# Computational model of lambda calculus

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# Contents

# 1 General Definitions

**Definition 1.1.** *An enumeration operator (or e-operator) $\Psi^A$ is a r.e. set. For any $A \subseteq \mathbb{N}$*

$$x \in \Psi^A \iff \exists u \, (finite \, D_u \subseteq A) \, \& \, ((x, u) \in \Psi) \tag{1}$$

**Definition 1.2.** *If $A$ is a r.e. set then $\Psi_A$ is the enumeration operator defined by it, namely*

$$x \in \Psi_A^B \iff \exists u \, (D_u \, is \, finite) \, \& \, ((x, u) \in A \wedge D_u \subseteq B) \tag{2}$$

**Definition 1.3.** *If $\theta$ is an enumeration operator then $G_\theta$ is a well-defined r.e. set defining it, namely*

$$(x, u) \in G_\theta \iff x \in \theta^{D_u} \tag{3}$$

**Lemma 1.1.** *If $\Phi$ is an enumeration operator, then $\Phi_{G_\Psi} = \Psi$*

*Proof.* ($\subseteq$)

$$x \in \Phi_{G_\Psi}^B \Rightarrow \exists u \, < x, u > \in G_\Psi \, \& \, D_u \subseteq B$$
$$\Rightarrow \exists u \, < x, u > \in \Psi^{D_u} \, \& \, D_u \subseteq B$$

since Du $\subseteq$ B & e-operator is monotone it follows that: $x \in \Psi^B$
($\supseteq$)
Since $\Psi$ is an e-operator, then $\forall B \, \exists e : \, \Psi = \Phi_e$, therefore:

$$x \in \Phi_e^B \Rightarrow \exists u \, < x, u > \in W_e \, \& \, D_u \subseteq B$$
$$\Rightarrow \exists u \, < x, u > \in W_e \, \& \, D_u \subseteq D_u, \, therefore$$
$$x \in \Psi^{D_u} \Rightarrow < x, u > \in G_\Psi$$
$$\Rightarrow \exists u \, < x, u > \in G_\Psi \, \& \, D_u \subseteq B$$
$$\Rightarrow x \in \Phi_{G_\Psi}$$

$\square$

**Definition 1.4.** *Let $\eta$ be an assignment of r.e. sets to the variables of lambda calculus. With every $\lambda$-term $E$ we inductively associate a r.e. set $[\![E]\!]_\eta$:*

   *1. $[\![x]\!]_\eta = \eta(x)$*

   *2. $[\![E_1 E_2]\!]_\eta = \Psi_{[\![E_1]\!]_\eta}([\![E_2]\!]_\eta)$*

   *3. $[\![\lambda x.E]\!]_\eta = G_{\lambda X.[\![E]\!]_{\eta[x:=X]}}$*

*Where $\lambda X.[\![E]\!]_{\eta[x:=X]}$ is a function*

$$A \in W \mapsto [\![E]\!]_{\eta[x:=A]} \tag{4}$$

**Lemma 1.2.** *For any environment $\eta$ and term t, $[\![t]\!]_\eta$ is an c.e. set.*

*Proof.* By structural induction on the definition of $[\![t]\!]_\eta$.

1. $[\![x]\!]_\eta = \eta(x)$ by definition

2. To show that $\Psi_{[\![E_1]\!]_\eta}([\![E_2]\!]_\eta)$ is a c.e. set we prove that $\Psi_A^B$ is an enumeration operator which follows from

$$n \in \Psi_A^B \iff \exists u(D_u \subseteq B \,\&\, <n, u> \in A) \tag{5}$$

3. Since the graph of an enumeration operator is an r.e. set, to show that $[\![\lambda_x u]\!]_\eta \overset{def 1.4.3}{=} G_{\Lambda X.[\![u]\!]_{\eta[x:=X]}}$ is an r.e. set we prove that $\Lambda X.[\![u]\!]_{\eta[x:=X]}$ is an e-operator. This can be done inductively on u:

   Let

$$[\![E]\!]_\eta[x := W_1 \; y := W_2] = {}^E\Psi_1^W \oplus W_2, where$$
$${}^E\Psi = \{< x, v > \mid x \in [\![E]\!]_{\eta[x:=L(D_v)\; y:=R(D_v)]}\}$$
$$x \in {}^E\Psi^W \iff \exists u \; <x, u> \in {}^E\Psi \,\&\, D_u \subseteq W$$

   - If u = x, then $\Lambda X.[\![u]\!]_{\eta[x:=X]}$ is the identity function $\Lambda X.X$, which is an enumeration operator.

   - If u = y $\neq$ x, then $\Lambda X.[\![u]\!]_{\eta[x:=X]}$ is the constant function $\Lambda X.\eta(y)$, which is an enumeration operator.

   - If u = $u_1 u_2$, then $\Lambda X.[\![u]\!]_{\eta[x:=X]}$ is $\Lambda X \Phi_{[\![u_1]\!]_{\eta[x:=X]}}([\![u_2]\!]_{\eta[x:=X]})$. Where by the i.h. $[\![u_1]\!]_{\eta[x:=X]}$ is an e operator, namely ${}^{u_1}\Psi$ and $[\![u_2]\!]_{\eta[x:=X]}$ is an e operator, namely ${}^{u_2}\Psi$ Therefore $\Phi_{u_1 \Psi}({}^{u_2}\Psi)$ is an e operator (by the proof from 1.2.2).

   - If u = $\lambda_y$ E. Then

$$[\![\lambda_y E]\!]_\eta[x := W_i] \overset{def 1.4.3}{=} G_{\Lambda Y.[\![E]\!]_\eta[x:=W_i, y:=Y]}$$
$$= \{< x, v > \mid x \in [\![E]\!]_\eta[x := W_i, y := D_v]\}$$
$$\overset{i.h.}{=} \{< x, v > \mid x \in {}^E\Psi\, W \oplus D_v\}$$
$$= \{< x, v > \mid \exists u \; <x, u> \in {}^E\Psi \,\&\, D_u \subseteq W \oplus D_v\}$$
$$\overset{def\, {}^E\Psi}{=} \{< x, v > \mid \exists u \; x \in [\![E]\!]_\eta[x := L(D_u), y := R(D_u)] \,\&\, D_u \subseteq W \oplus D_v\}$$
$$= \{<< x, v >, u > \mid x \in [\![E]\!]_\eta[x := L(D_u), y := R(D_u)] \,\&\, D_u \subseteq W \oplus D_v\}$$

   Since $[\![E]\!]_\eta[x := L(D_u), y := R(D_u)]$ is an e operator(i.h) (e operators are r.e. by definition) and $D_u \subseteq W \oplus D_v$ is r.e. set it follows that the above set is an e operator.

$\square$

**Lemma 1.3.** *For the following theorem we will need one lemma beforehand for better readability, namely*

$$\llbracket u \rrbracket_{\eta[x:=\llbracket v \rrbracket_\eta]} = \llbracket u[x \mapsto v] \rrbracket_\eta \tag{6}$$

*Proof.* Structural induction on u.

1. u = x, then:

$$\llbracket x \rrbracket_{\eta[x:=\llbracket v \rrbracket_\eta]} = \eta(x) = \llbracket v \rrbracket_\eta = \llbracket x[x \mapsto v] \rrbracket_\eta \tag{7}$$

2. u = y ≠ x, then:

$$\llbracket y \rrbracket_{\eta[x:=\llbracket v \rrbracket_\eta]} = \eta(y) = \llbracket y \rrbracket_\eta = \llbracket y[x \mapsto v] \rrbracket_\eta \tag{8}$$

3. u = pq, then:

$$\llbracket pq \rrbracket_{\eta[x:=\llbracket v \rrbracket_\eta]} \overset{def 1.4.2}{=} \Psi_{\llbracket p \rrbracket_{\eta[x:=\llbracket v \rrbracket_\eta]}}(\llbracket q \rrbracket_{\eta[x:=\llbracket v \rrbracket_\eta]})$$
$$\overset{ind.hyp.}{=} \Psi_{\llbracket p[x \mapsto v] \rrbracket_\eta}(\llbracket q[x \mapsto v] \rrbracket_\eta)$$
$$\overset{def 1.4.2}{=} \llbracket p[x \mapsto v]q[x \mapsto v] \rrbracket_\eta$$
$$\overset{def App}{=} \llbracket pq[x \mapsto v] \rrbracket_\eta$$

4. u = $\lambda_y$ p, then:

$$\llbracket \lambda_y p \rrbracket_{\eta[x:=\llbracket v \rrbracket_\eta]} \overset{def 1.4.3}{=} G_{\Lambda Y. \llbracket p \rrbracket_{\eta[x:]\llbracket v \rrbracket_\eta; y:=Y]}}$$
$$\overset{ind.hyp}{=} G_{\Lambda Y. \llbracket p[x \mapsto v] \rrbracket_{\eta[y:=Y]}}$$
$$\overset{def 1.4.3}{=} \llbracket \lambda_y p[x \mapsto v] \rrbracket_\eta$$

$\square$

**Theorem 1.4.** *If $E_1 \overset{\beta}{\twoheadrightarrow} E_2$ then $\llbracket E_1 \rrbracket_\eta = \llbracket E_2 \rrbracket_\eta$ for any $\eta$.*

*Proof.* We will prove one step of the $\beta$ reduction and then by induction the rest will follow

We have that $(\lambda x E_1)E_2 = E_1[x \mapsto E_2]$ and we will prove that

$$[\![(\lambda x E_1)E_2]\!]_\eta = [\![E_1[x \mapsto E_2]]\!]_\eta$$

$$
\begin{aligned}
[\![(\lambda x E_1)E_2]\!]_\eta \quad &\overset{def.1.4.2}{=} \quad \Psi_{[\![\lambda x E_1]\!]_\eta}([\![E_2]\!]_\eta)\\
&\overset{def.1.4.3}{=} \quad \Psi_{G_{\Lambda X [\![E_1]\!]_{\eta[x:=X]}}}([\![E_2]\!]_\eta)\\
&\overset{Lemma1.1}{=} \quad \Lambda X [\![E_1]\!]_{\eta[x:=X]}([\![E_2]\!]_\eta)\\
&\overset{def\Lambda}{=} \quad [\![E_1]\!]_{\eta[x:=[\![E_2]\!]_\eta]}\\
&\overset{Lemma1.3}{=} \quad [\![E_1[x \mapsto E_2]]\!]_\eta
\end{aligned}
$$

<div style="text-align:right">□</div>

**Definition 1.5.** *We define the fixed point combinator Y to be:*

$$Y \overset{def}{=} \lambda f.((\lambda x.f(x\ x))\,((\lambda x.f(x\ x)) \tag{9}$$

**Exercise 1.1.** *Show that any $\lambda$-term F has fixed point YF, such that*

$$YF \;=\; F(YF). \tag{10}$$

*Proof.*

$$
\begin{aligned}
YF &= ((\lambda x.F(x\ x))\,((\lambda x.F(x\ x))\\
&= F((\lambda x.F(x\ x))\,((\lambda x.F(x\ x))\\
&= F(\lambda f.((\lambda x.f(x\ x))\,((\lambda x.f(x\ x))F)\\
&= F(YF)
\end{aligned}
$$

<div style="text-align:right">□</div>

**Exercise 1.2.** *Let $\Psi$ be any e-operator. Show that $\Psi$ can be expressed as:*

$$\Psi \;=\; \Psi_{[\![x]\!]_\eta} \tag{11}$$

*for suitable choise of assigment $\eta$.*

*Proof.* Since $\Psi$ is an e-operator, then $\forall B\ \exists e:\ \Psi = \Phi_e$, therefore:

$$
\begin{aligned}
n \in \Psi^B &\iff \exists u\ <n,u> \in W_e\ \&\ D_u \subseteq B\\
&\iff \exists u\ <n,u> \in [\![x]\!]_\eta\ \&\ D_u \subseteq B,\ \text{where } \eta(x) = W_e\\
&\iff n \in \Psi^B_{[\![x]\!]_\eta}
\end{aligned}
$$

<div style="text-align:right">□</div>

**Exercise 1.3.** *Let $\Psi$ be any e-operator. Show that $\Psi$ has a r.e. fixed point W, such that:*

$$\Psi^W \;=\; W \tag{12}$$

*Proof.* Let $W = [\![Yx]\!]_\eta$, then:

$$[\![Yx]\!]_\eta = [\![x(Yx)]\!]_\eta = \Phi_{[\![x]\!]_\eta}([\![Yx]\!]_\eta) \tag{13}$$

$$\begin{aligned}
n \in \Phi_{[\![x]\!]_\eta}([\![Yx]\!]_\eta) &\iff \exists u \ < n, u > \in [\![x]\!]_\eta \ \& \ D_u \subseteq [\![Yx]\!]_\eta, \ let \ \eta(x) = W_e \\
&\iff \exists u \ < n, u > \in W_e \ \& \ D_u \subseteq [\![Yx]\!]_\eta \\
&\iff \eta \in \Psi^{[\![Yx]\!]_\eta}
\end{aligned}$$

$\square$

**Exercise 1.4.** *Show that for any assigment $\eta$:*

$$[\![Yx]\!]_{\eta[x:=\emptyset]} = \emptyset \tag{14}$$

*Proof.* We know that:

$$[\![Yx]\!]_{\eta[x:=\emptyset]} = [\![x(Yx)]\!]_{\eta[x:=\emptyset]} = \Phi_{[\![x]\!]_{\eta[x:=\emptyset]}}([\![Yx]\!]_{\eta[x:=\emptyset]}) \tag{15}$$

$$\begin{aligned}
n \in \Phi_{[\![x]\!]_{\eta[x:=\emptyset]}}([\![Yx]\!]_{\eta[x:=\emptyset]}) &\iff \exists u \ < n, u > \in [\![x]\!]_{\eta[x:=\emptyset]} \ \& \ D_u \subseteq [\![Yx]\!]_{\eta[x:=\emptyset]} \\
&\iff \exists u \ < n, u > \in \emptyset \ \& \ D_u \subseteq [\![Yx]\!]_{\eta[x:=\emptyset]} \\
&\Rightarrow \neg\exists n \ : \ n \in \Phi_{[\![x]\!]_{\eta[x:=\emptyset]}}
\end{aligned}$$

$\square$

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# References

[1] S. B. Cooper, *Computability Theory.* 2003.

[2] P. Odifreddi, *Classical recursion theory.* 1989.