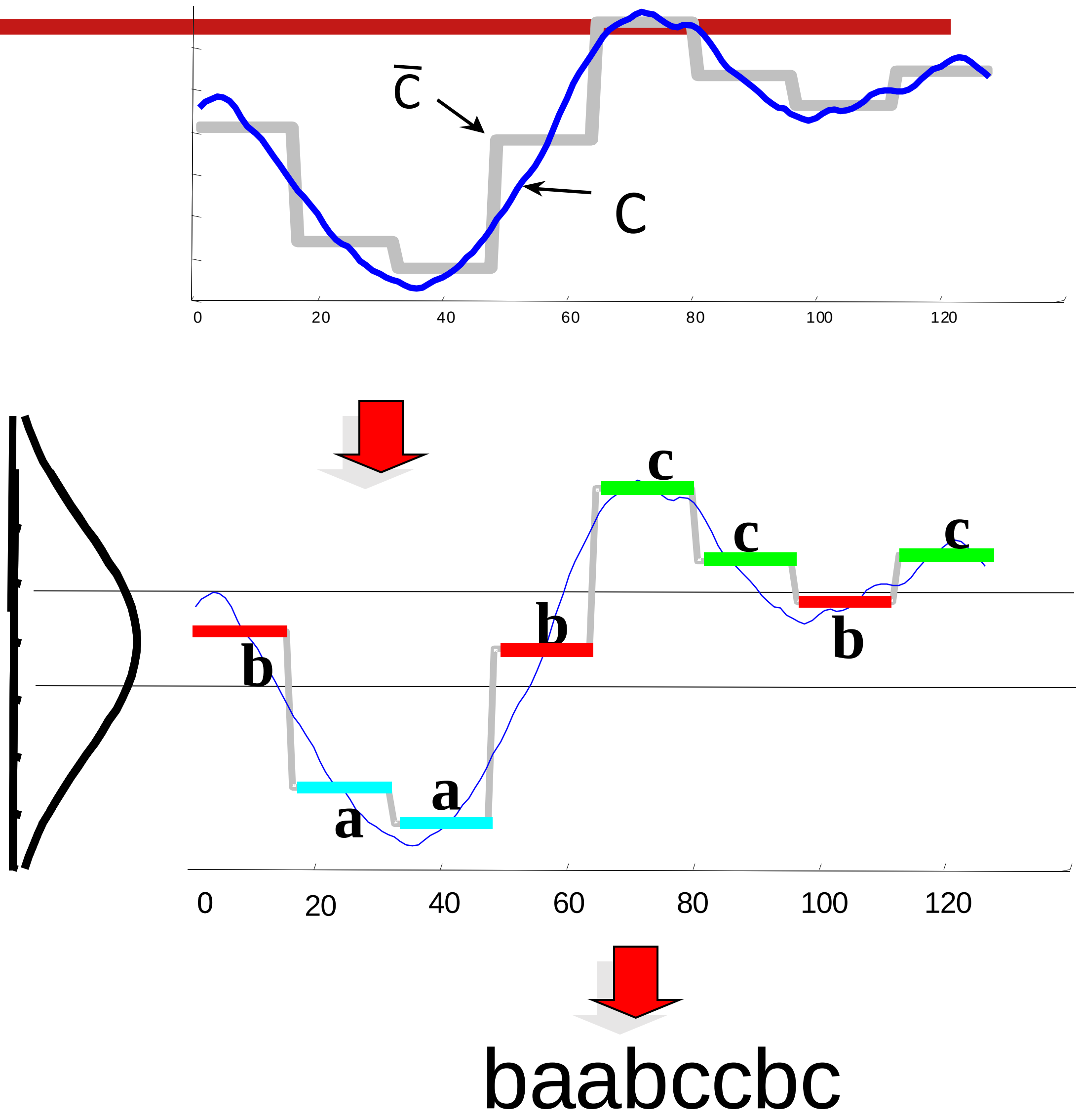
- SAX converts a TS into a discrete format using a small alphabet size such that every part of the representation contributes about the same amount of information about the shape of the TS.
- First converts the time series to PAA representation, then convert the PAA to symbols.



baabccbc

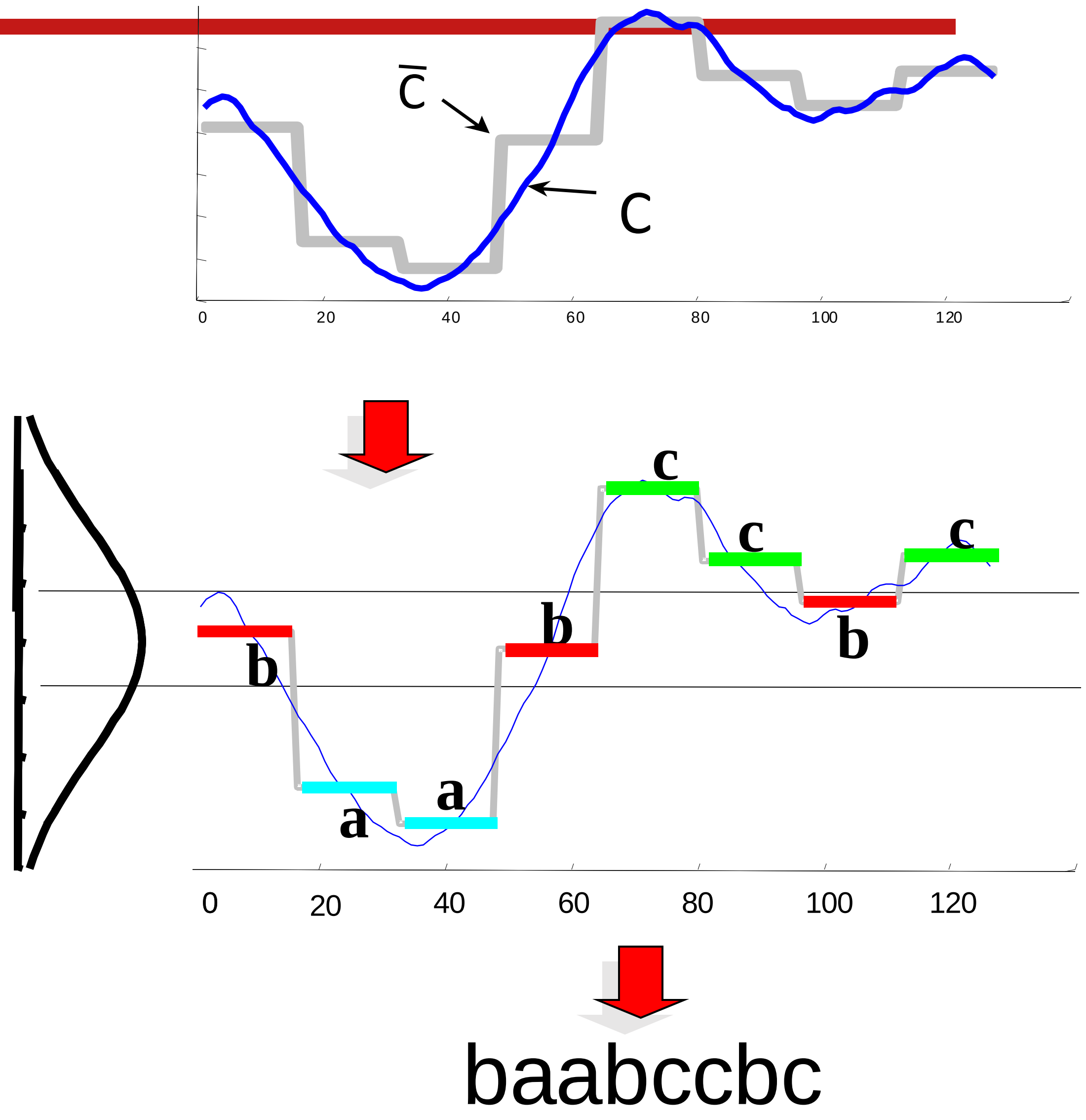
How do we obtain SAX?

- A time series *T* of length *n* is divided into *w* equal-sized segments; the values in each segment are approximated and replaced by their average.
- Next, we determine the breakpoints that divide the distribution space into *a* equiprobable regions, where *a* is the alphabet size specified by the user.
- The breakpoints are determined such that the probability of a segment falling into any of the regions is approximately the same.

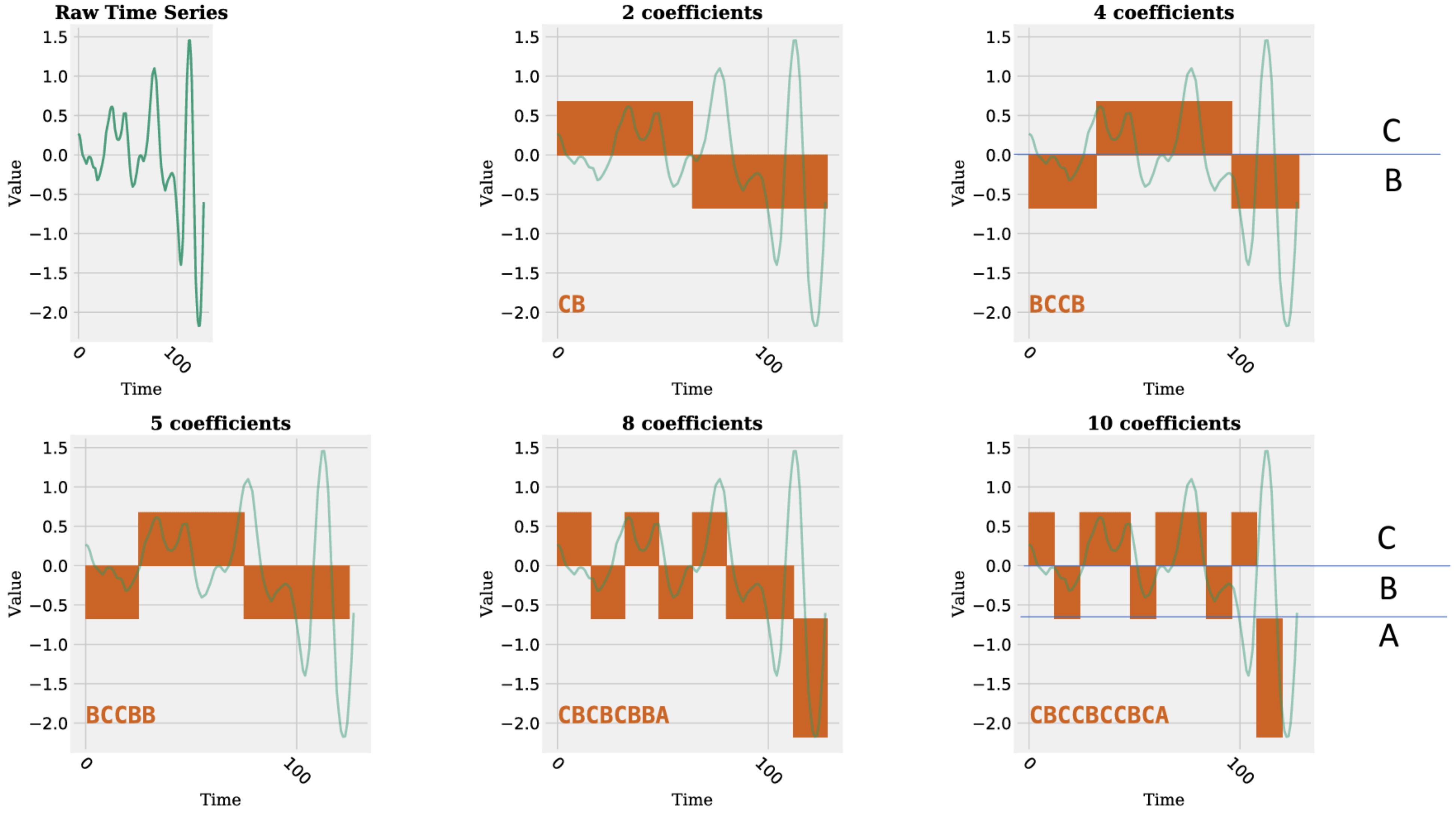


How do we obtain SAX?

- Once the breakpoints are determined, each region is assigned to a symbol and the PAA coefficients are mapped with the symbol corresponding to the region in which they reside.
- The symbols are assigned in a bottom-up fashion, i.e., the PAA coefficient that falls in the lowest region is converted to "a", in the one above to "b", and so forth.

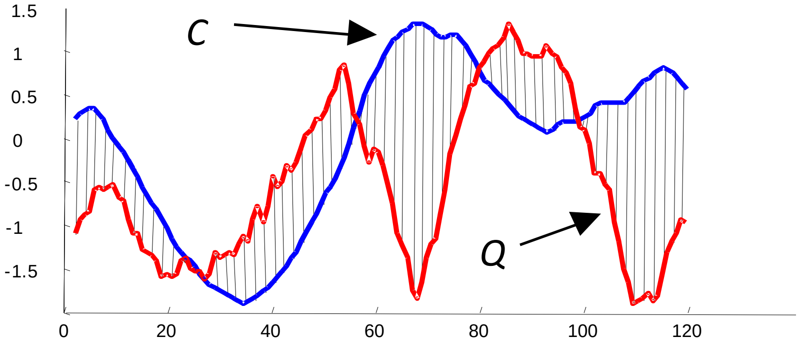


SAX - Example



Raw Time Series
2 coefficients
4 coefficients
5 coefficients
8 coefficients
10 coefficients
Value
Time
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
0
100
CB
BCCB
BCCBB
CBCBCBBA
CBCCBCCBCA
C
B
A

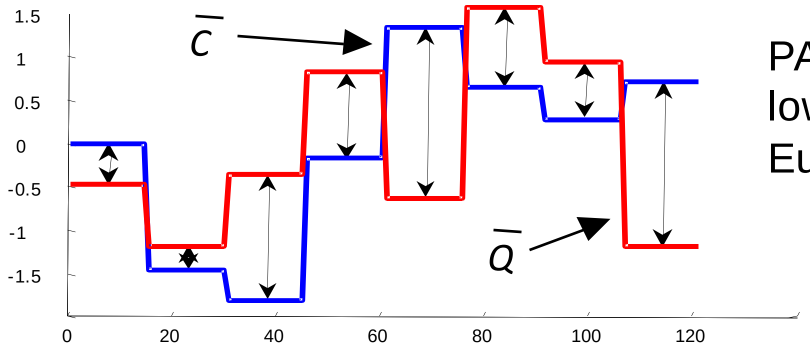
Distances and Approximations



$$D(Q,C) \equiv \sqrt{\sum_{i=1}^{n} (q_i - c_i)^2}$$

Euclidean Distance

$$DR(\overline{Q},\overline{C}) \equiv \sqrt{\tfrac{n}{w}}\sqrt{\sum_{i=1}^{w}\left(\overline{q}_i - \overline{c}_i\right)^2}$$



PAA distance lower-bounds the Euclidean Distance

$\hat{C}$ = **baabccbc**

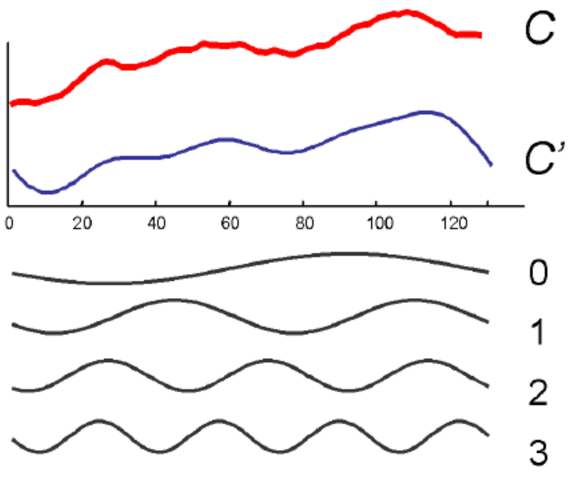
$\hat{Q}$ = **babcacca**

$$MINDIST(\hat{Q},\hat{C}) \equiv \sqrt{\tfrac{n}{w}}\sqrt{\sum_{i=1}^{w}\left(dist(\hat{q}_i,\hat{c}_i)\right)^2}$$

dist() can be implemented using a table lookup or by using for each symbol the mean value of its region.

Discrete Fourier Transform (DFT)

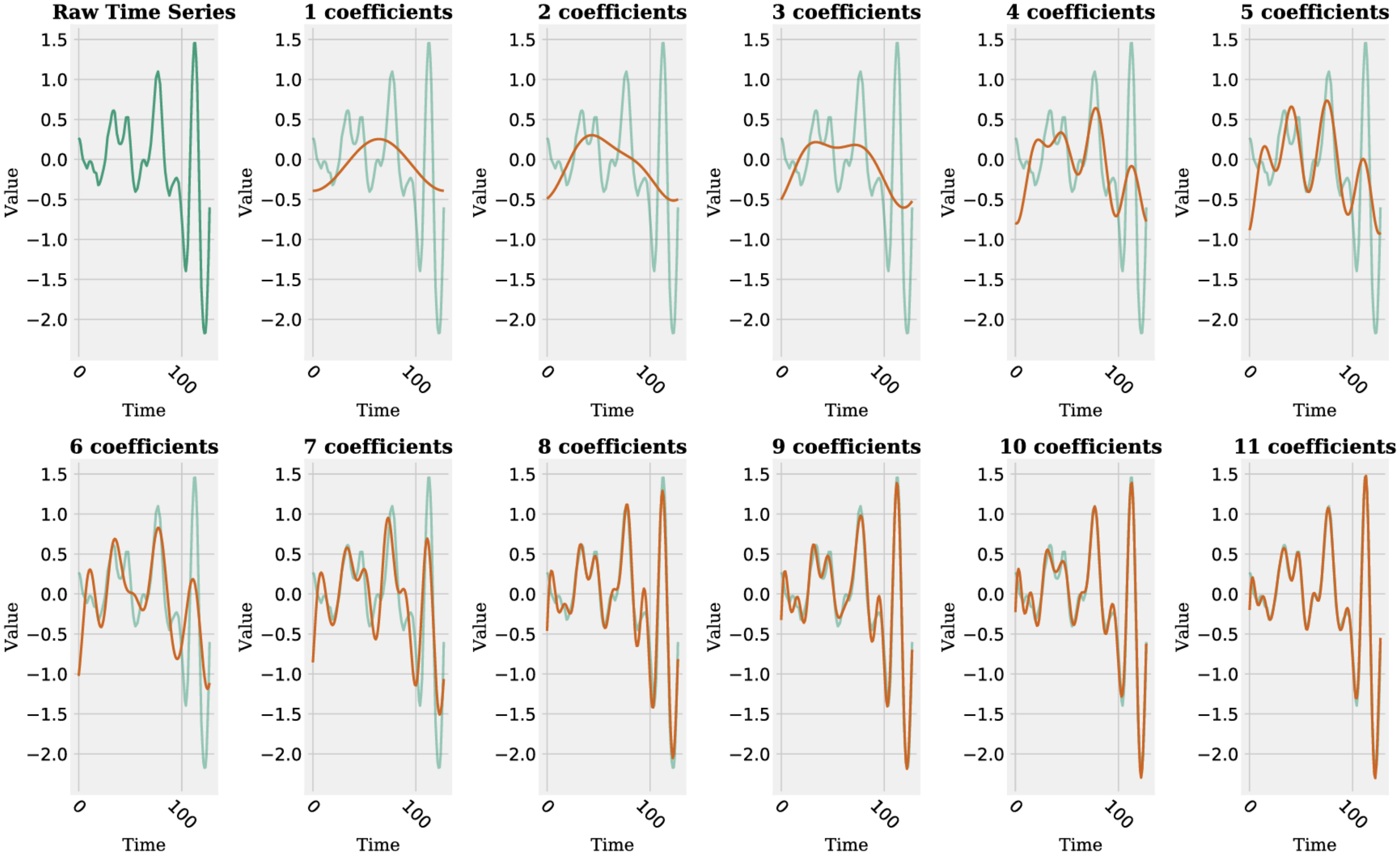
- Represent a TS of length *n* as a linear combination of *w* smooth periodic sinusoidal series (i.e., sines and cosines).
- Each wave is represented by a Fourier coefficient
- This representation is called Frequency Domain
- The DFT concentrates most of its energy in the first few Fourier coefficients
- Low-pass filter: approximate a TS by its first *w* Fourier coefficients.



How do we obtain DFT?

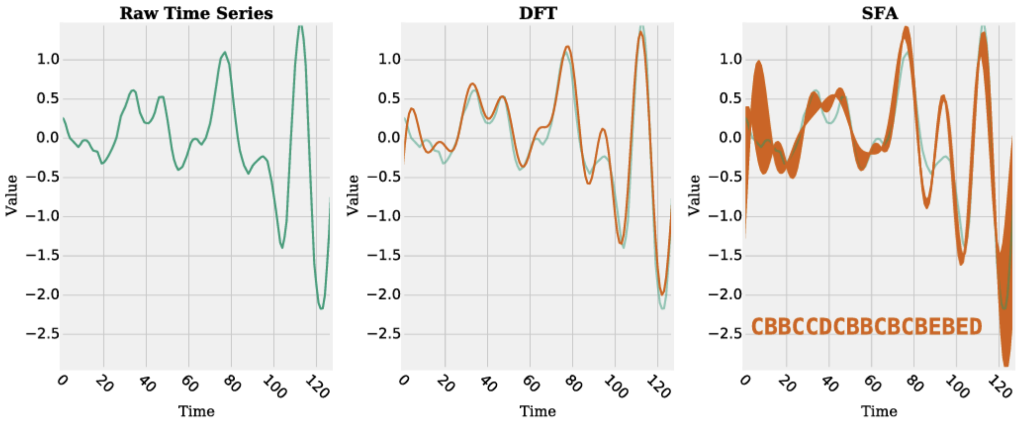
- The DFT decomposes a TS *T* of length *n* into a sum of *n* orthogonal basis functions using sinusoid waves represented with two numbers: amplitude and phase.
- A Fourier coefficient (sinusoid wave) is represented by the complex number: $X_u = (real_u, imag_u)$ $u = 0, 1, \dots, n\text{-}1$
- The *n-th* point DFT of a TS $T = \{x_1, \dots, x_n\}$ *is given by*
- $DFT(T) = X_0, \dots, X_{n-1} = \{real_0, imag_0, \dots, real_{n-1}, imag_{n-1}\}$
- with $X_u = \frac{1}{n} \sum_{i=1}^{n} x_i \, e^{-\frac{j2\pi i}{n}}$ $u \in [0, n), j = \sqrt{-1}$
- The first Fourier coefficient ($X_0 = \frac{1}{n} \sum_{i=1}^{n} x_i \, e^{0}$) is equal to the mean of a TS and can be discarded to obtain offset invariance, i.e., level removal.

DFT - Example


Raw Time Series
1 coefficients
2 coefficients
3 coefficients
4 coefficients
5 coefficients
6 coefficients
7 coefficients
8 coefficients
9 coefficients
10 coefficients
11 coefficients
Value
Time
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
0
100

Symbolic Fourier Approximation (SFA)

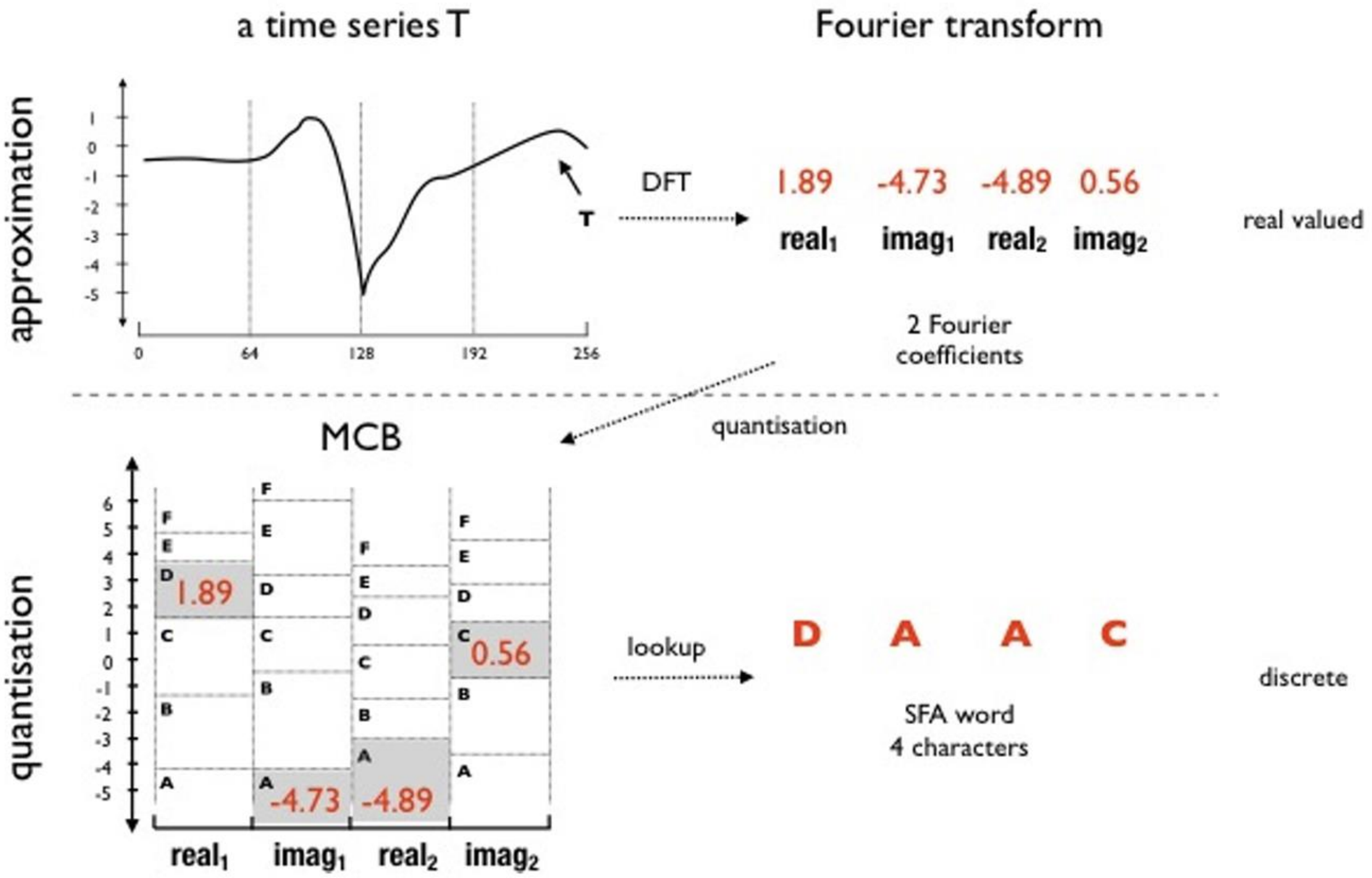
- SFA represents a TS with a word
- SFA is composed of
  - a) approximation using the Fourier transform
  - b) a data adaptive discretization
- The discretization intervals are learned from the Fourier transformed data distribution rather than using fixed intervals with Multiple Coefficient Binning (MCB)



Raw:	DFT	Discretization
0.2679	0	C
0.2480	-8.81	B
0.1828	-20.7	B
0.0817	-11.9	C
0.0051	-6.28	C
-0.023	-8.02	D
-0.052	-0.67	C
-0.082	15.31	B
-0.111	-18.7	B
-0.075	-18.36	C
-0.032	-5.67	B
-0.022	-16.84	C
-0.029	-8.919	B
[...]	[...]	[...]

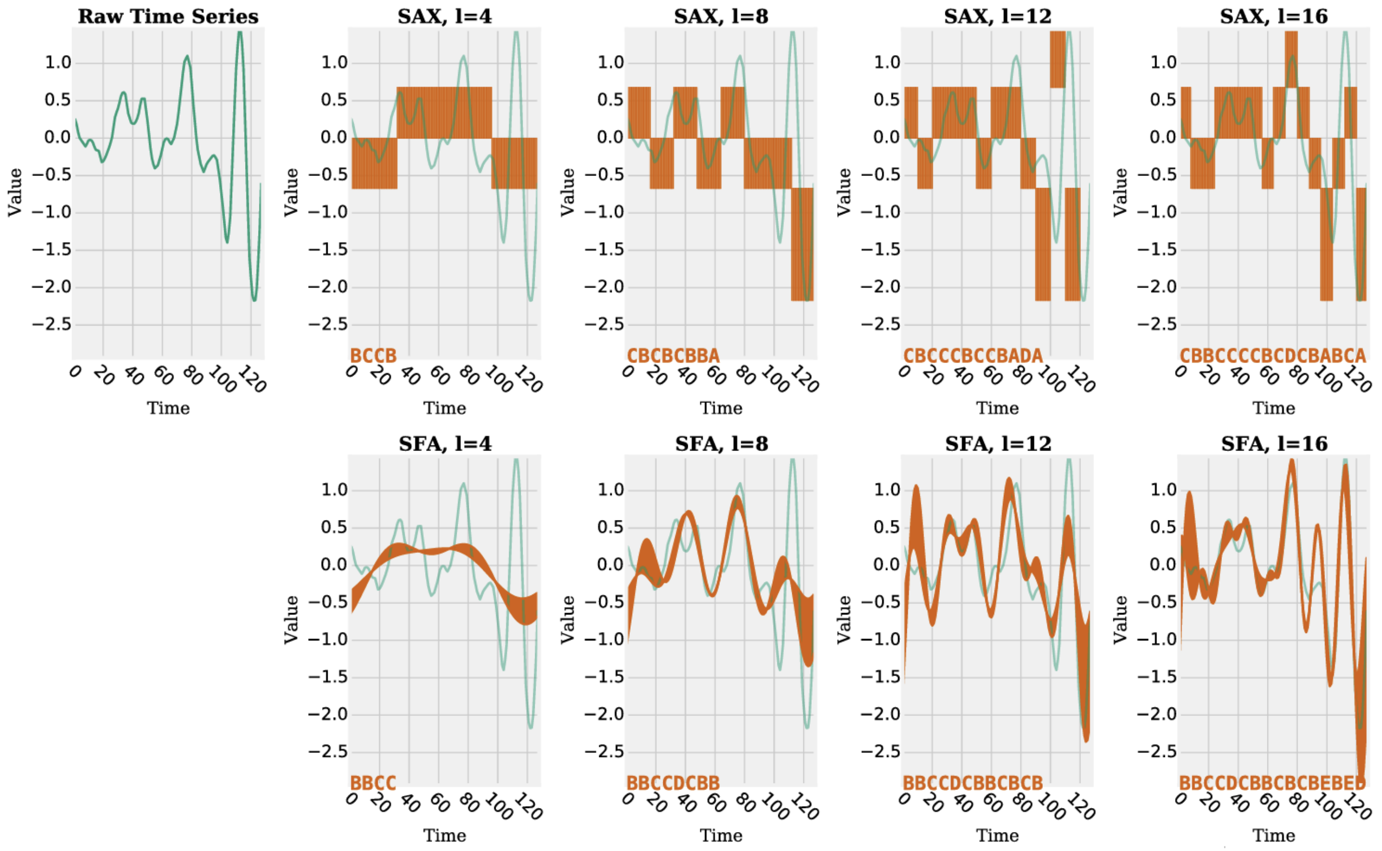
SFA Discretization

- The objective of MCB is to minimize the loss of information introduced by discretization.
- This is achieved by applying a discretization for each coefficient.
- MCB uses different breakpoints for each symbol on each coefficient.



Comparison of SFA and SAX

- Roughly we can say
  - SAX = PAA + Discretization
  - SFA = DFT + Discretization
- Approximation cause a loss of information
- Discretization augments the level of information loss.
- The higher the number of symbols and the alphabet size, the more exact is the representation.

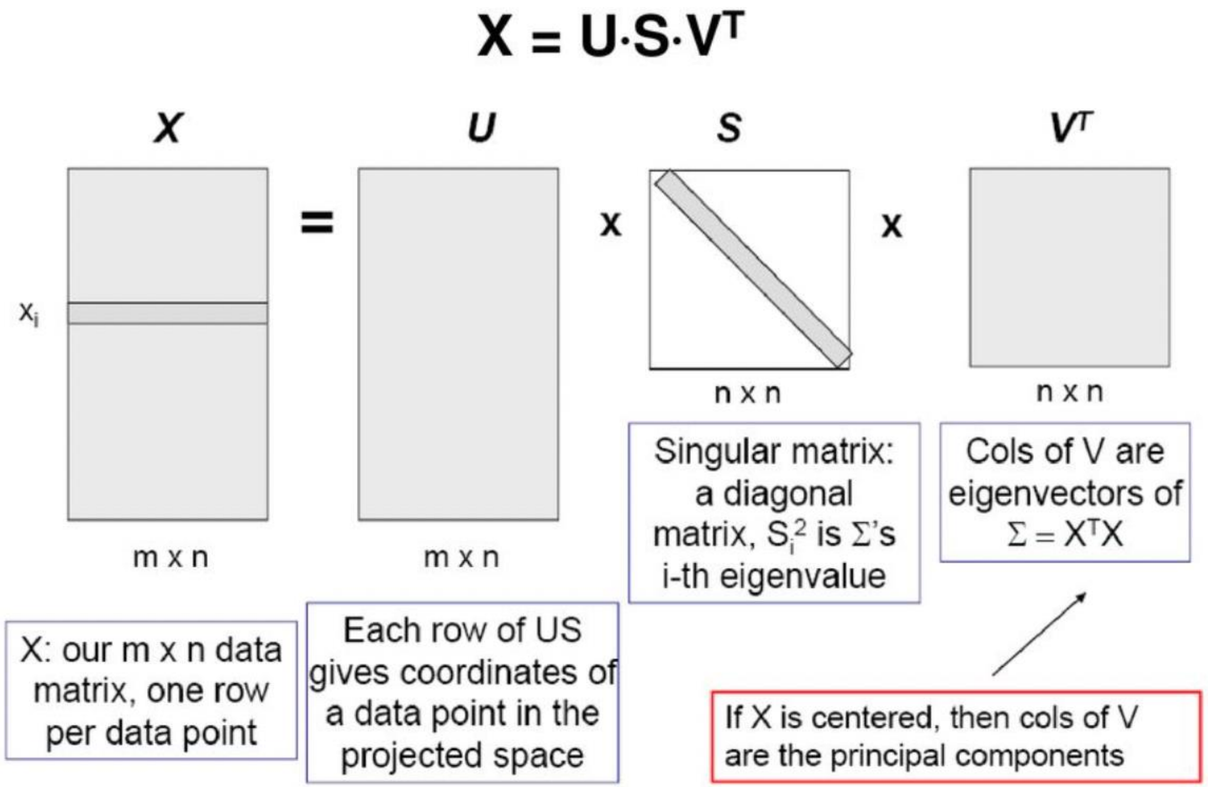


Properties of Symbolic Representations

- Noise removal: discretization
- String representations: allows for string domain algorithms like hashing or the bag-of-words to be applied
- Dimensionality reduction: allow for indexing high dimensional data
- Storage reduction: sequences have a much lower memory footprint than real-valued time series

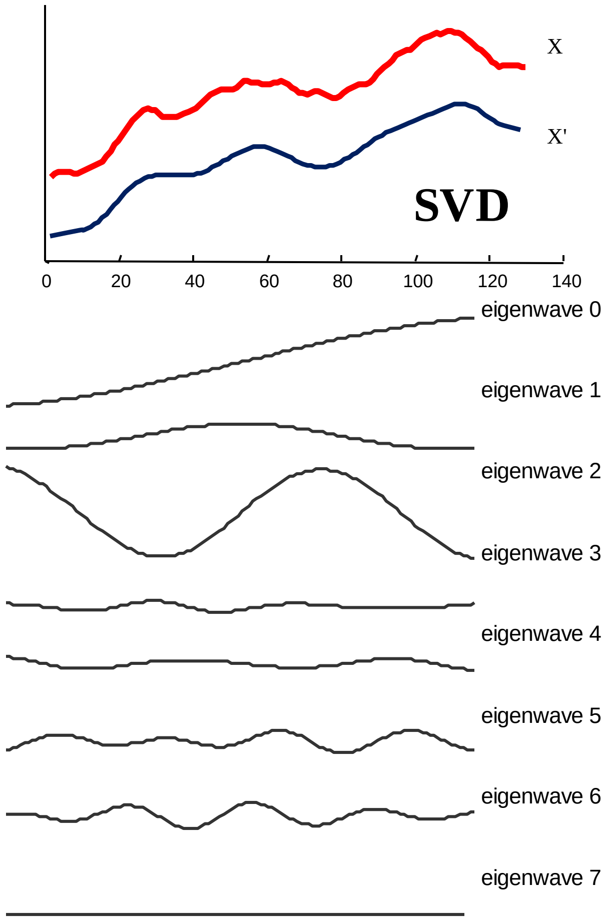
Singular Value Decomposition (SVD)

- SVD is a factorization method that decomposes a matrix into three other matrices: U, S, and V^T (transpose of V):
- $X = U\ S\ V^T$
- Input matrix X (m x n)
- U (m x m) contains orthogonal columns that represent the left singular vectors.
- S (m x n) is a diagonal matrix containing the singular values.
- V^T (n x n) contains orthogonal rows representing the right singular vectors.

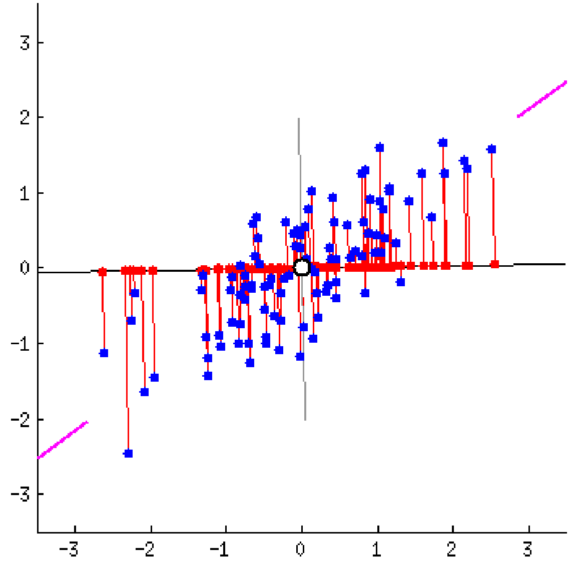


SVD for Time Series Approximations

- SVD is similar to DFT and SFT in that it represents the shape in terms of a linear combination of basis shapes.
- DFT and SFT are ***individual approximations*** as they examine one TS at a time. They are completely independent of the rest of the data.
- SVD is a ***global approximations*** as the entire dataset is examined and is then rotated such that the first axis has the maximum possible variance, the second axis has the maximum possible variance orthogonal to the first, the third axis has the maximum possible variance orthogonal to the first two, etc.
- The global nature of SVD is both a weakness and a strength.



Principal Component Analysis (PCA)



- PCA is a statistical procedure that aims to transform data into a new coordinate system where the axes are the principal components. These components are orthogonal and capture the maximum variance in the data.

1. **Standardize the Data**: Normalize the data X to have zero mean and unit variance matrix C.
2. **Calculate Covariance Matrix**: Calculate the covariance matrix $\Sigma = C^T C$ of the standardized data.
3. **Compute Eigenvectors and Eigenvalues**: Compute the eigenvectors and eigenvalues of the covariance matrix Σ.
4. **Select Principal Components**: Sort eigenvalues in descending order and choose the top-k eigenvalues to form principal components.
5. **Transform Data**: Project the original data X into the principal components to create a lower-dimensional representation.

Relationships between SVD and PCA

Dealing with data

- PCA primarily deals with the covariance structure of the data.
- SVD does not rely on a covariance matrix. It is a factorization that decomposes the original data without computing covariance.

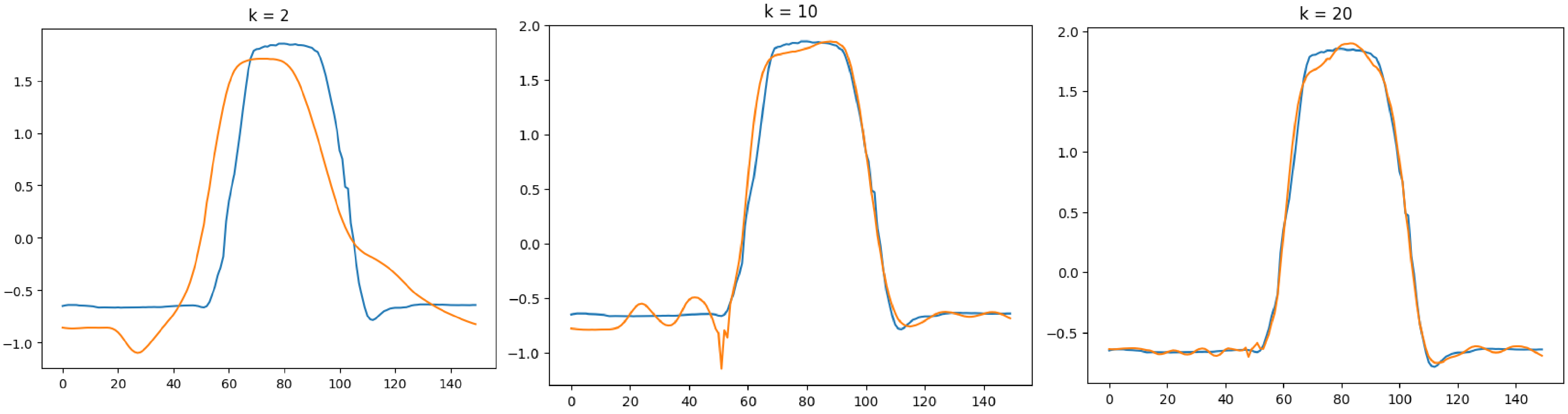
Computations

- Both PCA and SVD involve eigen-decomposition.
- PCA performs eigen-decomposition on the covariance matrix of the data which is a square symmetric matrix of size *m x m* where *m* is the number of time stamp.
- SVD performs eigen-decomposition on the data matrix itself of size *n x m* where *n* is the number of TS and m is the number of number of time stamps.

Relationships between SVD and PCA

- **PCA is a specific application of SVD**, primarily used for dimensionality reduction, while SVD is a more general matrix decomposition technique with broader applications in linear algebra and data analysis.
- PCA can be solved using SVD
- PCA focuses on the covariance structure and tries to maximize variance along orthogonal axes
- SVD focuses on matrix factorization and can handle cases where data is missing.
- **From an application perspective they are used interchangeably**.

PCA - Example



Remarks on Approximations

- None of the representations can be superior for all tasks.
- No research as proved how one should choose the best representation for the problem at hand and data of interest.
- The literature is not even consistent on nomenclature.
  - Example: Piecewise Aggregate Approximation
    - Piecewise Flat Approximation (Faloutsos et al., 1997)
    - Piecewise Constant Approximation
    - Segmented Means

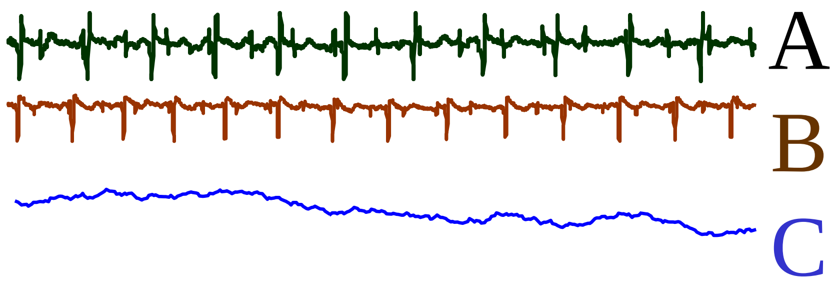
Approximations, Distances and Normalizations

- It does not make any sense to use a distance function accounting for time like DTW if a Time-Independent approximation is used.
- Thus, do not use DTW after DTF, SFA, SVD, PCA.
- It does make sense to use DTW after Time-Dependent approximations.
- Thus, you can use DTW after PAA or SAX.
- Normalizations can be applied before and/or after approximations depending on the objective of your TSA task.
- Time series normalizations does not make any sense after a Time-Independent approximation.
- In that case traditional “column-wise” normalizations like Min-Max scaling or Z-Score normalization should be used.

Global Structural Features

Structure-based Similarity

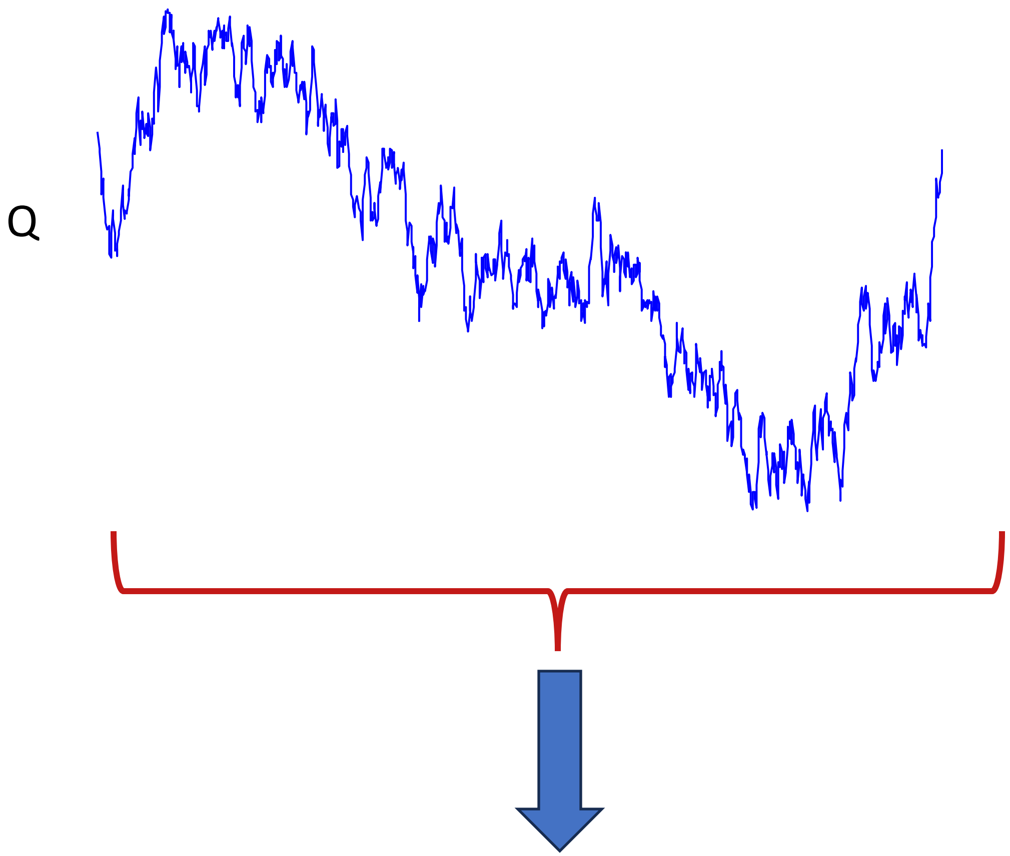
- For long time series, shape-based similarity typically give poor results.
- Structure-based similarity measure similarly of TS based on high level structure.
- The basic idea is to:
  1. extract ***global*** features from the time series,
  2. create a feature vector, and
  3. use it to measure similarity with Euclidean distance
- Example of features:
  - mean, variance, skewness, kurtosis,
  - 1^{st} derivative mean, 1^{st} derivative variance, …
  - parameters of regression, forecasting, Markov model



Feature\Time Series	A	B	C
Max Value	11	12	19
Mean	5.3	6.4	4.8
Min Value	3	2	5
Autocorrelation	0.2	0.3	0.5
…	…	…	…

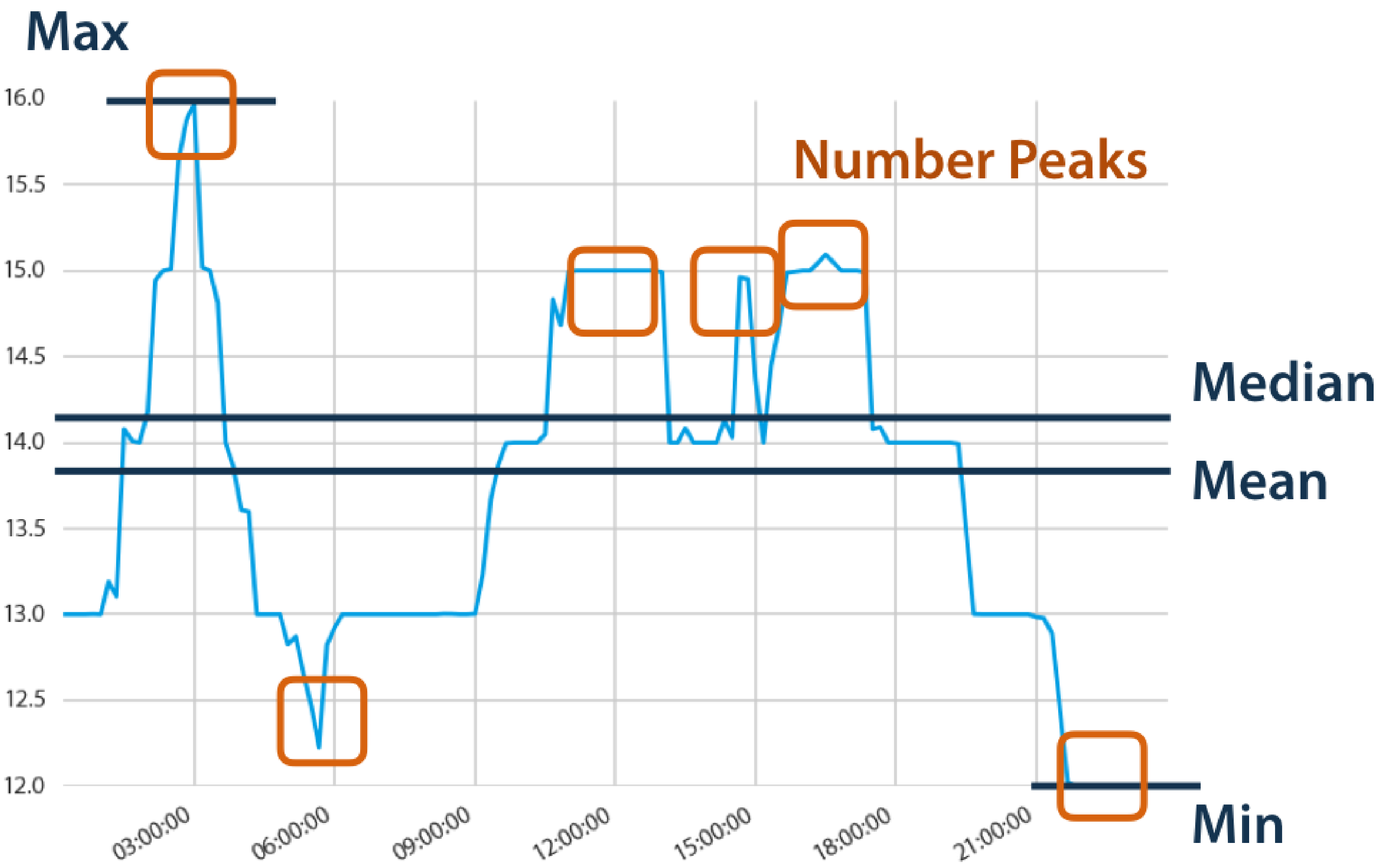
Simple Standard Features

- Mean
- Standard Deviation
- Variance
- Median
- 10th Percentile
- 25th Percentile
- 75th Percentile
- 90th Percentile
- IQR
- Covariance
- Skewness
- Kurtosis
- Min
- Max



	mean	std	var	med	…	max
Q	1.8	2.9	8.4	1.3	…	

TSFresh Features



TSFresh Features

- abs_energy - Returns the absolute energy of the time series which is the sum over the squared values
- absolute_maximum - Calculates the highest absolute value of the time series x.
- absolute_sum_of_changes - Returns the sum over the absolute value of consecutive changes in the series x
- agg_autocorrelation - Descriptive statistics on the autocorrelation of the time series.
- agg_linear_trend - Calculates a linear least-squares regression for values of the time series that were aggregated over chunks versus the sequence from 0 up to the number of chunks minus one.
- approximate_entropy - Implements a vectorized Approximate entropy algorithm.
- ar_coefficient - This feature calculator fits the unconditional maximum likelihood of an autoregressive AR(k) process.
- augmented_dickey_fuller - Does the time series have a unit root?
- autocorrelation - Calculates the autocorrelation of the specified lag
- benford_correlation - Useful for anomaly detection applications. Returns the correlation from first digit distribution when
- binned_entropy - First bins the values of x into max_bins equidistant bins.
- c3 - Uses c3 statistics to measure non linearity in the time series
- change_quantiles - First fixes a corridor given by the quantiles ql and qh of the distribution of x.
- cid_ce - This function calculator is an estimate for a time series complexity.
- count_above - Returns the percentage of values in x that are higher than t
- count_above_mean - Returns the number of values in x that are higher than the mean of x
- count_below - Returns the percentage of values in x that are lower than t
- count_below_mean - Returns the number of values in x that are lower than the mean of x
- cwt_coefficients - Calculates a Continuous wavelet transform for the Ricker wavelet, also known as the "Mexican hat wavelet" which is defined by
- energy_ratio_by_chunks - Calculates the sum of squares of chunk i out of N chunks expressed as a ratio with the sum of squares over the whole series.
- fft_aggregated - Returns the spectral centroid (mean), variance, skew, and kurtosis of the absolute fourier transform spectrum.

TSFresh Features

- fft_coefficient - Calculates the fourier coefficients of the one-dimensional discrete Fourier Transform for real input by fast fourier transformation algorithm
- first_location_of_maximum - Returns the first location of the maximum value of x.
- first_location_of_minimum - Returns the first location of the minimal value of x.
- fourier_entropy - Calculate the binned entropy of the power spectral density of the time series (using the welch method).
- friedrich_coefficients - Coefficients of polynomial, which has been fitted to the deterministic dynamics of Langevin model
- has_duplicate - Checks if any value in x occurs more than once
- has_duplicate_max - Checks if the maximum value of x is observed more than once
- has_duplicate_min - Checks if the minimal value of x is observed more than once
- index_mass_quantile - Calculates the relative index i of time series x where q% of the mass of x lies left of i.
- kurtosis - Returns the kurtosis of x.
- large_standard_deviation - Does time series have large standard deviation?
- last_location_of_maximum - Returns the relative last location of the maximum value of x.
- last_location_of_minimum - Returns the last location of the minimal value of x.
- lempel_ziv_complexity - Calculate a complexity estimate based on the Lempel-Ziv compression algorithm.
- length - Returns the length of x
- linear_trend - Calculate a linear least-squares regression for the values of the time series versus the sequence from 0 to length of the time series minus one.
- linear_trend_timewise - Calculate a linear least-squares regression for the values of the time series versus the sequence from 0 to length of the time series minus one.
- longest_strike_above_mean - Returns the length of the longest consecutive subsequence in x that is bigger than the mean of x
- longest_strike_below_mean - Returns the length of the longest consecutive subsequence in x that is smaller than the mean of x

TSFresh Features

- fast fourier matrix_profile - Calculates the 1-D Matrix Profile[1] and returns Tukey's Five Number Set plus the mean of that Matrix Profile.
- max_langevin_fixed_point - Largest fixed point of dynamics :math:argmax_x {h=0}` estimated from polynomial, which has been fitted to the deterministic dynamics of Langevin model
- maximum - Calculates the highest value of the time series x.
- mean - Returns the mean of x
- mean_abs_change - Average over first differences.
- mean_change - Average over time series differences.
- mean_n_absolute_max - Calculates the arithmetic mean of the n absolute maximum values of the time series.
- mean_second_derivative_central - Returns the mean value of a central approximation of the second derivative
- median - Returns the median of x
- minimum - Calculates the lowest value of the time series x.
- number_crossing_m - Calculates the number of crossings of x on m.
- number_cwt_peaks - Number of different peaks in x.
- number_peaks - Calculates the number of peaks of at least support n in the time series x.
- partial_autocorrelation - Calculates the value of the partial autocorrelation function at the given lag.
- percentage_of_reoccurring_datapoints_to_all_datapoints - Returns the percentage of non-unique data points.
- percentage_of_reoccurring_values_to_all_values - Returns the percentage of values that are present in the time series more than once.
- permutation_entropy - Calculate the permutation entropy.
- quantile - Calculates the q quantile of x.
- query_similarity_count - This feature calculator accepts an input query subsequence parameter, compares the query (under z-normalized Euclidean distance) to all subsequences within the time series, and returns a count of the number of times the query was found in the time series (within some predefined maximum distance threshold).

TSFresh Features

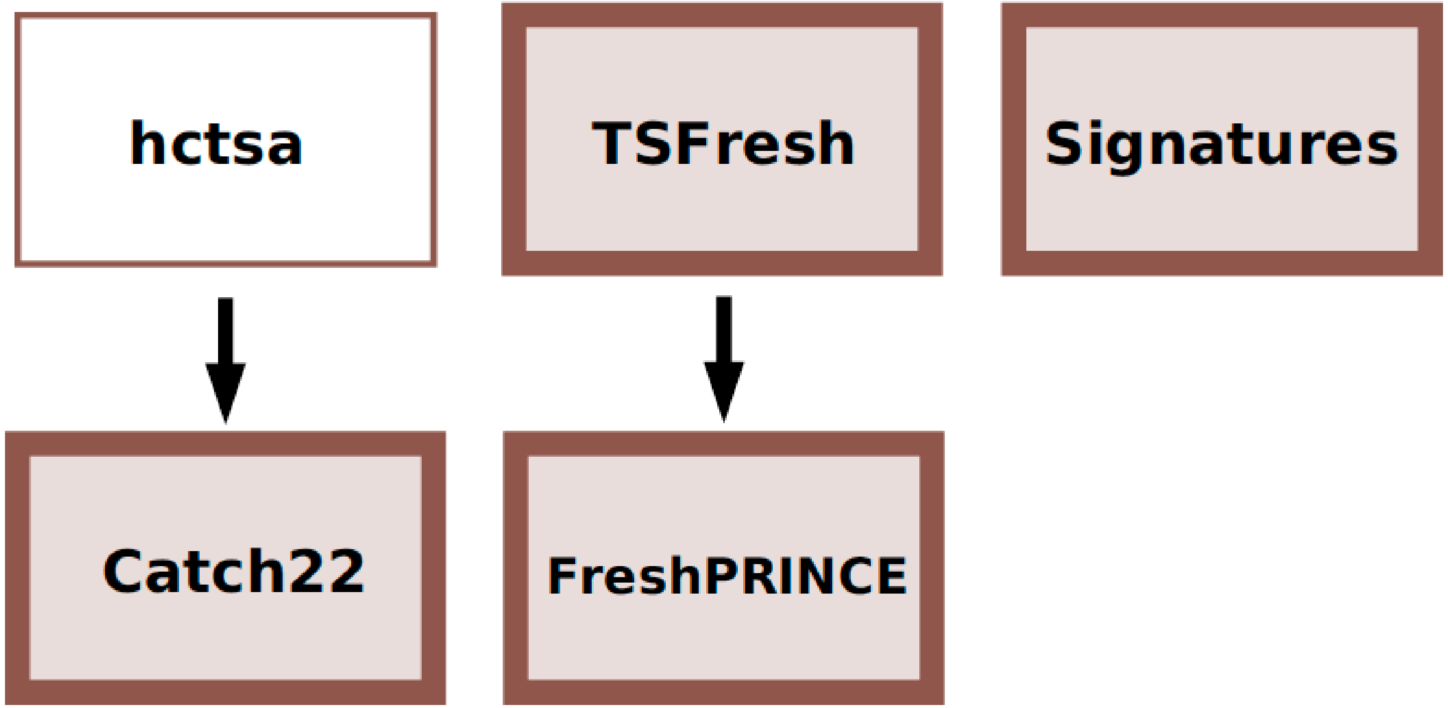
- range_count - Count observed values within the interval [min, max).
- ratio_beyond_r_sigma - Ratio of values that are more than r * std (so r times sigma) away from the mean of x.
- ratio_value_number_to_time_series_length - Returns a factor which is 1 if all values in the time series occur only once, and below one if this is not the case.
- root_mean_square - Returns the root mean square (rms) of the time series.
- sample_entropy - Calculate and return sample entropy of x.
- set_property - This method returns a decorator that sets the property key of the function to value
- skewness - Returns the sample skewness of x (calculated with the adjusted Fisher-Pearson standardized moment coefficient G1).
- spkt_welch_density - This feature calculator estimates the cross power spectral density of the time series x at different frequencies.
- standard_deviation - Returns the standard deviation of x
- sum_of_reoccurring_data_points - Returns the sum of all data points, that are present in the time series more than once.
- sum_of_reoccurring_values - Returns the sum of all values, that are present in the time series more than once.
- sum_values - Calculates the sum over the time series values
- symmetry_looking - Boolean variable denoting if the distribution of x looks symmetric.
- time_reversal_asymmetry_statistic - Returns the time reversal asymmetry statistic.
- value_count - Count occurrences of value in time series x.
- variance - Returns the variance of x
- variance_larger_than_standard_deviation - Is variance higher than the standard deviation?
- variation_coefficient - Returns the variation coefficient (standard error / mean, give relative value of variation around mean) of x

catch22: CAnonical Time-series CHaracteristics

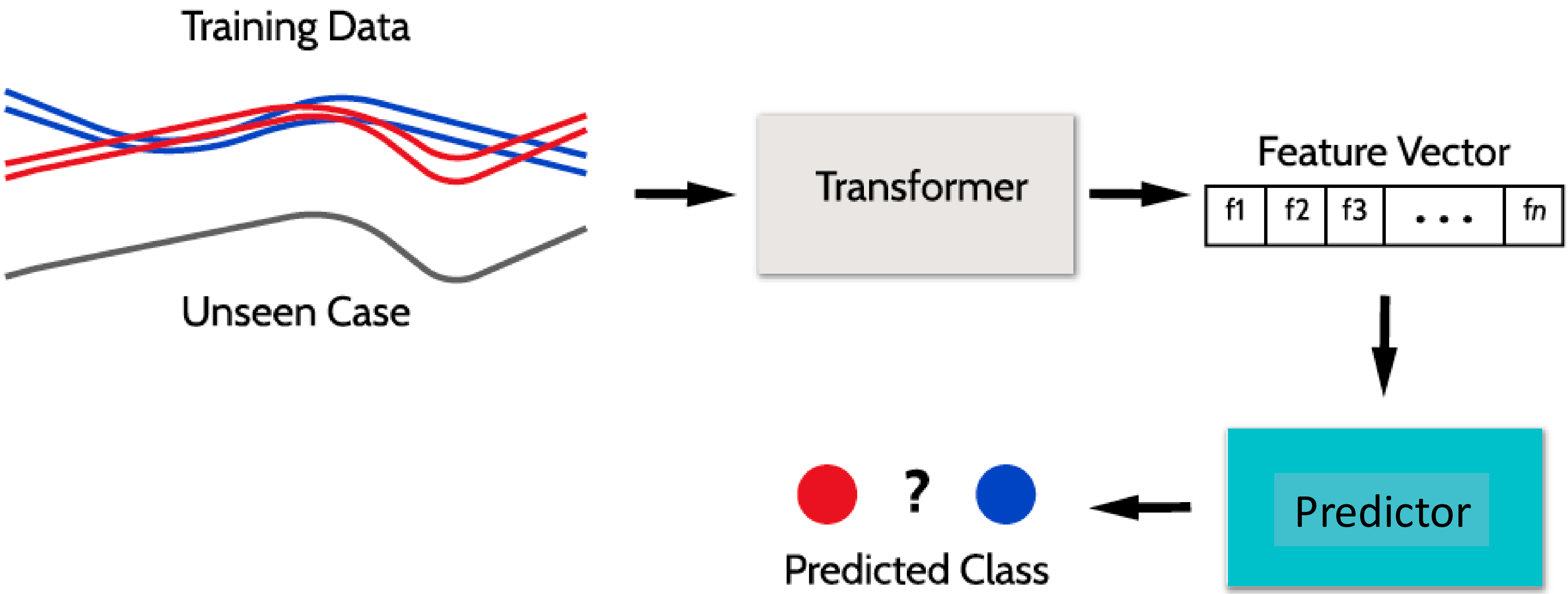
- The catch22 feature set spans a diverse range of time-series characteristics representative of the diversity of interdisciplinary methods for TSA.
- Features in catch22 capture TS properties of the distribution of values in the TS, linear and nonlinear temporal autocorrelation properties, scaling of fluctuations, and others.
- Selected by applying the procedure describe in [Lubba 2019] to a set of 93 datasets containing over 147k TS and using a filtered version of the HCTSA feature library (4791 features).
- The reduction from 4791 to 22 features is associated with a 1000-fold reduction in computation time and near linear scaling with TS length, despite an average reduction in classification accuracy of just 7%.

Distribution	
`DN_HistogramMode_5`	Mode of z-scored distribution (5-bin histogram)
`DN_HistogramMode_10`	Mode of z-scored distribution (10-bin histogram)
Simple temporal statistics	
`SB_BinaryStats_mean_longstretch1`	Longest period of consecutive values above the mean
`DN_OutlierInclude_p_001_mdrmd`	Time intervals between successive extreme events above the mean
`DN_OutlierInclude_n_001_mdrmd`	Time intervals between successive extreme events below the mean
Linear autocorrelation	
`CO_f1ecac`	First $1/e$ crossing of autocorrelation function
`CO_FirstMin_ac`	First minimum of autocorrelation function
`SP_Summaries_welch_rect_area_5_1`	Total power in lowest fifth of frequencies in the Fourier power spectrum
`SP_Summaries_welch_rect_centroid`	Centroid of the Fourier power spectrum
`FC_LocalSimple_mean3_stderr`	Mean error from a rolling 3-sample mean forecasting
Nonlinear autocorrelation	
`CO_trev_1_num`	Time-reversibility statistic, $\langle (x_{t+1} - x_t)^3 \rangle_t$
`CO_HistogramAMI_even_2_5`	Automutual information, $m = 2, \tau = 5$
`IN_AutoMutualInfoStats_40_gaussian_fmmi`	First minimum of the automutual information function
Successive differences	
`MD_hrv_classic_pnn40`	Proportion of successive differences exceeding 0.04σ [20]
`SB_BinaryStats_diff_longstretch0`	Longest period of successive incremental decreases
`SB_MotifThree_quantile_hh`	Shannon entropy of two successive letters in equiprobable 3-letter symbolization
`FC_LocalSimple_mean1_tauresrat`	Change in correlation length after iterative differencing
`CO_Embed2_Dist_tau_d_expfit_meandiff`	Exponential fit to successive distances in 2-d embedding space
Fluctuation Analysis	
`SC_FluctAnal_2_dfa_50_1_2_logi_prop_r1`	Proportion of slower timescale fluctuations that scale with DFA (50% sampling)
`SC_FluctAnal_2_rsrangefit_50_1_logi_prop_r1`	Proportion of slower timescale fluctuations that scale with linearly rescaled range fits
Others	
`SB_TransitionMatrix_3ac_sumdiagcov`	Trace of covariance of transition matrix between symbols in 3-letter alphabet
`PD_PeriodicityWang_th0_01`	Periodicity measure of [31]

Overview of Global Features and Relationships



Global Feature-based Predictor



Features, Approximations, Distances and Normalizations

- Normalizations can be applied before global features extraction depending on the objective of your TSA task.
- Time-Dependent approximations can be applied before global features extraction depending on the objective of your TSA task.
- It does not make any sense to use Time-Independent approximation after that global features have been extracted.
- It does not make any sense to use a distance function accounting for time like DTW after that global features have been extracted.

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