Diverse Mathematical Formulations of Friedmann's Equations for Explaining Expansion of Big Bang Universe in the Presence and Absence of Dark Energy

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Abstract

In cosmological theory, one of the most crucial principles is the ongoing, dynamic expansion of the universe, a concept substantiated by early 20th-century astronomical observations such as Edwin Hubble's redshift measurements of remote galaxies. Emerging from Albert Einstein's General Theory of Relativity, the Friedmann equations—conceived in 1920—constitute a mathematical framework governing cosmic evolution. These equations are derived from the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which presupposes a universe that is isotropic and homogeneous at macroscopic scales. The Friedmann equations consist of two distinct differential equations that factor in the spatial distribution of matter-energy constituents within the cosmos, thereby shaping its expansive behavior over time. This review paper delves into the intricate variations of Friedmann's equations, elucidating the underpinnings of cosmic expansion, while also probing the enigmatic role of dark matter in this expansive framework.

Keywords: Friedmann-Lemaître-Robertson-Walker (FLRW) metric, General Theory of Relativity, Cosmic expansion, redshift

1. Introduction

Derived through the Friedmann-Lemaître-Robertson-Walker (FLRW) metric which assumes that the universe is homogeneous and isotropic on large scales, the two Friedmann equations describe the evolution of the universe's scale factor (a) and its density as a function of time.[1] The Friedmann equations are "a set of equations in physical cosmology that govern the expansion of space in homogeneous and isotropic models of the universe within the context of general relativity", initially introduced by Alexander Friedmann.[2] They are a set of dynamical equations describing the expansion of the universe based on its matter content.[3] The first Friedmann equation is given as:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \tag{1}$$

where a is the scale factor of the universe, $\dot{a}$ represents the derivative of the scale factor a with respect to time, c is the speed of light, k represents the curvature of space, Λ is the cosmological constant, G is the gravitational constant, and ρ is the energy density of the universe, including both matter and energy.

Likewise, the second Friedmann equation is given as:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{2}$$

where p is the pressure of the cosmic constituents. The first equation also termed as "universe expansion rate" relates the expansion rate of the universe (Hubble parameter) to the energy density and curvature of the universe. The second equation termed as "universe acceleration rate" describes how the acceleration of the cosmic expansion ($\ddot{a}/a$) is related to the energy density (ρ) and pressure (p) of the matter and energy in the universe.[4]

2. Describing the expansion of the universe without dark energy

First Friedmann equation is given as:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{3}$$

where
a is a representation of the size of the universe, more accurately is an average distance between galaxies. Friedmann equations indicate how a evolves over time. $\ddot{a}/a$ is the speed of the expansion of the universe. Also the time derivative of a.

LHS of the Friedmann equation:

$$\frac{\dot{a}^2}{a^2} - \frac{8\pi G}{3}\rho \tag{4}$$

reflects the balance between the kinetic and gravitational potential energy. This Friedmann equation is an energy equation. $\dot{a}^2$ is analogous to the kinetic energy of expansion: how much outflowing energy the universe has. But that energy is resisted by the gravitational effect of all the matter and energy in the universe. p is density: how packed with stuff the universe is.

$$\frac{8\pi G}{3}\rho$$

represents the capacity of the universe to slow itself down, also analogous to the gravitational potential energy. Balance between these two energy-like terms denotes the fate of the universe.[5] If the kinetic energy of the expansion and the potential energy of the collapse are perfectly balanced i.e. if

$K.E. = P.E. = 0$, then, the universe will expand to a huge size and grind to a near halt. That is, if the expansion is exactly equal to the escape velocity. In that case, the LHS of the equation adds to a zero. But if the LHS is a little bit positive, that represents a tiny bit of extra energy. There will be some expansion energy remaining after gravity is diluted to nothing, and the universe will expand forever, never stopping. If the LHS is negative, in that case, there was never a high enough expansion rate to reach that extreme size. The universe will eventually fall back inwards, and we will see many distant galaxies up close as the universe undergoes expansion reversal.[6] We know the expansion speed (also termed Hubble's Constant), by measuring it from the redshifts of the galaxies as well as other methods. Until the late 90's, it was believed that the expansion or recollapsing of the universe depended on the measure of (the density of the universe). After weighing up the galaxies for decades, the density of the universe turns out to be too low, only about a quarter of what is needed to reverse the expansion.[7] The density or the recollapsing term is smaller than the expansion term, making the left side of the Friedmann equation positive and indicating that the universe will expand forever.

3. Describing the expansion of the universe with dark energy

Einstein's general theory of relativity allows us to describe the behavior of the universe on large scales, vastly beyond our capacity to physically explore and at a time billions of years into the future. But limits the prediction of this theory from the experimental observations, which leads to the display that the fate of the universe is governed by a mysterious force called "dark energy".[8]

To truly understand dark energy we have to look into the workings of Einstein's theory on a large cosmic scale.

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho = \frac{-kc^2}{a^2} \tag{5}$$

(expansion)-(density)=(curvature)[9]

In part II, the first Friedmann equation describing the comic balance between the outward expansion of the universe and the resistance to this expansion due to the gravitational effect of its contents was discussed. This cosmic conflict of expansion or contraction is actually won by expansion. This is true without even taking dark energy into account yet. By weighing up all of the matter in the universe, astronomers have found out that its volume is not sufficient enough to contract the universe back into its initial phase. This ultimately states the future of the universe is expanding, rather than contracting. There are two independent measurements that can test this prediction of the fate of the universe, one it being the "Geometry of the universe", which slightly points to an inconsistency in the first Friedmann equation.[10] We have discussed that the left side of the equation i.e. the sum of the expansion and density terms do not cancel out each other to a zero, rather they come out positive, referring that the expansion term is greater than the inward pulling density term or the contraction term.[11]

$$\left(\frac{\dot{a}}{a}\right)^2 > \frac{8\pi G}{3}\rho \tag{6}$$

The universe is too low density to recollapse.[12] The right side of the equation $\frac{kc^2}{a^2}$ describes the curvature of the space, pivoting on that k term. k is in a sense the shape of the universe. It's a spatial curvature as well as its spatial extent i.e.finity and infinity. The spatial geometry of the universe at a certain instant can be flat or curved. [13] The value of k can range between -1 to +1.

When $k = +1$, the universe has a positive curvature spatial geometry. Spatial snapshots of the cosmos at a given instant in time will be curved like the 3D surface of a 4D hypersphere. In a universe like that, the angles of triangles add up to more than 180 degrees and circles and spheres contain more internal area and volume than their circumferences should allow. But the total spatial volume of such a universe is finite, making the geometry a "closed geometry".

When $k = -1$, the universe has a negative curvature spatial geometry.The universe is a 3D version of a negatively curved hyperbolic plane. Triangles add up to less than 180 degrees and surfaces hold less volume than in flat space. Such a universe is infinitely large, making its geometry an "open geometry".

Likewise, $k = 0$ means the universe is flat. Zero spatial curvature at any given time. A flat universe is still infinite and open in all three spatial dimensions.

This gives us a way to independently verify what we got on the left side of the first Friedmann equation. When the measure of the density of the universe is taken into account the left side of the equation should be equal to its right i.e. the shape of the universe should be intrinsically tied to its fate. An overdense, recollapsing universe should have a spherical geometry. An underdense, infinitely expanding universe should be hyperbolic.[14] Only a universe with the right density, expanding at its escape velocity, slowing to a stop over infinite time, should be flat with its left and right hand side of the equation coming out to zero. This can be tested. By measuring the shape of the universe by checking how geometry works on a cosmic scale, k can be measured. With all the information in hand, the geometry should turn out to be hyperbolic, giving us a positive RHS to match the positive LHS we got from weighing our universe. Observations of the size of the cosmic microwave background(CMB) features allow us to verify that the longest triangles in the universe add up to exactly 180 degrees.[15] That is the geometry of a flat, Euclidean universe, flat to within 0.41 percent.[16] The right side of the first Friedmann equation has to be very close to zero, which is totally inconsistent with the level of positive curvature expected from an infinitely expanding universe.That is because when we tried to describe the universe by reducing the Einstein field equations into the Friedmann equations, we missed the cosmological constants$\Lambda g_{\mu\nu}$. When the first Friedmann equation is derived with the cosmological constant included, we end up with an addition on the LHS of the equation.

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho - \frac{\Lambda c^2}{3} = -\frac{kc^2}{a^2} \tag{7}$$

Assuming the cosmological constant is positive, it works on the side of the density term to help bring the left side down to zero. Even though the density is still too low to reverse the expansion, with the cosmological constant, geometry is no longer tied to the fate of the universe. A flat, $k = 0$ universe can expand forever. The expression "cosmological constant" is, as we interpret it, constant. As the universe expands, regular matter and energy get diluted. But the "energy" provided by the cosmological constant doesn't. Its density stays constant. So, the bigger the universe, the more the "energy". As there's more and more space, the energy increases, accelerating the expansion. In actuality, this "energy" is known as "dark energy" and is interpreted as "a type of energy that is simply part of empty space, an intrinsic property of spacetime". When the universe expands infinitely, the density of regular matter will drop below the level of this vacuum energy as described by the cosmological-constant term.[17] At that point, dark energy will govern expansion, something that is happening already.

Interpreting the cosmological constant as vacuum energy, a property of space itself, it will eventually dominate the evolution of the universe. As the galaxies expand very far away, the density of the universe becomes almost zero. The curvature of the flat universe is also zero. So, for far-future universe, the first friedmann equation can also be written as:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda c^2}{3} \tag{8}$$

Expansion will only depend on the cosmological constant. $a(t)$ will become a constant. This can also be called the Hubble Parameter(H) and could be thought as the rate at which the universe doubles in size. Substituting $a = \dot{H}$ to the first Friedmann equation, we get:

$$H^2 = \frac{\Lambda c^2}{3} \tag{9}$$

The Hubble Constant is just the Hubble Parameter for the present day. Typically, the Hubble Parameter becomes lower over time. But with a positive cosmological constant, the universe will eventually have a constant Hubble Parameter along with a constant doubling time. The doubling time is exponential growth, which looks like an accelerating expansion. Currently, there is still enough matter in the universe to influence the expansion rate, but the universe has already reached the point where dark energy dominates. The universe has been accelerating in its expansion for 6 billion years or so as seen in the white dwarf supernova measurements. If the cosmological constant stays constant, the exponential growth is expected to continue.

4. Summary and conclusions

The Friedmann equations play an important part in understanding the expansion of the universe. These equations explain the dynamic interplay between scale factor, energy density, curvature and the presence of dark matter and dark energy. Without the influence of dark energy, the universe expands eternally. The left side of the first Friedmann equation that represents the balance between kinetic and gravitational potential energy also reveals that the expansion energy of the universe overcomes the inward gravitational pull of its contents. Observations and measurements, including the cosmic microwave background, support the idea that our universe is flat and that it will expand indefinitely due to the insufficient density to reverse this expansion. However, the cosmological constant i.e. dark energy intricates this narrative. With the involvement of dark energy, the fate of the universe is decoupled from its geometry. Dark energy certifies that the universe's expansion continues to accelerate, leading to exponential growth in size and an ever-constant Hubble Parameter. In quintessence, the universe's fate is one of interminable expansion, accelerated by the presence of dark energy. The expansion of the cosmos will persist, and the universe will continue to evolve in ways that challenge our understanding, making cosmology a fascinating field of study that continues to unravel the mysteries of our vast universe.

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