0-Order Non-Linear Ordinary Differential Equations

Dead Duck or Phoenix?

Oscar I. Hernandez, Daniel J. Packer

ohernandez13@simons-rock.edu, dpacker14@simons-rock.edu; $\Lambda\Delta M$.
Division of Science, Mathematics & Computing, Bard College at Simon's Rock, USA.

Abstract. A common problem in the field of applied mathematics is to find solutions to ordinary differential equations of order n. Solutions to first and second order differential equations are well understood. For example, consider the homogenous linear second order differential equation, $\frac{d^2y}{dt^2} = -y$, which has the general solution $C_1 \sin t + C_2 \cos t$ for any $C_1, C_2 \in \mathbb{R}$. Many generalizations can also be made to differential equations of higher order. However, perhaps the most important class of differential equations are those of order zero, whose solutions have been overlooked in most undergraduate-level textbooks. For example, $y = e^x$ is a solution to the differential equation $\frac{\csc(x)}{\log_2(e)} \log_2(y^{(0)}(x)) = x\frac{\sec(x)}{\tan(x)}$. In this paper, we will explore and attempt to provide methods of solution to these equations.

Keywords: ordinary differential equations, non-linear, non-homogenous.

1 Introduction

Perhaps one of the largest fields in mathematics is ordinary differential equations. However, this discipline has failed to offer solutions to differential equations of order zero, for mentions of them in the literature are scarce and subtle. For convenience, we shall now refer to ordinary differential equations as ODEs.

The general form for an nth order ODE is

$$y^{(n)} + \sum_{i=0}^{n-1} p_i(t) y^{(i)} = g(t) \tag{1}$$

where paranthetical exponentiation (i) denotes the ith derivative. For example, the equation $y^{(1)}(t) + 17y = 37t + e^t - \sqrt[8]{\ln(\sin^2(\pi t))} - e^{\pi i} + 0$ is in the general form for a first order ODE. It follows that the general form for an ODE of order zero is $y^{(0)}(t) = g(t)$.

1.1 Overview

In Section 2, we will provide some methods of solution to ODEs of order zero. In Section 3, we will offer real applications of 0th order ODEs.

2 Methods of Solution

We ask the reader to look over any standard undergraduate textbook on differential equations. The methods we'll provide to solve ODEs of order zero should now seem familiar.

2.1 Integrating Factors

Perhaps the reader recalls that there exist standard methods for solving first order linear ODEs. Those are ODEs of the form $y^{(1)}(t) + p(t)y = g(t)$, and the method is that of integrating factors. The derivation will not be considered, but we shall remind the reader that the following is the general solution to the ODE above

$$y(t) = \frac{\int \mu(t)g(t) + c}{\mu(t)}$$

where $\mu(t) = e^{\int p(t)\ dt}$ is the integrating factor that facilitates integration.
Theorem 1. Let $\mu(t) = e^{\int \nu(t)}$ be the integrating factor for

$$\nu(t) = \frac{g^{(2)}(t)g(t) - (g^{(1)}(t))^2}{g^{(1)}(t)g(t)}.$$

Then, the solution to any 0th order linear ODE $y^{(0)}(t) = g(t)$ is

$$y(t) = \frac{g^{(1)}(t) \int \mu(t)g(t)\ dt}{\mu(t)g(t)}.$$

We leave the proof of this as an exercise to the reader.

2.2 Characteristic Polynomials

Homogeneous ODEs are ODEs of the form as in (1) for $g(t) = 0$. As the reader may remember, one of the general methods for solving homogenous higher order ODEs is that of the Characteristic Polynomial. This offers solutions to ODEs of the form from equation (1). For a 0th order ODE $y^{(0)}(t) = g(t)$, the following method may be used:

1. Assume y can be written in the form,

$$y(t) = \sum_{n=0}^{\infty} C_n g^{(n)}(t)$$

2. Since g is a constant function (by homogeneity), all derivatives, $g^{(n)}$ will vanish for $n > 0$. Thus,

$$y(t) = C_0 g^{(0)}(t) + \sum_{n=1}^{\infty} C_n \cdot 0 = C_0 g^{(0)}(t).$$

It is interesting to note that in the case of 0th order ODEs $C_0 = 0 = g^{(0)}(t)$. In Section 4, we examine the case where the 0th order ODE is non-homogenous

2.3 Undetermined Coefficients

The method of undetermined coefficients is used for finding particular solutions to non-homogenous ODEs. Suppose we have an ODE of the form $y^{(0)} = g(t)$. The following method may be used to solve for A.

1. We can assume the solution will be of the form $y(t) = Ag(t)$.
2. Plug this into our ODE and solve for A.

For example, consider the ODE $y^{(0)}(t) = e^t$. $Ae^t = e^t$. Setting the coefficients equal to each other[1], we see that $A = 1$. One can easily show that $A = 1$ works for all 0th order ODEs. Thus, $y(t) = 1 \cdot g(t)$ is a solution to all 0th order ODEs. It is worth noting that this method holds if $g(t) = 0$; in addition, if $g(t) = 0$, $A = 0$ also works, which makes $y(t) = 0 \cdot 0 = 0$ a solution.

3 Application

Solutions to 0th order ODEs are abundant in a multitude of scientific fields. Below are just a few examples.

Example 1. A classic problem in the field of physics is to solve a system at equilibrium. One of the most popular versions of this problem is to find the terminal velocity of a falling object. Let α be the coefficient of linear air resistance in units of $\frac{N \cdot s}{m}$, $v(t)$ be the velocity of the particle with units, $\frac{m}{s}$, and mg product of the mass and gravitational acceleration on the particle with units, $kg \cdot \frac{m}{s^2} = N$. Thus, the full equation for the force on the particle will be:

$$\begin{aligned} F &= ma \\ &= v(t)\alpha - mg \end{aligned}$$

However, if we assume that the particle is at equilibrium, then $F = 0$, so the equation becomes:

$$\begin{aligned} 0 &= v(t)\alpha - mg \\ v(t)\alpha &= mg \end{aligned}$$

As is clear to the most casual of observers, $v(t) = \frac{mg}{\alpha}$.

Example 2. Let G be a good at manufacturing cost $\$D$ and suppose a group of 100 people are willing to pay $\$\frac{(D-0.1D)}{D}$. At equilibrium, the item will sell when $D = \frac{D-0.1}{D}$. If we solve this 0th order ODE, we see that G will sell exactly when $D = 0$ or $D = 0.9$.

[1] We do this in order to not implicitly divide by 0 anywhere.

4 Open Problems and Additional Results

Recall the method of the characteristic polynomial developed in Section 2.2. The authors have noticed that for non-homogenous ODEs, $g^{(0)}(t)$ remains a solution, but have yet to provide a proof. One of the authors, Oscar Hernandez, offers \$4.20 to the first reader to prove the result for non-homogenous ODEs.

Recall the method of undetermined coefficients from Section 2.3. The authors have a strong suspicion that for a homogenous 0th order ODE, $y(t) = Ag(t)$ is a solution for any value of $A \in \mathbb{C}$. One of the authors, Daniel Packer, offers \$69.69 to the first reader to prove or disprove the statement above.

Over the years presenting on this topic, we have noticed the response to the lecture is usually:

> "Hey-o man! What a cool paper, lots'a stuff to think about. I knew about first, second, and higher-order ODEs, and I just learned about 0th order ODEs. What other kinds of ODEs could there be? Oh! What about negative-order ODEs?"

Our response is always as follows:

> "Please, get out of my house and stop wasting my time. Negative-order ODEs would have absolutely no merit to them."

That being said, we encourage the reader to consider fractional order differential equations; those are ODEs where the highest order derivative a non-negative rational number.

5 References

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