Introduction and Preprocessing

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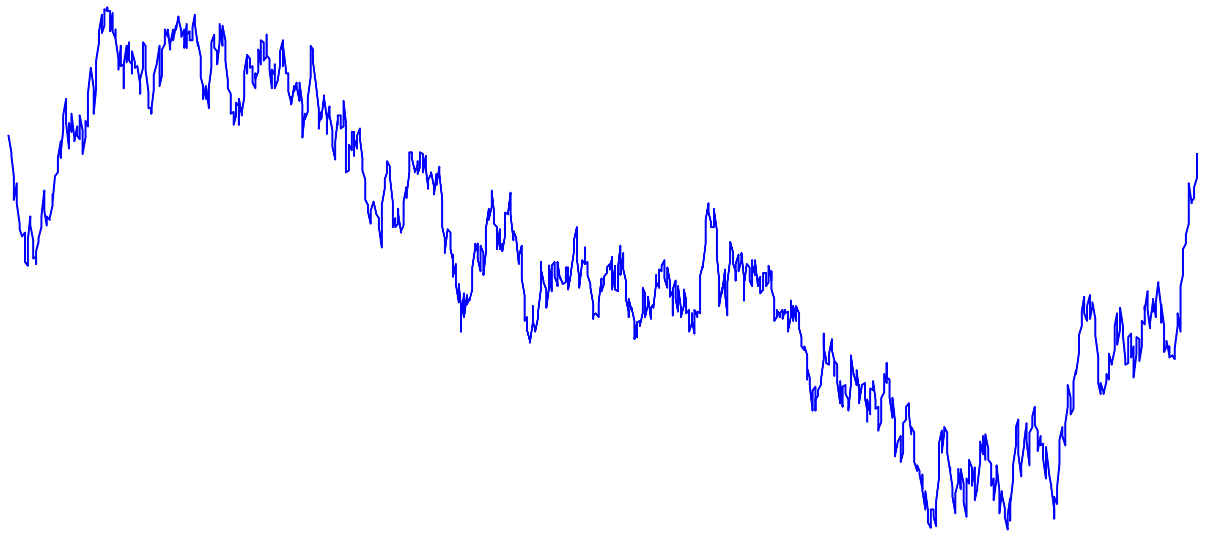
Syllabus

- Time Series, Datasets and Problems
- Missing Values
- Anomalies
- Normalizations
- Components
- Stationarity
- Autocorrelation



What is Time Series Analysis?

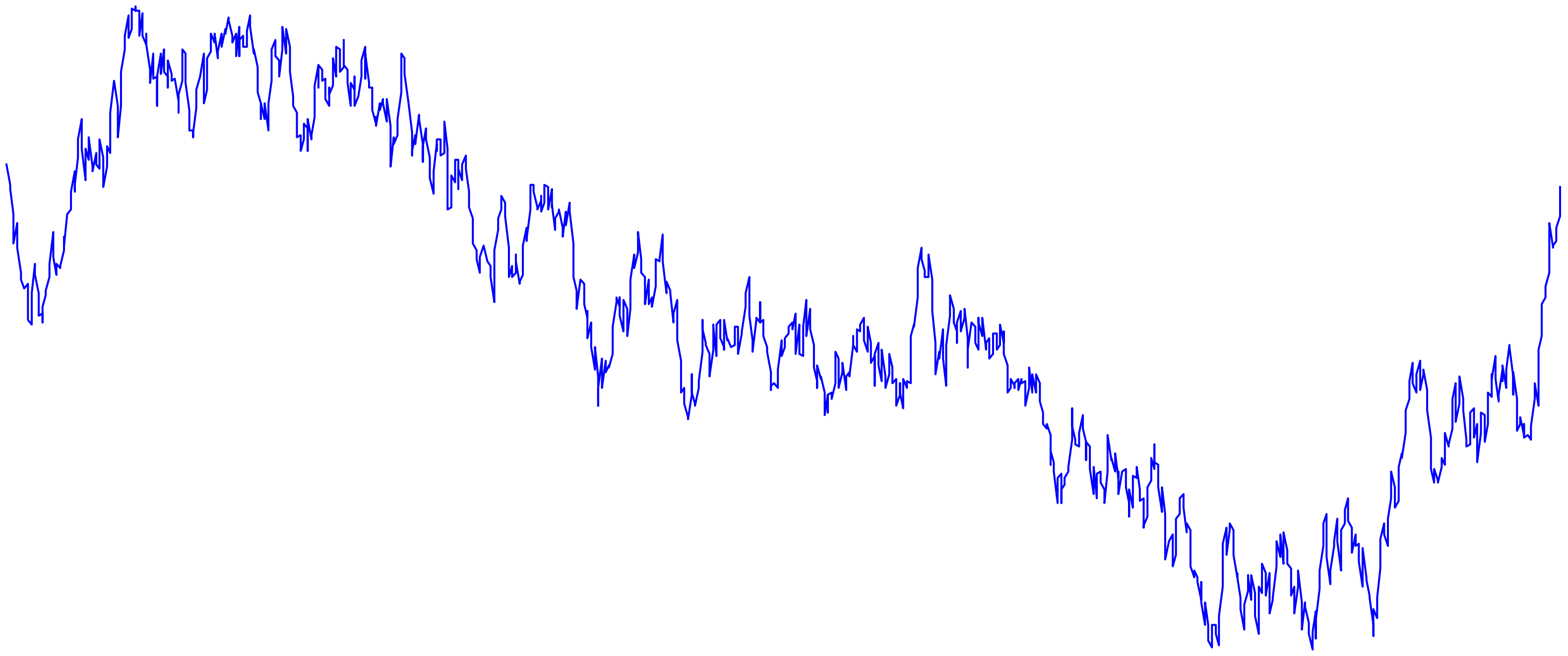
- Time Series Analysis (TSA):
  - TSA is a way of analyzing data points over an interval of time
  - Data points recorded at consistent intervals over a set period
  - Not just intermittent or random data collection
- Unique aspects of TSA:
  - Shows how variables change over time
  - Time is a crucial variable
  - Provides additional information and dependencies between data points
- Requirements for TSA:
  - Large number of data points for consistency and reliability
  - Ensures representative sample size
  - Helps cut through noisy data
  - Identifies trends, patterns, and accounts for seasonal variance
- Applications:
  - Forecasting: predicting future data based on historical data
  - Classification/Regression: predicting exogenous variable based on historical data

Why Companies Benefits From TSA?

- Benefits of time series analysis:
  - Understands underlying causes of trends or systemic patterns over time
  - Uses data visualizations to see seasonal trends and explore reasons behind them
  - Modern analytics platforms offer advanced visualizations beyond line graphs
- Predictive capabilities:
  - Analyzes data at consistent intervals for time series forecasting
  - Predicts likelihood of future events
  - Identifies seasonality and cyclic behavior for better understanding and forecasting
- Advantages of modern technology:
  - Ability to collect massive amounts of data daily
  - Easier to gather enough consistent data for comprehensive analysis

What is a Time Series?

- A time series $T = \{x_1, \ldots, x_m\}$ is a collection of m observations x_i made sequentially in time, generally at constant time intervals.



25.1750
25.2250
25.2500
25.2500
25.2750
25.3250
25.3500
25.3500
25.4000
25.4000
25.3250
25.2250
25.2000
25.1750
...
24.6250
24.6750
24.6750
24.6250
24.6250
24.6250
24.6750
24.7500

Intervals

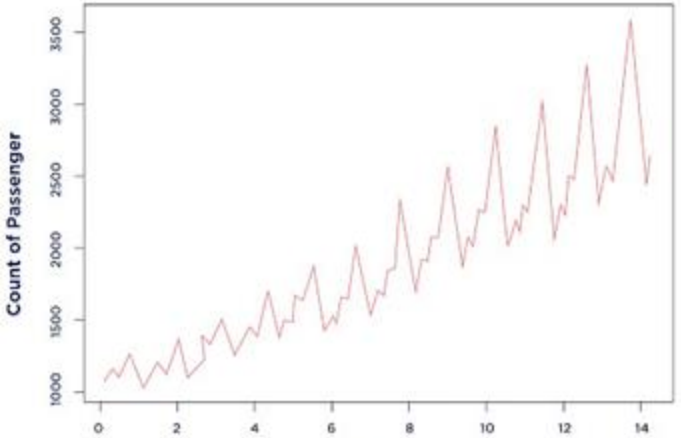
- Time: Milliseconds, Seconds, Minutes, Hours, Days, Months, Years, …
- Spatial: Locations, Positions, Machines, …
- Relative: one cm left, two cm left, …

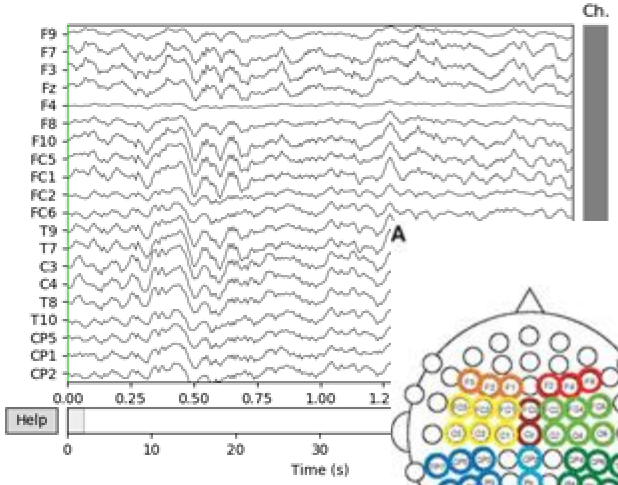
- The important point is to have an ordered variable that provides a direction for its values.
- Then from a given observation x_i it is easy to identify the past, i.e., what came before x_i, and the future, i.e., what comes after x_i

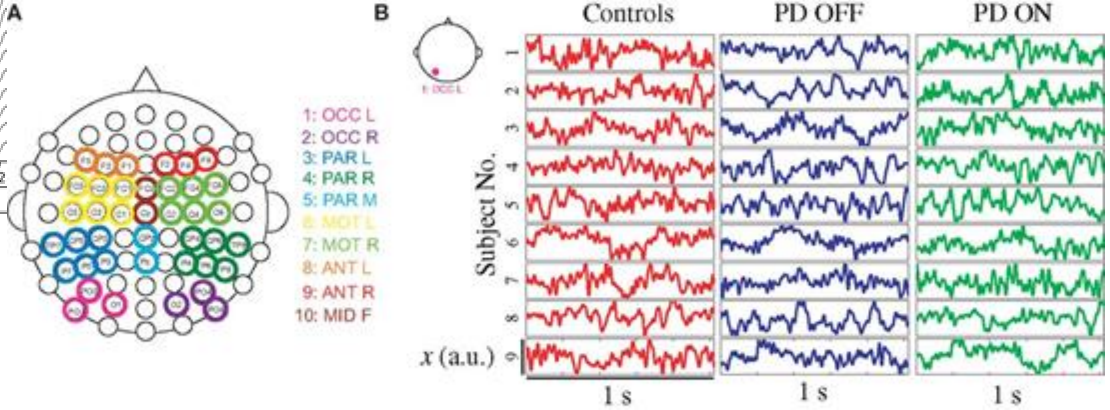
25.1750
25.2250
25.2500
25.2500
25.2750
25.3250
25.3500
25.3500
25.4000
25.4000
25.3250
25.2250
25.2000
25.1750
…
24.6250
24.6750
24.6750
24.6250
24.6250
24.6250
24.6750
24.7500

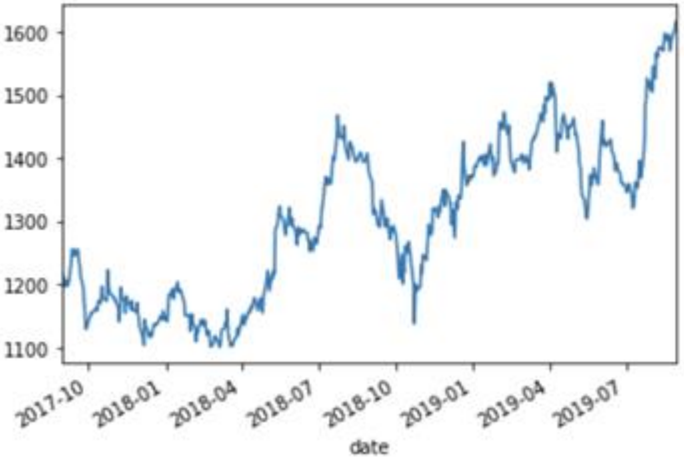
Time Series are Ubiquitous

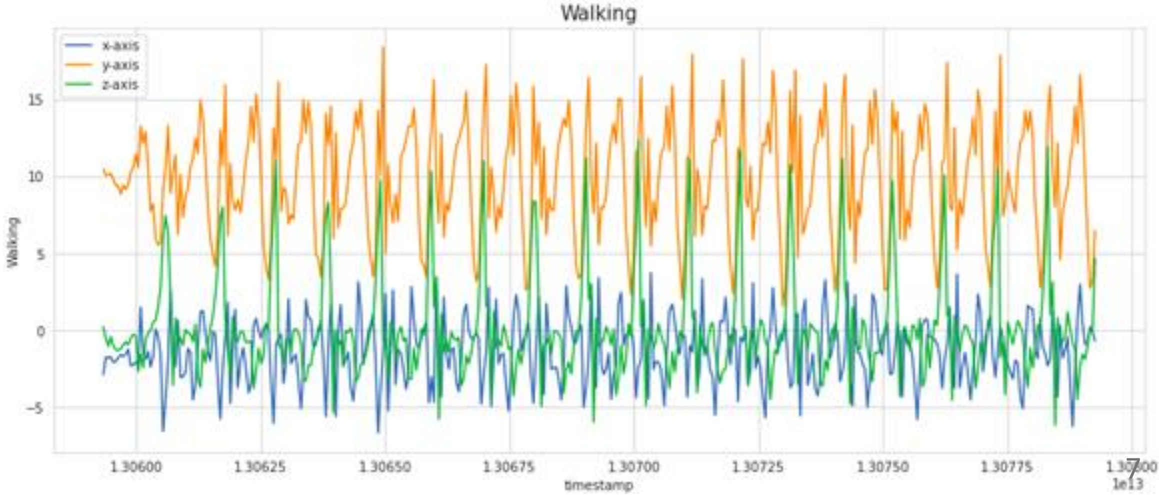
- Blood pressure
- Politics popularity rating
- The annual rainfall in Pisa
- Passengers of a company
- Accelerations on different axes
- The value of your stocks
- EEG and ECG





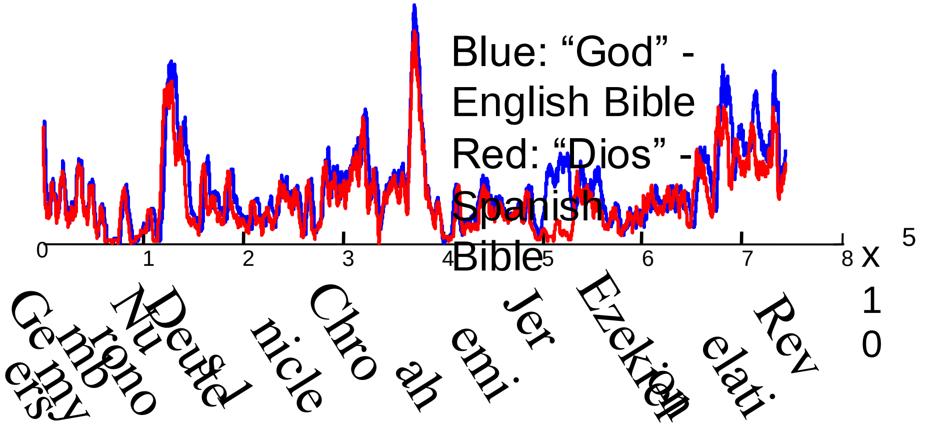


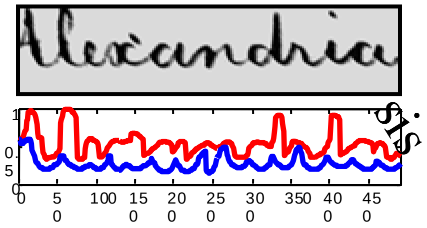


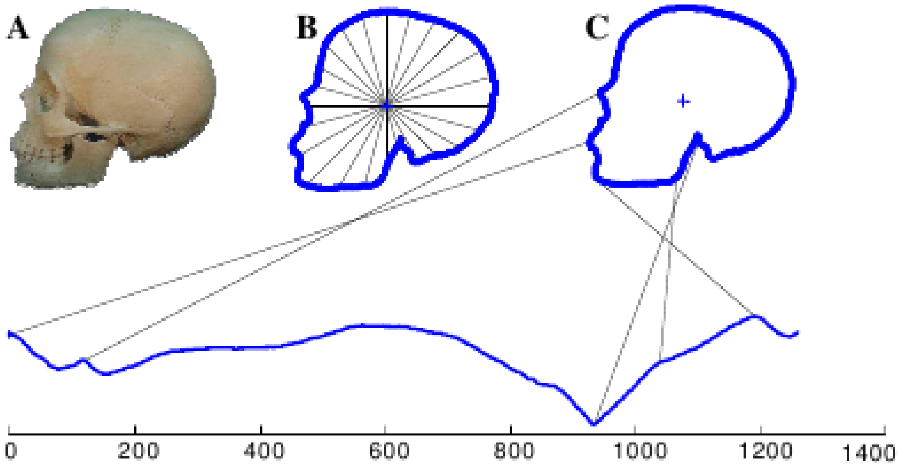


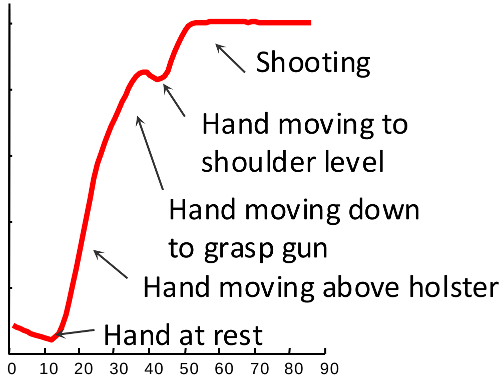
Time Series are Ubiquitous

- Other data types can be modeled as time series
  - Text data: words count
  - Images: edges displacement
  - Videos: object positioning





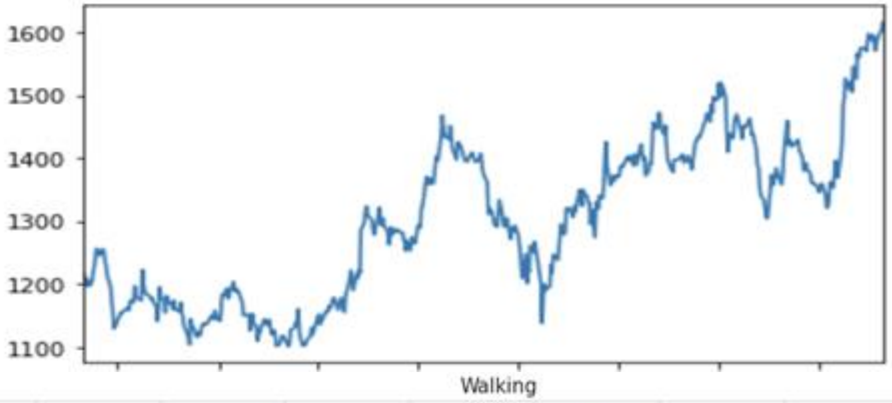




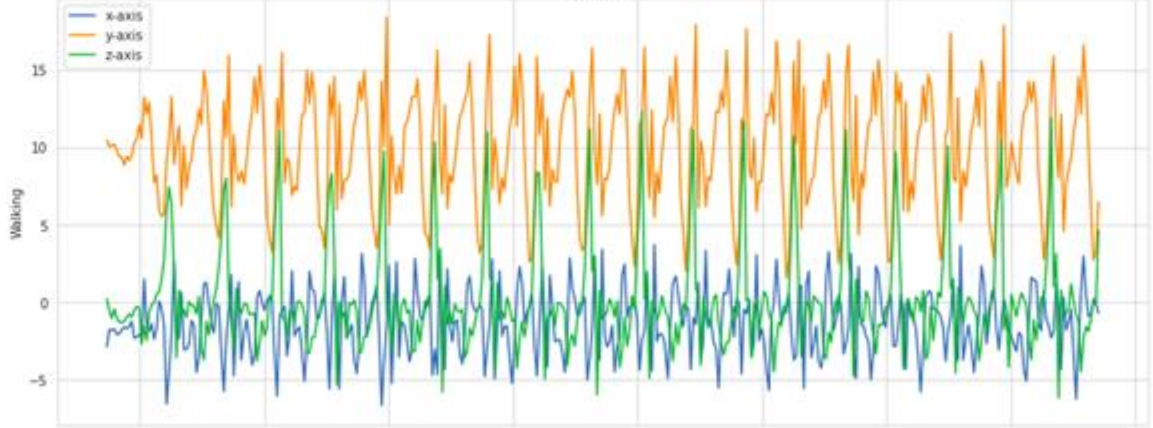
Time Series Data

- Time Series $\boldsymbol{T} = \{T_1, \ldots, T_c\}$ is a collection of *c* signals (or channels) each one with *m* observations x_i, i.e., $T_j = \{x_1, \ldots, x_m\}$

- Univariate Time Series *(c = 1)*

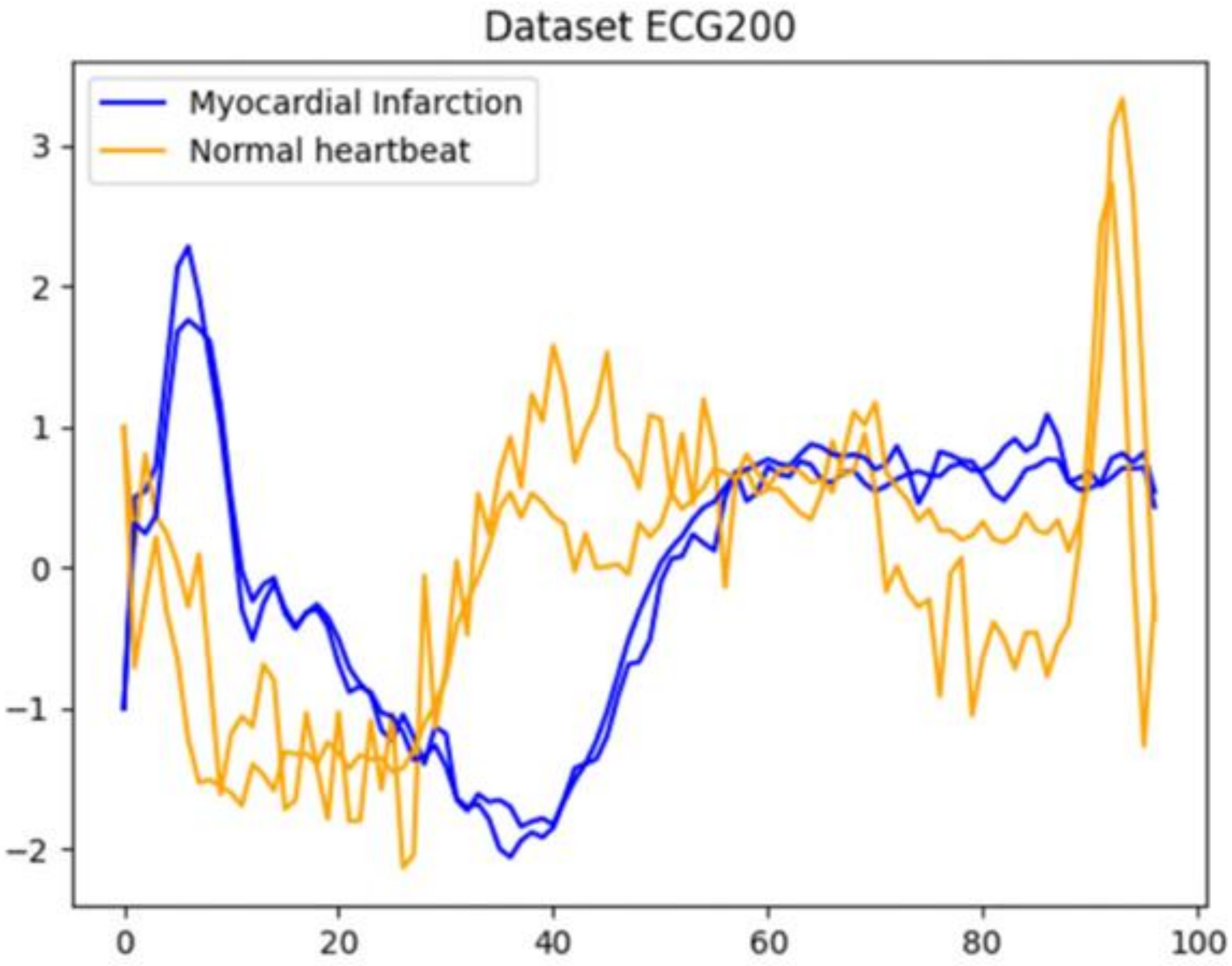


- Multivariate Time Serie *(c > 1)*



Time Series Data

- A Time Series Dataset is a collection of time series $X = \{\boldsymbol{T_1}, \dots \boldsymbol{T_n}\}$.
- Associated to each time series $\boldsymbol{T_i}$ we can have exogenous variables
  - Categorical Features
    - Accelerometers: Type of Movement, Crash vs NoCrash, Car Model, etc.
    - Machine Sensors: Failure vs NoFailure, Product in Production, Type of Machine, etc.
    - EEG: Syntoms, Seizure vs NoSeizure
    - Students Marks: Student Name, Course, Degree, University, Background
  - Continuous Features
    - Accellerometers: Age, Weight, Engine Temperature, etc.
    - Machine Sensors: Number of Products Realized, System Temperature, etc.
    - EEG: Age, Weight, Height, etc.
    - Students Marks: Age, Family Income, Weight, etc.


Dataset ECG200
Myocardial Infarction
Normal heartbeat
3
2
1
0
−1
−2
0
20
40
60
80
100

Time Series in Datasets

- Stored “horizontally”, typically univariate TS

id	t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
w4r5	111	110	111	112	120	123	116	118	119	123
e2t6	89	76	79	85	67	56	78	97	45	78
q1e4	91	95	89	87	110	120	125	130	111	90
w23t	110	111	112	122	123	112	118	119	123	119

Time Series in Datasets

- Stored “vertically”, typically multivariate TS. Wide Dataset

id	time	temp	speed	rot	product	failure	sys temp
w4r5	t_0	20.1	111	3	A	0	32.5
w4r5	t_1	18.6	110	4	A	0	32.5
w4r5	t_2	19.4	111	3	A	1	32.5
w4r5	t_3	20.4	112	5	A	0	32.5
w4r5	t_4	21.5	120	6	A	0	32.5
e2t6	t_0	12.7	89	29	B	0	34.6
e2t6	t_1	19.8	76	45	B	0	34.6
e2t6	t_2	17.4	69	34	B	0	34.6
e2t6	t_3	8.4	85	22	B	1	34.6
e2t6	t_4	7.9	65	19	B	1	34.6

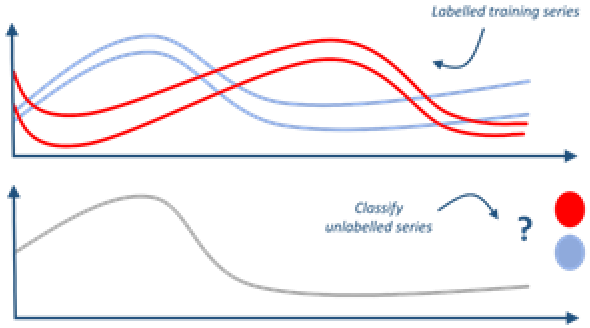
Time Series in Datasets

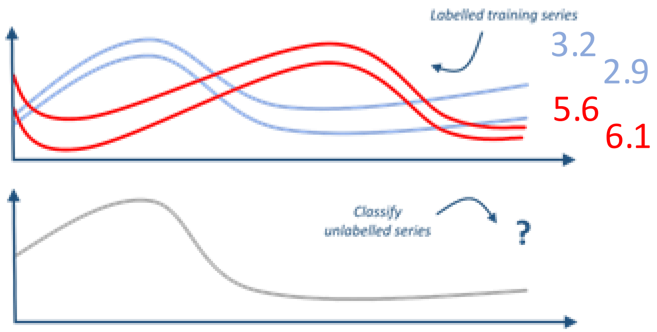
- Stored ‘‘vertically’’, typically multivariate TS.
- Long Dataset

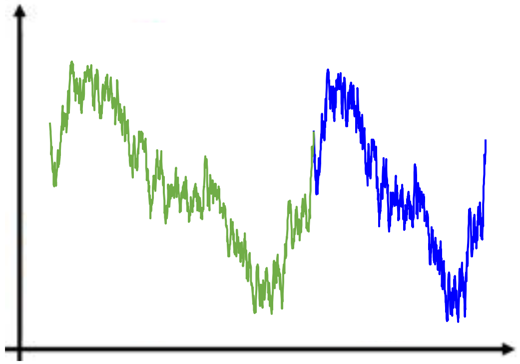
id	time	feat	value
w4r5	t_0	temp	20.1
w4r5	t_1	temp	18.6
w4r5	t_2	temp	19.4
w4r5	t_3	temp	20.4
w4r5	t_4	temp	21.5
w4r5	t_0	speed	111
w4r5	t_1	speed	110
w4r5	t_2	speed	111
w4r5	t_3	speed	112
w4r5	t_4	speed	120
…	…	…	…

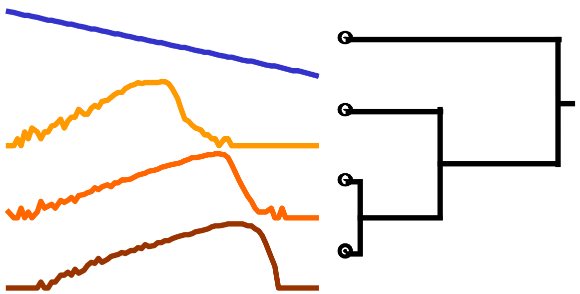
Time Series Analytics Tasks

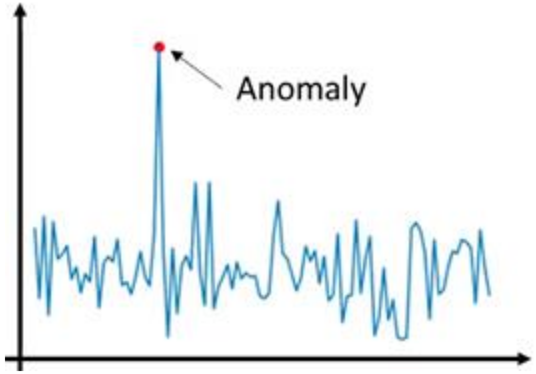
- Classification
- Regression
- Forecasting
- Clustering
- Anomaly Detection
- Pattern Mining

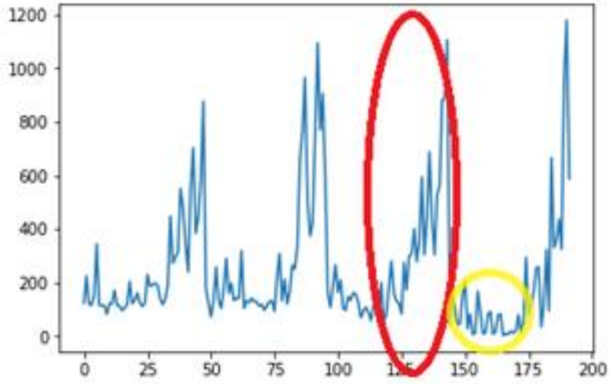






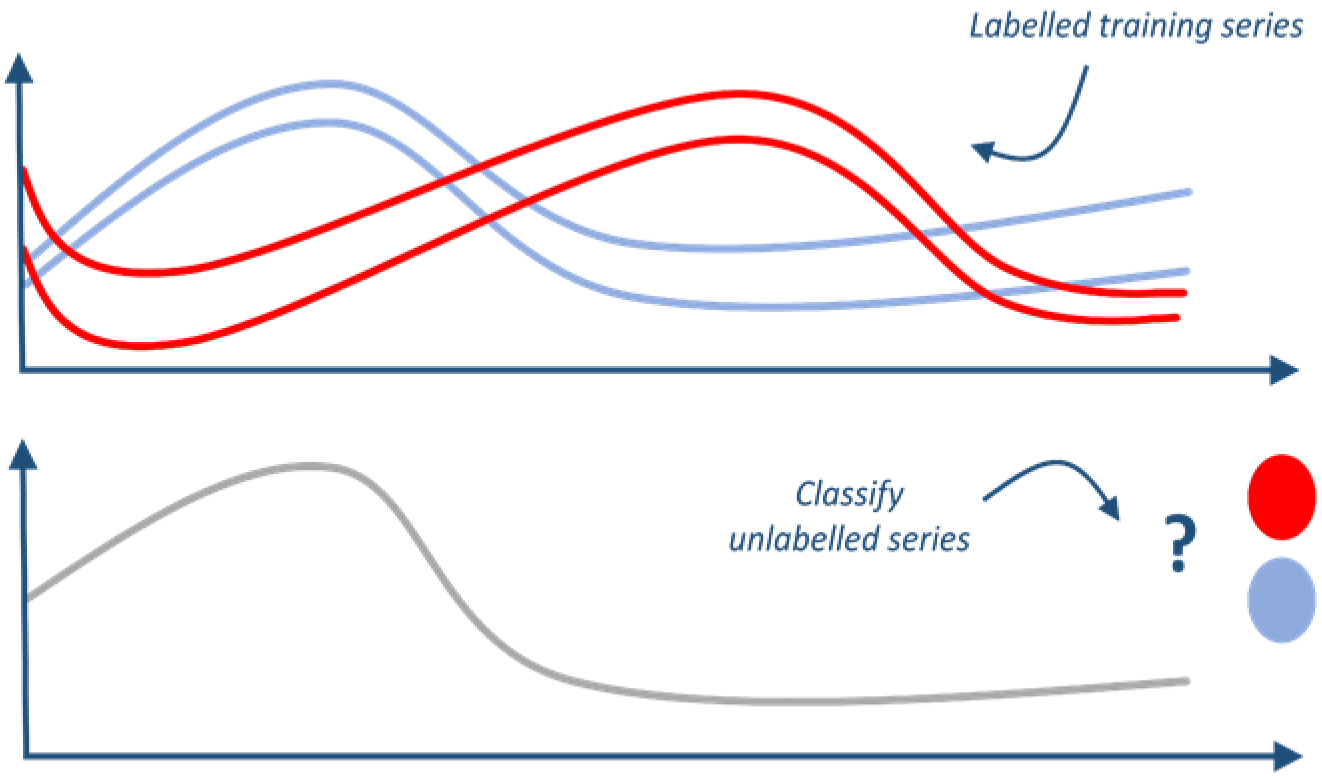






Time Series Classification - TSC

- Given a dataset $X = \{\boldsymbol{T_1}, \ldots \boldsymbol{T_n}\}$, TSC is the task of training a model f to predict an exogenous categorical output y for each time series T, i.e., $f(T) = y$.

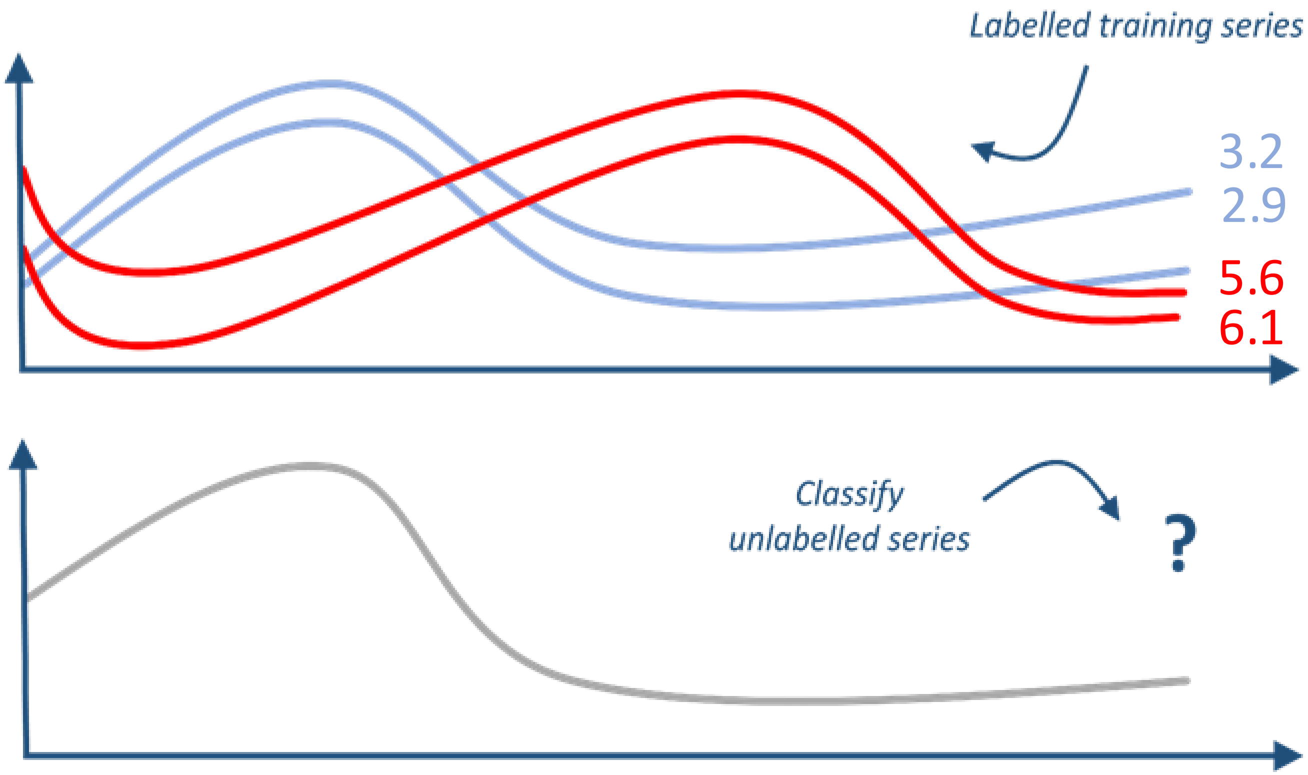


Time Series Classification - Examples

- ***Predictive variables***: multivariate time series as accelerometers on the x, y, z axes and speed coming from cars black boxes. ***Target variable***: means of transport, crash vs noCrash, car model, etc.
- ***Predictive variables***: multivariate time series as accelerometers on the x, y, z axes, heart rate and number of steps per sec coming from personal smart devices. ***Target variable***: type of movement, type of device, next location.
- ***Predictive variables***: multivariate time series as machine sensors such has temperature, humidity, number of rotations, accelerations, etc. ***Target variable***: failure vs nofailure, product in production, type of machine, etc.
- ***Predictive variables***: multivariate time series as EEG or ECG. ***Target variable***: symptoms, disease, treatment, etc.
- ***Predictive variables***: univariate or multivariate time series as students' marks. ***Target variable***: student background, university, etc.

Time Series Extrinsic Regression - TSER

- Given a dataset $X = \{\boldsymbol{T_1}, \ldots\ \boldsymbol{T_n}\}$, TSER is the task of training a model f to predict an exogenous <u>continuous</u> output y for each time series T, i.e., $f(T) = y$.

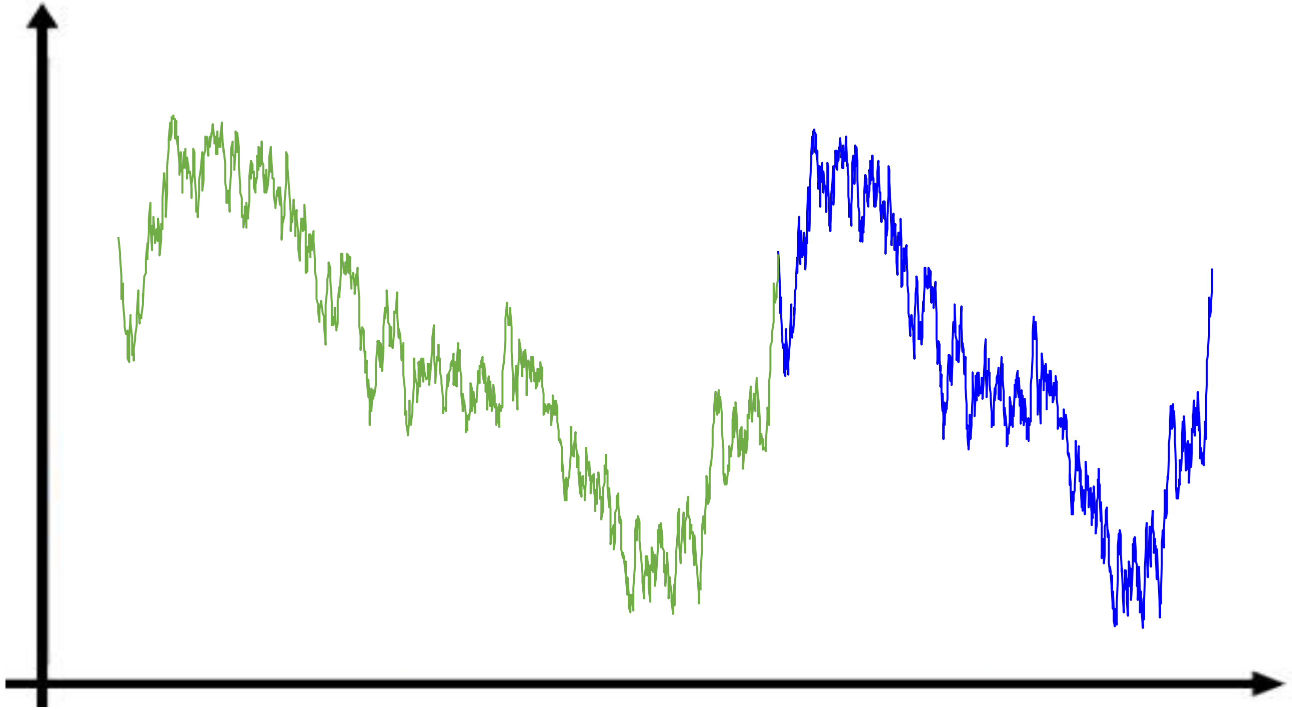


Time Series Regression - Examples

- ***Predictive variables***: multivariate time series as accelerometers on the x, y, z axes and speed coming from cars black boxes. **Target variable**: engine temperature, number of car repairs, etc.
- ***Predictive variables*** multivariate time series as accelerometers on the x, y, z axes, heart rate and number of steps per sec coming from personal smart devices. ***Target variable***: age, number of times gym is made.
- ***Predictive variables***: multivariate time series as machine sensors such has temperature, humidity, number of rotations, accelerations, etc. ***Target variable***: number of products realized, number of failures.
- ***Predictive variables***: multivariate time series as EEG or ECG. ***Target variable***: patient age, patient weight, etc.
- ***Predictive variables***: univariate or multivariate time series as students' marks. ***Target variable***: student age, family income.

Time Series Forecasting - TSF

- Given a dataset $X = \{\boldsymbol{T_1}, \ldots \boldsymbol{T_n}\}$, TSF is the task of training a model f to predict an endogenous continuous output y for each time series T, i.e., $f(T) = y$.

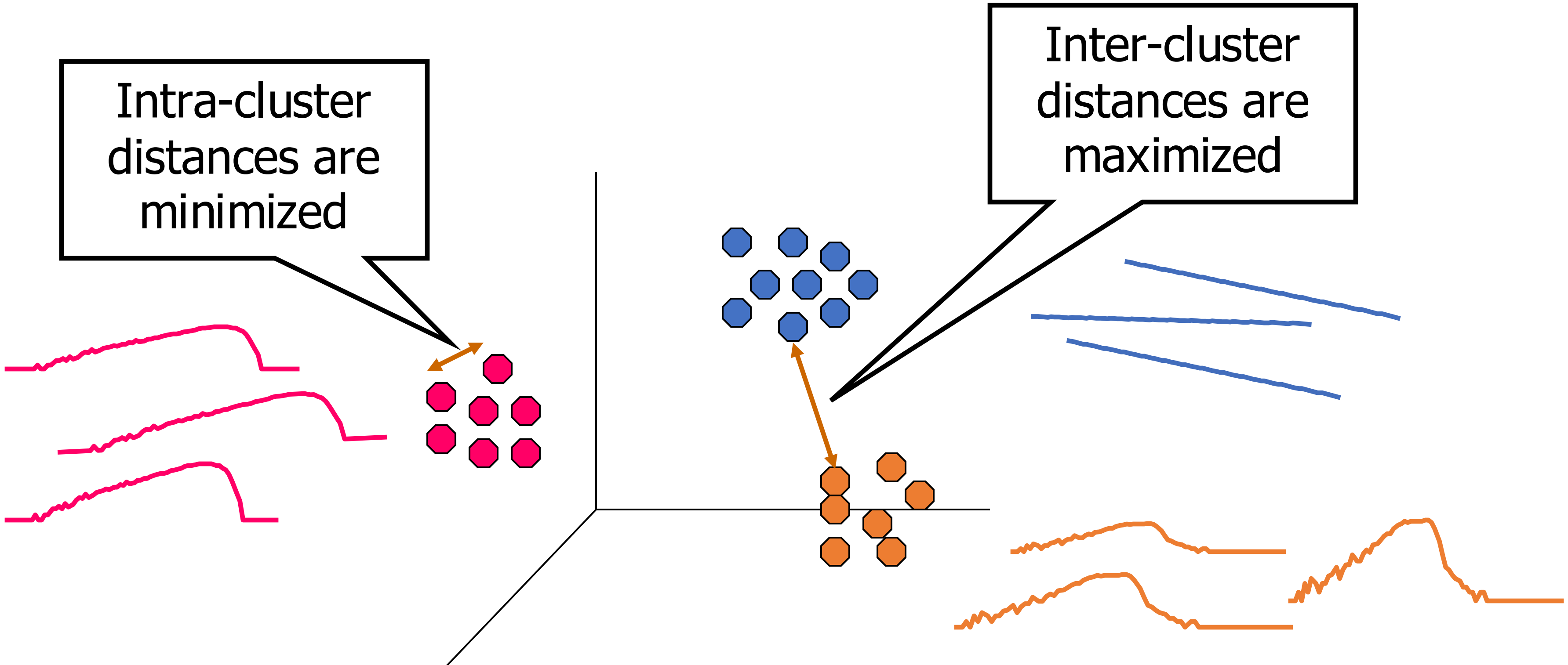


Time Series Forecasting - Examples

- ***Predictive variables***: multivariate time series as accelerometers on the x, y, z axes and speed coming from cars black boxes. ***Target variable***: time series as accelerometers on the x, y-, z axes and speed.
- ***Predictive variables***: multivariate time series as accelerometers on the x, y, z axes, heart rate and number of steps per sec coming from personal smart devices. ***Target variable***: time series as accelerometers on the x, y, z axes, heart rate and number of steps per sec .
- ***Predictive variables***: multivariate time series as machine sensors such has temperature, humidity, number of rotations, accelerations, etc. ***Target variable***: temperature, humidity, number of rotations, accelerations.
- ***Predictive variables***: multivariate time series as EEG or ECG. ***Target variable***: future values of EEG or ECG, etc.
- ***Predictive variables***: univariate or multivariate time series as students' marks. ***Target variable***: future students’ marks.

Time Series Clustering

- Given a dataset $X = \{\boldsymbol{T_1}, \ldots \boldsymbol{T_n}\}$, Time Series Clustering is the task of grouping similar time series such that those in a group are similar to one another and different from the time series in other groups.

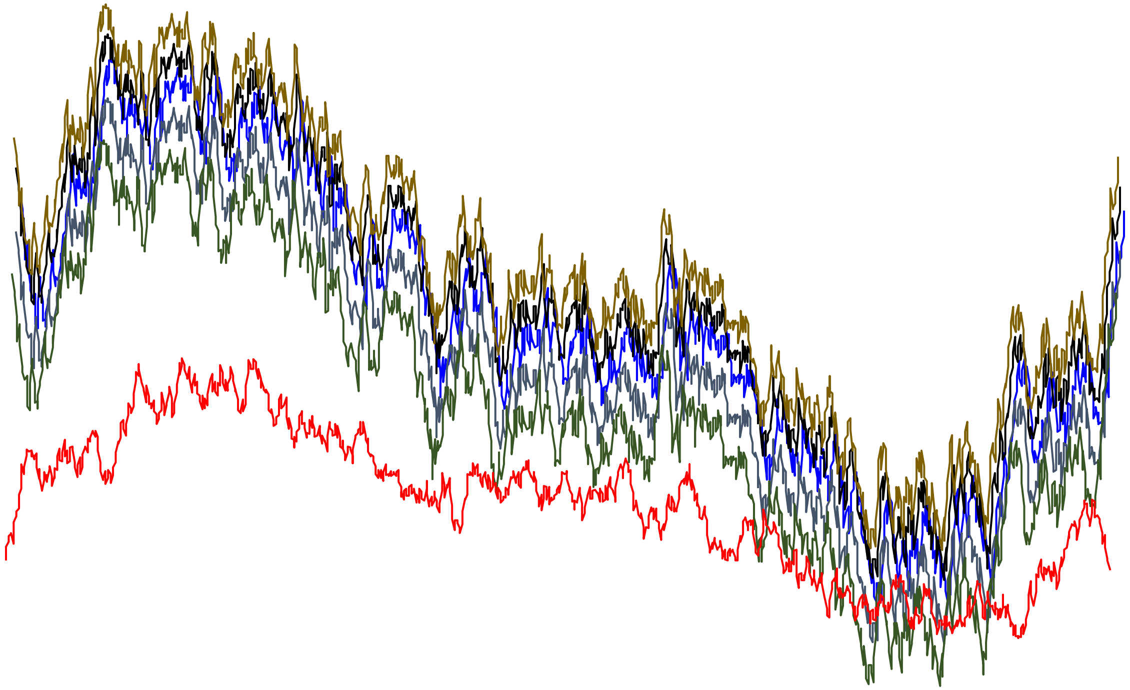


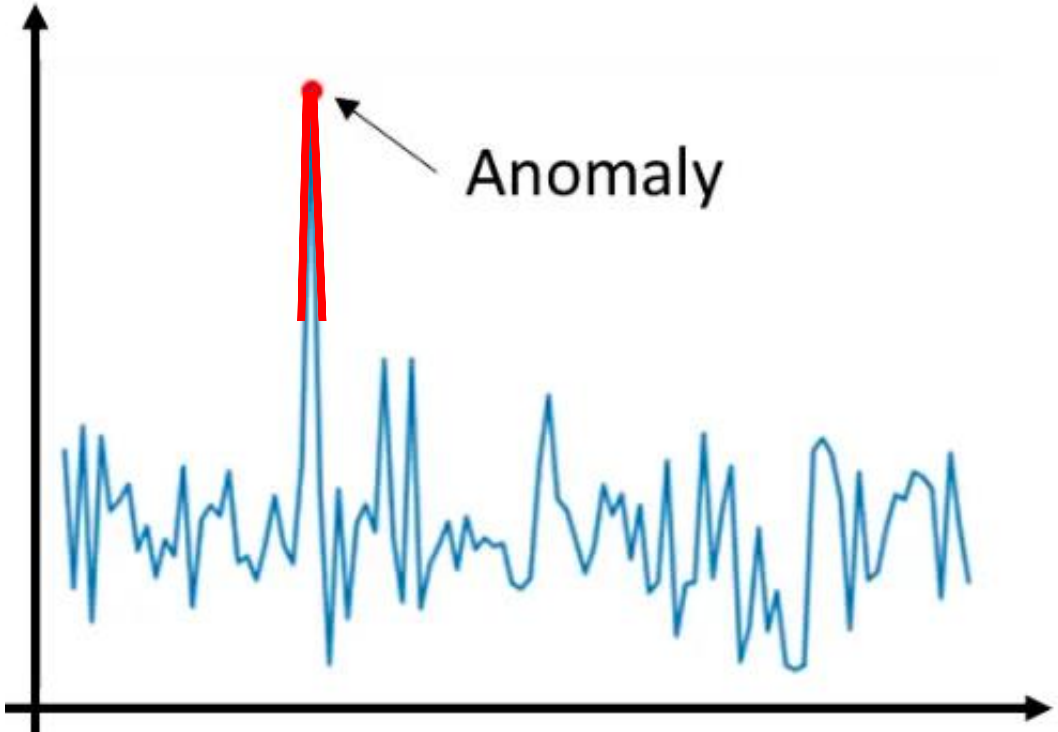
Time Series Clustering - Examples

- ***Variables***: multivariate time series as accelerometers on the x, y, z axes and speed coming from cars black boxes. ***Result***: groups of similar cars.
- ***Variables***: multivariate time series as accelerometers on the x, y, z axes, heart rate and number of steps per sec coming from personal smart devices. ***Result***: groups of similar users.
- ***Variables***: multivariate time series as machine sensors such has temperature, humidity, number of rotations, accelerations, etc. ***Results***: groups of similar products produced.
- ***Variables***: multivariate time series as EEG or ECG. ***Results***: groups of similar patients.
- ***Variables***: univariate or multivariate time series as students' marks. ***Results***: groups of similar students.

Anomaly Detection

- Given a dataset $X = \{\boldsymbol{T_1}, \ldots\ \boldsymbol{T_n}\}$, Anomaly Detection is the task of:
  a) identifying anomalous time series within the set X
  b) identifying anomalous time stamps for each time series T



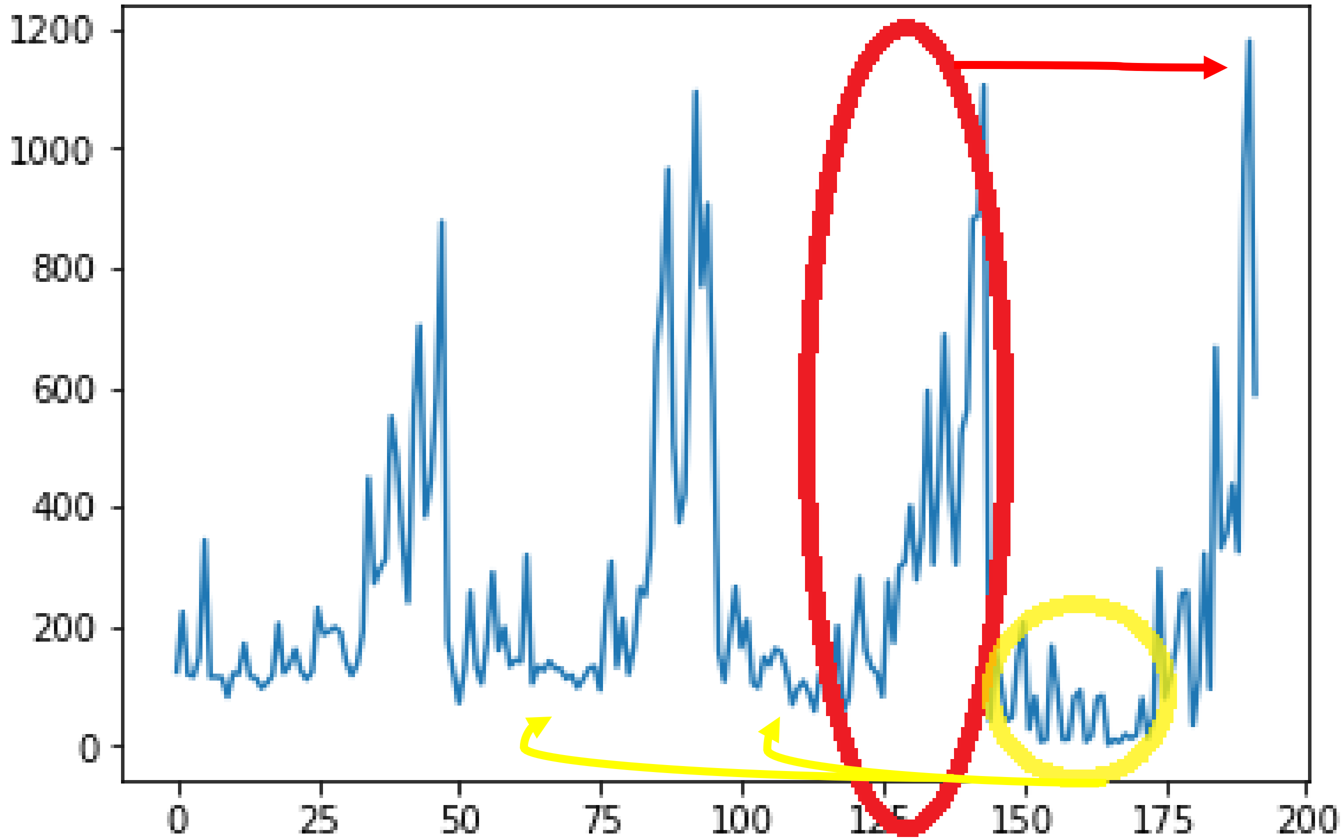


Time Series Anomaly Detection - Examples

- ***Variables***: multivariate time series as accelerometers on the x, y, z axes and speed coming from cars black boxes. ***Result***: anomalous cars or anomalous time stamps.
- ***Variables***: multivariate time series as accelerometers on the x, y, z axes, heart rate and number of steps per sec coming from personal smart devices. ***Result***: anomalous users or anomalous time stamps.
- ***Variables***: multivariate time series as machine sensors such has temperature, humidity, number of rotations, accelerations, etc. ***Results***: anomalous products/machines or anomalous time stamps.
- ***Variables***: multivariate time series as EEG or ECG. ***Results***: anomalous patients or anomalous time stamps.
- ***Variables***: univariate or multivariate time series as students' marks. ***Results***: anomalous students or anomalous time stamps.

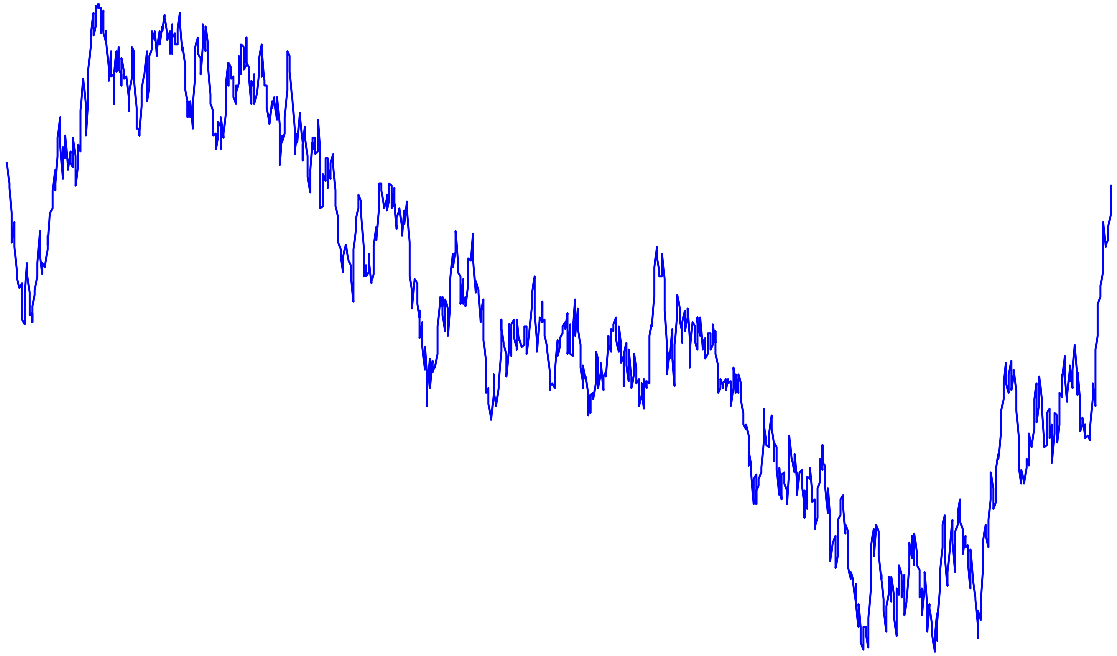
Pattern Mining

- Given a dataset $X = \{\boldsymbol{T_1}, \ldots \boldsymbol{T_n}\}$, Pattern Mining is the task of identifying repeated subsequences in each time series T.



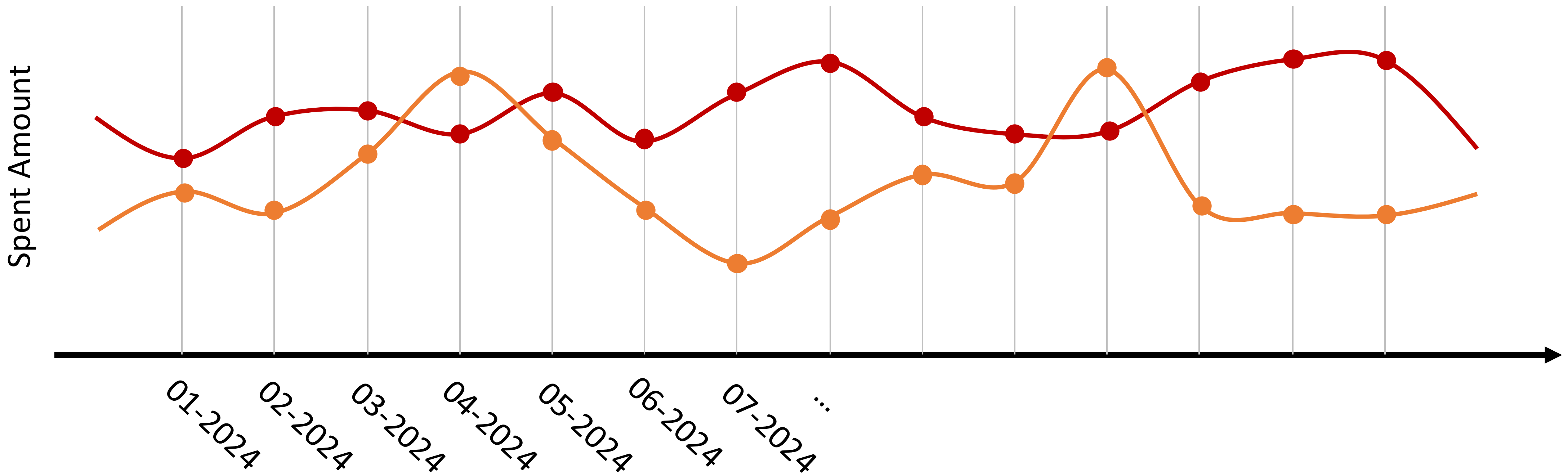
Problems in Working with Time Series

- Large amount of data
- Similarity is not easy to estimate
- Different data formats
- Different sampling rates
- Noise
- Missing values
- …



Assumption

We assume that for a given dataset $X = \{\boldsymbol{T_1}, \ldots\ \boldsymbol{T_n}\}$, all the time series have aligned time stamps, i.e., the *i-th* time stamp of a time series T_a corresponds to the *i-th* time stamp of another time series T_b and this is true for all the time series in X.



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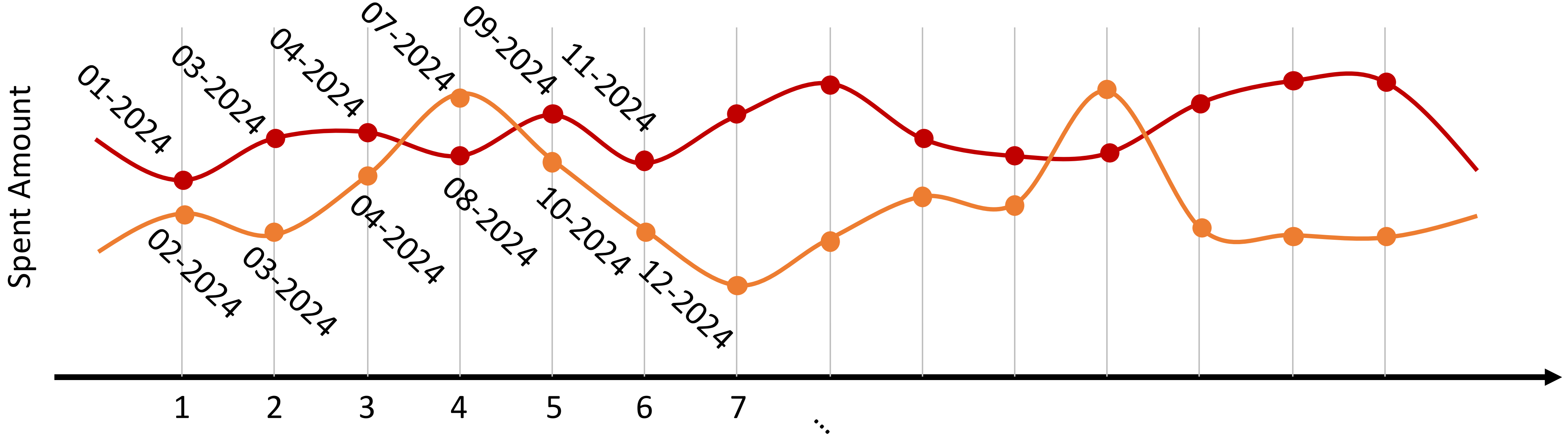


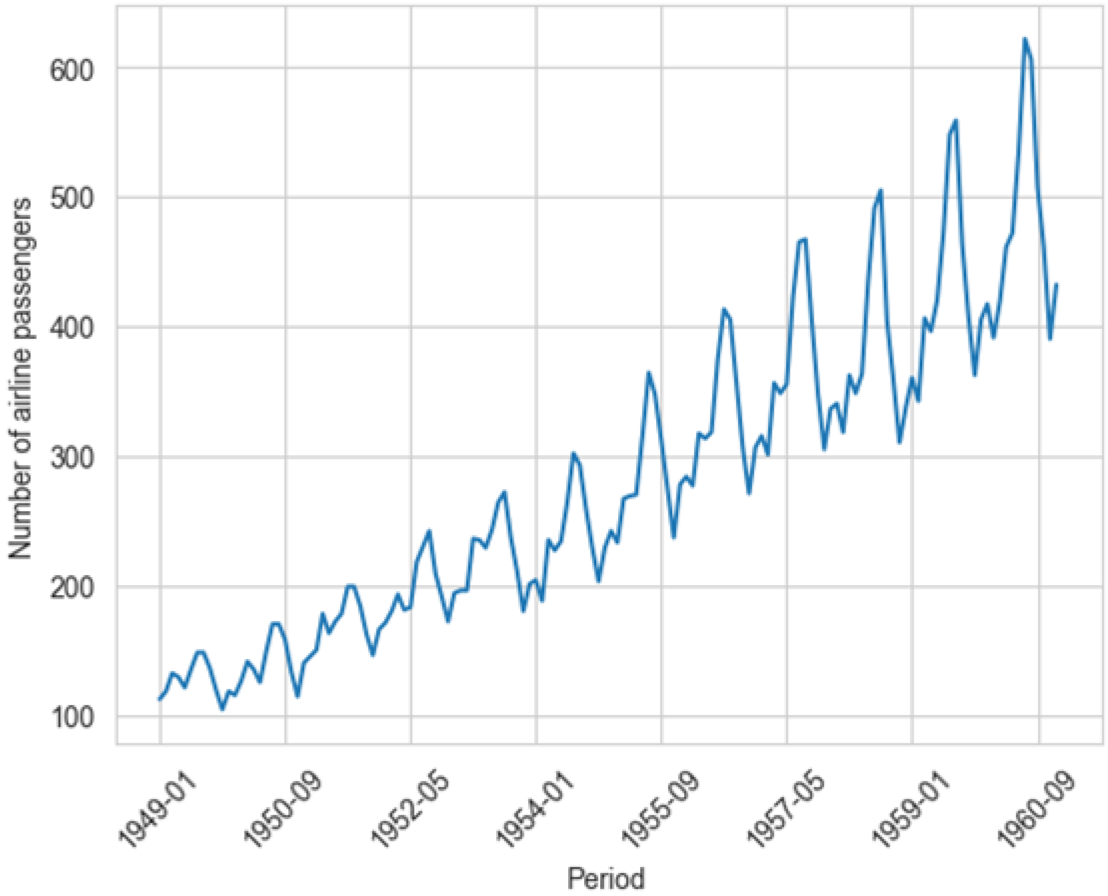
Time Series Visualization

A plot can reveal

- Trend: upward or downward pattern
- Periodicity: repetition of behavior in a regular pattern
- Seasonality: periodic behavior with a known period (hourly, weekly, monthly...)
- Heteroskedasticity: changing variance
- Dependence: positive (successive observations are similar) or negative (successive observations are dissimilar)
- Outliers: anomalous time stamps, anomalous subsequences
- Missing data: missing values at a certain time stamp or longer subsequences

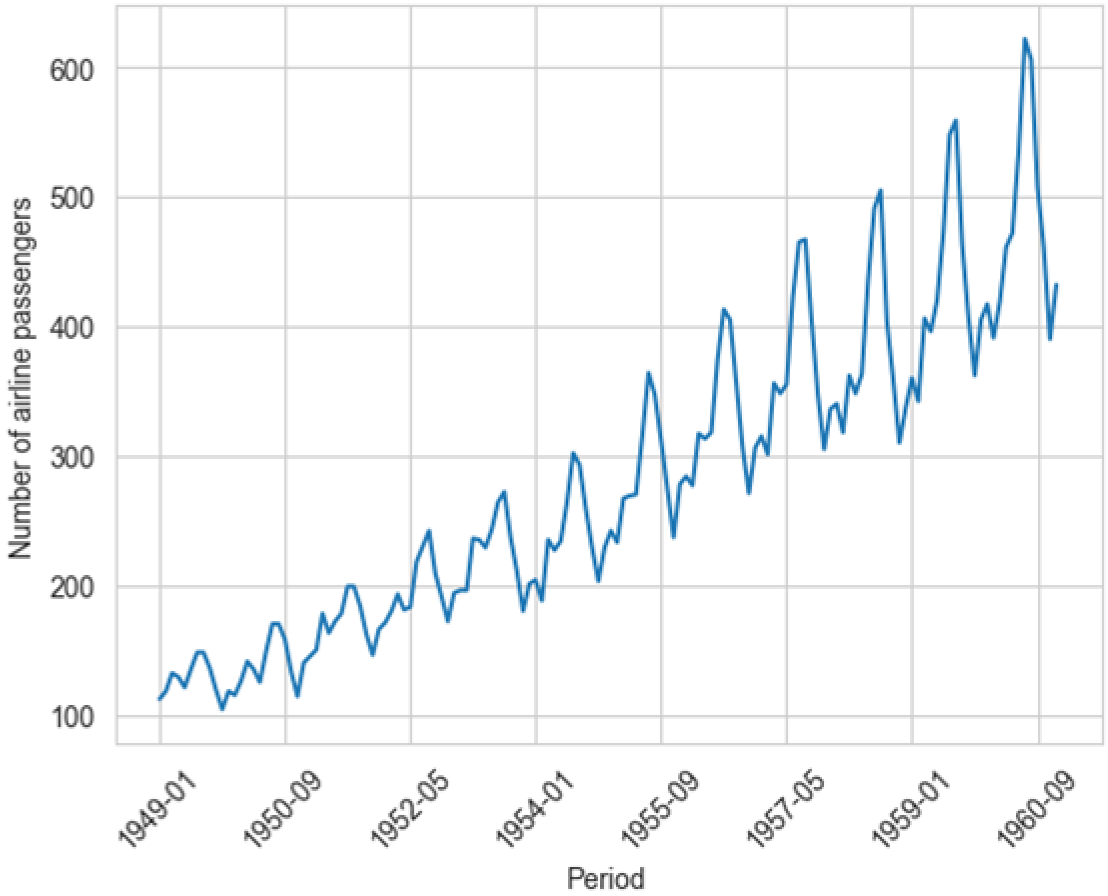
Example 1

- Airline passengers from 1949-1961



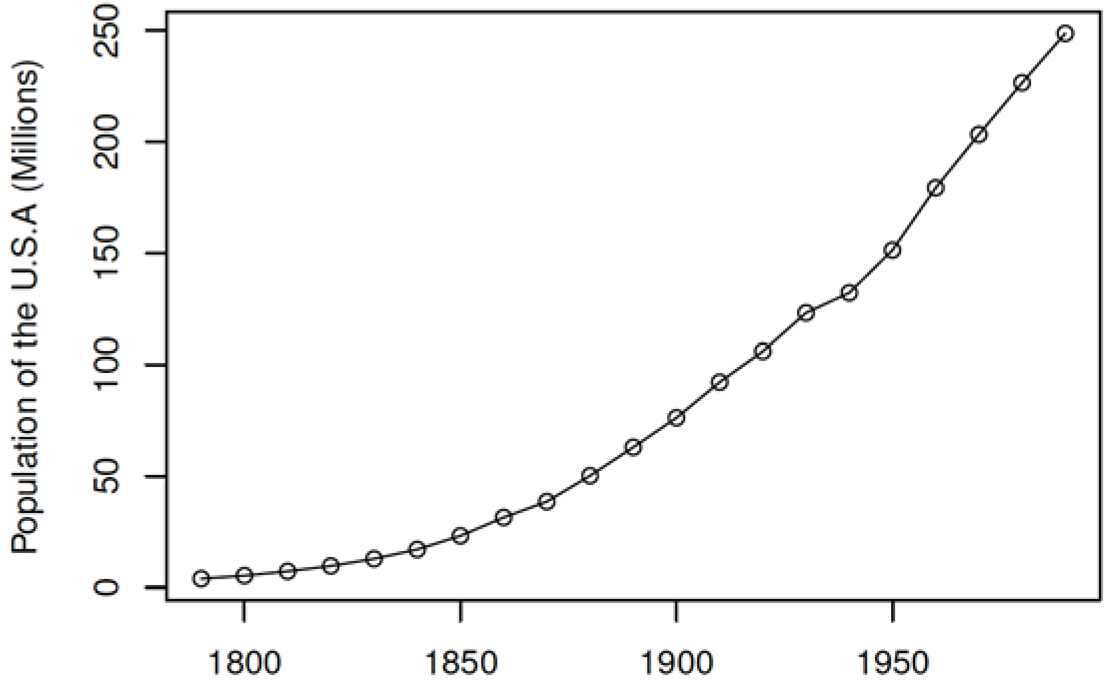
Example 1

- Airline passengers from 1949-1961
- Upward trend
- Seasonality on a 12 month interval
- Increasing variability



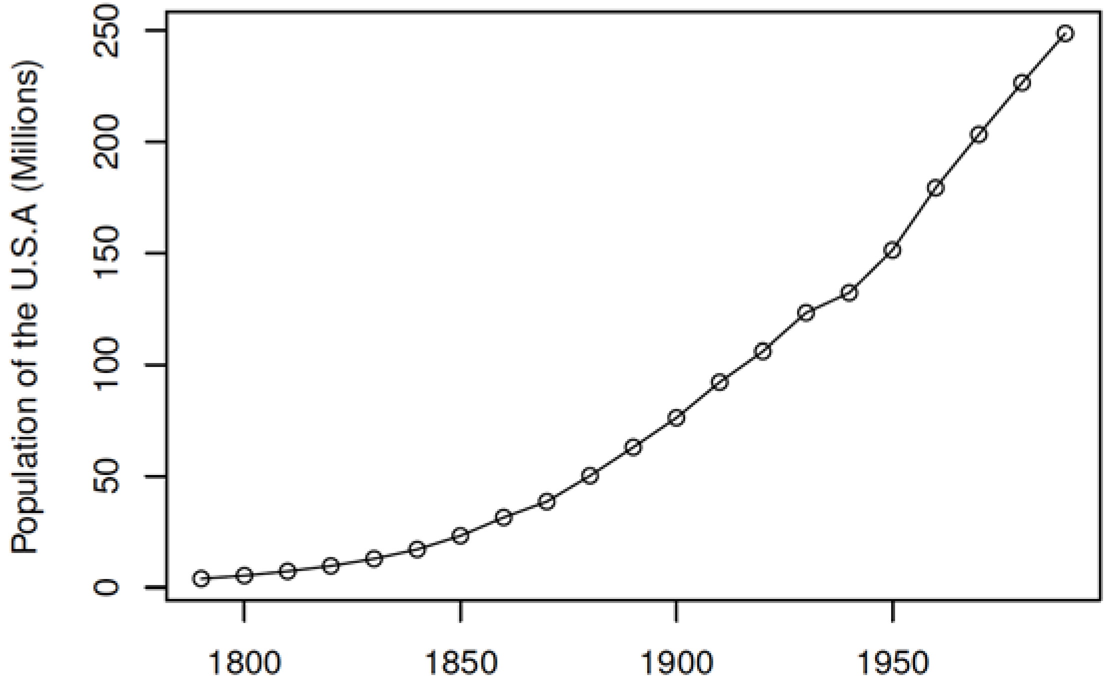
Example 2

- U.S.A. population at ten year intervals from 1790-1990



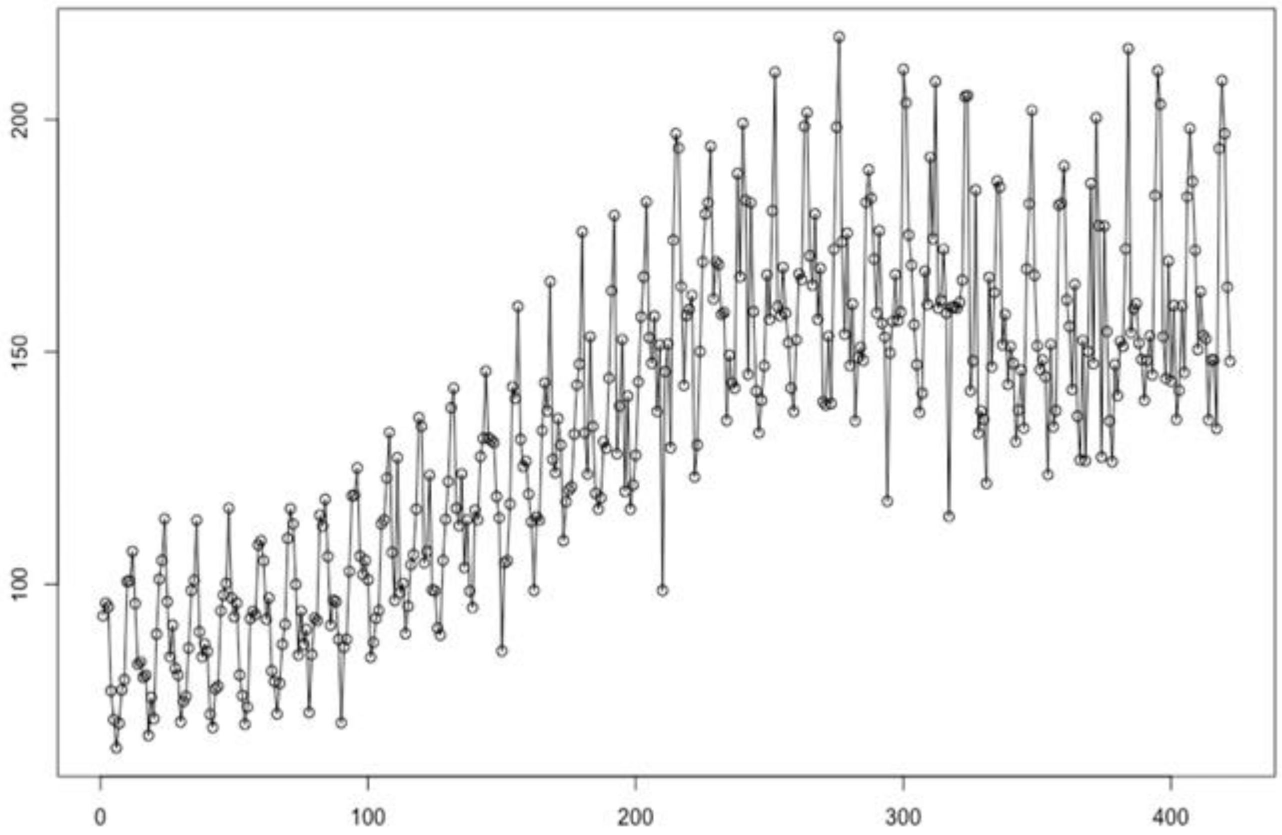
Example 2

- U.S.A. population at ten year intervals from 1790-1990
- Upward trend
- Slight change in shape/structure
- Nonlinear behavior



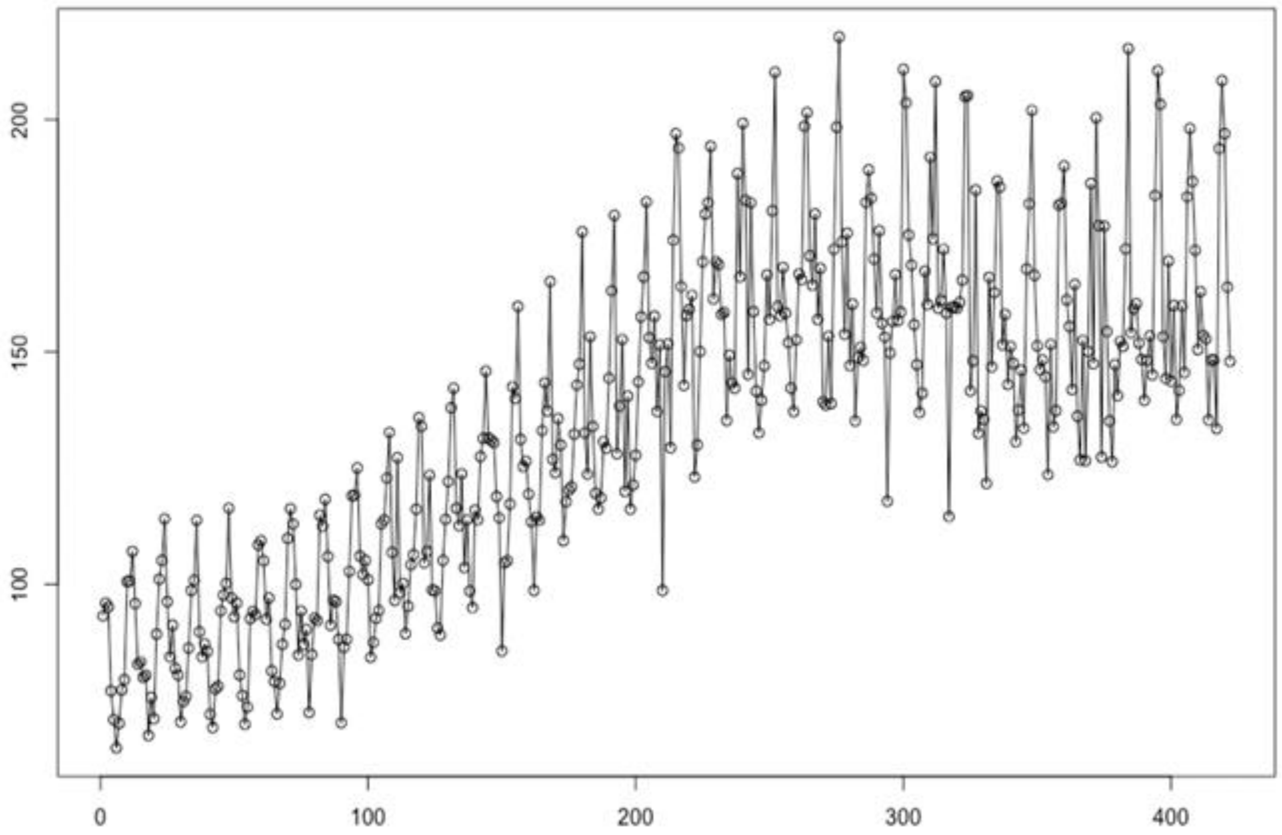
Example 3

- Monthly Beer Production in Australia



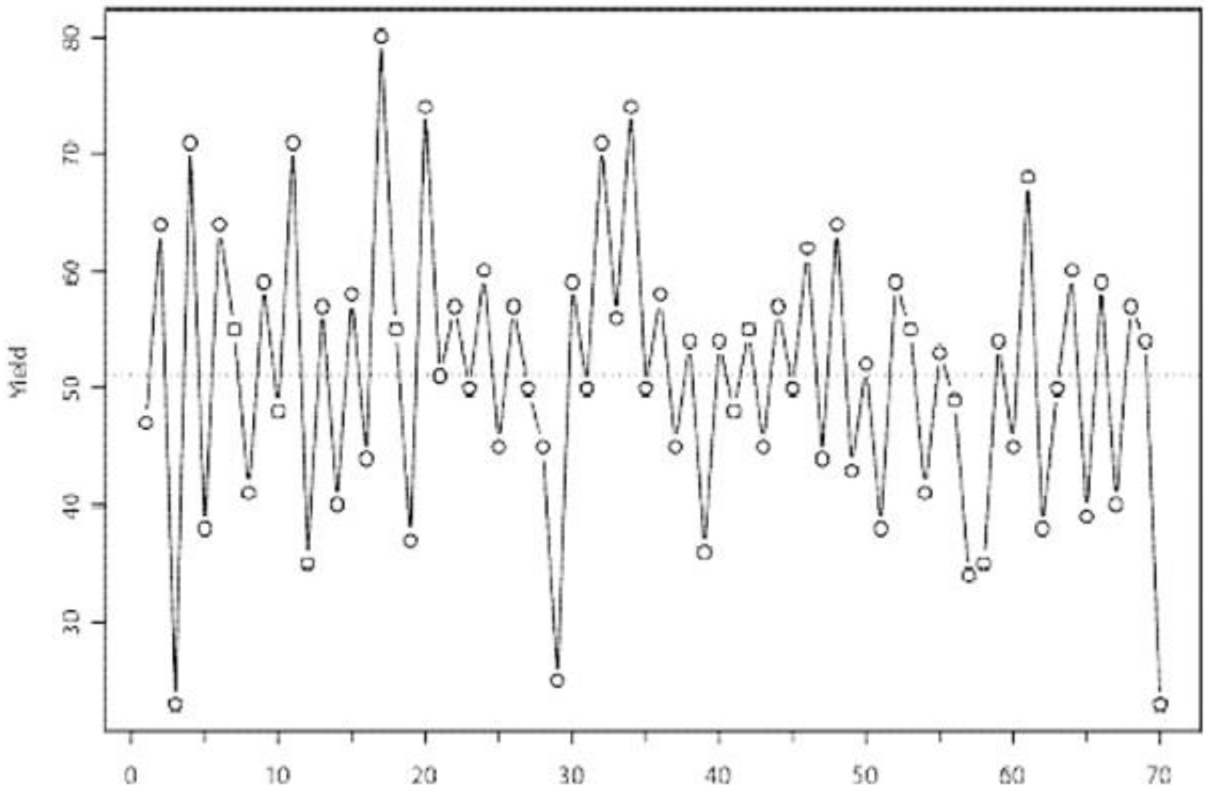
Example 3

- Monthly Beer Production in Australia
- No trend in last 100 months
- No clear seasonality



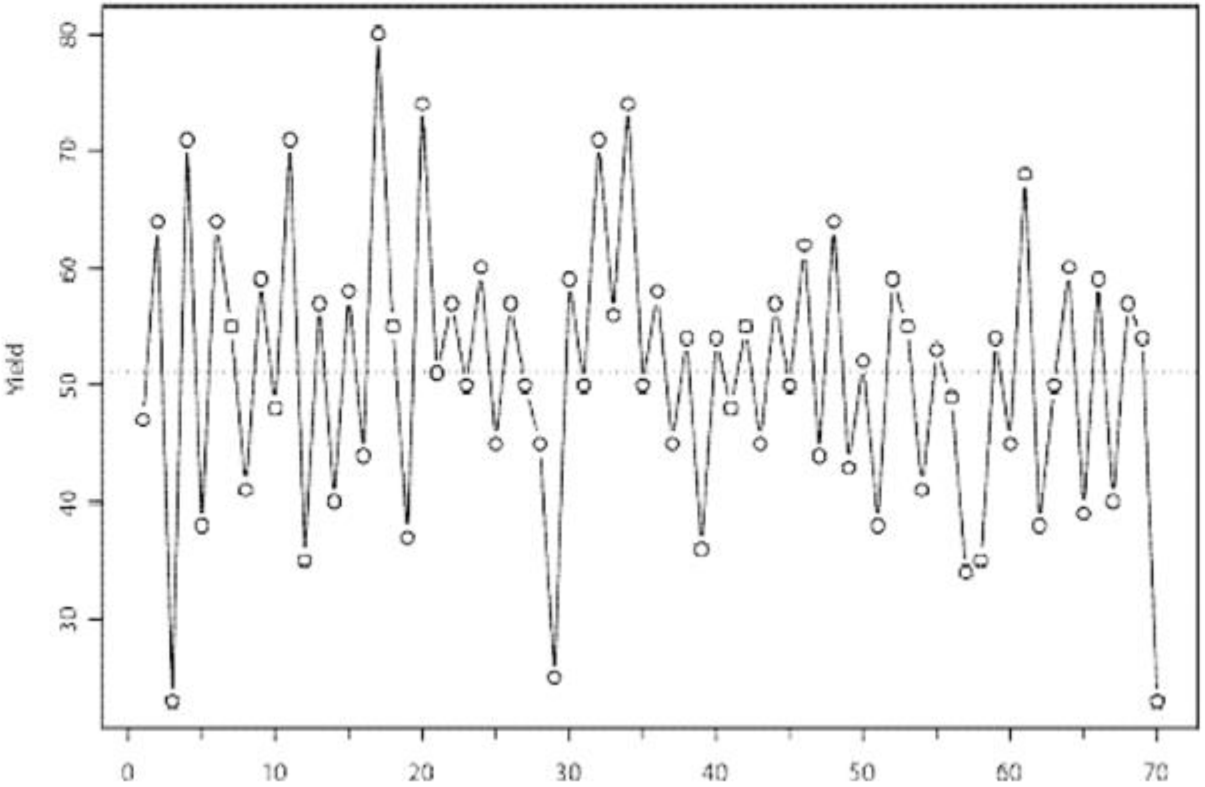
Example 4

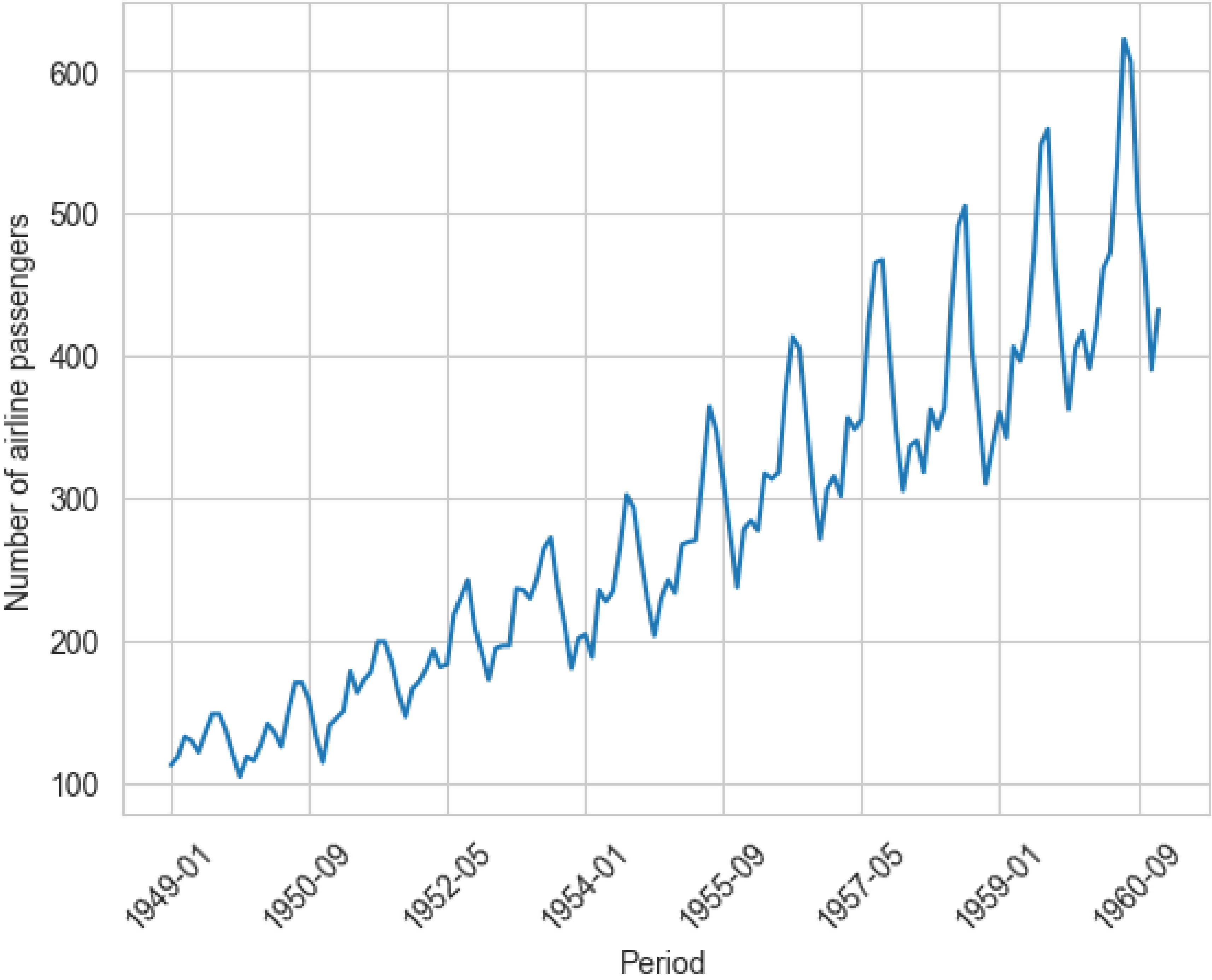
- Yield from a controlled chemical batch process

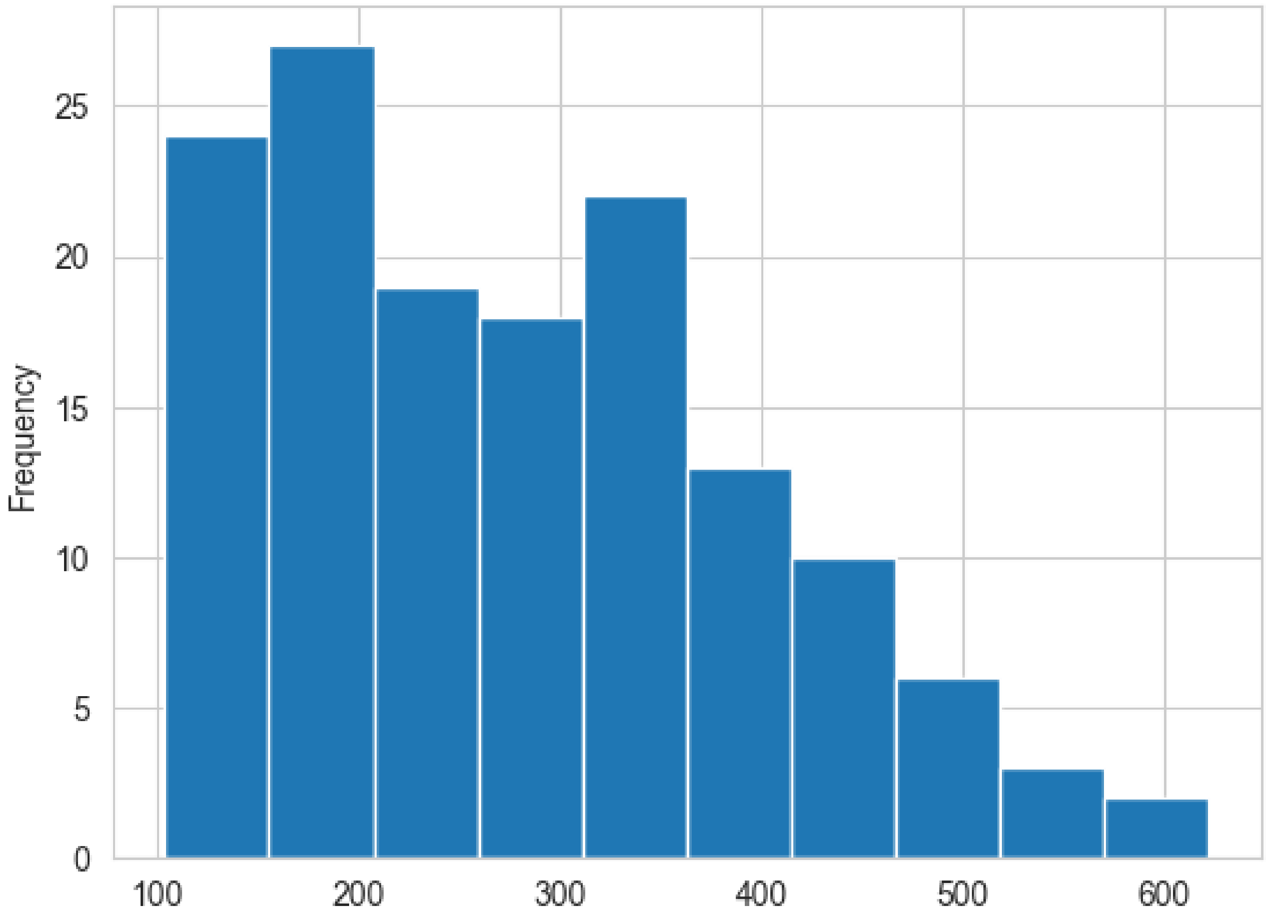


Example 4

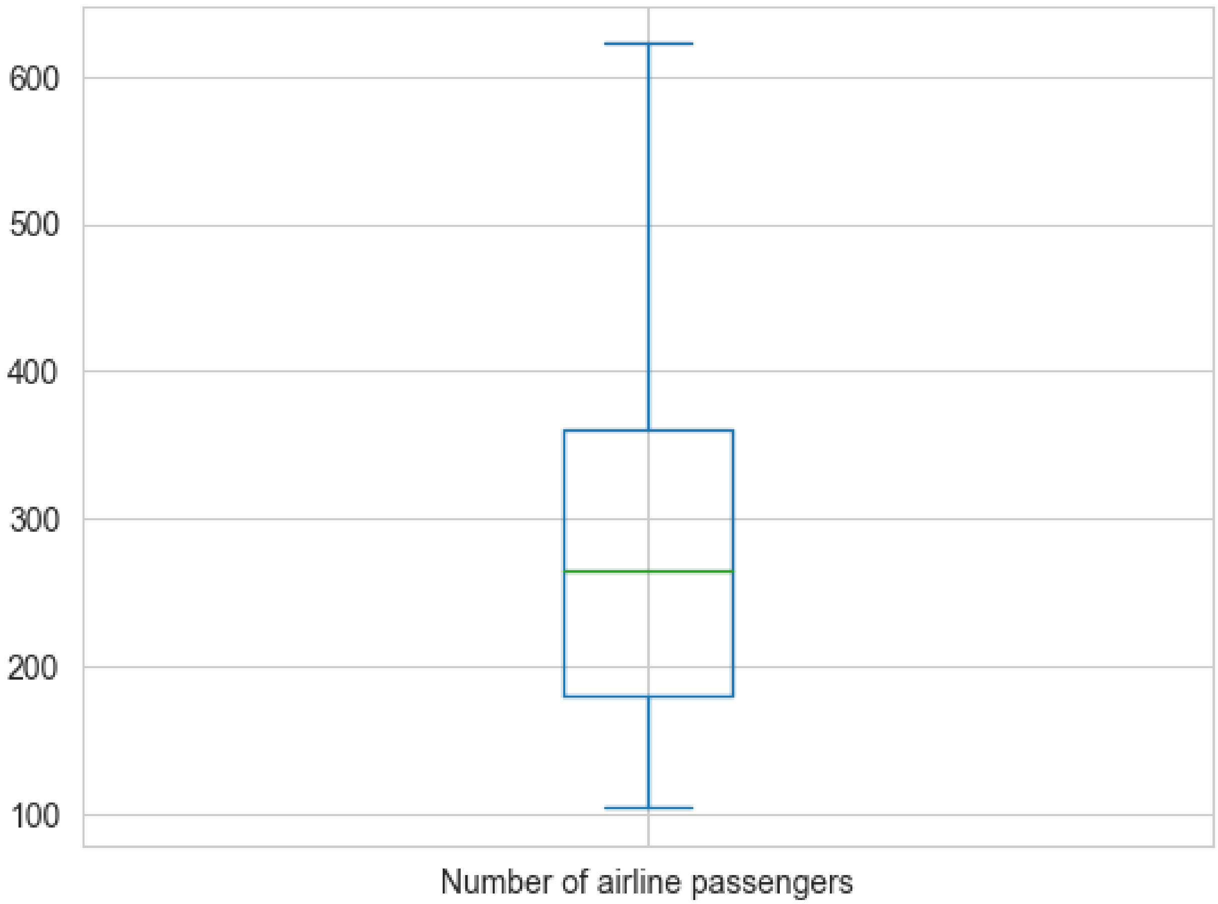
- Yield from a controlled chemical batch process
- Negative dependence: successive observations tend to lie on opposite sides of the mean.




600
500
400
300
200
100
Number of airline passengers
1949-01
1950-09
1952-05
1954-01
1955-09
1957-05
1959-01
1960-09
Period



TS Histogram



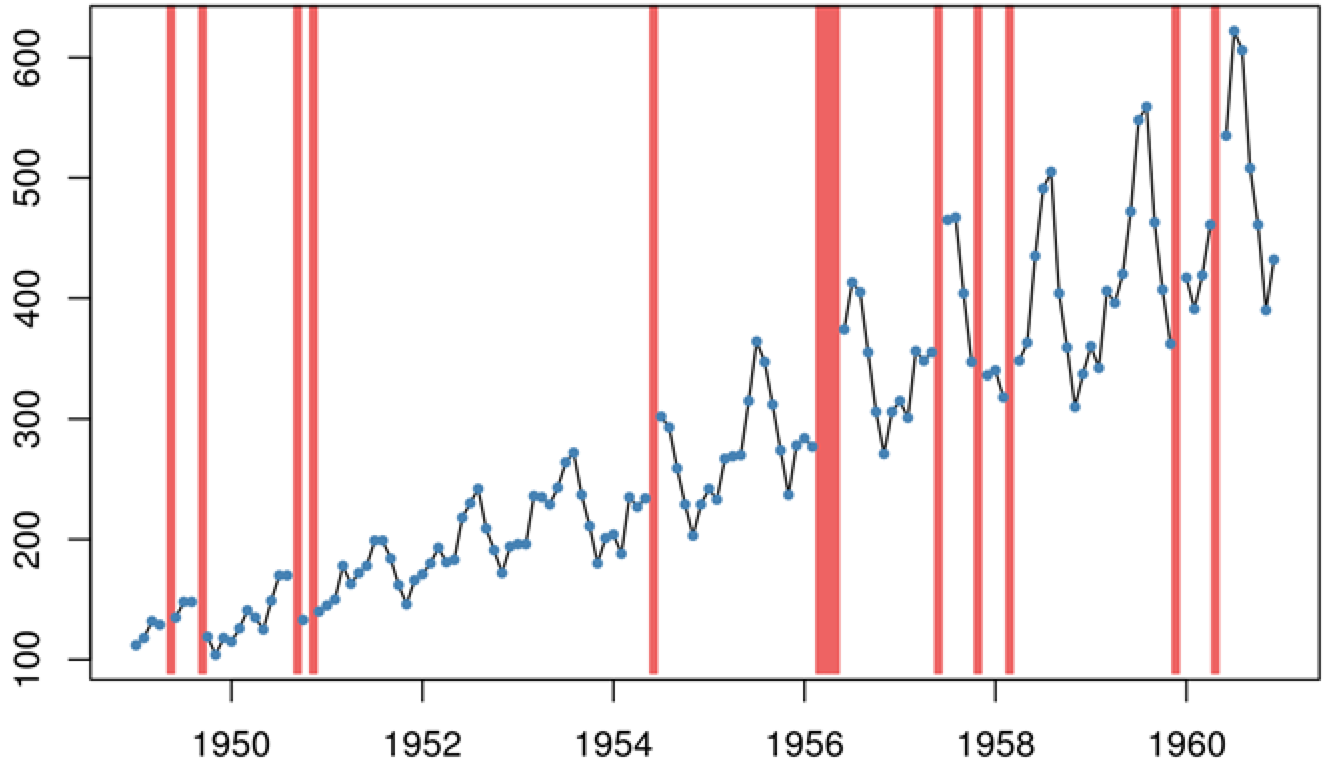
TS BoxPlot

Time in not considered in these plots!!!

Missing Values

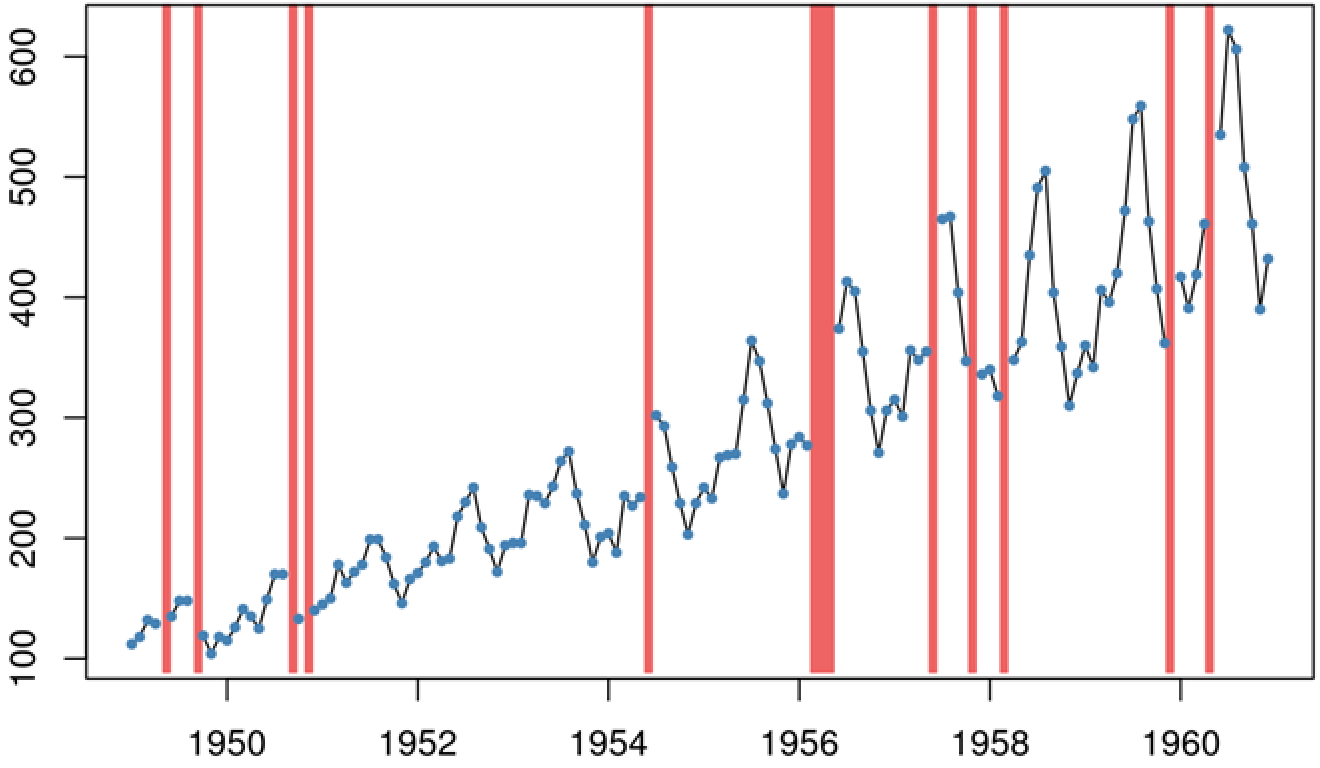
Missing Values

- Individual values for a single time stamp can be missing
- Contiguous values for sequential time stamps can be missing

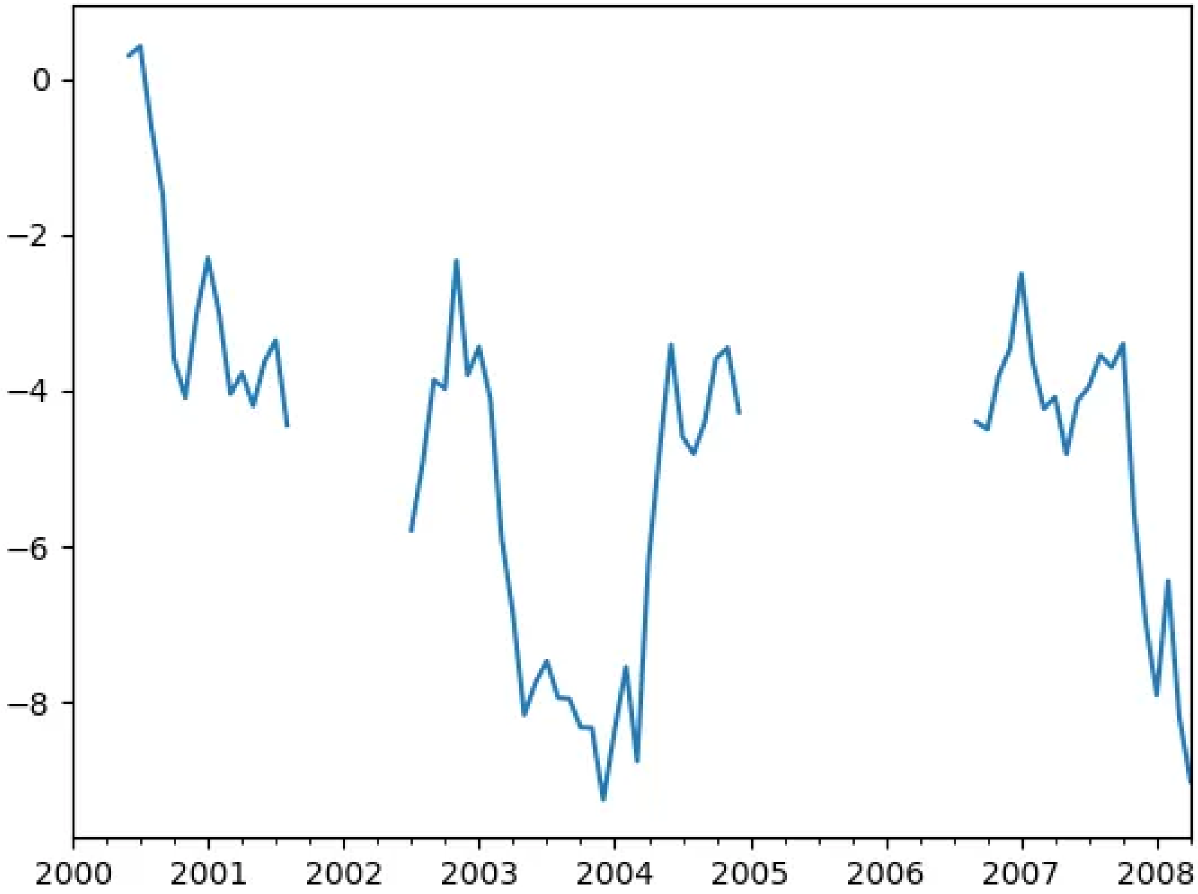


Missing Values Imputation Methods

- Fill with constant value
- Linear interpolation
- Forecasting
- Random

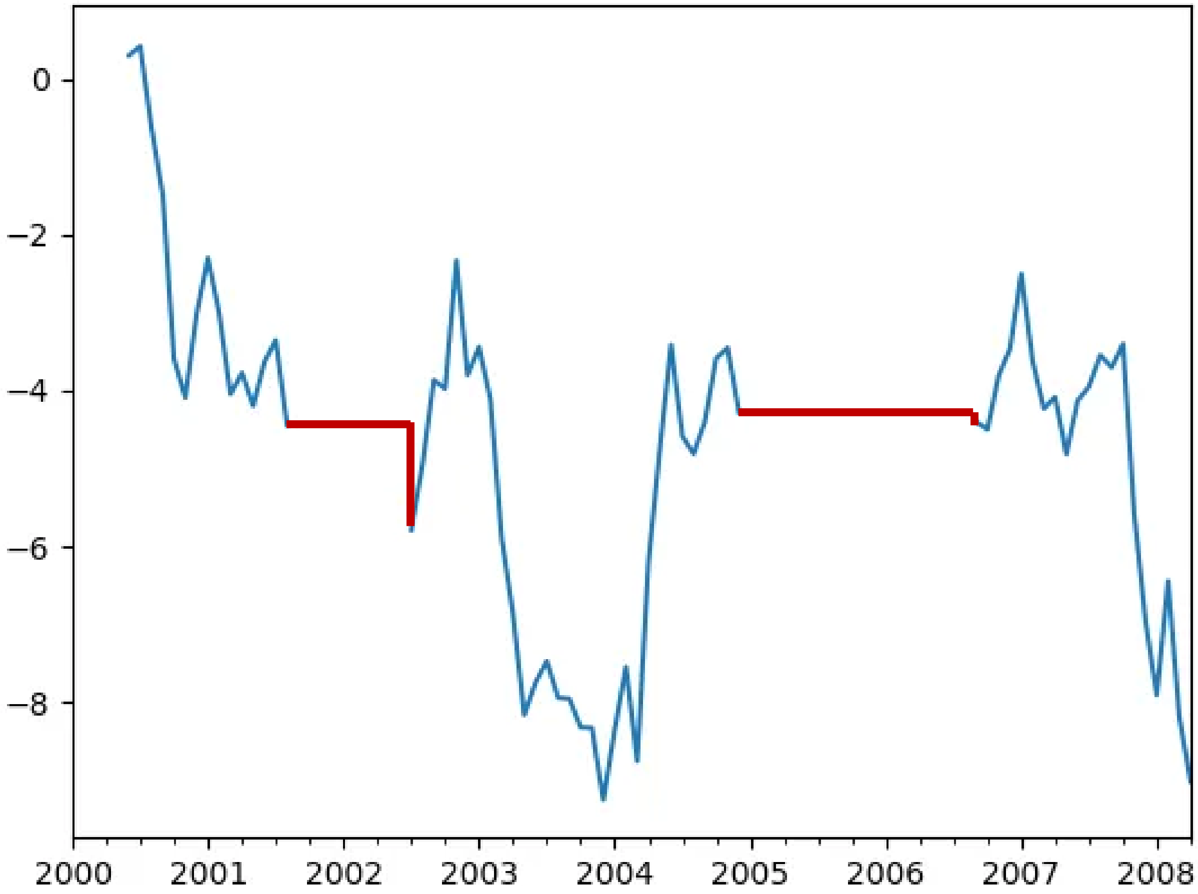


Missing Values Imputation



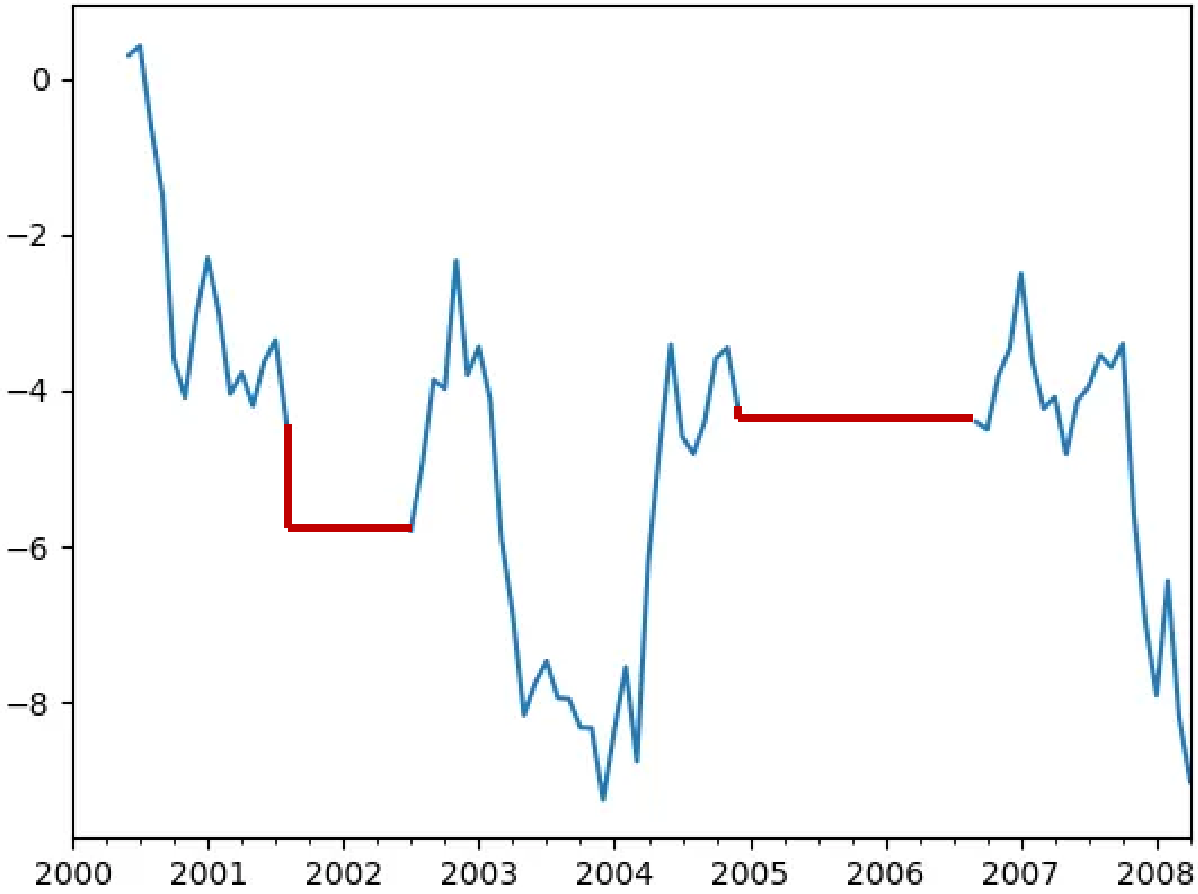
Missing Values Imputation: Padding

- Fill with constant value: **last** value (pad)



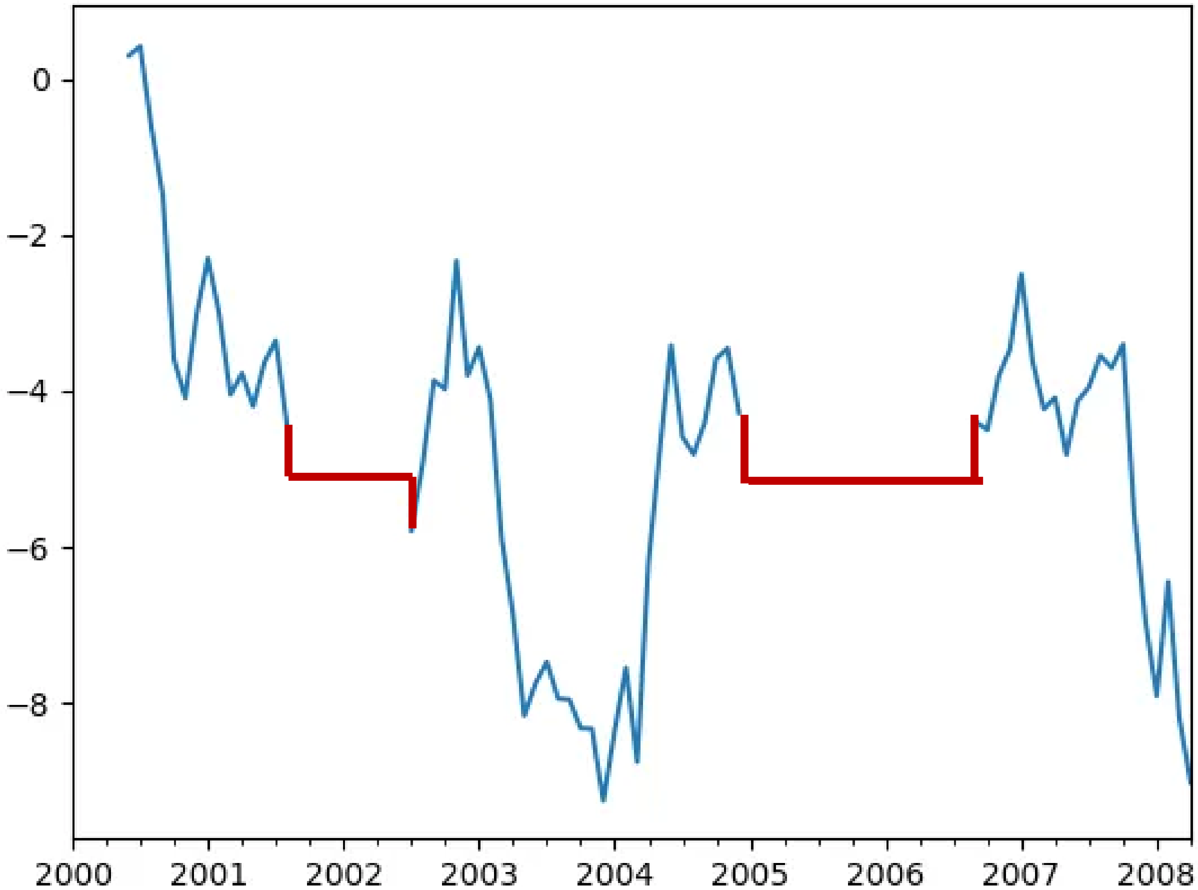
Missing Values Imputation: BackFilling

- Fill with constant value: **next** value (backfill)



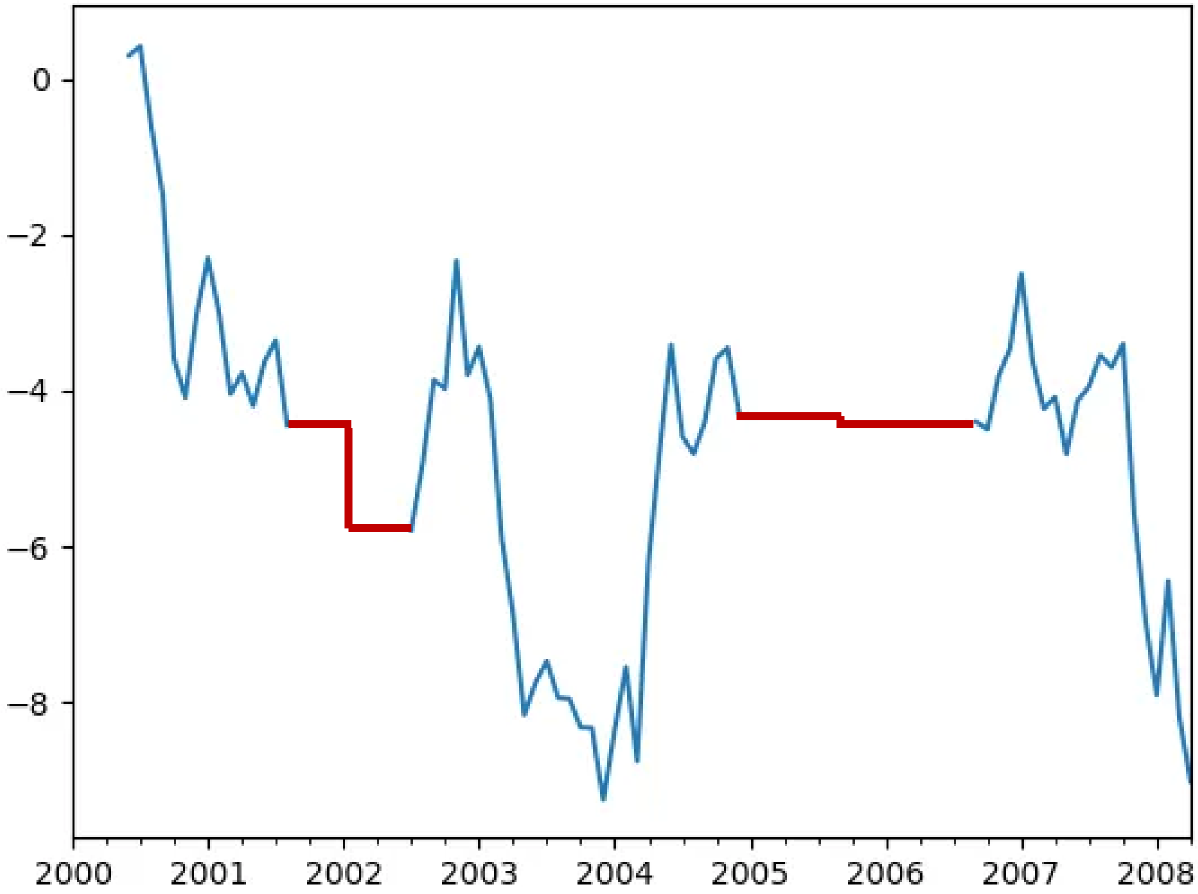
Missing Values Imputation: Mean

- Fill with constant value: **mean/median** value



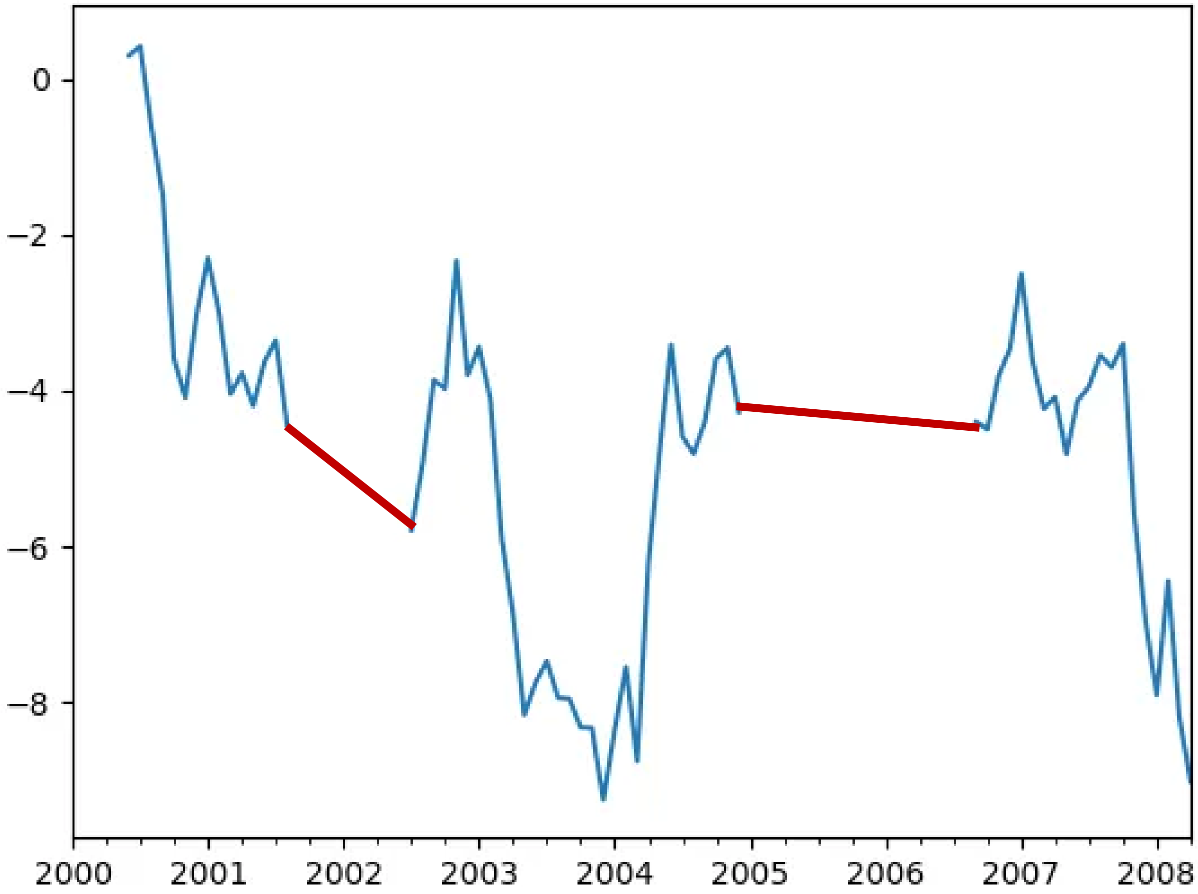
Missing Values Imputation: Nearest Value

- Fill with constant value: **nearest** value



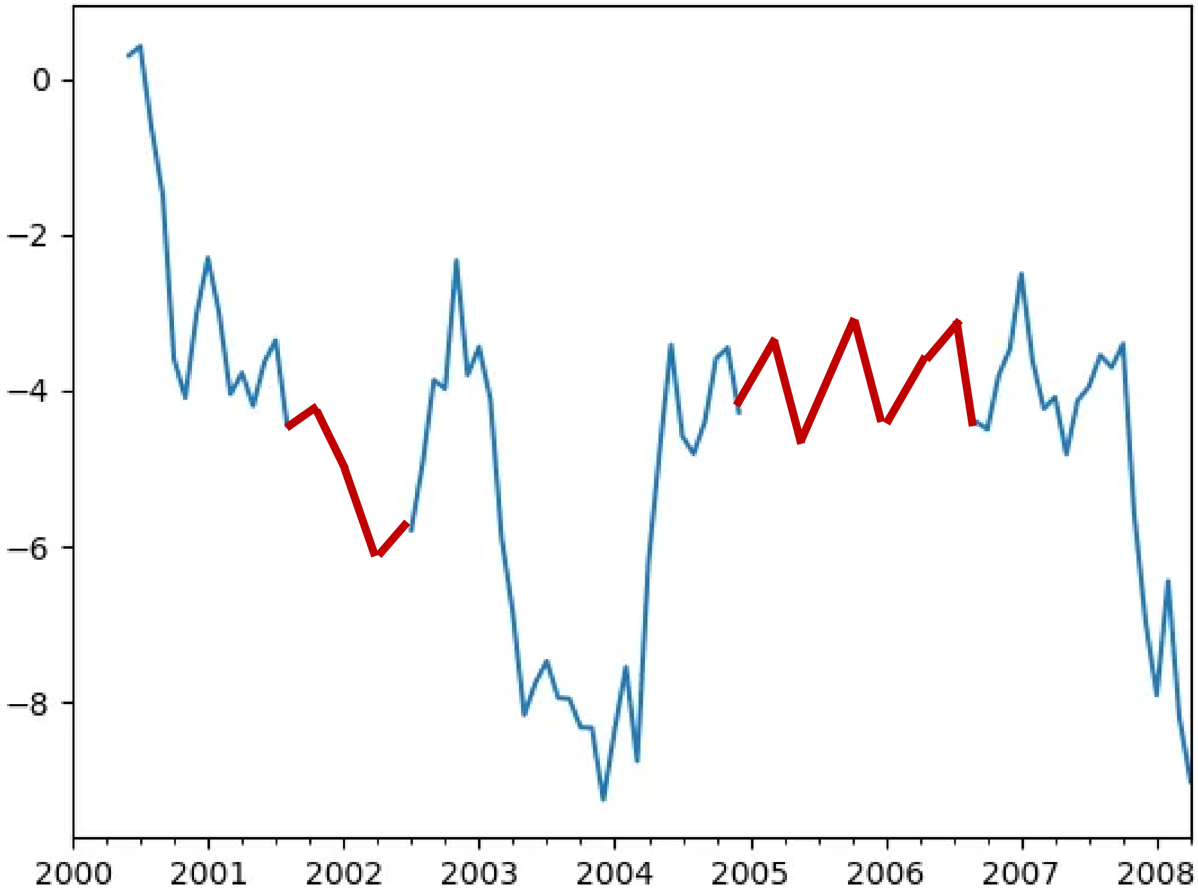
Missing Values Imputation: Interpolation

- Interpolate the last and first not missing values to get the missing ones.



Missing Values Imputation: Forecasting

- Interpolate using a forecasting or regressive model (see next lectures)



Anomalies

Anomalies and Outliers in Time Series

- An outlier (or anomaly) is a value or an observation that is distant from other observations, a data point that differ significantly from other data points.
- Outlier: “An observation which deviates so much from other observations as to arouse suspicions that it was generated by a different mechanism.” [Hawkins 1980]
- Sometimes it makes sense to formally distinguish two classes of outliers: Extreme values and mistakes.
- Mistakes can be considered as missing values and removed.
- Extreme values might be considered in the analysis.

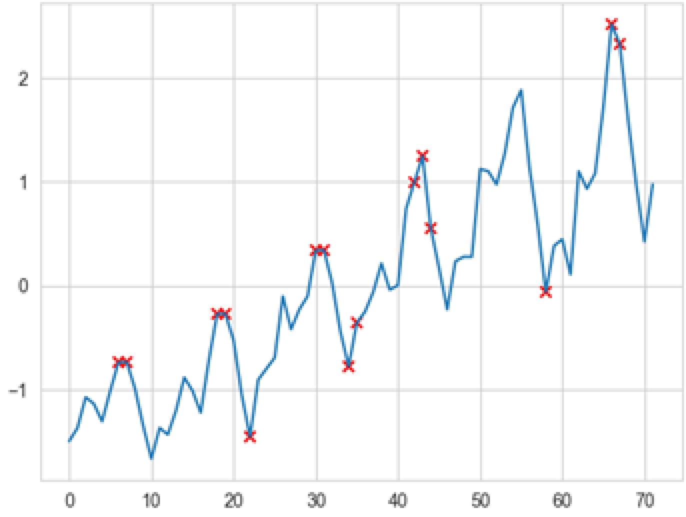
Outlier Detection Methods

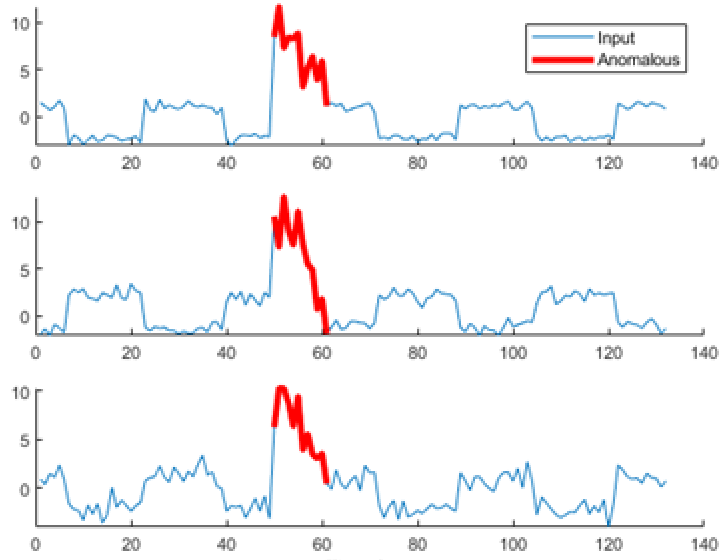
- Outlier detection methods **create a model of the normal patterns in the data**, and then compute an outlier score of a given data point based on the deviations from these patterns.
- Different models make different assumptions about the “normal” behavior.
- The outlier score of a data point is then computed by evaluating the quality of the fit between the data points and the model.
- In practice, the choice of the model is often dictated by the analyst‘s understanding of the kinds of deviations relevant to an application.

Outlier Types in Time Series

Outlier detection methods may differ depending on the type of outliers:

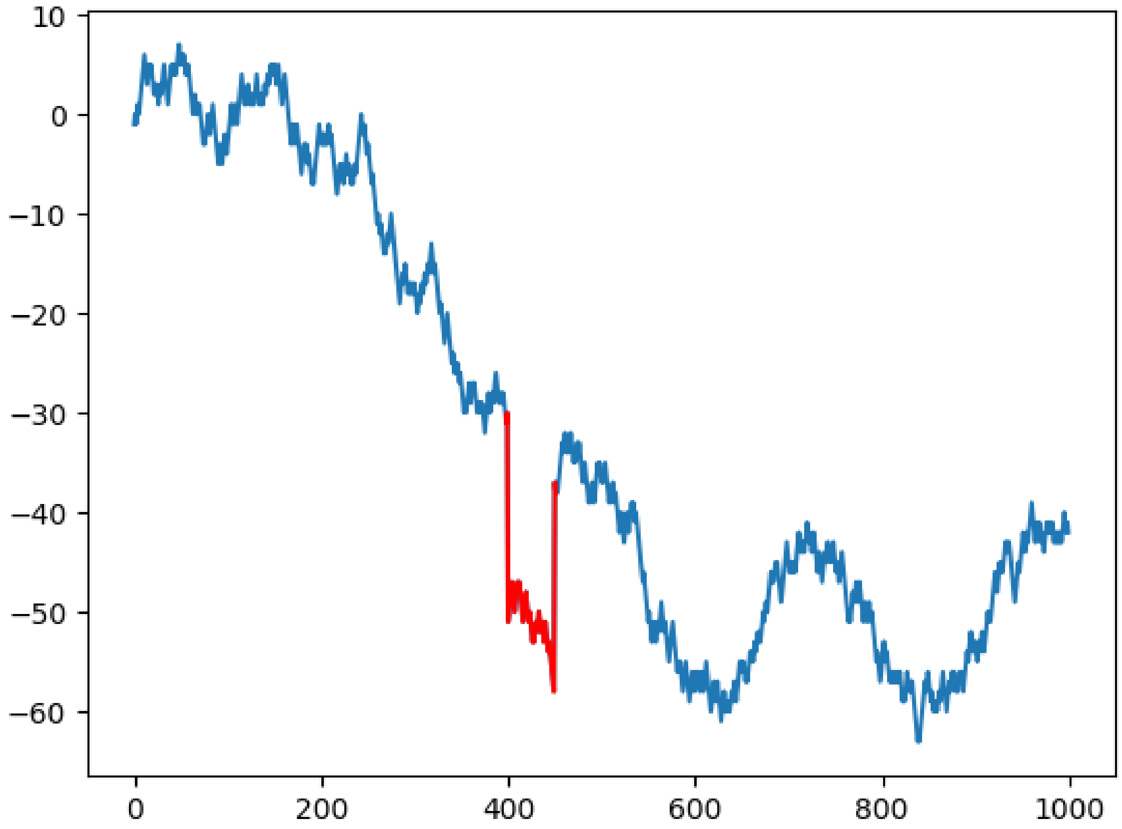
- **Point Outlier**: A value that behaves unusually in a specific time instant when compared either to the other values in the time series (global outlier) or to its neighboring points (local outlier).
- **Subsequences**: Consecutive points in time whose joint behavior is unusual, although each observation individually is not necessarily a point outlier
- **Instance**: Entire time series can also be outliers, but they can only be detected when the input data is a dataset of time series.





Examples of Anomalies in Time Series

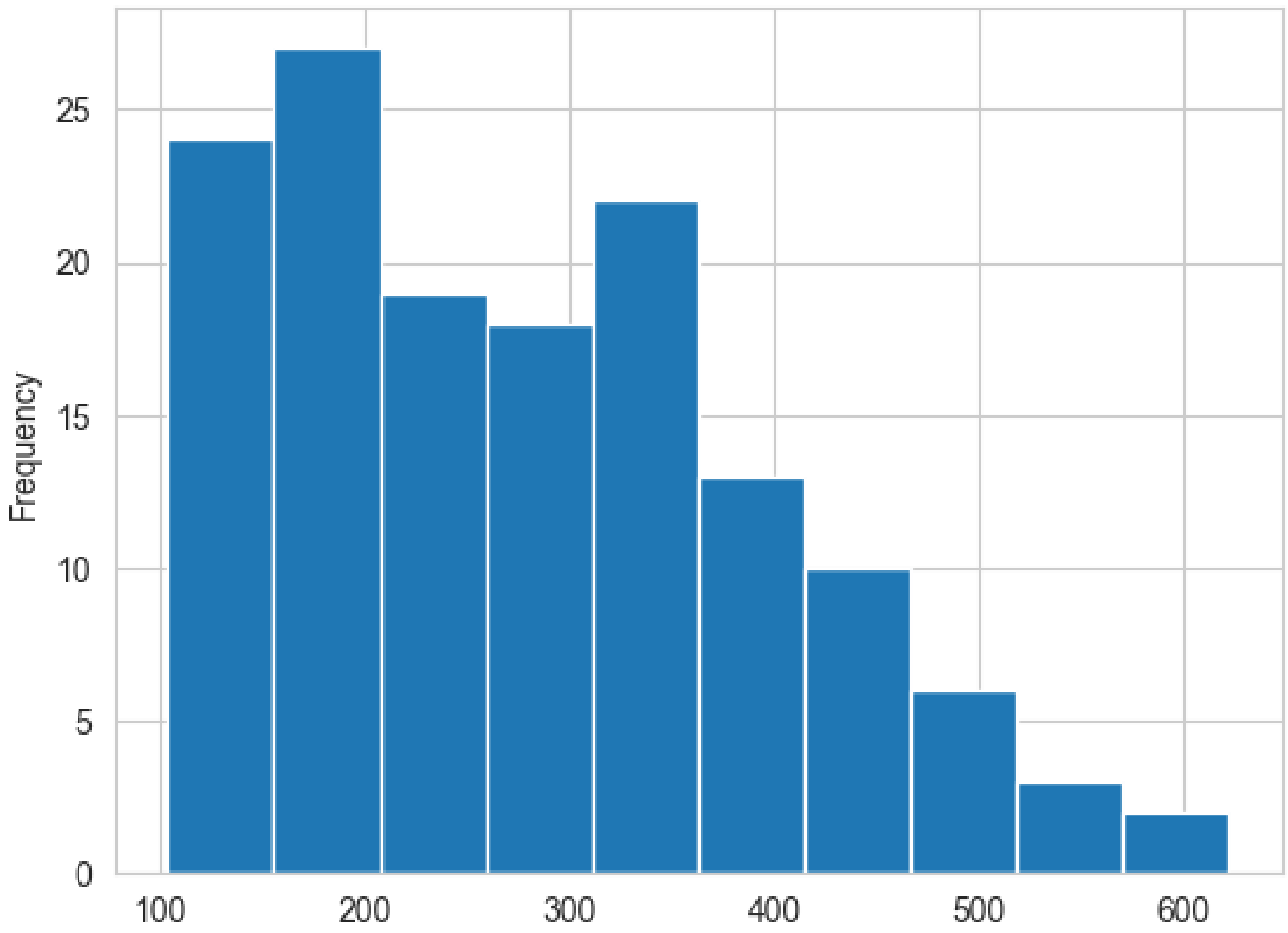
- **Level shifts:** when the signal moves to zero/default value for a short period of time.
- **Unexpected growth/decrease** in a short period of time that looks like a spike.



Outlier Detection Method

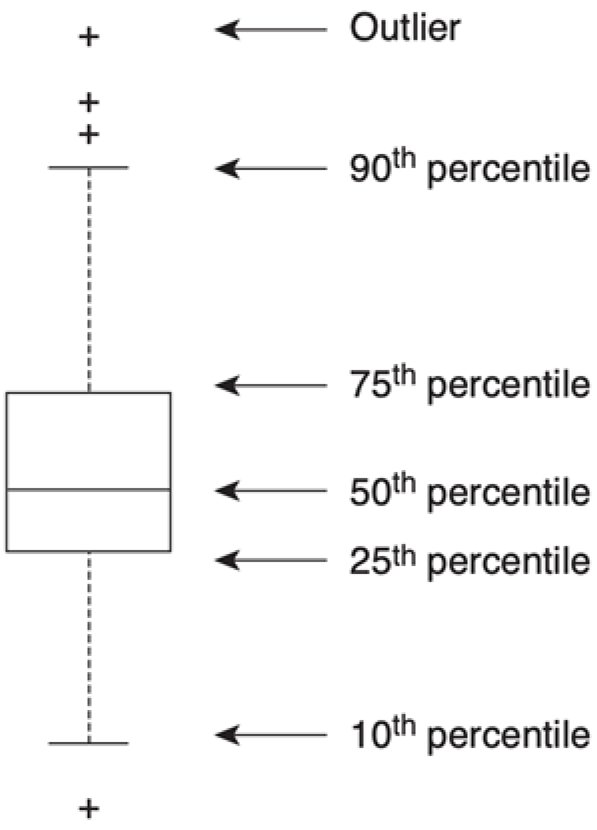
- An outlier is a point that significantly deviates from its expected value.
- Given a univariate time series *x*, a point at time *t* can be declared an outlier if the distance to its expected value is higher than a predefined threshold.
- **Estimation method**: if $\bar{x}_t$ is obtained using previous and subsequent observations to x_t (past, current, and future data).
- **Prediction method**: if $\bar{x}_t$ is obtained relying only on previous observations to x_t (past data).

Histogram



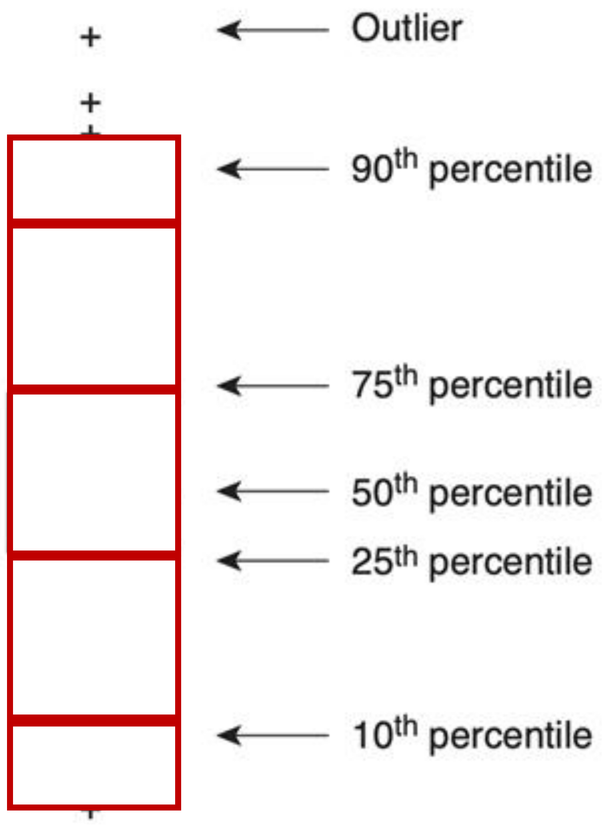
Boxplot

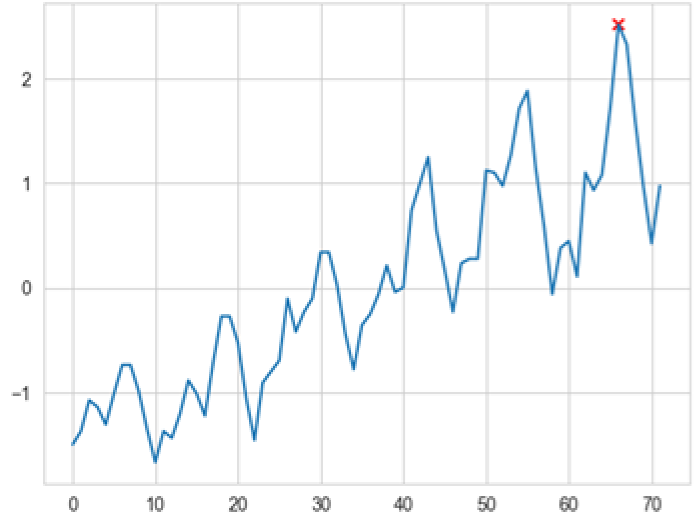
- Data represented with a **box**
- The ends of the box are at the
  - **Q1: 1st quartiles** (25%-quantile or 25th percentile)
  - **Q3: 3rd quartiles** (75%-quantile or 75th percentile)
- **Median**: value in the middle is the **Q2**: **2nd quartile** (50%-qua,, 50th perc.)
- The height of the box is **Interquartile range** (**IQR**): Q3 - Q1
- **Whiskers**: two lines outside the box extended from:
  - 1st, or 5th, or 10th percentile, or Q1 – k IQR (with k = 1.5)
  - 99th, or 95th, or 90th percentile, or Q3 + k IQR (with k = 1.5)
- **Outliers**: are points beyond whiskers
- In general, p%-quantile (0 < p < 100): Is the value *x* s.t. p% of the values are smaller and 100-p% are larger.



Inter-Quartile Range Filter

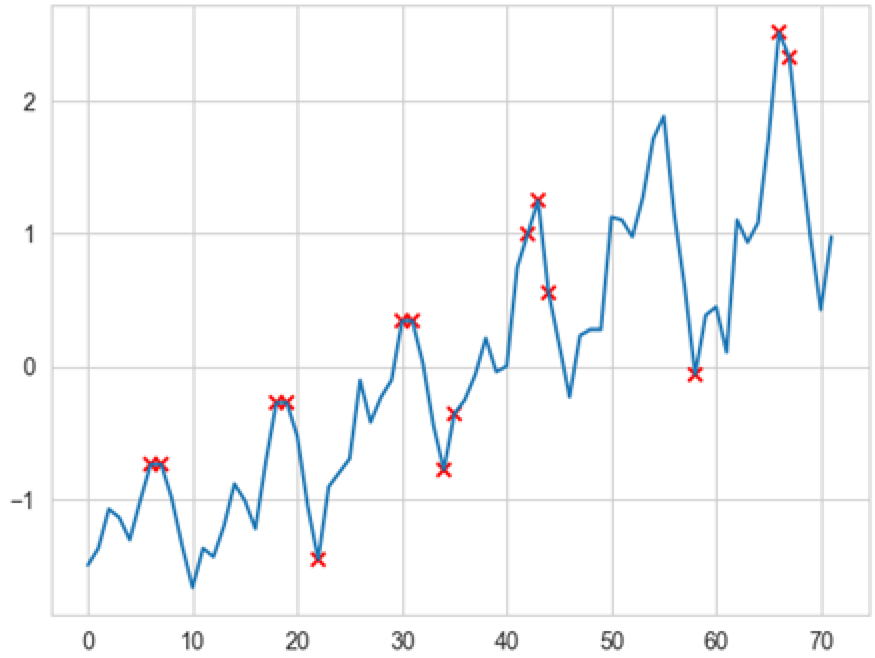
- The IQR criteria means that all observations
- above Q3 + 1.5 $*$ IQR or
- below Q1 − 1.5 $*$ IQR
- where Q3 and Q1 correspond to third ($q_{0.75}$) and first ($q_{0.25}$) quartile respectively, and IQR is the difference between the third and first quartile) are considered as potential outliers.



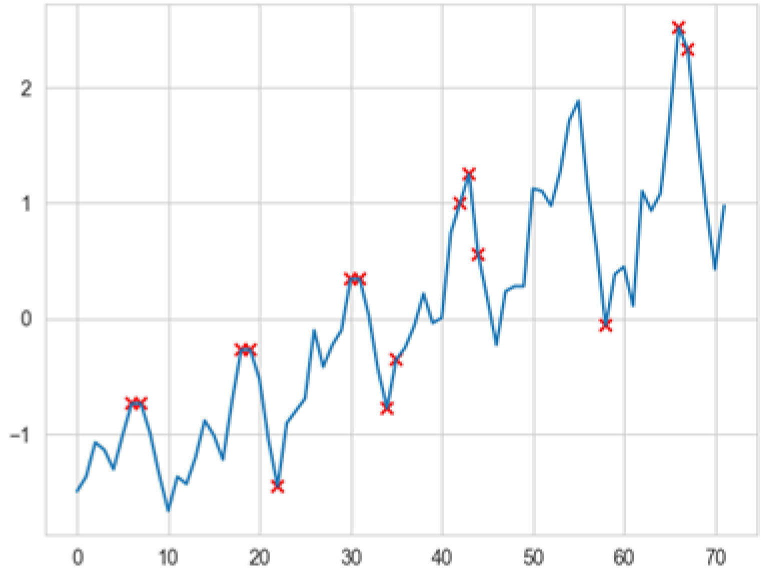


Hampel Filter

- Consider as outliers the values outside the interval (I) formed by the median, plus or minus 3 Median Absolute Deviations (MAD):
- I=[$median_X - 3 * MAD; median_X + 3 * MAD$]
- where MAD is the median absolute deviation and is defined as the median of the absolute deviations from the data's median, i.e.,
- $MAD = median(|x_i - median_X|)$



Grubbs' Test



- Detect outliers in univariate data
- Assume data comes from normal distribution
- Detects one outlier at a time, remove the outlier, and repeat
  - H_0: There is no outlier in data
  - H_A: There is at least one outlier
- Grubbs' test statistic:

mean

$$G = \frac{\max\left|X - \overline{X}\right|}{s}$$

std dev

one-sided test with alpha/N
two-sided test with alpha/2N

- Reject null hypothesis H_0 of no outliers if:

alpha significance
t – Student's distribution

$$G > \frac{(N-1)}{\sqrt{N}} \sqrt{\frac{t^2_{(\alpha/N,\,N-2)}}{N-2+t^2_{(\alpha/N,\,N-2)}}}$$

degrees of freedom

upper critical value of t-distribution

Handling Outliers in Time Series

Once that anomalies have been identified

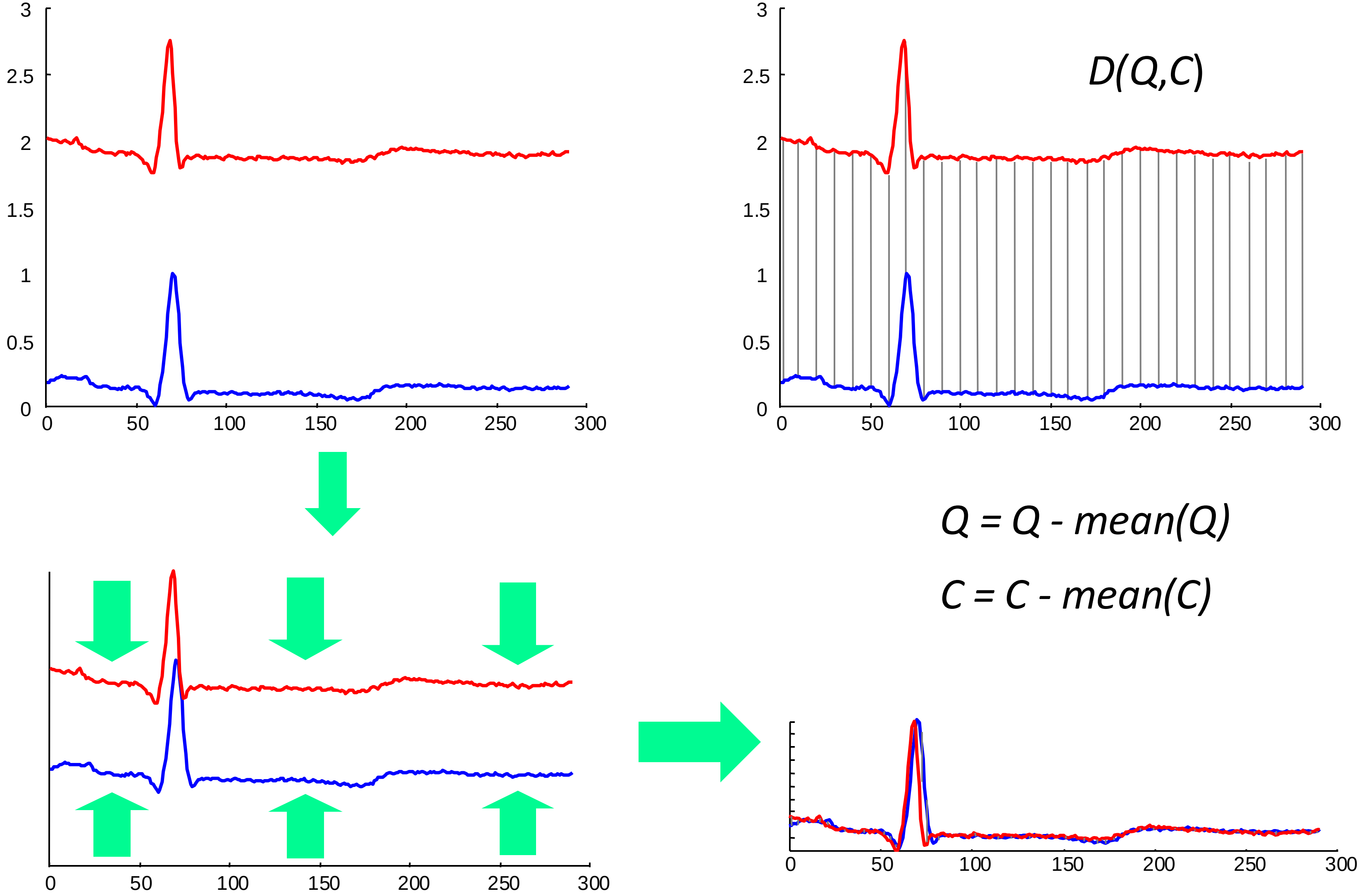
a) they can be removed and treated like missing values, i.e., replaced according to different strategies.
b) they can be maintained in the analysis.
c) they can be distinguished between “mistakes” and “extreme values” and treated according the most preferrable strategy.

Normalizations

Problem with Time Series Distortions

- A common tool of TSA consists in calculating distances among time series.
- Many ML-tools require that the input data is represented in the same range of values or that values are comparable.
- Distance calculations and ML models are very sensitive to “distortions” in the data.
- These distortions are dangerous and should be removed.
- Most common distortions:
  - Offset Translation
  - Amplitude Scaling
  - Linear Trend
  - Presence of Noise
- These distortions can be removed by using the appropriate normalizations.

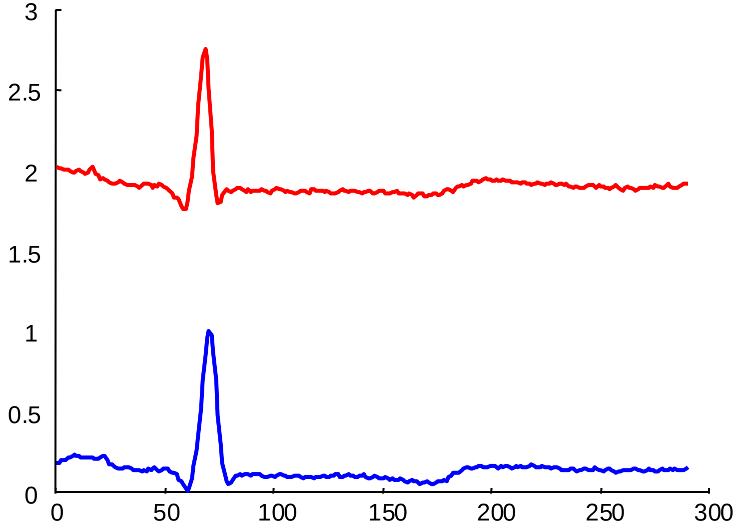
Offset Translation: Mean Removal

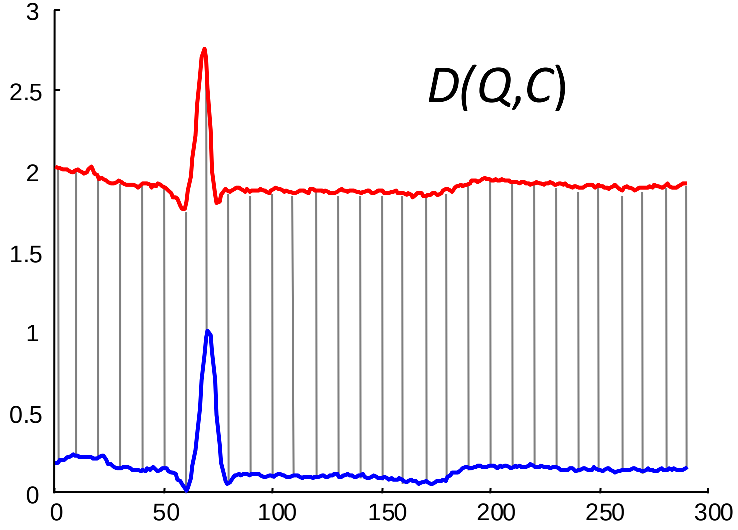


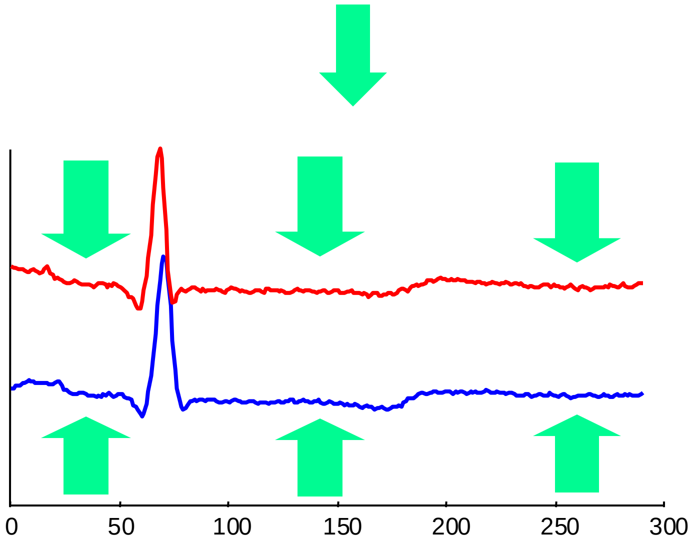
Q = Q - mean(Q)

C = C - mean(C)

Offset Translation: Min-Max Normalization

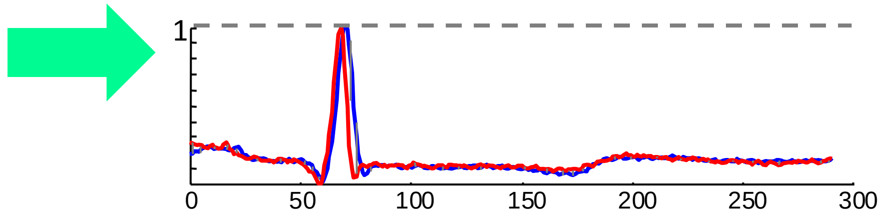




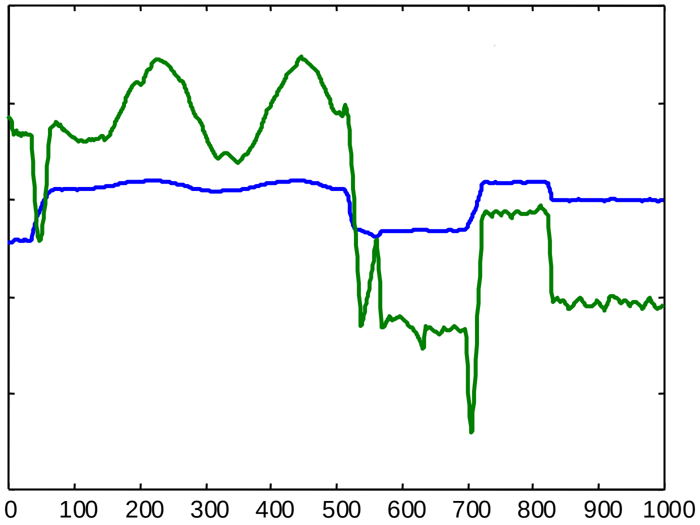


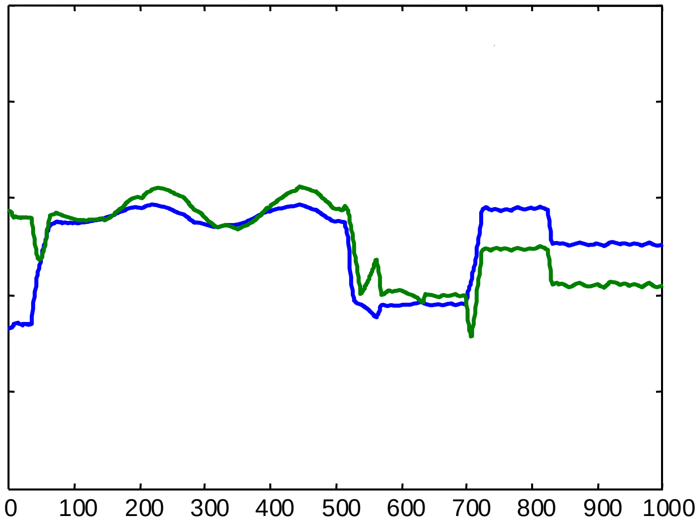
$$Q = (Q - min(Q)) \,/\, (max(Q) - min(Q))$$

$$C = (C - min(C)) \,/\, (max(C) - min(C))$$



Amplitude Scaling: Z-Score Normalization



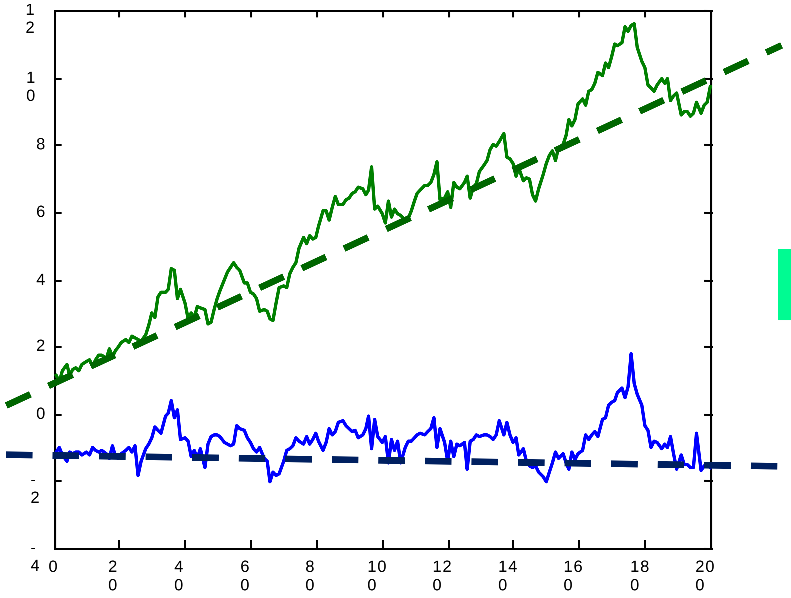


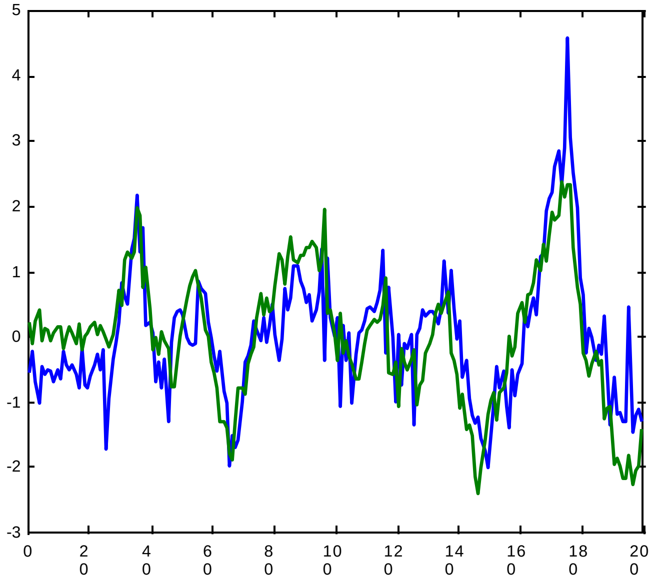
Q = (Q - mean(Q)) / std(Q)

C = (C - mean(C)) / std(C)

Linear Trend: Detrending

- Fit the best fitting straight line to the time series, then subtract that line from the time series.

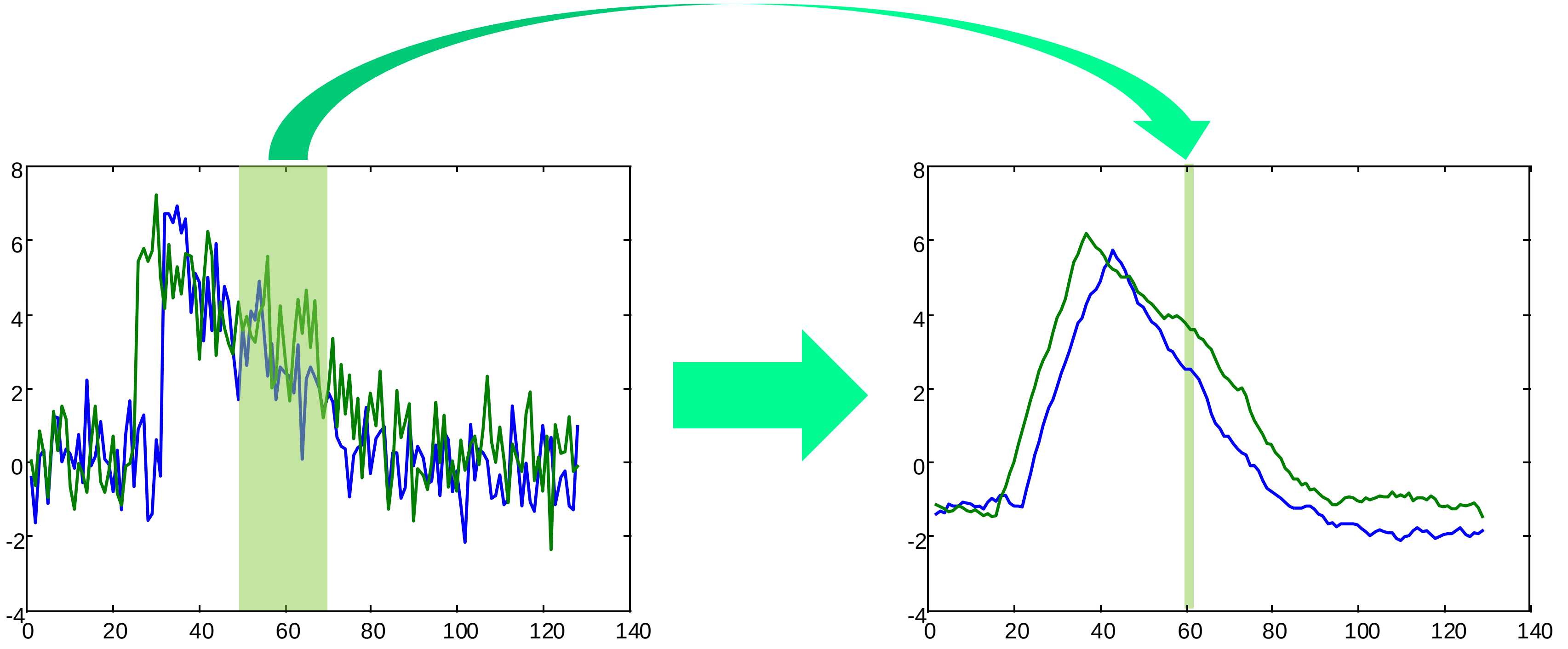




Removed linear trend, offset translation, amplitude scaling

Noise Removal: Mean Smoothing

- The intuition behind removing noise is to average each datapoints value with its neighbors.



$Q = smooth(Q)$

$C = smooth(C)$

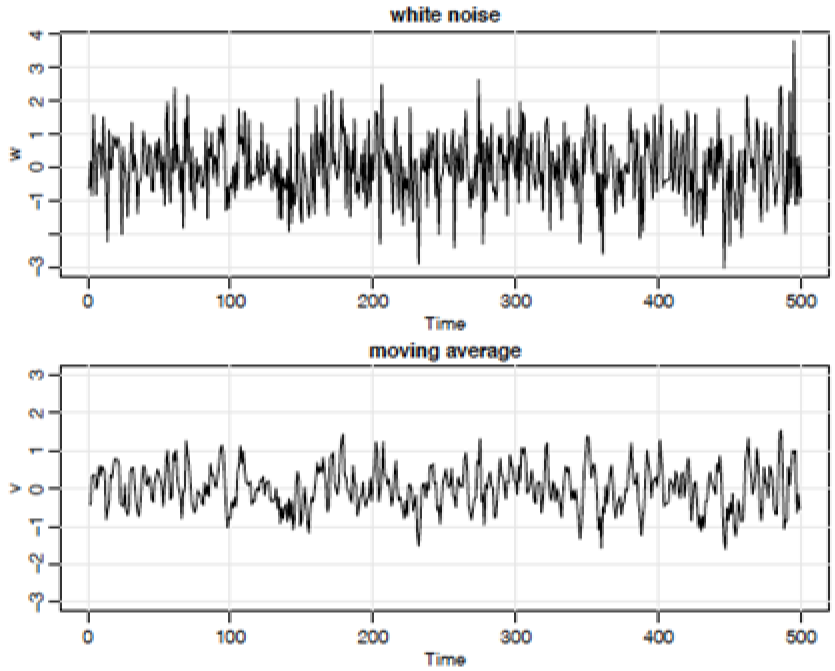
$D(Q,C)$

Moving Average

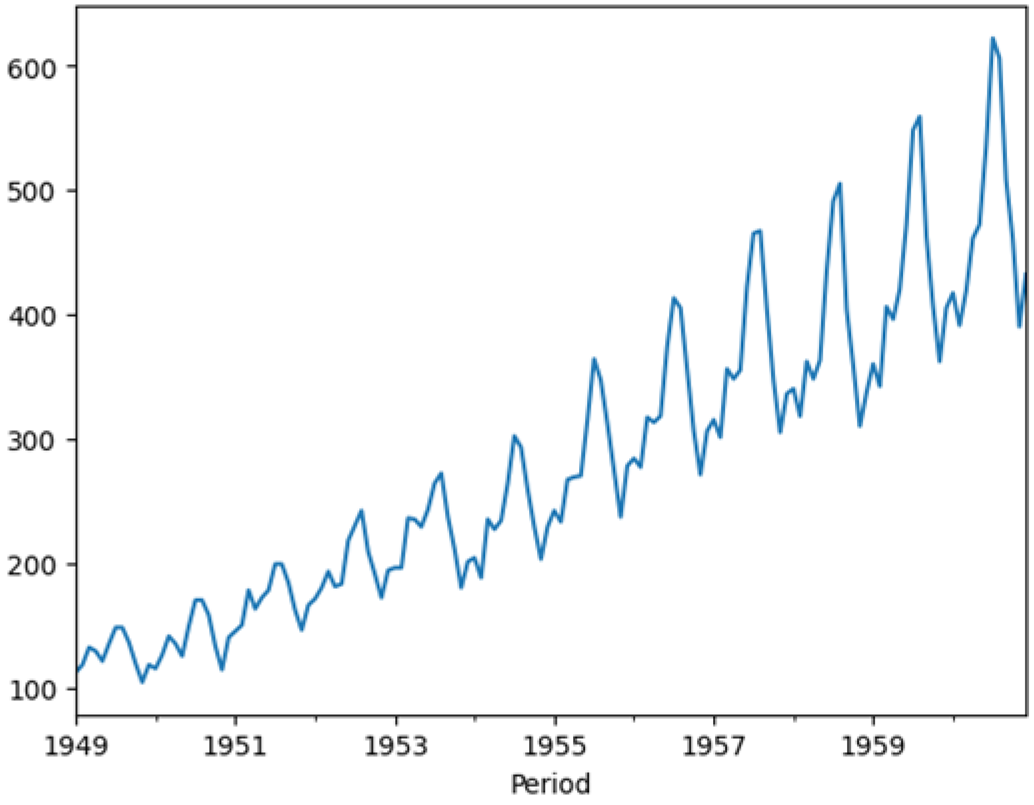
- Noise can be removed by a **moving average** (MA) that smooths the TS.
- Given a window of length *w* and a TS *t*, the MA is applied as follows
- $t_i = \frac{1}{w} \sum_{j=i-w/2}^{w/2} t_j$ for $i = 1, \ldots, n$
- For example, if w=3 we have
- $t_i = \frac{1}{3} \left(t_{i-1} + t_i + t_{i+1}\right)$

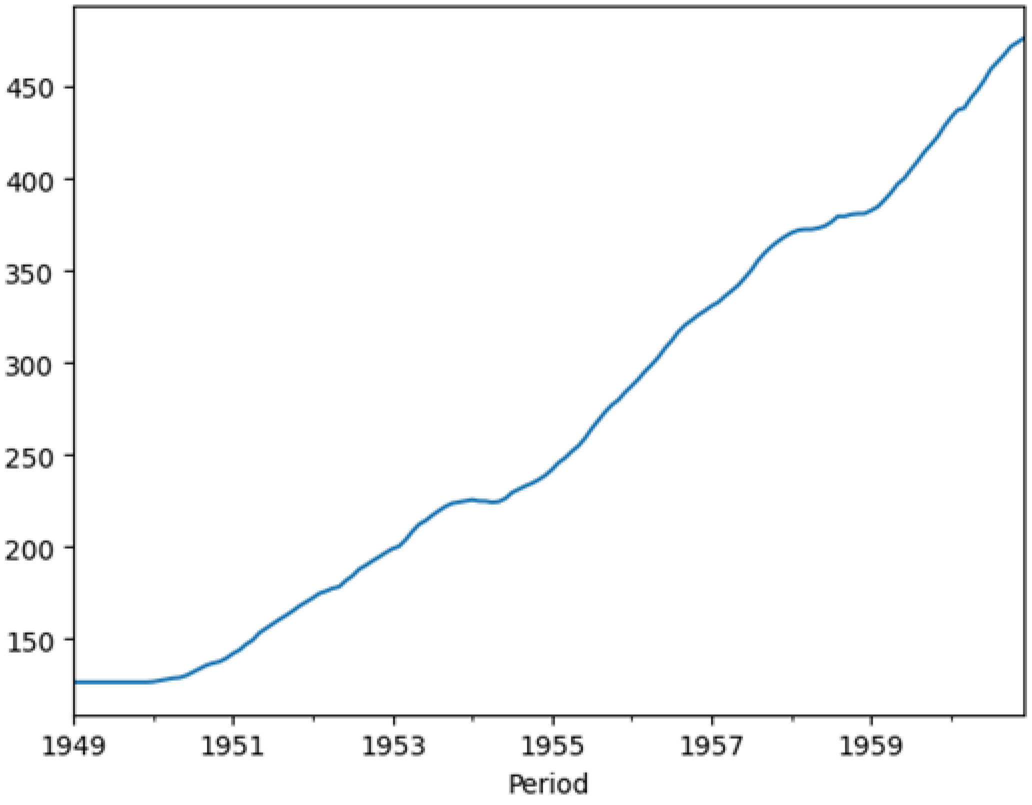
w=3

time	value	ma
t1	20	-
t2	24	22.0
t3	22	24.0
t4	26	24.3
t5	25	-



Moving Average Example



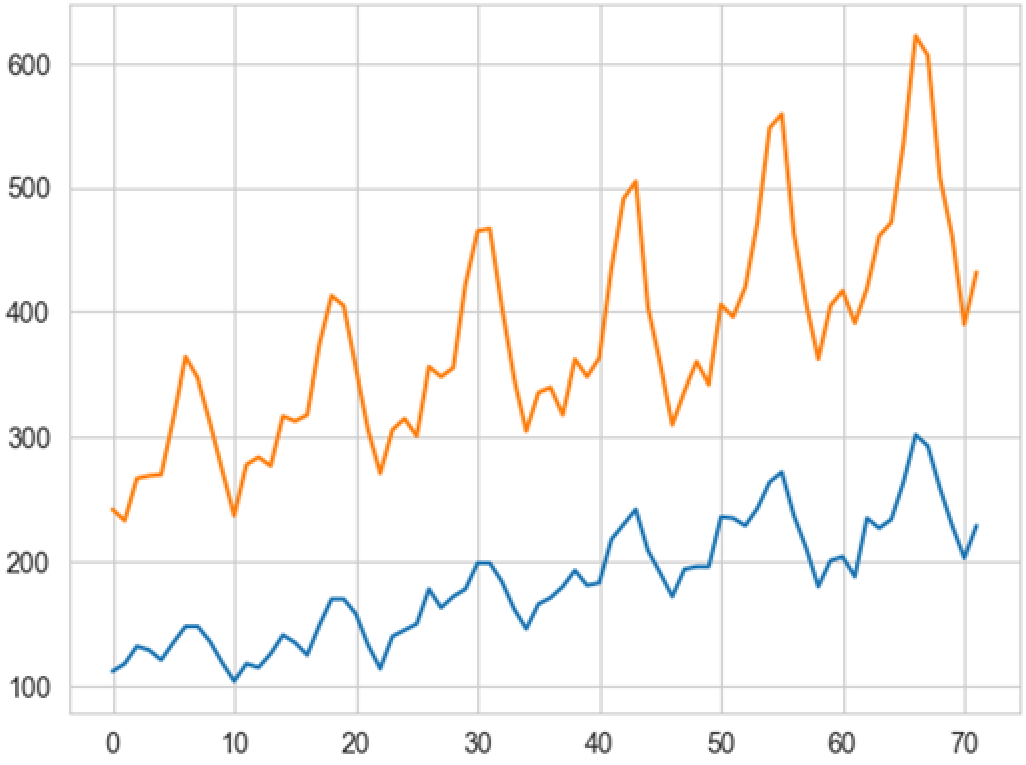


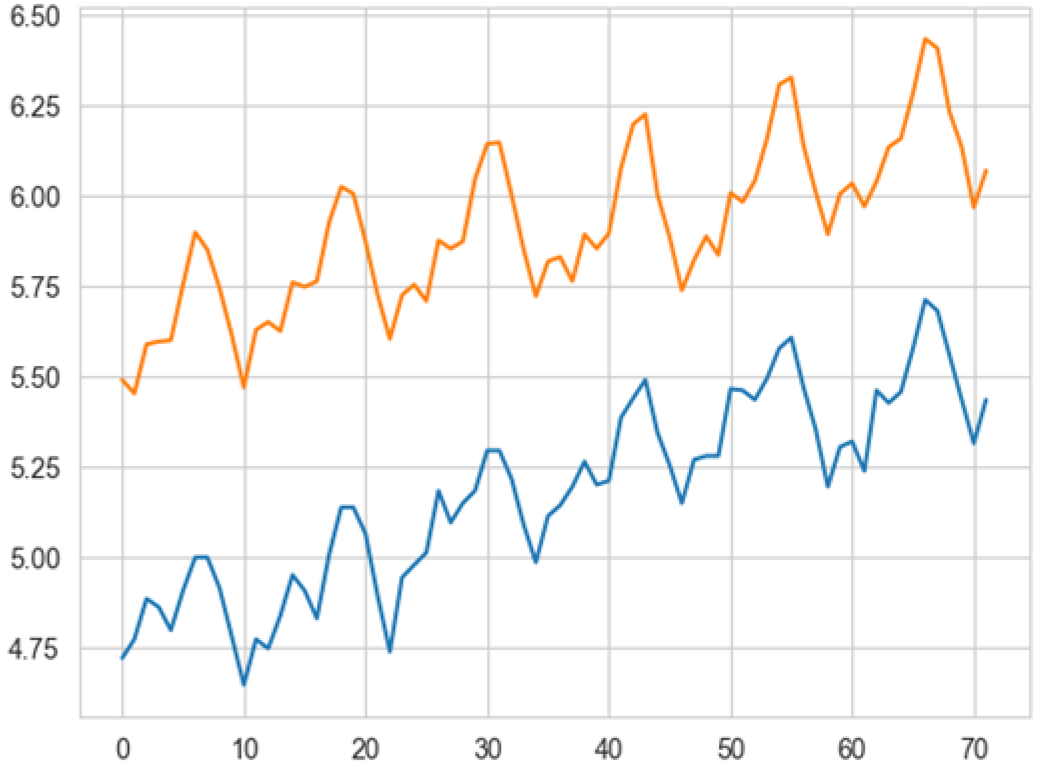
Log Transformation

- We apply the logarithm to each value of the TS.

Log
Q = log(Q)
C = log(C)

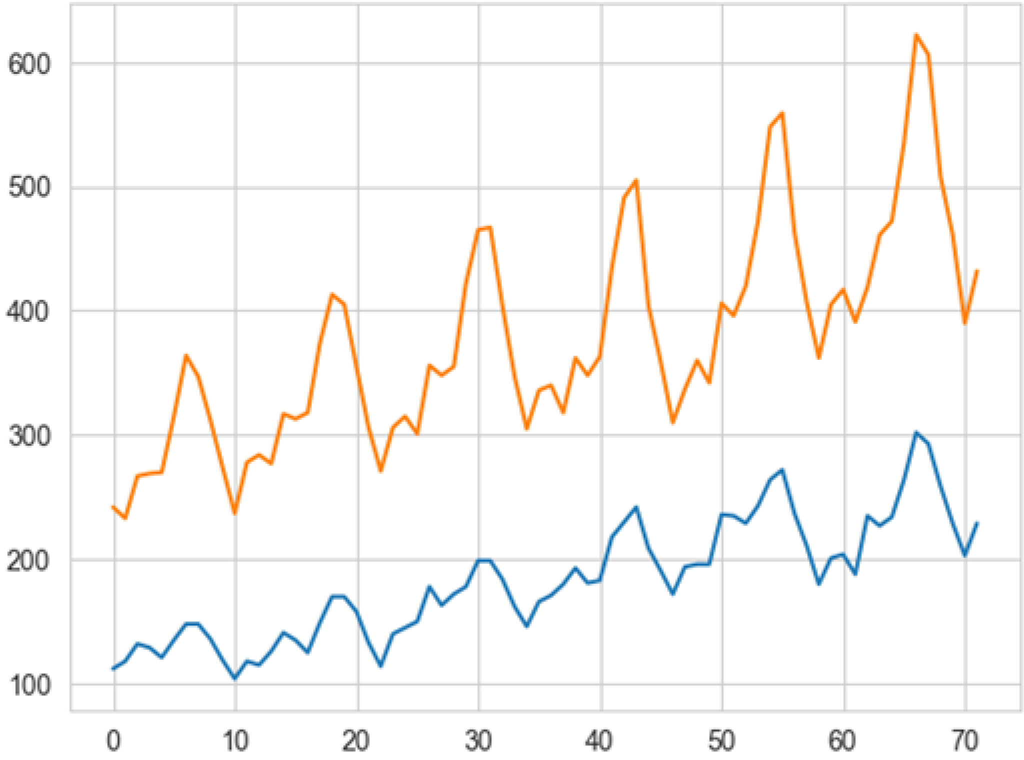
Log1p
Q = log(Q + 1)
C = log(C + 1)

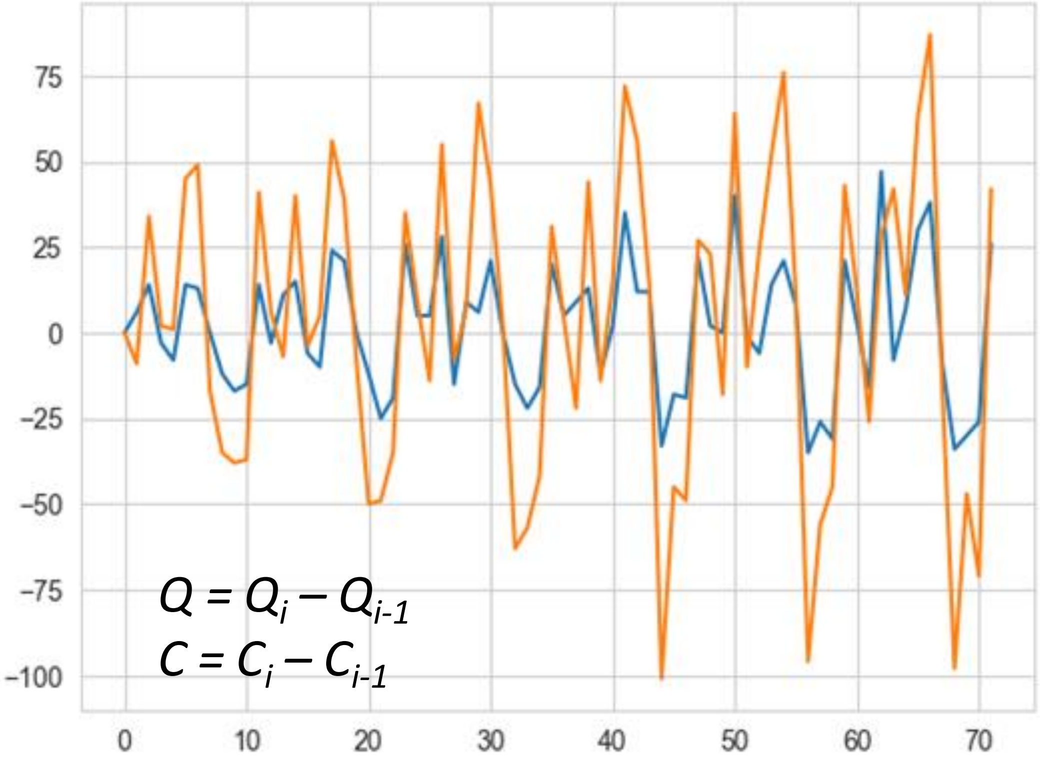




Differencing Transformation

- **Differencing**: we take the difference of the observation at a particular instant with that at the previous instant.





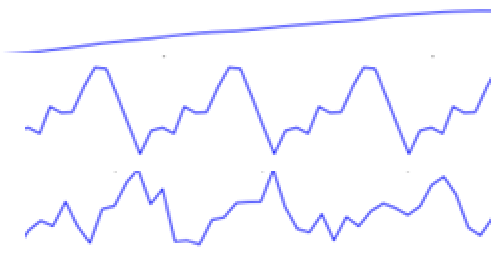
Time Series Components

Assumption

- Various Data Mining, Machine Learning and TSA models assume that variables are Independent and Identically Distributed (IID)
  - In times series values are (usually) not independent
  - Trend and seasonality might be present
  - Variance may change significantly (heteroskedasticity)
- A first goal in TSA is to reduce the time series to a simpler case:
  - Eliminate trend
  - Eliminate seasonality
  - Eliminate heteroskedasticity
- Then we model the remainder as dependent but identically distributed variables.

Time Series Components

- A given TS consists of three systematic components including level, trend, seasonality, and one non-systematic component called noise.
  - **Level**: The average value in the series.
  - **Trend**: The increasing or decreasing value in the series.
  - **Seasonality**: The repeating short-term cycle in the series.
  - **Noise**: The random variation in the series.



- A **systematic** component have consistency or recurrence and can be described and modeled.
- A **Non-Systematic** component cannot be directly modeled.

Time Series Components

- A TS can be modeled as an aggregate or combination of these four components.
- All series have a level and noise. The trend and seasonality components are optional.
- Level can be omitted if Offset Translation with Mean Removal is applied

- **Additive Model**: Y = Level + Trend + Seasonality + Noise/Residuals
  - Changes over time are consistently made by the same amount
  - A linear trend is a straight line.
  - A linear seasonality has the same frequency (width of cycles) and amplitude (height of cycles).

- **Multiplicative Model**: Y = Level * Trend * Seasonality * Noise/Residuals
  - A multiplicative model is nonlinear, such as quadratic or exponential.
  - Changes increase or decrease over time.
  - A nonlinear trend is a curved line.
  - A non-linear seasonality has an increasing/decreasing frequency and/or amplitude over time.

Time Series Components

Components meaning

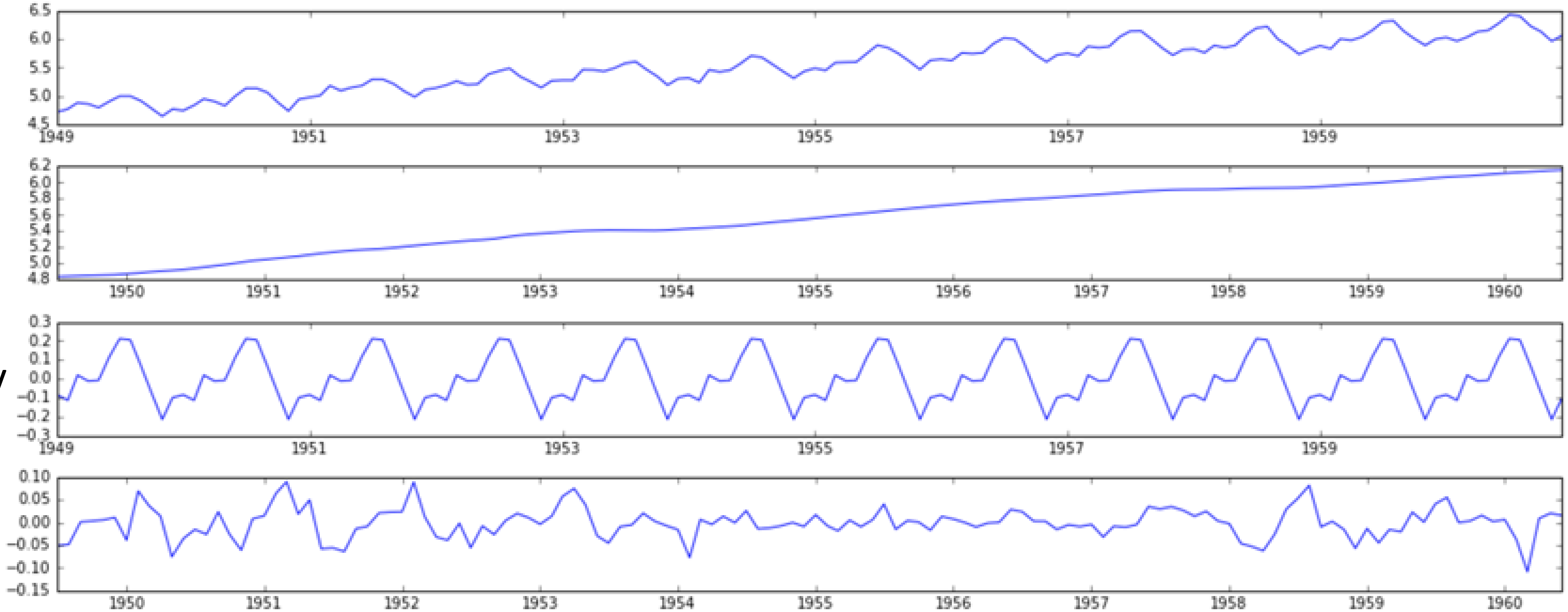
- Trend: upward or downward pattern that might be extrapolated into the future.
- Seasonality: periodic behavior with a known period (hourly, weekly, monthly...).
- Residuals: containing anything else, i.e., noise.

Y: TS

T: Trend

S: Seasonality

R: Residuals

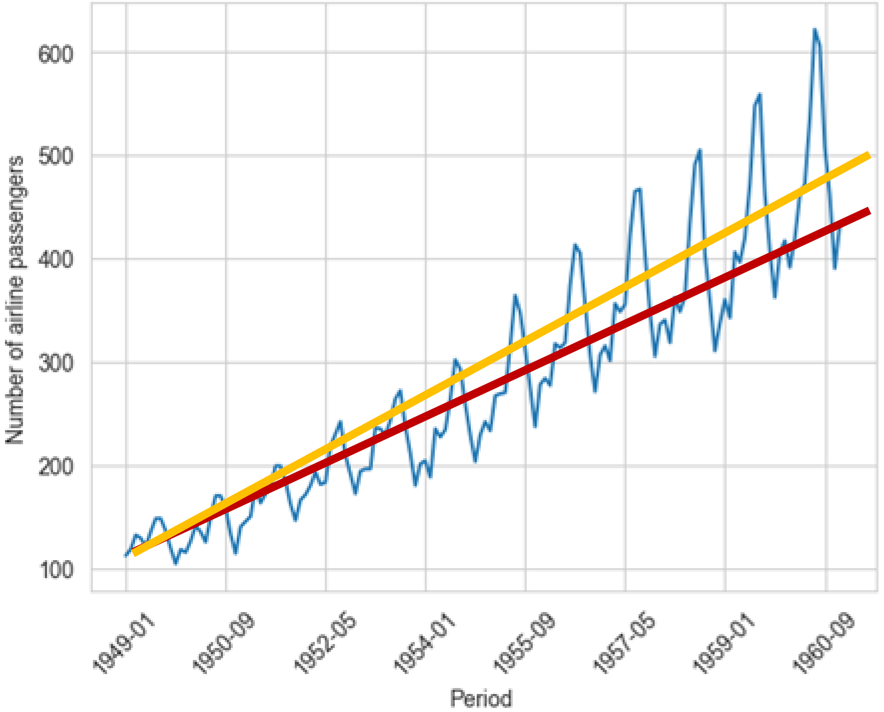


Additive Model

$$Y = T + S + R$$

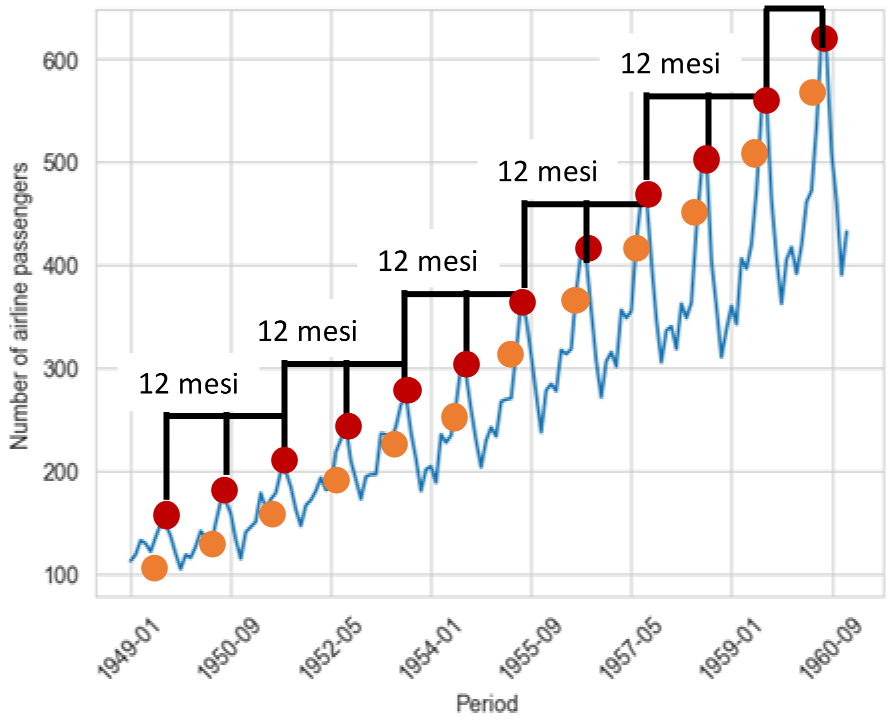
Time Series Decomposition: Trend

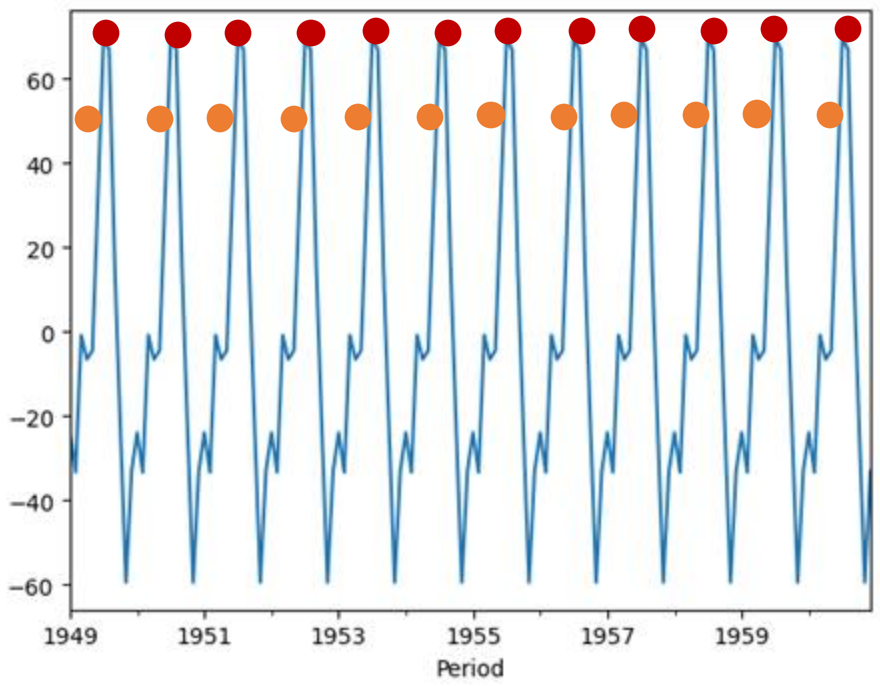
- Trend as a straight line between the first and the last point of the TS.
- Given a window *w*, trend as the moving average along TS with size *w*.
- Trend as the best fitting straight line obtained with a linear regression.



Time Series Decomposition: Seasonality

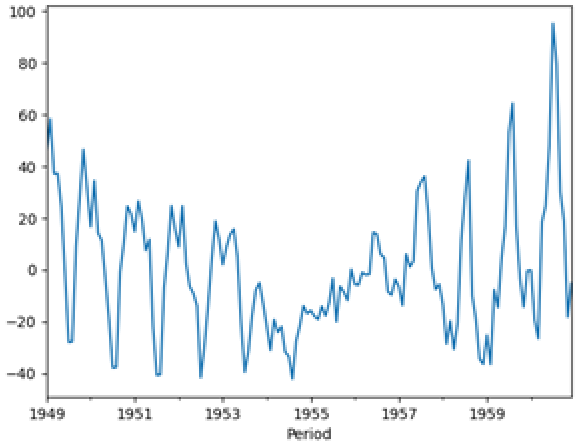
- Given the estimated number of periods *p*, and a detrended time series D = Y – T considering TS modeled with additive models, seasonality can be calculated as the mean value of for each time stamps among the various periods *p* repeated *p* times.





Time Series Decomposition: Residuals

- Given the time series Y, its trend T, its seasonality S, the residuals R are obtained as R = Y – T – S.



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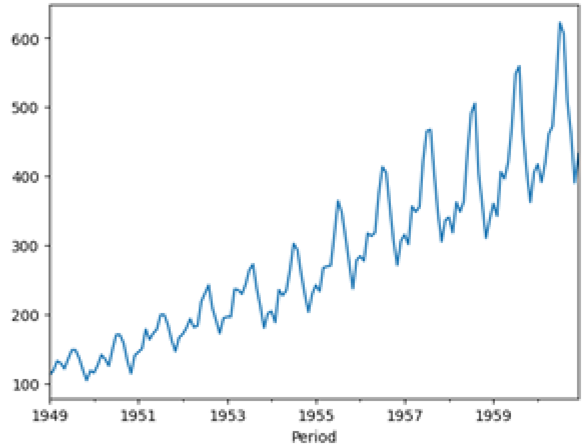

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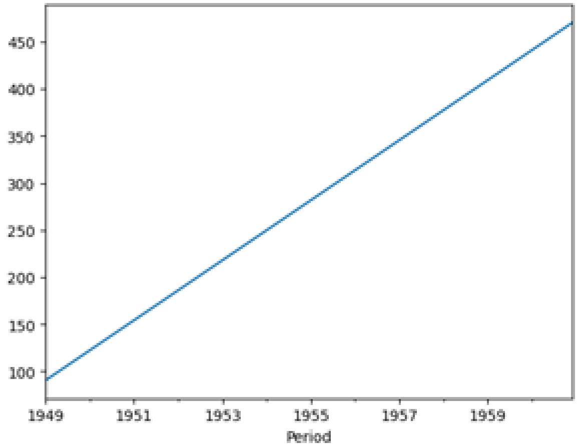

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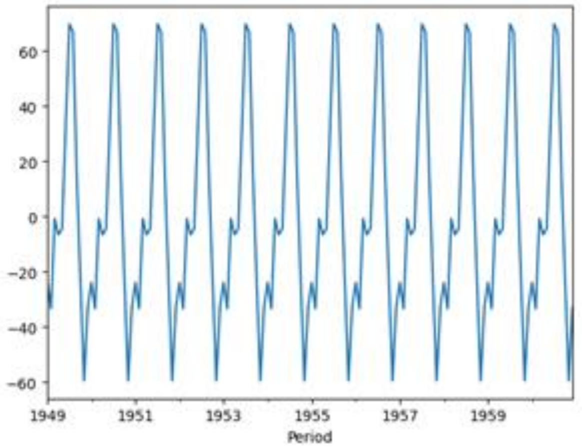


Component-based Normalizations

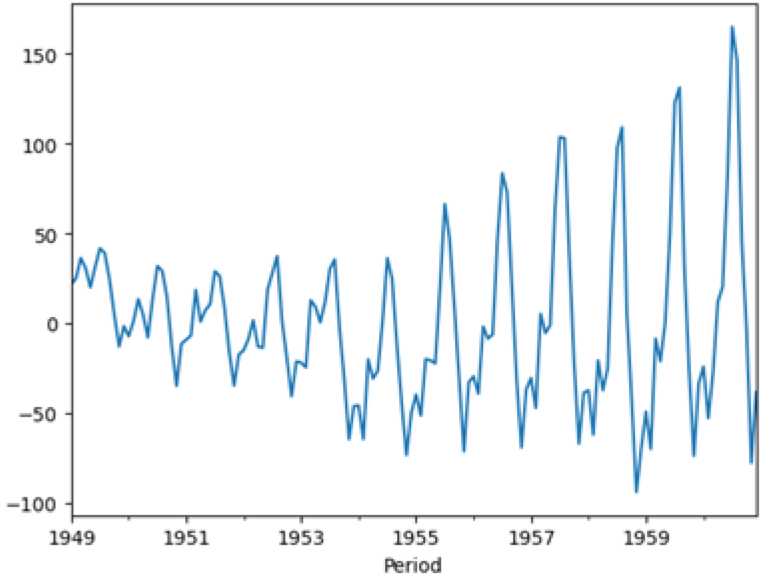
Detrend

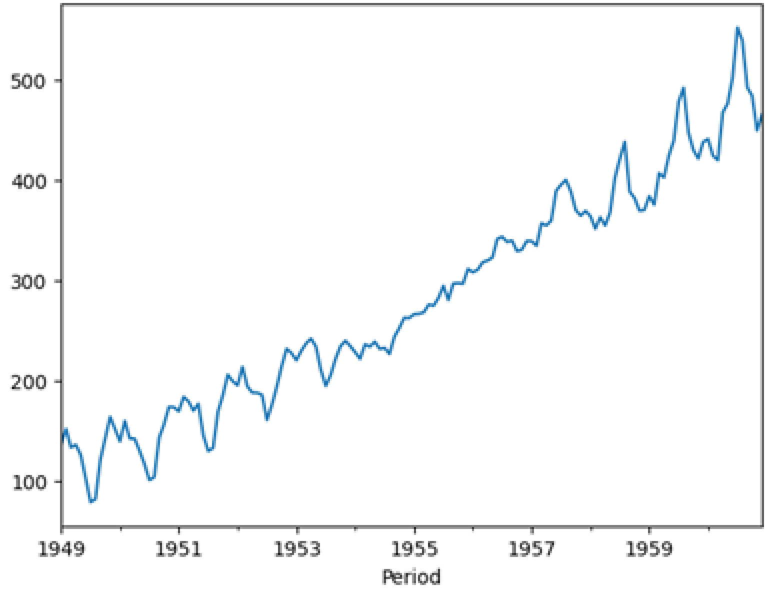
- Y' = Y – T

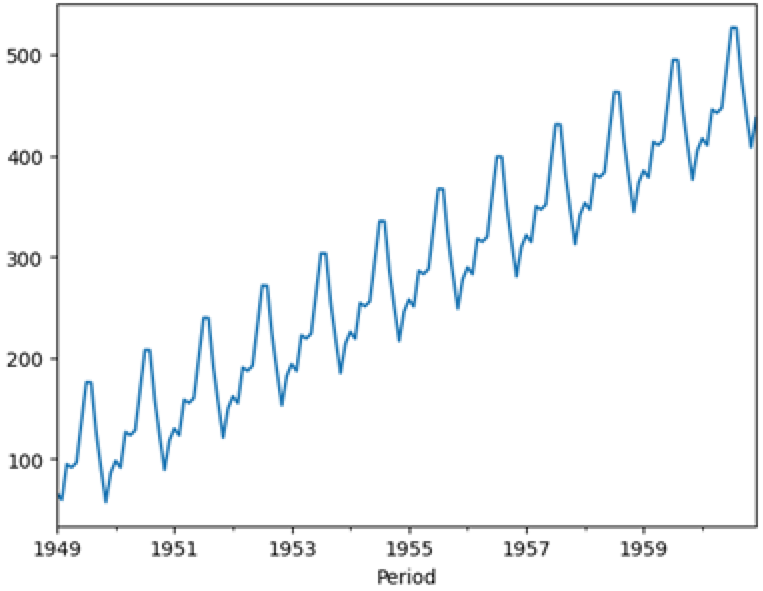
Deseasonalize

- Y' = Y – S

Denoise



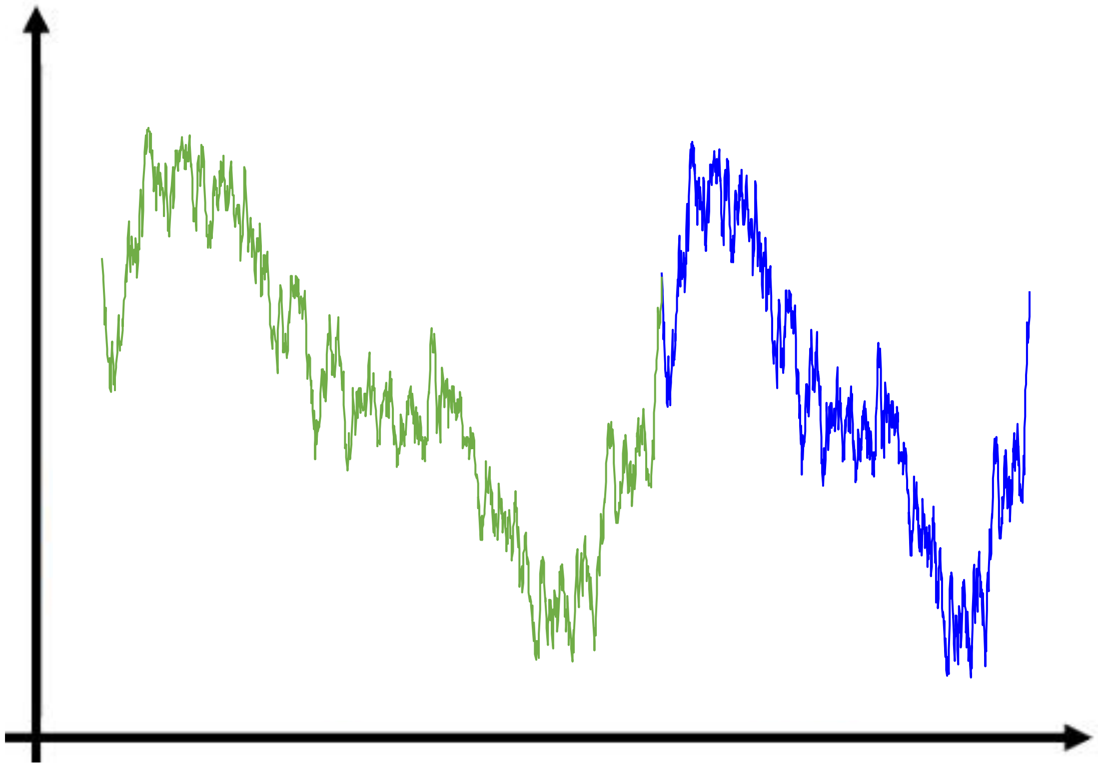




Stationarity

Why Do We Care About Stationarity?

- If your TS is not stationary, you cannot build reliable statistical TS predictive models.



Elementary Statistics

- **Expectation**: The expectation $E(x)$ of a variable x is its mean average value in the population. We denote the expectation of x by μ such that $E(x)=\mu$
- **Variance**: The variance of a variable is the expectation of the squared deviations of the variable from the mean, denoted by $\sigma^2(x)=E[(x-\mu)^2]$
- **Standard Deviation**: The standard deviation of a variable x, $\sigma(x)$, is the square root of the variance of x.
- **Covariance**: Covariance tells us how linearly related are two variables. Given two variables x and y with respective expectations μ_x and μ_y, the covariance is $\sigma(x,y)=E[(x-\mu_x)(y-\mu_y)]$.

Elementary Statistics

- **Sample Means:** In statistical situations, we estimate the covariance from a sample of population using $\bar{x}$ and $\bar{y}$.
- **Sample Covariance:**

$$Cov(x,y) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$$

- **Correlation:** Correlation is a dimensionless measure of how two variables vary together. It is a “covariance” of two variables normalized by their respective spreads.

$$Cor(x,y) = \frac{Cov(x,y)}{\sigma(x)\sigma(y)}$$

Probabilistic Model

- A complete probabilistic TS model can be specified as the **joint distribution function** of the sequence $x_1, x_2, \ldots, x_n$ of n random variables as the probability that the values of the series are jointly less than n constants $c_1, c_2, \ldots, c_n$.
  - $F(c_1, c_2, \ldots, c_n) = P(x_1 \leq c_1, x_2 \leq c_2, \ldots, x_n \leq c_n)$

- Although the joint distribution function describes the data completely, it is an unwieldy tool for analyzing TS data.

Stationary Time Series

- A **strictly stationary** TS is one for which the probabilistic behavior of every collection of values $\{x_1, x_2, \ldots, x_n\}$ is identical to that of the time shifted set $\{x_{1+h}, x_{2+h}, \ldots, x_{n+h}\}$
  - $P(x_1 \leq c_1, x_2 \leq c_2, \ldots, x_n \leq c_n) = P(x_{1+h} \leq c_1, x_{2+h} \leq c_2, \ldots, x_{n+h} \leq c_n)$
  - for all time points n > 0, for all numbers $c_1, c_2, \ldots, c_n$, for all time shifts h.

- If a TS is strictly stationary, then all of the distribution functions for subsets of variables must agree with their counterparts in the shifted set for all values of the shift parameter h.
- In other words, shifting the time axis does not affect the distribution.

Stationary Time Series

- The lack of independence between two subsequent values x_t and x_{t+h} can be assessed numerically using **covariance** and **correlation**.
- A **weakly stationary** TS x_t is one for which
  - the mean $\mu_t = \mu$ is independent of t, i.e., it is constant.
  - the autocovariance $\gamma(x_t, x_{t+h})$ is independent of t for each h.
- We typically use the term stationary to refer to weakly stationary.

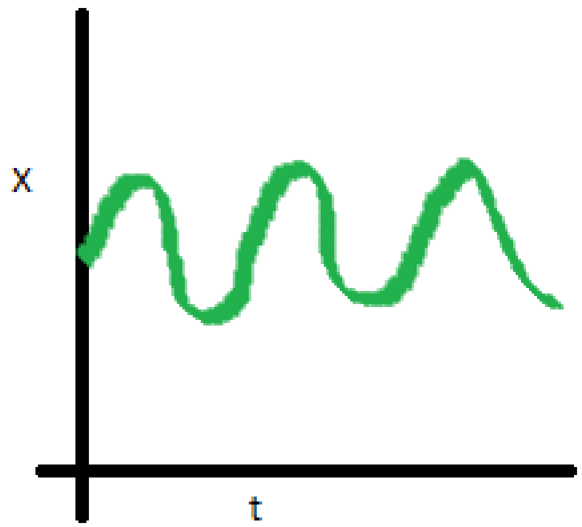
- In short, a TS with a certain trend or with a certain seasonality is not stationary.

Stationary Criteria

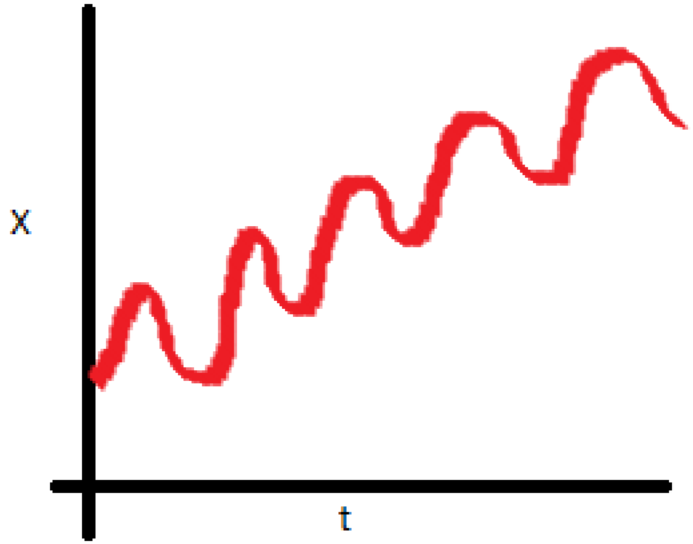
- There are three basic criterion for a TS to be stationary:
- The mean of the series should not be a function of time but should be a constant. We consider a time series as stationary in the mean if $\mu(t)$=μ, a constant.
- The variance of the series should not be a function of time. This property is called: homoscedasticity. Hence, a time series is stationary in the variance if $\sigma^2(t)$=σ^2, a constant.
- The covariance of x_t and x_{t+h} should not be a function of time.
- In short, a TS is stationary if it does not have time-dependent structure.

Stationary Time Series

- The mean of the series should not be a function of time, rather should be a constant.



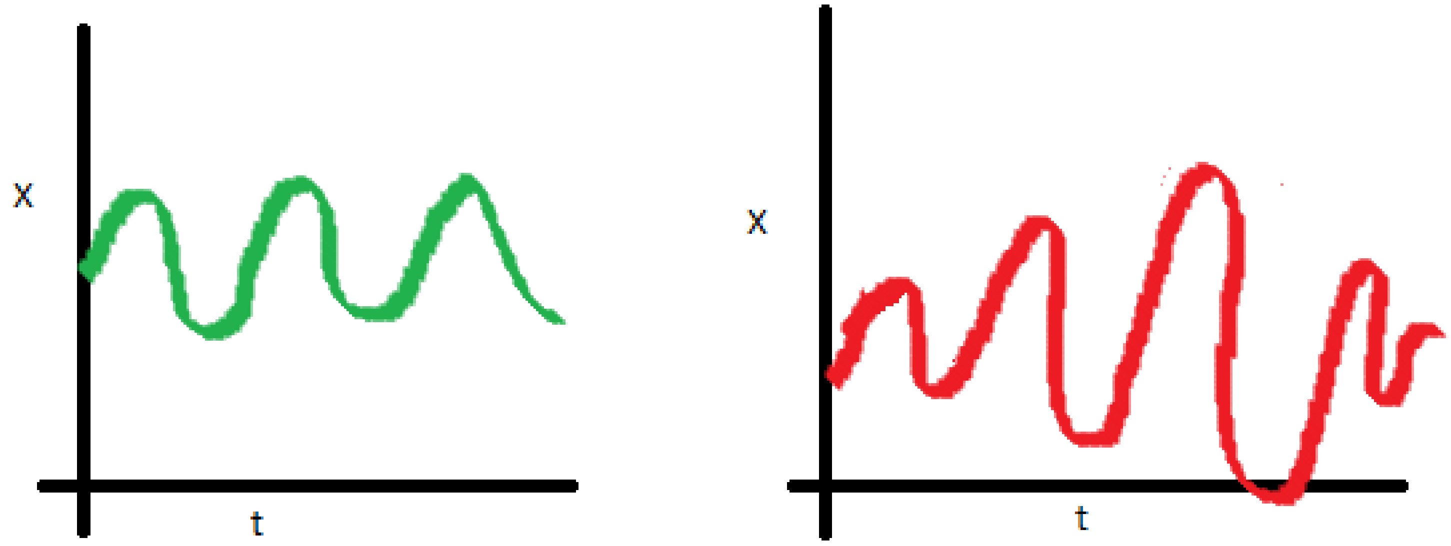
Stationary series



Non-Stationary series

Stationary Time Series

- The variance of the series should not a be a function of time, rather should be a constant.

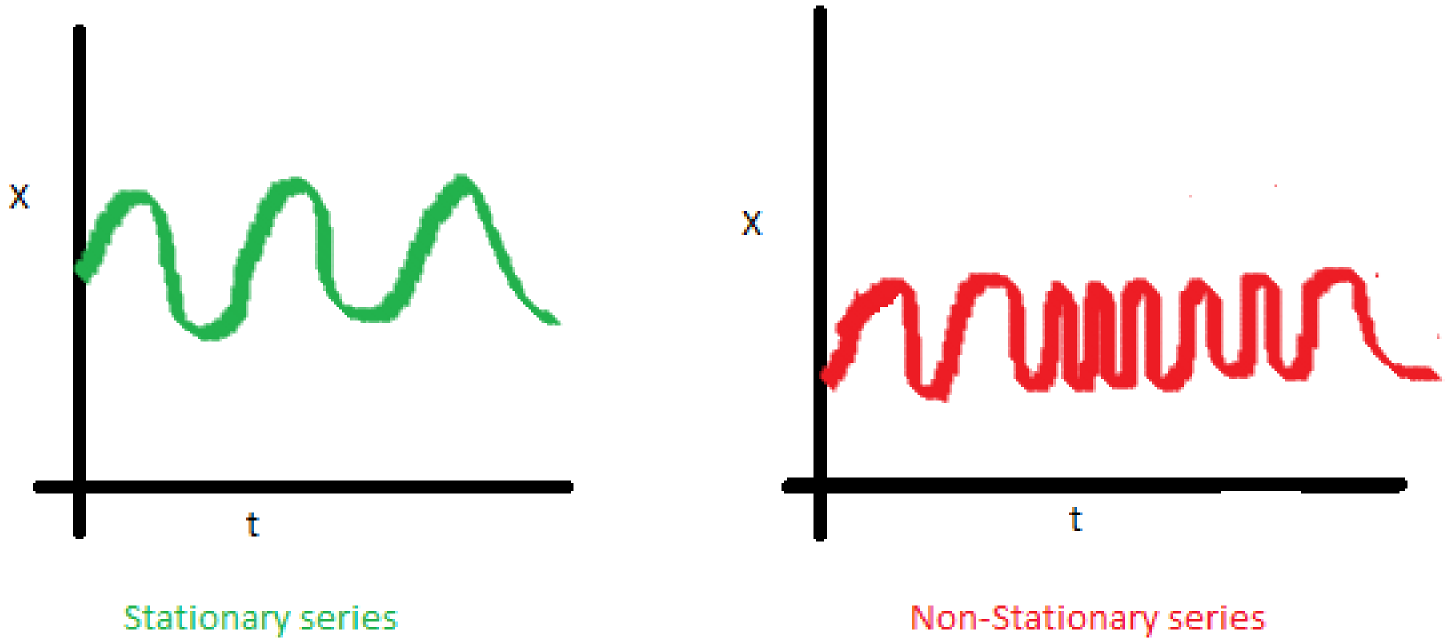


Stationary series

Non-Stationary series

Stationary Time Series

- The covariance of the *t-th* point and the *(t+h)-th* point should not be a function of time.

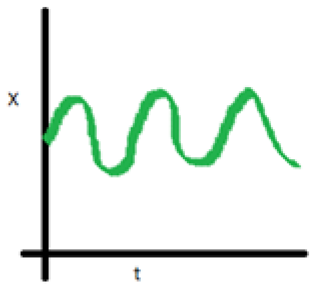


Make Stationarity a Non-Stationary Time Series

- There are various techniques to bring stationarity to a non-stationary time series.
- Detrending: removing the trend component from the TS by using
  - Log
  - Removing the smoothed mean
  - Linear regression
- Differencing: consider as TS the differences of the values instead of actual values
- Decomposition: remove all the schematic components and consider the residuals

Stationarity Test

- The Augmented Dickey-Fuller (ADF) test of stationarity results comprise of an ADF statistic and some Critical Values for different confidence levels.
- If the ADF statistic is less than the Critical Value, we can reject the null hypothesis and say that the series is stationary and does not have any time-dependent structure.
- ADF null hypothesis: a unit root is present in an autoregressive model.
- A unit root is a feature of some stochastic processes (such as random walks) that can cause problems in statistical inference involving TS models.



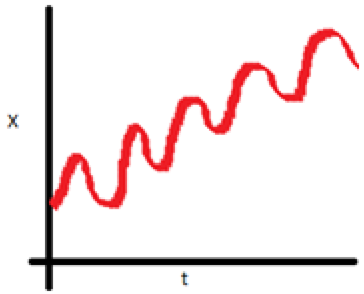
Stationary series

ADF Statistic: -4.808291
p-value: 0.000052
Critical Values:
5%: -2.870
1%: -3.449
10%: -2.571

Test NOT Passed
Null Hypotesis rejected

ADF Statistic: 0.815369
p-value: 0.991880
Critical Values:
5%: -2.884
1%: -3.482
10%: -2.579

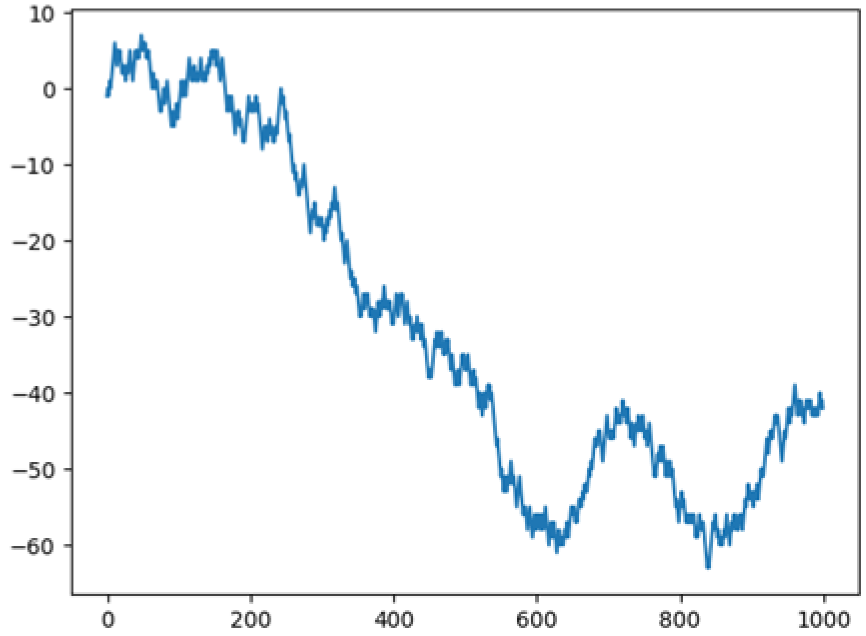
Test
Passed



Non-Stationary series

Random Walk

- Let $\{x_t\}$ with $t = 0, 1, 2, \ldots$ be a time series then $\{x_t\}$ is a **random walk** if $x_t = x_{t-1} + w_t$ where w_t is a white noise series.
- Substituting $x_{t-1} = x_{t-2} + w_{t-1}$ in $x_t = x_{t-1} + w_t$ and for others $t-i$
- We get $x_t = w_t + w_{t-1} + w_{t-2} + w_{t-3} + \ldots$
- If the series is not infinite, we have $x_t = w_1 + w_2 + w_3 + \ldots + w_t$
- where each w_i is IID noise.
- Random walk TS plays an important role as a building block for complex TS models



Random Walk

- Random walk models are widely used for non-stationary data, e.g. financial and economic data.
- Random walks typically have:
  - long periods of apparent trends up or down
  - sudden and unpredictable changes in direction.
- The forecasts for a random walk model are equal to the last observation, as future movements are unpredictable, and are equally likely to be up or down.

Autocorrelation

The Sample Autocorrelation Function

- In practical problems, we do not start with a model, but with observed data.
- To assess the **degree of dependence** in the data and to select a model, one of the important tools we use is the sample Autocorrelation Function (Sample ACF).
- Let $\{x_1, x_2, \ldots, x_n\}$ the observations of a time series.
- Sample mean $\bar{x}$ = 1/n $\sum_{t=1}^{n} x_t$
- Sample Autocovariance $\gamma(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+|h|} - \bar{x})(x_t - \bar{x})$
- It measures the linear dependence between lagged TS starting at two different time points on the same TS.
- Very smooth TS exhibit autocovariance that stay large even when *h* is large, whereas choppy TS tend to have autocovariance that are nearly zero for large separations. If $\gamma(h)$ = 0 are not linearly related
- Notice that if a TS is stationary autocovariance is not a function of time

The Sample Autocorrelation Function

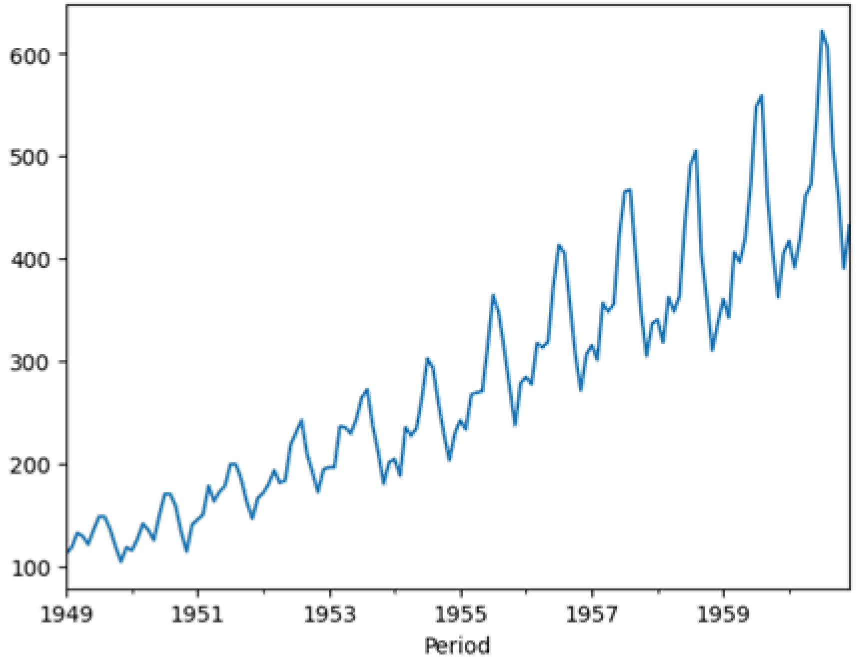
- **Sample Mean** $\bar{x}$ = 1/n $\sum_{t=1}^{n} x_t$
- **Sample Autocovariance** $\gamma(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+|h|} - \bar{x})(x_t - \bar{x})$
- **Sample Autocorrrelation** (ACF) $\rho(h) = \gamma(h) / \gamma(0) \in [-1,1]$
- ACF measures the linear relationship between lagged values of a TS.
- There are several autocorrelation coefficients, corresponding to each lag *h* = 1, 2, 3, …

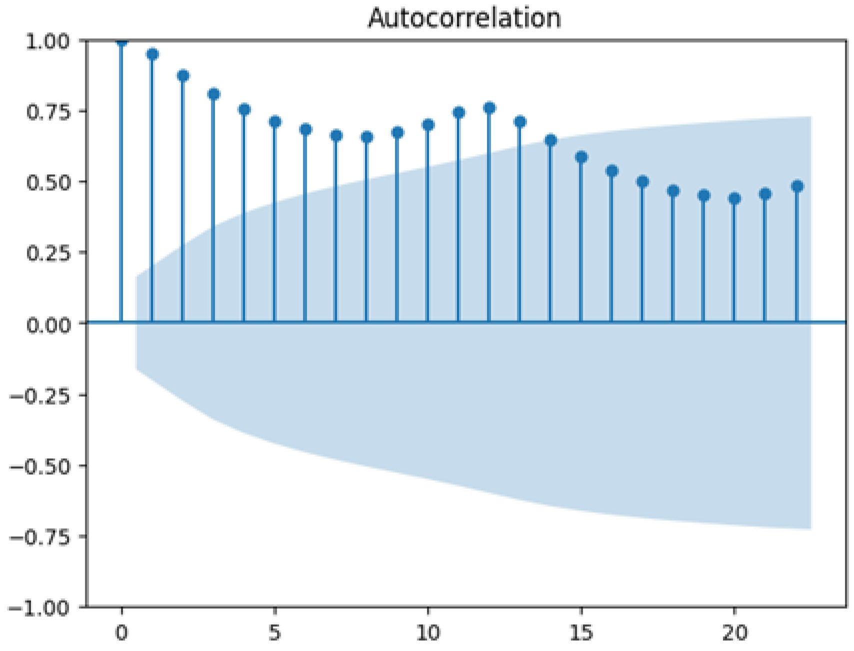
Autocorrelation - Remarks

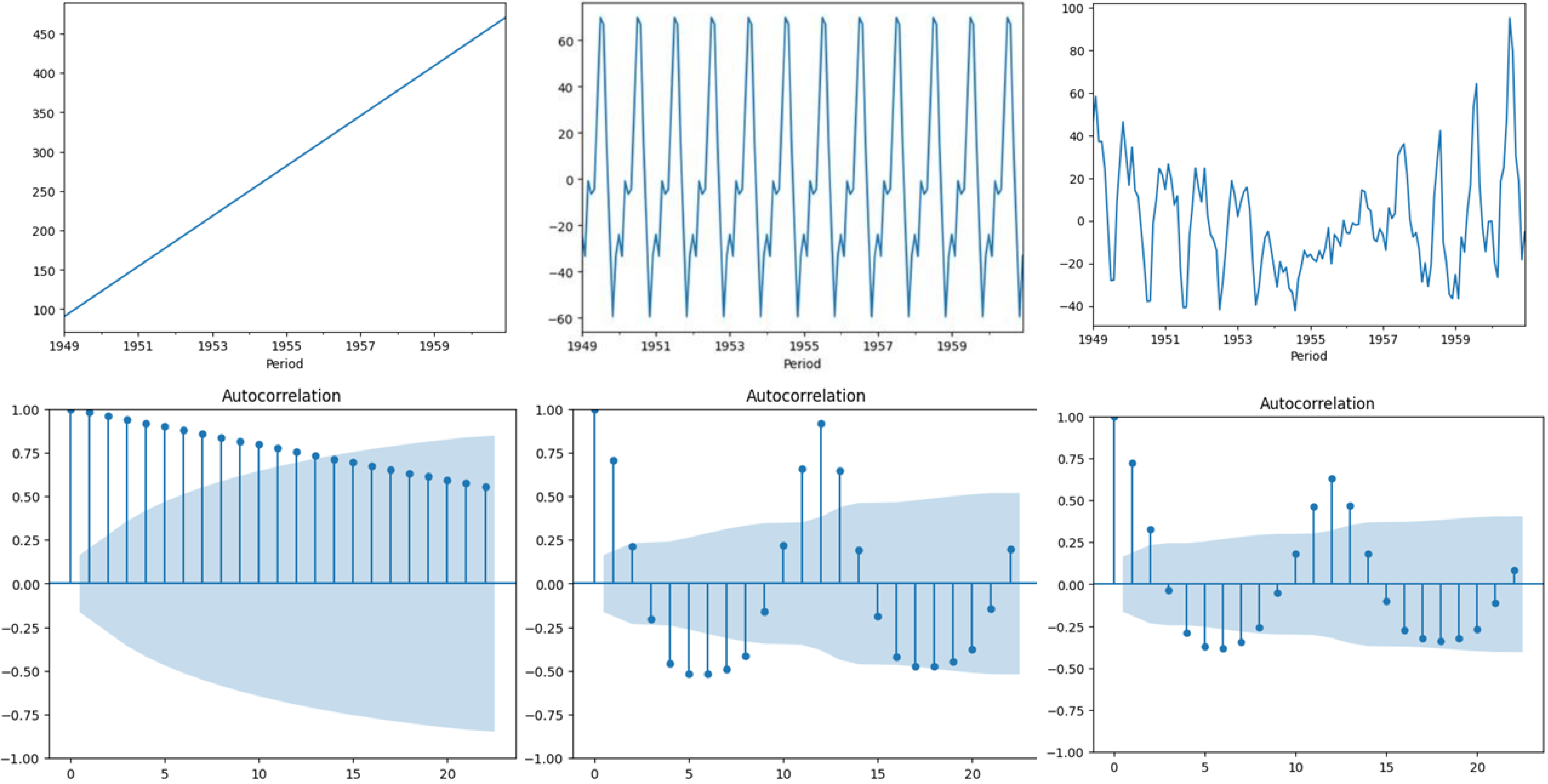
- Autocorrelations or lagged correlations are used to know whether a TS is dependent on its past, i.e., how sequential observations in a TS affect each other.
- ACF can be computed for any data set and is not restricted to stationary time series.
- For data containing a trend ACF will display slow decay as h increases.
- For data containing a deterministic periodic component ACF will exhibit similar behavior with the same periodicity.

ACF Plot

- An ACF plot is a correlogramn, i.e., a plot of the autocorrelation functions for sequential values of lag *h = 0, 1, ..., n*
- It shows the correlation structure in each lag.
- Confidence intervals are drawn as a cone.






450
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1959
Period
Autocorrelation
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Autocorrelation
1.00
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Autocorrelation
1.00
0.75
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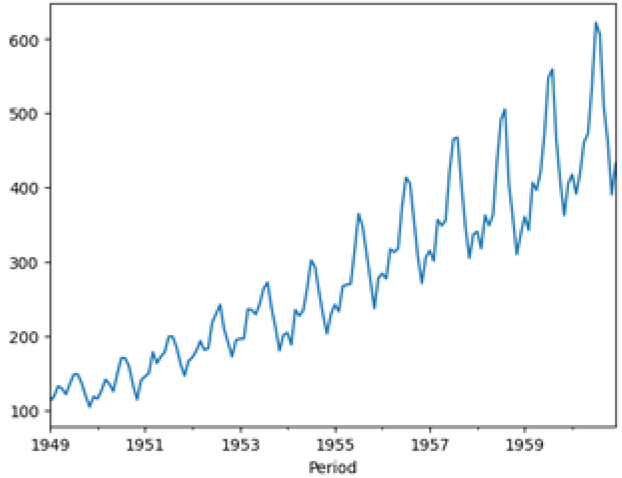
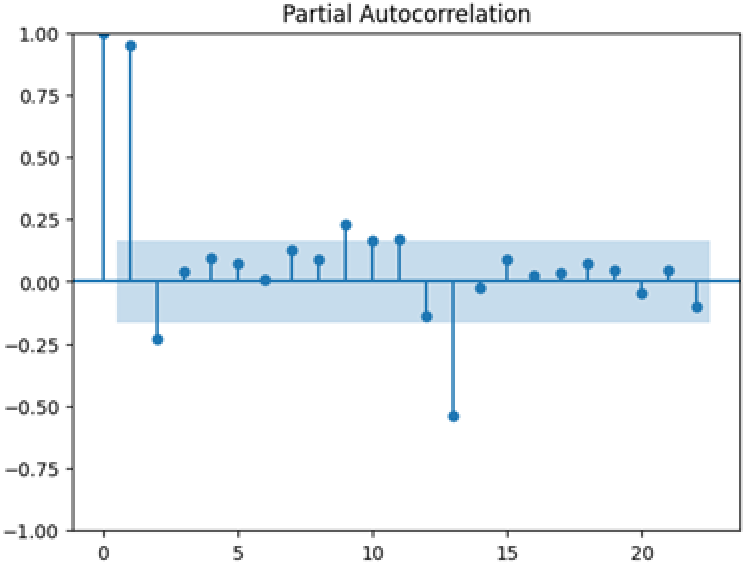
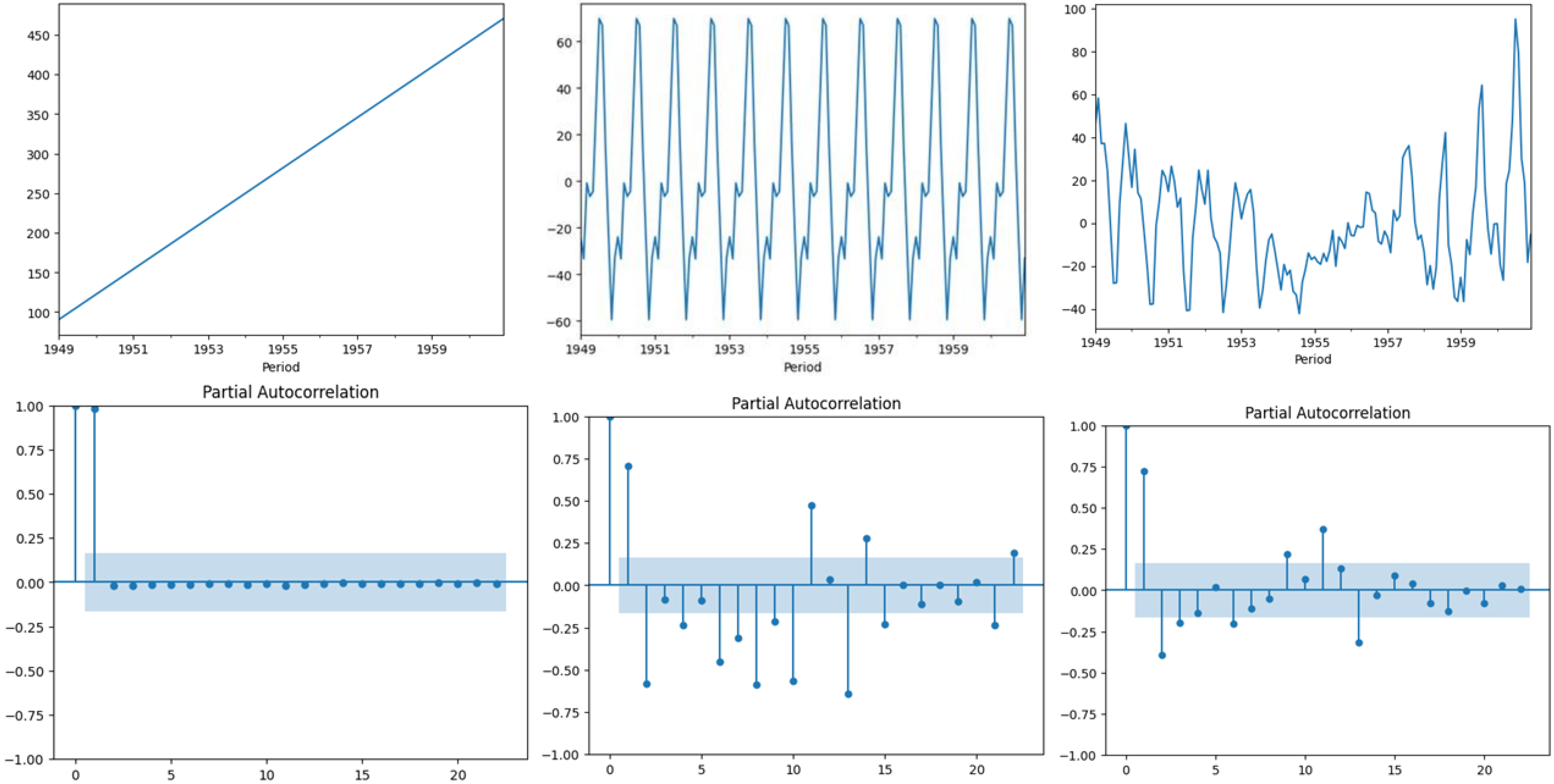
−0.25
−0.50
−0.75
−1.00
0
5
10
15
20

PACF plot

- A partial autocorrelation is a summary of the relationship between an observation in a TS with observations at prior time steps with the relationships of intervening observations *removed*.
- The partial autocorrelation at lag *h* is the correlation that results after removing the effect of any correlations due to the terms at shorter lags.
- The autocorrelation for an observation and an observation at a prior time step is comprised of *both the direct correlation and indirect correlations*.
- These indirect correlations are a linear function of the correlation of the observation, with observations at intervening time steps.
- These indirect correlations are removed by PACF.





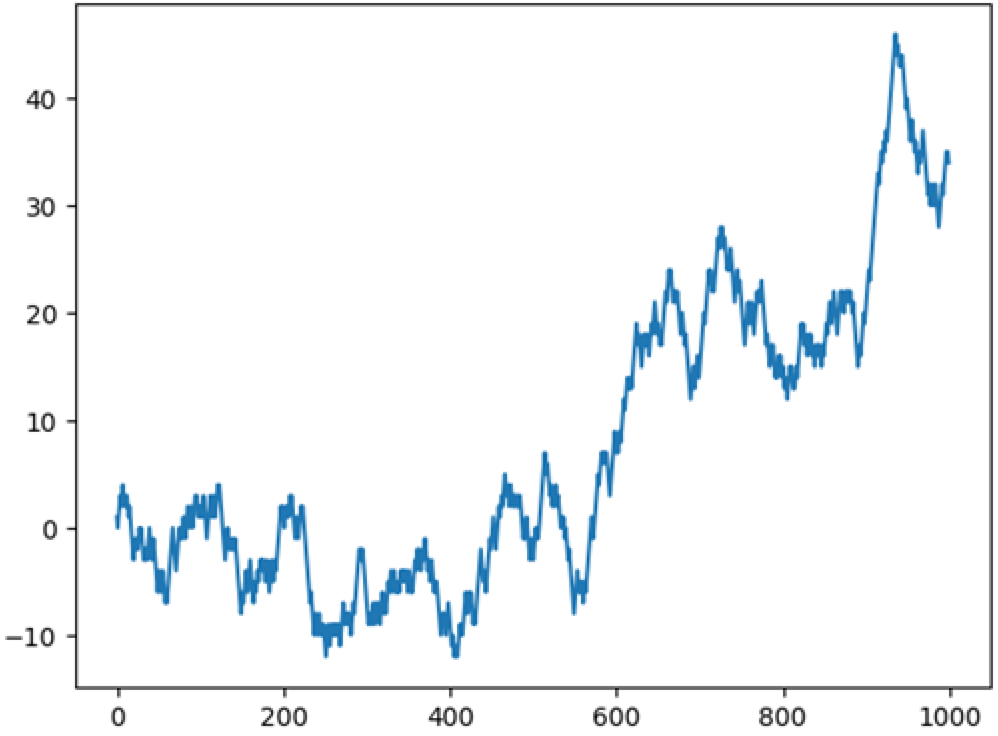

Period
Period
Period
Partial Autocorrelation
Partial Autocorrelation
Partial Autocorrelation

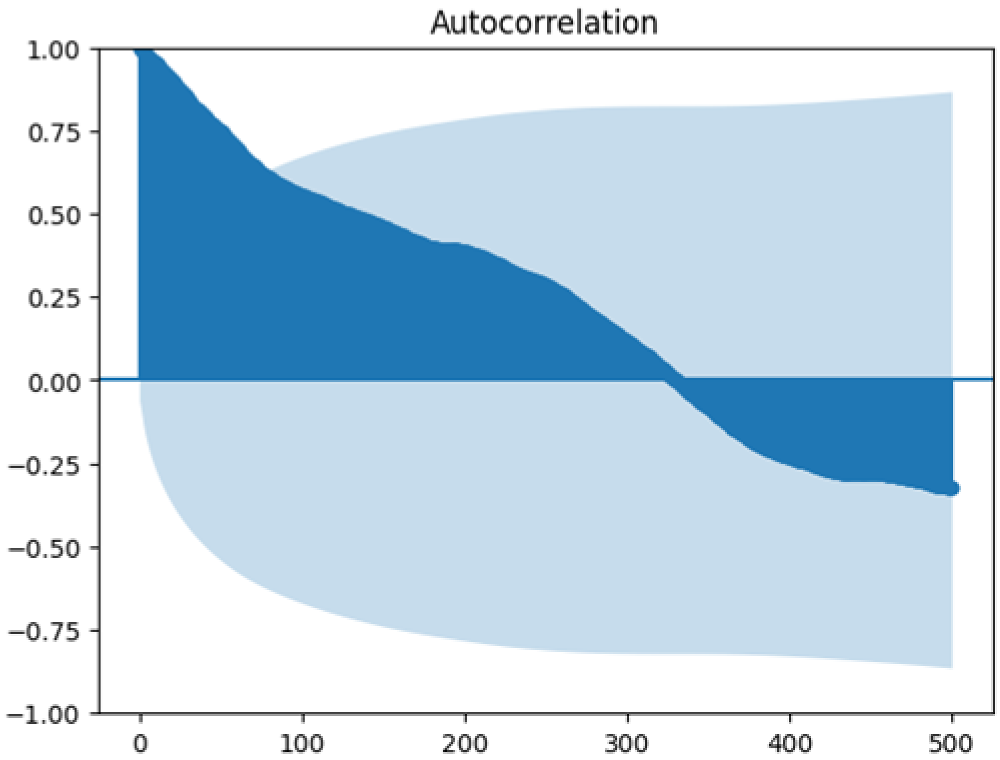
ACF and PACF Summary

- **Autocorrelation Function (ACF)**: It is a measure of the correlation between the TS with a lagged version of itself.
  - For instance, at lag *k=5*, ACF would compare TS at time instant $t_1 \ldots t_n$ with TS at instant t_1-5, …, t_2-5 (t_1-5 and t_n being end points).
- **Partial Autocorrelation Function (PACF):** This measures the correlation between the TS with a lagged version of itself but after eliminating the variations already explained by the intervening comparisons.
  - For instance, at lag *k=5*, would compare the correlation but remove the effects already explained by lags 1 to 4.

Random Walk and Autocorrelation

- Given the way that the random walk is constructed, we expect a strong autocorrelation with the previous observation and a linear fall off from there with previous lag values.





Random Walk and Stationarity

- Given the way that the random walk is constructed and the results of reviewing the autocorrelation, we know that the observations in a random walk are dependent on time.
- The current observation is a random step from the previous observation.
- Therefore, a random walk is a non-stationary TS.
- In fact, all random walk processes are non-stationary. Note that not all non-stationary time series are random walks.

ADF Statistic: -0.373987
p-value: 0.914352
Critical Values:
1%: -3.437
5%: -2.864
10%: -2.568

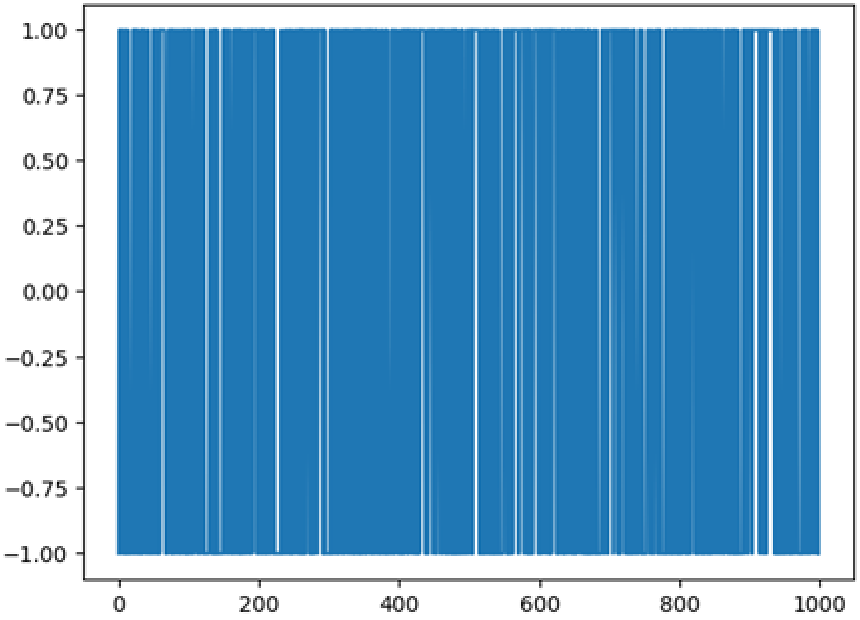
Test Passed

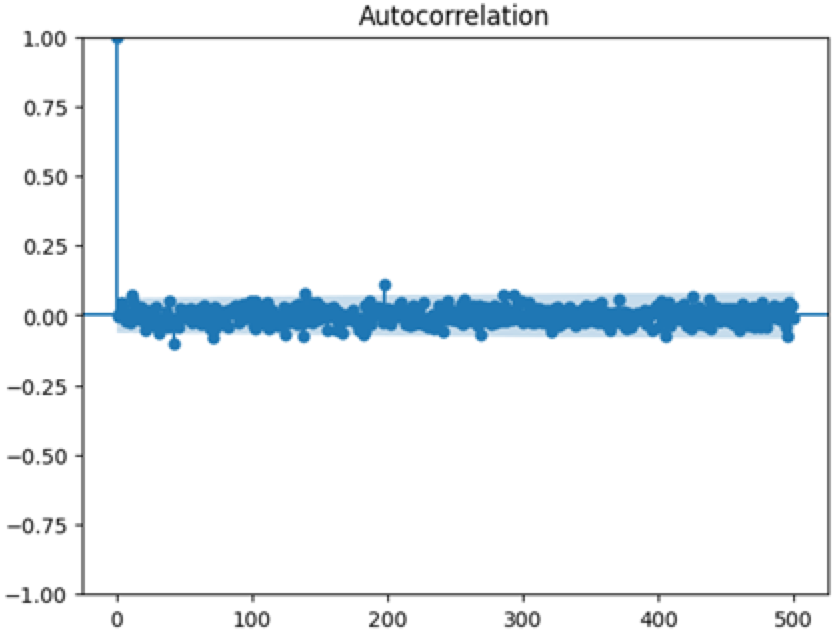
White Noise

- Time series that show no autocorrelation are called **white noise**.
- In other words, a white noise is made of random values with a given mean and standard deviation but not autocorrelation.
- When the differenced series is white noise, i.e. $\varepsilon_t = x_t - x_{t-1}$, where ε_t denotes white noise, than $x_t = x_{t-1} + \varepsilon_t$ is a **random walk model**

White Noise Autocorrelation and Stationarity

- We can obtain white noise by differencing a random walk.
- All correlations are small, close to zero and below the 95% and 99% confidence levels
- The time series is stationary.





Test NOT Passed
Null Hypotesis rejected

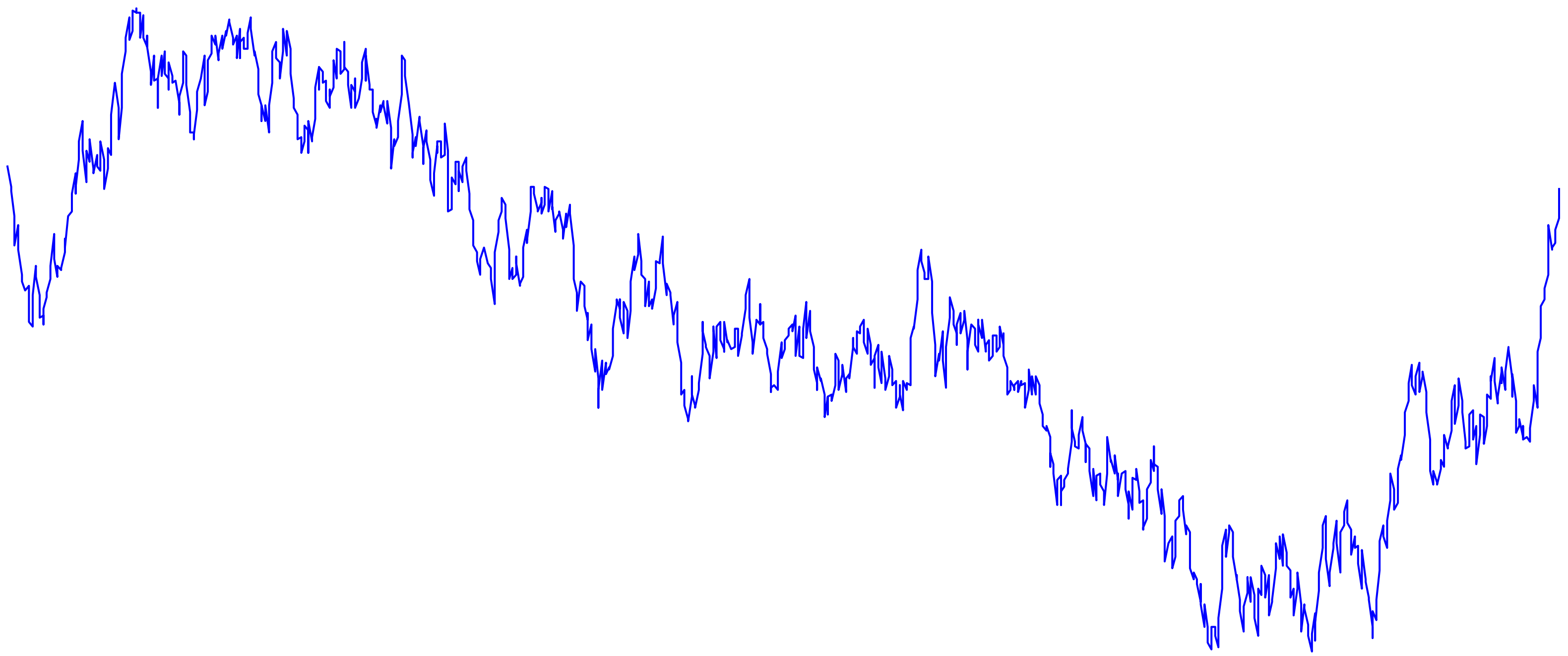
ADF Statistic: -31.659617
p-value: 0.000000
Critical Values:
1%: -3.437
5%: -2.864
10%: -2.568

Time Series Preprocessing for Forecasting

1. Plot the Time Series to examine the main characteristics (trend, seasonality, ...)
2. Remove the trend and seasonal components to get stationary residuals/models
3. Choose a model to fit the residuals using sample statistics (sample autocorrelation function)
4. Forecasting will be given by forecasting the residuals to arrive at forecasts of the original series X_t

Time Series Preprocessing Remarks

- Depending on the TSA task different normalizations might be required but not necessarily all of them!!!



References

- Forecasting: Principles and Practic. Rob J Hyndman and George Athanasaopoulus. (https://otexts.com/fpp2/)
- Time Series Analysis and Its Applications. Robert H. Shumway and David S. Stoffer. 4th edition.(http://www.stat.ucla.edu/~frederic/415/S23/tsa4.pdf)
- Time Series Analysis in R (https://s-ai-f.github.io/Time-Series/)
- Mining Time Series Data. Chotirat Ann Ratanamahatana et al. 2010. (https://www.researchgate.net/publication/227001229_Mining_Time_Series_Data)