A Solution to the Two-Body Problem

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Abstract

The two-body problem asks whether two massive academics undergoing mutual attraction can locate stable employment and lodging. We identify an important special case in which a solution exists, the first of its kind. We give a lower bound on net worth as a function of time. Given the significance of the result, we give a near-formal proof in Differential Dynamic Logic $d\mathcal{L}$ to increase confidence in the result, without the hard work of actually doing the proof correctly.

1 Introduction

The three-body problem has been known since Newton: Given three massive bodies mutually exerting gravity, compute their trajectories. It has also been known since the time of Poincareé that no analytic closed-form solution exists in the general case. Poincaré was not discouraged by this. Rather, (mostly because he and his wife were both on the academic job market at the same time), he did what any good mathematician would do and simplified the problem until there was some hope of solvability. Poincaré's simplification is what we now know as the *two-body problem*: "Given two massive academics in mutual attraction, determine stable employment and lodging". This simplification is of massive importance to academics even today, and as the modern academic knows, no general solution has yet been found.

2 Related Work and Previous Solutions

In extremely special cases, satisfactory solutions to the two-body problem have been found. The most notable special case is that of Blum and Blum who have found simultaneous full-professorships at a Research I university in an affordable city in related research areas. This is, clearly, the most difficult instance of the 2-body problem. However, the B^2 solution requires at least one Turing award, and thus does not scale to us mere mortals. It would have worked for Poincaré, I'm sure, but not for me. Who am I even kidding, the 1-body problem has long been solved.

Other approaches are more ambitious, seeking to provide an answer to us mere mortals by sidestepping the requirement for an analytic closed-form solution, satifying themselves with implicit or non-analytic solutions instead. The most notable approach is that of the Klein Four group, who take on the even more general three-body problem using nonanalytic functions such as a cappella: `https://www.youtube.com/watch?v=Aiq\_oaIvyak`

3 Our Approach

The key issue here, of course, is to identify a salary for two PhD's at the same time, as only so many employers are willing to employ the unemployable. In offline dating, it has been widely recognized that the problem becomes significantly easier as the distance in thesis topics increases. Thus, the Two-Body problem can be reduced to the Some-Body Problem as posed by Mercury: **Can anybody find me somebody to love? Who has a PhD in a technical field, but preferably not CS and definitely not formal methods, oh god please not formal methods?**

Our approach is centrally based on Mercury's reduction. Further, we make the simplifying assumption that the relationship between the bodies is *stable*, a common assumption which significantly simplifies the dynamics. Because we provide the first and only solution for such a seminal problem in bodyology, we provide a formal model and proof of our solution to the two-body problem in Differential Dynamic Logic, an established logic for verifying hybrid systems, many of which are Cyber-Physical Systems. This marks its first known use in verifying Cyber-Social-Physical-;-)-Academic Systems.

3.1 Model of Academics in Motion

In the two-body problem, we are given two rigid ;-) romantic bodies in motion and seek to show they can sustatin some minimum level of income over an extended period of time. As in the typical presentation of the problem, we assume the sole source of income is a university.

In this model, we consider two-dimensional rigid bodies, whose positions are described by x_1, y_1, x_2, y_2. Following standard simplifications, we assume the position of the university is somewhere between the academics in a way we will make precise soon. According to government statistics, the salary for each academic is inversely proportional to the distance from the university. We add another variable t to track the evolution of time.Putting this all together, we arrive at a 10-dimensional ODE describing the evolution of the system:

$$\begin{aligned}\alpha \equiv \{x_1' =& v_{x,1}, v_{x,1}' = -\frac{m_2 \cdot v_{y,1}}{d^2}, x_2' = v_{x,2}, v_{x,2}' = -\frac{m_1 \cdot v_{y,2}}{d^2},\\ y_1' =& v_{y,1}, v_{y,1}' = \frac{m_2 \cdot v_{x,1}}{d^2}, y_2' = v_{x,2}, v_{x,2}' = -\frac{m_1 \cdot v_{x,2}}{d^2},\\ \$' =& \frac{1}{d_{1,U}} + \frac{1}{d_{2,U}}, t' = 1\}\end{aligned}$$

Where we define $d \equiv \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $d_{1,U} \equiv \sqrt{(x_1 - x_U)^2 + (y_1 - y_U)^2}$ and $d_{2,U} \equiv \sqrt{(x_U - x_2)^2 + (y_U - y_2)^2}$ where the university is at a fixed location x_U, y_U.

Like all good proofs we have some preconditons. Specifically, the mass and distance are in harmony such that the bodies will maintain a stable orbit, and the university lie somewhere inside that orbit:

$$\begin{aligned}
Pre \equiv\ & v_{x,1}^2 + v_{y,1}^2 = v_{x,2}^2 + v_{y,2}^2 \\
& = \frac{m_1}{d} = \frac{m_2}{d} \wedge t = 0 \\
& \wedge (x_U - (x_1 + x_2)/2)^2 + (y_U - (y_1 + y_2)/2)^2 <= d^2
\end{aligned}$$

And our postcondition establishes a bound on income:

$$Post \equiv \$ \geq \$_0 + \frac{2 \cdot t}{d}$$

Putting these together, we arrive at our theorem-statement: **Stable Relationship Solution to the 2-Body Problem:**

$$Pre \to [\alpha]Post$$

Proof. We decompose the proof into its key lemmas, leaving the details of the proof as an exercise for the reader.

Lemma: Bounded Attraction. A stable relationship requires that the bodies be attracted to each other, but not so attracted to each other that they might lose their wits, or in our case, their orbits. The Bounded Attraction lemma asserts an upper bound on the attraction, or force:

$$[\alpha]v_{x,1}'^2 + v_{y,1}'^2 \leq \frac{m_2 \cdot v_1^2}{d^2}$$

And similar for the second body.

Lemma: Constant Separation. Next, we must prove the essential simplifying step that distance between the bodies is constant, without which the rest of the proof is intractible. Because distance makes the heart grow fonder, we call this lemma constant separation:

$$[\alpha]d' = 0$$

Lemma: Stable Relationship. Stability of relationships is essential to maintain the healthy life attitudes that allow one to succeed in academia. Thus we show the bodies are in a stable orbit at the radius $d/2$ which is centered around the origin, without loss of generality:

$$[\alpha]x_1^2 + y_1^2 = (d/2)^2$$

And so on for the other body.

Lemma: Home Sweet Home. In order to establish a bound on salary, we need a bound on the distance to the university. Because the university stays at its starting position inside the circle, we can bound this distance by the diameter:

$$[\alpha](x_1 - x_U)^2 + (y_1 - y_U)^2 \leq d^2$$

Lemma: Continuous Cash Flow These combine to give us a bound on the continuous rate at which cash changes, which leads directly to the theorem:

$$[\alpha]\$' \geq \frac{2 \cdot t}{d}$$

□

4 Conclusion

We have proven the first known lower bound on salary in the two-body problem under realistic situations. Practical applications of this solution abound for otherwise-broke academics on the job market. Our simple closed-form solution simplifies financial planning for the affected academic masses. We "formalized" the result for increased confidence, in case any... body should dispute its correctness.