Bilevel Langevin - Finding $Z(\lambda)$ and Z for simple problem

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What we want

We are given the probability density functions

$$p(\lambda) = \frac{1}{Z}\exp(-\beta\, C(x(\lambda), \lambda))$$

$$p(x \mid \lambda) = \frac{1}{Z(\lambda)}\exp(-\beta F(x, \lambda)/\epsilon)$$

for normalization constants Z, $Z(\lambda)$. We are interested in determining these constants.

Simple Bilevel Problem

Let us consider a simple quadratic problem:

$$\begin{aligned} C(x, \lambda) &= \frac{1}{2}(x-2)^2 \\ F(x, \lambda) &= \frac{1}{2}(x-1)^2 + \frac{1}{2}\lambda^2 x^2 \end{aligned}$$

Note that the inner function F is quadratic in x, hence $p(x|\lambda)$ is gaussian.

Finding $Z(\lambda)$

Note that

$$(x-1)^2 + \lambda^2 x^2 = (\lambda^2+1)\left(\left(x - \frac{1}{\lambda^2+1}\right)^2 - \frac{1}{(\lambda^2+1)^2} + \frac{1}{\lambda^2+1}\right) \tag{1}$$

We have that

$$p(x|\lambda) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) = \frac{1}{Z(\lambda)}\exp(-\frac{\beta}{\epsilon}(\frac{1}{2}(x-1)^2 + \frac{1}{2}\lambda^2x^2))$$

Then, using (1), we obtain

$$\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) = \frac{1}{Z(\lambda)}\exp\left(-\frac{1}{2}\frac{\beta}{\epsilon}\left(x - \frac{1}{\lambda^2+1}\right)^2(\lambda^2+1)\right)\exp\left(-\frac{1}{2}\frac{\beta}{\epsilon}\left(1 - \frac{1}{\lambda^2+1}\right)\right),$$

from which we have

$$\mu = \frac{1}{\lambda^2+1}, \quad \sigma = \sqrt{\frac{\epsilon}{\beta}\frac{1}{\lambda^2+1}}$$

and, finally,

$$\frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{Z(\lambda)}\exp\left(-\frac{1}{2}\frac{\beta}{\epsilon}\left(1 - \frac{1}{\lambda^2+1}\right)\right),$$

which, after rearranging, gives us

$$Z(\lambda) = \sqrt{\frac{\epsilon}{\beta}\frac{2\pi}{\lambda^2+1}}\exp\left(-\frac{1}{2}\frac{\beta}{\epsilon}\left(1 - \frac{1}{\lambda^2+1}\right)\right).$$

Finding Z

Note that we can find $x(\lambda)$ explicitly in our case. This is $x(\lambda) = \frac{1}{1+\lambda^2}$. Then,

$$p(\lambda) = \frac{1}{Z} \exp(-\beta\, C(x(\lambda), \lambda)) = \frac{1}{Z} \exp\left(-\frac{\beta}{2}\left(\frac{1}{1+\lambda^2} - 2\right)^2\right).$$

Then,

$$Z = \int_{-\infty}^{\infty} \exp\left(-\frac{\beta}{2}\left(\frac{1}{1+\lambda^2} - 2\right)^2\right) d\lambda.$$