20/4/19

* Matrices * (4 to 6 marks)

Matrix: An ordered rectangle array of numbers or functions is called "matrix".

element

$$\begin{array}{c|l}
\begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 8 \\ 4 & 1 & 0 \end{bmatrix} & \begin{array}{l} \to \text{row-1} \\ \to \text{row-2} \\ \to \text{row-3} \end{array} \\
\begin{array}{ccc} \downarrow & \downarrow & \searrow \\ \text{Col.1} & \text{Col.2} & \text{Col.3} \end{array} &
\end{array}$$

group of Horizontal elements are called rows

group of Vertical elements are called columns.

Positional number: (a_{ij})

i>j

$$\begin{bmatrix} 1_{11} & 0_{12} & 1_{13} \\ 2_{21} & 6_{22} & 1_{23} \\ 7_{31} & 2_{32} & 6_{33} \end{bmatrix}$$

i>j   i=j

$i \to$ row's number

$j \to$ column's number

$i = j \longrightarrow$ principle diagnol elements

$i < j$

$i > j$

Order of a matrix: It describes about no:of rows and no:of columns ~~and~~ in a matrix. If order of a matrix is $m \times n$; then

$$\boxed{\begin{array}{l} \text{i) no:of rows} = m \\ \text{ii) no:of columns} = n \end{array}}$$

→ No:of elements in each row = n (no:of columns)

→ No:of element in each column = m (no:of rows)

→ Total no:of elements in a matrix = $m \times n$

→ If order of a matrix is 25x16 ____

no:of element in each row = 16.

no:of elements in each column = 25

Total no:of elements in matrix = 400

→ If order of a matrix is 16x8 = then;

No:of elements in each row = 8

No:of elements in each column = 16.

Total no:of elements 128

General form of a matrix:

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

$$A = [a_{ij}]_{m\times n}$$

TYPES OF MATRIXES:

i) Based on order

→ If order of a matrix is $m \times n$, then

a) $m=n \longrightarrow$ square matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 6 & 0 & 1 \\ 5 & 6 & 2 \end{bmatrix} \quad \begin{matrix} m=3 \\ n=3 \end{matrix}$$

b) $m \neq n \longrightarrow$ Rectangular matrix

$$\begin{bmatrix} 2 & 6 & 5 \\ 8 & 1 & 0 \end{bmatrix} \quad \begin{matrix} m=2 \\ n=3 \end{matrix}$$

$m>n \longrightarrow$ vertical matrix

$$\begin{bmatrix} 2 & 5 \\ 1 & 0 \\ 6 & 1 \end{bmatrix} \quad \begin{matrix} m=3 \\ n=2 \end{matrix}$$

$m<n \longrightarrow$ horizontal matrix

$$\begin{bmatrix} 2 & 1 & 6 \\ 5 & 0 & 2 \end{bmatrix} \quad \begin{matrix} m=2 \\ n=3 \end{matrix}$$

c) $m=1 \longrightarrow$ row matrix

$$\begin{bmatrix} 2 & 1 & 6 & 5 & 6 \end{bmatrix} = (1 \times 5)$$

$m=1$
$n=5$

d) $n=1 \longrightarrow$ column matrix

$$\begin{bmatrix} 1 \\ 5 \\ 6 \\ 7 \end{bmatrix} = (4 \times 1)$$

$m=4$
$n=1$

ii) Based on nature of element:

a) Null matrix : ~~The every~~
{Zero matrix} If all the elements in a matrix are equal to zero, then it is called Null matrix. It is denoted by "O".

$A = [a_{ij}]_{m \times n}$ is null matrix if $a_{ij} = 0 \ \forall \ i \& j$

$\forall$ → for all, for every, for any.

b) Diagnol Matrix : A square matrix is said to be diagnol matrix, if all the elements = 0 except those of principle diagnols

$E = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}_{m \times m} = (2 \times 2)$ $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{m \times m} (3 \times 3)$

$A = [a_{ij}]_{m \times m}$ is diagnol matrix if

$$\boxed{\begin{array}{l} a_{ij} = 0 \ \forall \ i \neq j \\ a_{ij} \neq 0 \ \forall \ i = j \end{array}}$$

c) Scalar Matrix : A diagnol matrix is said to be scalar if all the diagnol elements are equal.

Eg: $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}_{(2 \times 2)}$ $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}_{(3 \times 3)}$

$A = [a_{ij}]_{m \times m}$ is scalar matrix if

$$\boxed{\begin{array}{l} a_{ij} = 0 \ \forall \ i \neq j \\ a_{ij} = k \ \forall \ i = j \end{array}}$$

iv) identity matrix or unit matrix: A square matrix If all the diagonal elements = 1; remaining all the elements is 0; then it is said to be unit or identity matrix. It is denoted by "I_n"

Eg: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{(3\times3)}$ (I_3) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$ I_2

$A = [a_{ij}]_{m\times m}$ is a unit matrix if

$$\boxed{\begin{array}{l} a_{ij} = 0 \ \forall \ i \neq j \\ \hline a_{ij} = 1 \ \forall \ i = j \end{array}}$$

***. properties of identity matrix :

→ $\boxed{A_{m\times n} \cdot I_n = A_{m\times n}}$

∴ I is called multiplicative identity

→ $I^n = I$

→ $(-I)^n$ — n is even → I; 'n' is odd → $-I$

→ $I^T = adj(I) = I^{-1} = I$

→ $|I| = 1$

v) upper triangular matrix: A square matrix is said to be upper triangular, if element $a_{ij} = 0$ for every $i > j$.

$A = [a_{ij}]_{m\times m}$ is upper triangular matrix if

$\boxed{a_{ij} = 0 \ \forall \ (i > j)}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{3\times3}$.

(vi) **lower triangular matrix**: A square matrix is said to be lower triangular matrix if $(a_{ij}=0)\ \forall\ (i<j)$

$A=[a_{ij}]_{m\times m}$ is said lower triangular matrix ; if

$a_{ij}=0\ \forall\ (i<j)$

$a_{ij}\neq 0\ \forall\ (i>j\ \&\ i=j)$.

→ **Algebra of matrices**:

i) **Equality of matrices**: Two matrices are said to be equal if

i) order is same

ii) corresponding elements are equal.

Eg: $\begin{bmatrix}1&2\\3&4\end{bmatrix}_{2\times 2}=\begin{bmatrix}1&2\\3&4\end{bmatrix}_{2\times 2}$

Eg:

→ If $\begin{bmatrix}x&y\\z&0\end{bmatrix}$ ~~and~~ $\begin{bmatrix}2&-3\\4&0\end{bmatrix}$ the i) $x+y+z$ ii) $x^2+y^2+z^2$ iii) $(x+y+z)^2$

i) $2+(-3)+4=3$

ii) $2^2+(-3)^2+(4)^2=29$

iii) $(3)^2=9$ //

→ If $\begin{bmatrix}x+y&3\\2&z\end{bmatrix}=\begin{bmatrix}4&3\\2&1\end{bmatrix}$ then i) $(x+y+z)$ ii) $(x+y+z)^2$

i) $4+1=5$

ii) $(5)^2=25$

→ If $\begin{bmatrix}xy^2&y^2\\0&2\end{bmatrix}=\begin{bmatrix}1&9\\0&2\end{bmatrix}$; $y<0$ then $x+y=\underline{-2}$, $x-y=\underline{4}$

$y<0$; $y=-3$

i) $1-3=-2$

ii) $1+3=4$

→ If $\begin{bmatrix} x+y & 0 \\ 0 & x-y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $x = $ ___ $y = $ ___

$$\begin{array}{r} \cancel{x}+y = \cancel{1} \\ -\cancel{x}-y = \cancel{1} \\ \hline 2y = 0 \end{array}$$

$\boxed{y = 0}$

$\boxed{x = 1}$

→ **Scalar Multiplication** let k be the scalar and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

→ If $A = \begin{bmatrix} 3 & -2 & 0 \\ 4 & -1 & 1 \end{bmatrix}$; then 1) $2A = \begin{bmatrix} 6 & -4 & 0 \\ 8 & -2 & 2 \end{bmatrix}$

2) $-3A = \begin{bmatrix} -9 & 6 & 0 \\ -12 & 3 & -3 \end{bmatrix}$

3) $4A = \begin{bmatrix} 12 & -8 & 0 \\ 16 & -4 & 4 \end{bmatrix}$

4) $10A = \begin{bmatrix} 30 & -20 & 0 \\ 40 & -10 & 10 \end{bmatrix}$

5) $-A$ or (Additive inverse) $= \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -1 \end{bmatrix}$

→ If $A = \begin{bmatrix} 2 & 4 \\ 8 & 1 \end{bmatrix}$ then 1) $3A = \begin{bmatrix} 6 & 12 \\ 24 & 3 \end{bmatrix}$

2) $-4A = \begin{bmatrix} 8 & -16 \\ 32 & -4 \end{bmatrix}$

3) $8A = $ ____ → $\begin{bmatrix} -16 & 32 \\ 64 & 8 \end{bmatrix}$

4) $-A = $ ____ $\begin{bmatrix} 2 & -4 \\ -8 & -1 \end{bmatrix}$

→ If $A = \begin{bmatrix} a+ib & a-ib \\ -a-ib & \cancel{-(a+ib)}\ -a+ib \end{bmatrix}$ then $-A$ or additive inverse =

$\begin{bmatrix} -a-ib & -a+ib \\ a+ib & a-ib \end{bmatrix}$

→ If

th

→ If $k, a,$

k

4 :

B =

* Matrix Add

and resultant

let $A = [$

→ If $A = [$

$\rightarrow$ If x, y, k are the scalars, $A = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$ and $KA = \begin{bmatrix} 3a & 4b \\ 0 & -24 \end{bmatrix}$

then k, a, b.

$4k = -24$
$\boxed{k = -6}$

$3a = -6 \times 2$
$3a = -12$
$\boxed{a = -4}$

$-6 \times 3 = -18 = 4b$
$\boxed{b = \frac{-18}{4} = \frac{-9}{2}}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\rightarrow$ If k, a, b are the scalars, $A = \begin{bmatrix} 0 & 4 \\ -2 & 1 \end{bmatrix}$ then $KA = \begin{bmatrix} 0 & 16 \\ 3A & 2B \end{bmatrix}$

$4k = 16$
$\boxed{k = 4}$

$k = 2B$
$4 = 2B$
$\boxed{B = 2}$

$\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$

* **Matrix Addition:** Addition of matrices is possible if order is same and resultant is obtained by adding corresponding elements.

let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$; then $A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$

$\rightarrow$ If $A = \begin{bmatrix} 1 & -3 & 5 \\ 7 & -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & 0 \end{bmatrix}$ then

1) $A+B = \begin{bmatrix} 3 & -3 & 6 \\ 6 & 3 & 4 \end{bmatrix}$
2) $A-B = \begin{bmatrix} -1 & -3 & 4 \\ 8 & -5 & 4 \end{bmatrix}$
3) $2A+3B = \begin{bmatrix} 8 & -6 & 13 \\ 11 & 10 & 8 \end{bmatrix}$
4) $3A-B = \begin{bmatrix} 1 & -9 & 14 \\ 23 & 1 & 12 \end{bmatrix}$
5) $2A-4B = \begin{bmatrix} -6 & -6 & 6 \\ 18 & 14 & 8 \end{bmatrix}$

re $\doteq$

→ If $A = \begin{bmatrix} 3 & -2 \\ 1 & -2 \end{bmatrix}$; $B = \begin{bmatrix} 4 & 0 \\ -2 & 0 \end{bmatrix}$ then

1) $A+B = \begin{bmatrix} 7 & -2 \\ -1 & -2 \end{bmatrix}$

2) $B-A = \begin{bmatrix} 1 & +2 \\ -3 & 2 \end{bmatrix}$

3) $2A+4B = \begin{bmatrix} 22 & -4 \\ -6 & -4 \end{bmatrix}$

4) $3A-3B = \begin{bmatrix} -3 & -6 \\ 9 & -6 \end{bmatrix}$

5) $A-2B = \begin{bmatrix} -5 & -2 \\ 5 & -2 \end{bmatrix}$

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

→ If A = diagonal (3,1,2) B = diagonal (1,2,3) then

$A+B = \text{—} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$A-B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

→ If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 7 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ -1 & 0 \end{bmatrix}$ and $A+B+4x = 0$ Then $x =$ ____

$A+B = \begin{bmatrix} 3 & 3 \\ 5 & 11 \\ 6 & 2 \end{bmatrix} + 4x = 0.$

$x = \frac{-1}{4} \times \begin{bmatrix} 3 & 3 \\ 5 & 11 \\ 6 & 2 \end{bmatrix}$

→ If $A = \begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$ and $A+2B+3x = 0$. then $x =$ ____

$2B = \begin{bmatrix} 6 & -4 \\ 2 & 8 \end{bmatrix}$

$A+2B = \begin{bmatrix} 8 & -3 \\ -2 & -1 \end{bmatrix}$

→ If $A+B = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix}$, $A-B = \begin{bmatrix} -2 & 1 \\ 0 & 2 \end{bmatrix}$ Then $A =$ ____ $B =$ ____

$$A+\cancel{B} = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix}$$

$$A-\cancel{B} = \begin{bmatrix} -2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$\boxed{B = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}}$$

→ If $A+2B = \begin{bmatrix} 4 & 1 \\ 7 & 0 \end{bmatrix}$, $2A-3B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ then i) $B =$ ____

~~A+2B~~

$A+2B$

$2A-3B$

$$2\cancel{A}+4B = \begin{bmatrix} 8 & 2 \\ 14 & 0 \end{bmatrix}$$

$$\cancel{2A}-3B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$7B = \begin{bmatrix} 6 & 1 \\ 14 & -1 \end{bmatrix}$$

$$B = \frac{1}{7}\begin{bmatrix} 6 & 1 \\ 14 & -1 \end{bmatrix}$$

→ If x, y are the scalars, $x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ then $x =$ ___ , $y =$ ___

***

$$\begin{bmatrix}x\\x\end{bmatrix} + \begin{bmatrix}y\\-y\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$x + y = 1$

$x - y = 1$

$2x = 2$

$\boxed{x = 1}$

$\boxed{y = 0}$

→ If m, n are the scalars and $m\begin{bmatrix}3,3\\3\end{bmatrix} + n\begin{bmatrix}3,4\\4\end{bmatrix} = (4,2)$

then $m = ?$ $n = ?$

$3m + 3n = 4$

$-3m + 4n = -2$

$-n = 2$

$\boxed{n = -2}$

$3m + 4n = 4$

$3m - 8 = 4$

$3m = 12$

$\boxed{m = 4}$

$12 - 6 = 4$

6

* **Properties of matrix Addition :-**

i) matrix addition is commutative i.e, $A+B = B+A$

ii) matrix addition is Associative, i.e, $A+(B+C) = (A+B)+C$

iii) $K(A+B) = KA + KB$

iv) $(A+O) = (O+A) = A$

∴ Null matrix is called Additive identity.

(v) $A + (-A) = (-A) + A = O$

∴ $(-A)$ is called Additive inverse of A.

(vi) If $A+B = A+C \Rightarrow \underline{B = C}$

~~$A.B \neq A$~~

$A.B = A.C \Rightarrow \boxed{B \neq C}$

* Matrix Multiplication *

→ Two matrices are comfortable with multiplication. If no. of columns in 1^{st} matrix = no. of rows in 2^{nd} matrix ;
The Resultant matrix having order $\boxed{\text{rows in } 1^{st} \text{ matrix} \times \text{columns in } 2^{nd} \text{ matrix}}$.

→ If 'A' is a matrix of order $m \times n$; 'B' is a matrix of order $p \times q$
then matrix multiplication :

1) $A \cdot B$ is possible when $\boxed{n = p}$
2) $B \cdot A$ is possible when $\boxed{q = m}$
3) The resultant of order ⇒ $(A \cdot B) = m \times q$
   $(B \cdot A) = p \times n$

→ If A is a matrix of 3×3 and B order $B \cdot A$ is 4×3.. then
Order B = $\underline{4 \times 3}$.

$$\underset{m \times n}{B} \cdot \underset{3 \times 3}{A} = (BA)_{4 \times 3}$$

$\boxed{n = 3}$

$\boxed{\begin{matrix} m = 4 \\ n = 3 \end{matrix}}$

→ If order of $A = (2 \times 3)$ and order of $A \cdot B = 2 \times 4$. then order of B =

$$\underset{2 \times 3}{A} \quad \underset{\substack{m \times n \\ 3 \quad 4}}{B} = AB_{2 \times 4}$$

$\boxed{m = 3}$ $\boxed{n = 4}$ $AB_{\boxed{3 \times 4}}$

→ If A is a matrix of order (3×3) and B is matrix of order 3×2 then
1) order of AB —— 2) order of BA

1) order $AB = \underset{3 \times 3}{A} \; \underset{3 \times 2}{B} = AB_{3 \times 2}$

2) Order $BA = \underset{3 \times 2}{B} \neq \underset{3 \times 3}{A}$ = BA not possible

→ If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ then $A.B =$ ____
$B.A =$ ____

$$\begin{bmatrix} 2(2)+3(0) & 2(1)+3(2) \\ 1(2)+2(0) & 1(1)+2(2) \end{bmatrix}$$

$$A.B. = \begin{bmatrix} 4 & 8 \\ 2 & 5 \end{bmatrix} \qquad B.A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 2 & 4 \end{bmatrix}$$

→ If $A \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ then $AB =$ ____
$BA =$ ____

$$A.B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(BA) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 0+0 & 2+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

$$\boxed{A.B \neq B.A}$$

→ If product of a ~~Null~~ Two matrix (A & B) is a Null matrix ; then A & B may or may not be null matrix

If $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ then $A^2 =$ ___ $A^3 =$ ___ $A^4 =$ ___ A^n ___

$$A^2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

$$\boxed{A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}}$$ { ∵ applicable for only diagnal matrix of any order }

Simil

A =

A =

A =

A =

A =

A =

→ If A

A^n

$A^n =$

Similarly:

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \Rightarrow A^4 = \begin{bmatrix} a^4 & 0 & 0 \\ 0 & b^4 & 0 \\ 0 & 0 & c^4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow A^{10} = \begin{bmatrix} 2^{10} & 0 \\ 0 & (-2)^{10} \end{bmatrix} = \begin{bmatrix} 1024 & 0 \\ 0 & 1024 \end{bmatrix} = 1024 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 1024\,[I]_3$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = A^4 = \begin{bmatrix} 81 & 0 \\ 0 & 256 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = A^5 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 241 & 0 \\ 0 & 0 & -32 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = A^4 = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \text{ then } A^{100} = \begin{bmatrix} i^{100} & 0 \\ 0 & -i^{100} \end{bmatrix} = \begin{bmatrix} (i^2)^{50} & 0 \\ 0 & ((-i)^2)^{50} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

→ If $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$; then $A^n =$

$$A^n = \begin{bmatrix} a^n & 0 \\ 0 & a^n \end{bmatrix} \text{ — (1)} \Rightarrow a^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{a^n I_2} \text{ — (2)}$$

$$A_1^n = a^{n-1} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \quad a^{n-1} A_2 \text{ — (3)}$$

→ If A is Scalar matrix ; then it is have 3 results.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow A^5 = \begin{bmatrix} 3^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 3^5 \end{bmatrix} = 3^5 . I = 3^{5-1} . A = 3^4 . A$$

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \Rightarrow A^4 = \begin{bmatrix} (-4)^4 & 0 & 0 \\ 0 & (-4)^4 & 0 \\ 0 & 0 & (-4)^4 \end{bmatrix} = (-4)^4 . I = (-4)^3 . A$$

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \rightarrow A^9 = \begin{bmatrix} (-2)^9 & 0 \\ 0 & (-2)^9 \end{bmatrix} = -2^9 . I = (-2)^8 . A$$

*** → If $A = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ then $A^{99} =$ ____ ; where ω is one of the complex cube roots of unity.

$$A^{99} = \begin{bmatrix} (\omega^3)^{33} & 0 \\ 0 & (\omega^3)^{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Complex cube roots of unity

↓

$1, \omega, \omega^2$

$$\boxed{\begin{array}{c} 1+\omega+\omega^2 = 0 \\ \hline \omega^3 = 1 \\ \hline \omega^{3n} = 1 \end{array}}$$

* $\boxed{I = I^n = I}$

* $-I = -I$ — n is odd

$= I$ — n is even //

→ If $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$ then $A^2 =$ ____ ; $A^3 =$ ____ ; $A^n =$ ____

$$A^2 = \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} a^2+a^2 & a^2+a^2 \\ a^2+a^2 & a^2+a^2 \end{bmatrix} = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}$$

$$= 2\begin{bmatrix} a^2 & a^2 \\ a^2 & a^2 \end{bmatrix} = 2a\begin{bmatrix} a & a \\ a & a \end{bmatrix} = \boxed{2a \times A}$$

$$= (2a)^{2-1} \cdot A.$$

$$A^3 = (2a)^{3-2} \cdot A.$$

$\cancel{A^3 \quad a \quad 2a^2 \quad 2a^2}$, $A^n = (\boxed{2a})^{n-1} \cdot A$

trace of A matrix

$$\boxed{A^n = (\text{sum of diagnal elements})^{n-1} \cdot A.}$$

↳ applicable for any square matrix of any order with all equal elements.

→ If all the elements in a square matrix are equal ; then $\boxed{A^n = (\text{trace of } A)^{n-1} \cdot A}$ for $\boxed{n \geq 2}$

where trace of A = Sum of the diagnol elements.

*. $A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ 7 $A^{2020} = \left(\frac{1}{2}+\frac{1}{2}\right)^{2020-1} \cdot A$

$= 1 \cdot A$

$= A.$

$A = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$ then $A^{100} = (1)^{99} \cdot A$

$= A$ //

→ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; then $A^2 - (a+d)A =$

$\cancel{(a+d)^{2-1}A - (a+d)A = 0}$:

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} \quad -① $$

$A^2 \cancel{(a+d)} A =$

$$(a+d)A = \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} -②$$

$$A^2-(a+d)A = \begin{bmatrix} bc-ad & 0 \\ 0 & bc-ad \end{bmatrix} = (bc-ad)\, I$$

$A^2-(a+d)A - (bc-ad)\cdot I$

$A^2 - (a+d)A + (ad-bc)I = 0$

$\boxed{A^2 - (\text{trace of } A)\, A + |A|\cdot I = 0}$ — applicable for only 2×2 matrix

If $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ the $A^2 =$ —— $\cancel{1)}A$ 2) $-A$ 3) I 4) $-A$.

$A^2 - (\text{trace of } A)\, A + |A|\cdot I = 0$

$A^2 - 1(A) + (0-0)\cdot I = 0$

$A^2 - A = 0$

$\boxed{A^2 = A}$

$\rightarrow A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = A^2 = A \Rightarrow \boxed{A^2 - (\text{trace of } A)A + |A|I = 0.}$

$\boxed{\text{If } A^2 = A \text{ then it is said to be "Idempondent matrix"}}$

$\rightarrow$ If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then $A^{1994} =$ ______ ; $A^{1995} =$ ______

$A^{1996} =$ ______ ; $A^{1997} =$ ______

$A^2 - 0(A) + (-1)I = 0$

$A^2 - I = 0$

$\boxed{A^2 = I}$

$A^{1994} = (A^2)^{997} = (I)^{997} = I$

$A^{1995} = A^{1994} \cdot A \Rightarrow I.A$

$A^{1996} = (A^2)^{998} = (I)^{998} = I$

$A^{1997} = A^{1996} \cdot A = I.A = A$

If $A^2 = I$; then the matrix A is said to be involutary matrix.

$n \rightarrow$ odd $= A.I = A$

$\downarrow$

even

$\shortparallel$

I

$\rightarrow$ If $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ $A^{100} =$ ______ ; $A^{101} =$ ______ ; $A^{97} =$ ______

$A^{66} =$ ______

$A^2 - 0 + (-1)I = 0$

$A^2 - I = 0$

$\boxed{A^2 = I}$ If $A^2 = I$; then it is said involutary matrix

$A^{100} = I$

$A^{101} = A$

$A^{97} = A$

$A^{66} = I.$

→ $A = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}$ then $A^{136} =$ ______ $A^{137} =$ ______

$A^{48} =$ ______ $A^{67} =$ ______

$A^2 - (0)A + (-1)I = 0 \Rightarrow A^2 - I = 0 \therefore A^2 = I$

$(A^2)^{68} = (I)^{68} = I$

$(A)^{136} A = AI = A$

$A^{48} = (A^2)^{24} = I^{24} = I$

$A^{67} = A^{66}(A) = A$

→ $A \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, the $A^{100} =$ ______ $A^{101} =$ ______ $A^{102} =$ ______

$A^{103} =$ ______

$A^2 - 0 \cdot A + (0+1)\bar{A} = 0$

$A^2 + I = 0$

$\boxed{A^2 = -I}$

$A^{100} = (A^2)^{50} = (-I)^{50} = I$

$A^{101} = A^{100} \cdot A = A \cdot I = A$

$A^{102} = (A^2)^{51} = (-I)^{51} = -I$

$A^{103} = A^{102} \cdot A = -I \cdot A = -A$

If $A = \cancel{\begin{bmatrix} -4 & -3 \\ 3 & 8 \end{bmatrix}}$ $A^2 - (-2$ $\begin{bmatrix} -5 & -26 \\ 1 & 5 \end{bmatrix}$

$A^{40} = \underline{I}$

$A^{41} = \underline{A}$

$A^{42} = \underline{-I}$

$A^{43} = \underline{-A}$

$\boxed{A^2 = -I}$

$(A^2)^{20} = (-I)^{20} = I$

$A^{40} \cdot A$

$I \cdot A$

$(-I)^{21} =$

→ If $A = \begin{bmatrix} -ab & -a^2 \\ b^2 & ab \end{bmatrix}$ then $A^2 =$ ____

$A^2 - 0(A) + (-a^2b^2 + a^2b^2) I = 0$

$\boxed{A^2 = 0}$

If $\boxed{A^n = 0}$ then matrix A is said to be Nypolant matrix.

$A^2 = 0$

$A^3 = 0$

$\vdots$

$\boxed{A^n = 0}$

n is called index of nypolant.

→ If $A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$ then $A^3 - A^2 =$ $\underline{A}$

$A^2 - 1(A) + (-1) I = 0$

$A^2 - \cancel{A} = I$

$\boxed{A^2 - A = I}$

$A(A^2 - A) = A \cdot I = A.$

$\boxed{A^3 - A^2 = A}$

→ If $A = \begin{bmatrix} 5 & 4 \\ -1 & 2 \end{bmatrix}$ & $A^2 - \beta_1 A + \beta_2 I = 0$; then $\beta_1 =$ $\underline{7}$ $\beta_2 =$ $\underline{14}$

$A^2 - (5+2)A + (10+4) I = 0$

$A^2 - 7A + 14I = 0$

$\boxed{\begin{matrix} \beta_1 = 7 \\ \beta_2 = 14 \end{matrix}}$

$\Rightarrow$ If $A = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}$ & $A^2 + k_1 A + k_2 I = 0$, $k_1 =$ ___ $k_2 =$ ___

$k_1 = 45$

$k_2 = 2$

* Types of matrices Based on Multiplication Operation *

Idempotent matrix A square matrix is said to be Idempodent if $A^2 = A$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$C = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

Involutary Matrix : A square matrix is said to be involutary if

$\boxed{A^2 = I}$

$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

→ If A is involutary then $A^n = I$; If n is even

$A^n = A$ if n is odd.

<u>Nypolent matrix</u>: A square matrix is said to be Nypolent, If $A^n = 0$; where n is positive integer

The least value of n at which $\boxed{A^n = 0}$ then it is called $\boxed{\text{index of Nypolent}}$

Eg: $\begin{bmatrix} -ab & a^2 \\ -b^2 & ab \end{bmatrix}$

→ If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ is a involutary matrix ; then $x =$

$A^2 = I$

$\cancel{\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = 0.$

$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$x^2+1 = 1$

$x^2 + \cancel{1} = \cancel{1}$

$\boxed{x = 0}$

→ **Properties of Matrix Multiplication:**

→ Matrix Multiplication is not commutative.

$$\boxed{A.B \neq B.A}$$

→ Matrix multiplication is Associative i.e $A.(BC) = (AB).C$.

→ Matrix multiplication is distributive i.e, $A(B+C) = AB + AC$

→ If $AB = O$; then either A or B need not be Null Matrix

→ If $AB = O$; then $B.A$ may or may not be equal to 'O' (Null matrix)

→ If $AB = AC$ then B and C need not be equal.

→ $A^m . A^n = A^{m+n}$.

→ $A^2 = A.A$

→ $A^3 = A.A.A$

→ $A^0 = I \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; If A is Non sigular

→ If $AB = A$; $BA = B$. ⇒ $A^2+B^2 =$ ______ ; A, B are ______

$A^2 = A.A$	$B^2 = BB$
$= (AB).A$	$B^2 = (B.A).B$
$= A(BA)$	$= B.(AB)$
$= AB$	$= B.A$
$A^2 = A$	$B^2 = B$

$A^2+B^2 = A+B$; then A and B are Idemponent matrices.

Commute matrices: Two matrices are said to be commute;

If $\boxed{A.B = B.A}$

Eg: $A = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

$A.B = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$; $B.A = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$

→ Matrix and it's adjoint, Matrix and its inverse are always commute.

→ If A, B are Square matrices of same order then $(A+B)^2 =$ ______

$(A-B)^2 =$ ______ ; $(A+B)(A-B) =$ ______ .

$(A+B)^2 = A^2 + A.B + B.A + B^2$

$(A-B)^2 = A^2 - AB - BA + B^2$

$(A+B)(A-B) = A^2 - AB + BA - B^2$

Special case:

If A, B are commute;

$AB = BA$;

$(A+B)^2 = A^2 + 2.A.B + B^2$

$(A-B)^2 = A^2 - 2A.B. + B^2$

$(A+B)(A-B) = A^2 - B^2$ //

→ If A, B are Idemponent matrix ; $\{A^2 = A;\ B^2 = B\}$ $\boxed{AB = BA = O}$; then

A+B = ______ matrix

1) Idemponent 2) Involutary 3) Nypolent 4) None.

$AB = BA = O$

A, B are idemponent matrix $\boxed{A^2 = A}$

$(A+B)^2 = A^2 + 2AB + B^2$

$= A + B$ → $(A+B)^2 = A+B$

∴ Idemponent matrix.

→ If $A(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $A(\beta) = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$ Then

$A(\alpha) \cdot A(\beta) =$ ______

1) $A(\alpha+\beta)$ 2) $A(\alpha\beta)$ 3) $A(\alpha-\beta)$ 4) None

$$A(\alpha) \cdot A(\beta) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\sin\alpha\cos\beta + \cos\alpha \cdot \sin\beta) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta + \sin\alpha \cdot \sin\beta \end{bmatrix}$$

→ If $\begin{bmatrix} x & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ -1 \end{bmatrix} = 0$. Then $x =$

$$\begin{bmatrix} 2x+4 & x-2 & 4 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2x^2+4x+4x-8-4 \end{bmatrix} = 0$$

$$2x^2+8x-12 = 0$$

$$2(x^2+4x-6) = 0$$

$$x^2+4x-6 = 0$$

$$\frac{-4 \pm \sqrt{16+24}}{2}$$

$$= \frac{-4 \pm \sqrt{40}}{2}$$

$$= \frac{-4 \pm \sqrt{4 \times 10}}{2} = -2 \pm \sqrt{10}$$

$$= 2 \pm \sqrt{10}$$

→ If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ then

1) $A^2 = B^2 = I$
2) $A^2 = I,\ B^2 = -I$
3) $A^2 = -I,\ B^2 = I$
4) $A^2 = -I,\ B^2 = -I$

$A^2 - (\text{trace of } A)A + |A|I = O$

$A^2 - (0)A + (1)I = 0$

$A^2 + I = 0$

$\boxed{A^2 = -I}$

→ If $A = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$

1) $AB = AC = O$
2) $AB \neq O$; $AC \neq O$
3) $AB = O$; $AC \neq O$
4) $AB \neq O$, $AC = O$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$

→ If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then $A^n =$ ____

1) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$

2) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$

3) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$

4) $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

let $n=0$

$A^n = \boxed{A^0 = A\ I.}$

put $n=0$ in the options

1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq I$

2) $\begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix} \neq I$

3) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq I$

~~4)~~ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

→ If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ then $A^{n+1} =$ ____

1) $\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}$

2) $n \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

~~3)~~ $2^n \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

4) $2^{n+1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

put $n=0$ $\boxed{A^{0+1} = A}$

~~1)~~ → A

2) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq A$

3) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow A$

~~4)~~ $2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq A$

let $n = 1$

$A^{1+1} = A^2$

$A^2 - (2)A + (0)I = 0$

$\boxed{A^2 = 2A}$

~~A=2~~ $\boxed{n=1 \longrightarrow 2A}$

1) option $\neq 2A$

~~3)~~ $2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$2A$

Trans

colu

→ It

→ If orde

→ Types c

a) Symm

Eg:

→

b) skew -

matrix i

② $\begin{bmatrix} 0 \\ -b \\ - \end{bmatrix}$

Transpose of a matrix : It is obtained by interchanging rows and columns in a matrix

→ It is denoted A^T

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 9 \end{bmatrix}$$

(2×3) $(m \times n)$ $\qquad$ (3×2) $(n \times m)$.

$$A = \begin{bmatrix} 7 & -1 & 2 \\ 6 & 2 & 4 \\ 2 & 6 & 1 \end{bmatrix} \quad = A^T = \begin{bmatrix} 7 & -6 & 2 \\ -1 & 2 & 6 \\ 2 & 4 & 1 \end{bmatrix}$$

→ If order of a matrix is $m \times n$; then order of its transpose is $n \times m$

→ Types of matrices based on Transpose Operation:

a) Symmetric matrix : A square matrix is said to be Symmetric ; if $\boxed{A^T = A}$

Eg: $A = \begin{bmatrix} 2 & 4 \\ 4 & 7 \end{bmatrix} \therefore A^T = \begin{bmatrix} 2 & 4 \\ 4 & 7 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 8 & 9 \\ 7 & 9 & 3 \end{bmatrix} \therefore A^T = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 8 & 9 \\ 7 & 9 & 3 \end{bmatrix}$$

→ $A = [a_{ij}]$ is Symmetric if $\boxed{a_{ij} = a_{ji}}$ $\forall i \& j$

b) Skew-Symmetric matrix : A square matrix is said to be skew symmetric matrix if $\boxed{A^T = -A}$

(1) $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$

$\Rightarrow -\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$

$\Rightarrow -A$

(2) $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} = -\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ $\qquad \boxed{A^T = -A}$

$= -[A]$

→ Sum of all the elements in a any ordered skew symmetric matrix = 0.

→ $A = [a_{ij}]$ is skew symmetric matrix if

$$\boxed{\begin{array}{l} a_{ij} = 0 \ \forall\ i = j \\ a_{ij} = -a_{ji} \ \forall\ i \neq j \end{array}}$$

Orthogonal matrix : A Square matrix is said to be orthogonal if $A \cdot A^T$

$= A^T \cdot A = I$.

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

To ~~be a orthogonal matrix~~ →

If A is orthogonal, then Sum of Squares of the any row or any column elements = 1.

1st row — $\cos^2\theta + \sin^2\theta = 1$

2nd row — $\sin^2\theta + \cos^2\theta = 1$

→ Determinant orthogonal matrix is = ±1

→ If A is orthogonal, then A^{-1} and A^T are also orthogonal.

→ $A = \begin{bmatrix} 3 & 4 & 7 \\ x & 2 & 1 \\ 7 & 1 & 3 \end{bmatrix}$ is a ~~ortho~~ Symmetric matrix then x = ——

$\boxed{a_{ij} = a_{ji} \ \forall\ i \& j}$

$x_{21} = a_{21} = 4$

$\boxed{x = 4}$

→ If $A = \begin{bmatrix} x & -3 & 7 \\ 3 & 0 & 8 \\ -7 & y & 0 \end{bmatrix}$ is a skew symmetric matric x = ——, y = ——

$x = 0$ (Diagnol elements = 0)

$y = -8$ ($a_{ij} = -a_{ji} \ \forall\ i \neq j$)

Properties of Transpose : ***.

→ $(A^T)^T = A$

→ $(I)^T = I$

→ $(kA)^T = k(A^T)$

→ $(A \pm B)^T = A^T \pm B^T$

→ $(AB)^T = B^T . A^T$

→ $(ABCD)^T = D^T . C^T . B^T . A^T$

→ If A is a square matrix, then 1) $\cancel{(A+B)}^T (A+A^T)$ Symmetric matrix

2) $A^T + A$ is Symmetric matrix

3) $A - A^T$ is Skew Symmetric matrix

4) $A^T - A$ is Skew symmetric.

5) $A . A^T$ is ______

6) $A^T . A$ is ______

Sum → Symmetric
difference → skew symmetric

① $(A+A^T)^T = A^T + A$

$\boxed{A^T = A}$

Symmetric matrix

② $\cancel{A^T}\ (A^T + A)^T = (A^T)^T + A^T$

$= A + A^T$

= symmetric matric

③ $(A - A^T)^T = A^T - (A^T)^T$

$= A^T - A$

$= -(A - A^T)$

= skew symmetric matrix

④ $(A^T - A)^T = (A^T)^T - A^T$

$= (A - A^T)$

$(A^T - A)^T = -(A^T - A)$

$\boxed{A^T = -A}$

→ If A is a Symmetric matrix, then K.A is ______

$(KA)^T = K \cdot A^T$

$(K.A)^T = K.A$

$\boxed{A^T = A}$ ∴ KA is Symmetric matrix.

If A is skew-Symmetric matrix,

$\boxed{A^T = -A}$

$(KA)^T = K.A^T$

$\boxed{(KA)^T \quad -KA}$

$\boxed{A^T = -A}$

∴ KA is skew symmetric matrix.

→ * Every Square matrix is expressed as sum of the symmetric and skew symmetric matrices.

$$\boxed{A = \underbrace{\frac{1}{2}(A+A^T)} + \underbrace{\frac{1}{2}(A-A^T)}}$$

→ If A,B are Symmetric matrices then A+B = Symmetric matrix
A−B = ________

$$(A+B)^T = A^T + B^T$$
$$\boxed{(A+B)^T = A+B}$$ — Symmetric matrix

$$(A-B)^T = A^T - B^T$$
$$\boxed{(A-B)^T = A-B}$$ — Symmetric matric

*→ Algebraic Sum of Symmetric matrices is a Symmetric matrices *

→ If A,B are skew symmetric matrices then A+B = ________
A−B ________

A,B skew symmetric matrices

$A^T = -A$, $B^T = -B$

$$(A+B)^T = A^T + B^T$$
$$(A+B)^T = -(A+B)$$
$$\boxed{A^T = -A}$$ skew symmetric matrix

$$(A-B)^T = A^T - B^T$$
$$(A+B)^T = -A+B$$

$$(A-B)^T = -(A-B)$$
$$\boxed{A^T = -A}$$ — skew symmetric matrix

*→ Algebraic Sum of Skew-Symmetric matrices is a skew-symmetric matrices
*

→ If A is symmetric matrix then $A^2 =$ Symmetric matrix

A is symmetric $A^3 =$ " matrix

$\boxed{A^T = A}$ $A^n =$ " matrix.

$A \cdot A = A^T \cdot A^T$

$\boxed{A^2 = (A^2)^T}$ Symmetric

$(A^3)^T = (A^2 \cdot A)^T$

$(A^3)^T = (A^2)^T \cdot (A)^T$

$\boxed{(A^3)^T = A^3}$ A^3 is Symmetric matrix

$\therefore$ $\boxed{A^n \text{ is Symmetric matrix}}$

→ $\boxed{\text{If A is Symmetric ; then } A^n \text{ is also Symmetric matrix}}$

→ If A is skew Symmetric matric then $A^2 =$ ______ matrix

$A^3 =$ ______ matrix

$A^n =$ ______ matrix

$A^T = -A$

$(A^2)^T = A^T \cdot A^T$

$(A^2)^T = +A^2$

$\boxed{A^T = +A}$ → Symmetric matrix

$(A^3)^T = (A \cdot A \cdot A)^T$

$= A^T \cdot A^T \cdot A^T$

$(A^3)^T = -A \cdot -A \cdot -A$

$(A^3)^T = -(A)^3$ — skew symmetric matrix

→ If A is skew Symmetric matrix then A^n → Symmetric ; n = even

→ skew symmetric ; n = odd

→ ~~I~~ ~~A~~: If A, B are skew symmetric matrices then

$A \pm B$; $A^2 \pm B^2$; $A^3 \pm B^3$; $A^4 \pm B^4$; $A^5 \pm B^5$; $A^6 \pm B^6$ is a

$$\left.\begin{matrix} A \pm B \\ A^3 \pm B^3 \\ A^5 \pm B^5 \end{matrix}\right\} \text{Skew symmetric} \qquad \left.\begin{matrix} A^2 \pm B^2 \\ A^4 \pm B^4 \\ A^6 \pm B^6 \end{matrix}\right\} \text{Symmetric}$$

→ If A, B are symmetric matrices then $AB + BA$ is ______

$AB - BA$ is ______

$$(AB+BA)^T = (AB)^T + (BA)^T$$

$$= \cancel{A^T B^T} +$$

$$= B^T A^T + A^T B^T$$

$$= BA + AB$$

$$\boxed{(AB+BA)^T = AB + BA}$$

$$(AB-BA)^T = (AB)^T - (BA)^T$$

$$= B^T A^T - A^T B^T$$

→ If A, B are skew symmetric matrices, then $AB + BA$ ______

$AB - BA$ ______

~~Square~~ **Trace of a matrix:** Sum of the diagnol elements in a Square matrix

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\boxed{\text{then } tr(A) = a_{11} + a_{22} + a_{33}}$

$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 7 & 1 \\ 1 & 1 & 3 \end{bmatrix} \Rightarrow tr(A) = 1+7+3 = 11$

properties of trace:

→ $tr(A) = tr(A^T)$

→ $tr(\text{Null matrix}) = 0.$

→ $tr(\text{skew symmetric matrix}) = 0.$

→ $tr(I_n) = n$

→ $tr(K.A) = K.\,tr(A)$

→ $tr(A \pm B) = tr(A) \pm tr(B)$

→ $tr(A + A^T) = tr(A) + tr(A^T)$
$= tr(A) + tr(A)$
$= 2\,tr(A)$

→ $tr(A - A^T) = tr(A) - tr(A^T)$
$= tr(A) - tr(A)$
$= 0.$

→ $tr(AB) \neq tr(A) \cdot tr(B)$

→ $tr(ABC) = tr(BAC) = tr(CAB)$ //

→ problems:

① If $tr(A) = 10$; then 1) $tr(A^T) = 10$

2) $tr(2A) = 2(10) = 20$.

3) $tr(A+A^T) = 2\,tr(A) = 20$ //

4) $tr(A-A^T) = 0$.

5) $tr(-4A) = -4(\text{trace of } A) = -40$

② If $tr(A) = -6$ then 1) $tr(A^T) = -6$

2) $tr(-3A) = 18$

3) $tr(6A) = -36$

4) $tr(A+A^T) = -12$

5) $tr(A-A^T) = 0$.

③ If $tr(A) = a+ib$ then $tr[(a-ib)A] =$ ____

$tr(kA) = k\,tr(A)$

$= (a-ib)(a+ib)$

$= a^2-(-1)b^2$

$= a^2+b^2$ //

④ If $tr(A) = 6$; $tr(A+B) = 10$; then $tr(B) =$

$tr(A) + tr(B) = 10$

$6 + tr(B) = 10$

$\boxed{tr(B) = 4}$

⑤ If $tr(A+B) = 10$; $tr(A-B) = 6$ then $tr(A) = ?$ $tr(B) =$ ____?

$tr(A) + \cancel{tr(B)} = 10$

$tr(A) - \cancel{tr(B)} = 6$

$2\,tr(A) = \frac{16}{2} = 8$

$tr(A) = 8$

$tr(B) = 2$ //

* Determinants and Inverse *

Minor of an element : Minor of an element a_{ij} is the determinant obtained after deleting i^{th} row and j^{th} column. It is denoted by (M_{ij})

let $A = \begin{bmatrix} 1 & 3 & 4 \\ 6 & 2 & 4 \\ 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

→ let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 7 \\ -1 & 4 & 6 \end{bmatrix}$ then $M_{12} = \rightarrow$ $M_{31} = \_\_\_\_$

$M_{12} = \begin{vmatrix} 3 & 7 \\ -1 & 6 \end{vmatrix} = 18+7 = 25$ //.

$M_{31} = \begin{vmatrix} 2 & 1 \\ 2 & 7 \end{vmatrix} = 14-\cancel{2} = 12$ //.

→ let $A = \begin{bmatrix} 1 & -2 & 3 \\ 8 & 9 & 1 \\ 3 & 7 & 0 \end{bmatrix}$ then $M_{11} + M_{22} + M_{33} = \underline{9}$

$= \begin{vmatrix} 9 & 1 \\ 7 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 8 & 9 \end{vmatrix}$

$= -7 + (-9) + (9+16)$

$= -7-9+25$

$= -16+25$

$= 9$ //

→ If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ then $M_{11} + M_{22} + M_{33} =$ 0

→ If $A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 1 & 3 \\ 2 & -1 & 3 \end{bmatrix} = M_{11} + M_{22} + M_{33} =$ ______

$$\left.\begin{aligned} M_{11} &= 0 \\ M_{22} &= -9 \\ M_{33} &= -3 \end{aligned}\right\} = -12$$

→ Cofactor of an element: cofactor of an element (a_{ij}) is the minor multiplied with $(-1)^{i+j}$.

$\therefore$ Cofactor of $a_{ij} = (-1)^{i+j}(M_{ij})$

→ If $A = \begin{bmatrix} 1 & 4 & -1 \\ 3 & -2 & 0 \\ 7 & -8 & 2 \end{bmatrix}$; then 1) Cofactor of 3 is
2) Cofactor of 0 is

$a_{21} = (-1)^3 (M_{21})$

$= (-1) \begin{vmatrix} 4 & -1 \\ -8 & 2 \end{vmatrix} = (-1)(8+8)$

$= \cancel{16} 0$

$a_{23} = (-1)^5 \begin{vmatrix} 1 & 4 \\ 7 & -8 \end{vmatrix} = (-1)(-8-28)$

$= 36$

$a_{13} = (-1)^4 \begin{vmatrix} 3 & -2 \\ 7 & -8 \end{vmatrix}$

$= (1)[-24+14]$

$= -10$

→ If $A = \begin{bmatrix} -1 & 7 & 0 \\ -3 & -2 & 1 \\ -4 & 4 & 8 \end{bmatrix}$; then 1) sum of cofactors of 7 and −4. 29.
2) product of cofactors of 0 and 1 96

Cofactor of 7 $= (-1)^3 \times \begin{vmatrix} -3 & 1 \\ -4 & 8 \end{vmatrix} = (-20) = 20$.

cofactor of −4 $= (-1)^4 \times \begin{vmatrix} 7 & 0 \\ -2 & 1 \end{vmatrix} = 9$.

Cofactor of 0 $= (-1)^4 \times \begin{vmatrix} -3 & -2 \\ -4 & 4 \end{vmatrix} = -4$

cofactor of 1 $= (-1)^5 \times \begin{vmatrix} -1 & 7 \\ -4 & 4 \end{vmatrix} = -24 = -24$

→ **cofactor matrix**: If every element in a matrix is replaced with its cofactor then it is called "cofactor matrix."

**

→ Transpose of cofactor matrix is called "Adjoint"

*. $\boxed{\text{Transpose of Adjoint is cofactor matrix}}$

** → If $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 3 \\ 4 & -1 & 1 \end{bmatrix}$

→ write a 4x4 matrix for given 3x3 matrix

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 3 \\ 4 & -1 & 1 \end{bmatrix} = \overset{\text{determinant}}{\begin{bmatrix} 2 & 3 & 0 & -2 \\ -1 & 1 & 4 & -1 \\ 1 & -1 & 2 & 1 \\ -2 & 3 & 0 & -2 \end{bmatrix}} = \overset{\text{Cofactor matrix:}}{\begin{bmatrix} 1 & 12 & 8 \\ 0 & 6 & 6 \\ 1 & -6 & -4 \end{bmatrix}}$$

→ If $A = \begin{bmatrix} -4 & 1 & 0 \\ 3 & -2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ then find cofactor matrix of A

4x4 matrix $= \overset{\text{determinant}}{\begin{bmatrix} -2 & 1 & 3 & -2 \\ 1 & 4 & 1 & 1 \\ 1 & 0 & -4 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix}} = $ Cofactor matrix: $\begin{bmatrix} -9 & -11 & 5 \\ -4 & -16 & 5 \\ 1 & 4 & 5 \end{bmatrix}$

Special cases:

**

→ If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$; then adjoint of A = ______

Adjoint matrix:

$$\begin{bmatrix} b0 & 0b \\ 0c & 00 \\ 00 & a0 \\ b0 & 0b \end{bmatrix} \cdot \begin{bmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{bmatrix}$$

observation:

$\boxed{\begin{matrix} a \to bc \\ b \to ac \\ c \to ab \end{matrix}}$

Similary:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ then } Adj(A) = \begin{bmatrix} 4\times5 & 0 & 0 \\ 0 & 3\times5 & 0 \\ 0 & 0 & 3\times4 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \text{ then } Adj(A) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -28 & 0 \\ 0 & 0 & -7 \end{bmatrix}.$$

$\rightarrow$ If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ then $Adj(A) = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$ $\rightarrow$ scalar matrix

{scalar matrix}

Similary;

If $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ then $Adj(A) = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $Adj(I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

---

(1) If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 6 \end{bmatrix}$ then $Adj(A) =$ ______

$Adj(A^T) =$ ______

$$\begin{bmatrix} 3 & 4 & 0 & 3 \\ 0 & 6 & 0 & 0 \\ 2 & 1 & 1 & 2 \\ 3 & 4 & 0 & 3 \end{bmatrix} = Adj(A) = \begin{bmatrix} 18 & -12 & 5 \\ 0 & 6 & -4 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 4 & 6 \end{bmatrix} \Rightarrow Adj(A^T) = \begin{bmatrix} 3 & 0 & 2 & 3 \\ 4 & 6 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 0 \\ -12 & 6 & 0 \\ 5 & -4 & 3 \end{bmatrix}$$

$$\boxed{Adj(A^T) = (adj A)^T}$$

$\rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix}$ {Symmetric matrix}

$$\begin{bmatrix} 3 & 4 & 2 & 3 \\ 4 & 4 & 3 & 4 \\ 2 & 3 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{bmatrix} = \text{adj of } A = \begin{bmatrix} -4 & 4 & -1 \\ 4 & -5 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

$\rightarrow A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then $Adj(A) =$ ______

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Adj of } A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

***

If :: $A = \begin{bmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & 0 \end{bmatrix}$ then $\boxed{Adj(A) = -A}$ For non diagonal matrix

$\rightarrow A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $Adj(A) =$ ______

$$\Rightarrow \begin{bmatrix} \cos\theta & 0 & -\sin\theta & \cos\theta \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & \sin\theta \\ \cos\theta & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Then $Adj(A) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

→ **Adjoint for (2x2) matrix:**

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; then Adj of(A) : $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

1st → interchange diagnol elements.

2nd → additive inverse of non diagnol elements.

Eg: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then adj (A) = $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$A = \begin{bmatrix} a+ib & a-ib \\ -a-ib & -a+ib \end{bmatrix}$ = $\begin{bmatrix} -a+ib & -a+ib \\ a+ib & a-ib \end{bmatrix}$

Note: Adjoint of :-

1) diagnol matrix ⇒ Diagnol matrix.
2) scalar matrix ⇒ Scalar matrix
3) Identity matrix ⇒ Identity matrix
4) Triangular matrix ⇒ Triangular matrix
5) Symmetric matrix ⇒ symmetric matrix

For upper Δ ⇒ upper Δ
For lower Δ ⇒ lower Δ

→ For skew Symmetric, the adjoint may be Symmetric or skew Symmetric.

Determinant of a matrix: Sum of the products of the elements of any row or column with their cofactors

$$A = \begin{bmatrix} a_{11}^{+} & a_{12}^{-} & a_{13}^{+} \\ a_{21}^{-} & a_{22}^{+} & a_{23}^{-} \\ a_{31}^{+} & a_{32}^{-} & a_{33}^{+} \end{bmatrix}$$

expanding w.r.t R_1 = $a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

→ Diagnol matrix: (3x3).

Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ then $|A| =$ abc.

$= a\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} - 0(\ ) + 0(\ )$

$\Rightarrow a(bc-0)$

$\Rightarrow abc.$

Similarly; $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} = |A| = 60$

→ If $A = \begin{bmatrix} x & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = -50$ then $x =$ ______

$x \times 2 \times 5 = -50$

$\boxed{x = -5}$

→ If $A = \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$ then $|A| = x^3$

→ If $A = \begin{bmatrix} x+y & 0 & 0 \\ 0 & x-y & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix}$ then $|A| = x^4+y^4$

∴ Determinant of a diagnol matrix$_{(3x3)}$ is product of diagnol elements

$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

$\boxed{|A| = a_{11} \times a_{22} \times a_{33}}$

$\Rightarrow$ If $A = \begin{bmatrix} +a & d & e \\ -0 & b & f \\ +0 & 0 & c \end{bmatrix}$ then $|A| =$ ______

$|A^T| =$ ______

$$|A| = a \begin{vmatrix} b & f \\ 0 & c \end{vmatrix} - 0(\ ) + 0(\ )$$

$$\boxed{|A| = abc}$$

$$A^T = \begin{bmatrix} \overset{+}{a} & \overset{-}{0} & \overset{+}{0} \\ d & b & 0 \\ e & f & c \end{bmatrix}$$

$$|A^T| = a \begin{vmatrix} b & 0 \\ f & c \end{vmatrix} - 0(\ ) + 0(\ )$$

$$\boxed{|A^T| = abc}$$

$$\boxed{|A| = |A^T|}$$

Eg: $\begin{bmatrix} 1 & 2014 & 2017 \\ 0 & 2 & 2018 \\ 0 & 0 & 3 \end{bmatrix}$ the $|A| =$

$\Rightarrow 1 \times 2 \times 3 = 6$

<u>NOTE</u>: Determinant of :

1) Diagnol matrix
2) Scalar matrix
3) Identity matrix
4) Triangular matrix (upper & lower)

is product of Diagnol elements.

$\rightarrow$ $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}_{2\times2} = 0-(-a)^2 = a^2$ //

$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}_{3\times3} \Rightarrow |A| = 0(\,) - a\begin{vmatrix} -a & c \\ -b & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix}$

$= -a(0+bc) + b(ac-0)$

$= -abc + abc$

$= 0$ //

NOTE: For skew symmetric matrix:

$|A| = [\;]_{2\times2} = \text{perfect square } (a^2)$ — even order.

$|A| = [\;]_{3\times3} = 0.$
$\downarrow$
odd order

$\rightarrow$ Determinant of even orderd $[(2\times2), (4\times4), (6\times6) \text{ etc}]$ skew symmetric matrices is perfect square.

$\rightarrow$ Determinant of odd ordered skew symmetric matrix is 0
$[(3\times3), (5\times5), (7\times7), (9\times9) \ldots \text{etc}]$

Exercise :

$\rightarrow$ $\begin{bmatrix} 0 & 1994 & 1995 \\ -1994 & 0 & 1996 \\ -1995 & -1996 & 0 \end{bmatrix}$ $|A| = 0.$

$\rightarrow$ $\begin{bmatrix} 0 & p-q & q-r \\ q-p & 0 & r-q \\ r-q & q-r & 0 \end{bmatrix} = |A| = 0.$

→ Determinant of $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \neq 0.$

$= a[bc-a^2] - b[b^2-ac] + c[ab-c^2]$

$= abc - a^3 - b^3 + abc + abc - c^3$

$= abc + abc + abc - a^3 - b^3 - c^3$

$= 3abc - a^3 - b^3 - c^3$

→ **Properties of Determinants:-**

→ Determinant of $A = |A^T|$

→ Determinant of cofactor matrix = Determinant of adjoint

$|\text{cofactor matrix}| = |\text{adjoint}|$

i.e, the property applicable for a row is also applicable for a column.

→ If all the elements in a row or column of a square matrix are equal to zero then its determinant value is 0.

$|A| = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_2 & b_2 & c_2 \end{bmatrix} = 0.$: $|A| = \begin{bmatrix} 1 & 4 & 0 \\ -2 & 9 & 0 \\ 1 & 3 & 0 \end{bmatrix} = 0$ //.

→ In a square matrix, If any two rows or coloums are identical, then it's determinant value is Zero.

$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_2 & c_2 \end{bmatrix} \Rightarrow |A| = 0.$ $A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} = 0.$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 7 & 2 & 2 \\ -8 & -1 & -1 \end{bmatrix} \Rightarrow |A| = 0.$

④ If the all elements in a row or column of a square matrix are multiplied by 'k', then its determinant value ~~of~~ is $k|A|$

$\boxed{kA \neq k|A|}$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ + & - & + \end{bmatrix} = a_{31}M_{31} - b_3 M_{32} + c_3 M_{33}$$

$$= \Delta$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix} = ka_3 M_{31} - kb_3 M_{32} + kc_3 M_{33}$$

$$= k(\Delta)$$

Eg: ① $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 100 & 200 & 300 \end{bmatrix} = 100 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = 0.$

② $\begin{bmatrix} 1/x & x^2 & yz \\ 1/y & y^2 & xz \\ 1/z & z^2 & xy \end{bmatrix} = xyz \begin{bmatrix} 1/x & x^2 & 1/x \\ 1/y & y^2 & 1/y \\ 1/z & z^2 & 1/z \end{bmatrix} = 0$ //

⑤ If all the elements in a row or column of a square matrix are $k \times$[elements of another (or) column]; then its determinant value is Zero

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{bmatrix} \quad |A| = 0.$$

Eg: $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -1 & -2 & -6 \end{bmatrix} = |A| = 0$

→ $\boxed{\text{Determinant } |kA| = k^n |A|}$

Ey: If A is a Square matrix of order (2×2) then $|kA| =$ $k^2|A|$

If A is a square matrix of order (3×3), then $|kA| =$ $k^3|A|$

If A is a Square matrix of order (4×4) then $|kA| =$ $k^4|A|$

If A is Square matrix of order (3×3), Determinant $|A| = 10$ then

1) $|2A| = 2^3|A| = 8\times10 = 80$
2) $|-3A| = (-3)^3|A| = -27\times10 = -270$
3) $|-6A| = (-6)^3(|A|) = -216\times10 = -2160$
4) $|-5A| = -5^3(|A|) = -125\times10 = 1250.$

If A is a square matrix of order (2×2), det $|A| = -4$; then
1) $|3A| = (3)^2|A| = 9(-4) = -36$
2) $|-4A| = (-4)^2|A| \Rightarrow 16(-4) = -64$
3) $|5A| = 25(-4) \Rightarrow -100$
4) $|10A| = 10^2(-4) = -400$

→ $\boxed{|AB| = |A|\cdot|B|}$

If A is orthogonal ; $A\cdot A^T = I$

$|A\cdot A^T| = |I|$

$|A|\cdot|A^T| = 1$

$|A|\cdot|A| = 1$

$(|A|)^2 = 1$

$\boxed{|A| = \pm1}$

$|A^2| = (|A|)^2$

$|A^3| = (|A|)^3$

$\boxed{|A^n| = (|A|)^n}$

Eg: If $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5 \\ 0 & 0 & 5 \end{bmatrix}$, $|A^2| = 25$; then $\alpha =$ ______

$|A^2| = 25$

$|A|^2 = 25$

$|A| = \sqrt{25}$

$\boxed{|A| = \pm 5}$

$25\alpha = 5$

$\boxed{\alpha = \pm 5/25 = \pm 1/5}$

→ In a square matrix, if any two rows or columns are ~~ident~~ interchanged then it's determinant value changes only in its sign.

If $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is Δ the $\begin{bmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{bmatrix}$ $\boxed{\Delta = -\Delta}$

If $\begin{bmatrix} 1 & 4 & 7 \\ 4 & 7 & 10 \\ 7 & 10 & 12 \end{bmatrix} = 9$; then $\begin{pmatrix} 7 & 10 & 12 \\ 4 & 7 & 10 \\ 1 & 4 & 7 \end{pmatrix} = -9.$

→ If all the elements in a row or column of a square matrix is sum of the two terms, then its determinant can be expressed as sum of the 2-determinants

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3+a_4 & b_3+b_4 & c_3+c_4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{bmatrix}$$

$(a_3+a_4)M_{31} + (b_3+b_4)M_{32} + (c_3+c_4)M_{33} = a_3M_{31} - b_3M_{32} + c_3M_{33} + a_4M_{31} - b_4M_{32} + c_4M_{33}$

$= (a_3+a_4)M_{31} - (b_3+b_4)M_{32} + (c_3+c_4)M_{33}$

$\boxed{L.H.S = R.H.S}$

Eg: $\begin{bmatrix} a_1 & la_1+mb_1 & b_1 \\ a_2 & la_2+mb_2 & b_2 \\ a_3 & la_3+mb_3 & b_3 \end{bmatrix}$

$$= \begin{bmatrix} a_1 & la_1 & b_1 \\ a_2 & la_2 & b_2 \\ a_3 & la_3 & b_3 \end{bmatrix} + \begin{bmatrix} a_1 & mb_1 & b_1 \\ a_2 & mb_2 & b_2 \\ a_3 & mb_3 & b_3 \end{bmatrix}$$

$= 0 + 0$

$= 0$ //

→ If a, b, c are different, determinant of $\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0.$

then 1) $abc = 1$ 2) $abc = -1$
3) $abc = 0$ 4) $abc = 2$

$$\begin{bmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{bmatrix} - \begin{bmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{bmatrix} = 0$$

$$abc\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} - \begin{bmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{bmatrix} = 0$$

$$abc\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} - (-)\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = 0$$

$$abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (abc-1) = 0$$

$abc - 1 = 0$

$\boxed{abc = 1}$

$$\begin{bmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{bmatrix} = 0$$

$$\begin{bmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{bmatrix} + \begin{bmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{bmatrix} = 0$$

$$abc\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} - \begin{bmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{bmatrix} = 0$$

$$abc\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} + \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = 0.$$

$$\begin{matrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{matrix} \left[(abc+1)\right] = 0$$

$abc + 1 = 0$

$\boxed{abc = -1}$

→ If all the elements in a row or column of a Square matrix are added with k(elements of another row or column) then its determinant value doesn't changes.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = A$$

$\boxed{R_2 \leftarrow R_2 + kR_1}$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+ka_1 & b_2+b_1k & c_2+c_1k \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta$$

Eg: $\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{bmatrix}$ $\quad x+y+z \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{bmatrix} = 0$

$= 0$

$$\begin{bmatrix} 1 & ab & c(a+b) \\ 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & ab & ab+bc+ca \\ 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \end{bmatrix} = 0$$

H/w →

$$\begin{vmatrix} 1 & ab & ab(a+b) \\ 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \end{vmatrix} = 0 \quad \underline{\text{prove it}}$$

$\to C_3 \leftarrow C_3 + abc\, C_1$

$$\begin{vmatrix} 1 & ab & a^2b + ab^2 \\ 1 & bc & b^2c + bc^2 \\ 1 & ca & c^2a + ca^2 \end{vmatrix} = \begin{matrix} 1 & ab(a \\ 1 & bc \\ 1 & ca \end{matrix}$$

$$\begin{bmatrix} 1994 & 1995 & 1996 \\ 1995 & 1996 & 1997 \\ 1996 & 1997 & 1998 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1994 & 1995 & 1996 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1994 & 1995 & 1996 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= 2(0)$$

$$= 0.$$

NOTE: If three rows or three columns are in Arithmetic or Geometric progression, then the determinant is zero.

$$\begin{bmatrix} 28 & 29 & 30 \\ 30 & 31 & 32 \\ 32 & 33 & 34 \end{bmatrix} = 0.$$

$$\begin{bmatrix} \log e & \log e^2 & \log e^3 \\ \log e^2 & \log e^3 & \log e^4 \\ \log e^3 & \log e^4 & \log e^5 \end{bmatrix}$$

$\boxed{\log e = 1}$

$$\boxed{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = 0 //}$$

→ If a, b, c are in arithmetic progression then determinant of

$$\begin{bmatrix} x+3 & x+4 & x+a \\ x+4 & x+5 & x+b \\ x+5 & x+6 & x+c \end{bmatrix} = 0$$

<u>Proof:</u>

$$\begin{bmatrix} x+3 & x+4 & x+a \\ 1 & 1 & b-a \\ 1 & 1 & c-b \end{bmatrix}$$

$b-a = c-b$ if a, b, c are in A.P.

$$\because \text{identical } R_2, R_3 \quad \begin{bmatrix} x+3 & x+4 & x+a \\ 1 & 1 & b-a \\ 1 & 1 & b-a \end{bmatrix} = 0.$$

→ If a, b, c are in arithmetic progression, then determinant of

$$\begin{bmatrix} x+3 & x+4 & x+ka \\ x+4 & x+5 & x+kb \\ x+5 & x+6 & x+kc \end{bmatrix} = 0$$

$$\begin{bmatrix} x+3 & x+4 & x+ka \\ 1 & 1 & kb-ka \\ 1 & 1 & kc-kb \end{bmatrix} = 0$$

<u>Determinant of:</u>

$$\begin{array}{l} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{array}{l} A.P \\ A.P = \\ A.P \end{array} \end{array}$$

$1(12-16) - 2(8-12) + 3(8-9)$

$1(-4) - 2(-4) + 3(-1)$

$-4 + 8 - 1 = 3$

$$\begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 10 \end{bmatrix} = \cancel{0}$$

(∵ ~~All rows are in Arithmetic progression~~ and ~~columns~~

$= 1(50-49) - 3(30-35) + 5(21-25)$

$= 1(1) - 1 + 15 - 20 = -6$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = \cancel{0 \ (\because A} \quad 0$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 4 & 7 & 0 \\ 7 & 10 & 12 \end{bmatrix} = 1(84-0) - 4(48-0) + 7(40-41)$$

$$= 84 - 152 - 7$$

$$= 84 - 169$$

$$= -85$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 4 & 7 & 10 \\ 7 & 10 & 14 \end{bmatrix} =$$

→ Determinant of $\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}$ = where $1, \omega, \omega^2$ are complex cube root of unity

$R_1 \to (R_1 + R_2) + R_3$

$$\begin{bmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ \omega+\omega^2+1 & \omega^2 & 1 \\ \omega^2+1+\omega & 1 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{bmatrix} = 0$$

→ $\begin{bmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{bmatrix} = |A| = \underline{0}$

→ $\begin{bmatrix} a-b & b-c & c-a \\ p-q & q-r & r-p \\ x-y & y-z & z-x \end{bmatrix}$ $C_1 \to C_1 + C_2 + C_3$ $= |A| = \underline{0}$

→ $\begin{bmatrix} p-q & m-n & a-b \\ q-r & n-o & b-c \\ r-p & o-p & c-a \end{bmatrix}$ $R_1 \to R_1 + R_2 + 3$

$|A| = 0$

** → Determinant of $\begin{bmatrix} x & a & a \\ a & x & a \\ a & a & x \end{bmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{bmatrix} x+2a & x+2a & x+2a \\ a & x & a \\ a & a & x \end{bmatrix}$$

$$x+2a \begin{bmatrix} 1 & 1 & 1 \\ a & x & a \\ a & a & x \end{bmatrix}$$

$C_2 \to C_2 - 1$

$= \quad C_3 \to C_3 - C_1$

$$= x+2a \begin{bmatrix} 1 & 0 & 0 \\ a & x-a & 0 \\ a & 0 & x-a \end{bmatrix}$$

$= \boxed{(x+2a)(x-a)^2}$ //

Eg: ① $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = |A| = 4$

$= (2+2)(2-1)^2$

$= 4$ //

② $\begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$ $|A| =$ ____

$(x+2a)(x-a)^2$

$= (2+8)(-2)^2$

$= 10(4) = 40$ //

If $\begin{bmatrix} 3x+8 & 3 & 3 \\ 3 & 3x+8 & 3 \\ 3 & 3 & 3x+8 \end{bmatrix} = 0$ then $x =$

$(3x+14)(3x+5)^2 = 0.$

$x = -14/3$ or $-5/3.$

If $\begin{bmatrix} 2x+1 & 3 & 3 \\ 3 & 2x+1 & 3 \\ 3 & 3 & 2x+1 \end{bmatrix}$ is 0 then $x =$

$(2x+7)(2x-2)^2 = 0$

$(2x+7)$ | $(2x-2)^2 = 0$

| | $2x-2 = 0$ |

$\boxed{x = -7/2}$ | $\boxed{x = 1}$

$$\begin{vmatrix} x+1 & 1-x & 1-x \\ 1-x & x+1 & 1-x \\ 1-x & 1-x & x+1 \end{vmatrix} = 0. \quad \text{then } x = \_\_\_$$

$(x+1+2-2x)\,(2x)^2$

$= (x+3)(2x^2)$

Types of matrices Based on Determinant Value :-

a) Singular Matrix : If the determinant of square matrix is zero then it is said to be singular Matrix

$\boxed{|A| = 0}$

b) Non Singular Matrix : A square matrix is said to be Non singular if its determinant value is not equal to zero. i.e. $\boxed{|A| \neq 0}$

→ If $A = \begin{bmatrix} x & 3 \\ 4 & 2 \end{bmatrix}$ is a singular matrix then x = ______

$|A| = 0$

$2x - 12 = 0$

$2x = 12$

$\boxed{x = 6}$

→ $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a singular matrix. then θ = ______

$+0(\ ) \ -0(\ ) \ +1\left(\cos^2\theta - \sin^2\theta\right) = 0$

$0 + (\cos 2\theta) = 0$

$\cos 2\theta = 0 \Rightarrow 2\theta = 90$

$\boxed{\theta = 45^\circ} = \pi/4$

H/W:

* If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ -2 & -4 & \lambda \end{bmatrix}$ is singular matrix; then the $\lambda =$ ____.

* If $A = \begin{bmatrix} 1 & 2 & x \\ 4 & -1 & 7 \\ 2 & 4 & -6 \end{bmatrix}$ is singular; then $x =$ ____

* If $\begin{bmatrix} a & a & x \\ m & m & m \\ b & x & b \end{bmatrix} = 0$ then $x =$ ____

HINT $\Rightarrow$ $\boxed{C_2 \leftarrow C_2 - C_1}$

** $|A| = \begin{bmatrix} a-b-c & 2b & 2c \\ 2b & b-c-a & 2c \\ 2a & 2b & c-a-b \end{bmatrix} =$

1) $a+b+c$ 2) $(a+b+c)^2$ 3) $(a+b+c)^3$ 4) $(a+b+c)^4$

* If each row or column have same degree homogeneous functions, then Determinant may be number (or) function.

If it is a function; it's degree is equal to degree of product of diagnol elements

Note : Homogeneous function :

Eg: $x^2 + xy + y^2$

2 2 2

$\boxed{x^3 + x^2y + y^3}$

3 3 3

Determinant A -

$$\Rightarrow \begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{bmatrix}$$

degree of $a\, b^2\, ab = 5$

1) $(a-b)(b-c)(c-a)$

2) $(a-b)(b-c)(c-a)(a+b+c)$

3) $(a-b)(b-c)(c-a)\,abc$

~~4)~~ $(a-b)(b-c)(c-a)(ab+bc+ca)$ degree $= 5$

→ Determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

degree of $c^2 \cdot b \cdot 1 = 3$

a) $a+b+c$

b) $(a+b+c)^2$

~~c)~~ $(a+b+c)^3$ — deg $= 3$

d) $(a+b+c)^4$

→ Determinant of

$$\begin{bmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{bmatrix}$$

1) xyz 2) $2xyz$ 3) $3xyz$ 4) $4xyz$

put $\boxed{x = y = z = 1}$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = 4.$$

$\therefore 4xyz$

(or) put $x = 1$; $y = 2$; $z = 3$

$$\begin{bmatrix} 5 & 1 & 1 \\ 2 & 4 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

1st → $|A| = 5(6) - 1(0) + 1(-6)$

$= 30 - 6$

$= \boxed{24}$

2nd → Also substitute in options (x, y, z values)

1) 6

2) 12

3) 18

4) 24

* Inverse of a matrix (or) multiplicative inverse *

→ For every Non singular matrix 'A', there exist a another matrix B; such that $\boxed{AB = BA = I}$; then B = multiplicative inverse of A.

→ It is denoted by A^{-1}.

$\therefore \boxed{A \cdot A^{-1} = A^{-1} \cdot A = I}$

$$\boxed{A^{-1} = \frac{1}{|A|} \times [\text{Adjoint of } A]}$$

Necessary conditions for existence of Inverse :

→ Matrix should be Non- singular.

→ $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

→ $A^{-1} = \frac{1}{|A|}$ adjoint of A

→ adjoint of A = $|A| . A^{-1}$ = $|A|^{n-1}$

→ $|\text{adjoint of } A| = |A|^{n-1}$

$= |\text{cofactor matrix}|$

→ $A^{-1} = \frac{1}{|A|} \text{adj}(A)$.

$= \frac{A . A^{-1}}{I} = \frac{1}{|A|} A \text{adj}.(A)$

$= A . \text{adj}(A) = |A| . I$.

$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$

****.

→ let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ {diagnol matrix}

⟹ $A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$ {diagnol matrix}

→ let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$; the $A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1/a \end{bmatrix}$

→ let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I^{-1}$ then $I^{-1} \Rightarrow I$.

Problems:

let $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 8 \end{bmatrix}$

57 3 5
78 57
35 13
57 35

$\text{adj } A = \begin{bmatrix} -9 & 11 & -4 \\ 11 & -17 & 8 \\ -4 & 8 & -4 \end{bmatrix}$

$\text{det of } A = \boxed{(1\times(-9)) + (3\times 11) + 5(\times(-4))}$

$= -9 + 33 - 20 = 4.$

$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$

$= \frac{1}{4} \begin{bmatrix} -9 & 11 & -4 \\ 11 & -17 & 8 \\ -4 & 8 & -4 \end{bmatrix}$

** $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow$ then $A^{-1} =$

**.

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

$\text{Adj}(A) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$= |A| = [0 + 0 - 1] = -1 \Rightarrow -1 \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

---

* $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow$ then $A^{-1} =$.

$\text{Adj}(A) = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

$$\begin{matrix} -3 & 1 & 2 & -3 \\ 1 & 2 & 1 & 1 \\ 5 & 7 & 3 & 5 \\ -3 & 1 & 2 & -3 \end{matrix}$$

$|A| = -21 - 15 + 35 = -36 + 35 = -1$

$A^{-1} = -1 \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

---

* $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \Rightarrow$ then $A^{-1} =$ $\quad |A| = 1($

$\begin{bmatrix} 14 & 14 \\ 13 & 13 \\ 13 & 13 \\ 14 & 14 \end{bmatrix}_{4\times4}$ then $\text{Adj}(A) = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

$A^{-1} = |A| [\text{Adj of } A]$

$=$

* $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $A^{-1} =$ ____

Inverse of (2x2) Matrix :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1}$$

$$A^{-1} = \frac{1}{|A|} \times \text{adjoint}(A)$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

Eg: $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ $\quad A^{-1} = $ —

$$= \frac{-1}{5} \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$$

Eg: $A = \begin{bmatrix} 7 & 1 \\ 4 & 1 \end{bmatrix}$ $\quad A^{-1} =$

$$= \frac{1}{3} \begin{bmatrix} 1 & -1 \\ -4 & +7 \end{bmatrix}$$

$A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ ~~then~~; $a^2+b^2+c^2+d^2 = 1$; then $A^{-1} = ?$

$$= \frac{1}{(a^2+b^2) + (c-id)(c+id)} \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$$

$$\frac{1}{\underbrace{a^2+b^2+c^2+d^2}_{=1}} \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$$ //

Properties of Inverse:

1. $[A^{-1}]^{-1} = A$
2. $|A|^{-1} = \frac{1}{|A|}$
3. $A \cdot A^{-1} = A^{-1} \cdot A = I$
4. $I^{-1} = I$
5. $(A^{-1})^T = (A^T)^{-1}$
6. $(A^{-1})^m = (A^m)^T$
7. $(AB)^{-1} = B^{-1} \cdot A^{-1}$
8. $(ABCD)^{-1} = D^{-1} \cdot C^{-1} \cdot B^{-1} \cdot A^{-1}$

→ If $A = \begin{bmatrix} ab & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & b^2 \end{bmatrix}$ then $|A^{-1}| =$

$|A^{-1}| = \frac{\cancel{1}}{|A|^{-1}}$ $\quad |A^{-1}| = |A|^{-1} = \frac{1}{|A|}$

$|A| = ab(a^2b^2) - 0(\_) + 0(\_)$

$|x| = a^3b^3$

$$\boxed{|A^{-1}| = \frac{1}{a^3b^3}}$$

→ If $A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ $= (B^{-1} \cdot A^{-1})^{-1} \Rightarrow (A^{-1})^{-1} \cdot (B^{-1})^{-1}$

$= A \cdot B$

$AB = \begin{bmatrix} 6 & 1 \\ 2 & 0 \end{bmatrix}$

→ If A, B are square matrices of same order $B = \cancel{[A]^{-1}} B.A$
$-A^{-1}B.A$. then i) $(A+B)^2$ =

$B = -A^{-1}B.A$

$AB = -A^{-1}A.B.A$

$\boxed{AB = -B.A}$

$\boxed{AB \neq BA}$

$(A+B)^2 = A^2 + AB + BA + B^2$

$= A^2 - \cancel{BA} + \cancel{BA} + B^2$

$= A^2 + B^2$

Similarly = $(A-B)^2 = (A-B)(A-B)$

$A^2 - AB - BA + B^2$

→ A square matrix is satisfying the relation $\cancel{a^3 - 3a^2 + 2a + 4I}$ $A^3 - 3A^2 + 2A + 4I = 0$ then A^{-1}.

$A^3 - 3A^2 + 2A + 4I = 0$.

$A^{-1}\left[A^3 - 3A^2 + 2A + 4I\right] = A^{-1}(0)$.

$A^2 - 3A + 2I + 4A^{-1} = 0$.

$4A^{-1} = -(A^2 - 3A + 2I)$

$\boxed{A^{-1} = \frac{-1}{4}(A^2 - 3A + 2I)}$

→ A square matrix satisfying the relation ~~then~~ $\cdot A^3+2A-I=0$.
then $A^{-1}=$ ______

$A^3+2A-I=0$

$A^{-1}[A^3+2A-I]=0$

$A^2+2I-A^{-1}=0$

$\boxed{A^{-1}=A^2+2I}$

→ A square matrix is satisfying the relation $A^3+I=0$ then $A^{-1}=$ ______

$A^{-1}(A^3+I)=0$

$A^2+A^{-1}=0$ $\boxed{A^{-1}=-A^2}$

→ If $A=\begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & x & 1\end{bmatrix}$ & $A^{-1}=\begin{bmatrix}1/2 & -1/2 & 1/2\\ -4 & 3 & y\\ 5/2 & -3/2 & -1/2\end{bmatrix}$ then $x=$ ______

$y=$ ______

Such that $\boxed{A.A^{-1}=I}$ $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$3\frac{1}{2}+x(-4)+\frac{5}{2}=0.[a_{13}]$

$-4x+4=0$

$-4x=-4$

$\boxed{x=1}$

→ If A is a Square matrix of order 3×3 then $A^3 - \beta_1 A^2 + \beta_2 A - \beta_3 I = 0$

where $\beta_1 =$ trace of A.

$\beta_2 = m_{11} + m_{22} + m_{33}$

$\beta_3 = |A|$.

→ If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ the $A^3 =$ ______ a) A^2 b) $2A^2$ c) $3A^2$ d) (A^2)

$\beta_1 = 1+1+1 = 3$

$\beta_2 = 0$

$\beta_3 = 0$.

$A^3 - 3A^2 + 0 - 0 = 0$.

$A^3 - 3A^2 = 0$.

$\boxed{A^3 = 3A^2}$

→ $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ then $A^3 =$ ______

1) 0

2) A

3) $(a^2+b^2+c^2)A$

4) $-(a^2+b^2+c^2)A$.

given Skew Symmetric matrix

$\beta_1 = 0$

$\beta_2 = \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} + \begin{vmatrix} 0 & -b \\ b & 0 \end{vmatrix} + \begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix}$

$\beta_2 = +a^2 + b^2 + c^2$

$\beta_3 = 0$.

$A^3 - 0(A^2) + (a^2+b^2+c^2)A - 0(I) = 0$.

$\boxed{A^3 = -(a^2+b^2+c^2)A.}$

→ If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ then $A^3 =$ ______ 1) A 2) $-A$ ~~3)~~ 0 4) I

$B_1 = 0.$

$B_2 = \begin{vmatrix} 2 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ -2 & -3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} \Rightarrow 0 + 3 + 7 = 10$

$B_3 = |A| = 1(0) - 1(-3) + 3(-1) = \cancel{+3} \cancel{-3} = 0.$

$\boxed{A^3 - B_1A^2 + B_2A - B_3I = 0}$

$\boxed{A^3 = B_1A^2 - B_2A + B_3I} \Rightarrow 0 + 10(A) + 0 = 10A.$

→ If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ then $A^{-1} =$ ______

1) $A^2 - 6A + 9I$
~~2)~~ $\frac{1}{4}(A^2 - 6A + 9I)$
3) $\frac{-1}{4}(A^2 - 6A + 9I)$
4) $\frac{1}{4}(A^2 + 6A + 9I)$

$B_1 = 6$

$B_2 = 3 + 3 + 3 = 9$

$B_3 = 2(3) + 1(-1) + 1(-1)$

$B_3 = 4.$

$A^{-1}(A^3 - 6A^2 + 9A - 4I = 0.)$

$A^2 - 6A + 9I - 4A^{-1} = 0.$

$A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$

→ If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then $A^2 - 2A =$ ______ 1) A^{-1} 2) $2A^{-1}$ ~~3)~~ $-A^{-1}$ 4) $-2A^{-1}$

$B_1 = 2$

$B_2 = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \cancel{-1} + 0 + \cancel{1} = 0$

$B_3 = -1$ $\boxed{A^3 - B_1A^2 + B_2A - B_3I = 0}$

$A^3 - 2A^2 + I = 0$

$A(A^2 - 2A) + I = 0 \Rightarrow \frac{-I}{A} = -IA^{-1} = -A^{-1}$

→ **properties of Adjoint of a Matrix :**

1) $A \cdot adj(A) = (adj A) \cdot A = |A| I$

2) If A is Singular matrix $|A| = 0$

then $A \cdot adj(A) = (adj A) A = O$

3) $|adj(A)| = |\text{cofactor matrix}|$

$= |A|^{n-1}$

4) $|adj \cdot (adj(A))| = |adj(A)|^{n-1}$

$= [[|A|]^{n-1}]^{n-1}$

$= [|A|]^{(n-1)^2}$

5) $|adj, adj \ldots r \text{ times of } adj A| = [|A|]^{(n-1)^r}$

6) $adj(A^T) = [adj(A)]^T$

7) $adj(kA) = k^{n-1} \cdot adj \cdot A$

8) $Adj(AB) = (adj \cdot B)(adj A)$

9) $Adj(A^n) = (Adj A)^n$

10) $adj[Adj(A)] = |A|^{n-2} \cdot A$

→ If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ then $A \cdot adj(A) =$

$= |A| \cdot I$

$= -5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$ //

→ If $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A \cdot adj A = K \cdot I$; $K = ?$

$|A| = \cos^2\theta - \sin\theta(-\sin\theta)$

$|A| = \cos^2\theta + \sin^2\theta = 1$

$\boxed{K = 1}$

→ If $A \cdot adj A = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$ then $|A| = \underline{10}$

$|adj A| = \underline{|A|^{n-1}} = 10^{3-1} = 100$

$|A^T| = \underline{|A| = 10}$

$|A^{-1}| = \underline{\frac{1}{|A|} = \frac{1}{10}}$

$|3A| = \underline{3^3 |A|} = 3^3 |10| = 270$

→ If $A \cdot adj(A) \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ $|A| = -4$

$|adj A| = \underline{(-4)^2 = 16}$

$|A^T| = \underline{|A| = -4}$

$|A^{-1}| = \underline{\frac{-1}{4}}$

$|3A| = \underline{27(-4) - 108}$

→ If $A \cdot adj(A) \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ then $|A| = \underline{5}$

$|Adj A| = \underline{|5|^{2-1} = 5}$

$|A^T| = \underline{|A| = 5}$

$|A^{-1}| = \underline{\frac{1}{|A|} = \frac{1}{5}}$

$|3A| = \underline{9 \times 5} = 45$

→ If A is a Square matrix of order 4×4, $|adj(A)| = -27$ then $|A| =$ ____

$-27 = |A|^{4-1}$

$-27 = |A|^3$

$\boxed{|A| = -3}$

→ $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the Adjoint of (3×3) matrix A ; ~~then~~ $|A| = 4$

then $\alpha =$

Since P is adjoint of A

$|P| = |adj\ of\ A|$.

$1(0) - \alpha(-2) + 3(-2) = |A|^{n-1}$

$2\alpha + 6 = (4)^2$

$2\alpha - 6 = 16$

$\alpha = \frac{22}{2}$

$\boxed{\alpha = 11}$

→ The matrix obtained after replacing all the elements of A with their cofactors is $\begin{bmatrix} 1 & 4 & 7 \\ 4 & 7 & 10 \\ 7 & 10 & 12 \end{bmatrix}$ the $|A| =$ ____

given matrix is cofactor matrix ; ∴ $\begin{vmatrix} 1 & 4 & 7 \\ 4 & 7 & 10 \\ 7 & 10 & 12 \end{vmatrix} = |A|^{n-1}$

$1(-16) - 4(-22) + 7(-9) = |A|^2$

$-16 + 88 - 63 = |A|^2$

$= -79 + 88 = |A|^2$

$= 9 = |A|^2$

$\boxed{|A| = \pm 3}$

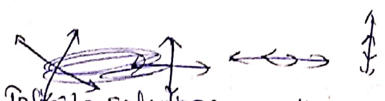
* Solution of Linear Equations:

Two variable linear Equations represent straight lines.

→ straight lines are 3-types.

1) parallel lines 

- no common point ⇒ no solution.
- equations differ only in constant terms Eg: $3x+4y=7$; $3x+4y=8$
- The equations $a_1x+b_1y=c_1$; $a_2x+b_2y+c_2=0$. represents parallel lines if $\boxed{\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}}$

2) Similar or coincident lines 

- Infinite common points ⇒ Infinite solutions. equations are same.

  Eg: $\{x+2y=3\ ;\ x+2y=3\}$

- The equations $a_1x+b_1y=c_1$; $a_2x+b_2y=c_2$ represent Similar lines $\left[\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\right]$

3) Intersecting lines :

- 1 common point ⇒ 1 solution,

  Eg: $x-y=1$
  $3x+2y=4$.

- The equations $a_1x+b_1y=c_1$; $a_2x+b_2y=c_2$ represent intersecting lines if $\left[\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\right]$

→ If given set of equations has at least one solution, they are said to be consistent otherwise inconsistent

3-variable linear equations → represents plane Surfaces

* Non homogeneous system of equations

Consider the set of three linear Non-homogeneous equations in 3-variables

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

They can be expressed in matrix form:

$$\underbrace{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{D}$$

It is in the form $\boxed{AX = D}$

$$\boxed{X = D \cdot A^{-1}}$$

* If A is Non Singular matrix $\boxed{|A| \neq 0}$ then No. of solutions $= 1$.
  for System of equations

* If A is Singular Matrix $\boxed{|A| = 0}$ the system of equations has no solution or infinite solutions.

  In particular, If $Adj(A)\,[D] = 0$. the System possess infinite solution

  - If $Adj(A)\,[D] \neq 0$, the System possess no Solution.

PRBLMS:

→ The System of equations:

$$\left.\begin{array}{l} x+y+z=6 \\ x+2y+\lambda z=0 \\ x+2y+3z=10 \end{array}\right\}$$ has no solution.

then $\lambda = \_\_\_$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & \lambda \\ 1 & 2 & 3 \end{bmatrix}$$

$|A| = 0$.

$$6 - 2\lambda - 3 + \lambda + 0 = 0$$
$$6 - 2\lambda - 3 + \lambda = 0$$
$$-\lambda + 3 = 0$$
$$\boxed{\lambda = 3}$$

→ The System of equations $x+y+z=2$ has unique solution
$2x+y-z=3$
$3x+2y+kz=4$,

1) $k=0$ 2) $k\neq0$ 3) $k<1$ 4) $-2<k<2$.

$|A|\neq0$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{bmatrix} \neq 0.$$

$\boxed{\cancel{k=2}}$ $\cancel{1+2}+2k+3+1\neq0$

$-2k+1\neq0$

$1\neq2k$

$2k\neq0$

$\boxed{k\neq0}$

→ The system of equations $x+y+z=6$; $2x-y+z=3$; $x+2y-z=2$.
① ② ③

①+③ : $2x+3y=8$ } has only one solution
②+③ $3x+y=5$
→ Intersecting lines.

→ The System of equation $x+2y+z=3$ — ① has — solution
$2x+3y+z=3$ — ②
$3x+5y+2z=1$ — ③

② − ① $x+y=0$.

multiply ① eq with ② ⇒ $4x+6y+4z=6$ — ④

④ − ③ : $\boxed{x+y=5}$

∴ Two lines are parallel : has no solution

→ The System of equations $x-y+2z=4$, $3x+y+4z=6$, $x+y+z=1$ have ______ solutions

$\cancel{\infty}\ 2$

→ The system of equations
$$\left.\begin{aligned} 4x+y+2z&=5 \\ x-3y+3z&=10 \\ 9x-3y+7z&=20 \end{aligned}\right\}$$
has ______ solution.

→ The System of equation
$$\left.\begin{aligned} x+y+z&=6, \\ x+2y+3z&=10 \\ x+2y+\lambda z&=\mu \end{aligned}\right\}$$
is incosistent

1) $\lambda=3$ $\mu=10$ 2) $\lambda\neq3, \mu=10$ 3) $\lambda\neq3$, $\mu\neq10$ 4) $\lambda\neq3$; $\mu\neq10$

② − ① = $y+2z=4$
$y+(\lambda-1)z=\mu-6$.

given inconsistent.

$$\frac{1}{1}=\frac{2}{\lambda-1}\neq\frac{4}{\mu-6}$$

$$\boxed{\begin{array}{c|c} \lambda-1=2 & \frac{1}{1}\neq\frac{4}{\mu-6} \\ \lambda=3 & \mu-6\neq4 \\ & \mu\neq10 \end{array}}$$

→ The System of equations $\left.\begin{array}{l} x+y+z=6 \\ x+2y+3z=10 \\ x+2y+\lambda z=\mu \end{array}\right\}$ has infinite solutions.

1) $\lambda=3$ $\mu=10$ 2) $\lambda\neq 3$ $\mu=10$ 3) $\lambda\neq 3$; $\mu\neq 10$

4) $\lambda=3$ $\mu\neq 10$.

Homogeneous System of equations:

→ consider the set of 3 linear homogeneous equations in 3 variables

$$a_1x+b_1y+c_1z=0$$
$$a_2x+b_2y+c_2z=0$$
$$a_3x+b_3y+c_3z=0.$$

→ They can be expressed in matrix form $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

→ It is in the form $\boxed{A\cdot X=0}$

→ If A is non singular matrix ~~then~~ $\boxed{|A|\neq 0}$; System of equations has only one solution i.e, at origin. called as trivial solution.

→ If A is singular matrix $\boxed{|A|=0}$; System of equations has infinite no. of solutions i.e, System posses non-trivial solution also

→ Homogeneous System of equations are always consistent

~~Exercise~~ : problems :

The system of equations $\left.\begin{aligned} x+ky+3z &= 0 \\ 3x+ky-2z &= 0 \\ 2x+3y+3z &= 0 \end{aligned}\right\}$ has a non trivial solution

then $k =$ ____

$\therefore$ Since Non trivial solution ; $|A| = 0$.

$$\begin{bmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -3 \end{bmatrix} = 0.$$

→ The system of equations $\left.\begin{aligned} 3x-2y+2z &= 0 \\ 2x-14y+15z &= 0 \\ x+2y+3z &= 0 \end{aligned}\right\}$ has non trivial solution

$|A| = 0$

$$\begin{bmatrix} 3 & -2 & 2 \\ 2 & -14 & 15 \\ 1 & 2 & 3 \end{bmatrix} = 0.$$

→ The System of equations $x-y+z=0$; $x+2y-z=0$; $2x+y+3z=0$.

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix} \neq 0$$

$7+5-3 \neq 0$

$\boxed{9 \neq 0}$

$① \times 2 \Rightarrow$ $2x - \cancel{2y} + 2z = 0$
$x + \cancel{2y} - z = 0$

$\boxed{3x + z = 0}$

$① + 3 = \boxed{3x + 4z = 0}$

→ ~~The system of equations~~ 2?

→ The solution of the system of equations $x+y+z=2$

$$x+2y+3z=1$$
$$3x+y-5z=4$$

1) $(3,1,1)$ 2) ~~4, 3, 1~~ 3) $(1,1,1)$ 4) $(7,-10,4)$

Substitute

→ Solution of $x+y+z=7$

$$x+2y+3z=16$$
$$x+3y+4z=22$$

1) $(1,-3,2)$ 2) $(1,2,3)$ 3) ~~$(1,3,3)$~~ 4) $(-1,4,4)$

*. **Different methods for finding solutions:**

Consider the system of linear equations $a_1x+b_1y+c_1z=d_1$,

$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

They can be expressed in matrix form as :

$$\underbrace{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{D}$$

$$\boxed{AX=D}$$

$$\boxed{X = D.A^{-1}}$$

1) **Cramer's Rule:**

→ used if **A** is Non singular matrix.

→ By cramers rule $x = \frac{\Delta_1}{\Delta}$ $y = \frac{\Delta_2}{\Delta}$ $z = \frac{\Delta_3}{\Delta}$

→ Matrix Inversion method :

- used if A is Nonsingular matrix.
- By matrix inversion method $\boxed{X = A^{-1}D}$

→ Gauss Jordan Method: let $a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

~~Agumented matrix~~

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \rightarrow \text{Agumented matrix}$$

It can reduced to the form of $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$ by using elementary row operations then $x = \alpha$; $y = \beta$; $z = \gamma$.

→ An elementary row transformation is an operation of any one of the following types.

1) The interchange of any two rows.
2) The multiplication of the elements of any row by a Non-zero number.
3) The addition to the elements of a row, the corresponding elements of another row multiplied by a ~~m~~ number

→ If the matrix $A \times \begin{bmatrix} -1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{matrix} -4 & 7 \\ 7 & 7 \end{matrix}$

then ~~A=~~ !

$AB = C$

$A^{-1}(AB) = A^{-1}C$

$B = A^{-1}C$

$A = C \cdot B^{-1}$

→ The matrix $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

25/4/19.

* Functions, Limits and Continuity *

Function: The relation between two variables or more than 2 variables.

$y = f(x)$ → independent

y ↓ dependent variable

Domain: The set of values assigned to the independent variable and accepted by the function

Co Domain: The set of values obtained from dependent variable.

Range: The interval between minimum value and maximum value of co Domain

Real valued Function: If function gives real values; then it is said to be real valued Function.

Eg: If a function $f(x) = \frac{1}{x}$; is R.V.F then domain is

$\boxed{x \neq 0}$ $\boxed{x \in R - \{0\}}$

→ $f(x) = \frac{1}{(x-1)(x-2)}$ is R.V.F then domain = ____

$x \in R - \{1, 2\}$

→ If $f(x) = \sqrt{x-1}$ is R.V.F then domain is

$x \in R - [1, \infty)$.

$\boxed{\begin{array}{l} x = 1 \\ f(x) = 0 \end{array}}$

Types of functions :

Transcedental function: Except all algebraic functions, the remaining all functions are called transcedental functions.

$f(x) = x^2+x+1$ ✗ $f(x) = x + 3\sin x$ ✓

$f(x) = \log x$ ✓ $f(x) = e^x$ ✓

Identity function: If $f: A \to A$; then it is called Identity function.

i.e, domain of and co domain should be equal,

$f(x) = x$ ✓

$1 = 1$

domain = codomain

$f(x) = \sin x$ ✗

polynomial function: $P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots a_m$

$a_0, a_1, a_2, \dots a_m \to$ Real numbers.

$f(x) = x^2+x+1$ ✓ ; $f(x) = \frac{x^2}{3} - \frac{1}{2}x + 5/3$ ✓

$f(x) = x^3 + \sqrt{x} + 3$ ✗

$f(x) = x^{-3} + x + x^4$ ✗

Even & odd Function:

If $f(-x) = f(x) \Rightarrow$ even function

$f(-x) = -f(x) \Rightarrow$ odd function

odd

- x, x^3, x^5
- x^3+5
- x^5+x^3
- $\sin x, \tan x, \cot x, \operatorname{cosec} x$
- $\sinh x, \tanh x, \coth x, \operatorname{cosech} x$
- $\log\left(\frac{a+x}{a-b}\right)$

odd × even = odd

$\frac{odd}{even}$ = odd

$\frac{even}{odd}$ = odd

even

$x^2, x^4, \ldots x^{2n} \ldots$

- x^2+1, x^4+x^2+2

- $\sqrt{a^2-x^2}$

- $\cos x, \sec x$ Eg: $\boxed{\begin{array}{l} \cos(-x) = \cos x \\ \sec(-x) = \sec x \end{array}}$

  $\cosh x$, $\operatorname{sech} x$

$(\text{odd})^{\text{even power}} = \text{even}$ Eg: $5^4 \Rightarrow \text{even}$

odd × odd = even $\frac{x^5}{x^3} = x^2 \Rightarrow$ even

$\frac{\text{odd}}{\text{odd}}$ = even $x^5 \cdot x^3 = x^8$ //

* Step function: This function eliminates the decimal value and gives absolute value:

Eg: $f(x) = [x]$

$f(1.9) = [1.9] = 1$

$f(2.1) = [2.1] = 2$

$f(0.5) = [0.5] = 0$

** $f(-1.5) = [-1.5] = -2$

$f(-3.5) = [-3.5] = -4$

$f(-2.8) = [-2.8] = -3$

$f(-10.2) = [-10.2] = -11$

$f(-3.0) = [-3.0] = -3$ //

* Problem: If $f(x) = x - [x]$ then $f(2.3) = $ ___

*

$f(2.3) = 2.3 - [2.3]$

$= 2.3 - 2$

$= 0.3$ //

Modulus Function: Modulus function removes negative value and gives positive value of that function.

$$\boxed{\begin{array}{lll} |f(x)| = f(x) & & f(x)>0 \\ |f(x)| = -f(x) & & f(x)<0 \\ |f(x)| = 0 & & f(x)=0. \end{array}}$$

Eg.
$$\boxed{\begin{array}{lll} |ax-b| = & ax-b & ; x > b/a \\ |ax-b| = & -(ax-b) & ; x < b/a \\ |ax-b| = & 0 & ; x = b/a \end{array}}$$

Eg. If $x > b/a$ then $\frac{|ax-b|}{ax-b} = \frac{ax-b}{ax-b} = 1.$

If $x < b/a$; then $\frac{|ax-b|}{ax-b} = -1.$

Eg: If $x<0$ then $\frac{7x+10|x|}{x+9|x|} = \frac{7x-10x}{x-9x} = \frac{-3x}{-8x} = 3/8$

If $x>0$ then $\frac{7x+10|x|}{x+9|x|} = \frac{17x}{10x} = \frac{17}{10}$

If $x=0$ then $\frac{0}{0}$ = indeterminant //

Inverse Function:

If $f: A \xrightarrow{\text{domain}} B$ codomain

then $f^{-1}: B \longrightarrow A$ – codomain
domain

If $y = f(x)$
$\boxed{x = f^{-1}(y)}$

Eg: $f(x) = ax+b$
$y = ax+b$
$x = \frac{y-b}{a}$
$f^{-1}(x) = \frac{x-b}{a}$

$f(x) = \frac{ax+b}{cx-a}$

$y = \frac{ax+b}{cx-a}$

$ycx - ya = ax+b$

$ycx - ax = b+ya$

$x(yc-a) = b+ya \Rightarrow x = \frac{b+ya}{yc-a} = f^{-1}(x) = \frac{ax+b}{cx-a}$

$f(x)$	$f^{-1}(x)$
$ax+b$	$\frac{x-b}{\cancel{a}}$
$\sqrt{a^2-x^2}$	$\sqrt{a^2-x^2}$
a^x	$\log_a^x$
$\cancel{\sqrt{a}}\ x^n$	$\sqrt[n]{x}$
$\frac{ax+b}{cx-a}$	$\frac{ax+b}{cx-a}$

$f(x) = \log_a^x$

$y = \log_a^x$

$x = \cancel{y^a}\ a^y$

$\boxed{f^{-1}(x) = a^x.}$

$y = a^x.$

$x = \log_a y$

$\boxed{f^{-1}(x) = \log_a^x.}$

__logarthmic functions:__

$\boxed{f(x) = \log_a^x \ ; \ a>0}$

$f(x) = \text{not defined} \quad (-\infty < x < 0)$

$f(x) = -\infty \quad (x=0)$

$f(x) = -\text{ve value} \quad (0<x<1)$

$f(x) = 0 \quad (x=1$

$f(x) = +\text{ve value} \quad (x>1)$

$\boxed{\log_1 1 = \text{undetermined}}$

$\log(mn) = \log m + \log n$

$\log m/n = \log m - \log n$

$\log m^n = n \log m$

$\log_{b^n} a^m = \frac{m}{n} \log_b a$

$\log_b^a = \frac{\log a}{\log b}$

$a^{\log_a f(x)} = f(x).$

$\boxed{\text{If } \log_a^x = b \text{ the } \boxed{x = a^b}}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh^{-1} x = \log\left(x + \sqrt{1+x^2}\right)$

$\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$

$\tanh^{-1} x = \log\sqrt{\frac{1+x}{1-x}}$

$\coth^{-1} x = \log\sqrt{\frac{x+1}{x-1}}$

$\text{sech}^{-1} x = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

$\text{cosech}^{-1} x = \log\left(\frac{1+\sqrt{1+x^2}}{x}\right).$

Composite Functions: Function of functions are called composite functions.

$$f \circ g(x) = f(g(x))$$

$$f \circ g \circ \phi(x) = f(g(\phi(x)))$$

Eg: If $f(x) = a^x$, $g(x) = \log_a x$ then $f \circ g^{-1}(x) =$ ____

$$f(x) a^x = g^{-1}(x) = a^x$$

$$f \circ g^{-1}(x) = a^{a^x}$$

*Eg: If $f(x) = \frac{ax+b}{cx-a}$ then $f \circ f(x) = \boxed{x}$

$$\boxed{f \circ f^{-1}(x) = x}$$

proof: If $f(x) = \frac{3x+2}{x-3}$ then $f \circ f^{-1}(x) =$?

$$\frac{3(3x+2) + 2(x-3)}{3x+2+3(x-3)} = \frac{11x}{11} = x$$

→ If $f(x) = \frac{x}{\sqrt{1-x^2}}$, $g(x) = \frac{x}{\sqrt{1+x^2}}$ then $f \circ g(x) =$ ____

$$f(g(x)) = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1-\frac{x^2}{(1+x^2)}}} = \frac{\frac{x}{\sqrt{1+x^2}}}{\frac{1}{\sqrt{(1+x^2)}}} = \frac{x}{\sqrt{1+x^2}}\sqrt{(1+x^2)}$$

$$= x$$

Limits and continuity :-

I) If $\underset{x \to \infty \ (or) \ x \to -\infty}{Lt} \dfrac{f(x)}{g(x)}$ where $f(x), g(x) \in$ algebraic functions

Step-1: Select highest degree term.

Sol: $\dfrac{\text{pick up coeff of Highest degree Term in Numerator}}{\text{pick up coeff of Highest degree term in Denominator.}}$

Eg:

① $\underset{x \to \infty}{Lt} \dfrac{x^2+x+1}{x+2}$;

H.D.T $= x^2$

Sol. $\dfrac{1}{0} = \infty.$

② $\underset{x \to \infty}{Lt} \dfrac{x^3+x+5}{x^4+1}$

H.D.T $= x^4$ =

Sol. $\dfrac{0}{1} = 0.$

③ $\underset{x \to \infty}{Lt} \dfrac{3x+2}{5-7x}$

H.D.T $= x$

Sol. $= \dfrac{3}{-7} = \dfrac{-7}{3}.$

II) $\underset{x \to \infty}{Lt} \dfrac{f(x)}{g(x)}$, where $f(x), g(x)$ are exponential functions.

Step-1: select highest base term:

Sol: $\dfrac{\text{Coeff of H.B.T in Numerator}}{\text{Coeff of H.B.T. in Denominator}}$

Eg: ① $\underset{x \to \infty}{Lt} \dfrac{e^x+1}{2^x-1}$ $\boxed{e^x > 2^x}$

H.B.T $= e^x$ $= \dfrac{1}{0} = \infty.$

② $\lim_{x \to \infty} \dfrac{2^x + 1}{3^x + 5}$.

H.B.T $= 3^x$

Sol: $\dfrac{0}{1} = 0.$

③ $\lim_{x \to \infty} \dfrac{2^{x+1} + 1}{2^{x+5} + 3} = \dfrac{2^x \cdot 2 + 1}{2^x \cdot 2^5 + 3}$

H.B.T $= 2^x$

Sol: $= \dfrac{2}{2^5} = \dfrac{1}{2^4} = \dfrac{1}{16}$ //

III) $\lim_{n \to \infty} \dfrac{1^k + 2^k + 3^k + 4^k + \ldots + n}{n^{k+1}} = \dfrac{1}{k+1}$ $\boxed{k \in \text{posite integers}}$

Eg: $\lim_{n \to \infty} \dfrac{1 + 4 + 9 + 16 \ldots + n^n}{n^3} = \dfrac{1}{2+1} = \dfrac{1}{3}.$

$\lim_{n \to \infty} \dfrac{1 + 2 + 3 + 4 + \ldots n^n}{n^2} = \dfrac{1}{1+1} = \dfrac{1}{2}$

$\lim_{n \to \infty} \dfrac{1^{50} + 2^{50} + 3^{50} + \ldots + n^n}{n^{51}} = \dfrac{1}{50+1} = \dfrac{1}{51}$ //

(IV) $\underset{\substack{x \to \infty \\ f(x) \to \infty}}{lt} \left(1 + \frac{a}{f(x)}\right)^{b(f(x))} = e^{ab}$

Eg: $\underset{x \to \infty}{lt} \left(1 + \frac{3}{x^2}\right)^{4(x^2)} = e^{12}$

$\underset{x \to \infty}{lt} \left(1 - \frac{1}{x^3+1}\right)^{x^3+1} = e^{-1} = \frac{1}{e}$ //

$\underset{x \to \infty}{lt} \left(1 + \frac{3}{\tan x}\right)^{2\tan x} = e^{6}$ //

(V) $\underset{x \to \infty}{lt} \left(1 + \frac{1}{ax+b}\right)^{cx+d} = e^{c/a}$

Eg: $\underset{x \to \infty}{lt} \left(1 + \frac{1}{5x+2}\right)^{10x+1} = e^{2}$

(VI) $\underset{x \to \infty}{lt} \left(\frac{x+a}{x+b}\right)^{x+c} = e^{a-b}$

Eg: $\underset{x \to \infty}{lt} \left(\frac{x}{x+3}\right)^{x} = e^{-3}$

$\underset{x \to \infty}{lt} \left(\frac{x+1}{x-1}\right)^{x} = e^{2}$ //

(VII) If $x \to \phi$ $\quad \phi - \begin{Bmatrix} \sin x, \tan x, \sin^{-1}x, \tan^{-1}x \\ \sinh x, \tanh x, \sinh^{-1}x, \tanh^{-1}x \end{Bmatrix}$

$$\lim_{x\to 0} \frac{p f_1(ax) \pm q f_2(bx)}{r f_3(cx) \pm s f_4(dx)} = \frac{pa \pm qb}{rc \pm sd}$$

① $$\lim_{x\to 0} \frac{3\sin 3x + 5\tan^{-1}x}{5\sinh x + 3\tanh^{-1}x} = \frac{(3\times 3) + (5\times 1)}{(5\times 1) + (3\times 1)} = \frac{14}{8}$$

② $$\lim_{x\to 0} \frac{4\sin x - 3\tan x}{3\sin x + 4\tan x} = \frac{4-3}{3+4} = \frac{1}{7}$$

③ $$\lim_{x\to 0} \frac{\sin ax}{x} = a \qquad \lim_{x\to 0} \frac{\sin x^\circ}{x} = \frac{\sin \pi/180\, x}{x} = \pi/180$$

④ $$\lim_{x\to 0} \frac{\tan ax}{x} = a \qquad \lim_{x\to 0} \frac{\tan x^\circ}{x} = \frac{\sin \pi/180\,(x)}{x} = \pi/180$$

⑤ $$\lim_{x\to 0} \frac{\sin^{-1}3x}{x} = 3$$

$\to$ $$\lim_{x\to 0} \frac{1-\cos mx}{x^2}$$

$$\lim_{x\to 0} \frac{2\sin^2\frac{mx}{2}}{x^2}$$

$$\boxed{\lim_{x\to a} f(g(x)) \quad f\left[\lim_{x\to a} g(x)\right]}$$

$$2\left[\lim_{x\to 0}\left(\frac{\sin\frac{mx}{2}}{x}\right)^2\right]$$

$$\Rightarrow 2\,\frac{m^2}{4} = m^2/2 \,//$$

$$\lim_{x \to 0} \frac{1-\cos mx}{1-\cos nx}$$

$$\lim_{x \to 0} \frac{\cancel{2}\sin^2 \frac{mx}{2}}{\cancel{2}\sin^2 \frac{nx}{2}}$$

$$\lim_{x \to 0} \left[\frac{\sin mx/2}{\sin nx/2}\right]^2$$

$$= \left[\frac{\frac{m}{\cancel{2}}}{\frac{n}{\cancel{2}}}\right]^2$$

$$= \frac{m^2}{n^2}$$

$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2} = \frac{(1-\cos bx) - (1-\cos ax)}{x^2}$$

$$= \frac{1-\cos bx}{x^2} - \frac{1-\cos ax}{x^2}$$

$$= \frac{b^2}{2} - \frac{a^2}{2} = \frac{b^2-a^2}{2} \,//$$

$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} = \frac{\frac{b^2-a^2}{\cancel{2}}}{\frac{c^2-d^2}{\cancel{2}}} = \frac{b^2-a^2}{c^2-d^2} \,//$$

(VIII) $\lim_{x\to 0} f(x)^{g(x)} = 1$

Where $f(x), g(x) \in \phi \begin{cases} \sin x, \tan x, \sin^{-1}x, \tan^{-1}x \\ \sinh x, \tanh x, \sinh^{-1}x, \tanh^{-1}x \end{cases}$

$\lim_{x\to 0} (\sin x)^{x} = 1$

$\lim_{x\to 0} \cos(\sec^{-1}x)^{\sin x} = 1.$

(IX) $\lim_{x\to 0} \left[1 + a f(bx)\right]^{\frac{c}{g(dx)}} = e^{\frac{abc}{d}}$

$\boxed{\text{where } f(x), g(x) \in \phi}$

① $\lim_{x\to 0} \left[1+x\right]^{1/x} = e^{\frac{1 \times 1 \times 1}{1}} = e$

② $(1+\tan x)^{3\cot x} = e^{3}$

③ $(1+3\sin^{-1}2x)^{\frac{2}{\sin^{-1}3x}} = e^{12/3} = e^{4}$

(X) $\lim_{x\to 0} \dfrac{\sqrt[m]{1+f_1(ax)} - \sqrt[n]{1+f_2(bx)}}{x} = \dfrac{a}{m} - \dfrac{b}{n}$

$\boxed{\text{where } f_1, f_2 \in \phi}$

$\lim_{x\to 0} \dfrac{\sqrt[5]{1+\sin 3x} - \sqrt[4]{1+\tan 3x}}{x} = \dfrac{3}{5} - \dfrac{3}{4}$

$\lim_{x\to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x} = \dfrac{1}{2} + \dfrac{1}{2} = \dfrac{2}{2} = 1$

$$\lim_{x\to 0} \frac{\sqrt{1+\sin x} - \sqrt[3]{1+\tan x}}{\sin x}$$

$$\frac{\lim_{x\to 0} \dfrac{\sqrt{1+\sin x} - \sqrt[3]{1+\tan x}}{x}}{\lim_{x\to 0} \dfrac{\sin x}{x}} = \frac{\frac{1}{2} - \frac{1}{3}}{1} = \frac{1}{6}$$

(XI) L'Hospital Rule :

When

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ r } \frac{\infty}{\infty}$$

Sol: $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ (until we get definite value).

Ex: ① $\lim_{x\to 1} \frac{\log x}{x-1} \left\{ \frac{0}{0} \right\}$

$$= \frac{1}{x} = 1$$

② $\lim_{x\to a} \frac{x^n - a^n}{x-a} = \frac{nx^{n-1} - 0}{}$

$$= n.a^{n-1}$$

③ $\frac{x^m - a^m}{x^n - a^n} = \frac{mx^{m-1}}{n.x^{n-1}} = \boxed{\frac{m}{n} a^{m-n}}$

$x\to a$

④ $\lim_{x\to 0} \frac{a^x - 1}{b^x - 1} = \frac{a^x \log a - 0}{b^x \log b} = \frac{\log a}{\log b} = \log_b a$

⑤ $\lim_{x\to 0} \frac{a^x + b^x - 2}{x} = \frac{a^x \log a + b^x \log b - 0}{}$

$$= \log a + \log b$$

$$= \log(ab)$$

6) $\lim_{x\to 0} \dfrac{a^x+b^x-2}{c^x+d^x-2}$

$$\frac{a^x \log a + b^x \log b - 0}{c^x \log c + d^x \log d - 0}$$

$\boxed{\frac{d}{dx} a^x = a^x \log a}$

$$= \frac{\log a + \log b}{\log c + \log d} = \frac{\log ab}{\log cd} = \boxed{\log_{cd} ab}$$

7) $\lim_{x\to 0} \dfrac{e^{\sin 2x}-1}{x}$

$$\lim_{x\to 0} \frac{e^{\sin 2x} \cdot \cos 2x \cdot (2)}{1}$$

$$= e \cdot (1)\, 2$$

$$= 2$$

(XII) $\lim_{x\to a} \sqrt[n]{f(x)} = D.E$ if $f(a) = 0$.

Eg: $\lim_{x\to} \sqrt{4-x^2} = D.E$

(XIII) $\lim_{x\to a} \dfrac{1}{(f(x))^n} = D.E$; if $f(a)=0$ & n is odd

$\boxed{\text{if } n \text{ is even } = \infty}$

Eg. $\lim_{x\to 0} \dfrac{1}{x^3}$; D.E

(XIV) $\lim_{x\to} [f(x)] = D.E$ $\boxed{\text{if } f(a) \text{ is integer}}$

Eg: $\lim_{x\to} [x] = f(3)$ D.E

(XV) $\underset{x \to a}{Lt} \frac{|f(x)|}{f(x)} = D.E$ if $f(a) = 0$.

Eg: $\underset{x \to 1}{Lt} \frac{|x-1|}{x-1} = D.E$, $\boxed{f(a) = 0}$

(XVI) $\left.\begin{array}{l} Lt \ \tan x \\ Lt \ \sec x \end{array}\right\}$ D.E for $(2n+1)\pi/2$, $n \in Z$

$\{-3\pi/2, -\pi/2, \pi/2, 3\pi/2\}$

Eg: $\underset{x \to \pi/2}{Lt} \tan x = D.E$

(XVII) $\left.\begin{array}{l} Lt \ \cot x \\ Lt \ \operatorname{cosec} x \end{array}\right\}$ D.E for $n\pi$, $n \in Z$

$(\pi, 2\pi, 3\pi, 4\pi, \ldots n\pi)$

Eg: $\boxed{\underset{x \to 2\pi}{Lt} \cot x = D.E}$

$\boxed{\underset{x \to 0}{Lt} \cot x = D.E}$

(XVIII) If $\underset{x \to a}{Lt} f(x)$ doesn't exist then $f(a)$ is discontinuous at $x = a$

(XIX) If $f(x)$ is continuous at $x = a$ then $\boxed{\underset{x \to a}{Lt} f(x) = f(a)}$

$f(x)$	The points at which $f(x)$ is discontinuous	The points at which $f(x)$ is continuous
Polynomial	—	R
$\lvert f(x)\rvert$	—	R
$\sin x, \cos x$	—	R
$\tan x, \sec x$	$(2n+1)\pi/2$; $n \in Z$	$R - \{(2n+1)\pi/2\}$
$\cot x, \operatorname{cosec} x$	$n\pi$; $n \in Z$	$R - \{n\pi\}$
$[f(x)]$	$f(x)$ is integer	$f(x)$ value is non integer
$\frac{\lvert f(x)\rvert}{f(x)}$	$f(a) = 0$	$R - \{x \text{ at } f(a) = 0\}$

→ If $f(x) = (1+x)^{\frac{1}{x}}$ is continuous at $x = 0$
then $f(0) =$ ____

$$f(0) = \lim_{x \to 0} (1+x)^{1/x} = e$$

→ If $f(x) = \frac{\sin 3x}{x}$ is continuous at $x = 0$

then $f(0) =$ ____

$$\lim_{x \to 0} \frac{\sin 3x}{x} = 3$$

→ The $f(x) = [x]$ is ___d___

a) continuous at $x = 3$ c) dis continuous at $= 4$
b) continuous at $x = 10$ d) discontinuous $x = 2.5$