DIFFERENTIATION

Objectives

- ☐ geometrical meaning of Derivative
- ☐ standard Derivatives
- ☐ chain Rule
- ☐ Theorms
- ☐ Implicit functions
- ☐ Parametric form
- ☐ one function w.r.t other function
- ☐ Inverse trigonometric, exponential, logarithmic, Hyperbolic
- ☐ Partial Derivatives

Representation of Derivative

$$y = f(x)$$

1st order principle

$$\frac{dy}{dx} = \frac{d}{dx}(f(x)) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{x_2 \to x} \frac{y_2 - y}{x_2 - x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y' = f'(x)$$

$$= D(y) = D(f(x))$$

NOTE

$$\frac{dy}{dx} \neq \frac{dy}{1} \times \frac{1}{dx} = \frac{d}{dx} \times y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

if $x = 1$
$y = 2$
(so)
$y = 1$
$x = 1/2$

Problem

Find $\frac{dy}{dx}$ in the following cases by using 1st order principle.

(i) $y = \sin x$

$= \lim_{h \to 0} \frac{(\sin(x+h)) - \sin x}{h}$

$= \lim_{x \to 0} \frac{2\sin\left(\frac{x+h-x}{2}\right)\cos\left(\frac{x+h+x}{2}\right)}{h}$

$= \lim_{x \to 0} \frac{\sin\left(\frac{x+h-x}{2}\right)}{h/2} \cdot \cos(x + h/2)$

$= 1 \cdot \cos x = \cos x$

(ii) $y = \ln(x)$

$= \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$

$= \lim_{x \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{x \to 0} \frac{\ln(1 + h/x)}{h}$

$= \frac{h/x}{h} = \frac{1}{x}$

$\ln(A) \to 1$

$f'(x)$

$= A - 1$

(iii) $y = \sqrt{x}$

$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} = \lim_{x \to 0} \frac{\frac{x+h-x}{\sqrt{x+h}+\sqrt{x}}}{h}$

$= \frac{1}{2\sqrt{x}}$

nciple

STANDARD FORMULAS	S.NO:	Function	Derivative
	1)	k	0
	2)	x^n	nx^{n-1}
	3)	e^x	e^x
	4)	a^x	$a^x \cdot \ln a$
	5)	$\ln x$, $\ln\lvert x\rvert$	$1/x$
Imp —	6.)	$\log_a x$	$1/x \ln a$
	7)	$\sin x$	$\cos x$
	8)	$\cos x$	$-\sin x$
	9)	$\tan x$	$\cancel{\frac{1}{1+x^2}}$ $\sec^2 x$
	10)	$\cot x$	$-\operatorname{cosec}^2 x$
	11)	$\sec$	$\sec x \cdot \tan x$
	12)	cosec	$-\operatorname{cosec} x \cdot \cot x$
	13)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
	14)	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
	15)	$\tan^{-1} x$	$\frac{1}{1+x^2}$
	16)	$\cot^{-1} x$	$\frac{-1}{1+x^2}$
	17)	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
	18)	$\operatorname{cosec}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
	19)	$\{x\}$ – Fractional part	1
	20)	$[x]$ – greatest integer	0
Imp —	21)	x^x	$x^x(1+\ln x)$
	22)	$\sqrt{x^2+a^2}$	$\frac{x}{\sqrt{x^2+a^2}}$

23) $\sqrt{x^2-a^2}$ — $\frac{x}{\sqrt{x^2-a^2}}$ (or) $\sqrt{a^2-x^2} = \frac{-x}{\sqrt{a^2-x^2}}$

24) $\ln(x+\sqrt{x^2+a^2})$ — $\frac{1}{\sqrt{x^2+a^2}}$

25) $\ln(x+\sqrt{x^2-a^2}$ — $\frac{1}{\sqrt{x^2-a^2}}$

26 $\sqrt{x}$ — $\frac{1}{2\sqrt{x}}$

27) $\sqrt{f}$ — $\frac{f'}{2\sqrt{f}}$

28) $\frac{1}{x}$ — $\frac{-1}{x^2}$

29) $\frac{1}{f}$ — $\frac{-f'}{f^2}$

Theorems of Differentiation

1) $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

2) $\frac{d}{dx}(u \pm v) = \frac{d}{dx}u \pm \frac{d}{x}v$

3) $\frac{d}{dx}(k f x) = k\frac{d}{dx}f(x)$

Product (or) uv rule

4) $\frac{d}{dx}(uv) = u'v + v'u$

Ex: $y = x^3 \cdot \sin x$
$u \quad v$

$u - x^3 \qquad v - \sin x$
$u' - 3x^2 \qquad v' = \cos x$

$\therefore \frac{dy}{dx} = \cancel{u'v + v'u}\ uv' + vu'$
$= 3x^2 \cdot \sin x + \cos x \cdot x^3$

5) $\frac{d}{dx}\left(\frac{u}{v}\right) = \cancel{\frac{uv' - v'u}{v^2}}\ \frac{v'u - uv'}{v^2}$

$uv' - vu'$

$Ex: y = \frac{3^x \cdot \log_5}{x^2+4}$

$u = 3^x \cdot \log_5 x$ $\quad v = x^2+4$

~~$u' = 3^x \cdot \log 3$~~ $\quad v' = 2x$

$u' = 3^x \cdot \frac{1}{x \ln 5} + \frac{3^x \cdot \ln 5}{\ln 5} \cdot \log_5 x$

$$\frac{dy}{dx}\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{\left(3^x \cdot \frac{1}{x \ln 5} + 3^x \ln 5 \cdot \log_5 x\right)(x^2+4) - 2x\left(3^x \log_5 x\right)}{(x^2+4)^2}$$

6) CHAIN RULE

$$\to \frac{dv}{du} = \left(\frac{dv}{dw}\right)\left(\frac{dw}{du}\right)$$

(or)

$$= \frac{dv}{dw} \times \frac{dw}{ds} \times \frac{ds}{du}$$

(or)

$$= \frac{dv}{dw} \times \frac{dw}{ds} \times \frac{ds}{dt} \times \frac{dt}{du}$$

If $y = f(g(x))$

$$\frac{dy}{dx} = \frac{d\,f(g(x))}{dx} = \frac{d\,f(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$$

$= f'g(x) \cdot g'(x)$

1) If $y = \sin(x^3)$

$$\frac{dy}{dx} = \cos x^3 \cdot 3x^2$$

2) $y = \tan^{-1}(\sin^{-1}(x))$

$$\frac{dy}{dx} = \frac{1}{1+(\sin^{-1}x)^2} \cdot \frac{1}{\sqrt{1-x^2}}$$

3) $y = \sin^3(e^{x^2})$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin e^{x^2})^3 \quad \cos(e^{x^2})$$

$$= 3(\sin e^{x^2})^2 \cdot \cancel{\frac{1}{\sqrt{1+(e^{x^2})^2}}} \cdot e^{x^2} \cdot 2x$$

4) $y = \ln^5(\cos^2(8x))$

$$\cancel{\frac{dy}{dx} = 5(\ln \cos^2 8x)^4 \cdot \frac{1}{\cos^2(8x)}}$$

$$\frac{dy}{dx} = (2\ln \cos 8x)^5$$

$$= \cancel{10}(\ln \cos 8x)^4 \cdot 2 \cdot \frac{1}{\cos 8x} \cdot -\sin 8x \cdot 8$$

$$= \frac{-80(2\ln \cos 8x)^4 \cdot \sin 8x}{\cos 8x}$$

chain rule can also be used to differentiate $f(x)$ w.r.t $g(x)$

$$\frac{d(f(x))}{dg(x)} = \frac{\frac{df(x)}{dx}}{\frac{dg(x)}{dx}}$$

(i) $\sin x$ w.r.t $\cos x$

$$\frac{d(\sin x)}{d(\cos x)} = \frac{\frac{d}{dx}\sin x}{\frac{d}{dx}\cos x} = \frac{\cos x}{-\sin x} = -\cot x$$

(ii) $\tan x$ w.r.t $\tan^{-1}x$

$$\frac{d(\tan x)}{d(\tan^{-1}x)} = \frac{\sec^2 x}{\frac{1}{1+x^2}} = \sec^2 x + \sec^2 x \cdot x^2$$

(iii) $\tan^{-1}x^4$ w.r.t $\sin^3 x^5$

$$\frac{d(\tan^{-1}x^4)}{d(\sin^3 x^5)} = \frac{\frac{1}{1+(x^4)^2}\cdot 4x^3}{3(\sin x^5)^2\cdot\cos x^5\cdot 5x^4}$$

$$= \frac{4}{15}\,\frac{\frac{x^3}{1+x^8}}{\sin^2 x^5\cdot\cos x^5\cdot x^4}$$

(iv) $\ln(\sin(5\cot^{-1}(3x^2)))$ w.r.t $\sin(5\cot^{-1}(3x^2))$

$$= \frac{d\,\ln(\sin(5\cot^{-1}(3x^2)))}{d\,\sin(5\cot^{-1}(3x^2))}$$

$$\cancel{= \frac{1}{\sin(5\cot^{-1}}} = \frac{d\ln(t)}{dt}$$

$$= \frac{1}{t} = \frac{1}{\sin(5\cot^{-1}(3x^2))}$$

(v) $y = \frac{1}{z^2+1}$ at $z = 1$

$x = z^2+5z+6$

$$\frac{dy}{dx} = \frac{\frac{d}{dz}(z^2+1)^{-1}}{\frac{d}{dz}\, z^2+5z+6}$$

$$= \frac{(-1)(z^2+1)^{-2}\cdot 2z}{2z+5}$$

$$= \frac{-2z}{2z+5(z^2+1)^2} = \frac{-2(1)}{2+5(4)} = \frac{-2}{28}$$

$$= \frac{-1}{14}$$

Problems

1) $y = x\tan^{-1}x$

$$\frac{dy}{dx} = x \cdot \frac{1}{1+x^2} + \tan^{-1}x \cdot (1)$$

2) $y = \frac{1}{\sqrt{x^2+a^2} + \sqrt{x^2+b^2}}$

$$\frac{dy}{dx} = \frac{d}{dx}\frac{\sqrt{x^2+a^2} - \sqrt{x^2+b^2}}{x^2+a^2 - x^2-b^2}$$

$$= \frac{d}{dx}\frac{\sqrt{x^2+a^2} - \sqrt{x^2+b^2}}{a^2-b^2}$$

$$= \frac{1}{a^2-b^2}\left(\frac{d}{dx}\sqrt{x^2+a^2} - \frac{d}{dx}\sqrt{x^2+b^2}\right)$$

$$= \frac{1}{a^2-b^2}\left(\frac{1}{2\sqrt{x^2+a^2}} - \frac{1}{2\sqrt{x^2+b^2}}\right)$$

$$= \frac{\cancel{2}x}{\cancel{2}a^2-2b^2}\left(\frac{1}{\sqrt{x^2+a^2}} - \frac{1}{\sqrt{x^2+b^2}}\right)$$

3) $y = \log_a(\log_a \ln x)$

$$\frac{dy}{dx} = \frac{1}{\log_a x \cdot \ln a} \cdot \frac{1}{x \cdot \ln a}$$

$$= \frac{1}{x \log_a x \cdot \ln^2 a}$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$= \frac{1}{x \log_a x \cdot \frac{\log_e x}{\log_e a} \cdot \ln^2 a}$$

$$= \frac{1}{x \ln a \cdot \ln x}$$

4) $y = \sin(x + 60^\circ)$

$$\frac{dy}{dx} = \frac{d}{dx}\sin\left(\frac{\pi}{180}x + 60^\circ\right)$$

$$= \cos\left(\frac{\pi}{180}x + 60\right) \cdot \frac{\pi}{180}$$

5) $y = \cos(2\sin^2 x^3)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos\left(2\sin(x^3)\right)^2\right)$$

$$= \left(-\sin\left(\left(2\sin(x^3)\right)^2\right)\right) \cdot 4\sin(x^3) \cdot \cos(x^3) \cdot 3x^2$$

Parametric Form → For a circle

$x^2+y^2=r^2$

then any point on circle can be of the form $(r\cos\theta, r\sin\theta)$ - Parametric Point

$y = r\cos\theta$
$x = r\sin\theta$] - Parametric Equation

→ For a parabola

$y = x^2$

(t, t^2) - parametric point

$y = t^2$
$x = t$] - Parametric equation

→ For any curve

$y = g(t) \quad x = f(t)$

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

Examples (i) $y = \cos\theta$; $x = \sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}\cos\theta}{\frac{d}{d\theta}\sin\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

(ii) $y = 3t^2+5 \quad x = t^3-8 \quad$ at $t = 4$

$(x^3)' \cdot 3x^2$

$$\frac{dy}{dx} = \frac{\frac{d}{dt}3t^2+5}{\frac{dt}{dt}t^3-8}$$

$$\frac{\cancel{6t}^{2}}{\cancel{3t^2}_{t}} = \frac{2}{t} = \frac{2}{1} = 2$$

(iii) $x = z^2+5z+6 \quad y = \frac{1}{z^2+1}$ at $z=1$

$$\frac{dy}{dx} = \frac{\frac{d}{dz}\left(\frac{1}{z^2+1}\right)}{\frac{d}{dz}(z^2+5z+6)}$$

$$= \frac{\frac{d}{dz}(z^2+1)^{-1}}{\frac{d}{dz}(z^2+5z+6)}$$

$$= \cancel{\frac{2z}{2z+5}} = \frac{(-1)(z^2+1)^{-2}\cdot 2z}{2z+5}$$

$$= \frac{-2z}{(2z+5)(z^2+1)^2}$$

$\frac{dy}{dx}$ represents slope of the curve

2) Find Equation of tangent to $y = t^3+t^2+1$ and $x = t^2+4t-1$ at $(11,13)$

$$\frac{dy}{dx} = \frac{3t^2+2t}{2t+4}$$

$y = t^3+t^2+1 = 13 \Rightarrow t = 2$

$x = t^2+4t-1 = 11 \Rightarrow t^2+4t-12 = 0$

$t^2+6t-2t-12 = 0$

$t(t+6)-2(t+6) = 0$

$t = 2 \text{ (or) } -6$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{3(2)^2+2(2)}{2(2)+4} = \frac{16}{8} = 2$$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - 13 = 2(x - 11)$

$2x - y = -13+22$

$= 9$

continued...

(6) ~~$y = \log(2\sin^2$~~

$$y = \log_{10} x + \log_x 10 + \log_x x + \log_{10} 10$$

$$\frac{d}{}y = \frac{1}{\log_{10} x} + \log_{10} x + 1 + 1$$

$$= (\log_{10} x)^{-1} + \log_{10} x + 2$$

$$\frac{dy}{dx} = (-1)(\log_{10} x)^{-2} \cdot \frac{1}{x \ln 10} + \frac{1}{x \ln 10}$$

$$= \frac{1}{x \ln 10}\left(\frac{-1}{(\log_{10} x)^2} + 1\right)$$

(7) $y = \log_x (\ln x)$

$$\log_a b = \frac{\log_c b}{\log_c a} \qquad y = \frac{\log_e \ln x}{\log_e x}$$

$$\cancel{\frac{dy}{dx} = \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{\frac{1}{x}}}$$

$$\frac{dy}{dx} = \frac{\ln(\ln x)}{\ln(x)}$$

$$= \frac{\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)\ln x - \ln(\ln x)\left(\frac{1}{x}\right)}{\ln^2 x}$$

$$= \frac{\frac{1}{x}(1 - \ln(\ln x))}{\ln^2 x}$$

⑧ $y = \tan(x^n) + \tan^n x$

$$\cancel{\frac{dy}{dx} = \frac{1}{1+x^{2n}} \cdot nx^{n-1} + n(\tan x)^{n-1} \cdot \frac{1}{1+\tan x^2}}$$

$$\frac{dy}{dx} = \sec^2(x^n) \cdot nx^{n-1} + n(\tan x)^{n-1} \cdot \sec^2 x$$

⑨ $y = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)$

$$\frac{dy}{dx} = \frac{1}{2}\left(x \cdot \frac{x}{\sqrt{x^2-a^2}} + \sqrt{x^2-a^2}\right) - \frac{a^2}{2}\left(\frac{1}{\sqrt{x^2-a^2}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2-a^2}{\sqrt{x^2-a^2}} + \sqrt{x^2-a^2}\right)$$

$$= \frac{1}{2}\left(\frac{x^2-a^2+x^2-a^2}{\sqrt{x^2-a^2}}\right) = \frac{1}{2}\left(\frac{2(x^2-a^2)}{\sqrt{x^2-a^2}}\right)$$

$$= \sqrt{x^2-a^2}$$

⑩ $y = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x(-x)}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2} + a^2\left(\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right)\frac{1}{a}\right)$$

$$\frac{a}{\sqrt{1-\frac{x^2}{a^2}}} = \frac{a}{\frac{\sqrt{a^2-x^2}}{a}} = \frac{a^2}{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\left(\frac{-x^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2} + \frac{a^2}{\sqrt{a^2-x^2}}\right)$$

$$= \frac{1}{2}\left(\frac{a^2-x^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}\right)$$

$$= \sqrt{a^2-x^2}$$

(11) $$y = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$(a\sin bx)' = ab\cos bx$

$(b\cos bx)' = -ab\sin bx$

$$\frac{dy}{dx} = \frac{1}{a^2+b^2}\left(e^{ax}(\cancel{ab}(ab\cos bx + b^2\sin bx)) + ae^{ax}(a^2\sin bx - b\cos bx)\right)$$

$$= \frac{1}{a^2+b^2}\left(ae^{ax}(\cancel{b\cos bx} + b^2\sin bx + a^2\sin bx - \cancel{b\cos bx}\right)$$

$$= \frac{ae^{ax}}{a^2+b^2}(a^2\sin bx + b^2\sin bx)$$

$$= \frac{e^{ax}}{\cancel{a^2+b^2}}(\cancel{a^2+b^2})(\sin bx) = e^{ax}\cdot\sin bx$$

Differentiation of Implicit function

$y = \sin^2 x + e^x$ — Explicit

$e^x + 10^y = 8$ — Implicit

All implicit functions need not to be ~~converted into~~ explicit functions

like $y^3 - \tan^{-1}y + e^y = x^5$

Examples

1) $y + y^3 = e^x + \sin(x)$

$$\frac{dy}{dx} + 3y^2\cdot\frac{dy}{dx} = e^x + \cos x$$

$$\frac{dy}{dx} = \frac{e^x + \cos x}{1+3y^2}$$

(ii) $\tan^{-1} y^3 + 3^y = x^5 + 9^x$

$$\frac{1}{1+(y^3)^2} \cdot 3y^2 \cdot \frac{dy}{dx} + 3^y \cdot \ln 3 \cdot \frac{dy}{dx}$$

$$= 5x^4 + 9^x \cdot \log 9$$

$$\frac{dy}{dx} = \frac{5x^4 + 9^x \log 9}{\frac{3y^2}{1+y^6} + 3^y \cdot \ln 3}$$

1) If $x^2 + y^2 + xy = 2$ find $\frac{dy}{dx}$

$$2x + 2y \cdot \frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{-2x - y}{2y + x}$$

3) $\sin(xy) + \cos(xy) = \tan(x+y)$

$$= \cos(xy)\left(x\frac{dy}{dx} + y\right) + -\sin xy\left(\frac{x\,dy}{dx} + y\right)$$

$$\sec^2(x+y)^2$$

$$= \frac{1}{1+(x+y)^2}\left(1+\frac{dy}{dx}\right)$$

$$= x\frac{dy}{dx} + y\,(\cos xy - \sin xy) = \frac{1+\frac{dy}{dx}}{1+(x+y)^2}\ \sec(x+y)$$

$$= \frac{1+\frac{dy}{dx}}{x\frac{dy}{dx} + y} = \frac{\cos xy - \sin xy}{\sec(x+y)}$$

=

$$\Rightarrow x\cos xy \cdot \frac{dy}{dx} + y\cos xy - x\cdot\frac{dy}{dx}\sin xy - y\sin xy$$

$$= \sec^2(x+y) + \sec^2(x+y)\frac{dy}{dx}$$

$$x\cos xy\cdot\frac{dy}{dx} - x\cdot\frac{dy}{dx}\cdot\sin xy - \sec^2(x+y)\frac{dy}{dx}$$

$$= -y\cos xy + y\sin xy + \sec^2(x+y)$$

$$\frac{dy}{dx} = \frac{y\sin xy + \sec^2(x+y) - y\cos xy}{x\cos xy - x\sin xy - \sec^2(x+y)}$$

3) $2^x + 2^y = 2^{x+y}$

$$2^x\cdot\ln 2 + 2^y\cdot\ln 2\cdot\frac{dy}{dx} = 2^{x+y}\cdot\ln 2\cdot\left(1+\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{2^y\cdot\ln 2 - 2^{x+y}\cdot\ln 2}{2^{x+y}\ln 2 - 2^x\ln 2}$$

$$= \frac{2^y - 2^{x+y}}{2^{x+y} - 2^x}$$

(correct but need to reverse)

4) $\sin y = x\sin(a+y)$ then express $\frac{dy}{dx}$ as a function of a) x b) y

$$\cancel{\cos y\cdot\frac{dy}{dx} = x\cdot\cos(a+y)\left(1+\frac{dy}{dx}\right)}$$

$$\cancel{\frac{dy}{dx} = \frac{x\cos(a+y)}{\cos y - x\cos(a+y)}}$$

$\sin y = x\sin a\cos y + x\cos a\sin y$

$$\frac{1}{x} = \frac{\sin a \cos y + \cos a \sin y}{\sin y}$$

$$\frac{1}{x} = \sin a \cot y + \cos a \qquad \text{(1)}$$

$$\frac{-1}{x^2} = -\sin a \operatorname{cosec}^2 y \cdot y' \cancel{+ \cos a}$$

$$y' = \frac{1}{x^2(\sin a \operatorname{cosec}^2 y)}$$

Now we need to convert y into x

From (1)

$$\frac{1 - x\cos a}{x \sin a} = \cot y$$

$$\left(\frac{1 - x\cos a}{x\sin a}\right)^2 = \cot^2 y$$

$$\operatorname{cosec}^2 y = 1 + \left(\frac{1 - x\cos a}{x\sin a}\right)^2$$

$$y' = \frac{1}{x^2 \sin a\left(1 + \left[\frac{1 - x\cos a}{x \sin a}\right]^2\right)}$$

(2) From (1)

$$\frac{1}{x} = \sin a \cot y + \cos a$$

$$x = \frac{1}{\sin a \cot y + \cos a}$$

$$y' = \frac{1}{\dfrac{\sin a \operatorname{cosec}^2 y}{(\sin a \cot y + \cos a)^2}}$$

$$= \frac{(\sin a \cot y + \cos a)^2}{\sin a \operatorname{cosec}^2 y}$$

(OR)

$\sin y = x \sin(a+y)$

$$x = \frac{\sin y}{\sin(a+y)}$$

$$1 = \frac{-\sin y \cos(a+y)(\cancel{a}+y') + \sin(a+y)\cos y\, y'}{\sin^2(a+y)}$$

Mistakes

$\frac{u}{v} = \frac{vu - u'v}{v^2}$

&

$\frac{d}{dx}(a+y) = 0+y$

$\sin^2(a+y) = y'\{\sin y \cos(a+y) + \sin(a+y)\cos y\}$

$\cancel{+\sin y \cos(a+y)}$

$\sin^2(a+y) = y'(\sin(\cancel{y-a+y})(a+\cancel{y}-\cancel{y}))$

$$y' = \frac{\sin^2(a+y)}{\sin a}$$

5) $x\sqrt{1+y} + y\sqrt{1+x} = 0$ prove that

$\frac{dy}{dx} = \frac{-1}{(1+x)^2}$ (only in terms of x)

$x\sqrt{1+y} = -y\sqrt{1+x}$

$x^2(1+y) = y^2(1+x)$

$x^2 + x^2y = y^2 + xy^2$

$2x + 2x\cdot y + y'x^2 = 2y\cdot y' + x\cdot 2y\cdot y' + y^2$

$x\sqrt{1+y} = -y\sqrt{1+x}$

$x^2(1+y) = y^2(1+x)$

$x^2 - y^2 + x^2y - xy^2 = 0$

$(x-y)(x+y) + xy(x-y) = 0$

$x - y(x + y + xy) = 0$

$x - y = 0 \Rightarrow$ if $x = y$, existense of 2 solutio
so

$x + y + xy = 0$

$y(1 + x) = -x$

$y = \dfrac{-x}{1+x}$

$= \dfrac{-x + 1 - 1}{1+x}$

$= -1 + \dfrac{1}{1+x}$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(1+x)^{-1}$

$= \dfrac{-1}{(1+x)^2}$

New

6) $y = \cfrac{x}{2 + \cfrac{x}{2 + \cfrac{x}{2 + \dots \infty}}}$

Find $\dfrac{dy}{dx}$ in ① x only ② y only

①

$\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x}{2+y}$ $\qquad y = \dfrac{x}{2+y}$

$= \cancel{(x+y)(1)} -$ $\qquad y(2+y) = x$

$2y + y^2 = x$

$2y' + 2yy' = x$

$2y'(1+y) = x$

$y' = \dfrac{x}{2(1+y)} = \cancel{\dfrac{y(2+y)}{2(1+y)}}$

Logarithmic Differentiation

1) log's can be applied to convert product and division into sum or difference

2) $[\ln(f)]' = \frac{1}{f} \cdot f' = \frac{f'}{f}$

The presence of type $\left(\frac{f'}{f}\right)$ may imply to use log

3) $y = f_1 \cdot f_2 \cdot f_3 \cdots f_n$

$$\frac{y'}{y} = \frac{f_1' \cdot f_2 \cdot f_3 \cdots f_n + f_1 \cdot f_2' f_3 \cdots f_n + f_3' f_1 f_2 f_4 \cdots f_n \cdots}{f_1 f_2 f_3 \cdots f_n}$$

$$\frac{y'}{y} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3} \cdots \frac{f_n'}{f_n}$$

(or)

Same

By applying log

$\log y = \log f_1 + \log f_2 + \log f_3 + \cdots \log f_n$

Differentiate $\frac{y'}{y} = \frac{f'}{f} + \frac{f_2'}{f_2} + \frac{f_3'}{F_3} \cdots \frac{f_n'}{F_n}$

$$y' = y\left(\sum_{K=1}^{K} \frac{f_K'}{f_K}\right)$$

1) Find y' at $x=1$ if $\frac{(x^2+1)(x^2+3)}{(x^2+2)(\cos\pi x)} = y$

$\ln y = \ln(x^2+1) + \ln(x^2+3) - \ln(x^2+2) - \ln(\cos\pi x)$

$$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{2x}{x^2+3} - \frac{2x}{x^2+2} - \frac{-\pi\sin\pi x}{\cos\pi x}$$

At $x=1$

$$\frac{y'}{y} = \frac{2}{2} + \frac{2}{4} - \frac{2}{3} + 0$$

$y' = y\left(\frac{5}{6}\right) = \frac{-8}{3} \times \frac{5}{6} = \frac{-20}{9}$

2) Find y'' at $x=2$ if

$y=(x+3)(x+2)(x+1)(x-2)(x-3)(x-4)$

$\cancel{\frac{dy}{dx} = \log}$

$\log y = \log(x+3) + \log(x+2) + \log(x+1) + \log$
$+ \log(x-3) + \log(x-4)$

$$\frac{y'}{y} = \frac{1}{x+3} + \frac{1}{x+2} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x-3} + \frac{1}{x-4}$$

At $x=2$ $\frac{1}{x-2} = 0$

at 0 log do not used

⟹ For example if we have

$y=(\sin x)^4$ $\qquad$ $y = (4)^{\sin x}$

$y' = 4(\sin x)^3 \cdot \cos x$ $\qquad$ $y' = 4^{\sin x} \log 4 \cdot \cos x$

But if we have of the form $(fx)^{gx}$ then we will apply log

$y = \cancel{\sin x}\, (fx)^{gx}$

$\log y = \log f(x)^{gx}$

$\log y = g(x) \log_e f(x)$

$$\frac{y'}{y} = g(x) \cdot \frac{f'(x)}{f(x)} + g'x \cdot \ln f(x)$$

$$y' = y\left(g'(x) \cdot \ln f(x) + g(x) \cdot \frac{f'(x)}{f(x)}\right)$$

$$y' = (f(x))^{g(x)}\left(g'x \ln f(x) + g(x) \frac{f'(x)}{f(x)}\right)$$

For example

$$y = x^x$$

$$\log_e y = x \log_e x$$

$$\frac{y'}{y} = x \cdot \frac{1}{x} + \ln(x)$$

$$y' = y(1+\ln(x)) = x^x(1+\ln x)$$

(or)

$$= x^x(\ln ex)$$

so)

Memorize $\frac{d}{dx} x^x = x^x(1+\ln x) = x^x(\ln ex)$

$\Rightarrow y = (\sin x)^{\sin x}$

From chain rule $\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dx}$

$$\frac{dy}{dx} = \frac{d}{d\sin x}(\sin x)^{\sin x} \cdot \frac{d\sin x}{dx}$$

$$= (\sin x)^{\sin x}(1+\ln \sin x) \cdot \cos x$$

Exponential use for $(f(x))^{g(x)}$

$$y = (f(x))^{g(x)}$$

$$y = e^{\ln f(x)^{g x}}$$

$$= e^{g(x)\ln f(x)}$$

$$y' = e^{g(x)\ln f(x)} \cdot \left(g(x)\frac{f'(x)}{f(x)} + g'(x) \cdot \ln f(x)\right)$$

so from exponential

$$y' = (f(x))^{g(x)}\left(g'(x)\ln f(x) + g(x) \cdot \frac{f'(x)}{f(x)}\right)$$

from log

$$y' = (f(x))^{g(x)}\left(g'(x)\ln f(x) + g(x) \cdot \frac{f'(x)}{f(x)}\right)$$

Magic of $(f(x))^{g(x)}$

let treat $(f(x))^{g(x)}$ as x^n

then $y' = \cancel{f^g} \cdot g f^{g-1} \cdot f' = g f' \cdot \frac{f^g}{f}$

let treat $(f(x))^{g(x)}$ as a^x

$$y' = (f(x))^{g(x)} \cdot \ln f(x) \cdot g'(x)$$

But actual one is

$$y' = (f(x))^{g(x)} \left(g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right)$$

so sum of x^n & a^x

Fo

1) $y = (x^3 \ln x)^{\tan x}$

$\cancel{y' = (x^3 \ln x)^{\tan x} \left(3x^2 \ln x + \frac{x^3}{x}\right) \ln(\ln(x)) + \frac{}{x^3 \cdot \ln x}}$

1) $e^{x \sin(x^3)} + (\tan x)^x = y$ find y'

$$y' = e^{x \cdot \sin(x^3)} \cdot \left(x \cdot \cos(x^3) \cdot 3x^2 \cancel{(3\sin^2 x \cos x)} + \sin^3 x \right) +$$

$$(\tan x)^x \left(\ln(\tan x) + x \frac{\cancel{1} \sec^2 x}{\tan x} \right)$$

2) If $y = a^{x^{a^{x^{\cdots}}}}$ prove that

$$\frac{dy}{dx} = \frac{y^2 \log y}{x(1 - y\log x \cdot \log y)}$$

If $y = a^{x^{a^{x^{\cdots}}}}$

$y = a^{x^y}$

$y' = a^{x^y} \log a \left(x^y\left(y' \ln x + y \cdot \frac{1}{x}\right)\right)$

If $y = a^{x^y}$

$\log y = \cancel{y \log a^x}\ x^y \ln a$

$y' = y \log y \left(y' \ln x + \frac{y}{x}\right)$

$y' = y \log y\, y' \ln x + \frac{y^2 \cdot \log y}{x}$

$y'(1 - \log y \cdot \ln x) = \frac{y^2 \log y}{x}$

$y' = \frac{y^2 \cdot \log y}{x(1 - \log y \cdot \lg x)}$

$\frac{\sec^2 x}{\cdots x})$

5) $y = 1 + \frac{c_1}{x-c_1} + \frac{c_2 x}{(x-c_1)(x-c_2)} + \frac{c_3 x^2}{(x-c_1)(x-c_2)(x-c_3)}$

T.P.T

$$\frac{dy}{dx} = \frac{y}{x}\left(\frac{c_1}{c_1-x} + \frac{c_2}{c_2-x} + \frac{c_3}{c_3-x}\right)$$

$$y = 1 + \frac{c_1}{x-c_1} + \frac{c_2 x}{(x-c_1)(x-c_2)} + \frac{c_3 x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$$= \frac{x}{x-c_1} + \frac{c_2 x}{(x-c_1)(x-c_2)} + \frac{c_3 x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$$= \frac{x}{x-c_1}\left(1 + \frac{c_2 x}{x-c_2}\right) + \frac{c_3 x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

PROBLEMS

from cengage

Ex: 4.5) If $y=(1+x^{1/4})(1+x^{1/2})(1-x^{1/4})$ then find $\frac{dy}{dx}$

$y' = (1+x^{1/4})(1-x^{1/4})(1+x^{1/2})$

$= (1-x^{1/4 \times 2})(1+x^{1/2})$

$= 1-x$

$= 1-x$

$y' = -1$

Ex: 4.6 If $f(x) = x|x|$ then prove that $f'(x) = 2|x|$

$f(x) = \begin{cases} -x^2, & x<0 \\ x^2, & x\geq 0 \end{cases}$

$f'(x) = \begin{cases} -2x, & x<0 \\ 2x, & x>0 \end{cases}$

$f'(x) = 2|x|$

4.7) If $y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$; $x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

then find $\frac{dy}{dx}$

$y = \sqrt{\frac{1-(1-2\sin^2 x)}{1+(\cancel{1-2\sin^2 x}\ 2\cos^2 x-1)}}$

$y = \sqrt{\frac{2\sin^2 x}{2\cos^2 x}}$

Ex: 4.8)

4.9)

$y = \sqrt{\tan^2 x} = \tan x \Rightarrow y' = \sec^2 x$

Ex:- 4.8) If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

then show that $\frac{dy}{dx} - y + \frac{x^n}{n!} = 0$

$$\frac{dy}{dx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^{n-1}}{(n-1)!}$$

$$= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^{n-1}}{(n-1)!}\right) - \left(\frac{x^n}{n!}\right)$$

$$= y - \frac{x^n}{n!}$$

$$\frac{dy}{dx} - y + \frac{x^n}{n!} = 0$$

4.9) Find $\frac{dy}{dx}$ for $y = \sin^{-1}(\cos x)$ where $x \in (0, 2\pi)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\cos^2 x}} \times -\sin x$$

$\left(\frac{\pi}{2}, \pi\right)$