Neo-Classical Logic
Une Logique non classique*

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Resume.

Standard logics, such as *classical logic* or *intuitionist logic* suffer from flaws that remained unaddressed as of today (like philosophical questions about using the axiom of *excluded middle*, which draws the line between the two logic mentioned previously). In this paper, we introduce *neo-classical logic*, in an attempt to reconciles classical and intuitionistic logics, as well as to capture some real-world event which can not be explained by previous logics, due to their lack of expressivity.

After introducing its syntax and semantique, we study the meta-theory of this new logic, and show multiple interesting results, such as the *law of contradiction*, or that validity is decidable, which states that a proposition can simultaneously hold and not hold.

1 Introduction

Motivation. Our paper is motivated by strong empirical evidences that some events, concepts, objects can simultaneously exist and don't exist (see Figure 1, but also [6, 8, 10]).

Despite strong evidence, to the best of our knowledge, no logic seems to be expressive enough to prove statements as simple as « Exists a such that a does not exist. » This absence (or possibly presence) shows that there exists a gap between current theoretical models and the actual reality.

Contribution. Our main contribution in this paper is to introduce a new logic, *neoclassical logic*[1], which reduces this gap, allowing us to express more faithfully the subtleties of the world.

Moreover, we demonstrate how we use neo-classical logic to reconcile two important logic theories, namely classical logic and intuitionistic logic, with a *constructive* proof of the *Law of Excluded Middle* in our neo-classical logic. Neo-classical logic subsumes classical logic and intuitionistic logic.

Outline. In Sections 2 et 3, we discuss the limitations of the two most popular logics, namely classical and intuitionistic logic. From that, we conclude that there is a need to fill the gap left opened by those two logic systems; in order to capture real-life cases that can not be addressed by the two logic aforementioned. In Section 4, we introduce our proposal, which we call *neo-classical logic*, by successively presenting both its syntax and its semantique. In Section 5, we discuss the meta-theory of the logic. Finally we present an actual use case in Section 6, showing that our logic allows to capture complex arguments while remaining decidable. Finally, in Section 7, we conclude by discussing some limitations, presenting some unrelated work and highlight some interesting research paths for future work.

2 Classical Logic is Not Modern

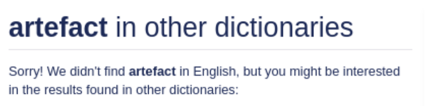
We look up the dictionary entry of the word « classical » in Cambridge dictionary again [9], and see what it could mean. The main usages include « traditional in style or form, or based on methods developed over a long period of time, and considered to be of lasting value », and « used to describe something that is attractive because it has a simple, traditional style. »

We also note that this sentence was provided as an exemple:

> Does she study classical ballet or modern ballet?

*A paper on logic must have some French, but we don't know how to typeset accents, so here it is.

[1] Also spelled « neo-classical » due to our lack of motivation to check the consistency across the paper.



(a) Evidence of the non existence of the word « Artefact » [10]

(b) Evidence of existence of the word « Artefact » [8]

Figure 1: Empirical evidence of the simultaneous existence and non-existence of the word « Artefact » in English, according to the Cambridge Dictionary.

This sentence gives a strong hint that *classical* ballet is *not modern*. We are also confident to conclude that classical logic is not modern, via a simple application of substitution. A modern-day logic should be modern, in order to catch up with contemporary development of real life events and technological advancements. Moreover, it is an unfortunate fact that classical logic cannot represent the simultaneous existence and non-existence of a word (cf. Figure 1) — it cannot model contemporary dictionaries in a modern world.

3 Intuitionistic Logic is Not Intuitive

In the first courses of computer science, students learn classical logic due to its simplicity. Some mathematicians and theoretical computer scientists, however, prefer intuitionistic logic.

Unfortunately, we are unable to find an entry for « intuitionistic » in Cambridge dictionary, which is a strong hint that it is not intuitive. Think about that, why would computer scientists call something intuitive using a complicated word that looks like « intuitive », but decide against its use?

4 Neo-Classical Logic

In order to address the unsatisfactory deficiency of both classical logic and intuitionistic logic, we introduce *neo-classical logic*.

4.1 Formules

$$
\begin{array}{llll}
\mathcal{A} & ::= & a, b, c, \ldots & \textit{Atomic propositions} \\
\mathcal{P}, \mathcal{Q} & ::= & \mathcal{A} & \textit{Atomic proposition} \\
 & | & \neg\mathcal{P} & \textit{Negation of P} \\
 & | & \mathcal{P} \wedge \mathcal{Q} & \textit{And}
\end{array}
$$

Notice that, together with the semantique we define below, and in particular with the traditional encoding of $P \vee Q$ as $\neg(\neg P \wedge \neg Q)$ and $P \Rightarrow Q$ as $\neg(P \wedge \neg Q)$, we can have usual constructs. Therefore, we allow ourself to use those elements as needed.

The syntax of formules does not differ much from classical or intuitionistic logic — this is why we think our logic is neo-classical: it looks classical, but is quite modern. We explain the modernness in detail when we discuss the semantique.

We fix a set $\mathcal{A}$ of atomic propositions, ranging over $a, b, c, \ldots$. Atomic propositions form the basic form of neo-classical logic formules, and we also recognise standard logical operators: negation, conjunction, disjunction, and implication.

4.2 Semantique

Let $\vdash$ be a (binary) predicate defined according to the rules in Figure 2. We use an infix notation: $\Gamma \vdash P$ where Γ is an environment and P a formula of the neo-classical logic. We write $\Gamma \nvdash P$ for $\neg(\Gamma \vdash P)$.

$$\frac{a \notin \Gamma}{\Gamma \vdash \neg a}\,(\neg\text{Exists}) \qquad \frac{a \in \Gamma}{\Gamma \vdash a}\,(\text{Exists}) \qquad \frac{\Gamma_1 \vdash P \qquad \Gamma_2 \vdash Q}{\Gamma_1 \cup \Gamma_2 \vdash P \wedge Q}\,(\wedge) \qquad \frac{\Gamma \vdash \neg P}{\Gamma \vdash \neg(P \wedge Q)}\,(\text{DeMorganL})$$

$$\frac{\Gamma \vdash \neg Q}{\Gamma \vdash \neg(P \wedge Q)}\,(\text{DeMorganR}) \qquad \frac{\Gamma \vdash P}{\Gamma \vdash \neg\neg P}\,(\neg\neg\mathrm{I})$$

Figure 2: Inference rules of the $\vdash$ predicate

The novelty of neo-classical logic lies in the rules ($\neg$Exists) and (Exists). Those two rules establish a strong correspondence between a witness (or observation) of an event and establishing the existence of that event. On the one hand, rule ($\neg$Exists) states that if we have no evidence of an event, we can conclude that this event does not exist. This is, informally, motivated by the fact that having no evidence of an event is indistinguishable from that event not existing. Indeed, if there is a way to distinguish two possible worlds, one with the event, one without; then the distinctive element actually serves as witness of the event. Therefore, from Occam's razor (often stated « Pluralitas non est ponenda sine necessitate »[2], and widely spread in the literature, see e.g. [7, Book I, 4, 188a17], but also [4, Prima Pars, Q.2 art.3 -AG2, &c.]), we should not suppose the existence of an event we have no evidence of; which ends the justification of that rule.

Conversely for (Exists), if we have an observation of an event a, we can deduce the existence of that event. This is trivially motivated.

Exemple 1 (Artefact exists and does not exist).

$$\frac{\dfrac{}{\text{artefact} \vdash \text{artefact}}\,(\text{Exists}) \qquad \dfrac{}{\varnothing \vdash \neg\text{artefact}}\,(\neg\text{Exists})}{\text{artefact} \vdash \text{artefact} \wedge \neg\text{artefact}}\,(\wedge)$$

Theoreme 2 (Consistency). *The deduction system is consistent, i.e.* $\varnothing \nvdash \bot$.

Preuve. It is not possible to introduce $\bot$. Voilà.

Remarque 3 (Absence of Absolute Truths And Absolute Falsities). We note that $\top$ and $\bot$ are not formules of neo-classical logic. This is important since there are no absolute truths and absolute falsities in the modern world. We make a conscious effort to model this observation in our logic.

5 Meta[3]-Theory of *Neo-Classical Logic*

Lemme 4 (To tree or not to tree).

$$\forall \Gamma \,.\, \forall P \,.\, \Gamma \vdash P \vee \Gamma \nvdash P$$

Preuve. First, we need to clarify the statement. It is obviously not a neo-classical formula (typically, neo-classical logic do not have quantifier). Instead, it is a formula from first-order classical logic, where $\vdash$ is a (binary) predicate; and $\vee$ is the disjunction of classical logic.

The result then follows directly from the law of excluded-middle of classical logic. Voilà.

Lemme 5 (If no proof, then proof of negation).

$$\forall \Gamma \,.\, \forall P \,.\, \Gamma \nvdash P \Rightarrow \Gamma \vdash \neg P$$

Preuve. By induction on P.

Case $P = a$: from the premisses of (Exists), we have that $a \notin \Gamma$. Therefore, rule ($\neg$Exists) applies.

Case $P = \neg Q$: neither ($\neg$Exists), (DeMorganL), (DeMorganR) nor ($\neg\neg$I) apply. By case analysis on Q:

[2] « entities should not be multiplied beyond necessity », translation Wikipedia.

[3] Not to be confused with Meta Platforms Inc, "metaverse", or whatever.

Case Q **is** a: since ($\neg$Exists) does not apply, we have that $a \in \Gamma$, therefore $\Gamma \vdash Q$, therefore $\Gamma \vdash \neg P$ by applying ($\neg\neg$I).

Case Q **is** $\neg Q'$: since ($\neg\neg$I) does not apply, we have that $\Gamma \nvdash Q'$. From the induction hypothesis, we have that $\Gamma \vdash \neg Q'$. Therefore, $\Gamma \vdash \neg P$ (i.e. $\Gamma \vdash \neg\neg\neg Q'$) by applying ($\neg\neg$I).

Case Q **is** $Q_1 \wedge Q_2$: since neither (DeMorganL) nor (DeMorganR) apply, we have that both $\Gamma \nvdash \neg Q_1$ and $\Gamma \nvdash \neg Q_2$. Therefore, from the induction hypothesis, both $\Gamma \vdash \neg\neg Q_1$ and $\Gamma \vdash \neg\neg Q_2$. From the premises of ($\neg\neg$I) on those two statements, we have that both $\Gamma \vdash Q_1$ and $\Gamma \vdash Q_2$. Therefore, $\Gamma \vdash Q_1 \wedge Q_2$ (i.e. $\Gamma \vdash Q$), by applying ($\wedge$). Therefore, we can prove $\Gamma \vdash \neg P$, i.e. $\Gamma \vdash \neg\neg Q$, by applying ($\neg\neg$I).

Case $P = Q_1 \wedge Q_2$: Since rule ($\wedge$) does not apply, we have that for any subsets Γ_1, Γ_2 of Γ, neither $\Gamma_1 \vdash Q_1$ nor $\Gamma_2 \vdash Q_2$. I.e., $\forall \Gamma' \in \Gamma \,.\, \Gamma \nvdash Q_1$ (resp. Q_2). Therefore, from the induction hypothesis, we have that for any subset Γ' of Γ, $\Gamma' \vdash \neg Q_1$ (resp. Q_2).

Therefore, the proof finishes by showing that $\Gamma \vdash \neg P$, i.e. $\Gamma \vdash \neg Q_1 \wedge Q_2$ by applying either (DeMorganL) or (DeMorganR). Voilà.

5.1 Law of Contradiction

Lemme 6 (Empiricism brings knowledge).

$$\forall P \,.\, \exists \Gamma \,.\, \Gamma \vdash P$$

Preuve. By induction on P.

Case $P = a$: let $\Gamma = \{a\}$, apply (Exists).

Case $P = \neg P'$: By case analysis on P':

Case $P' = a$: Let $\Gamma = \varnothing$, and the result holds directly from rule ($\neg$Exists).

Case $P' = Q_1 \wedge Q_2$: From the induction hypothesis, we know that there exist Γ_1 such that $\neg Q_1$ holds. Therefore, we can take $\Gamma = \Gamma_1$ and apply (DeMorganL).

Case $P' = \neg Q$: In that case, we have to prove $\neg\neg Q$. The result holds directly from the induction hypothesis and by applying the rule ($\neg\neg$I).

Case $P = Q_1 \wedge Q_2$: From the induction hypothesis, we know that there exists a Γ_1 (resp. Γ_2) such that $\Gamma_1 \vdash Q_1$ (resp. $\Gamma_2 \vdash Q_2$). We can take $\Gamma = \Gamma_1 \cup \Gamma_2$ and apply the rule ($\wedge$). Voilà.

Theoreme 7 (Law of contradiction).

$$\forall P \,.\, \exists \Gamma \,.\, \Gamma \vdash P \wedge \neg P$$

Preuve. This Theoreme is a corollaire of lemme 6. Voilà.

5.2 Excluded-Middle

Theoreme 8 (Excluded-Middle).

$$\forall \Gamma \,.\, \forall P \,.\, \Gamma \vdash P \vee \neg P$$

Preuve. As we said above, $P \vee \neg P$ is encoded in our calculus as $\neg(\neg P \wedge \neg\neg P)$. Therefore, we actually have to prove that this formula holds for all P and Γ.

From lemme 4, $\Gamma \vdash P$ or $\Gamma \nvdash P$. By case analysis:

Case $\Gamma \vdash P$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash P}\ \text{Hypothesis}}{\Gamma \vdash \neg\neg P}\ (\neg\neg\mathrm{I})}{\Gamma \vdash \neg(\neg P \wedge \neg\neg P)}\ (\text{DeMorganL})$$

Case $\Gamma \nvdash P$: From lemme 5, $\Gamma \vdash \neg P$.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \neg P}\text{ Hypothesis}}{\Gamma \vdash \neg\neg\neg P}\ (\neg\neg\mathrm{I})}{\Gamma \vdash \neg(\neg P \wedge \neg\neg P)}\ (\text{DeMorganR})$$

Voilà.

5.3 Decidability of Validity

Finally, we want to show that deciding whether a formula P is valid under a context Γ is decidable. For that, we first show how to compute two sets: first, the set of event that *must* be in Γ, and second, the set of event that must not be in Γ, in order to satisfy the formula. Finally, we compare Γ with those two sets to decide whether P can be satisfied.

Set of required events. Let $\mathbb{R}_P$ be the set of required events to satisfy P. Notice that there might be multiple ways to satisfy a single formula, i.e. multiple sets of required events are possible. Therefore, $\mathbb{R}_P$ is actually a set of sets of events. $\mathbb{R}_P$ can easily be defined as follows:

$$\mathbb{R}_P \stackrel{\text{def}}{=} \begin{cases} \emptyset & \text{if } P = \neg a,\ \forall a \\ \{\{a\}\} & \text{if } P = a,\ \forall a \\ \{\mathbb{P}_{Q_1} \cup \mathbb{P}_{Q_2} | \mathbb{P}_{Q_1} \in \mathbb{R}_{Q_1} \wedge \mathbb{P}_{Q_2} \in \mathbb{R}_{Q_2}\} & \text{if } P = Q_1 \wedge Q_2 \\ \mathbb{R}_{\neg Q_1} \cup \mathbb{R}_{\neg Q_2} & \text{if } P = \neg(Q_1 \wedge Q_2) \\ \mathbb{R}_Q & \text{if } P = \neg\neg Q \end{cases}$$

Lemme 9 (Required is correct). *For all Γ, for all P, if $\nexists \mathbb{P} \in \mathbb{R}_P$ such that $\mathbb{P} \subseteq \Gamma$, then $\Gamma \nvdash P$.*

Preuve. The proof is direct, by induction on P. Each possible case of corresponds to a case of $\mathbb{R}_P$, corresponds to the premises of a rule (or two rules for the two variants of (DeMorgan)). Voilà.

Lemme 10. *For all P, $\mathbb{R}_P$ is computable.*

Preuve. Let $n_1(P)$ be the number of occurrences of $\neg(Q_1 \wedge Q_2)$ in P. Let $n_2(P)$ be the number of occurrences of $\neg Q$ in P. Let $\prec$ be an order relation on formules defined as follows:

$$P \prec Q \text{ if and only if } (n_1(P) < n_1(Q)) \text{ or } (n_1(P) = n_1(Q) \text{ and } n_2(P) < n_2(P))$$

We easily show, by induction, that, for each recursive computation of $\mathbb{R}_Q$ to compute $\mathbb{R}_P$, $Q \prec P$. Also, there is a finite number of recursive call at each step.

Therefore, the computation eventually terminates. Voilà.

Set of forbidden events. Let $\mathbb{F}_P$ be the set of forbidden events to satisfy P. $\mathbb{F}_P$ can easily be defined as follows:

$$\mathbb{F}_P \stackrel{\text{def}}{=} \begin{cases} \{a\} & \text{if } P = \neg a,\ \forall a \\ \emptyset & \text{if } P = a,\ \forall a \\ \mathbb{F}_{Q_1} \cap \mathbb{F}_{Q_2} & \text{if } P = Q_1 \wedge Q_2 \\ \mathbb{F}_{\neg Q_1} \cup \mathbb{F}_{\neg Q_2} & \text{if } P = \neg Q_1 \wedge Q_2 \\ \mathbb{F}_Q & \text{if } P = \neg\neg Q \end{cases}$$

Lemme 11 (Forbidden is correct). *For all Γ, for all P, if $\Gamma \cap \mathbb{F}_P \neq \emptyset$, then $\Gamma \nvdash P$.*

Preuve. The proof is direct by induction on P. Each possible case of corresponds to a case of $\mathbb{F}_P$, corresponds to the premises of a rule (or two rules for the two variants of (DeMorgan)). Voilà.

Lemme 12. *For all P, $\mathbb{F}_P$ is computable.*

Preuve. The proof is similar to the proof of lemme 10. Voilà.

Theoreme 13. *For all* Γ*, for all* P*,* $\Gamma \vdash P$ *if and only if:*

1. $\exists \mathbb{P} \in \mathbb{R}_P \,.\, \mathbb{P} \subseteq \Gamma$*; and*
2. $\Gamma \cap \mathbb{F}_P = \emptyset$*.*

Preuve. The if direction is a direct consequence of lemmes 9 et 11, by contraposition.
We have to show the other direction, that is the two conditions are sufficient to have $\Gamma \vdash P$.
The result is direct by induction on P. Voilà.

Corollaire 14 (The validity of a formula is decidable)**.** *For any* Γ*, for any* P*,* $\Gamma \vdash P$ *is decidable.*

6 Practical Applications

As logicians, we have a moral duty to remain close to practical applications of our works. Indeed, the formal study of arguments allows every citizen to cast a light and understand actual facts on the world that surrounds us. In order to show that our work has indeed some practical use, in this section, we show an actual exemple taken from everyday's life.

While police violence is a well-documented issue in modern democracies [1,5], we still have statements that police violence does not exist [2,3] (e.g. « Ne me parlez pas de [...] violences policières, ces mots sont inacceptables dans un État de droit. »[4], Macron *et al.*, reported at 0:40 in [3], and in [2]).

In the following, we show that our logic captures such arguments, as they are actually derivable. Let v_p an actual observation of police violence. In the following, we show how we can simultaneously claim that such event occurs and does not occurs, given such observation.

$$\dfrac{\dfrac{\dfrac{}{v_p \in \{v_p\}}\text{ Set theory}}{\{v_p\} \vdash v_p}\text{ (Exists)} \qquad \dfrac{\dfrac{}{v_p \notin \varnothing}\text{ Set theory}}{\emptyset \vdash \neg v_p}\text{ } (\neg\text{Exists})}{\{v_p\} \vdash v_p \wedge \neg v_p}\ (\wedge)$$

7 Conclusion, Unrelated and Future Work

In this paper, we presented new semantique for the propositional calculus. We show that this new semantique have interesting properties. For instance, the standard law of excluded-middle holds in our calculus, or that the validity of a formula is decidable. In addition, we show that new, previously unexplored[5] laws, such as the *Law of contradiction* also holds for our calculus.

Overall, as stated in the motivation, our calculus captures intuitive notions of proofs based on the notion of evidence. Our calculus finally brings the formal basis needed for reasoning on the ontological[6] notion of existence, which is a problem that was left opened since almost 2.5 millennia; yet still relevant as of today.

Unrelated Work. Very little is unrelated to this one, as this work regards the relation between evidence and knowledge. Therefore, all evidence-based sciences are related to this work. Furthermore, as it aims to give a formal basis to ontological[7] arguments, it relates to philosophical works, which we therefore have to exclude from unrelated works as well.

Future Work. The current main limitation is that it is based on propositional logic; and therefore lacks the expressiveness needed to capture more complex argumentations. Future work could include extending this paper up to higher-order logic. Such extension could benefit from an extended expressiveness, possibly being expressive enough to encode arithmetics.

The exemple shown in Section 6 illustrate that our logic allows a fine characterisation of the reasoning that leads to some claims. For instance, we saw that the authors of the sentence above (Macron *et al.*, reported in [2,3]) loose some information in their reasoning. The current work is limited in that it does not give explanation for this loss. Intuitively, we can hypothesize various causes, ranging from plain lack of knowledge to deliberate ignorance. Future works is needed to elaborate on such hypothesis.

[4] « Don't say [...] "police violence" to me, those words are unacceptable in a Rechtsstaat. » Translation by the authors.
[5] To the best of the author's knowledge.
[6] Ditto.
[7] or « epistemic », as you can guess, the authors are not philosophers and are just using those big words to impress reviewer #2.

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