A Formal Treatment of ${}^{k}/_{n}$ Power-Hours

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1 Related Work

Power Hour is the name of a drinking game in which you take one shot of beer every minute, and while this is fun, sometimes you may prefer to drink slower. This requires a generalization of the Power Hour, however sophisticated algorithms for this are prone to memory leaks. This problem is addressed in [1] with a category of algorithms called ${}^{k}/_{n}$ Power-Hours.

Further studies of the Power Hour were made in [2] where the author correctly points out that more games were possible then reported in [1]. He further studies the effects of adding multiple players.

We present here a pictorial representations to address further memory leaks, a generalization over states, along with a series of lemmas that suggest an extra possiblity not considered earlier in the ${}^{k}/_{n}$ Power-Hours. We also include an appendix containing graphs over all the possible fair 4 state, one person games.

2 Introduction

Let's start with some notation. We have adapted the notation of [1] where $\uplus$ means full, $\cup$ means empty, and $\cap$ means flipped. Further we have extended it with the symbol $+$ to shorten: fill and drink.

The rules of a ${}^{k}/_{n}$ Power-Hour are as follows: every minute you have to perform one or more actions ending in a (possibly new) state. Due to memory concerns the actions should be uniquely determined based on the current state of your glass. If you have a full glass you have to drink it, before executing the actions. You will end up drinking k shots in n minutes.

Now we are equipped to take a first look at our pictorial representation of a configuration, specifically the ${}^{2}/_{60}$ as described by [1]. [1]

$$\begin{tikzcd} \uplus \arrow[rr] & & \cup \arrow[dl, "+"] \\ & \cap & \end{tikzcd}$$

Figure 1: ${}^{2}/_{n}$

It reads: on $\uplus$ empty it and leave $\cup$. On $\cup$ fill and drink, then leave $\cap$.

3 A Multitude of States

In this section we present our generalized lemmas, along with examples.

The first generalization we need to discuss is the number of states, while the conventional 3 are neat we suggest a fourth $\subset$ (lying), that can be used without conflict with the others. How-

[1] We usually omit identity arrows, as here from $\cap$ to $\cap$, unless we only use one state.

ever we advice against using it together with $\supset$, as this will cause confusion.

Having offered an extra state we can only assume other people will do so as well. Therefore we will write ${}^{k}/^{s}_{n}$ for a ${}^{k}/_{n}$ game with s states. If nothing is indicated we assume a 3 state game.

Definition 1. a fractional game is a ${}^{k}/^{s}_{n}$ where k is some fraction $\frac{f(n)}{d}$. A *pure* fractional game has $s = d$.

Specifically this means a game with a graph where there is a cycle, the longest circle is d steps, and $\uplus$ is in it.

Lemma 2. *for a ${}^{k}/^{s}_{n}$ power hour game there are s trivial fractional configurations; ${}^{\frac{n}{1}}/_{n}, {}^{\frac{n}{2}}/_{n}, \ldots, {}^{\frac{n}{s}}/_{n}$.*

Proof. by induction on the number of states. □

With the conventional 3 states they look like this:

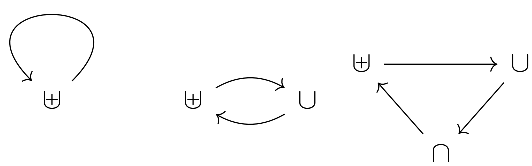

Figure 2: ${}^{\frac{n}{1}}/_{n}, {}^{\frac{n}{2}}/_{n}, {}^{\frac{n}{3}}/_{n}$

An interesting aspect is that we can add $+$ to any of these arrows, the effect of which is are presented in the following lemma:

Lemma 3. *for a fractional ${}^{\frac{kn}{d}}/^{s}_{n}$ game with $k \leq d$, can make it into a ${}^{\frac{(k+1)n}{d}}/^{s}_{n}$ game.*

Proof. we add 1 drink pr. d steps, in a k step game, thus we have $\frac{1}{d}n + \frac{kn}{d} = \frac{n}{d} + \frac{kn}{d} = \frac{n+kn}{d} = \frac{(k+1)n}{d}$. □

We can now make the more interesting ${}^{2n}/_{n}$ and ${}^{\frac{2n}{3}}/_{n}$ games;

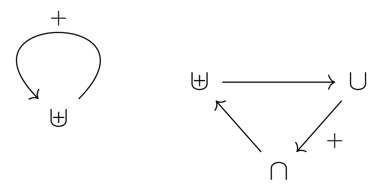

Figure 3: ${}^{2n}/_{n}, {}^{\frac{2n}{3}}/_{n}$

In fact;

Theorem 4. *all possible pure fractional games ${}^{\frac{kn}{d}}/^{s}_{n}$ can be constructed from the previous two lemmas.*

Proof. left as an exercise to the reader. □

Corollary 5. all possible pure fractional games can be written on the form ${}^{\frac{kn}{s}}/^{s}_{n}$, with $k \leq s+1$.

4 Constant Amount Games

In this section we look at ${}^{c}/^{s}_{n}$ games where c is not affected by n. Intuitively this means that we cannot have (non-identity) cycles in the graphs. We already saw the ${}^{2}/_{n}$ game. However which games are possible? In [1] the authors claim that a ${}^{3}/_{60}$ game is impossible[2], however we claim that it is possible, and even stronger:

Lemma 6. *with s states, and $n \geq s$ there are $s+1$ lower constant games: ${}^{0}/^{s}_{n}, {}^{1}/^{s}_{n}, \ldots, {}^{s}/^{s}_{n}$.*

Proof. by induction on the number of states. □

[2] they were conducting relevant research while writing it, so it is understandable.

Continuing with the classical examples they are[3]:

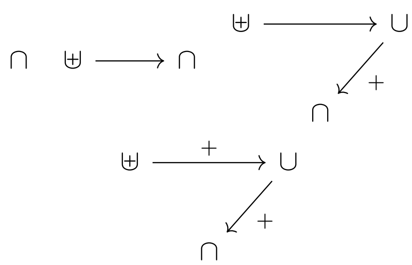

Figure 4: ${}^{0}/_{n}, {}^{1}/_{n}, {}^{2}/_{n}, {}^{3}/_{n}$

At this point you may think that we are done with constant games, however as noted by ... there is some symmetry in this game, and we can take advantage of this here;

Lemma 7. *with s states and $n \geq s$ there are s upper constant games: ${}^{n-s+1}/^{s}_{n}, {}^{n-s+2}/^{s}_{n}, \ldots, {}^{n}/^{s}_{n}$.*

Proof. by induction on the number of states. □

You may have already figured out how they look, but here they are;

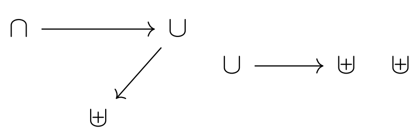

Figure 5: ${}^{n-2}/_{n}, {}^{n-1}/_{n}, {}^{n}/_{n}$

5 Having a Party

Definition 8. a game ${}^{k}/^{s}_{n}$ with $k \in \mathbb{N}$ is called fair.

Now it is time to mix it all together:

[3]Notice that we end on a $\cap$ to indicate that it does not need refilling.

Theorem 9 (Soundness)**.** *there is a fractional ${}^{\frac{kn-c}{d}}/^{s}_{n}$ game if $k \leq d+1, d+|c| \leq s$, and $kn-c \geq 0$.*

The equation may look complicated, but it is quite logical when you look at a picture:

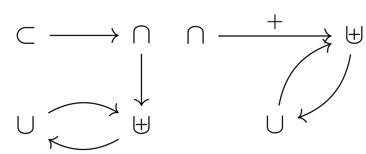

Figure 6: ${}^{29}/^{4}_{60}, {}^{\frac{59}{2}}/_{60} \cong {}^{31}/_{60}$[2]

And finally we are ready to express the master lemma:

Theorem 10 (Completeness)**.** *all possible fractional games can be written as ${}^{\frac{kn-c}{d}}/^{s}_{n}$ with $k \leq d+1, d+c \leq s$ and $kn-c \geq 0$.*

Which captures all the intuitions we have built.

Now that we have that in order, should invite some friends:

Lemma 11. *if $d-1$ players take turns playing with the same cup, and there is a fair game ${}^{\frac{kn-c}{d}}/^{s}_{n}$ with $p \mid c$ then there is a fair ${}^{\frac{kn-c}{pd}}/^{s}_{n}$ game.*

This means that with 2 girls, 1 cup, and an hour the possible games are:

$$ {}^{\frac{60k}{2\cdot 3}}/^{3}_{60} = {}^{10k}/^{3}_{60} $$

for $k \leq 4$ (due to proposition 5), giving the new configuration ${}^{10}/_{60}$. If we also have a die we could even get: ${}^{10k-1}/^{9}_{60}$ for $k \leq 4$. And with 3 people we could play ${}^{5k}/^{4}_{60}$ for $k \leq 5$, giving the new (fair) games: ${}^{5}/^{4}_{60}, {}^{25}/^{4}_{60}$.

6 Bibliography

[1] Ben Blum, Dr. William Locas, Chris Martens, Dr. Tom Murphy VII, *Algorithms for k/n Power-Hours*, SIGBOVIK 2012

[2] Dr. Tom Murphy VII, *New results in* $^{k}/_{n}$ *Power-Hours*, SIGBOVIK 2014

A 4 State Games

A.1 Trivial Fractional Games

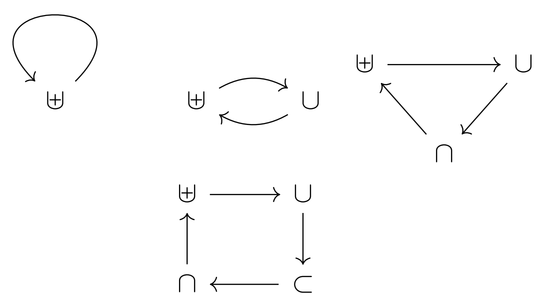

Figure 7: $^{\frac{n}{1}/4}/_{n}$, $^{\frac{n}{2}/4}/_{n}$, $^{\frac{n}{3}/4}/_{n}$, $^{\frac{n}{4}/4}/_{n}$

A.2 Other Fractional Games

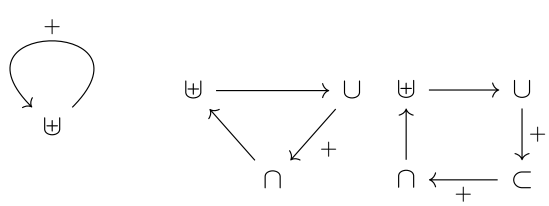

Figure 8: $^{2n/4}/_{n}$, $^{\frac{2n}{3}/4}/_{n}$, $^{\frac{3n}{4}/4}/_{n}$

A.3 Lower Constant Games

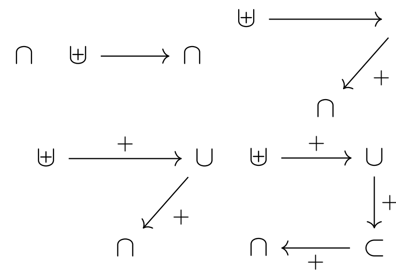

Figure 9: $^{0/4}/_{n}$, $^{1/4}/_{n}$, $^{2/4}/_{n}$, $^{3/4}/_{n}$, $^{4/4}/_{n}$

A.4 Upper Constant Games

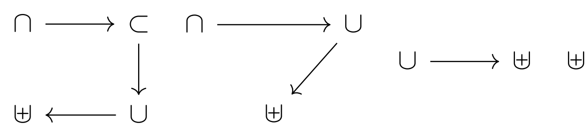

Figure 10: $^{n-3}/_{n}$, $^{n-2}/_{n}$, $^{n-1}/_{n}$, $^{n}/_{n}$

A.5 Mixed Games

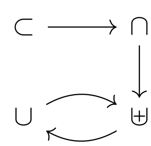

Figure 11: $^{\frac{n-2}{2}/4}/_{n}$