Computational model of lambda calculus

Martin Stoev, Anton Dudov

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1 General Definitions

Definition 1.1. *An enumeration operator (or e-operator) Ψ^A is a r.e. set. For any $A \subseteq \mathbb{N}$*

$$x \in \Psi^A \iff \exists u\,(finite\ D_u \subseteq A)((x,u) \in \Psi) \tag{1}$$

Definition 1.2. *If A is a r.e. set then Ψ_A is the enumeration operator defined by it, namely*

$$x \in \Psi_A^B \iff \exists u\,(D_u isfinite)((x,u) \in A \wedge D_u \subseteq B) \tag{2}$$

Definition 1.3. *If θ is an enumeration operator then G_θ is a well-defined r.e. set defining it, namely*

$$(x,u) \in G_\theta \iff x \in \theta^{D_u} \tag{3}$$

Lemma 1.1. *If Ψ is an enumeration operator, then $\Psi_{G_\Psi} = \Psi$*

Proof. TO BE DONE @anton ☐

Definition 1.4. *Let η be an assignment of r.e. sets to the variables of lambda calculus. With every λ-term E we inductively associate a r.e. set $[\![E]\!]_\eta$:*

1. $[\![x]\!]_\eta = \eta(x)$
2. $[\![E_1 E_2]\!]_\eta = \Psi_{[\![E_1]\!]_\eta}([\![E_2]\!]_\eta)$
3. $[\![\lambda x.E]\!]_\eta = G_{\lambda X.[\![E]\!]_{\eta[x:=X]}}$

Where $\lambda X.[\![E]\!]_{\eta[x:=X]}$ is a function

$$A \in W \mapsto [\![E]\!]_{\eta[x:=A]} \tag{4}$$

Lemma 1.2. *For any environment η and term t, $[\![t]\!]_\eta$ is an c.e. set.*

Proof. By structural induction on the definition of $[\![\mathrm{t}]\!]_\eta$.

1. $[\![x]\!]_\eta = \eta(x)$ by definition
2. To show that $\Psi_{[\![E_1]\!]_\eta}([\![E_2]\!]_\eta)$ is a c.e. set we prove that Ψ_A^B is an enumeration operator which follows from

$$n \in \Psi_A^B \iff \exists u(D_u \subseteq B < n, u >\in A) \tag{5}$$

3. To be done.

☐

Lemma 1.3. *For the following theorem we will need one lemma beforehand for better readability, namely*

$$[\![u]\!]_{\eta[x:=[\![v]\!]_\eta]} = [\![u[x \mapsto v]]\!]_\eta \tag{6}$$

Proof. Structural induction on u.

1. u = x, then:

$$[\![x]\!]_{\eta[x:=[\![v]\!]_\eta]} = \eta(x) = [\![v]\!]_\eta = [\![x[x \mapsto v]]\!]_\eta \tag{7}$$

2. u = y $\neq$ x, then:

$$[\![y]\!]_{\eta[x:=[\![v]\!]_\eta]} = \eta(y) = [\![y]\!]_\eta = [\![y[x \mapsto v]]\!]_\eta \tag{8}$$

3. u = pq, then:

$$\begin{aligned}
[\![pq]\!]_{\eta[x:=[\![v]\!]_\eta]} &\overset{def1.4.2}{=} \Psi_{[\![p]\!]_{\eta[x:=[\![v]\!]_\eta]}}([\![q]\!]_{\eta[x:=[\![v]\!]_\eta]}) \\
&\overset{ind.hyp.}{=} \Psi_{[\![p[x\mapsto v]]\!]_\eta([\![q[x\mapsto v]]\!]_\eta])} \\
&\overset{def1.4.2}{=} [\![p[x \mapsto v]q[x \mapsto v]]\!]_\eta \\
&\overset{defApp}{=} [\![pq[x \mapsto v]]\!]_\eta
\end{aligned}$$

4. u = λ_y p, then:

$$\begin{aligned}
[\![\lambda_y p]\!]_{\eta[x:=[\![v]\!]_\eta]} &\overset{def1.4.3}{=} G_{\Lambda Y.[\![p]\!]_{\eta[x:][\![v]\!]_\eta;y:=Y]}} \\
&\overset{ind.hyp}{=} G_{\Lambda Y.[\![p[x\mapsto v]]\!]_{\eta[y:=Y]}} \\
&\overset{def1.4.3}{=} [\![\lambda_y p[x \mapsto v]]\!]_\eta
\end{aligned}$$

□

Theorem 1.4. *If* $E_1 \overset{\beta}{\twoheadrightarrow} E_2$ *then* $[\![E_1]\!]_\eta = [\![E_2]\!]_\eta$ *for any* η.

Proof. We will prove one step of the β reduction and then by induction the rest will follow

We have that $(\lambda x E_1)E_2 = E_1[x \mapsto E_2]$ and we will prove that $[\![(\lambda x E_1)E_2]\!]_\eta = [\![E_1[x \mapsto E_2]]\!]_\eta$

$$\begin{aligned}
[\![(\lambda x E_1)E_2]\!]_\eta &\overset{def.1.4.2}{=} \Psi_{[\![\lambda x E_1]\!]_\eta}([\![E_2]\!]_\eta) \\
&\overset{def.1.4.3}{=} \Psi_{G_{\Lambda X[\![E_1]\!]_{\eta[x:=X]}}}([\![E_2]\!]_\eta) \\
&\overset{Lemma1.1}{=} \Lambda X[\![E_1]\!]_{\eta[x:=X]}([\![E_2]\!]_\eta) \\
&\overset{def\Lambda}{=} [\![E_1]\!]_{\eta[x:=[\![E_2]\!]_\eta]} \\
&\overset{Lemma1.3}{=} [\![E_1[x \mapsto E_2]]\!]_\eta
\end{aligned}$$

□

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References

[1] S. B. Cooper, *Computability Theory.* 2003.

[2] P. Odifreddi, *Classical recursion theory.* 1989.