Definitions

Recall the following definitions. Let $\Omega = [0,1]$ denote the unit interval. Given any *even* integer $M > 0$ let Ω^h denote the *periodic* grid on Ω,

$$\Omega^h \stackrel{\text{def}}{=} [\, x_0 = 0,\ x_1,\ \ldots,\ x_{M-1},\ x_M = 1\,]\,, \tag{1}$$

where $h = M^{-1}$, $x_i = i\,h$ for $i = 0,\ 1,\ \ldots,\ M$, and given *any* integer l we identify the points x_i and $x_{i+l\,M}$; i.e., $x_{i+l\,M} = x_i$ for every integer $l \in \mathbb{Z}$.

Define the grid function u on Ω^h by

$$u \stackrel{\text{def}}{=} (u_0,\ u_1,\ u_2,\ \ldots,\ u_{M-1}) \tag{2}$$

where $u_i \in \mathbb{R}$ for $i = 0,\ 1,\ \ldots,\ M-1$ are M *real* numbers. When $u : [0,1] \to \mathbb{R}$ is a function defined for all $x \in \Omega = [0,1]$ and the u_i in (2) are given by $u_i = u(x_i)$ we will usually use the notation u^h to distinguish the grid function defined on Ω^h from the function u defined on all of $\Omega = [0,1]$.

For $k = -\frac{M}{2}+1,\ -\frac{M}{2}+2,\ \ldots,\ \frac{M}{2}-1,\ \frac{M}{2}$, define the kth discrete Fourier basis function on Ω^h by

$$w^{(k)} \stackrel{\text{def}}{=} \left(w_0^{(k)},\ w_1^{(k)},\ \ldots,\ w_{M-1}^{(k)} \right) \tag{3}$$

with the ith component $w_i^{(k)}$ of $w^{(k)}$ given by

$$w_i^{(k)} \stackrel{\text{def}}{=} W^{(k)}\,(x_i)$$

where

$$W^k(x) \stackrel{\text{def}}{=} e^{2\,\pi\,\iota\,k\,x}$$

and $\iota = \sqrt{-1}$. The *discrete Fourier coefficients* b_k of u are defined by

$$\hat{u}_k \stackrel{\text{def}}{=} \langle\, u\,,\, w^{(k)}\,\rangle\,, \tag{4}$$

where $\langle\, \cdot\,,\, \cdot\,\rangle$ denotes the discrete inner product on the grid Ω^h, which is defined by

$$\langle\, u\,,\, u'\,\rangle \stackrel{\text{def}}{=} \frac{1}{M} \sum_{i=0}^{M-1} u_i\, \overline{u_i'} \tag{5}$$

for any two grid functions $u = (u_0,\ u_1,\ u_2,\ \ldots,\ u_{M-1})$ and $u' = \left(u_0',\ u_1',\ \ldots,\ u_{M-1}'\right)$ on Ω^h.

NOTE: It is important to remember that in the definition of the discrete inner product in (5) $\overline{u_i'}$ denotes the *complex conjugate* of u_j'; i.e., if $u_i' = \xi_i + \iota\,\eta_i$, then $\overline{u_i'} = \xi_i - \iota\,\eta_i$.

Problems

Problem 01 (20 points total) Prove the discrete form of Parseval's equality:

$$\| \, u \, \|_2^2 \, = \, \| \, \hat{u} \, \|_2^2 \, ,$$

or equivalently,

$$\frac{1}{M} \sum_{i=0}^{M-1} |u_i|^2 \, = \sum_{k=-\frac{M}{2}+1}^{\frac{M}{2}} \hat{u}_k \, \overline{\hat{u}}_k$$

where u is the grid function defined in (2) and

$$\hat{u} \stackrel{\text{def}}{=} \left(\hat{u}_{-\frac{M}{2}+1}, \, \hat{u}_{-\frac{M}{2}+2}, \, \ldots, \, \hat{u}_{\frac{M}{2}} \right)$$

is the *Discrete Fourier Transform* of u.

HINT: You *may* have to use some of the ideas from the proof in the Lecture Notes that the $w^{(k)}$ for $k = -\frac{M}{2}+1, -\frac{M}{2}+2, \ldots, \frac{M}{2}-1, \frac{M}{2}$ are a basis for $\mathbb{C}^M$

Problem 02 (20 points total) Given a positive even integer M and any *real-valued* function $f : [0,1] \to \mathbb{R}$, define the grid function f^h on Ω^h by $f^h = (f_0^h, \, f_1^h, \, \ldots, \, f_{M-1}^h)$ where $f_i^h = f(x_i)$, $x_i = i \, h$, and $h = M^{-1}$ as above. Now define the discrete Fourier coefficients $\hat{f}_k^h$ of f^h as in (4) above,

$$\hat{f}_k^h \stackrel{\text{def}}{=} \langle \, f^h \, , \, w^{(k)} \, \rangle \tag{6}$$

and define the grid function $\hat{\hat{f}}^h$ on Ω^h by

$$\hat{\hat{f}}^h \stackrel{\text{def}}{=} \sum_{k=-\frac{M}{2}+1}^{\frac{M}{2}} \hat{f}_k^h \, w^{(k)} \, .$$

Prove that the ith component of $\hat{\hat{f}}^h$ equals the ith component of f^h,

$$\hat{\hat{f}}_i^h = f_i^h \, .$$

Conclude that this implies the *infinitely differentiable* function $\tilde{f}^h(x)$ defined by

$$\tilde{f}^h(x) \stackrel{\text{def}}{=} \sum_{k=-\frac{M}{2}+1}^{\frac{M}{2}} \hat{f}_k^h \, W^{(k)}(x) \, , \qquad \text{for all } x \in [0,1] \, , \tag{7}$$

interpolates the function f at the gridpoints x_i in $[0,1]$.

Problem 03 (20 points total) Given the same setup as in **Problem 02** above so, in particular, the discrete Fourier coefficients $\hat{f}_k^h$ of f associated with the grid Ω^h are defined as in (6) above,

$$\hat{f}_k^h \stackrel{\text{def}}{=} \langle\, f^h \,,\, w^{(k)} \,\rangle\,,$$

answer the following two questions.

(a) (10 points) Consider the function $\tilde{f}^h\,(x)$ defined in (7) above. Is $\tilde{f}^h\,(x)$ a real-valued function? In other words, is $\tilde{f}^h(x) \in \mathbb{R}$ for all $x \in [0,1]$?

(b) (10 points) Now consider the function $F^h\,(x)$ defined for $x \in [0,1]$ by

$$F^h\,(x) \stackrel{\text{def}}{=} \sum_{k=0}^{M-1} \hat{f}_k^h \, W^{(k)}(x)\,,$$

where for $k = 0,\, 1,\, \ldots,\, \frac{M}{2}$ the coefficients f_k^h are defined as in (6) above, and similarly, for $k = \frac{M}{2}+1,\, \frac{M}{2}+2,\, \ldots,\, M-1$,

$$\hat{f}_k^h \stackrel{\text{def}}{=} \langle\, f^h \,,\, w^{(k)} \,\rangle$$

where $w^{(k)}$ is defined as in (3), albeit with $k = \frac{M}{2}+1,\, \frac{M}{2}+2,\, \ldots,\, M-1$. Is F a real-valued function? In other words, is $F(x) \in \mathbb{R}$ for all $x \in [0,1]$?

Problem 04 (20 points total) Consider the downwind method,

$$(Lu)_i \stackrel{\text{def}}{=} u_i^n \,+\, \sigma\left(u_i^n \,-\, u_{i+1}^n\right)\,. \tag{8}$$

(a) (10 points) What is the symbol $\lambda\,(\beta)$ of L?

(b) (10 points) Let $M > 0$ be an even integer. For $k = -\frac{M}{2}+1,\, -\frac{M}{2}+2,\, \ldots\,,\, \frac{M}{2}-1\,,\, \frac{M}{2}$, find all of the modes $w^{(k)}$ amplified by the linear finite difference operator L.

Problem 05 (20 points total) Let $\sigma \stackrel{\text{def}}{=} a\Delta t/h$ and let L denote the Lax-Friedrichs method

$$u_j^{n+1} \,=\, (Lu^n)_j \stackrel{\text{def}}{=} \frac{1}{2}\,\left(u_{j-1}^n \,+\, u_{j+1}^n\right) \,+\, \frac{\sigma}{2}\,\left(u_{j-1}^n \,-\, u_{j+1}^n\right)$$

(a) (05 points) Is the Lax-Friedrichs method stable in the discrete L^2 norm,

$$\|\, Lu^n \,\|_2 \,\le\, \|\, u^n \,\|_2\,? \tag{9}$$

Is there a constraint you must place on the parameter σ in order for (9) to hold? If so, what is it?

(b) (05 points) Show that Lax-Friedrichs is a *consistent* discretization of the model problem,

$$\frac{\partial v}{\partial t} + a\frac{\partial v}{\partial x} = 0, \tag{10a}$$

with initial data

$$v(x,0) = v^0(x), \tag{10b}$$

on the unit interval $\Omega = [0,1]$ with periodic boundary conditions,

$$v(0,t) = v(1,t), \quad \text{for all } t \geq 0. \tag{10c}$$

(c) (10 points) If the answer to item **Problem 05 (a)** above is 'yes', then prove Lax-Friedrichs converges to the exact solution of the model problem (10). Prove this directly, rather than by applying the Lax Equivalence Theorem. In other words, use the same ideas I used to prove that consistency plus stability implies convergence in the Lax-Equivalence Theorem.