(b) Use Calculus to find the exact point $x_{min}=(x_{min}, y_{min})$ at which attains its relative minimum, as well as its minimum value f_{min}. Why is this also the absolute minimum?

Let's calculate $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$

$f(x,y) = x^2+y^2+xy-5x-7y+20$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 2x+y-5 \\ \frac{\partial f}{\partial y}(x,y) = 2y+x-7 \end{cases} \Rightarrow \nabla f(x,y) = (2x+y-5, 2y+x-7)$$

We Know that (x,y) is a critical point $\Leftrightarrow \nabla f(x,y) = (0,0)$

$$\nabla f(x,y) = (0,0) \Leftrightarrow \begin{cases} 2x+y=5 \\ x+2y=7 \end{cases} \Leftrightarrow \begin{cases} 4x+2y=10 \\ x+2y=7 \end{cases} \Leftrightarrow 3x=3 \Leftrightarrow x=1 \Rightarrow y=\frac{7-x}{2}=3$$

Then $(1,3)$ is a critical point. Let's check that $p=(1,3)$ is a minimum calculating the Hessian matrix.

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x}(x,y) & \frac{\partial^2 f(x,y)}{\partial x \partial y} \\ \frac{\partial^2 f(x,y)}{\partial y \partial x} & \frac{\partial^2 f(x,y)}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

We have that $\Delta_1 > 0$, $\Delta_2 > 0$, therefore the associated quadric form of the matrix is positive definite so $(1,3)$ is a relative minimum.

We have that $(1,3)$ is the only critical point therefore $(1,3)$ is an absolute minimum.