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A Short Proof of the Pi Theorem of Dimensional Analysis

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The Pi Theorem of Vaschy and Buckingham [8, 4] has been proved in many different ways, varying considerably in complexity and generality [1–8]. We present here a short proof somewhat akin to that of Birkhoff [1], but expressed more concisely by the simple device of taking logarithms from the start to convert it into a (virtually trivial) linear problem.

By a *relation* between physical quantities $x_0, x_1, \ldots, x_n$ we mean a statement expressible in the form

$$(x_0, x_1, \ldots, x_n) \in E,$$

where E is some subset of $\mathbb{R}^{n+1}$. We do *not* assume that E is a smooth hypersurface in $\mathbb{R}^{n+1}$, nor in general do we assume that E is the graph of a (single-valued) function of n variables (but see Theorem 2 below).

Theorem 1 (The Pi Theorem). *A relation between positive dimensional quantities x_i with dimensions $M^{\alpha_i}L^{\beta_i}T^{\gamma_i}$ (where $i = 0, 1, \ldots, n$, α_i, β_i and γ_i are real but not necessarily rational, and M, L and T denote mass, length and time respectively) that is, independent of the choice of units of mass, length and time, may be expressed as a relation between dimensionless products of powers of the x_i (monomials).*

Proof: In terms of new variables $\xi_i = \ln x_i$, the relation takes the form

$$\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_n) \in E,$$

where E is a subset of $\mathbb{R}^{n+1}$. Now, if the units of mass, length and time are divided by μ, λ and τ respectively, then x_i gets multiplied by $\mu^{\alpha_i}\lambda^{\beta_i}\tau^{\gamma_i}$, so ξ_i increases by an amount $a\alpha_i + b\beta_i + c\gamma_i$, where $a = \ln \mu$, $b = \ln \lambda$, $c = \ln \tau$. Since the relation is independent of units, the set E must be invariant under all translations by vectors of the form $a\boldsymbol{\alpha} + b\boldsymbol{\beta} + c\boldsymbol{\gamma}$, where $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, etc., i.e. by vectors in the subspace V of $\mathbb{R}^{n+1}$ spanned by $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ (this group of translations being isomorphic with the dimensional group [1]). Thus E must be of the form $A + V$, where A is a subset of the orthogonal complement W of V with respect to the usual scalar product on $\mathbb{R}^{n+1}$. Hence $\boldsymbol{\xi} \in E$ if and only if $P\boldsymbol{\xi} \in A$, where P is the orthogonal projection of $\mathbb{R}^{n+1}$ onto W. In terms of an orthonormal basis $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_m$ of W, $\xi \in E$ if and only if $(\mathbf{e}_0 \cdot \boldsymbol{\xi}, \mathbf{e}_1 \cdot \boldsymbol{\xi}, \ldots, \mathbf{e}_m \cdot \boldsymbol{\xi})$ belongs to a certain subset B of $\mathbb{R}^{m+1}$ (N.B. $n - 3 \leq m \leq n$). Thus the original relation is equivalent to a relation between the $m + 1$ quantities $\mathbf{e}_j \cdot \boldsymbol{\xi}$, or equivalently between their exponentials

$$\Pi_j = \exp \mathbf{e}_j \cdot \boldsymbol{\xi} \quad (j = 0, 1, \ldots, n),$$

each of which is a dimensionless product of powers of the x_i.

In practical applications of dimensional analysis, the following theorem is often more useful than the Pi Theorem itself.

Theorem 2. *If the relation of Theorem* 1 *takes the form*

$$x_0 = f(x_1, \ldots, x_n)$$

for some function f, then f must be of the form

$$f(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i^{\delta_i}\right) g(\Pi_1, \Pi_2, \ldots, \Pi_m),$$

where the δ_i are real numbers and the Π_j dimensionless products of powers of $x_1, \ldots, x_n$.

Proof: We write vectors in $\mathbb{R}^{n+1}$ in the form $\boldsymbol{\xi} = (\xi_0, \tilde{\boldsymbol{\xi}})$, where $\tilde{\boldsymbol{\xi}} \in \mathbb{R}^n$ consists of the last n components of $\boldsymbol{\xi}$. Let $\tilde{V}$ be the subspace of $\mathbb{R}^n$ spanned by $\tilde{\boldsymbol{\alpha}}$, $\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\gamma}}$, $\tilde{W}$ its orthogonal complement, $\tilde{P}$ and $\tilde{Q}$ the orthogonal projections of $\mathbb{R}^n$ onto $\tilde{W}$ and $\tilde{V}$ respectively. We have $\boldsymbol{\xi} \in E$ if and only if $\xi_0 = \phi(\tilde{\boldsymbol{\xi}})$, where

$$\phi(\xi_1, \ldots, \xi_n) \doteq \ln f(e^{\xi_1}, \ldots, e^{\xi_n}).$$

Since E is invariant under translations by vectors in V, ϕ satisfies

$$\phi(\tilde{\boldsymbol{\xi}} + a\tilde{\boldsymbol{\alpha}} + b\tilde{\boldsymbol{\beta}} + c\tilde{\boldsymbol{\gamma}}) = \phi(\tilde{\boldsymbol{\xi}}) + a\alpha_0 + b\beta_0 + c\gamma_0$$

for any real a, b and c. Thus

$$\phi(\tilde{\boldsymbol{\xi}} + a\tilde{\boldsymbol{\alpha}} + b\tilde{\boldsymbol{\beta}} + c\tilde{\boldsymbol{\gamma}}) - \phi(\tilde{\boldsymbol{\xi}}) = l(a\tilde{\boldsymbol{\alpha}} + b\tilde{\boldsymbol{\beta}} + c\tilde{\boldsymbol{\gamma}}),$$

where l is a linear form on $\tilde{V}$, which is independent of $\tilde{\boldsymbol{\xi}}$. Hence, for any $\tilde{\boldsymbol{\xi}}$ in $\mathbb{R}^n$, we have

$$\phi(\tilde{\boldsymbol{\xi}}) = \phi(\tilde{P}\tilde{\boldsymbol{\xi}} + \tilde{Q}\tilde{\boldsymbol{\xi}}) = \phi(\tilde{P}\tilde{\boldsymbol{\xi}}) + l(\tilde{Q}\tilde{\boldsymbol{\xi}}),$$

in which the last term on the right-hand side is a linear form in $\tilde{\boldsymbol{\xi}}$ on $\mathbb{R}^n$ vanishing on the subspace $\tilde{W}$, and is therefore of the form $\tilde{\boldsymbol{\delta}} \cdot \tilde{\boldsymbol{\xi}}$, where $\tilde{\boldsymbol{\delta}}$ is a vector in $\tilde{V}$. Thus, if $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_m$ is an orthonormal basis of $\tilde{W}$, we have

$$\begin{aligned}\phi(\tilde{\boldsymbol{\xi}}) &= \phi\left[\sum_{j=1}^{m} (\tilde{\mathbf{e}}_j \cdot \tilde{\boldsymbol{\xi}})\tilde{\mathbf{e}}_j\right] + \tilde{\boldsymbol{\delta}} \cdot \tilde{\boldsymbol{\xi}} \\ &= \psi(\Pi_1, \ldots, \Pi_m) + \tilde{\boldsymbol{\delta}} \cdot \tilde{\boldsymbol{\xi}},\end{aligned}$$

say, where

$$\Pi_j = \exp(\tilde{\mathbf{e}}_j \cdot \tilde{\boldsymbol{\xi}})$$

is a dimensionless product of powers of $x_1, \ldots, x_n$. Hence

$$\begin{aligned}f(x_1, \ldots, x_n) &= \left(\prod_{i=1}^{n} x_i^{\delta_i}\right) \exp \psi(\Pi_1, \ldots, \Pi_m) \\ &= \left(\prod_{i=1}^{n} x_i^{\delta_i}\right) g(\Pi_1, \ldots, \Pi_m),\end{aligned}$$

say, where the δ_i are the components of the vector $\tilde{\boldsymbol{\delta}}$.

Corollary. *A relation of the kind considered in Theorem 2 can only be independent of units if the dimensions of the quantities $x_0, x_1, \ldots, x_n$ are such that the $(n + 1) \times 3$ matrix whose columns are $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ has the same rank as the $n \times 3$ matrix whose columns are $\tilde{\boldsymbol{\alpha}}$, $\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\gamma}}$.*