Computational model of lambda calculus

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2018-9-28

1 General Definitions

Definition 1.1. *An enumeration operator (or e-operator) Ψ^A is a r.e. set. For any $A \subseteq \mathbb{N}$*

$$x \in \Psi^A \iff \exists u\,(finite\ D_u \subseteq A)\ \&\ ((x,u) \in \Psi) \tag{1}$$

Definition 1.2. *If A is a r.e. set then Ψ_A is the enumeration operator defined by it, namely*

$$x \in \Psi_A^B \iff \exists u\,(D_u is finite)\ \&\ ((x,u) \in A \wedge D_u \subseteq B) \tag{2}$$

Definition 1.3. *If θ is an enumeration operator then G_θ is a well-defined r.e. set defining it, namely*

$$(x,u) \in G_\theta \iff x \in \theta^{D_u} \tag{3}$$

Lemma 1.1. *If Φ is an enumeration operator, then $\Phi_{G_\Psi} = \Psi$*

Proof. ($\subseteq$)

$$\begin{aligned} x \in \Phi^B_{G_\Psi} &\Rightarrow \exists u\ < x, u > \in G_\Psi\ \&\ D_u \subseteq B \\ &\Rightarrow \exists u\ < x, u > \in \Psi^{D_u}\ \&\ D_u \subseteq B \end{aligned}$$

since Du $\subseteq$ B & e-operator is monotone it follows that: $x \in \Psi^B$
($\supseteq$)
Since Ψ is an e-operator, then $\forall B\ \exists e :\ \Psi = \Phi_e$, therefore:

$$\begin{aligned} x \in \Phi^B_e &\Rightarrow \exists u\ < x, u > \in W_e\ \&\ D_u \subseteq B \\ &\Rightarrow \exists u\ < x, u > \in W_e\ \&\ D_u \subseteq D_u,\ therefore \\ x &\in \Psi^{D_u}\ \Rightarrow < x, u > \in G_\Psi \\ &\Rightarrow \exists u\ < x, u > \in G_\Psi\ \&\ D_u \subseteq B \\ &\Rightarrow x \in \Phi_{G_\Psi} \end{aligned}$$

□

2 Computable Lambda Model

Definition 2.1. *Let η be an assignment of r.e. sets to the variables of lambda calculus. With every λ-term E we inductively associate a r.e. set $[\![E]\!]_\eta$:*

1. $[\![x]\!]_\eta = \eta(x)$
2. $[\![E_1 E_2]\!]_\eta = \Psi_{[\![E_1]\!]_\eta}([\![E_2]\!]_\eta)$
3. $[\![\lambda x.E]\!]_\eta = G_{\lambda X.[\![E]\!]_{\eta[x:=X]}}$

Where $\lambda X.[\![E]\!]_{\eta[x:=X]}$ *is a function*

$$A \in W \mapsto [\![E]\!]_{\eta[x:=A]} \tag{4}$$

Lemma 2.1. *For any environment* η *and term* t, $[\![t]\!]_\eta$ *is an c.e. set.*

Proof. By structural induction on the definition of $[\![\mathrm{t}]\!]_\eta$.

1. $[\![x]\!]_\eta = \eta(x)$ by definition
2. To show that $\Psi_{[\![E_1]\!]_\eta}([\![E_2]\!]_\eta)$ is a c.e. set we prove that Ψ_A^B is an enumeration operator which follows from

$$n \in \Psi_A^B \iff \exists u(D_u \subseteq B \;\&\; < n, u > \in A) \tag{5}$$

3. Since the graph of an enumeration operator is an r.e. set, to show that $[\![\lambda_x u]\!]_\eta \overset{def1.4.3}{=} G_{\Lambda X.[\![u]\!]_{\eta[x:=X]}}$ is an r.e. set we prove that $\Lambda X.[\![u]\!]_{\eta[x:=X]}$ is an e-operator. This can be done inductively on u:

   Let

$$[\![E]\!]_\eta[x := W_1 \; y := W_2] = {}^E\Psi_1^W \oplus W_2, where$$
$${}^E\Psi = \{< x, v > \mid x \in [\![E]\!]_{\eta[x:=L(D_v) \; y:=R(D_v)]}\}$$
$$x \in {}^E\Psi^W \iff \exists u \; < x, u > \in {}^E\Psi \;\&\; D_u \subseteq W$$

   - If u = x, then $\Lambda X.[\![u]\!]_{\eta[x:=X]}$ is the identity function $\Lambda X.X$, which is an enumeration operator.
   - If u = y $\neq$ x, then $\Lambda X.[\![u]\!]_{\eta[x:=X]}$ is the constant function $\Lambda X.\eta(y)$, which is an enumeration operator.
   - If u = u_1u_2, then $\Lambda X.[\![u]\!]_{\eta[x:=X]}$ is $\Lambda X \Phi_{[\![u_1]\!]_{\eta[x:=X]}}([\![u_2]\!]_{\eta[x:=X]})$. Where by the i.h. $[\![u_1]\!]_{\eta[x:=X]}$ is an e operator, namely ${}^{u_1}\Psi$ and $[\![u_2]\!]_{\eta[x:=X]}$ is an e operator, namely ${}^{u_2}\Psi$ Therefore $\Phi_{{}^{u_1}\Psi}({}^{u_2}\Psi)$ is an e operator (by the proof from 1.2.2).
   - If u = λ_y E. Then

$$\begin{aligned}
[\![\lambda_y E]\!]_\eta[x := W_i] &\overset{def1.4.3}{=} G_{\Lambda Y.[\![E]\!]_\eta[x:=W_i, y:=Y]} \\
&= \{< x, v > | x \in [\![E]\!]_\eta[x := W_i, y := D_v]\} \\
&\overset{i.h.}{=} \{< x, v > | x \in {}^E\Psi \; W \oplus \; D_v\} \\
&= \{< x, v > | \exists u \; < x, \; u > \in {}^E\Psi \;\&\; D_u \subseteq W \oplus D_v\} \\
&\overset{def\,{}^E\Psi}{=} \{< x, v > | \exists u \; x \; \in \; [\![E]\!]_\eta[x := L(D_u), y := R(D_u)] \;\&\; D_u \subseteq W \oplus D_v\} \\
&= \{<< x, \; v >, \; u > | x \; \in \; [\![E]\!]_\eta[x := L(D_u), y := R(D_u)] \;\&\; D_u \subseteq W \oplus D_v\}
\end{aligned}$$

     Since $[\![E]\!]_\eta[x := L(D_u), y := R(D_u)]$ is an e operator(i.h) (e operators are r.e. by definition) and $D_u \subseteq W \oplus D_v$ is r.e. set it follows that the above set is an e operator.

$\square$

Lemma 2.2. *For the following theorem we will need one lemma beforehand for better readability, namely*

$$[\![u]\!]_{\eta[x:=[\![v]\!]_\eta]} = [\![u[x \mapsto v]]\!]_\eta \tag{6}$$

Proof. Structural induction on u.

1. u = x, then:

$$[\![x]\!]_{\eta[x:=[\![v]\!]_\eta]} = \eta(x) = [\![v]\!]_\eta = [\![x[x \mapsto v]]\!]_\eta \tag{7}$$

2. u = y $\neq$ x, then:

$$[\![y]\!]_{\eta[x:=[\![v]\!]_\eta]} = \eta(y) = [\![y]\!]_\eta = [\![y[x \mapsto v]]\!]_\eta \tag{8}$$

3. u = pq, then:

$$\begin{aligned} [\![pq]\!]_{\eta[x:=[\![v]\!]_\eta]} &\overset{def1.4.2}{=} \Psi_{[\![p]\!]_{\eta[x:=[\![v]\!]_\eta]}}([\![q]\!]_{\eta[x:=[\![v]\!]_\eta]}) \\ &\overset{ind.hyp.}{=} \Psi_{[\![p[x\mapsto v]]\!]_\eta([\![q[x\mapsto v]]\!]_\eta]}) \\ &\overset{def1.4.2}{=} [\![p[x \mapsto v]q[x \mapsto v]]\!]_\eta \\ &\overset{defApp}{=} [\![pq[x \mapsto v]]\!]_\eta \end{aligned}$$

4. u = λ_y p, then:

$$\begin{aligned} [\![\lambda_y p]\!]_{\eta[x:=[\![v]\!]_\eta]} &\overset{def1.4.3}{=} G_{\Lambda Y.[\![p]\!]_{\eta[x:][\![v]\!]_\eta;y:=Y]}} \\ &\overset{ind.hyp}{=} G_{\Lambda Y.[\![p[x\mapsto v]]\!]_{\eta[y:=Y]}} \\ &\overset{def1.4.3}{=} [\![\lambda_y p[x \mapsto v]]\!]_\eta \end{aligned}$$

$\square$

Theorem 2.3. *If $E_1 \overset{\beta}{\twoheadrightarrow} E_2$ then $[\![E_1]\!]_\eta = [\![E_2]\!]_\eta$ for any η.*

Proof. We will prove one step of the β reduction and then by induction the rest will follow

We have that $(\lambda x E_1)E_2 = E_1[x \mapsto E_2]$ and we will prove that

$[\![(\lambda x E_1)E_2]\!]_\eta = [\![E_1[x \mapsto E_2]]\!]_\eta$

$$\begin{aligned}
[\![(\lambda x E_1)E_2]\!]_\eta &\overset{def.1.4.2}{=} \Psi_{[\![\lambda x E_1]\!]_\eta}([\![E_2]\!]_\eta) \\
&\overset{def.1.4.3}{=} \Psi_{G_{\Lambda X [\![E_1]\!]_{\eta[x:=X]}}}([\![E_2]\!]_\eta) \\
&\overset{Lemma 1.1}{=} \Lambda X [\![E_1]\!]_{\eta[x:=X]}([\![E_2]\!]_\eta) \\
&\overset{def \Lambda}{=} [\![E_1]\!]_{\eta[x:=[\![E_2]\!]_\eta]} \\
&\overset{Lemma 1.3}{=} [\![E_1[x \mapsto E_2]]\!]_\eta
\end{aligned}$$

□

Definition 2.2. *We define the fixed point combinator Y to be:*

$$Y \overset{def}{=} \lambda f.((\lambda x.f(x\,x))\,((\lambda x.f(x\,x)) \tag{9}$$

3 Exercises

Further details on this exercises can be found in [1], [2].

Exercise 3.1. *Show that any λ-term F has fixed point YF, such that*

$$YF \;=\; F(YF). \tag{10}$$

Proof.

$$\begin{aligned}
YF &= ((\lambda x.F(x\,x))\,((\lambda x.F(x\,x)) \\
&= F((\lambda x.F(x\,x))\,((\lambda x.F(x\,x)) \\
&= F(\lambda f.((\lambda x.f(x\,x))\,((\lambda x.f(x\,x))F) \\
&= F(YF)
\end{aligned}$$

□

Exercise 3.2. *Let Ψ be any e-operator. Show that Ψ can be expressed as:*

$$\Psi \;=\; \Psi_{[\![x]\!]_\eta} \tag{11}$$

for suitable choise of assigment η.

Proof. Since Ψ is an e-operator, then $\forall B\, \exists e :\ \Psi = \Phi_e$, therefore:

$$\begin{aligned}
n \in \Psi^B &\iff \exists u\ < n, u > \in W_e \,\&\, D_u \subseteq B \\
&\iff \exists u\ < n, u > \in [\![x]\!]_\eta \,\&\, D_u \subseteq B,\ \ where\ \ \eta(x) = W_e \\
&\iff n \in \Psi^B_{[\![x]\!]_\eta}
\end{aligned}$$

□

Exercise 3.3. *Let Ψ be any e-operator. Show that Ψ has a r.e. fixed point W, such that:*

$$\Psi^{W} \;=\; W \tag{12}$$

Proof. Let $W = [\![Yx]\!]_\eta$, then:

$$[\![Yx]\!]_\eta = [\![x(Yx)]\!]_\eta = \Phi_{[\![x]\!]_\eta}([\![Yx]\!]_\eta) \tag{13}$$

$$\begin{aligned} n \in \Phi_{[\![x]\!]_\eta}([\![Yx]\!]_\eta) &\iff \exists u \ < n, u > \in [\![x]\!]_\eta \;\&\; D_u \subseteq [\![Yx]\!]_\eta, \; let\, \eta(x) = W_e \\ &\iff \exists u \ < n, u > \in W_e \;\&\; D_u \subseteq [\![Yx]\!]_\eta \\ &\iff \eta \in \Psi^{[\![Yx]\!]_\eta} \end{aligned}$$

□

Exercise 3.4. *Show that for any assigment η:*

$$[\![Yx]\!]_{\eta[x:=\emptyset]} = \emptyset \tag{14}$$

Proof. We know that:

$$[\![Yx]\!]_{\eta[x:=\emptyset]} = [\![x(Yx)]\!]_{\eta[x:=\emptyset]} = \Phi_{[\![x]\!]_{\eta[x:=\emptyset]}}([\![Yx]\!]_{\eta[x:=\emptyset]}) \tag{15}$$

$$\begin{aligned} n \in \Phi_{[\![x]\!]_{\eta[x:=\emptyset]}}([\![Yx]\!]_{\eta[x:=\emptyset]}) &\iff \exists u \ < n, u > \in [\![x]\!]_{\eta[x:=\emptyset]} \;\&\; D_u \subseteq [\![Yx]\!]_{\eta[x:=\emptyset]} \\ &\iff \exists u \ < n, u > \in \emptyset \;\&\; D_u \subseteq [\![Yx]\!]_{\eta[x:=\emptyset]} \\ &\Rightarrow \neg\exists n \;:\; n \in \Phi_{[\![x]\!]_{\eta[x:=\emptyset]}} \end{aligned}$$

□

References

[1] S. B. Cooper, *Computability Theory.* 2003.

[2] P. Odifreddi, *Classical recursion theory.* 1989.