Let L denote a linear finite difference method of the form

$$\left(Lu^n\right)_i = \sum_s c_s\, u^n_{j+s}\,. \tag{1}$$

Problem 01 (20 points) Prove that the linear finite difference method L is consistent if and only if

$$\sum_s c_s = 1 \qquad \text{and} \tag{2a}$$

$$\sum_s c_s\, s = -\sigma\,. \tag{2b}$$

HINT: For a linear finite difference method to be consistent, the first two terms in the Taylor series expansion of the symbol must be equal to the first two terms in the Taylor expansion for $e^{-\iota\sigma\beta}$.

Problem 02 (10 points total) Prove that if the linear finite difference method L in (1) is second-order accurate, then

$$\sum_s c_s\, s^2 = \sigma^2\,. \tag{3}$$

Problem 03 (20 points) Prove that only even powers of β appear in the Taylor series expansion in β about $\beta = 0$ of $|\lambda(\beta)|$, the magnitude of the symbol of L.

HINT: Start by showing this must be true for $|\lambda(\beta)|^2$.

Problem 04 (20 points total) What is the relationship between the order $2q$ of the leading order term in β of this expansion and the order of accuracy p of the truncation error? You must provide a convincing argument for your answer.

Problem 05 (30 points) (The zero average phase error property of Fromm's method)

Explain the convergence rate you observed in Computing Homework 02 when you used Fromm's method on smooth initial data (e.g., the Gaussian pulse) with a CFL number of $\sigma = 0.5$.

There are (at least) two ways to do this problem, either of which will suffice for full credit. In other words, just do one of either Part **(a)** or Part **(b)** below.

(a) (Truncation Error Analysis of Fromm's method) Carry out a truncation error analysis for Fromm's method,

$$u_{j+\frac{1}{2}}^{n+\frac{1}{2}} \;=\; u_j^n \;+\; \frac{1}{2}\,(1 \;-\; \sigma)\,\Delta u_j^n\,, \tag{4a}$$

$$u_j^{n+1} \;=\; u_j^n \;+\; \sigma\left(u_{j-\frac{1}{2}}^{n+\frac{1}{2}} \;-\; u_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right)\,, \tag{4b}$$

where

$$\Delta u_j^n \;\stackrel{\text{def}}{=}\; \frac{1}{2}\,\left(u_{j+1}^n - u_{j-1}^n\right)\,. \tag{5}$$

Explicitly include *all* terms up to fourth order in your Taylor series expansions. In other words, retain all terms of the form $C_{pq}\,h^p\,\Delta t^q$ where $p+q \le 4$ and C_{pq} is a constant that will depend on the CFL number σ, one or more of the derivatives of the exact solution $v(x,t)$ up to fourth order,

$$\frac{\partial^p \partial^q v}{\partial x^p \partial t^q} \qquad \text{where } p+q \;\le\; 4\,,$$

but not on h or Δt. Now set $\sigma = 0.5$.

(b) (Fourier Analysis of the Amplitude and Phase Error of Fromm's method) Carefully examine the amplitude and phase errors for Fromm's method presented in §1.4.1 "The Amplitude Error" and §1.4.2 "The Phase Error" of the Lecture Notes. Use these results to construct an argument that Fromm's method should converge like $O\left(h^3\right)$ for $\sigma \;=\; 0.5$. You may use the analytical results presented in §1.4.1 and §1.4.2. In other words, you do not have to rederive them for this assignment.