MAT 228A Theory Homework 1

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0.1.1

THEOREM.

$$\left|\frac{f(x_i)-f(x_{i-1})}{h}-f'(x_i)\right| \leqslant Ch \tag{0.1}$$

where C is independent of $h = x_i - x_{i-1}$, $f \in C^2[x_{i-1}, x_i]$

PROOF. By Lagrange's theorem, the Taylor series of $f(x)$ about x_{i-1} can be expressed in the following way:

$$f(x_{i-1}) = f(x_i - h) = f(x_i) - f'(x_i)h + \frac{f''(\theta)}{2}h^2 \tag{0.2}$$

where $\theta \in [x_{i-1}, x_i]$. From (0.1) it follows that

$$\left|\frac{f(x_i)-f(x_{i-1})}{h}-f'(x_i)\right| = \left|\frac{f''(\theta)}{2}\right| h \tag{0.3}$$

from which the following estimate is true

$$\left|\frac{f(x_i)-f(x_{i-1})}{h}-f'(x_i)\right| \leqslant Ch \tag{0.4}$$

where $C = \sup\limits_{\theta\in[x_{i-1},x_i]} \left|\frac{f''(\theta)}{2}\right| = \max\limits_{\theta\in[x_{i-1},x_i]} \left|\frac{f''(\theta)}{2}\right|$. The supremum is necessarily attained, since f'' is a continuous function on a compact $[x_{i-1}, x_i]$.

$\square$

0.1.2

THEOREM.

$$\frac{(f(x_{i+1})+E(x_{i+1})h^2)-(f(x_{i-1})+E(x_{i-1})h^2)}{2h} = \\ f'(x_i)+O(h^2)+E'(x_i)h^2+O(h^4) \tag{0.5}$$

where $h = x_i - x_{i-1} = x_{i+1} - x_i,\ f, E \in C^3[x_{i-1}, x_{i+1}]$

PROOF. By Lagrange's theorem, the Taylor series of $F(x) \in C^3[x_{i-1}, x_{i+1}]$ about x_{i+1} and x_{i-1} can be expressed in the following way:

$$F(x_{i+1}) = F(x_i) + F'(x_i)h + \frac{F''(x_i)}{2}h^2 + \frac{F'''(\theta_1)}{6}h^3 \tag{0.6}$$

$$F(x_{i-1}) = F(x_i) - F'(x_i)h + \frac{F''(x_i)}{2}h^2 - \frac{F'''(\theta_2)}{6}h^3 \tag{0.7}$$

where $\theta_1, \theta_2 \in [x_{i-1}, x_i]$. Subtracting (0.7) from (0.6) and dividing by $2h$ results in

$$\frac{F(x_{i+1}) - F(x_{i-1})}{2h} = F'(x_i) + \frac{F'''(\theta_1) + F'''(\theta_2)}{12}h^2 \tag{0.8}$$

Considering that $F(x) = f(x) + E(x)h^2$, (0.8) becomes

$$\frac{(f(x_{i+1}) + E(x_{i+1})h^2) - (f(x_{i-1}) + E(x_{i-1})h^2)}{2h} = \\ f'(x_i) + E'(x_i)h^2 + \frac{f'''(\theta_1) + f'''(\theta_2)}{12}h^2 + \frac{E'''(\theta_1) + E'''(\theta_2)}{12}h^4 \tag{0.9}$$

from which follows the estimate

$$\left| \frac{(f(x_{i+1}) + E(x_{i+1})h^2) - (f(x_{i-1}) + E(x_{i-1})h^2)}{2h} - f'(x_i) - E'(x_i)h^2 \right| \leqslant \\ C_1h^2 + C_2h^4 \tag{0.10}$$

where $C_1 = \sup\limits_{\theta \in [x_{i-1}, x_{i+1}]} \left| \frac{f'''(\theta)}{12} \right|$, $C_2 = \sup\limits_{\theta \in [x_{i-1}, x_{i+1}]} \left| \frac{E'''(\theta)}{12} \right|$.

The suprema are necessarily attained, since f''', E''' are continuous functions on a compact $[x_{i-1}, x_{i+1}]$.

But the equation (0.10) is simply the definition of

$$\frac{(f(x_{i+1}) + E(x_{i+1})h^2) - (f(x_{i-1}) + E(x_{i-1})h^2)}{2h} = \\ f'(x_i) + E'(x_i)h^2 + O(h^2) + O(h^4) \tag{0.11}$$

□

0.2.1

THEOREM. $(\forall K \in \mathbb{R}_+) \wedge (\forall n \in \mathbb{N})$

$$1 + K\Delta t \leqslant e^{K\Delta t} \implies (1 + K\Delta t)^n \leqslant e^{Kt} \tag{0.12}$$

where $t = n\Delta t$.

PROOF. Since $K\Delta t \geqslant 0$,

$$1 + K\Delta t \leqslant 1 + K\Delta t + \sum_{k=2}^{\infty} \frac{(K\Delta t)^k}{k!} = \sum_{k=0}^{\infty} \frac{(K\Delta t)^k}{k!} = e^{K\Delta t} \tag{0.13}$$

i.e.

$$1 + K\Delta t \leqslant e^{K\Delta t} \tag{0.14}$$

and, hence,

$$(1 + K\Delta t)^n \leqslant e^{Kn\Delta t} = e^{Kt} \tag{0.15}$$

Since $t = n\Delta t$.

□

0.2.3

THEOREM. *Heun's method*

$$y^{n+1} = y^n + \tfrac{\tau}{2}[f(y^n) + f(y^n + \tau f(y^n))]$$

is a stable and consistent discretization of the initial value problem

$$y_t = f(y),\ \ y(0) = y_0$$

when $f \in C^2$

CONSISTENCY. The truncation error is

$$\frac{1}{\tau}[y(t^{n+1}) - y(t^n) - \frac{\tau}{2}[f(y(t^n)) + f(y(t^n) + \tau f(y(t^n)))]] \tag{0.16}$$

Remark that, since $f \in C^2 \implies y \in C^3$,

$$y(t^{n+1}) = y(t^n + \tau) = y(t^n) + y_t(t^n)\tau + \frac{y_{tt}(t^n)\tau^2}{2} + O(\tau^3) \tag{0.17}$$

and that

$$f(y(t^n) + \tau f(y(t^n))) = f(y(t^n)) + f_y(y(t^n))f(y(t^n))\tau + O(\tau^2) \tag{0.18}$$

Relations (0.17) and (0.18) imply, after reordering of terms, that the truncation error is

$$[y_t(t^n) - f(y(t^n))] + \frac{\tau}{2}[y_{tt}(t^n) - f_y(y(t^n))f(y(t^n))] + O(\tau^2) \tag{0.19}$$

However, it is true that

$$y_t(t^n) - f(y(t^n)) = 0 \tag{0.20}$$

which is the statement of the initial value problem, and that

$$\begin{aligned} y_{tt}(t^n) = f_t(y(t^n)) = f_y(y(t^n))y_t(t^n) = f_y(y(t^n))f(y(t^n)) \\ \implies y_{tt}(t^n) - f_y(y(t^n))f(y(t^n)) = 0 \end{aligned} \tag{0.21}$$

Thus, from (0.20) and (0.21) it follows that the truncation error is of second order:

$$O(\tau^2) \tag{0.22}$$

and, therefore, Heun's method is consistent.

□

Stability. Let

$$F(y) = \frac{f(y) + f(y + \tau f(y))}{2} \tag{0.23}$$

Also, $f \in C^2 \implies F \in C^2$.

Since $\mathbb{R}$ is locally compact, any two states $y_1 < y_2 < \infty$ belong to some compact $K \subset \mathbb{R}$. If the solution is sought only for such time intervals during which the solution does not escape the compact K, then $F \in C^2(K) \implies F \in Lip(K)$, which means that

$$\forall y_1, y_2 \in K,\ |F(y_2) - F(y_1)| \leqslant C|y_2 - y_1| \tag{0.24}$$

where the Lipschitz constant $C = \sup\limits_{\theta \in K}|F_y(\theta)|$, by Lagrange's theorem. The supremum is attained because F_y is continuous on a compact.

Now, consider how the approximation operator "repels" the two states:

$$\begin{aligned} |L_\tau[y_2] - L_\tau[y_1]| = |(y_2 - y_1) + \tau(F(y_2) - F(y_1))| \leqslant |y_2 - y_1| + \tau|F(y_2) - F(y_1)| \\ \leqslant (1 + \tau C)|y_2 - y_1| \end{aligned} \tag{0.25}$$

But that is exactly the definition of stability. Thus, the notion of stability of an approximation operator is the same as the notion of compactness of an operator: an operator is called *compact* if the image of a bounded set is a totally bounded set (i.e. the image of a bounded set is precompact). Since precompact sets are bounded, compact operators preserve boundedness of sets. But that is exactly how stability is understood: it is when a bounded region of uncertainty in input remains bounded after the approximation operator is applied.

□

CONVERGENCE. Since Heun's method is stable and consistent when $f \in C^2$, by Theorem 0.2.2 it converges to the exact solution of the initial value problem stated in the theorem. By the same theorem, the convergence of Heun's method is of *second order*, which is the same order as the order of the truncation error.

$\square$

1.1.1

THEOREM. *The centered difference method* $u_i^{n+1} = u_i^n + \frac{\sigma}{2}(u_{i-1}^n - u_{i+1}^n)$ *is unstable in the discrete max norm* $\|\cdot\|_\infty$

PROOF. The discrete max norm and the discrete L^2 norm are related thus:

$$\|x\|_\infty \leqslant \|x\|_2 \leqslant \sqrt{n}\|x\|_\infty \tag{0.26}$$

where $x \in \mathbb{R}^n$. This relationship also means that the discrete max norm and the discrete L^2 norm induce the same topology on $\mathbb{R}^n$. In fact, in finite dimensional linear spaces, all norms induce the same topology. Thus, the stability of a finite difference scheme is a topological property. This make sense, because the spectrum of the approximation operator, which determines the stability of a finite difference scheme, is a topological invariant.

Since the centered difference method is unstable in the L^2 norm, from (0.26) it follows that it is also unstable in the discrete max norm $\|\cdot\|_\infty$.

NOTE: is the centered difference scheme *always* unstable?

Counter-example:

$$\max_{\phi\in S}|\lambda(\phi)| = \sqrt{1+\sigma^2} = \sqrt{1+(\frac{a\tau}{h})^2} \tag{0.27}$$

If $\tau = Ah^2$, $A \in R_+$ while $\tau, h \to 0$, then

$$\max_{\phi\in S}|\lambda(\phi)| = \sqrt{1+(\frac{a\tau}{h})^2} = \sqrt{1+a^2A\tau} = 1 + \frac{a^2A\tau}{2} + O(\tau^2) \tag{0.28}$$

$$\max_{\phi\in S}|\lambda(\phi)| = 1 + \frac{a^2A\tau}{2} + O(\tau^2) \leqslant e^{C\tau} \tag{0.29}$$

where $C < \infty$ is a constant. Therefore, by spectral stability criterion, the centered difference scheme must be stable if $\tau = Ah^2$, $A \in R_+$ while $\tau, h \to 0$. However, this scheme is not preferred practically, since the stability requirement on the time step $\tau = O(h^2)$ is too rigid.

$\square$