INTEGRATION

→ Integration is the reverse process of differentiation

For ex:-

$$\frac{d}{dx} f(x) = g(x)$$

$$\boxed{\int} g(x)dx = f(x) + c$$

Integral sign

function to be integrated [Integrand]

antiderivative (or) Primitive

arbitrary constant

For ex:- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + c$

$= \frac{\pi}{2} - \cos^{-1}x + c = -\cos^{-1}x + C$

→ Integral of basic functions

	$f(x)$	$\int f(x)$
1)	c	c
2)	e^x	e^x
3)	$\frac{1}{x}$	$\ln\lvert x\rvert + c$
4)	x^n	$\frac{x^{n+1}}{n+1} + c \ (n \neq -1)$
5)	$\sin x$	$-\cos x$
6)	$\cos x$	$\sin x$
7)	$\sec^2 x$	$\tan x$
8)	$\operatorname{cosec}^2 x$	$\cancel{\operatorname{cosec} x}\ -\cot x$
9)	$\sec x \cdot \tan x$	$\sec x$
10)	$\operatorname{cosec} x \cdot \cot x$	$-\operatorname{cosec} x$

11)	a^x	$\frac{a^x}{\log_e a}$
12)	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x+c$ (or) $-\cos^{-1}x+c$
13)	$\frac{1}{1+x^2}$	$\tan^{-1}x+c$ (or) $-\cot^{-1}x+c$
14)	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\sec^{-1}x+c$ (or) $-\csc^{-1}x+c$
15)	K	$Kx+c$

Algebra of Integration

① If $\int f(x)\,dx = g(x)+c$
then $\int f(t)\,dt = g(t)+c$

For ex:- $\int \cos x^2\,dx^2 = \sin x^2 + c$

② If $\int f(x)\,dx = g(x)+c$
then $\int f(ax+b)\,dx = \frac{g(ax+b)}{a}+c$

Note:- $\int e^{3x^2+2}\,dx = \frac{1}{3}e^{3x^2+c}+c$ ✗ since $3x^2+2$ is not linear $(ax+b)$

$\int e^{7x-9}\,d(7x-9) = e^{7x-9}+c$

③ $\int k\,f(x)\,dx = k\int f(x)\,dx$

④ $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

⑤ $\int (fx+k)\,dx = \int fx\,dx + \int k\,dx$

⑥

7) $\int \frac{dx}{x^5}$ $\quad \int x^{-5}dx = \frac{x^{-4}}{-4} = \frac{-1}{4x^4} + C$

8) $\int \frac{dx}{3x+7} = \cancel{(3x+7)^{-2}}$

$= \frac{\log|3x+7|}{3} + C$

9) $\int \frac{dx}{3x^2+16}$

$= \frac{1}{16} \int \frac{dx}{1 + \frac{3}{16}x^2}$

$= \frac{1}{16} \frac{dx}{1+\left(\frac{\sqrt{3}x}{4}\right)^2} = \frac{1}{16} \frac{\tan^{-1}\left(\frac{\sqrt{3}x}{4}\right)}{\frac{\sqrt{3}}{4}} + C$

10) $\int \frac{dx}{(4x+3)^2} = \int (4x+3)^{-2}dx$

$= \frac{(4x+3)^{-2+1}}{-2+1} = \frac{(4x+3)^{-1}}{-1} \cdot \frac{1}{4} + C$

11) $\int (e^{3x+2})^5 dx = \int e^{15x+10} dx$

$= \frac{e^{15x+10}}{15}$

12) $\int (2\cdot 3^x + 5^{x+1})^2 dx$

$= \int 4 9^x + 5^{2(x+1)} + 4 3^x 5^{x+1}$

$\frac{4\cdot 9^x}{\log 9} + \frac{25^{(x+1)}}{\log_e 25} + \frac{20\ 15^x}{\log_e 15} + C$

13) $\int \sec^2(4x-5)\,dx$

$= \frac{\tan(4x-5)}{4} + c$

14) $\int \operatorname{cosec}^2[2x+7]\,d(2x+7)$

$= -\cot(2x+7) + c$

*** 15) $\int \cos(3x+2)\,d(2x+3)$

$d(f(x)) = g(x)\cdot d(x)$

$= \int \cos(3x+2)\,2\cdot dx$

$= -\frac{2}{3}\sin(3x+2) + c$

16) $\int x\cdot\sqrt[5]{x+1}\,dx$

$= \int (x+1-1)\cdot(x+1)^{1/5}\,dx$

$= \int (x+1)^{6/5} - (x+1)^{1/5}\,dx$

$= \int (x+1)^{6/5}\,dx - \int (x+1)^{1/5}\,dx$

$= \frac{5}{11}(x+1)^{11/5} - \frac{5}{6}(x+1)^{6/5} + c$

*** 17) $\int \frac{3x+5}{(x+4)(2x+1)}\,dx$

$= \int \frac{2x+1+x+4}{(x+4)(2x+1)}\,dx$

$= \int \frac{1}{x+4}\,dx + \int \frac{1}{2x+1}\,dx$

$= \frac{\log|x+4|}{1} + \frac{\log|2x+1|}{2} + c$

18) $\int \sin^2 x \, dx$

$= \int \frac{1-\cos 2x}{2} dx$

$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$

19) $\int \frac{x^2}{(x+1)} dx$

$= g(x) \cdot d(x)$

$= \int \frac{x^2 - 1 + 1}{x+1} dx$

$= \int \frac{(x-1)(x+1) + 1}{x+1}$

$= \int x - 1 \, dx + \int \frac{1}{(x+1)} dx$

$= \cancel{\log|x-1|} + 1$

$= \frac{x^2}{2} - x + \log|x+1| + c$

20) $\int 3^{4\log_9 (x) + 2} dx$

$= \int \cancel{3\log} \, 3^{\log_{3^2} (x+2) + 2} dx$

$= \int 3^{\frac{4}{2}\log_3 (x+2) + 2} dx = \int x^2 \cdot 3^2$

$= \cancel{\int (x+2)^2 dx} = \int x^2 \cdot 9$

$= \frac{9x^3}{3} = 3x^2 + c$

21) $\int \tan^2 x \, dx$

$= \int (\sec^2 x - 1) dx$

$= \tan x - x + c$

22) $\int \cos^3 x\, dx$

$\cos 3x = 4\cos^3 x - 3\cos x$

$$\cos^3 x = \frac{\cos 3x - 3\cos x}{4}$$

$$\int \frac{\cos 3x - 3\cos x}{4}\, dx = \frac{1}{4}\left[\frac{\sin 3x}{3} - 3\sin x\right] + C$$

23) $\int \frac{dx}{(x+1)(x+5)}$

$$= \frac{1}{4}\int \frac{4}{(x+1)(x+5)}\, dx$$

$$= \frac{1}{4}\int \frac{(x+5)-(x+1)}{(x+1)(x+5)}\, dx$$

$$= \frac{1}{4}\int \frac{1}{x+1} - \frac{1}{x+5}\, dx$$

$$= \frac{1}{4}\left[\log|x+1| - \log|x+5|\right] + C$$

24 $\int \tan x \cdot \sec x \cdot d(x^\circ)$

$$\frac{d\left(\frac{\pi}{180}x\right)}{dx} = \frac{\pi}{180}$$

$$d\left(\frac{\pi}{180}\cdot x\right) = \frac{\pi}{180}\cdot dx$$

$$\int \tan x \cdot \sec x \cdot \frac{\pi}{180}\cdot dx$$

$$= \frac{\pi}{180}\sec x + C$$

25) $\int \sin x \cdot \sin 5x \; dx$

$= \frac{1}{2}\int (\cos 4x - \cos 6x)\, dx$

$= \frac{1}{2}\left(\frac{\sin 4x}{4} - \frac{\sin 6x}{6}\right) + c$

26) $\int \tan^2 x \cdot \sin^2 x \; dx$

$\tan^2 \cdot \sin^2 x = \tan^2 x\,(1 - \cos^2 x)$

$= \tan^2 x - \sin^2 x$

$= \int (\tan^2 x - \sin^2 x)\, dx$

$= \int (\sec^2 x - 1) - \left(\frac{1 - \cos 2x}{2}\right)$

$= \tan x - x - \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$

27) $\int \frac{x^2 + \cos^2 x}{(x^2+1) \cdot \sin^2 x} dx$

$= \int \frac{x^2 + 1 - 1 + \cos^2 x}{(x^2+1) \cdot \sin^2 x} dx$

$= \int \frac{x^2 + 1 + \sin^2 x}{(x^2+1) \cdot \sin^2 x} dx$

$= \int \frac{1}{\sin^2 x} + \frac{1}{x^2+1}$

$= \int (\operatorname{cosec}^2 x - \frac{1}{1+x^2}\, dx$

$= -\cot x - \tan^{-1} h + c$

28) $\int 3x+5\sqrt{5x+3}\,dx$

$= \int \frac{3}{5}(5x+3)\sqrt{5x+3}\,dx$

$= \frac{3}{5}(5x+3)^{\frac{3}{2}+1}\,dx$

$= \frac{3}{5} \times \frac{2}{5}(5x+3)^{5/2}$

$\int \frac{(5x+3)3}{5} + 5 - \frac{9}{5}\sqrt{5x+3}\,dx$

$\frac{3}{5}\int \frac{3}{5}(5x+3)^{3/2} + \frac{16}{5}(5x+3)^{1/2} + C$

$\frac{3}{5}\frac{(5x+3)^{5/2}}{5 \times \frac{5}{2}} + \frac{16}{5}\frac{(5x+3)^{3/2}}{3/2 \times 5} + C$

$\frac{6}{125}(5x+3)^{5/2} + \frac{32}{75}(5x+3)^{3/2} + C$

29) $\int \frac{x^6}{x^2+1}\,dx$

$= \int \frac{(x^2)^3}{x^2+1}\,dx$

$= \int \frac{(x^2)^3+1-1}{x^2+1}\,dx$

a^3+b^3
$=(a+b)(a^2+b^2-ab)$

$= \int \frac{(x^2)^3+1^3}{x^2+1} - \frac{1}{x^2+1}$

$= \int \frac{\cancel{(x^2+1)}(x^4+1^2-x^2)}{\cancel{x^2+1}}\,dx - \int \frac{1}{x^2+1}\,dx$

$= \frac{x^5}{5} + x - \frac{x^3}{3} - \tan^{-1}x + C$

30) $\int \cos x \cdot \cos 2x \cdot \cos 4x \cdot \sin x \, dx$

$= \frac{1}{2}\int \sin 2x \cdot \cos 2x \cos 4x \, dx$

$= \frac{1}{4}\int \sin 4x \cdot \cos 4x \, dx$

$= \frac{1}{8}\int \sin 8x \, dx$

$= \frac{-\cos 8x}{64} + C$

(B) Using substitution method:-

we know that $\int f(x)\, dx = g(x) + C$

But if

$I = \int f(h(x)) \cdot h'x \, dx$

let $h(x) = t \Rightarrow d(h(x)) = dt$

$h'(x) \cdot dx = dt$

$I = \int f(t) \cdot dt$

$\int f(t)\, dt = g(t) + C$

Ex:- 1) $\int \sin(\cos x) \cdot \sin x \, dx$

let $\cos x = t$

$d(\cos x) = d(t)$

$-\sin x \cdot dx = dt$

$-\int \sin t \cdot dt$

$= \cos t + C$

$= \cos(\cos x) + C$

$\int \frac{1}{x^2+1} dx$

$+C$

(2) $\int e^{e^x+x}\,dx$

$= \int e^{e^x} \cdot e^x\,dx$

let $e^x = t$

$e^x \cdot dx = dt$

$= \int e^t \cdot dt = e^t + c = e^{e^x} + c$

(3) $\int \frac{x}{1+9x^4}$

$= \int \frac{x}{1+(3x^2)^2}\,dx$ let $t = 3x^2$

$dt = 6x \cdot dx$

$= \frac{1}{6} \int \frac{6x\,dx}{1+(3x^2)^2}$

$= \frac{1}{6} \int \frac{dt}{1+t^2}$

$= \frac{1}{6} \tan^{-1}(3x^2) + c$

(4) $\int \tan^5 x \cdot \sec^2 x\,dx$

$\int t^5 \cdot dt$ let $t = \tan x$

$dt = \sec^2 x \cdot dx$

$= \frac{t^6}{6} = \frac{\tan^6 x}{6}$

(5) $\int \frac{x}{1+\sqrt{x}}\,dx$

~~$\int$ let $t = \sqrt{x}$~~

~~$dt = \frac{-1}{\sqrt{x}}\,dx$~~

$\int \frac{\sqrt{x}}{}$

let $t = \sqrt{x}$

~~$dt = \frac{1}{2} x^{1/2 - 1}$~~

$dt = \frac{1}{2\sqrt{x}} dx$

$2\sqrt{x}\, dt = dx \qquad (t^2 = \sqrt{x})$

$2t^2 dt = dx$

$= \int \frac{t^2 \cdot 2t\, dt}{1+t} = 2\int \frac{t^3}{1+t} dt$

$= 2\int \frac{t^3+1-1}{1+t}$

~~$= \int \frac{t^3+1\left((t^3)^2+1^2-t^3\right)}{1}$~~

$t+1)+2 \quad = 2\int \frac{\cancel{(t+1)}(t^2+1-t)}{\cancel{t+1}} + \int \frac{1}{1+t}$

$= 2\left(\frac{t^3}{3} + t - \frac{t^2}{2} - \log|1+t| + c\right)$

~~$= 2\left(\frac{\tan^3 x}{5} + \tan x - \frac{\tan^2 x}{2} - \ln|1+\tan x| + c\right)$~~

$= 2\left(\frac{(\sqrt{x})^3}{3} + \sqrt{x} - \frac{(\sqrt{x})^2}{2} - \ln|1+\sqrt{x}| + c\right)$

Note:- Try to find $f'(x)$ so that we can place it in t

5) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

let $t = \sqrt{x}$

$\cancel{dt =} t^2 = x$

$2t \cdot dt = dx$

$= \int \frac{2t \cdot dt}{t + t^3}$

$= 2\int \frac{t}{t+t^3} dt$ $\quad \sqrt[3]{x} = x^{1/3}$ not $x^{3/2}$

$= 2 \int \frac{\cancel{t}}{\cancel{t}(1+t^2)} dt$

$= 2\tan^{-1} t + C$

$= 2\tan^{-1}(\sqrt{x}) + C$

$\cancel{2x}\ t^6 = x$

$6t^5 \cdot dt = dx$

$= 6\int \frac{t^5}{t^3+t^2} dt$

$= 6\int \frac{t^3}{1+t} dt$

$= 6\int \frac{t^3+1-1}{1+t} dt$

$= 6\int t^2+1-t - \frac{1}{1+t} dt$

$= 6\left(\frac{(\sqrt[6]{x})^3}{3} + \sqrt[6]{x} - \sqrt[6]{x} - \ln|1+\sqrt[6]{x}|\right.$

Points to remember

① use substitution we will try to simplify Integrand

② Search for f and f' pair where f'(x) should be present with dx
If so $t = f(x)$

③ If f(x) is present with dx then we can also think of substituting $f(x) = t$

④ Using substitution, we will get rid of radical sign, if present

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NOTE:- If f'(x) present with dx then in rest of the expression f(x) can be treated as x for the purpose of integration

Ex:- 6) $\int \frac{x^3}{1+x^4}$

$= \frac{1}{4} \ln|1+x^4| + c$

7) $\int x^3 \sqrt[3]{1+5x^3}\, dx$

$= \frac{1}{15} \int 15x^3 (1+5x^3)^{1/3} dx$

$\quad\; f' \qquad\quad f$

$= \int (1+x)^{1/3} dx$

$\boxed{f(x) = x}$

$= \frac{3(1+x)^{4/3}}{4} + c$

$= \frac{1}{\cancel{15}_5} \frac{\cancel{3}}{4} (1+5x^3)^{4/3} + c$

$= \frac{1}{20} (1+5x^3)^{4/3} + c$

$\ln|1+\sqrt[6]{x}| + c$

8) $\int e^{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}}dx$ $\left[f' = \frac{1}{\sqrt{1-x^2}} \quad f = \sin^{-1}x \right.$ treating

$= \int e^x dx = e^x + C = e^{\sin^{-1}x} + C$

9) $\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$

$= -\cos(\tan^{-1}x) + c$

10) $\int \tan x \, dx$

M-1) $\int \frac{\sin x}{\cos x}dx$

$= -\int \frac{-\sin x}{\cos x}dx$ $\quad f'x = -\sin x$

$\quad f(x) = \cos x = x$

$= -\ln|\cos x| + c$

$= \ln\left|\frac{1}{\cos x}\right| + c = \boxed{\ln|\sec x| + c}$ ★★★

M-2) $\int \frac{\sec x \cdot \tan x}{\sec x}dx$

$= \ln|\sec x| + c$ $\quad f'x = \sec x \cdot \tan x$

11) $\int \cot x \, dx$

M-1) $\int \frac{\cos x}{\sin x}dx$

$= \ln|\sin x| + c$

$= -\ln|\operatorname{cosec}| + C$

M-2) $-\int \frac{\operatorname{cosec} x \cdot \cot x}{\operatorname{cosec} x}$

$= -\ln|\operatorname{cosec} x| + c$

★★★ $= \boxed{\ln|\sin x| + c}$

12) $\int \sec x \cdot dx$

$= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx \qquad f'(x) = \sec^2 x + \sec x \cdot \tan x$

$= \boxed{\ln|\sec x + \tan x| + c = -\ln|\sec x - \tan x| + c}$

$\left[s+t = \frac{(s+t)(s-t)}{s-t} = \frac{s^2-t^2}{s-t} = \frac{1}{s-t} \right)$

(or)

$\sec x + \tan x = \frac{1+\sin x}{\cos x}$

$= \frac{(\sin x/2 + \cos x/2)^2}{\cos^2 \frac{x}{2} - \sin^2 x/2}$

$= \frac{(\sin x/2 + \cos x/2)^{\cancel{2}}}{(\cos \frac{x}{2} - \sin x/2)\cancel{(\cos x/2 + \sin x/2)}}$

$= \frac{\sin x/2 + \cos x/2}{\cos x/2 - \sin x/2}$

$= \frac{1+\tan x/2}{1-\tan x/2}$

$= \tan\left(\frac{\pi}{4} + x\right)$ or $\cot\left(\frac{\pi}{4} - x\right)$

$\Longrightarrow \int \sec x\, dx = \ln|\sec x + \tan x| = -\ln|\sec x - \tan x|$

$= \ln\left|\tan\left(\frac{\pi}{4} + x\right)\right| = \ln\left(\cot\left(\frac{\pi}{4} - x\right)\right)$

Ex 13:- $\int \csc(x)\,dx$

$= \int \frac{\operatorname{cosec}x(\operatorname{cosec}x-\cot x)}{\operatorname{cosec}x-\cot x}\,dx$

$= \ln|\operatorname{cosec}x-\cot x|+c$

$= -\ln|\operatorname{cosec}x+\cot x|+c$

$\cancel{= \ln|\cot}$

14) $\int \frac{\sin(x+\alpha)}{\sin x}\,dx$

$= \int \frac{\sin x\cdot\cos\alpha+\cos x\sin\alpha}{\sin x}$

$= \int \cos\alpha+\sin\alpha\cdot\cot x\,dx$

NOTE $= x\cos\alpha+\sin\alpha\ln|\sin x|+c$

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15) $\int \frac{\sin(x+B)}{\cos(x+\alpha)}\,dx$

$\cancel{= \int \frac{\sin x\cos B+\cos x\sin B}{\sin}}$

$= \int \frac{\sin(x+\alpha+B-\alpha)}{\cos(x+\alpha)}\,dx$

$= \frac{\sin(x+\alpha)\cos(B-\alpha)+\cos(x+\alpha)\sin(B-\alpha)}{\cos(x+\alpha)}$

$= \int \tan(x+\alpha)\cos(B-\alpha)+\sin(B-\alpha)\,dx$

15) $\int \frac{e^{2x}-1}{e^{2x}+1}\,dx$

$t = e^{x}+1$

$dt = e^{x}dx$

~~$\frac{e^{x}dx - dx}{e^{x}+1} = \frac{dt - \frac{1}{t}}{t}$~~

~~$t = e^{2x}-1$~~

~~$dt = 2e^{2x}dx$~~

~~$dx = \frac{dt}{2e^{2x}}$~~

~~$\int \frac{t^{2x}-1}{t} \cdot \frac{dt}{2e^{2x}}$~~

$= \cancel{\int \frac{-(e^{x}+1)}{(e^{x}+1)}\,dx} =$

$=$ Multiplying by e^{-x}

M-1

$= \int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx$

let $t = e^{x}+e^{-x}$

$dt = e^{x}-e^{-x}\,dx$

$= \int \frac{dt}{t} = \ln|t| + c = \ln|e^{x}+e^{-x}| + c$

$= \ln|e^{2x}+1| - x + c$

M-2 $\int \frac{e^{2x}-1}{e^{2x}+1}\,dx = \int \frac{e^{2x}-1-1+1}{e^{2x}+1}\,dx = \int \frac{e^{2x}+1-2}{e^{2x}+1}\,dx$

$= \int 1 - \frac{2}{e^{2x}+1}\,dx = \int 1\,dx - \int \frac{2e^{-2x}}{1+e^{-2x}}$

$= \cancel{x - 2\ln|e^{2x}+1| + c} = x + \ln|1+e^{-2x}| + c$

16) $\int \tan^{5}x \cdot \sec x\,dx$

$= \int \tan^{4}x \cdot \tan x \cdot \sec x\,dx$

$= \int (\tan^{2}x)^{2} \cdot \tan x \sec x\,dx$

$= \int (\sec^{2}x-1)^{2} \cdot \tan x \cdot \sec x\,dx$

f f' Shortcut

$\int (\sec^{4}x - 2\sec^{2}x + 1)\tan x \sec x$

$= \frac{\sec^5 x}{5} + \frac{2\sec^3 x}{3} + \sec x + C$

17) $\int \sin^5 x \, dx$

$= \int \sin x \cdot \sin^4 x \, dx$

$= \int (\sin^2 x)^2 \cdot \sin x \, dx = \int (1 - \cos^2 x)^2 \cdot \sin x \, dx$

$f \quad f'$

$= \cancel{\int (\cos^4 x + 2\cos^2 x + 1) \cdot \sin x \, dx}$

shortcut

$= \frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} + \cos x + C$

$= \int (\cos^4 x - 2\cos^2 x + 1) \sin x \, dx$

$= \int (-\cos^4 x + 2\cos^2 x - 1)(-\sin x) \, dx$

$= -\frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C$

$\ln|e^x + e^{-x}| + C$

$\ln|e^{2x} + 1| - x + C$

$= \int \frac{e^{2x} + 1 - 2}{e^{2x} + 1} dx$

$\int 1 \, dx - \int \frac{2e^{-2x}}{1 + e^{-2x}}$

$= x + \ln|1 + e^{-2x}| + C$

shortcut

INTEGRATION USING BY PARTS

$(f \cdot g)' = f'g + g'f$

$f'g = (f \cdot g)' - g'f$

$\int f'g = \int (f \cdot g)' - \int g'f$

$\underset{\text{II}}{\int f'} \underset{\text{I}}{g} = \underset{\int \text{II}}{f} \cdot \underset{\text{I}}{g} - \underset{\text{I}' \int \text{II}}{\int g'f}$

$$\boxed{\int \text{I} \cdot \text{II} = \text{I} \cdot \int \text{II}\, dx - \int (\text{I})' \int \text{II}\, dx}$$

— write I function as it is and integrate II fun. Integrate (differentiate I & Integrate II)

Ex:- 1) $\int \underset{\text{I}}{x} \cdot \underset{\text{II}}{\sin 3x}\, dx$

$= -x \cdot \frac{\cos 3x}{3} - \int 1 \cdot -\frac{\cos 3x}{3}\, dx$

$= -\frac{x \cos 3x}{3} + \frac{\sin 3x}{9} + c$

2) $\int \underset{\text{I}}{(x^2 - 5x)} \cdot \underset{\text{II}}{e^{2x}}\, dx$

$= \frac{(x^2 - 5x) \cdot e^{2x}}{2} - \int \frac{(2x-5)(e^{2x})}{2}\, dx$

$= \frac{(x^2 - 5x) \cdot e^{2x}}{5} - \left(\frac{(2x-5)e^{2x}}{4} - \int \frac{\cancel{2} \cdot e^{2x}}{\cancel{2} \cdot \cancel{4}\, 2} \right)$

$= \frac{x^2 - 5x \cdot e^{2x}}{5} - \frac{(2x-5)e^{2x}}{4} + \frac{e^{2x}}{4} + c$

But how can we decide I & II ? - ILATE

I - Inverse
L - logarithmic
A - Algebraic
T - Trigonometric
E - exponential

3) $\int x^3 \cdot \ln(x)\,dx$
II I

$\int uv = u\int v\,dx - \int u' v_1\,dx$

u	$v\,dx$	$\ln x$	$x^3\,dx$
u'	v_1	$\frac{1}{x}$	$\frac{x^4}{4}$

$$\int x^3 \cdot \ln x\,dx = \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{x \cdot 4}\,dx$$

$$= \ln x \cdot \frac{x^4}{4} - \cancel{\frac{x^4}{16} + c} - \frac{x^4}{16} + C$$

NOTE :-

(i) By parts try to convert dissimilar functions into similar functions
(ii) If functions contains inverse (or) logarithmic then try to apply By parts

Ex:- $\int \ln x\,dx$ $= \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx$

$u - \ln x$ $\quad 1\,dx$ $\quad = \boxed{x\ln x - x + c}$ Memorise

$\frac{1}{x}$ $\quad x$ $\quad = x(\ln x - 1) + c$

2) $\int \sin^{-1}x\,dx \;=\; x\sin^{-1}x - \int \frac{1}{\sqrt{1-x^2}}\,x\,dx$

$\sin^{-1}x \quad 1$

$\frac{1}{\sqrt{1-x^2}} \quad x$

$= x\sin^{-1}x$

$= x\sin^{-1}x - \frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}$

$= x\sin^{-1}x - \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}$

$\boxed{= x\sin^{-1}x + \sqrt{1-x^2} + C}$ Memorise

3) $\int \underset{v}{e^x}\cdot \underset{u}{\sin x}\,dx \;=\; \sin x\cdot e^x - \int \cos x\cdot e^x\,dx$

$\sin x \quad e^x$

$\cos x \quad e^x$

$= e^x\sin x - \left[\cos x\cdot e^x - \int -\sin x\cdot e^x\right]$

$= e^x\sin x - \cos x\cdot e^x - \int e^x\cdot\sin x\,dx$

Question: I

$I = e^x\sin x - \cos x\cdot e^x - I$

$2I = e^x\sin x - \cos x\cdot e^x$

$I = \frac{e^x(\sin x - \cos x)}{2}$

4) $I = \int \sin(3\ln(5x))\,dx$

$= \sin(3\ln(5x))\cdot x - \int \cos(3\ln 5x)\cdot\frac{3}{\cancel{5x}}\cdot 5\cdot x\,dx$ $\quad \sin(3\ln 5x) \quad 1$ $\quad x$

$= x\sin(3\ln(5x)) - 3\int\cos(3\ln 5x)\,dx$

$= x\sin(3\ln(5x)) - 3\left(\cos(3\ln 5x)\cdot x - \int -\frac{\sin(3\ln 5x)\cdot 3\cdot 5x}{5x}\right)$

$= x\sin(3\ln 5x) - x3\cos(3\ln 5x) - 9\int\sin(3\ln(5x))$

$I + 9I = x\left(\sin(3\ln(5x)) - \cos(3\ln(5x))\right)$

$I = \frac{x}{10}\left(\sin(3\ln(5x)) - \cos(3\ln(5x))\right) + C$

Also can be solved by

let $t = 3\ln(5x)$

$dt = \frac{3}{5x} \cdot 5\, dx$

$x\, dt = 3\, dx$

$e^t = 25x^3$, $x = e^t$

$\int \sin(3\ln(5x))\, dx = \frac{1}{3}\int 3 \sin t\, dx$

$= \frac{1}{3}\int x \cdot \sin t\, dt$

$\frac{1}{3}\int \underset{I}{x}\, \underset{II}{\sin t}\, dt = -x\cos t - \int 1 \cos t\, dt$

$= -x\cos t - \sin t + c$

3) $\int \sec^3 x\, dx$

$= \int \underset{(1)}{\sec x} \cdot \underset{(2)}{\sec^2 x}\, dx$

$= \sec x \cdot \tan x - \int \sec x \tan x \cdot \tan x\, dx$

$= \sec x \cdot \tan x - \int \sec x \tan^2 x\, dx$

$= \sec x \cdot \tan x - \cancel{\tan^2 x} \int (\sec x \cdot (\sec^2 x - 1))\, dx$

$= \sec x \cdot \tan x - \left(\int \sec^3 x\, dx - \int \sec x\, dx\right)$

$\cancel{2I} = \sec x \cdot \tan x + \sec x \cdot \tan x$

$I = \cancel{\sec x \cdot \tan x}$

$2I = \sec x \cdot \tan x + \ln|\sec x + \tan x|$

$I = \frac{1}{2}\left(\sec x \cdot \tan x + \ln|\sec x + \tan x|\right) + C$

Types of problems till now solved:-

1. Normal By parts problems like $\int e^x \cdot x^2$
2. complex one like $\int 1 \cdot \sin^{-1} dx$, $\int \ln(x)\, dx$
3. using I problems like $\int e^x \sin x$, $\int \sec^3 x$, $\int \sin(3\ln(5x))$
4. using Integral By parts expansion type

$$\int \mathrm{I}\,\mathrm{II}\,dx = \mathrm{I}\int \mathrm{II}\,dx - \mathrm{I}'\iint \mathrm{II}\,dx + \mathrm{I}''\iiint \mathrm{II}\,dx - \mathrm{I}'''\iiiint \mathrm{II}\,dx + \dots$$

INTEGRATION BY PARTS EXPANSION TRICK

Ex:- 1) $\int \underset{\mathrm{I}}{(x^3-5x^2+7x+9)}\, \underset{\mathrm{II}}{e^{3x}}\, dx$

$$= (x^3-5x^2+7x+9)\frac{e^{3x}}{3} - (3x^2-10x+7)\cdot\frac{e^{3x}}{9} + (6x-10)\frac{e^{3x}}{27} - 6\left(\frac{e^{3x}}{3^4}\right) + C$$

By expansion rule

By Normal method

$\int (x^3-5x^2+7x+9)\, e^{3x}\, dx$

$$= (x^3-5x^2+7x+9)\cdot\frac{e^{3x}}{3} - \int (3x^2-10x+7)\cdot\frac{e^{3x}}{3}\, dx$$

$$= \frac{(x^3-5x^2+7x+9)\cdot e^{3x}}{3} - \frac{1}{3}\left[\frac{(3x^2-10x+7)\cdot e^{3x}}{3} - \int (6x-10)\dots\right.$$

$$= \frac{(x^3-5x^2+7x+9)\, e^{3x}}{3} - \frac{(3x^2-10x+7)\cdot e^{3x}}{9} + \frac{1}{3}\left((6x-10)\cdot e^{3x}\dots\right.$$

$$= \frac{x^3-5x^2+7x+9}{3}\cdot e^{3x} - \frac{(3x^2-10x+7)e^{3x}}{9} + \frac{1}{9}\left[(6x+10)\cdot\frac{e^{3x}}{3} - 6\int\frac{e^{3x}}{3}\right] + c$$

$$= \frac{x^3-5x^2+7x+9}{3}\cdot e^{3x} - \frac{(3x^2-10x+7)\cdot e^{3x}}{9} + \frac{(6x+10)\cdot e^{3x}}{27} - \frac{6e^{3x}}{9\times 6\,3^4} + C$$

Same result

2) $\int (x^2-5x)(\sin 7x)\,dx$ (I, II)

$$= (x^2-5x)\left(\frac{-\cos 7x}{7}\right) - (2x-5)\left(\frac{-\sin 7x}{49}\right) + 2\left(\frac{\cos 7x}{7^3}\right) + C$$

Shortcut

$\tan x \sec x$	$\sec x$
f'	$f \to x$

3) $\int x^2 \cdot \tan x \cdot \sec^2 x\,dx$ (I, II)

$$= \int x^2 \cdot (\tan x \cdot \sec x \cdot \sec x)\,dx$$

$$= \frac{x^2\cdot \sec^2 x}{2} - \frac{2x\cdot \tan x}{4} + \frac{2\ln|\sec x|}{4} + C$$

→ INTEGRATION BY SUBSTITUTION OF TRIGONOMETRIC FUNCTIONS

$\sqrt{x+3}$ let $t^2 = x+3$ $dx = 2t\,dt$

$t^2 - 3 = x$

$\sqrt{x^2-4}$ let $t^2 = x^2-4$ $\cancel{2}x\,dx = \cancel{2}t\,dt$

$x^2 = t^2 - 4$

$$\frac{x^5}{\sqrt{x^2-4}}$$

$$= \frac{x\cdot(x^2)^2\cdot dx}{\sqrt{x^2-4}}$$

$$= \frac{\cancel{t}dt\cdot(t^2-4)^2}{\cancel{t}}$$

can be solved if there is a odd power in numer.

$$\frac{x^4}{\sqrt{x^2-4}}$$

can not be solved by normal subsustitution

↓

Here we use trigono fun substitution

$\cos^2 x = 1-\sin^2 x$. $\tan^2 x + 1 = \sec^2 x$ $\sec^2 x - 1 = \tan^2 x$

$$\int\frac{x^4}{\sqrt{x^2-4}}\,dx =$$ let $x = 2\sec\theta$

$dx = 2\sec\theta\cdot\tan\theta$

$$\int\frac{2^4\sec^4\theta}{\sqrt{4(\sec^2\theta-1)}}\,dx = 2^3\int\frac{\sec^4\theta}{\tan\theta}\,d\theta\,(2\sec\theta\cdot\tan\theta)$$

$$= 2^4\int \sec^5\theta$$

$$= 2^3 \quad \frac{1}{2}\int 2^5\sec^5\theta = \frac{1}{2}\cdot\frac{x^6}{6}$$

Expression	Suggested Substitution
$\sqrt{a^2-x^2}$	$x = a\cos\theta$ (or) $a\sin\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$ (or) $a\cot\theta$
$\sqrt{x^2-a^2}$	$x = a\,\mathrm{cosec}\,\theta$ (or) $a\sec\theta$
$\sqrt{\frac{a+x}{a-x}}$, $\sqrt{a+x}$, $\sqrt{a-x}$, $\sqrt{a+x}+\sqrt{a-x}$	$x = a\cos 2\theta$
$\sqrt{(x-a)(b-x)}$	$x = a\sin^2\theta + b\cos^2\theta$
$\sqrt{(a-x)(b-x)}$ (or) $\sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$

$x = a\sin^2\theta + b\cos^2\theta$

$x - a = a\sin^2\theta + b\cos^2\theta - a \quad = a(-\cos^2\theta) + b\cos^2\theta = (b-a)c^2\theta$

$x - a = \cancel{a(\sin^2\theta - 1)} + b\cos^2\theta - a\sin^2$

$x - a = c(\cos^2\theta(a-b) \quad bc^2 - a(1-s^2)$ $\quad (b-a)c^2$

$b - x = b - a\sin^2\theta - b\cos^2\theta$

$= b(1-\cos^2\theta) - a\sin^2\theta$

$= b\sin^2\theta - a\sin^2\theta$

$= \sin^2\theta(b-a)$

$\sqrt{(x-a)(x-b)} = \sqrt{(b-a)^2 \cdot \sin^2\theta \cdot \cos^2\theta}$

$= \boxed{|b-a| \sin\theta \cdot \cos\theta}$

In the same way

$x = a\sec^2\theta - b\tan^2\theta$

$x - a = a\sec^2\theta - b\tan^2\theta - a$

$x - a = -b\tan^2\theta + a(\sec^2\theta - 1)$

$= \tan^2\theta\,(a-b)$

$x - b = a\sec^2\theta - b\tan^2\theta - b$

$= a\sec^2\theta - b(\tan^2\theta + 1)$

$= \sec^2\theta\,(a-b)$

$\sqrt{(x-a)(x-b)} = \sqrt{(a-b)^2 \cdot \sec^2\theta \cdot \tan^2\theta}$

$\boxed{= |a-b| \cdot \sec\theta \cdot \tan\theta}$

Ex:- i) $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$ $\quad \sin\theta = \frac{x}{a}$

$\theta = \sin^{-1}\left(\frac{x}{a}\right)$

$x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$

$\displaystyle\int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \theta + c = \sin^{-1}\left(\frac{x}{a}\right) + c$

2) $\displaystyle\int \frac{dx}{\sqrt{x^2 - 9}}$

let $x = 3\sec\theta \Rightarrow \sec\theta = \frac{x}{3}$ $\quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$dx = 3\sec\theta \cdot \tan\theta\, d\theta$ $\quad \Big|\ \sqrt{9\sec^2\theta - 9} = 3\tan\theta$

$\displaystyle\int \frac{3\sec\theta \cdot \tan\theta\, d\theta}{3\tan\theta} = \int \sec\theta \cdot d\theta$

$= \ln|\sec\theta + \tan\theta| + c$

$= \ln\left|\frac{x}{3} + \sqrt{\frac{x^2}{9} - 1}\right| + c$

$= \ln|x + \sqrt{x^2 - a}| + \underbrace{\ln 3| + c}_{C}$

3) $\int \sqrt{x^2+4}\, d\theta$ $\qquad x = 2\tan\theta$

$\qquad dx = 2\sec^2\theta \cdot d\theta$

$= \int 2\sec\theta \cdot 2\sec^2\theta\, d\theta$ $\qquad \sqrt{4\tan^2\theta + 4} = 2\sec\theta$

$= 4\int \sec^3\theta\, d\theta$

$= 4\int \sec\theta \cdot \sec^2\theta \cdot d\theta$

$4\left[\sec\theta \cdot \tan\theta - \int \sec\theta \cdot \tan\theta \cdot \tan\theta\, d\theta\right]$

$= 4\left[\sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta\right.$

$= 4\left[\sec\theta\tan\theta - \left[\int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta\right]\right.$

$I - 4\sec\theta\tan\theta - 4I + 4\sec\theta\tan\theta$

$I = \frac{4}{5}\sec$

$= 4\int \sec\theta \cdot \sec^2\theta\, d\theta$

$= 4\left[\sec\theta \cdot \tan\theta - \int \sec\theta \cdot \tan^2\theta\, d\theta\right.$

$= 4\left[\sec\theta \cdot \tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta\right.$

$= 4\left[\sec\theta \cdot \tan\theta - \int \sec^3\theta - \int \sec\theta\, d\theta\right.$

$\left[\int \sec^3\theta\, d\theta = \frac{1}{2}\left(\sec x \cdot \tan x + \ln|\sec x + \tan x|\right)\right) + c$

$\left(\frac{x}{a}\right)$

$3\tan\theta$

$+c$

$= 4\sec\theta\tan\theta - 2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| - \sec\theta\tan\theta$

$= \cancel{3}\,\cancel{S}\ \sec\theta\tan\theta - 2\ln\left|\sec\theta+\tan\theta\right|$

~~$\sec\theta$~~ $\boxed{\tan\theta = \frac{x}{2} \quad \sec\theta = \frac{\sqrt{x^2+4}}{2}}$

$= \frac{x}{2}\cdot\frac{\sqrt{x^2+4}}{2} - 2\ln\left|\sqrt{x^2+4}+x\right| + \ln 2 + C$

4) $\int \frac{x^3\,dx}{\sqrt{2-x^2}}$ $\quad x = \sqrt{2}\sin\theta$, $dx = \sqrt{2}\cos\theta\,d\theta$ $\quad\Big|\quad \sqrt{2-2\sin^2\theta} = \sqrt{2}\cos\theta$

$= \int \frac{(\sqrt{2})^3\sin^3\theta\cdot\sqrt{2}\cos\theta\,d\theta}{\sqrt{2}\cos\theta}$

$= 2\sqrt{2}\int \sin^3\theta\,d\theta$

$= 2\sqrt{2}\int \sin\theta(1-\cos^2\theta)\,d\theta$

$= 2\sqrt{2}\int \overset{f}{(\cos^2\theta-1)}\overset{f'}{(-\sin\theta)}\,d\theta$

$= 2\sqrt{2}\left[\frac{\cos^3\theta}{3} - \cos\theta\right] + C$

$\sin\theta = \frac{x}{\sqrt{2}}$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-\frac{x^2}{2}}$

$= \frac{\sqrt{2-x^2}}{\sqrt{2}}$

$= 2\sqrt{2}\left[\left(\sqrt{\frac{2-x^2}{2}}\right)^3 - \sqrt{\frac{2-x^2}{2}}\right]$

Also this problem can be solved by substitution $x^2 = t$

STANDARD ALGEBRAIC INTEGRAL FORMULAS

tan

$\cancel{\int \frac{dx}{\sqrt{a}}}$ 9 standard integrals

1) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c$

2) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a-x}{a+x}\right| + c$

$\frac{2+c}{c}$

3) $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

4) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

5) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2+a^2}\right| + c$

6) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + c$

7) $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

8) $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

9) $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$

So we can integrate functions of type $\frac{1}{Q}$, $\frac{1}{\sqrt{Q}}$, $\sqrt{Q}$ by above 9.S.I

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Ex: 1) $\int \frac{dx}{3x^2+6x-7}$

$= \frac{1}{3}\int \frac{dx}{x^2+2x-\frac{7}{3}} = \frac{1}{3}\int \frac{dx}{(x+1)^2-1-\frac{7}{3}}$

$= \frac{1}{3}\int \frac{dx}{(x+1)^2-\left(\sqrt{\frac{10}{3}}\right)^2}$

$\cancel{= \ln\left|x+1\sqrt{}\right.}$ $= \frac{1}{3}\ln\left|\frac{x+1-\sqrt{\frac{10}{3}}}{x+1-\sqrt{\frac{10}{3}}}\right| + c$

2) $\int \sqrt{8x-x^2}\, dx$

$= \int \sqrt{-(x^2-8x)}\, \cancel{dx}\, dx$

$= \int \sqrt{-(x^2-8x)-16}\, dx = \int \sqrt{(x-4)^2-16}\, dx$

$= \int \sqrt{4^2-(x^2-4x)^2}$

$= \frac{x}{2}\sqrt{4^2-(x-8)^2} + 8\sin^{-1}\left(\frac{x-8}{4}\right) + c$

PROBLEMS

1) $\int \underset{\text{II}}{x}\,\underset{\text{II}}{3^x}\,dx$

$\cancel{= \frac{3^x x^2}{2} - \int 3^x \cdot \ln|3|}$

$= x \cdot \frac{3^x}{\log_e|3|} - \int \frac{3^x}{\ln|3|} dx$

$= \frac{x \cdot 3^x}{\ln|3|} - \frac{3^x}{\ln|3| \ln|3|} + C$

2) $\int \ln(x^2+1)\,dx$

$= \int \underset{\text{II}}{1} \cdot \underset{\text{I}}{\ln(x^2+1)}\,dx$

$= x\ln(x^2+1) - \int \frac{x \cdot 2x}{x^2+1} dx$

$= x\ln(x^2+1) - \int \frac{2x^2}{x^2+1} dx$

$= x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1} dx$

$= x\ln(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1} dx$

$= x\ln(x^2+1) - 2\left[x - \tan^{-1}x\right] + C$

3) $\int \underset{\text{II}}{x^n}\,\underset{\text{I}}{\ln x}\,dx$

$\cancel{= \frac{\ln x \cdot x^{n+1}}{n+1} - \int \frac{x^{n+1}}{x(n+1)} dx}$

$\cancel{= \ln x \cdot \frac{x^n}{x} - \int \frac{n x^{n-1}}{x} dx}$

6) $\int \sin(\ln x)\,dx = \int \underset{II}{1}\cdot \underset{I}{\sin(\ln x)}\,dx$

$= \sin(\ln x)\cdot x - \int \frac{\cos(\ln x)}{\cancel{x}}\cdot \cancel{x}\,dx$

$= x\sin(\ln x) + \left[\cos(\ln x)\cdot x - \int \frac{\sin(\ln x)}{\cancel{x}}\cdot\cancel{x}\,dx\right.$

$I = x\sin(\ln x) + x\cos(\ln x) - I$

$$\boxed{I = \frac{x}{2}\left(\sin(\ln x) + \cos(\ln x)\right) + c}$$

7) $\int \frac{e^{2x}}{\sqrt[4]{e^x+1}}\,dx$

let $e^x + 1 = t^4 \Rightarrow e^x = (t^4 - 1)$

$e^x\,dx = 4t^3\cdot dt$

$dx = \frac{4t^3\cdot dt}{e^x}$

$\frac{1}{4} - 1$

$\frac{-3}{4}$

$\int \frac{\cancel{e^x}\cdot e^x}{\cancel{t}}\cdot\left(\frac{4t^{\cancel{3}\,2}\cdot dt}{\cancel{e^x}}\right)$

~~$= \frac{4t^3}{3} + c = \frac{4(e^x+1)^{\frac{3}{4}}}{3} + c$~~

$= \int (t^4 - 1)\cdot 4t^2\,dt \qquad = \int (4t^6 - 4t^2)\,dt$

$= 4\left(\frac{t^7}{7} - \frac{t^3}{3}\right) + c$

$= 4\left(\frac{(e^x+1)^{\frac{7}{4}}}{7} - \frac{(e^x+1)^{\frac{3}{4}}}{7}\right) + c$

8) $\int \sqrt{1+\cos^2 x} \cdot \sin 2x \cdot \cos 2x \, dx$

let $t^2 = 1+\cos^2 x$

$2t\,dt = -2\sin x \cos x$

$2t\,dt = -\sin 2x \, dx$

$\cos^2 x = t^2 - 1$

$\cos 2x = 2\cos^2 x - 1$
$= 2(t^2-1)-1$
$= 2t^2 - 3$

$= \int \sqrt{1+(t^2-1)} \cdot (-2t\,dt) \cdot (2t^2-3)$

$= \int t(-2t)\cdot(2t^2-3)\,dt$

$= 4\int(4t^4 - 6t^2)\,dt = \dfrac{4t^5}{5} - \dfrac{6t^3}{3} + c$

$= 2\left(\dfrac{3t^3}{3} - \dfrac{2t^5}{5}\right) + c$

$= 2\left(\dfrac{3\left(\sqrt{1+\cos^2 x}\right)^3}{3} - \dfrac{2(1+\cos^2 x)^{\frac{5}{2}}}{5}\right)$

9) $\int \dfrac{e^{\sqrt{x}}}{\quad} \, dx$

let $t^2 = x$

$2t\,dt = dx$

$\int e^t \cdot 2t\,dt = 2\int e^t \cdot t\,dt$

$= 2\left[t \cdot e^t - e^t\right] + c = 2(\sqrt{x}\, e^{\sqrt{x}} - \dots)$

10) $\displaystyle\int \frac{x^2 \cdot \tan^{-1}(x)}{1+x^2}\,dx$

let $t = \tan^{-1} x \Rightarrow \tan t = x$

$dt = \frac{1}{1+x^2}\,dx \qquad x^2 = (\tan t)^2$

$(1+\cancel{x^2})\,dt = dx$

$= \displaystyle\int \underset{\text{II}}{(\tan t)^2} \cdot \underset{\text{I}}{t}\,dt = \int (\sec^2 t - 1)\, t\, dt$

$= t(\tan t - t) - \int (\tan t - t)\,dt$

$I = t(\tan t - t) - \cancel{\int \tan t\, d}$

$\left(\ln|\sec t| - \frac{t^2}{2}\right) + c$

$= t(\tan t - t) - \ln|\sec t| + \frac{t^2}{2} + c$

$= \tan^{-1} x\,(x - \tan^{-1} x) - \ln|\sec(\tan^{-1} x)| + \frac{(\tan^{-1} x)^2}{2} + C$

Mixed Problems

1) $\int \frac{1-\tan^2 x}{1+\tan^2 x} dx$

$= \int \cos 2x \quad = \frac{\sin 2x}{2} + C$

2) $\int x^2 \sqrt[5]{x^3+2}\, dx$

$t^5 = x^3+2$

$5t^4 \cdot dt = (3x^2 \cancel{+2})dx$

$\frac{5t^4}{3} dt = x^2 dx$

$\frac{5}{3}\int t^4 \cdot t\, dt \quad = \frac{5}{3}\int t^5 = \frac{5}{3}\left[\frac{t^6}{6}\right] + C$

$= \frac{5}{3}\left[\frac{(x^3+2)^{\frac{6}{5}}}{6}\right] + C$

3) $\int \sec^2 x \cdot \csc^2 x\, dx$

$= \int \cancel{\sec^2 x \cdot \cos} \quad = \int \frac{1}{\sin^2 x \cdot \cos^2 x} dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx$

$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx$

$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$

$= \tan x - \cot x + C$

$$\boxed{\sec^2 x \cdot \operatorname{cosec}^2 x = \sec^2 x + \operatorname{cosec}^2 x}$$

4) $\displaystyle\int \frac{2x - \sqrt{\sin^{-1} x}}{\sqrt{1-x^2}}\, dx$

let $t^2 = \sin^{-1} x \Rightarrow x = \sin t^2$

$2t \cdot dt = \dfrac{1}{\sqrt{1-x^2}}\, dx$

$= \int (\sin t^2 - t) \cdot 2t\, dt$

$= \int \underset{\mathrm{I}}{2t}\, \underset{\mathrm{II}}{\sin(t^2)}\, dt - \int 2t^2\, dt$ try with division

$= 2\left[-t \cos(t^2) \cdot 2t - \int -\cos(t^2) \cdot 2t\, dt\right] - \int 2t^2$

$= -4t^2 \cos(t^2) + \int 2t \cos(t^2)\, dt - \int 2t^2$

$= -4t^2 \cos(t^2) + 2\int t \cos(t^2)\, dt - \int 2t^2$

$= -4t^2 \cos(t^2) + 2\,[$

$= \displaystyle\int \frac{2x}{\sqrt{1-x^2}}\, dx - \int \frac{\sqrt{\sin^{-1} x}}{\sqrt{1-x^2}}\, dx$

$= -\displaystyle\int \frac{-2x}{\sqrt{1-x^2}}\, dx$ ($-2x$ — f', $\sqrt{1-x^2}$ — f) $- \displaystyle\int \frac{\sqrt{\sin^{-1} x}}{\sqrt{1+x^2}}$ ($\sqrt{\sin^{-1}x}$ — f, $\sqrt{1+x^2}$ — f')

$= \dfrac{(\sqrt{1-x^2})^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \dfrac{(\sin^{-1} x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}$

$$= \frac{\left(\sqrt{1-x^2}\right)^{\frac{1}{2}}}{\frac{1}{2}} - \frac{\left(\sin^{-1}x\right)^{\frac{3}{2}}}{3/2} + c$$

5)

$$\int \frac{x}{x^2+2x+1}\,dx$$

$$= \int \frac{x}{(x+1)^2} = \int \frac{x+1-1}{(x+1)^2}\,dx$$

$$= \int \frac{1}{x+1} - \int \frac{1}{(x+1)^2}$$

$$= \ln|x+1| - \frac{(x+1)^{-1}}{-1} + c$$

$$= \ln|x+1| + \frac{1}{x+1} + c$$

6)

$$\int \log\left(x+\sqrt{x^2+a^2}\right)dx$$

$$= \int \underset{\mathrm{II}}{1} \cdot \underset{\mathrm{I}}{\log\left(x+\sqrt{x^2+a^2}\right)}\,dx$$

$$= x\log\left(x+\sqrt{x^2+a^2}\right) - \frac{1}{2}\int \frac{2x}{\sqrt{x^2+a^2}}\,dx$$

NOTE:- $\frac{d}{dx}\log\left(x+\sqrt{x^2+a^2}\right) = \frac{1}{\sqrt{x^2+a^2}}$

$$= x\log\left(x+\sqrt{x^2+a^2}\right) - \frac{\cancel{1}}{\cancel{2}} \frac{\left(x^2+a^2\right)^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + c$$

7) $\int \sqrt{\frac{x-1}{x+1}} \cdot \frac{dx}{x^2}$

try to find logic by formula $\sqrt{\frac{a-x}{a+x}}$

$x = a\cos 2\theta$

$= \int \sqrt{\frac{1-\frac{1}{x}}{1+\frac{1}{x}}}\, dx$

$(a=1)$

$\frac{1}{x} = \cos 2\theta$

$= \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \cdot \frac{dx}{x^2}$

$-\frac{1}{x^2} = -2\sin 2\theta$

$= \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \cdot 2\sin 2\theta\, d\theta$

$= \int \frac{\sin\theta}{\cos\theta} \cdot 2\sin 2\theta\, d\theta$

$= 4\int \frac{\sin\theta \cdot \sin\theta \cdot \cos\theta}{\cos\theta}\, d\theta$

$= 4\int \sin^2\theta = 2\int 2\sin^2\theta$

$= 2\int (1-\cos 2\theta)\, d\theta$

$= 2\left[\theta - \frac{\sin 2\theta}{2}\right] + C$

$= 2\left[\ 2\theta - \frac{\sin 2\theta}{2}\right. + C$

$= \cos^{-1}\left(\frac{1}{x}\right) - \frac{\sqrt{1-\cos^2(2\theta)}}{2}$

$= \cos^{-1}\left(\frac{1}{x}\right) - \frac{\sqrt{1-\left(\frac{1}{x}\right)^2}}{2} + C$

8) $\int \frac{\sin^{-1}x}{x^2}dx$

$= \int \underset{II}{\frac{1}{x^2}} \cdot \underset{I}{\sin^{-1}x}\; dx$

$= \sin^{-1}x\left(\frac{-1}{x}\right) + \int \cancel{\frac{1}{\sqrt{1-}}}\, \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{x}\, dx$

$= \frac{-\sin^{-1}x}{x} + \int I_1\, dx$

$\int I_1\, dx = \int \frac{1}{x\sqrt{1-x^2}}\, dx$

let $t^2 = 1-x^2 \quad \Rightarrow \quad x^2 = \cancel{t^2}\; 1-t^2$

$\cancel{2}t\,dt = -\cancel{2}x\,dx$

$= \int \frac{x}{x^2\sqrt{1-x^2}}\,dx \quad = -\cancel{\int \frac{t\,dt}{(t^2-1)\cdot t}} \quad = -\int \frac{1\cdot t}{(t^2-1)\cdot t}$

$= \cancel{-\int \frac{1}{t^2-1}} \quad = -\int \frac{t\,dt}{(1-t^2)t}$

$= \frac{-1}{2}\ln\left|\frac{1-\sqrt{1+x^2}}{1+\sqrt{1+x^2}}\right| + C$

substituting $\int I_1$

$= \frac{-\sin^{-1}x}{x} - \frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1+x^2}}\right| + C$

9) $\int \frac{x^6-1}{x^2+1}\,dx$

$= \int \frac{(x^2)^3-1^3}{x^2+1} = \int x^2$

$\left[a^3-b^3 = (a+b)^3 + 3ab(a-b)\right]$

9) $\int \frac{x^6-1}{x^2+1}\,dx$

$= \int \frac{x^6-1+1-1}{x^2+1}\,dx = \int \frac{x^6+1-2}{x^2+1}\,dx = \int \frac{(x^2)^3+1-2}{x^2+1}\,dx$

$= \int \frac{(x^2)^3+1^3}{x^2+1}\,dx + \int \frac{-2}{x^2+1}\,dx$

$= \int \frac{(x^2+1)(x^4+1-x^2)}{\cancel{x^2+1}}\,dx + \int \frac{-2}{x^2+1}\,dx$

$= \frac{x^5}{5} + x - \frac{x^3}{3} - 2\tan^{-1}x + c$

10) $\int x^2 \sqrt{4-x^2}\cdot dx$

let $x = 2\sin\theta$ $\qquad dx = 2\cancel{\sin\theta}\cos\theta\, d\theta$

$= \int 4\sin^2\theta \sqrt{4-4\sin^2\theta}\; 2\cos\theta\, d\theta$

$= 16 \int \sin^2\theta \cos^2\theta\, d\theta$

$= 4 \int 4\sin^2\theta\cos^2\theta\, d\theta = 4\int (2\sin\theta\cos\theta)^2\, d\theta$

$= 4\int \sin^2(2\theta)\, d\theta = \cancel{4}^{2}\int\left[\frac{1-\cos(4\theta)}{2}\right] d\theta$

$= \theta - \frac{\sin 4\theta}{4} = \sin^{-1}\left(\frac{x}{2}\right) - \frac{\sin 4(\sin^{-1}\frac{x}{2})}{2}$

11) $\int \frac{1+\cos 4x}{\tan x - \cot x} dx$

$= \int \frac{2\cos^2(2x)}{\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}} dx$

$= \int \frac{2\cos^2 2x}{\frac{\sin^2 x - \cos^2 x}{\cos x \cdot \sin x}} dx$

$= \int \frac{2\cos^{\cancel{2}}(\cancel{2x})}{\frac{-\cos 2x}{\cancel{\frac{1}{2}\sin(\frac{x}{2})}} \sin x \cdot \cos x}$

$= \cancel{-\int 2\cos^2(2x) \cdot \sin(\frac{x}{2}) dx}$

$= -\int 2\sin x \cdot \cos x \cdot \cos(2x) dx$

$= -\int \sin(2x)\cos(2x) dx$

$= -\frac{1}{2}\int \sin 4x \, dx = +\frac{1}{2} \cdot \frac{\cos 4x}{4} + C$

12) $\int x \sin x \cdot \sec^3 x \, dx$

$= \int \underset{I}{x} \cdot \underset{II}{\tan x \cdot \sec^2 x} dx$

$= x \int \tan x \cdot \sec^2 dx - \int \int \tan x \cdot \sec^2 x \, dx$

$= x\left(\frac{\tan^2(x)}{2}\right) - \int \frac{\tan^2 x}{2}$ $(\sec^2 x - 1)$

$= \frac{x}{2}\tan^2 x - \left(\frac{\tan x - x}{2}\right) + C$