Purification of noisy state preparation and measurements

Demonstration of

1. Turn on the IBMQ and find the errors, f, ϵ, q.
2. With the parameter f, ϵ, q, run the purification protocol with your computer and check out the graph of purification.
3. Turn on the IBMQ and apply the purification protocol, after which, run the protocol to find f, ϵ, q. See how the errors are suppressed.

In quantum computing, one desires to prepare the noiseless state $|0\rangle\langle 0|$ for each qubit system. However, in practice, one can prepare a noisy initial state ρ with the fidelity $f = \langle 0|\rho|0\rangle$, instead of preparing $|0\rangle\langle 0|$,

$$\begin{aligned} \rho &= f|0\rangle\langle 0| + (1-f)|1\rangle\langle 1| && (1) \\ &= (1-\varepsilon_{dep}^{SP})|0\rangle\langle 0| + \varepsilon_{dep}^{SP}\frac{\mathbb{1}}{2} && (2) \end{aligned}$$

where $\varepsilon_{dep}^{SP} = 2(1-f)$.

A measurement is desired to be noiseless as well, $M_i = |i\rangle\langle i|$. In practice, however, a measurement contains noise. A noisy measurement with the noise fraction $q = \langle \bar{i}|\tilde{M}_i|\bar{i}\rangle$ can also be described by a noisy measurement under the depolarizing noise as follows,

$$\begin{aligned} \tilde{M}_i &= (1-q)|i\rangle\langle i| + q|\bar{i}\rangle\langle \bar{i}| && (3) \\ &= (1-\varepsilon_{dep}^{meas})|i\rangle\langle i| + \varepsilon_{dep}^{meas}\frac{\mathbb{1}}{2} && (4) \end{aligned}$$

where $\varepsilon_{dep}^{meas} = 2q$. A POVM element for n-qubit system can be written as $\tilde{M}_{\mathbf{i}} = \bigotimes_{k=1}^{n} \tilde{M}_{i_k}$ for $\mathbf{i} = (i_1 i_2 ... i_n)$.

The CNOT operation on a quantum state σ is defined as follows,

$$\begin{aligned} V_2^{AB}(\sigma) &= V_2^{AB}\sigma V_2^{AB\dagger}, && (5) \\ V_2^{AB} &= |0\rangle\langle 0|_A \otimes \mathbb{1}_B + |1\rangle\langle 1|_A \otimes X_B && (6) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. && (7) \end{aligned}$$

If the depolarizing noise with the noise strength ε_{dep}^{CNOT} applies to a noiseless CNOT operation V_2 then a noisy CNOT operation $\tilde{V}_2$ is described as follows,

$$\tilde{V}_2(\sigma) = (1-\varepsilon_{dep}^{CNOT})V_2\sigma V_2^\dagger + \varepsilon_{dep}^{CNOT}\mathrm{tr}[\sigma]\frac{\mathbb{1}}{2}\otimes\frac{\mathbb{1}}{2}. \quad (8)$$

Here $\mathrm{tr}[\sigma]$ in the second term is to enforce the operation is trace-preserving (TP). The collective CNOT operation on (n+1)-qubit registers $SA_1A_2...A_n$ is defined as follows,

$$V_n^{SA_1...A_n} = \prod_{k=1}^{n} V_2^{SA_k}. \quad (9)$$

To look into how the collective CNOT operation affect a noisy quantum state $\gamma = \rho^{\otimes n+1}$, let us consider a partial state γ_{SA_i} of γ:

$$\begin{aligned} (\gamma_{SA_i})_{ab,xy} &= {}_{SA_i}\langle ab|\gamma|xy\rangle_{SA_i} && (10) \\ &= (\gamma)_{1+a\cdot 2^n+b\cdot 2^{n-i+1},1+x\cdot 2^n+y\cdot 2^{n-i+1}}, && (11) \end{aligned}$$

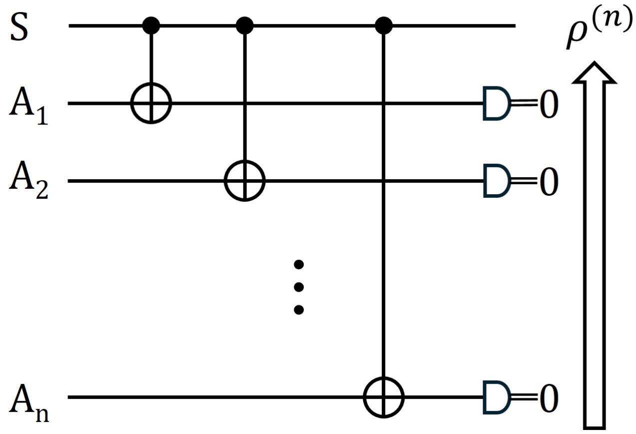

Figure 1. Circuit for the purification

where $\gamma_{SA_i} = \begin{pmatrix} (\gamma_{SA_i})_{00} & (\gamma_{SA_i})_{01} \\ (\gamma_{SA_i})_{10} & (\gamma_{SA_i})_{11} \end{pmatrix}$. If the collective CNOT operation is noiseless, then the state after the operation is

$$V_n^{SA_1 \ldots A_n}(\gamma) = V_n^{SA_1 \ldots A_n} \gamma V_n^{SA_1 \ldots A_n \dagger}. \tag{12}$$

However, it becomes more complicated if the depolarizing noise is applied to each CNOT operation. Recall a noisy CNOT operation in Eq.(8), a noisy collective CNOT operation is described as follows,

$$\tilde{V}_n^{SA_1 \ldots A_n}(\sigma) = \circ_{k=1}^{n} \tilde{V}_2^{SA_k}(\sigma). \tag{13}$$

This can be decomposed into layer-by-layer, let us consider i-th CNOT operation on a state $\sigma^{(i-1)}$ after $(i-1)$-th CNOT operation is applied,

$$\sigma_{SA_i}^{(i)} = (1 - \varepsilon_{dep}^{CNOT}) V_2^{SA_i} \sigma_{SA_i}^{(i-1)} V_2^{SA_i \dagger} + \varepsilon_{dep}^{CNOT} \mathrm{tr}[\sigma_{SA_i}^{(i-1)}] \frac{\mathbb{1}_S}{2} \otimes \frac{\mathbb{1}_{A_i}}{2}, \tag{14}$$

the partial state $\sigma_{SA_i}^{(i)}$ is defined in Eq.(11).

In the state purification protocol, we accept the quantum state of the system register S only when the outcome of measurements on the n-additional registers is identical to 0^n. For the final state after the collective CNOT operation, $\gamma = \tilde{V}_n^{SA_1 \ldots A_n}(\rho)$, the success probability of the state purification with n-additional qubits is defined as follows,

$$p_{succ}^{(n)} = \mathrm{tr}\left[(\mathbb{1}_S \otimes \tilde{M}_{0^n})\gamma\right] \tag{15}$$

$$= p(0^n), \tag{16}$$

$$\tilde{M}_{0^n} = \bigotimes_{k=1}^{n} \tilde{M}_0. \tag{17}$$

Let the unnormalized state after the measurement on the additional registers be $\Gamma^{(n)} = \mathrm{tr}_{A_1 \ldots A_n}\left[(\mathbb{1} \otimes \tilde{M}_{0^n})\gamma\right]$, then the success probability $p_{succ}^{(n)}$ becomes $\mathrm{tr}\Gamma^{(n)}$ and the normalized purified state is

$$\rho^{(n)} = \frac{\Gamma^{(n)}}{p_{succ}^{(n)}}. \tag{18}$$

The fidelity of the purified state with n-additional registers is $f^{(n)} = \langle 0 | \rho^{(n)} | 0 \rangle = \Gamma_{00}^{(n)} / p_{succ}^{(n)} = \rho_{00}^{(n)}$.

[1]