MAT 228A Theory Homework 2

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Problem 1

THEOREM. *$\|f\|_2^2 = \|\hat{f}\|_2^2$, where $\hat{f}$ is the discrete Fourier transform of f.*

PROOF. Let $\mathbb{C}^M$ inherit the structure of a Hilbert space from the inner product defined in equation (5). The discrete Fourier transform is a linear operator in this space:

$$\hat{f} = \begin{pmatrix} \langle f^h, w^{(k_1)} \rangle \\ \cdots \\ \langle f^h, w^{(k_M)} \rangle \end{pmatrix} = F^* f \tag{0.1}$$

Fourier matrix $F = \left(w^{(k_1)} \dots w^{(k_M)} \right)$ is unitary, since the column vectors of F form an orthonormal basis of $\mathbb{C}^M$. Naturally, the matrix F^* is also unitary. By definition, unitary operators preserve inner products and, therefore, norms:

$$\|\hat{f}\|_2^2 = \langle \hat{f}, \hat{f} \rangle = \langle F^* f, F^* f \rangle = \langle f, f \rangle = \|f\|_2^2 \tag{0.2}$$

$\square$

Problem 2

THEOREM. *Prove that $\hat{f^h} = f^h$, where $\hat{f^h} = \sum\limits_k \langle f^h, w^{(k)} \rangle w^{(k)}$*

PROOF. Fourier matrix F is unitary: $FF^* = F^*F = I \implies$

$$\hat{f^h} = \sum_k \langle f^h, w^{(k)} \rangle w^{(k)} = \left(w^{(k_1)} \dots w^{(k_M)} \right) \begin{pmatrix} \langle f^h, w^{(k_1)} \rangle \\ \cdots \\ \langle f^h, w^{(k_M)} \rangle \end{pmatrix} = FF^* f^h = f^h \tag{0.3}$$

The function

$$\hat{f}(x) = \sum_k \langle f^h, w^{(k)} \rangle w^{(k)}(x) \tag{0.4}$$

is infinitely differentiable, as a finite linear combination of infinitely differentiable functions, and is equal to the original function f at the grid points. Therefore, $\hat{f}$ interpolates f (trigonometric interpolation).

□

Problem 3

Lemma 0.1.

$$\overline{c_k} = c_{-k} \Longleftrightarrow \sum_{|k|\leqslant n} c_k e^{2\pi ikx} \in \mathbb{R}\ \forall x \in [0,1]$$

Sufficiency. Let $\overline{c_k} = c_{-k},\ |k| \leqslant n$. Then

$$S_n = \sum_{|k|\leqslant n} c_k e^{2\pi ikx} = c_0 + \sum_{k=1}^{n}(c_k e^{2\pi ikx} + c_{-k}e^{-2\pi ikx}) = c_0 + \sum_{k=1}^{n}(c_k e^{2\pi ikx} + \overline{c_k e^{2\pi ikx}})$$

which is invariant under complex conjugation:

$$\overline{S_n} = S_n \Longrightarrow S_n \in \mathbb{R}$$

□

Necessity. Let $S_n \in \mathbb{R}$ Then

$$S_n - \overline{S_n} = 0 \Longleftrightarrow \sum_{|k|\leqslant n} (c_k - \overline{c_{-k}})e^{2\pi ikx} = 0$$

Since $e^{2\pi ikx}$ are linearly independent, the identity holds for all x if and only if

$$c_k - \overline{c_{-k}} = 0$$

□

(a). Since

$$\hat{f}(x) = \sum_{-\frac{M}{2}<k\leqslant\frac{M}{2}} c_k e^{2\pi ikx} = \sum_{|k|\leqslant\frac{M}{2}} c_k e^{2\pi ikx} - c_{-\frac{M}{2}} e^{-M\pi ix} = \Sigma(x) + r(x)$$

and since $\Sigma(x) \in \mathbb{R}$, by the lemma, and, $r(x) \in \mathbb{C}$,

$$S(x) + r(x) = \hat{f}(x) \in \mathbb{C}$$

i.e. the interpolation is not strictly real-valued (N must be odd in order to obtain real-valued trigonometric interpolation in the complex form).

(b). Let $\hat{f}(x)$ be strictly real-valued. Then

$$\hat{f}(x) - \overline{\hat{f}(x)} = \sum_{k\leqslant M-1} (c_k e^{2\pi ikx} - \overline{c_k}e^{-2\pi ikx}) \equiv 0 \Longleftrightarrow c_k = 0$$

since complex exponents with different wavenumbers k are linearly independent.

But then $\hat{f}(x) \equiv 0$, which means that $\hat{f}(x)$ is, in general, complex.

Problem 4

(a).

THEOREM. *Compute the symbol for the downwind method:*

$$(Lu)_i = u_i^n + \sigma(u_i^n - u_{i+1}^n)$$

PROOF.

$$L = \sum_{s=0}^{1} c_s S^s = 1 + \sigma(1 - S^1) = (1+\sigma) - \sigma S^1$$

The symbol is

$$\lambda(\beta) = \sum_{s=0}^{1} c_s e^{i\beta s} = (1+\sigma) - \sigma e^{i\beta}$$

□

(b).

THEOREM. *Find amplification factors for all modes, $k \in \overline{[-\frac{M}{2}+1, \frac{M}{2}]}$*

PROOF. The amplification factor for k-th mode is $|\lambda(\beta(k))|$.

$$\begin{aligned}(0.5)\quad |\lambda(\beta)|^2 = \lambda(\beta)\overline{\lambda(\beta)} &= ((1+\sigma) - \sigma e^{i\beta})((1+\sigma) - \sigma e^{-i\beta}) = \\ &(1+\sigma)^2 + \sigma^2 - 2\sigma(1+\sigma)\cos\beta = 1 + 2\sigma(1+\sigma)(1-\cos\beta) \\ &\implies |\lambda(\beta(k))| = \sqrt{1 + 2\sigma(1+\sigma)(1-\cos 2\pi kh)}\end{aligned}$$

Thus, *all modes are amplified* except

$$\cos\beta(k) = \cos 2\pi kh = 0 \iff k = 0$$

Since almost all modes are amplified by the finite difference operator, the downwind method is inherently unstable.

□

Problem 5

(a).

THEOREM. *Lax-Friedrichs method is stable in L^2 norm.*

PROOF. The symbol of Lax-Friedrichs finite difference operator is:

$$L = \sum_{s=-1}^{1} c_s S^s = \frac{1}{2}(S^{-1} + S^1) + \frac{\sigma}{2}(S^{-1} - S^1) \implies$$

$$\begin{aligned}(0.6)\quad \lambda(\beta) = \frac{1}{2}(e^{i\beta} + e^{-i\beta} - \sigma(e^{i\beta} - e^{-i\beta})) &= \cos\beta - i\sigma\sin\beta \implies \\ |\lambda(\beta)|^2 &= \cos^2\beta + \sigma^2\sin^2\beta = 1 - (1-\sigma^2)\sin^2\beta\end{aligned}$$

$$\max_{\beta} |\lambda(\beta)| \leqslant 1 \Longleftrightarrow \sigma \in [0,1]$$

Thus, by spectral stability criterion, Lax-Friedrichs method is stable in L^2 norm whenever $\sigma \in [0,1]$

□

(b).

THEOREM. *Lax-Friedrichs method is consistent.*

PROOF. The local truncation error is:

$$\tau_i^n = \frac{u_i^{n+1} - (Lu^n)_i}{\Delta t} = \frac{1}{\Delta t}(u_i^{n+1} - \frac{1+\sigma}{2}u_{i-1}^n - \frac{1-\sigma}{2}u_{i+1}^n)$$

It is convenient to represent the values of a function and its derivatives in operator form, as a vector in which the n-th component is $(n-1)$-th term of the Taylor series (i.e. as k-jets, where $k=2$ in our case, i.e. up to the first derivative).

Thus,

$$\frac{u_i^{n+1}}{\Delta t} = \begin{pmatrix} 1 + O(\Delta t) \\ \partial_t \end{pmatrix}, \ \frac{u_{i-1}^n}{\Delta t} = \begin{pmatrix} 1 + O(h) \\ -\frac{a}{\sigma}\ \partial_x \end{pmatrix}, \ \frac{u_{i+1}^n}{\Delta t} = \begin{pmatrix} 1 + O(h) \\ \frac{a}{\sigma}\ \partial_x \end{pmatrix}$$

and

$$\tau_i^n = \begin{pmatrix} 1 - (\frac{1+\sigma}{2} + \frac{1-\sigma}{2}) + O(\Delta t) \\ \partial_t + \frac{a(1+\sigma)}{2\sigma}\partial_x - \frac{a(1-\sigma)}{2\sigma}\partial_x \end{pmatrix} = \begin{pmatrix} O(\Delta t) \\ \partial_t + a\partial_x \end{pmatrix} = O(\Delta t)$$

Due to triangle inequality for norms,

$$\|\tau^n\| = \frac{\|u^{n+1} - (Lu^n)\|}{\Delta t} \leqslant \max_i |\tau_i^n| = O(\Delta t) \tag{0.7}$$

$$\|\tau^n\| = O(\Delta t) \longrightarrow 0, \ t \longrightarrow 0$$

Thus, Lax-Friedrichs method is consistent.

□

(c).

THEOREM. *Lax-Friedrichs method converges to the exact solution of the linear transport equation.*

PROOF. Convergence of a consistent stable linear finite difference method is essentially the consequence of triangle inequality property of norms, the definition of an operator norm which implies $\|Lx\| \leqslant \|L\|\|x\|$, the fact that $\|L\| \leqslant 1$ for stable Lax-Friedrichs operator, and, finally, linearity of the latter:

(0.8)
$$\begin{aligned}\|u(t^n) - u^n\| = \|u(t^n) - Lu^{n-1}\| = \|u(t^n) - Lu(t^{n-1}) + Lu(t^{n-1}) - Lu^{n-1}\| \leqslant \\ \leqslant \|u(t^n) - Lu(t^{n-1})\| + \|Lu(t^{n-1}) - Lu^{n-1}\| \leqslant \\ \leqslant \|u(t^n) - Lu(t^{n-1})\| + \|L(u(t^{n-1}) - u^{n-1})\| \leqslant \\ \leqslant \|u(t^n) - Lu(t^{n-1})\| + \|L\|\|u(t^{n-1}) - u^{n-1}\| \leqslant \\ \leqslant \|u(t^n) - Lu(t^{n-1})\| + \|u(t^{n-1}) - u^{n-1}\|\end{aligned}$$

Since Lax-Friedrichs is consistent and its truncation error is $O(\Delta t)$

$$\|u(t^n) - Lu(t^{n-1})\| = O(\Delta t^2) \Longrightarrow \\ \|u(t^n) - u^n\| = O(\Delta t^2) + \|u(t^{n-1}) - u^{n-1}\| \tag{0.9}$$

Since $\|u(t^n)-u^n\|$ is defined recursively through $\|u(t^{n-1})-u^{n-1}\|$, by induction we find that

$$\|u(t^n) - u^n\| = O(N\Delta t^2) + \|u(t^0) - u^0\| = O(\Delta t)$$

Thus, consistency and stability imply *first-order* convergence of Lax-Friedrichs method:

$$\|u(t^n) - u^n\| \longrightarrow 0, \ \Delta t \longrightarrow 0$$

□