A Comment on "On the Completely Monotone Conjecture for Rényi Entropy"

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In [1], we made the conjecture that the entropy of $h(Y_t)$ is completely monotone in t (Gaussian Completely Monotone Conjecture, GCMC), where $Y_t = X + \sqrt{t}Z$, and X is independent of $Z \sim \mathcal{N}(0, 1)$. We have proved that GCMC holds for the first 4 derivatives by obtaining the exact expressions with signs. We have no other choice but to conduct hard and involved calculus.

In [2], the authors asserted that in some neighbourhood of $\alpha = 1$, GCMC can be generalized from Shannon entropy to Renyi entropy, including GCMC as the special case where $\alpha = 1$.

They added a new parameter α to GCMC and mimicked what we have done in [1] for $\alpha \neq 1$. They have also verified it up to the 4th derivative by partially finding the corresponding intervals. The extra α in the expression is very complicated to deal with. They need to make α very close to 1 to reduce its influence. In principle, we can replace Shannon entropy with Renyi entropy in all the existing models in Shannon theory and solve it as an exercise in maths.

However, they do not need to verify it from scratch. The existence of such intervals can be trivially obtained from [1]. Thus, the authors in [2] made a much stronger conjecture without showing any new evidence compared with [1].

Two basic results in mathematics.

1 In information theory, it is well known that Renyi entropy converges to Shannon entropy as $\alpha \to 1$; i.e.,

$$h_\alpha(X) \to h(X),$$

as $\alpha \to 1$. The continuity holds for the derivatives of $h_\alpha(Y_t)$.

2 In calculus, if a continuous function $f(x)$ satisfies that $f(a) > 0$, then there exists an interval $(a - \epsilon, a + \epsilon)$, $f(x) > 0$.

I have the following comments:

The existence of the intervals in [2] can be trivially obtained from [1] by the result 1 and 2: As $h_\alpha(X_t) \to h(X_t)$, when $\alpha \to 1$, and in [1], the signs of the 1st-4th derivatives had already been obtained, the continuity can be used to show the existence of the intervals $(1-\epsilon_k, 1+\epsilon_k)$ for the first 4 derivatives of Renyi entropy. We don't need to calculate an interval for each derivative by hand as done in [2], which is hard to check. In fact, if GCMC holds for n-th derivative, then we can find the interval $(1-\epsilon_n, 1+\epsilon_n)$ for Renyi entropy by continunity.

The work [2] didn't show any new evidence to make such a much stronger conjecture. To be completely monotone, we need to find the intersection of all the intervals $(1-\epsilon_k, 1+\epsilon_k)$, k = 1, 2, ... For Renyi entropy, even if they have found an interval $(1-\epsilon_k, 1+\epsilon_k)$ for the k-th derivative to be monotone, the limiting intersection of each order $\cap_{k=1}^{\infty}(1-\epsilon_k, 1+\epsilon_k)$ may converge to the point $\alpha = 1$, not an interval; i.e., $\cap_{k}^{\infty}(1-\epsilon_k, 1+\epsilon_k) = \{1\}$. Then it is exactly the GCMC in [1]. It is the key challenge to rule out $\{1\}$ to generalize from Shannon entropy to Renyi entropy. Unfortunately, the authors did not discuss and provide any evidence at this point. In fact, the first four intervals in [2] have become smaller and smaller and the 4th derivative interval $(0.93, 1.76)$ is very close to 1 at the left endpoint.

In the literature of Shannon theory, we have found many elegant connections between entropy and Gaussian distribution, dating back to Shannon's 1948 paper. Generally speaking, Shannon entropy is the limiting form of Renyi entropy and is unique and much more "nice". Renyi entropy is not as fundamental as Shannon entropy. Renyi entropy and Gaussian distribution do not have a very deep connection in the literature. To introduce a much deeper conjecture without any new evidence on Renyi entropy is unacceptable.

On the other hand, if the Renyi version is correct, then there will be an interval (α, β), where $\alpha < 1 < \beta$ are two constants for Gaussian distribution. Furthermore, Renyi entropy and Shannon entropy will be regarded as almost equivalent in such an interval. There is a result that looks like it violates such an observation: Gaussian distribution maximizes entropy but for Renyi entropy with $\alpha \neq 1$, the maximizer is not Gaussian. Complete monotonicity is much deeper than "maximum entropy". So it is very likely to be wrong.

Also, there are some problems in the literature review.

In [2]: "This conjecture, known as the completely monotone conjecture, was explicitly stated by Cheng and Geng in [4]. Cheng and Geng investigated the completely monotone conjecture for the higher order derivatives of $h(p)$. By extending McKean's techniques, they showed that the completely monotone conjecture holds for the third and fourth order derivatives. "

In fact, we didn't extend McKean's work. As the title of our paper indicates, we followed Costa's work. In history, the research interest on Gaussian distribution in Shannon theory society was ignited by Shannon's 1948 paper on entropy power inequality. How to generalize Shannon EPI has become a very traditional topic since then. In mathematical physics, researchers studied the heat equation which is also identical to the Gaussian channels in information theory. There are two independent research lines in mathematical physics and information theory.

Though McKean studied a similar problem in 1966, it was unknown to Shannon theory society until [1]. Section IX. in [1] has clearly pointed out the connection with McKean's work. The history of McKean's work and GCMC can also be found at [3]. Before [1], McKean's "conjecture" was not regarded as a serious conjecture even in mathematics because no evidence to support it and almost nobody followed this "conjecture". It is impossible to make the completely monotone conjecture based on only the signs of the first two derivatives. McKean's paper was forgotten by his society as there was no evidence to support his "conjecture". Before [1], in the work of Costa, Dembo, etc., nobody cited McKean's paper in the literature of Shannon theory.

In technique, McKean's conjecture is almost the same with GCMC. Because it is widely known that Gaussian distributions maximize entropy. If we are sure that GCMC is true, it is natural to study its maximizer.

It is the obligation of the authors of [2] to provide some new evidence to make a much deeper conjecture. In principle, **IEEE Transactions on Information Theory** should not publish a paper without any new evidence to overwrite an existing paper in its own journal. Without any new evidence, the conjecture in [2] is misleading and will cause chaos in the future study of GCMC.

References

[1] F. Cheng and Y. Geng, "Higher Order Derivatives in Costa's Entropy Power Inequality," IEEE Trans. Inform. Theory, vol. 61, no. 11, pp. 5892-5905, Nov. 2015.

[2] H. Wu, L. Yu, L. Guo, "On the Completely Monotone Conjecture for Rényi Entropy," IEEE Trans. Inform. Theory, 2025 (https://ieeexplore.ieee.org/document/10966505), https://arxiv.org/abs/2312.01819.

[3] F. Cheng,"A Reformulation of Gaussian Completely Monotone Conjecture: A Hodge Structure on the Fisher Information along Heat Flow," https://arxiv.org/abs/2208.13108.