Objectionability: A Computational View of Mathematical Computation

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Abstract

Any student in an "engineering" discipline (that is, the fields of math, computer science, mechanical engineering, underwater basket weaving, et al) can sympathize with looking at a equation or problem and realizing that pages of unintelligible scribbles would be required to reach a satisfactory conclusion. Similarly, professors often add allowances for "algebraic grunt work" that is often assumed to be correct in grading rubrics. However, such terms are often, at most, vacuously defined and it is up to the subjective preferences of whoever is looking to determine whether four pages of equational reasoning is sufficient or to remove points for omitting a further two pages of integration by parts, second-order substitution and a re-derivation of the Cauchy-Schwarz inequality.

In this paper, we attempt to provide a mathematical model to describe the "grossness" of any particular mathematical problem P (the precise formal definition of which is semi-intentionally left vacuous). In particular, we define $\mathbb{G}(P)$ to be the *computational grossness* of P and prove several further useless results that come of direct consequence. As such, we also establish several *grossness classes* for problems.

Finally, we introduce the concept of *objectionability*, whether any *arbitrarily gross* problem is uncomputable by a non-idealized human. Among other things, we will find that O, determing whether a problem is objectionable, is itself objectionable.

1 Introduction

In 1936, Alan Turing introduced the concept of a *Turing Machine* that could be used to compute formally-defined mathematical functions. In the same vein, the idea of *decidability* was introduced, determining whether a given problem was solvable with pen-and-paper mathematical manipulations.

We see, though, that some problems, while perhaps decidable, are simply "gross"- that is to say, they are difficult to generate solutions to. This phenomenon has even been referenced by the famed Jay-Z, referring to "99 problems" that seemed to be of trivial value compared to the problems that the song's subject had been presented with.

However, in reasoning about decidability, little to no thought is given to whether anyone would *want* to solve such problems using pen-and-paper mathematical methods – with the advent of computers, many of these "gross" computations can be done with little to no effort on the part of a human. Even so, the fact remains that such "gross" problems exist. Further thought reveals that (without loss of generality) the problem of writing Python code to solve something may be "too gross".

This paper attempts to resolve this issue by defining $\mathbb{G}(P)$, the grossness of P for a problem P in $\mathbb{E}$ (the set of all problems) then discuss several schemes by which $\mathbb{G}$ may be estimated. We also briefly discuss how we might determine whether a problem is *objectionable*, that someone may reasonably object to being asked to solve it.

2 Repugnance Theory

We begin by defining $\mathbb{G}(P)$ for some $P \in \mathbb{E}$ as the minimum grossness for any solution to P. As we will see, however, this is difficult to determine in a general case.

Consider the following problem:

Problem 1. Fix some alphabet Σ and other arbitrary set Π. Define the following sets:

$$\Gamma = \big\{(\mu, n, \tau) \in \{w \in \Sigma^* : w \text{ every proper prefix of } w \text{ is in } \mu\} \times \mathbb{N} \times \{\zeta \in \Pi : \langle\zeta\rangle \geq n\} : |\langle\tau\rangle^{|\mu|}\langle n\rangle| \leq |\mu^{|\langle n\rangle|}| + |\langle\tau\rangle|\big\}$$

$$\kappa = \big\{(\Sigma^*, \iota, \psi) : \psi \in \{f : \Sigma^* \times \Sigma^* \to \Sigma^* \mid \forall x, y \in \Sigma^*, f(x,y) = f(y,x)\} \text{ and } \forall x \in \Sigma^*, \psi(\iota, x) = x = \psi(x, \iota)\big\}$$

where $\langle x \rangle$ is an arbitrary encoding of an object x over Σ^*.

Let Q be the subproblem of "Given a string S over Σ^*, is S equivalent to $\langle(\nu, \chi, \omega)\rangle$ where $(\nu, \chi, \omega) \in \Gamma \cap \kappa$?".

Is Q decidable?

A naive approach might have a student attempt to determine what each set *means* and to reason about the properties and behavior of each. Note, however, that $\langle P \rangle$ over the alphabet of unicode characters occurring in this paper (which we will take to be the problem statement as given) contains 39 characters when any symbols that are not considered "common English characters" are removed. This, too, is a vacuous definition, however, so we will define a "common English character" to be anything that has a standard unicode codepoint less than 95. With this interpretation, then, we could say that $\mathbb{G}(P) = 39$, as any solution to this problem must begin by reading the problem statement. We will call this approach the "character counting" approach.

An alternative approach, however, might involve inspecting the types of the sets involved – μ is a string, n is a natural number and τ takes some type t, where t is the type of "things in Π". This is not to start a discussion of type theory, so we suffice to say that this apprach leads to finding that $\mathbb{G}(P) = T$, where T is the set of all types that fulfill the given criteria. It then follows that the "character counting" method is only sufficient to provide an upper bound on $\mathbb{G}$.

A real type-chasing approach, however, would note that the definition of κ is actually a monoid, giving us that any element of κ is also a monoid in the category of endofunctors and thus $\Gamma \times \kappa = \varnothing$. In this case, though, the problem cannot possibly be less gross than this observation, giving us that $\mathbb{G}(P) \geq$ `"glasgow"`.

3 Voodoo

In this paper, we will be concerned mostly with the lower bound of grossness as established via another measure we have provisionally named *voodoo*, denoted with $\mathcal{V}(P)$ defined as follows:

Take a problem $P \in \mathbb{E}$. If P can be reduced to some P' via an application of some subroutine Q, $\mathcal{V}(P) \geq n \cdot \mathbb{G}(P')$. It naturally falls out, then, that $\mathcal{V}(P) = \sum_{P \text{ reduces to } P'} \mathbb{G}(P')$. From here, we can apply the Rauen Incompleteness Principle to see that $\mathcal{V}$ is a complete but undefined function.

From this, however, we can establish the following lemma:

Lemma 1 (Wong's Lemma)**.** *For all $P \in \mathbb{E}$, let L be the upper bound of $\mathbb{G}(P)$ as determined by the character counting method. Additionally, define $\daleth = \mathbb{G}(P')$, where P' is the derivative of P. Then,*

$$\mathbb{G}(P) \geq \mathcal{V}(P)^{\daleth} \ (\text{mod } L) \tag{1}$$

Proof. Fix a problem P. $\mathbb{G}(P')$ is trivially bounded from above by 1114016[†] and below by $\mathbb{N}$. We can then express $\mathcal{V}(P)$ type-theoretically as a combinatorial 4-form, which means that, by the Chinese remainder theorem, $\mathcal{V}(P)^{\daleth}$ cannot be greater than a type-theoretic solution to P modulo L. But wait, we already established that $\mathbb{G}(P)$ is bounded from below by the set of type-theoretic solutions to P, so the claim immediately follows. □

Note that this does not contradict our earlier statement about character-counting; the modulo ensures that we remain under our upper bound.

We now present the following result:

Theorem 1 (Reed's Theorem)**.** *For all $P, Q \in \mathbb{E}$,*

$$\mathbb{G}(P) \circ \mathbb{G}(Q) \geq \frac{\mathcal{V}(\overrightarrow{P \cdot \mathbb{G}(Q)} \times \overrightarrow{\mathbb{G}(P) \cdot Q})}{\mathcal{V}(P \circ Q)^{\daleth_Q}} \tag{2}$$

where $\circ$ is the composition operator.

Proof. We begin by applying Wong's Lemma to the right hand side of the equation.

$$\frac{\mathcal{V}(\overrightarrow{P \cdot \mathbb{G}(Q)} \times \overrightarrow{\mathbb{G}(P) \cdot Q})}{\mathcal{V}(P \circ Q)^{\daleth_Q}} \geq \frac{\daleth_{\overrightarrow{P \cdot \mathbb{G}(Q)} \times \overrightarrow{\mathbb{G}(P) \cdot Q}} \mathbb{G}(\overrightarrow{P \cdot \mathbb{G}(Q)} \times \overrightarrow{\mathbb{G}(P) \cdot Q})}{\mathbb{G} P \circ Q \circ P} \ (\text{mod } L)$$

Note that, at this point, all operators are now all near-computable with sufficient paper (and we can thus assume that our human prover has done so).

However, on attempting to apply the same thing to the left hand side, we get that

$$\mathbb{G}(P) \circ \mathbb{G}(Q) \leq \mathcal{V}(P)^{\daleth_P} \circ \mathcal{V}(Q)^{\daleth_Q} > (\mathcal{V}(P) \circ \mathcal{V}(Q))^{\daleth_{P \circ Q}}$$

From here, however, the authors of this paper object (see section 2) to showing the remaining computations. Without lots of Generality,

[†] This is the difference between the highest defined unicode codepoint and 95

we see that the two quantities thus have well-ordered grossness. Specifically, something that is near-computable cannot be greater than anything that is not. Because the LHS is ultimately too gross to reduce, then, it must be greater than the idealized RHS. □

Because of this, we can also establish the following corollary

Corollary 1 (Falk Corollary)**.** $(\mathbb{E}, \mathbb{G}, \circ)$ is a well-defined field.

4 The Principle of Repugnant Exclusion

Consider the 3-Body Problem ($\therefore$). Notice that, despite being somewhat simple to state, it is difficult to intuit a value of $\mathbb{G}(\therefore)$.

Suppose P is as follows: *Given* $\perp \Rightarrow lv$, *find* lv. Simply enough, $\mathbb{G}(P) = \perp$. Note that P has triviality and $\therefore$ has pure nontriviality, so we can apply the law of excluded middle to see that $\mathbb{G}(\therefore) = \top$. We will refer to this relation by saying that $\therefore$ is *excluded* from P. Without loss of generality, we can generalize this proof to say that, given a problem P that can be excluded from a problem Q such that $\mathbb{G}(Q) = \perp$ must have $\mathbb{G}(P) = \top$ (and vice versa). This is the Principle of Repugnant Exclusion.

5 Objectability

We end by opening a discussion on *objectability*, the discussion of whether a problem is *objectionable*. Consider the 3-Body Problem used to establish the Principle of Repugnant Exclusion. This problem involves several abstractions, reductions, equations and several calculus and algebraic impossibilities[‡]. It would be reasonable to say that no reasonable person could sit down and solve this problem, so we would call this problem *objectionable*. We consider this to be a simple enough definition with profound consequences. Formally, we believe that any problem P for which $\mathbb{G}(P)$ is greater than $|\langle P \rangle|^{|\langle P \rangle}$ is objectionable, but we do not have a proof at the current time (we offer this as an open problem).

By way of example, consider the problem *Given a problem P, is P objectionable?* (this is the problem O). By applying earlier results from the Rauen Incompleteness Principle and Wong's Lemma, we see that O is undecidable. Any resulting decider M, then, cannot be encoded via any finite alphabet Σ, which means that it must contain an infinite number of non-common English symbols (thus immediately failing the character-counting test). Furthermore, attempting to analyze the types of $L(O)$ (the set of $\langle P \rangle$ for all P that are objectionable) is objectionable without a well-defined proof of the objectionability of O (see citation 2). If such a proof existed, then, it must itself be objectionable (and so we cannot assume that the proof was ever written or read by any human author). We, as humans, must thus conclude that O itself is objectionable.

References

1. A. M. Turing. On computable numbers, with an application to the entscheidungsproblem. Proceedings of the London Mathematical Society, s2-42(1):230–265, 1937
2. You thought there would be a citation, but it was me, Dio Brando!
3. "Anime was a mistake" - Hayao Miyazaki

[‡]It is also already unsolvable.