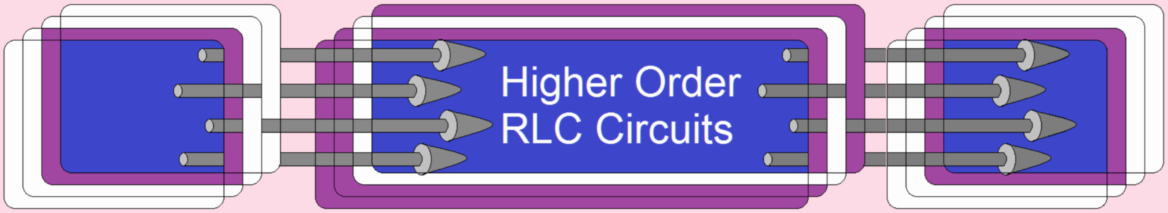
Solved Problems In Transfer Functions of RLC circuits.
Resource: Solutions & Problems of Control Systems, 2nd ed - AK Jairath.

Level: Intermediate.

Circuiting PrerequisitesTo Laplace Transform Electric Circuits.



Transfer Functions

(Intermediate Level)

Apologies for any errors and omissions.

August 2020.

I selected AK Jairath textbook because it goes back to 1992, when this engineer first published this book. 2nd edition in 1994, and reprinted in 1996.

Solutions & Problems in Control System. May not be in circulation now. Its a small book. Concise similar to Schaums (Supplementary), its not a main textbook. Chapter 1 is Transfer Functions. All the problems in chapter 1 are are made up of R L C components. So this was in line with my/our starting plan to stay within the electric circuits corridor. First keep things simple. So if you asked why, thats the reason I selected this chapter. We did some theory-examples in transfer function at end of Part B, so its best to do them first since these are fresh in minds.

Got an oppurtunity to work with RLC components in the transfer function and secondly control systems context, why waste it. So I did these few example problems.

AK Jairath: The transfer function of a system is the ratio of Lapalce transforms of the output and input quantities, initial conditions being zero. When a physical system is analysed, a mathematical model is prepared by writing differential equations with the help of various laws. An equation describing a physical system has integrals and differentials. The response can be obtained by solving such equations.

The **steps involved** in obtaining the transfer function are:

1. Write differential equations of the system.

2. Replace terms involving $\frac{d}{dt}$ by s and $\int \square\, dt$ by 1/s. **<--- Applies to L & C.**
L and C from RLC was worked in electric circuits.
See notes bottom next page.

3. Eliminate all but the desired variable.

See figure next page.

$v(t) = e^{st}$ OR $i(t) = e^{st}$ $\quad L\frac{d(e^{st})}{dt} = sL \cdot e^{st}$ $\quad \frac{1}{C}\int e^{st}\,dt = \frac{1}{sC} \cdot e^{st}$

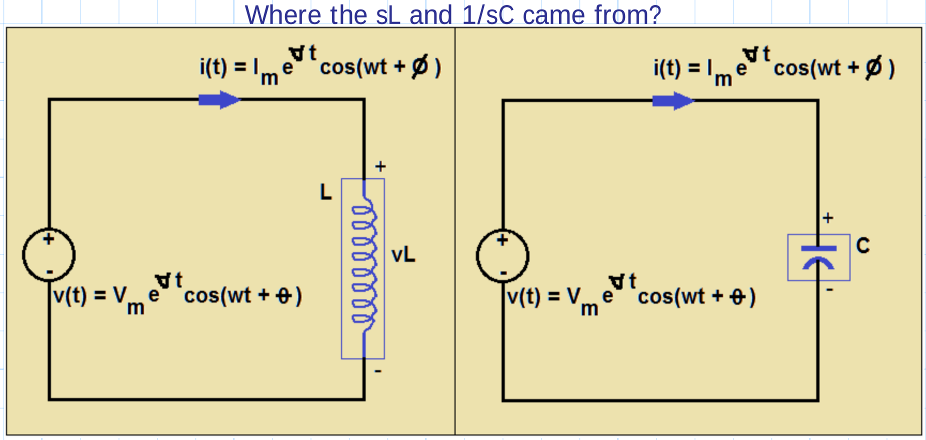
^ Here*. ^ Here*.

$L\left(\frac{di}{dt}\right)$ $\quad L \;:\; sL \quad \frac{di}{dt} : \; I(s)$ Inductor current derivative of i(t) - time domain. Its equivalent frequency domain: I(s).

$\frac{1}{C}\int i(t)\,dt \quad \frac{1}{C} : \frac{1}{sC} \quad \int i(t)\,dt \;:\; I(s)$ Capacitor current integral over a limit t - time domain. Its equivalent frequency domain: I(s).

*Figure and notes below for reference.

Where the sL and 1/sC came from?



$$v(t) = V_m e^{\sigma t}\cos(\omega t + \theta) \qquad i(t) = I_m e^{\sigma t}\cos(\omega t + \phi)$$

$$s = \sigma + j\omega \qquad\qquad s = \sigma + j\omega$$

$$v(t) = V_m e^{\sigma t}\cos(\omega t + \theta) \qquad\qquad i(t) = I_m e^{\sigma t}\cos(\omega t + \phi)$$

$$= \mathrm{Re}\left(V_m e^{j\theta} e^{st}\right) \qquad\qquad = \mathrm{Re}\left(I_m e^{j\theta} e^{st}\right)$$

Taking the real part of v(t)/i(t)
See notes in Part 3 A and B.

$$= \mathrm{Re}\left(V_m e^{st}\right) \qquad\qquad = \mathrm{Re}\left(I_m e^{st}\right)$$

Inductor:

$$v(t) = L\frac{d\left(I_m e^{st}\right)}{dt} = sL\,I_m e^{st}$$

$$\mathrm{Re}\left(V_m e^{st}\right) = sL\,I_m e^{st}$$

$$V_m e^{st} = sL\,I_m e^{st}$$

$$V_m = sL\,I_m$$

$$V = sL \cdot I$$

$$V(s) = sL \cdot I(s) \quad \text{<----}$$

Capacitor:

$$v(t) = \frac{1}{C}\int I_m e^{st}\,dt = \frac{1}{sC} I_m e^{st}$$

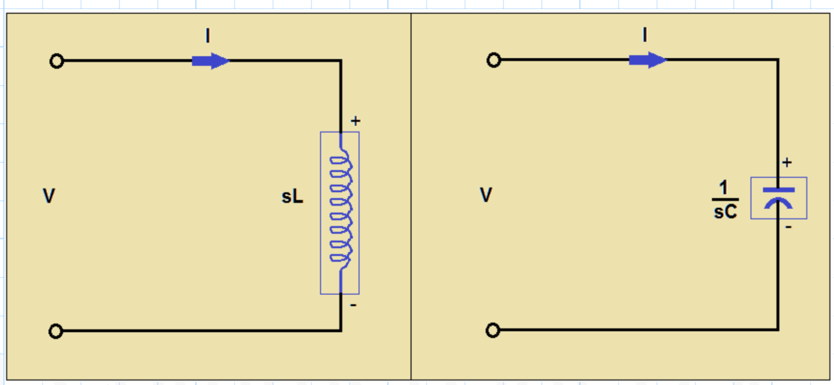
$$\mathrm{Re}\left(V_m e^{st}\right) = \frac{1}{sC} I_m e^{st}$$

$$V_m e^{st} = \frac{1}{sC} I_m e^{st}$$

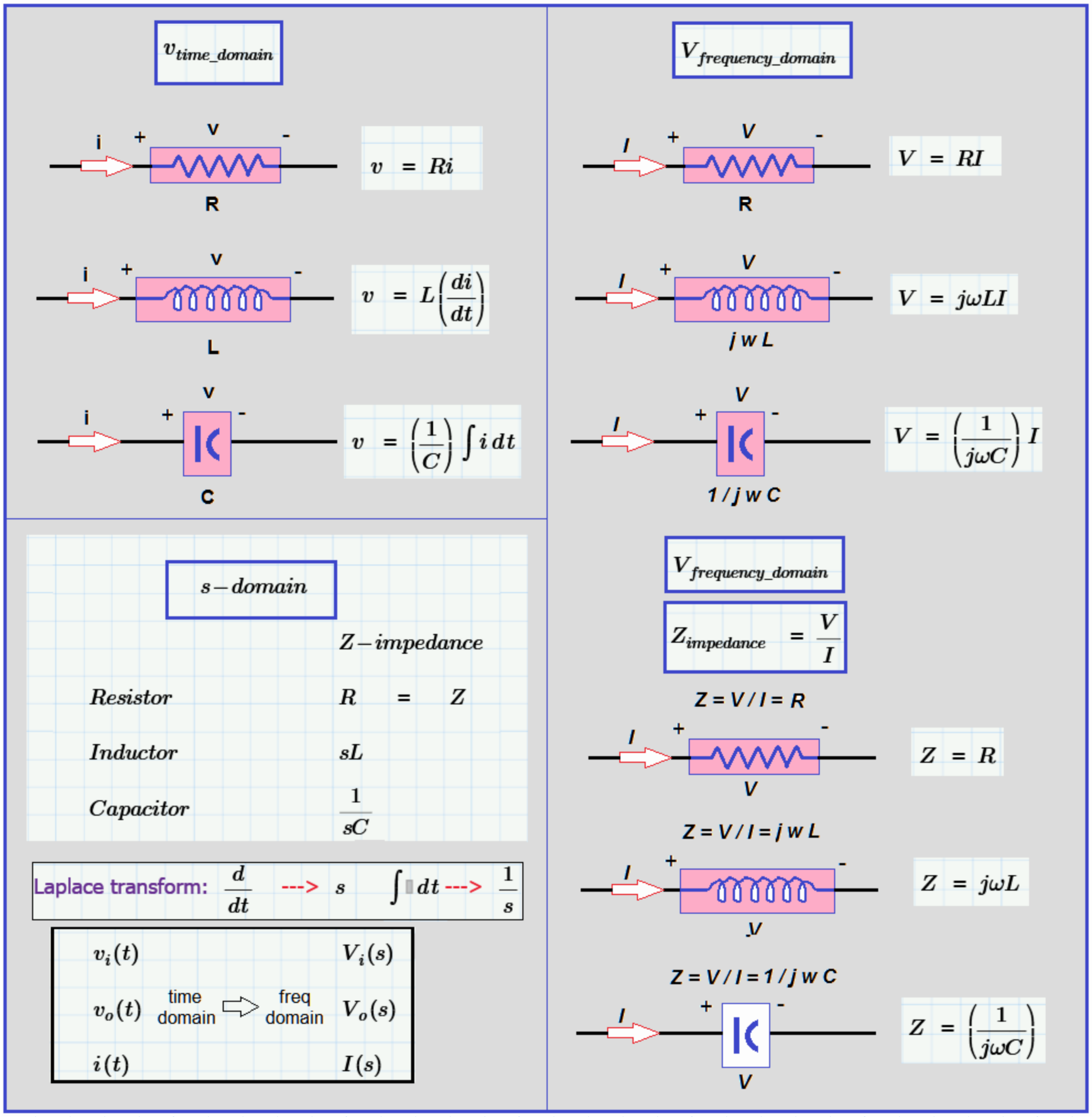
$$V_m = \frac{1}{sC} I_m$$

$$V = \frac{1}{sC} I$$

$$V(s) = \frac{1}{sC} I(s) \quad \text{<----}$$



Chapter 6 Part C. Solving Problems (Examples and Exercises). Source of study material: 1). Electric Circuits 6th Ed Schaums - Nahvi & Edminister. 2). Solutions & Problems of Control System - AK Jairath. 3). Engineering Circuits Analysis - Hyat & Kemmerly.
My Homework. This is a pre-requisite study for Laplace Transforms in circuit analysis.
Karl S. Bogha.



Derivative and Integral substitues for s and 1/s for the component L and C respectively.

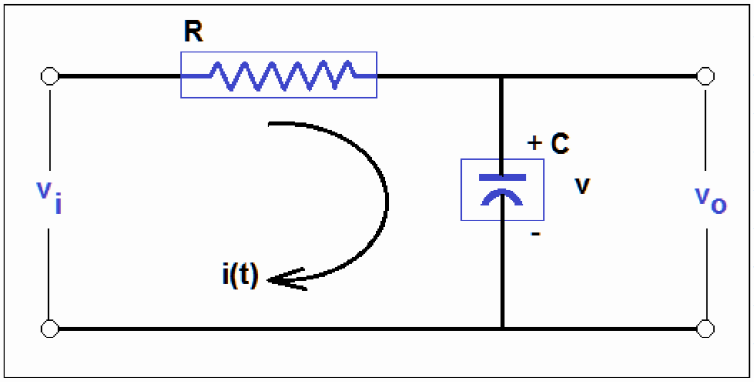
$$\frac{d}{dt} \dashrightarrow s \qquad\qquad \int . \, dt \dashrightarrow \frac{1}{s}$$

Inductor: $v_L(t) = L \cdot \frac{di}{dt}$ Capacitor: $v_C(t) = \frac{1}{C}\int i\,dt$

---> $V_L(s) = Ls \cdot I(s)$ <--- ---> $V_C(s) = \frac{1}{sC} \cdot I(s)$ <---

Engineering college year 2 course of 4 year program OR year 1 of 3 year program. Re-fresher OR Self Study. Graduate Study Review.
May be used in New Zealand, US, Malaysia, India, Pakistan, UK, and other Common Wealth Country engineering colleges.
Any errors and omissions apologies in advance.

Chp 1 Problem 1-1:



Derive the transfer function of the circuit shown in figure to the left.

Solution:

First thing is its a series circuit. We do a voltage conservation. Meaning the sum of voltages add to zero. You call that Kickoff's OR Kickout's Law.

The output is across the capacitor terminals.
The input is supply voltage for the resistor and capacitor.

$v\_i(t) \quad = \quad R \cdot i(t) + v\_C(t)$ i(t) is the circuit's current.

Set $v\_o(t) \quad = \quad v\_C(t) \quad = \quad \frac{1}{C}\int i\,dt$

$v\_i(t) \quad = \quad R \cdot i(t) + v\_o(t)$

Now we convert the expression above to the s-domain.
Which in control systems textbook they say 'Taking the Laplace transform'.
Laplace Transforms starts with transfer functions in the s-plane or in terms of complex frequency. So, thats why we used a Controls textbook. Same.

$V_i(s) \quad = \quad R\,I(s) + V_0(s)$

Vo(s) is that voltage across the capacitor C terminals, which we can set this in the s-domain of the capacitor.

$V_o(s) \quad = \quad \frac{1}{sC} \cdot I(s)$ <--- C: 1/sC, and i(t): I(s).

Its more than forming a loop equation, we want to all the required variables in the expression so we can form that Vo(s)/Vi(s).

How do we know what all terms and their forms we need before we can get to forming a transfer function?

Keep working in more and more example problems, partially looks like a guess, but after a few examples we get the general idea.

The Electrical Engineering expressions for defining components are formed in such a way that they have a future in advanced math where they can be manipulated in various ways to take full benefit of the math resulting in some output that serves a circuit's purpose - Karl Bogha.

$$V_o(s) = \frac{1}{sC} \cdot I(s)$$

$$I(s) = V_o(s) \cdot sC$$

$$V_i(s) = RI(s) + V_0(s)$$

<--- Lets plug in or if you prefer substitute the expression we got into this expression we formed earlier.

$$V_i(s) = R\langle V_o(s) \cdot sC \rangle + V_o(s)$$

$$= sRC\langle V_o(s) \rangle + V_o(s)$$

$$V_i(s) = V_o(s) \cdot (sRC + 1)$$

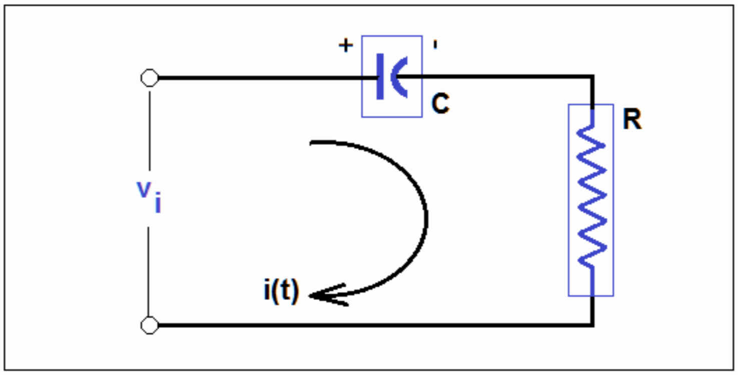
$$= V_o(s) \cdot (1 + sRC)$$

<--- How would we had known that?
Surely had to work examples.
Keep clear of people and peers who say dont do the example go to the end of chapter problems, they lie so they have the edge - Engineer.
In the work place you never ever get problems to solve like hard end of chapter problems in hard core engineering textbooks, fake, it rarely help, most time you got all the time in the world - Karl Bogha.

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}$$

Answer.

Chp 1 Problem 1-2:



We seek the transfer function I(s)/Vi(s) ?

Solution:

First thing is its a series circuit. We do a voltage conservation, meaning the sum of voltages add to zero. You call that *Kickoff's* Law!

$$v\_i(t) = R \cdot i(t) + v\_C(t)$$

i(t) is the circuit's current.

$$v\_C(t) = \frac{1}{C}\int i\,dt = \frac{1}{sC} I(s)$$

$$v\_i(t) = R \cdot i(t) + v\_C(t)$$

$$V_i(s) = R I(s) + \frac{1}{sC} I(s)$$

$$= R I(s) + \frac{1}{sC} I(s)$$

$$= I(s)\left(R + \frac{1}{sC}\right)$$

$$\frac{I(s)}{V_i(s)} = \frac{1}{\left(R + \frac{1}{sC}\right)}$$

Simplify this term, multiply by sC/R.

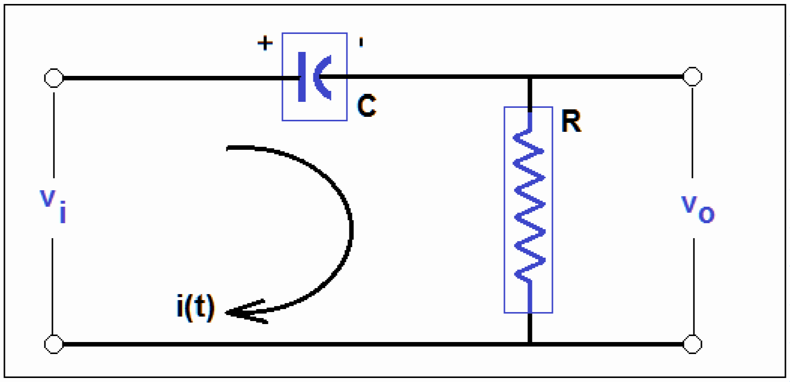
$$\frac{I(s)}{V_i(s)} = \frac{\left(\frac{sC}{R}\right)}{\frac{sC}{R}\left(R + \frac{1}{sC}\right)} = \frac{\left(\frac{sC}{R}\right)}{sC + \frac{1}{R}} = \left(\frac{1}{sC + \frac{1}{R}}\right)\frac{sC}{R} = \left(\frac{sC}{sCR + 1}\right)$$

$$\frac{I(s)}{V_i(s)} = \left(\frac{sC}{1 + sCR}\right)$$

Answer.

Good if we can work the final form of expression like this instead of the one a few steps before. It takes some extra effort to get it in a neat form that is more electric circuit friendly and meaningful.

Chp 1 Problem 1-3:



We seek the transfer function Vo(s)/Vi(s) ?

Solution:

Conservation of voltage means something else?
I am not sure, when conserved it would remain the same.
So the sum equal zero in a loop. That for me is conserved.
Maybe they used it for something else. Usuall I am not the first.

We kickoff with the voltage conservation.

$$V_i(t) = R\,i(t) + \frac{1}{C}\int i(t)\,dt$$

$$\frac{1}{C}\int i(t)\,dt = \frac{1}{sC}\cdot I(s)$$

$$V_i(s) = R\,I(s) + \frac{I(s)}{sC}$$

Our circuit identifies voltage across resistor terminals as Vo(t) which now becomes? Vo(s) for the frequency domain.

$$V_o(s) = R\,I(s)$$

$$I(s) = \frac{V_o(s)}{R}$$ Substitute in here: $$V_i(s) = R\,I(s) + \frac{I(s)}{sC}$$

$$V_i(s) = R\left(\frac{V_o(s)}{R}\right) + \left(\frac{V_o(s)}{R}\right)\cdot\frac{1}{sC}$$ Isolate Vo(s)

$$V_i(s) = V_o(s)\cdot\left(1 + \frac{1}{sCR}\right)$$

Next for the required transfer function:

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\left(1 + \frac{1}{sCR}\right)}$$

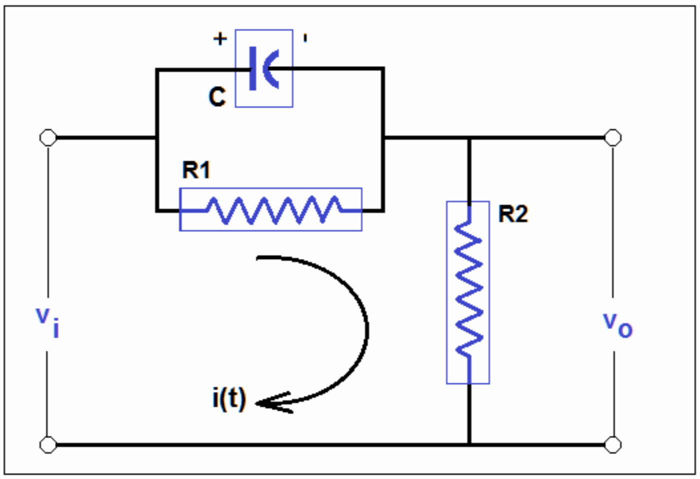
<--- This can be simplified.
Its awkward, that is why we simplify these awkward terms.

Multiply by sCR:

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\left(1 + \frac{1}{sCR}\right)} \cdot \frac{sCR}{sCR} = \frac{sCR}{(sCR + 1)}$$

Answer.

Chp 1 Problem 1-4:



We seek the transfer function, Vo(s)/Vi(s), of the electrical network shown to the left in phase lead form ?

Solution:

Z1 is the parallell of C and R1:

$$\frac{1}{Z1} = \frac{1}{R1} + \frac{1}{\frac{1}{j\,\omega C}} = \frac{1}{R1} + \frac{1}{\frac{1}{sC}}$$

$$s = \sigma + j\,\omega$$
$$\sigma = 0$$
$$s = j\,\omega$$

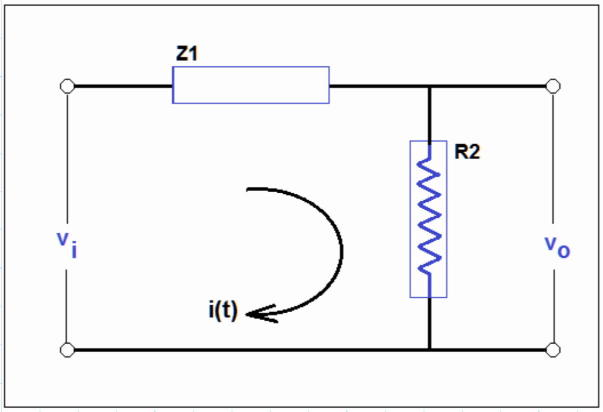
We are concerned with frequency, so we can set sigma = 0.

$$\frac{1}{Z1} = \frac{1}{R1} + sC = \frac{1}{R1} + \frac{sC}{1}$$ multiply by R1

$$\frac{1}{Z1} = \frac{R1}{R1} + \frac{sCR1}{1} = \frac{R1 + sCR1}{1 \cdot R1} = \frac{R1 + sCR1}{R1}$$

$$\frac{1}{Z1} = \frac{R1}{R1} + \frac{sCR1}{R1} = 1 + \frac{sCR1}{R1} = \frac{1 + sCR1}{R1}$$

$$Z1 = \frac{R1}{1 + sCR1}$$ After inverting.



We kickoff with the voltage conservation.

$$v_i(t) = Z1I(s) + R2I(s)$$

$$v_o(t) = R2I(s)$$

Taking the Laplace Transform of the above 2 equation:

$$V_i(s) = Z1I(s) + R2I(s)$$

$$V_o(t) = R2I(s)$$ Plug in equation above

$$I(s) = \frac{V_o(s)}{R2}$$ Plug in equation above

$$V_i(s) = Z1\left(\frac{V_o(s)}{R2}\right) + V_o(s)$$

$$V_i(s) = V_o(s) \cdot \left(\frac{Z1}{R2} + 1\right)$$ Plug in Z1

$$V_i(s) = V_o(s) \cdot \left(\frac{\left(\frac{R1}{1 + sCR1}\right)}{R2} + 1\right)$$

$$= V_o(s) \cdot \left(\frac{\left(\frac{R1}{1 + sCR1}\right)}{R2} + \frac{R2}{R2}\right)$$

$$= V_o(s)\cdot\left(\frac{\left(\frac{R1}{1+sCR1}\right)+R2}{R2}\right)$$

Next rearrange and multiply by ---> $\frac{1+sCR1}{1+sCR1}$

$$= \frac{V_o(s)}{R2}\cdot\left(\left(\frac{R1}{1+sCR1}\right)+\frac{R2\cdot(1+sCR1)}{(1+sCR1)}\right)$$

$$= \frac{V_o(s)}{R2}\cdot\left(\frac{R1+R2+sCR1R2}{1+sCR1}\right)$$

$$= V_o(s)\left(\frac{R1+R2}{R2}\right)\left(\frac{1+sCR1R2}{1+sCR1}\right)$$

...not finished yet in this expression.

$$V_i(s) = V_o(s)\left(\frac{R1+R2}{R2}\right)\left(\frac{1+\frac{sCR1R2}{R1+R2}}{1+sCR1}\right)$$

Place $\frac{1}{R1+R2}$ in there so it cancels the middle term (R1+R2)/R2 when multiplied.

$$\frac{V_i(s)}{V_o(s)} = \left(\frac{R1+R2}{R2}\right)\left(\frac{1+\frac{sCR1R2}{R1+R2}}{1+sCR1}\right)$$

Next invert both sides.

$$\frac{V_o(s)}{V_i(s)} = \left(\frac{R2}{R1+R2}\right)\left(\frac{1+sCR1}{1+\frac{sCR1R2}{R1+R2}}\right)$$

As provided in textbook.

$$\frac{V_o(s)}{V_i(s)} = \left(\frac{R2}{R1+R2}\right)\left(\frac{1+sCR1}{1+\left(\frac{R2}{R1+R2}\right)sCR1}\right)$$

Transfer function.

We can simplify a little. Make RC the time constant in a series circuit = tau, and make the constant R2/(R1+R1) = a. OR just any constant T.

$$T = CR1$$

$$a = \frac{R2}{R1+R2}$$

$$\frac{V_o(s)}{V_i(s)} = a\left(\frac{1+sT}{1+asT}\right)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{a\cdot(1+sT)}{(1+a\ s\ T)}$$

Answer.

Comment: Previous example problems used T for RC in the final transfer functions.
I left it out because my aim was the approach on how to get the transfer functions. T is not necessarily a time constant for this circuit.
You can verify. We could use P of Q but since its RLC, T or tau makes more sense.

Took time with the algebra otherwise a good easy example for most.

Chap 1 Problem 1.7 :

I jump to problem 1.7 because its the same circuit. This provides a continuity and not having to return later after several problems.

Derive the transfer function of the circuit shown (same circuit of problem 1.4).
If v_i(t) = 8 sin(10t) V, R1 = 50 k Ohms, R2 = 5 k Ohms and C = 1 uF.

Calculate the output voltage in magnitude and phase angle relative to input voltage?

Solution:

Gain $G(s) = \dfrac{V_o(s)}{V_i(s)} = \left(\dfrac{R2}{R1+R2}\right)\left(\dfrac{1+sCR1}{1+\left(\dfrac{R2}{R1+R2}\right)sCR1}\right)$

$k := 10^3$ $\quad M := 10^6$ $\quad u := 10^{-6}$

$R1 := 50\ k$ $\quad R2 := 5\ k$ $\quad C := 1\ u$

Substitute into transfer function:

$$G(s) = \frac{V_o(s)}{V_i(s)} = R2\cdot\frac{(1+sCR1)}{(R1+R2)+(sCR1R2)}$$

$$= 5000\left(\frac{1+0.05\ s}{55000+250\ s}\right)$$ Divide numerator and denominator by 55,000.

$$= 0.091\left(\frac{1+0.05\ s}{1+0.0045\ s}\right)$$

$$G(s) = 0.01\frac{(1+0.05\ s)}{(1+0.0045\ s)}$$ Constant 0.091 rounded off to 0.01

Zero: $(1+0.05\ s)$
Pole: $(1+0.0045\ s)$

We are interested in s = 0 + jw, where sigma =0.
Hence we can analyse the frequency response.
Substitute s for jw in transfer function.
Now we have 1+0.05s and 1+0.0045s, this gives us the magnitude and angle for both. Since we have a real and imaginary part.

$s = \sigma + j\ \omega$
$\sigma = 0$
$s = 0 + j\ \omega = 0$

$$G(j\omega) = 0.01\frac{(1+0.05\ j\ \omega)}{(1+0.0045\ j\ \omega)}$$

Before we can calculate the angles we need the value of w ?

$v(t) = 8\sin(10 \cdot t)$ ---> $A\sin(\omega t)$

$\omega = 10$

Zero: $(1 + 0.05\ j10) = 1 + 0.5j$
Pole: $(1 + 0.0045\ j10) = 1 + 0.045j$

$Z\_Ang\_G\_s := \operatorname{atan}\left(\frac{0.5}{1}\right) = 26.5651$ **deg**

$P\_Ang\_G\_s := \operatorname{atan}\left(\frac{0.045}{1}\right) = 2.5766$ **deg**

$Ang\_G(s) = 26.565 - 2.577 = 23.988$ degrees. Answer.

Now for the magnitude of the transfer function,
here is where the constant 0.1 is applied.

Magnitude of zero: $\sqrt{1^2 + 0.5^2} = 1.118$

Magnitude of pole: $\sqrt{1^2 + 0.045^2} = 1.001$

Magnitude of G(s): $(0.1) \cdot \left(\frac{1.118}{1.001}\right) = 0.1117$

The input signal is vi(t) = 8 sin (wt)
From which we can obtain the amplitude is 8 V maximum.
We next multiply the magnitude of G(s) to 8V for the maximum output voltage.

$Amplitude := 8.0 \quad Mag\_G(s) := 0.1117$

$V_o := Amplitude \cdot Mag\_G(\mathbf{s}) = 0.894$ V. Answer.

Good example. Can be found in most circuits and all controls textbook.

Chap 1 Problem 1.8 :

Problem 1.8 is next here because it works on the same transfer function of problem 1.4. This is indicated in the problem statement, exact same circuit.

If C = 1uF in the circuit of problem 1.4.
What values of R1 and R2 will give T = 0.6 sec, and a = 0.1

Solution:

$$G(s) = \left(\frac{R2}{R1+R2}\right)\left(\frac{1+sCR1}{1+\left(\frac{R2}{R1+R2}\right)sCR1}\right) \qquad T = CR1 \qquad a = \frac{R2}{R1+R2}$$

$$G(s) = \frac{a\cdot(1+sT)}{(1+asT)} \qquad C := 1\ u\ F \qquad T := 0.6 \qquad a := 0.1$$

CR1 = 0.6, solve for R1: $CR1 = 0.6$

$$(1\ uF)\ R1 = 0.6$$

$$R1 = \frac{0.6}{1\cdot u} = 6\cdot 10^{5}\ \text{Ohm.} = 0.6\cdot M\ \text{Ohm.}$$ Answer.

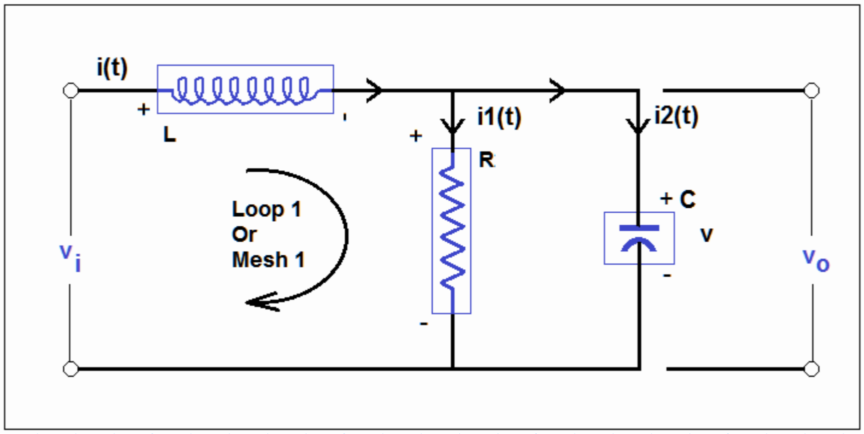
$$a = \frac{R2}{R1+R2} \quad \text{--->} \quad 0.1 = \frac{R2}{600000+R2} \quad \text{--->} \quad 0.1\ (600000+R2) = R2$$

$$60000 + 0.1\ R2 = R2$$

$$0.9\ R2 = 60000$$

$$R2 = \frac{60000}{0.9} = 66666.7 = 0.066\ \text{M Ohms.}$$ Answer.

Chp 1 Problem 1-5:



We seek the transfer function, Vo(s)/Vi(s), of the electrical network shown to the left in phase lead form ?

Solution:

Current at node: $i(t) = i1(t) + i2(t)$

Voltage conservation in loop at left side:

$$v_i(t) = L\frac{di}{dt} + Ri1(t)$$

Next, in a cleaver way, we pull in the v_o(t) relationship thru the capacitor voltage, where C is voltage across resistor R, and we know v_o(t) is the voltage across the capacitor.

$$v_o(t) = Ri1(t) = \frac{1}{C}\int i2(t)\,dt$$

$$v_i(t) = L\frac{di}{dt} + Ri1(t)$$

$$V_i(s) = sLI(s) + RI1(s)$$

$$RI1(s) = \frac{1}{C}\int i2(t)\,dt = \frac{1}{sC}I2(s)$$

$$RI1(s) = V_o(s) = \frac{I2(s)}{sC}$$

Voltage across R: $V_o(s) = \dfrac{I2(s)}{sC} = R\,I1(s)$ thus $I1(s) = \dfrac{V_o(s)}{R}$

We update our I(s)expression here $V_i(s) = sL\,I(s) + R\,I1(s)$

$$i(t) = i1(t) + i2(t)$$

$$I(s) = I1(s) + I2(s)$$

$$V_i(s) = sL\,(I1(s) + I2(s)) + R\,I1(s)$$

$$V_i(s) = sL\,(I1(s) + I2(s)) + V_o(s)$$

Substitute voltage across C for R: $V_o(s) = \dfrac{I2(s)}{sC}$

$$sCV_o(s) = I2(s)$$

$$V_i(s) = sL\left(\frac{V_o(s)}{R} + sCV_o(s)\right) + V_o(s)$$

$$V_i(s) = V_o(s) + sL\left(\frac{V_o(s)}{R} + sCV_o(s)\right)$$

$$V_i(s) = V_o(s) + V_o(s)\cdot\left(\frac{sL}{R} + sCsL\right)$$

$$V_i(s) = V_o(s)\cdot\left(1 + \frac{sL}{R} + s^2\ LC\right)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\left(1 + \frac{sL}{R} + s^2\ LC\right)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\left(s^2\ LC + \frac{sL}{R} + 1\right)}$$

Answer. Lots of substitutions.
A compact answer below.

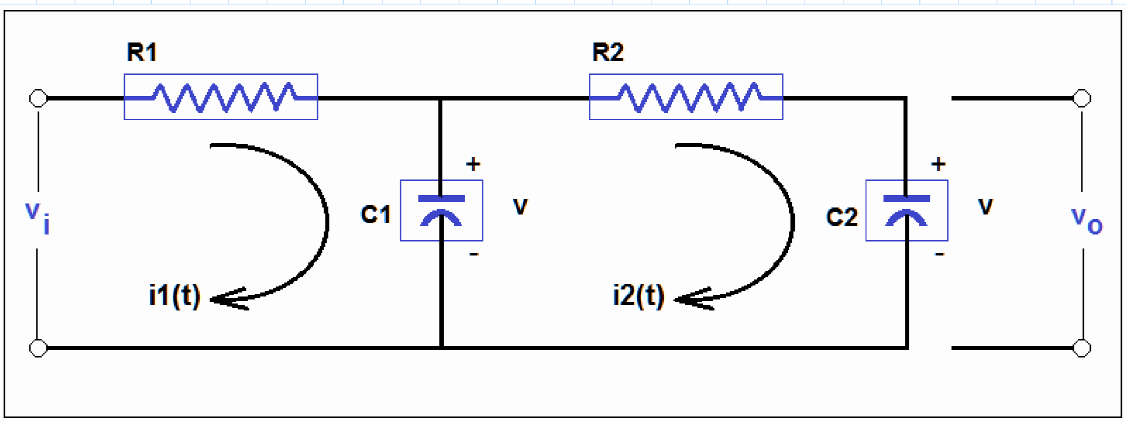
The Engineer makes the expression simpler in appearance, quadratic expression, thru the use of variable T1 and T2. T1 = L/R maybe a time contant but not here. T2 = CR which is NOT a time constant, you verify should it be of concern.

$$T1 = \frac{L}{R} \qquad T2 = CR$$

$$T1T2 = \left(\frac{L}{R}\right)(CR) = LC$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\left(T1T2s^2 + T1s + 1\right)}$$ Answer.

Chp 1 Problem 1-6:



We seek the transfer function, Vo(s)/Vi(s), of the electrical network shown above ?

Solution:

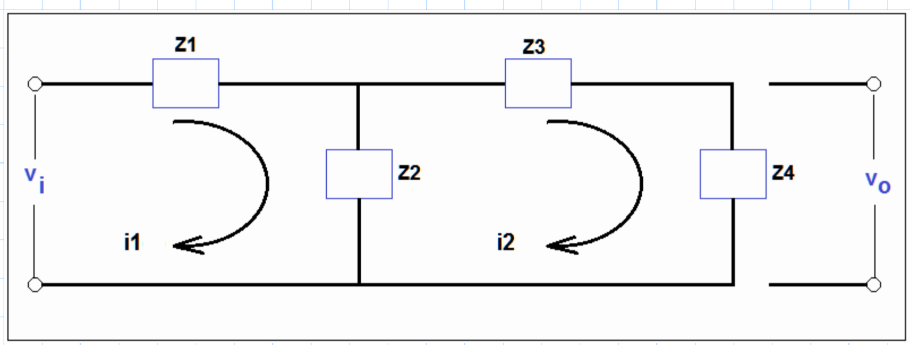
Set up the impedance Z for each component:

$$Z1 = R1$$

$$Z2 = \frac{1}{sC1}$$

$$Z3 = R2$$

$$Z4 = \frac{1}{sC2}$$



Voltage mesh/loop equations in Laplace:

Left loop:

$$V_i(s) = Z1I1(s) + Z2(I1 - I2)$$

$$V_i(s) = I1(s)\ (Z1 + Z2) - Z2I2 \qquad \text{...Eq 1}$$

Right loop:

$$0 = Z2(I2 - I1) + Z3I2(s) + Z4I2(s)$$

$$0 = -Z2I1 + I2(s)\ (Z2 + Z3 + Z4) \qquad \text{...Eq 2}$$

Next we form an expression for Vo:

$$V_o(s) = Z4I2(s) \qquad \text{...Eq 3}$$

If I am correct, from these few examples we seen, we want to place one expression for current, into the the other equation, then work towards the transfer function, provided we have Vo(s) and Vi(s) in that expression to work with.

Here, I1(s) looks the better simpler choice to place in Eq 2.
Because we do not have a voltage source on the RHS.
Then we set Vo(s) for Z4I2(s).
Then work with the equation which can fit-in Vo, Vi, and I1 and I2 in it.
If we dont have it yet continue re-hashing.
What you think, that's the plan? *Of course!*

$$0 = -Z2I1 + I2(s)\ (Z2 + Z3 + Z4) \qquad \text{...Eq 2}$$

$$I1(s)\ Z2 = I2(s)\ (Z2 + Z3 + Z4)$$

$$I1(s) = \frac{I2(s)\ (Z2 + Z3 + Z4)}{Z2}$$

$$V_i(s) = I1(s)\ (Z1 + Z2) - Z2I2 \qquad \text{...Eq 1, substitute I1(s)}$$

$$V_i(s) = \frac{I2(s)\ (Z2 + Z3 + Z4) \cdot (Z1 + Z2)}{Z2} - Z2I2(s)$$

Fix for Z2 at very right of numerator.

$$V_i(s) = \frac{I2(s) \cdot ((Z2 + Z3 + Z4) \cdot (Z1 + Z2))\ Z2}{Z2} - \frac{Z2Z2I2(s)}{Z2}$$

$$V_i(s) = \frac{I2(s) \cdot \left(((Z2 + Z3 + Z4) \cdot (Z1 + Z2))\ Z2 - Z2^2\right)}{Z2}$$

$$\frac{V_i(s)}{I2(s)} = \frac{((Z2 + Z3 + Z4) \cdot (Z1 + Z2))\ Z2 - Z2^2}{Z2}$$

For me this is new, not a twist but certainly new I dont remember doing a substitution on the LHS! Ok Not typical. Hope I am gaining skills here.

$$V_o(s) = Z4I2(s) \qquad \text{...Eq 3}$$

$$I2(s) = \frac{V_o(s)}{Z4}$$

Substitute this in the expression Vi(s)/I2(s)

$$\frac{V_i(s)}{\left(\dfrac{V_o(s)}{Z4}\right)} = \frac{\left((Z2 + Z3 + Z4) \cdot (Z1 + Z2) - Z2^2\right)}{Z2}$$

$$\frac{V_i(s)}{V_o(s)} = \frac{\left((Z1 + Z2) \cdot (Z2 + Z3 + Z4) - Z2^2\right)}{Z2 \cdot Z4}$$

Invert the expression so we get Vo(s) in the numerator.

$$\frac{V_o(s)}{V_i(s)} = \frac{Z2 \cdot Z4}{\left((Z1 + Z2) \cdot (Z2 + Z3 + Z4) - Z2^2\right)}$$

Lets expand the denominator expression:

$$(Z1 + Z2) \cdot (Z2 + Z3 + Z4) = Z1Z2 + Z1Z3 + Z1Z4 + Z2Z2 + Z2Z3 + Z2Z4$$

Now for the full denominator expression:

$$= Z1Z2 + Z1Z3 + Z1Z4 + Z2Z2 + Z2Z3 + Z2Z4 - Z2Z2$$

$$= Z1Z2 + Z1Z3 + Z1Z4 + Z2Z3 + Z2Z4$$

$$\frac{V_o(s)}{V_i(s)} = \frac{Z2 \cdot Z4}{Z1Z2 + Z1Z3 + Z1Z4 + Z2Z3 + Z2Z4}$$ The transfer function.

Next we substitute the values of impedances Z1...Z4:

$$\frac{V_o(s)}{V_i(s)} = \frac{Z2 \cdot Z4}{Z1Z2 + Z1Z3 + Z1Z4 + Z2Z3 + Z2Z4}$$

$$Z1 = R1 \qquad Z2 = \frac{1}{sC1} \qquad Z3 = R2 \qquad Z4 = \frac{1}{sC2}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{sC1} \cdot \frac{1}{sC2}}{\frac{R1}{sC1} + R1R2 + \frac{R1}{sC2} + \frac{R2}{sC1} + \frac{1}{s^2\ C1C2}}$$

As usual these types expressions are made simpler, especially in electric circuits.
It helps in building the physical circuit. *Which I almost forgot the true purpose here. We are building circuits and components are to be put together on a bread board for testing. Hello?..true purpose? Why not?*

$$= \left(\frac{1}{\frac{R1}{sC1} + R1R2 + \frac{R1}{sC2} + \frac{R2}{sC1} + \frac{1}{s^2\ C1C2}}\right) \cdot \frac{1}{sC1} \cdot \frac{1}{sC2}$$

$$= \frac{1}{R1sC2 + R1R2 \cdot s^2\ C1C2 + R1sC1 + R2sC2 + 1}$$ Multiplied by sC1 sC2 top and bottom.

$$= \frac{1}{sR1C2 + s^2\ R1R2 \cdot C1C2 + sR1C1 + sR2C2 + 1}$$

$$= \frac{1}{sR1C2 + sR1C1 + sR2C2 + 1 + s^2\ R1R2 \cdot C1C2}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + s(R1C2 + R1C1 + R2C2) + s^2\ R1R2 \cdot C1C2}$$ Answer.

The denominator is a neat 2nd order expression.
The circuit is also a practical circuit for application in electric circuits, electronics and other electrical/electronic applications.

Chp 1 Problem 1.11:

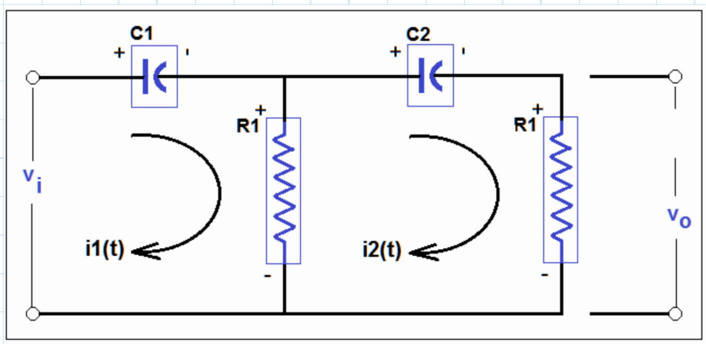
Problem 1.11 comes here because this problem has a similar transfer function to problem 1.6. As indicated in the problem statement of 1.11. The changes being only in the arrangement of components, that being the swap between R and C.

Determine the transfer function relation Vo(s) to Vi(s) for the circuit.
Calculate output voltage t>>0 for a unit step voltage input at t=0.

Solution:

In 1.6 we used the impedance Z to construct the transfer function. Later we plugged in the values for Z's. So thats why this transfer function is relevant.

$$\frac{V_o(s)}{V_i(s)} = \frac{Z2 \cdot Z4}{Z1Z2 + Z1Z3 + Z1Z4 + Z2Z3 + Z2Z4}$$ <--- From problem 1.6



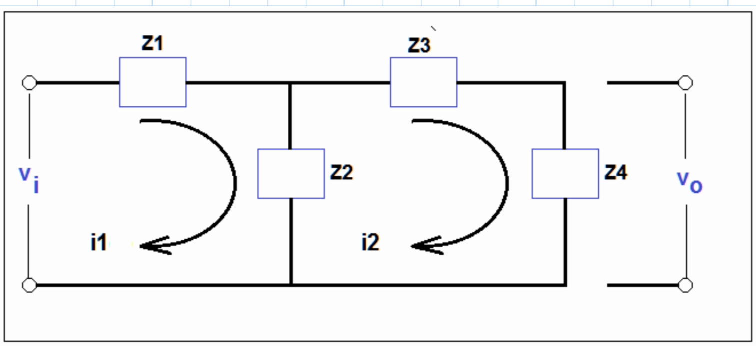
<--- This is the circuit for problem 1.8.

$u := 10^{-6}$ $M := 10^{6}$

$C1 := 1\ u\ \ F$ $C2 := 0.5\ u\ \ F$

$R1 := 1\ M$ $R2 := 1\ M$

The Z impedance circuit becomes:



We make 10^6 the common multiplier for resistors and 10^-6 for capacitor. Now we only need work with the simple numbers.

$Z1 = \frac{1}{s}$ $Z2 = 1$

$Z3 = \frac{1}{0.5\ s}$ $Z4 = 1$

$$\frac{V_o(s)}{V_i(s)} = \frac{Z2 \cdot Z4}{Z1Z2 + Z1Z3 + Z1Z4 + Z2Z3 + Z2Z4}$$

$$= \frac{1 \cdot 1}{\left(\frac{1}{s}\right) \cdot 1 + \left(\frac{1}{s}\right)\left(\frac{1}{0.5\ s}\right) + \left(\frac{1}{s}\right)(1) + (1)\left(\frac{1}{0.5\ s}\right) + (1)\ (1)}$$

$$= \frac{1}{\frac{1}{s} + \left(\frac{1}{0.5\, s^2}\right) + \left(\frac{1}{s}\right) + \left(\frac{1}{0.5\, s}\right) + 1}$$ Multiply by s^2

$$= \frac{s^2}{s + 2 + s + 2\, s + s^2} = \frac{s^2}{2 + 4\, s + s^2} = \frac{s^2}{s^2 + 4\, s + 2}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{s \cdot s}{s^2 + 4\, s + 2}$$

Unit step voltage comes on at t=0 and is of unit value, ie 1. Vi(s) must equal 1.

$$V_o(s) = \frac{V_i(s) \cdot s \cdot s}{s^2 + 4\, s + 2} = \frac{1 \cdot s \cdot s}{s^2 + 4\, s + 2}$$

$$s_{z1} := 1 \quad V_o(s) = \frac{s}{s^2 + 4\, s + 2}$$

$$ax^2 + bx + c \quad : \quad s^2 + 4\, s + 2$$

$$s1 = \frac{-b - \sqrt{b^2 - 4\, ac}}{2\, a} = \frac{-4 - \sqrt{4^2 - 4\, 1\, 2}}{2\, 1} = -3.4142$$

$$s2 = \frac{-b + \sqrt{b^2 - 4\, ac}}{2\, a} = \frac{-4 + \sqrt{4^2 - 4\, 1\, 2}}{2\, 1} = -0.5858$$

We solved the denominator for the poles. Which math wise were the roots but electrical wise these are the poles.

$$V_o(s) = \frac{s^2}{(s + 3.414) \cdot (s + 0.586)}$$

The poles going back in the transfer function with the opposite sign.

For the pole to be maximum s1 and s2? -3.414 and -0.586

What about the numerator what any value to solve?
Its NOT the numerator its the COEFFICIENTS of Vo(s) and those same for time domain.

At t<0 Vo(<0) = 0, and t>0 Vo(>0) = 0, but for t>>0 Vo(>>0) = 1u(t).
At -0 its near same as 0+ equal 0. So we use continuity here?
No, basically math. To solve for coefficients using the?
Method of proper fractions OR Equating coefficients of like powers.

Next calculate the coefficients.

$$V_o(s) = \frac{s^2}{(s+3.414)\cdot(s+0.586)}$$ Split LHS to solve for coefficients.

$$\frac{s\cdot s}{(s+3.414)\cdot(s+0.586)} = \frac{A}{(s+3.414)} + \frac{B}{(s+0.586)}$$ 2nd order eq.

$$A(s+0.586) + B(s+3.414) = As + 0.586\ A + Bs + B3.414$$

Arrange like terms:

*s below is numerator term in transfer function - s*s split to s*s.*
One 's' for 1 equation (As+Bs) = 1 <---coefficient of s = 1. Like terms.

$As + Bs = s$ ---> $A + B = 1$ Eq 1
$0.586\ A + 3.414\ B = 0$ ---> $0.586\ A + 3.414\ B = 0$ Eq 2

$0.586\ A + 0.586\ B = 0.586$ Eq 1 x 0.586...Eq 3
$0.586\ A + 3.414\ B = 0$ Eq 2

$(0.586 - 3.414)\ B = 0.586$ Eq 3 - 2
$(0.586 - 3.414) = -2.828$
$-2.828\ B = 0.586$

$$B = \frac{0.586}{-2.828} = -0.2072$$

$A + B = 1$
$A - 0.207 = 1$
$A = 1 + 0.207 = 1.207$

The circuit s-domain: $$V_o(s) = \frac{A}{(s+3.414)} + \frac{B}{(s+0.586)}$$

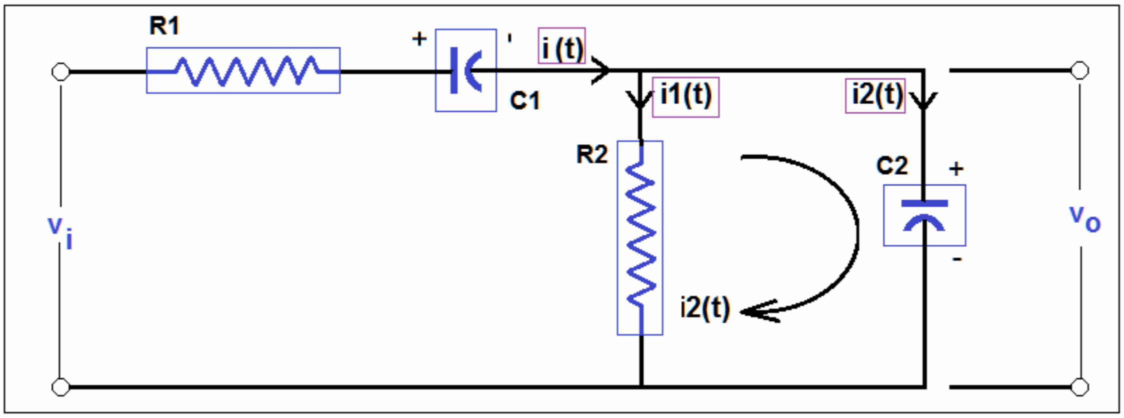
$$V_o(s) = \frac{1.21}{(s+3.414)} - \frac{0.21}{(s+0.586)}$$

The general form of v_o(t): $Ae^{-s1\cdot t} + Be^{-s1\cdot t}$

$$v_o(t) = 1.21\ e^{-3.414\cdot t} - 0.21\ e^{-0.586\cdot t}$$ Answer.

Interesting solution math wise. What math can do for determining coefficients by 'equating coefficients of like powers'.

Chp 1 Problem 1.9:



Find the transfer function of the network shown in figure above.
Plot its poles and zeros for R1 = R2 = 1, and C1 = C2 = 1.

Solution:

Current equation at node:

$i(t) \quad = \quad i1(t) + i2(t)$ Note: Current thru R1 and C1 is i(t).

Voltage mesh equations:

$$v_i(t) \quad = \quad R1i(t) + \frac{1}{C1} \cdot \int i\, dt + R2 \cdot ((i1(t) - i2(t)))$$

Deviation here: $R2 \cdot ((i1(t) - i2(t)))$ we neglect i2(t) leaving ---> $R2i2(t)$
Shown later.

Voltage across R2 is Vo(t).
Form the voltage mesh equation using Vo(t).

$$v_o(t) \quad = \quad R2 \cdot ((i2(t) - i1(t))) + \frac{1}{C2} \cdot \int i2(t)\, dt$$

We may not need this mesh equation.

We did this just so we see the time domain equations, we could have started with to s-domain as we did in other example(s).

Now for converting to s-domain,
in other words taking the Laplace Transform:

$$I(s) \quad = \quad I1(s) + I2(s)$$

$$V_i(s) = R1I(s) + \frac{1}{sC1} I(s) + R2\cdot(I1(s) - I2(s))$$

$$V_o(s) = R2\cdot(I2(s) - I1(s)) + \frac{1}{sC2}\cdot I2(s)$$ We may not need this equation.

The voltage across C2 is the same across R2.
This is the voltage v_o(t) or Vo(s).
We can use this voltage expression and plug into the Vi(s) equation.
Obviously we want to plug in for R2I1(s).

$$v_o(t) = \frac{1}{C2}\int i2(t)\,dt$$

$$= R2\cdot i1(t)$$ Here we do not do a mesh method on the current thru R2. We simply identify it to i1(t), since its the voltage across the resistor terminals equated to v_o(t).

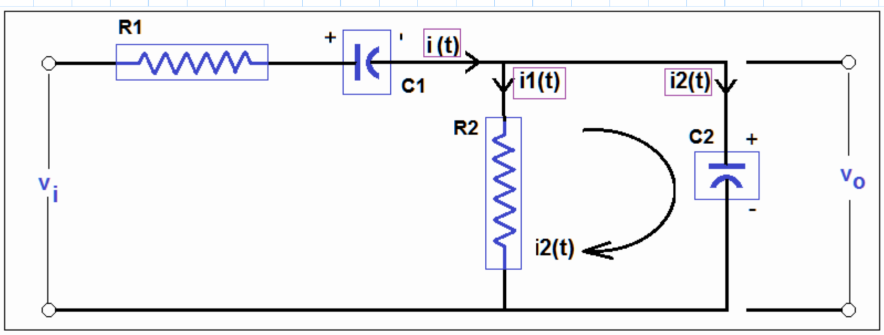
Their Laplace transform:

$$Vo(s) = \frac{1}{sC2} I2(s)$$

$$= R2\cdot I2(s)$$

$$V_i(s) = R1I(s) + \frac{1}{sC1} I(s) + R2\cdot(I1(s) - I2(s))$$ The main equation now.

$$V_i(s) = R1I(s) + \frac{1}{sC1} I(s) + R2\cdot I1(s) - R2I2(s)$$ Plug in Vo at R2I1(s)



$$V_i(s) = R1I(s) + \frac{1}{sC1} I(s) + V_o(s) - R2I2(s)$$

Mesh or voltage loop problem, stated earlier below.
Deviation here: $R2\cdot((i1(t) - i2(t)))$ we neglect i2(t) leaving ---> $R2i2(t)$

Few attempts to find a substitute for R2I2(s) was not obtained.
The equation, voltage conservation, by the author-engineer did not include the i2(t) expression for R2. The engineer is taking i1(t) as a known current or on its own.
So there is no need for R2(i1(t) - i2(t)), rather just R2i1(t).

The engineer's solution stated the assumption current distribution as shown below.
I did it taking two loops, mesh equations, until I knew why. Otherwise the assumption would not been clear to me. Thus I leave it as it is, with correction continued below.

The improved or updated voltage equation becomes:

$$v_i(t) = R1i(t) + \frac{1}{C1}\cdot\int i\,dt + R2\cdot((i1(t) - i2(t)))$$

$$v_i(t) = R1i(t) + \frac{1}{C1}\cdot\int i\,dt + R2\cdot i1(t)$$

$$V_i(s) = R1I(s) + \frac{1}{sC1}I(s) + R2I1(s)$$

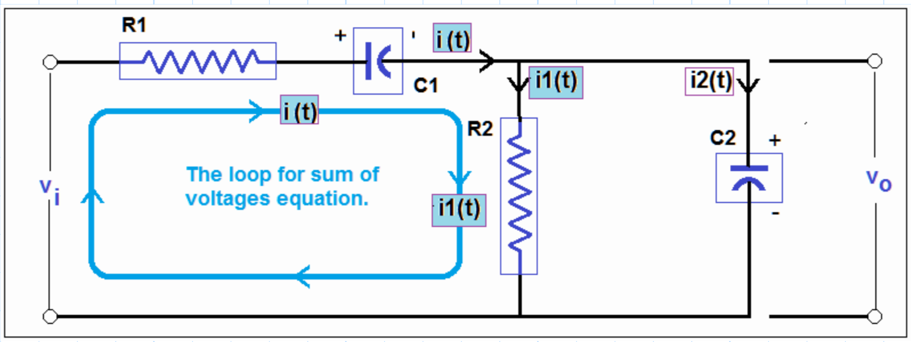


Figure to left is the voltage loop given i1(t) and i2(t) are known values.

$$V_o(t) = R2\cdot I1(s) \quad \text{Plug in equation above.}$$

$$V_i(s) = R1I(s) + \frac{1}{sC1}I(s) + V_o(s)$$

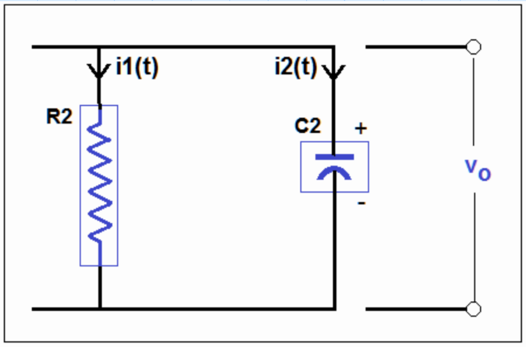
$$I(s) = I1(s) + I2(s) \quad \text{Plug in equation below.}$$

$$V_i(s) = R1(I1(s) + I2(s)) + \frac{1}{sC1}(I1(s) + I2(s)) + V_o(s)$$

Rearranging:

$$V_i(s) = (I1(s) + I2(s))\left(R1 + \frac{1}{sC1}\right) + V_o(s)$$

I cannot find a Vo(s)/Vi(s) from the above expression.
Cleaver engineer does a substitution for I1(s) and I2(s).



$$V_o(s) = R1\,I1(s)$$

$$I1(s) = \frac{V_o(s)}{R1}$$

$$V_o(s) = \frac{1}{sC2}\,I2(s)$$

$$I2(s) = sC2\,V_o(s)$$

Substitute the expressions for I1(s) and I2(s) into the Vi(s) equation.

$$V_i(s) = (I1(s) + I2(s))\left(R1 + \frac{1}{sC1}\right) + V_o(s)$$

$$V_i(s) = \left(\frac{V_o(s)}{R1} + sC2\,V_o(s)\right)\left(R1 + \frac{1}{sC1}\right) + V_o(s)$$

$$V_i(s) = V_o(s)\cdot\left(\frac{1}{R1} + sC2\right)\left(R1 + \frac{1}{sC1}\right) + V_o(s)$$

Another new trick, maybe not, but not common divide by Vo(s)

$$\frac{V_i(s)}{V_o(s)} = \left(\frac{1}{R1} + sC2\right)\left(R1 + \frac{1}{sC1}\right) + 1$$

Multiply the parenthesis: $\frac{R1}{R1} + \frac{1}{sC1R1} + sC2R1 + \frac{sC2}{sC1}$ Note: $C1 = C2$

$$= 1 + \frac{1}{sC1R1} + sC2R2 + 1$$

$$\frac{V_i(s)}{V_o(s)} = 1 + \frac{1}{sC1R1} + sC2R2 + 1 + 1$$

$$\frac{V_i(s)}{V_o(s)} = \frac{1}{sC1R1} + sC2R2 + 3$$

Invert for Vo(s)/Vi(s): $\frac{V_o(s)}{V_i(s)} = \frac{1}{\frac{1}{sC1R1} + sC2R2 + 3}$

Next simplify this expression for the purpose of attaining an expression in s form.

The s form of expression we seek where we can identify zeros and poles.

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\frac{1}{sC1R1} + sC2R2 + 3}$$

Multiply top and bottm by sC1R1

$$= \frac{sC1R1}{1 + (sC1R1)\,(sC2R2) + 3\,(sC1R1)}$$

$$= \frac{sC1R1}{1 + \left(s^2 \cdot C1C2R1R2\right) + 3\,sC1R1}$$

Let C = C1 = C2 = 1
R = R1 = R2 = 1

$$\frac{V_o(s)}{V_i(s)} = \frac{sCR}{1 + \left(s^2 \cdot C^2 \cdot R^2\right) + 3\,sCR}$$

Since R1=R2=C1=C2=1
We substitute for 1.

$$\frac{V_o(s)}{V_i(s)} = \frac{s}{1 + s^2 + 3\,s} = \frac{s}{s^2 + 3\,s + 1}$$

Answer for transfer function.

Zero: 0 Answer.

Pole(s): Solve quadratic equation $s^2 + 3\,s + 1$

$As^2 + Bs + C$

$$s1\ s2 = \frac{-B \quad +/- \quad \sqrt{B^2 - 4\,AC}}{2\,A}$$

$$s1 = \frac{-3 + \sqrt{3^2 - (4 \cdot 1 \cdot 1)}}{2 \cdot 1} = -4 \cdot 10^{-1}$$

$$s2 = \frac{-3 - \sqrt{3^2 - (4 \cdot 1 \cdot 1)}}{2 \cdot 1} = -3$$

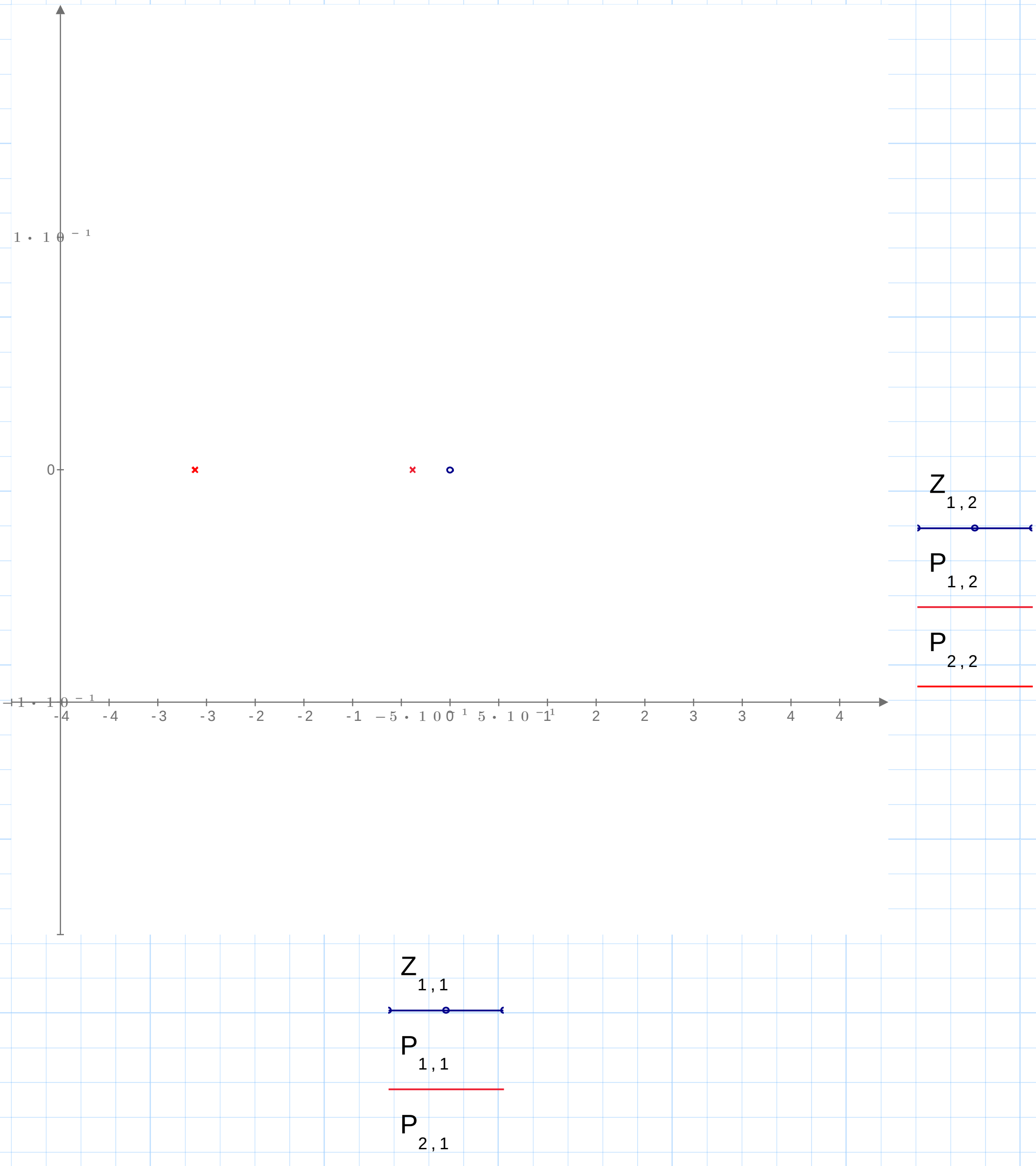
Poles: -0.382 and -2.618 Answer.

The manual plot is easy, real x-axis and imaginary y-axis.
Here all the zero and ploes are on the x-axis at 0, -0.382, and -2.618.

Lets try plotting the functions, numerator and denominator.

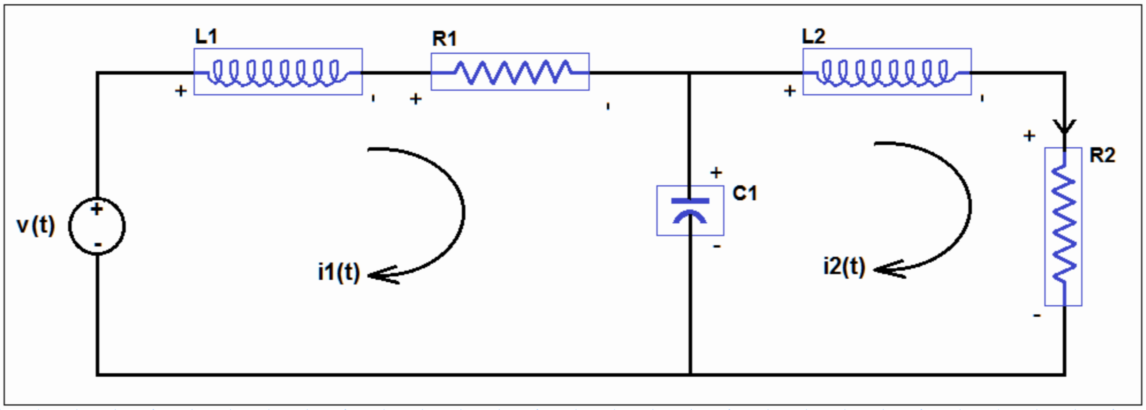
Origin(1,1) Set start of matrix at 1,1.

$Z := \begin{bmatrix} 0 & 0 \end{bmatrix}$ $\quad P := \begin{bmatrix} -0.382 & 0 \\ -2.618 & 0 \end{bmatrix}$ Using matriz Z for zero and P for poles.



Answer. Plot above.

Chp 1 Problem 1.10:



Write the differential equations for the electrical circuit above.

Solution:

I kickoff with the sum of voltage around a loop equal zero.
I do an equation for each loop.

Loop i1(t):

$$v_i(t) = L1\left(\frac{di1(t)}{dt}\right) + R1i1(t) + \frac{1}{C1}\cdot\int i1(t)\,dt - \frac{1}{C1}\cdot\int i2(t)\,dt$$

Loop i2(t):

$$0 = L2\left(\frac{di2(t)}{dt}\right) + R2i2(t) + \frac{1}{C1}\cdot\int i2(t)\,dt - \frac{1}{C1}\cdot\int i1(t)\,dt$$

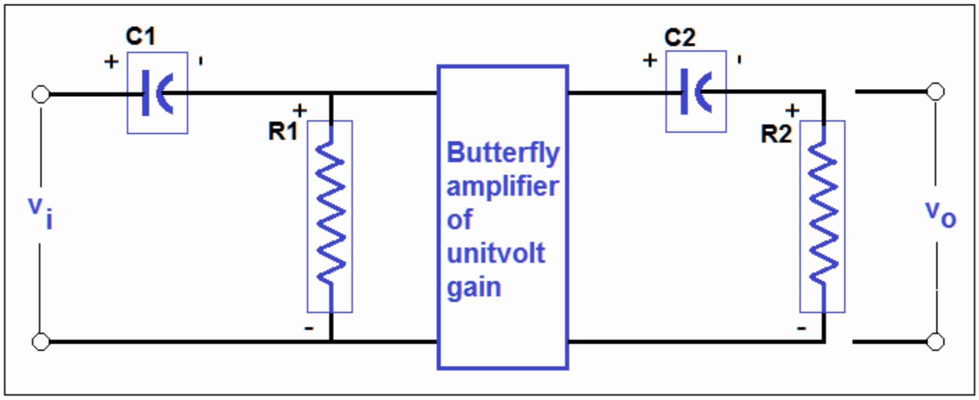
Answer.

Not part of the question but how if I did a s-domain on the time domain, what the typical controls engineering course will say is taking the Laplace transform? You verify.

$$V_i(s) = sL1\cdot I1(s) + R1\cdot I1(s) + \frac{1}{sC1}\cdot I1(s) - \frac{1}{sC1}\cdot I2(s)$$

$$0 = sL2\cdot I2(s) + R2\cdot I2(s) + \frac{1}{sC1}\cdot I2(s) - \frac{1}{sC1}\cdot I2(s)$$ Answer.

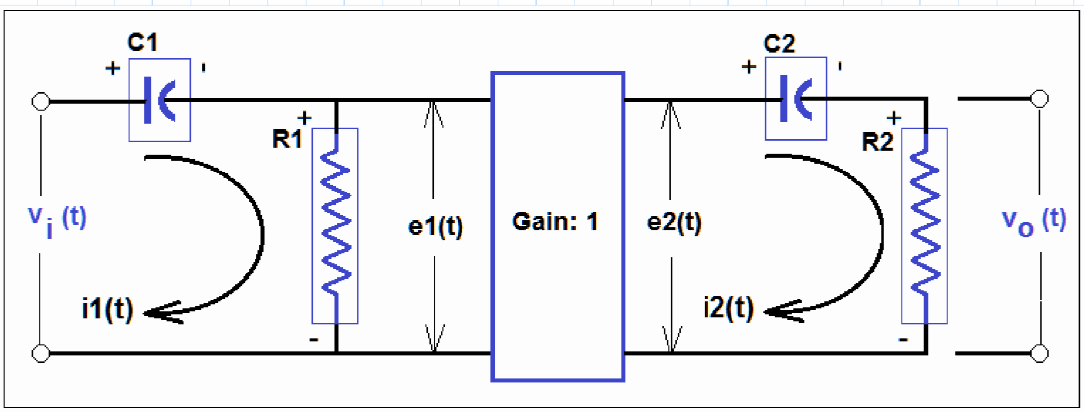
Chp 1 Problem 1.12:



Determine the transfer function relating Vo(s) to Vi(s) for network above.
Calculate the output voltage, t>>0, for a unit step voltage input at t=0, when C1 = 1 uF, R = 1 M Ohm, C2 = 0.5 uF and R2 = 1 M Ohm.

Solution:

Circuit re-sketched for applying sum of voltage in a loop method.
Kickoff's Voltage Law, KVL, usually what the electrical engineer calls.



Amplifier gain e2(t)/e1(t) = 1. Therefore e1(t) = e2(t).
The circuit has a voltage input v_i(t), and to the output side of the amplier is a voltage gained e2(t) this is similar to supplying voltage to the circuit to the right of the ampliffier.

We proceed with KVLoop on the left and right, and we equate the resistor R1 voltage for e1(t).

$$v_i(t) = \frac{1}{C1}\int i1(t)\,dt + R1 i1(t)$$

$$e2(t) = \frac{1}{C2}\int i2(t)\,dt + R2 i2(t)$$

$$e1(t) = R1 i1(t)$$ Amplifier left side voltage.

$$v_o(t) = R2 i2(t)$$ Amplifier right side voltage.
This being the voltage output v_o(t)

Now we take the Lapalce transforms of the expressions above.
Call it what you want, La Place or No Place, its converting to s-domain.

$$V_i(s) = \frac{I1(s)}{sC1} + R1 \cdot I1(s) \qquad \text{Eq 1}$$

$$E2(s) = \frac{I2(s)}{sC2} + R2 \cdot I2(s) \qquad \text{Eq 2}$$

$$E1(s) = R1 \cdot I1(s) \qquad \text{Eq 3}$$

$$V_o(s) = R2 \cdot I2(s) \qquad \text{Eq 4}$$

METHOD 1:
This by building interconnected relationship, as I done in the past problems here.
The long way and the answer is same as the textbook answer.
If I had not done this then it may remain a mystery!
You may verify.
Method 2 is easy, which was my first re-action to the problem. Just place Vo/Vi, after forming their expression without the usual inter-related quations.
I will do method 2 after method 1 completion.

Rearrange Eq 1:

$$V_i(s) = I1(s) \cdot \left(\frac{1}{sC1} + R1\right) \qquad \text{Eq 5}$$

Rearrange Eq 2:

$$E2(s) = I2(s) \cdot \left(\frac{1}{sC2} + R2\right) \qquad \text{Eq 6}$$

Rearrange Eq 4:

$$I2(s) = \frac{V_o(s)}{R2} \qquad \text{Eq 7...substitute in Eq 6}$$

$$E2(s) = \frac{V_o(s)}{R2}\cdot\left(\frac{1}{sC2}+R2\right) \qquad \text{Eq 8}$$

E2(s) = E1(s): $\quad E1(s) = R1\cdot I1(s) = E2(s)$

Next substitute E1(s) for E2(s) in Eq 8.

$$E1(s) = E2(s) = R1\cdot I1(s) = \frac{V_o(s)}{R2}\cdot\left(\frac{1}{sC2}+R2\right) \qquad \text{Eq 9}$$

Substitute Eq 9 for R1I1(s) in Eq 1.

$$V_i(s) = \frac{I1(s)}{sC1}+\frac{V_o(s)}{R2}\cdot\left(\frac{1}{sC2}+R2\right) \qquad \text{Eq 10}$$

How do I substitute for I1(s), try Eq 5, then substitute into eq 10:

$$V_i(s) = I1(s)\cdot\left(\frac{1}{sC1}+R1\right) \qquad \text{Eq 5}$$

$$I1(s) = \frac{V_i(s)}{\left(\frac{1}{sC1}+R1\right)} \qquad \text{Eq 11.....substitute in Eq 10.}$$

$$V_i(s) = \frac{\frac{V_i(s)}{\left(\frac{1}{sC1}+R1\right)}}{sC1}+\frac{V_o(s)}{R2}\cdot\left(\frac{1}{sC2}+R2\right) \qquad \text{Eq 12...looks messy may do it.}$$

$$V_i(s)-\frac{\frac{V_i(s)}{\left(\frac{1}{sC1}+R1\right)}}{sC1} = \frac{V_o(s)}{R2}\cdot\left(\frac{1}{sC2}+R2\right)$$

$$V_i(s)-\frac{V_i(s)}{\left(\frac{1}{sC1}+R1\right)}\cdot\frac{1}{sC1} = \frac{V_o(s)}{R2}\cdot\left(\frac{1}{sC2}+R2\right)$$

$$V_i(s)\cdot\left[1-\frac{1}{\left(\frac{1}{sC1}+R1\right)\cdot sC1}\right] = \frac{V_o(s)}{R2}\cdot\left(\frac{1}{sC2}+R2\right)$$

$$V_i(s)\cdot\left(1-\frac{1}{(1+sC1R1)}\right) = V_o(s)\cdot\left(\frac{1}{sC2R2}+1\right)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\left(1-\frac{1}{(1+sC1R1)}\right)}{\left(\frac{1}{sC2R2}+1\right)}$$ Transfer function. Need simplifying.

$$\frac{V_o(s)}{V_i(s)} = \frac{\left(\frac{(1+sC1R1)}{(1+sC1R1)}-\frac{1}{(1+sC1R1)}\right)}{\left(\frac{1+sC2R2}{sC2R2}\right)} = \frac{\left(\frac{(sC1R1)}{(1+sC1R1)}\right)}{\left(\frac{1+sC2R2}{sC2R2}\right)}$$

$$= \left(\frac{(sC1R1)}{(1+sC1R1)}\right)\cdot\left(\frac{sC2R2}{1+sC2R2}\right)$$

$$= \left[\frac{\left(s^2\cdot C1C2R1R2\right)}{\left(1+sC2R2+sC1R1+s^2\cdot C1C2R1R2\right)}\right]$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\left(s^2\cdot C1C2R1R2\right)}{\left(1+s(C1R1+C2R2)+s^2\cdot(C1C2R1R2)\right)}$$

Let $A = C1C2R1R2$ $\quad B = C1R1 \quad C = C2R2$

$$\frac{V_o(s)}{V_i(s)} = \frac{\left(A\cdot s^2\right)}{\left(1+(B+C)\cdot s+A\cdot s^2\right)}$$ One Transfer Function - METHOD 1.

$C1 := 1\cdot10^{-6} \quad C2 := 0.5\cdot10^{-6} \quad R1 := 1\cdot10^{6} \quad R2 := 1\cdot10^{6}$

$A := C1\cdot C2\cdot R1\cdot R2 = 0.5$ Or fraction: $\frac{1}{2}$ $\quad B := C1\cdot R1 + C2\cdot R2 = 2$

$$\frac{V_o(s)}{V_i(s)} = \frac{\left(\frac{1}{2}\right)\cdot s^2}{1+\left(\frac{3}{2}\right)\cdot s+\left(\frac{1}{2}\right)\cdot s^2}$$ Multiply by 2.

$$\frac{V_o(s)}{V_i(s)} = \frac{2\left(\frac{1}{2}\right)\cdot s^2}{2\cdot\left(1+\left(\frac{3}{2}\right)\cdot s+\left(\frac{1}{2}\right)\cdot s^2\right)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{s^2}{2+3\cdot s+s^2} = \frac{s^2}{s^2+3\cdot s+2}$$ Answer. SAME AS TEXTBOOK!

Calculate the output voltage, t>>0, for a unit step voltage input at t=0:

Since its unit step voltage input the initial conditions for t<0 = 0.
So i(-0) = i(0+...just near 0) = 0 and
v(-0) = v(0+...just near 0) = 0 v(++) = 1
Comment: How do I get the numerator (zero) = 1 for t>>0 so the Vi(s) = 1 or greater; u(t=0 or t>0) = 1 or u(t) = 1x Constant. But NOT equal 0.

$$\frac{V_o(s)}{V_i(s)} = \frac{s^2}{(s+2)\,(s+1)} \qquad V_o(s) = V_i(s)\cdot\frac{s^2}{(s+2)\,(s+1)}$$

$$V_o(s) = 1\cdot\frac{s\cdot s}{(s+2)\,(s+1)} = \frac{A}{(s+2)}+\frac{B}{(s+1)}$$

To solve for coefficients using the? Method of proper fractions OR Equating coefficients of like powers.

$$s = A(s+1)+B(s+2) = As+A+Bs+2\,B$$

$$s = s(A+B)+(A+2\,B)$$

Arrange for like terms:

$$s \quad : \quad s(A+B) \quad \text{--->} \quad A+B = 1 \quad \text{Eq 1}$$

$$0 \quad : \quad (A+2\,B) \quad \text{--->} \quad A+2\,B = 0 \quad \text{Eq 2}$$

$$B = -1 \quad \text{Eq 2-1}$$

Substitute B in Eq 1.

$$A+B = 1$$
$$A-1 = 1$$
$$A = 2$$

$$V_o(s) = \frac{A}{(s+2)} + \frac{B}{(s+1)} = \frac{2}{(s+2)} - \frac{1}{(s+1)}$$

Now with the coefficients, zeros, and poles I can form the voltage output in time domain. This will be an exponential equation because the voltage source is a step function, unity, or constant.

$$V_o(s) = Ae^{s1t} + Be^{s2t}$$

$$V_o(s) = -2\,e^{-2\,t} - 1\,e^{-1\,t}$$

Now to convert from s-domain to time domain:

$$v_o(t) = -2\,e^{-2\,t} - 1\,e^{-1\,t}$$

$$v_o(t) = -2\,e^{-2\,t} - e^{-t}$$

Answer. Same as textbook.

Please verify the solution steps and reasoning on the voltage output equation where Vi(s) = 1.

METHOD 2:

Now for Method 2, the supposed to be simpler and shorter solution.

$$V_i(s) = \frac{I1(s)}{sC1} + R1 \cdot I1(s) \qquad \text{Eq 1}$$

$$E2(s) = \frac{I2(s)}{sC2} + R2 \cdot I2(s) \qquad \text{Eq 2}$$

$$E1(s) = R1 \cdot I1(s) \qquad \text{Eq 3}$$

$$V_o(s) = R2 \cdot I2(s) \qquad \text{Eq 4}$$

Set up the transfer function, Vo(s)/Vi(s) based on their respective equations directly:

$$\frac{V_o(s)}{V_i(s)} = \frac{R2 \cdot I2(s)}{\frac{I1(s)}{sC1} + R1 \cdot I1(s)} = \frac{I2(s) \cdot R2}{I1(s) \cdot \left(R1(s) + \frac{1}{sC1}\right)}$$

Eq 5...maybe I2(s) and I1(s) substituion may help.

$$I1(s) = \frac{E1(s)}{R1} \qquad I2(s) = \frac{E2(s)}{\left(R2 + \frac{1}{sC2}\right)}$$

From Eq 2 above.

$$\frac{I2(s)}{I1(s)} = \frac{\dfrac{E2(s)}{\left(R2+\dfrac{1}{sC2}\right)}}{\dfrac{E1(s)}{R1}} = \frac{E2(s)}{\left(R2+\dfrac{1}{sC2}\right)}\cdot\frac{R1}{E1(s)}$$

Gain = 1, E2(s)/E1(s) = 1, therefore E1(s) = E2(s).

$$E1(s) = E2(s)$$

Now the current ratio equation becomes: $\dfrac{I2(s)}{I1(s)} = \dfrac{R1}{\left(R2+\dfrac{1}{sC2}\right)}$

Returning to Eq 5 substitute for I2(s)/I1(s):

$$\frac{V_o(s)}{V_i(s)} = \frac{I2(s)\cdot R2}{I1(s)\cdot\left(R1+\dfrac{1}{sC1}\right)} \qquad \text{Eq 5}$$

$$= \frac{R1}{\left(R2+\dfrac{1}{sC2}\right)}\cdot\frac{R2}{\left(R1+\dfrac{1}{sC1}\right)}$$

$$= \frac{R1R2}{R1R2+\dfrac{R2}{sC1}+\dfrac{R1}{sC2}+\dfrac{1}{s^2\ C1C1}}$$

Let: $A = R1\cdot R2 = 1\cdot 10^{12}$

$B = \dfrac{R2}{C1} = 1\cdot 10^{12}$ $\qquad C = \dfrac{R1}{C2} = 2\cdot 10^{12}$

$D = \dfrac{1}{C1\cdot C2} = 2\cdot 10^{12}$

$$\frac{V_o(s)}{V_i(s)} = \frac{1\cdot 10^{12}}{1\cdot 10^{12}+\dfrac{1\cdot 10^{12}}{s}+\dfrac{2\cdot 10^{12}}{s}+\dfrac{2\cdot 10^{12}}{s^2}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1+\dfrac{1}{s}+\dfrac{2}{s}+\dfrac{2}{s^2}} = \frac{1}{1+\dfrac{3}{s}+\dfrac{2}{s^2}}$$

Multiply by s^2 top and bottom.

$$\frac{V_o(s)}{V_i(s)} = \frac{(s^2)\cdot 1}{(s^2)\cdot\left(1+\frac{3}{s}+\frac{2}{s^2}\right)}$$

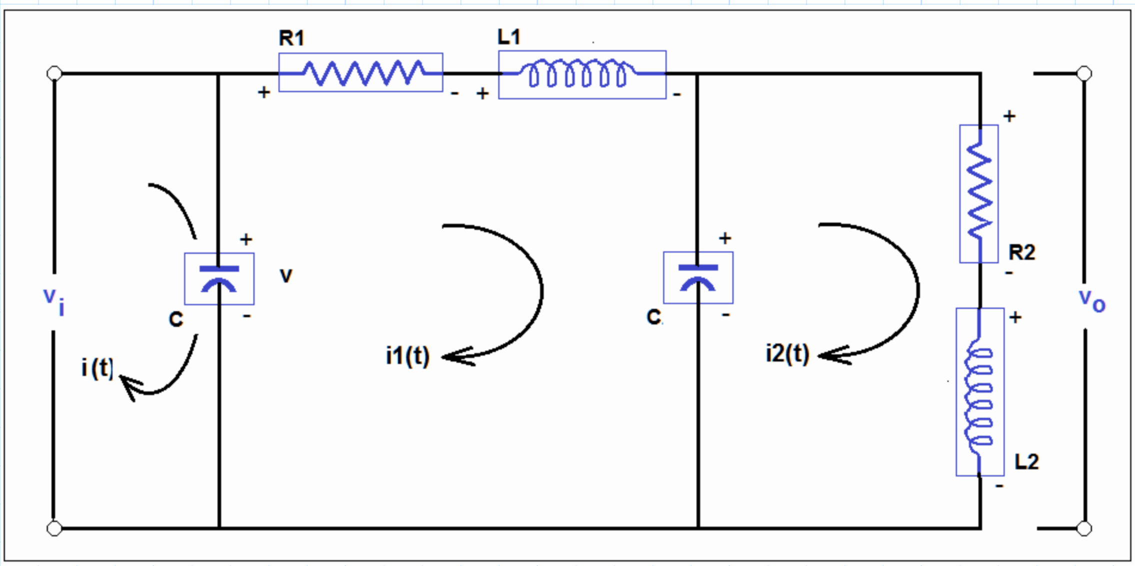
$$\frac{V_o(s)}{V_i(s)} = \frac{s^2}{s^2+3\,s+2}$$

Answer.
Same method used by engineer the faster method.

The short method may give the impression there is no relationship with the components like that established in the longer method. However, the transfer function's definition is just that, output divided by input. Do consider the circuit's components and connections, and carefully construct the equations.

The remaining part on the output voltage same as completed following the long method of the transfer functions.

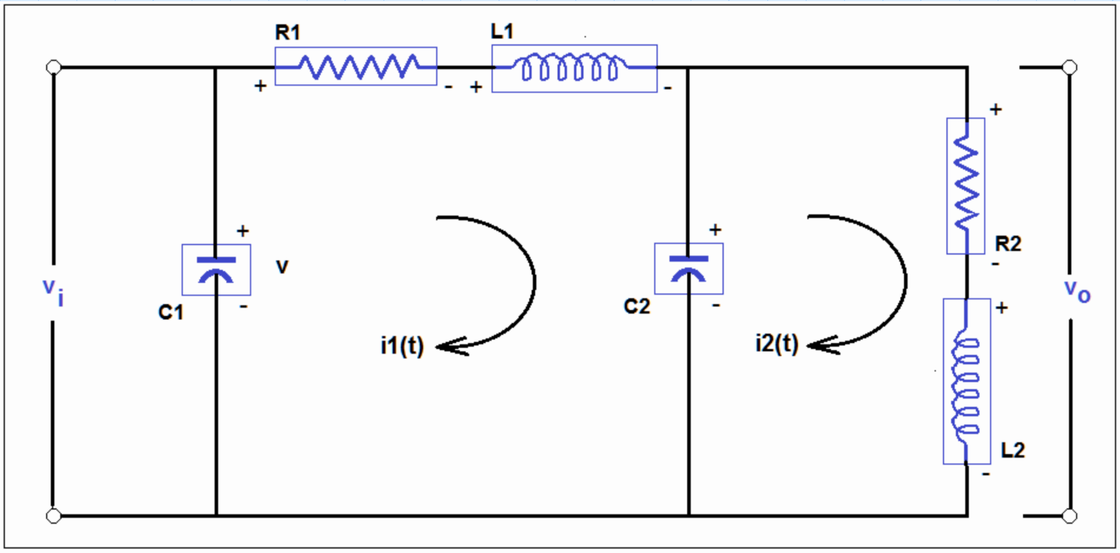
Chp 1 Problem 1.13:



Determine the transfer function of the electrical network above:

Solution:

C1 = C2, the question did not show C1 and C2, rather C.
To ease tracking the solution they were made into C1 and C2.



The basic steps we first started with provided here again, these steps were much the same to what we did in the previous problems.

The **steps involved** in obtaining the transfer function are:

1. Write differential equations of the system.

2. Replace terms involving $\frac{d}{dt}$ by s and $\int \blacksquare \, dt$ by 1/s, for inductor and capacitor respectively.

3. Eliminate all but the desired variable.

Step 1:

Check current flow direction. Coming out of C1 -ve terminal -ve voltage.

v_i(t) :

$$v_i(t) = -\frac{1}{C1} \cdot \int i(t)\,dt$$

Check current flow direction. Coming out of C1 -ve terminal -ve voltage (left loop).

$$\frac{1}{C1} \cdot \int (i1(t) - i(t))\,dt = \frac{1}{C1} \cdot \int i1(t)\,dt - v_i(t)$$

This can be written as:

$$v_i(t) + \frac{1}{C1} \cdot \int i1(t)\,dt = 0$$

Sum of voltages, next vi(t) to the LHS, resulting in the same.

$$-v_i(t) = \frac{1}{C1} \cdot \int i1(t)\,dt \quad \text{OR} \quad v_i(t) = -\frac{1}{C1} \cdot \int i1(t)\,dt$$

Step 1:

v_i(t) :

$$v_i(t) = -\frac{1}{C1}\cdot\int i(t)\,dt$$

Check current flow direction. Coming out of C1 -ve terminal -ve voltage.

$$v_i(t) = \frac{1}{C1}\cdot\int i1(t)\,dt$$

Check current flow direction. Coming out of C1 -ve terminal -ve voltage.

This can be written as: $v_i(t) + \frac{1}{C1}\cdot\int i1(t)\,dt = 0$

Sum of voltages, next vi(t) to the LHS, resulting in the same.

$$-v_i(t) = \frac{1}{C1}\cdot\int i1(t)\,dt \quad \text{OR} \quad v_i(t) = -\frac{1}{C1}\cdot\int i1(t)\,dt$$

Centre loop:

$$0 = \frac{1}{C1}\cdot\int i1(t)\,dt + R1i1(t) + L1\left(\frac{di1(t)}{dt}\right) + \frac{1}{C2}\cdot\int (i1(t) - i2(t))\,dt$$

$$-v_i(t) = \frac{1}{C1}\cdot\int i1(t)\,dt$$

Substitute in equation above.

$$0 = -v_i(t) + R1i1(t) + L1\left(\frac{di1(t)}{dt}\right) + \frac{1}{C2}\cdot\int (i1(t) - i2(t))\,dt$$

$$v_i(t) = R1i1(t) + L1\left(\frac{di1(t)}{dt}\right) + \frac{1}{C2}\cdot\int (i1(t) - i2(t))\,dt \qquad \text{Eq 1}$$

v_o(t):

$$v_o(t) = R2\cdot i2(t) + L2\left(\frac{di2}{dt}\right) \qquad \text{Eq 2}$$

Right loop:

$$0 = \frac{1}{C2}\cdot\int (i2(t) - i1(t))\,dt + R2i2(t) + L2\left(\frac{di2(t)}{dt}\right) \qquad \text{Eq 3}$$

Substitute v_o(t) into Right Loop.

Step 2:

Assuming all initial conditions for L and C are zero.
We proceed with taking the? *Laplace Transform*. Convert to s-domain.

$$v_i(t) = R1i1(t) + L1\left(\frac{di1(t)}{dt}\right) + \frac{1}{C2}\cdot\int (i1(t) - i2(t))\,dt \qquad \text{Eq 1}$$

$$V_i(s) = R1I1(s) + sL1\ I1(s) + \frac{1}{sC2}\cdot(I1(s) - I2(s))$$

$$V_i(s) = R1I1(s) + sL1\ I1(s) + \frac{1}{sC2}\cdot(I1(s) - I2(s)) \qquad \text{Eq 4 <---Same as textbook.}$$

$$v_o(t) = R2\cdot i2(t) + L2\left(\frac{di2}{dt}\right) \qquad \text{Eq 2}$$

$$V_o(t) = R2\cdot I2(s) + sL2I2(s) \qquad \text{Eq 5 <----Same as textbook.}$$

$$0 = \frac{1}{C2}\cdot\int (i2(t) - i1(t))\,dt + R2i2(t) + L2\left(\frac{di2(t)}{dt}\right) \qquad \text{Eq 3}$$

$$0 = \frac{1}{sC2}(I2(s) - I1(s)) + R2I2(s) + sL2\cdot I2(s) \qquad \text{Eq 6}$$

Step 3:

$$\frac{V_o(t)}{V_i(s)} = \frac{R2\cdot I2(s) + sL2I2(s)}{R1I1(s) + sL1\ I1(s) + \frac{1}{sC2}\cdot(I1(s) - I2(s))}$$

Same as textbook.
Past this point you have to solve it for the best possible form.

$$\frac{V_o(t)}{V_i(s)} = \frac{I2(s)\cdot(R2 + sL2)}{I1(s)\cdot\left(R1 + sL1 + \frac{1}{sC2}\right) - \frac{1}{sC2}\ I2(s)}$$

Find a substitute for I2 in terms of I1 for the denominator.

$$0 = \frac{1}{sC2}(I2(s) - I1(s)) + R2I2(s) + sL2 \cdot I2(s) \qquad \text{Eq 6}$$

Solve for I2 above:

$$\frac{1}{sC2} I2(s) + R2I2(s) + sL2 \cdot I2(s) = \frac{I1(s)}{sC2} \qquad \text{Multiply by sC2.}$$

$$I2(s) + R2I2(s)\ sC2 + sL2 \cdot I2(s)\ sC2 = I1(s)$$

$$I2(s) \cdot \left(1 + sC2R2 + s^2\ C2L2\right) = I1(s) \qquad \text{Eq 7}$$

$$I2(s) = \frac{I1(s)}{\left(1 + sC2R2 + s^2\ C2L2\right)}$$

$$\frac{V_o(t)}{V_i(s)} = \frac{I2(s) \cdot (R2 + sL2)}{I1(s) \cdot \left(R1 + sL1 + \frac{1}{sC2}\right) - \frac{1}{sC2}\ I2(s)}$$

Substitute I2(s) in denominator above.

$$\frac{V_o(t)}{V_i(s)} = \frac{I2(s) \cdot (R2 + sL2)}{I1(s) \cdot \left(R1 + sL1 + \frac{1}{sC2}\right) - \frac{1}{sC2} \cdot \frac{I1(s)}{\left(1 + sC2R2 + s^2\ C2L2\right)}}$$

$$\frac{V_o(t)}{V_i(s)} = \frac{I2(s) \cdot (R2 + sL2)}{I1(s) \cdot \left(\left(R1 + sL1 + \frac{1}{sC2}\right) - \left(\frac{1}{sC2 + s^2\ C2^2\ R2 + s^3\ C2^2\ L2}\right)\right)}$$

$$\frac{V_o(t)}{V_i(s)} = \left(\frac{I2(s)}{I1(s)}\right) \cdot \frac{(R2 + sL2)}{\left(R1 + sL1 + \frac{1}{sC2}\right) - \left(\frac{1}{sC2 + s^2\ C2^2\ R2 + s^3\ C2^2\ L2}\right)}$$

Find an equation for I2(s)/I1(s) .... Eq 7 below.

$$I2(s) \cdot \left(1 + sC2R2 + s^2\ C2L2\right) = I1(s) \qquad \text{Eq 7}$$

$$\frac{I2(s)}{I1(s)} = \frac{1}{\left(1 + sC2R2 + s^2\ C2L2\right)}$$

Substitute $\frac{I2(s)}{I1(s)}$ in transfer function

$$\frac{V_o(t)}{V_i(s)} = \left(\frac{1}{1 + sC2R2 + s^2\ C2L2}\right)\left[\frac{(R2 + sL2)}{\left(R1 + sL1 + \frac{1}{sC2}\right) - \left(\frac{1}{sC2 + s^2\ C2^2\ R2 + s^3\ C2^2\ L2}\right)}\right]$$

$$\frac{V_o(t)}{V_i(s)} = \frac{(R2 + sL2)}{\left(1 + sC2R2 + s^2\ C2L2\right)\cdot\left(R1 + sL1 + \frac{1}{sC2}\right) - \left[\frac{\left(1 + sC2R2 + s^2\ C2L2\right)}{sC2\left(1 + s\ C2R2 + s^2\ C2\ L2\right)}\right]}$$

Set C1 = C2 = C, as given.

$$\frac{V_o(t)}{V_i(s)} = \frac{(R2 + sL2)}{\left(1 + sCR2 + s^2\ CL2\right)\cdot\left(R1 + sL1 + \frac{1}{sC}\right) - \left(\frac{1}{sC}\right)}$$

Expand the left side terms at the bottom, and set equal to A.
Then the bottom right side term's denominator set to B.

$$\left(1 + sCR2 + s^2\ CL2\right)\cdot\left(R1 + sL1 + \frac{1}{sC}\right) =$$

$$R1 + sL1 + \frac{1}{sC} + sCR1R2 + s^2\ CR2L1 + R2 + s^2\ CR1L2 + s^3\ CL1L2 + sL2 = A$$

$$s^3\ (CL1L2) + s^2\cdot(CR2L1 + CR1L2) + s\left(L1 + \frac{1}{s^2\ C} + CR1R2 + L2\right) + (R1 + R2) = A$$

$$s^3\ (L1L2) + s^2\cdot C\,(R2L1 + R1L2) + s\left(L1 + L2 + CR1R2 + \frac{1}{s^2\ C}\right) + (R1 + R2) = A$$

$$\frac{1}{sC} = B$$

$$\frac{V_o(t)}{V_i(s)} = \frac{(R2 + sL2)}{A - \left(\frac{1}{B}\right)}$$

$$s^3 (L1L2) + s^2 \cdot C(R2L1 + R1L2) + s\left(L1 + L2 + CR1R2 + \frac{1}{s^2 C}\right) + (R1 + R2) = A$$

In my a equation above there is (1/s^2C) this is not in the textbook anwer.
Textbook answer below does not have B term (1/sC) maybe this was negligible to the overall function because it becomes huge in the denominator, and when it divides the numerator its small or negligible. Usually C is in microFarad units. This may also be the case for (1/s^2C) in the A term. Except for this my result is the same.

$$\frac{V_o(t)}{V_i(s)} = \frac{R2 + sL2}{s^3 (L1L2) + s^2 \cdot C(R2L1 + R1L2) + s\left(L1 + L2 + CR1R2 + \frac{1}{s^2 C}\right) + (R1 + R2) - \left(\frac{1}{sC}\right)}$$

Neglecting (1/sC) and (1/s^2 C):

$$\frac{V_o(t)}{V_i(s)} = \frac{R2 + sL2}{s^3 (L1L2) + s^2 \cdot C(R2L1 + R1L2) + s(L1 + L2 + CR1R2) + (R1 + R2)}$$

My Answer.
You can verify this answer correct it, or present your own. *Here this is as far as I am going.*

Textbook Answer:

$$\frac{V_o(s)}{Vi(s)} = \frac{R2 + sL2}{s^3\ CL1L2 + s^2\ C(R1L2 + L1R2) + s(L1 + L2 + CR1R2) + (R1 + R2)}$$

Transfer function above does look tidy!
You solve it for yourself if you see a need.

You can sort it with your local lecturer/engineer.
Apologies for any errors and omissions.

This brings to end the 13 example problems.
Next Schaum's Chapter 8 Solved Problems.