Lec 5 Unitary freedom in ensemble of density matrices

Suppose $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_j q_j |\phi_j\rangle\langle\phi_j|$

Define: $|\tilde{\psi}_i\rangle = \sqrt{p_i}\,|\psi_i\rangle$, $|\tilde{\phi}_j\rangle = \sqrt{q_j}\,|\phi_j\rangle$

$$\rho = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j| .$$

Thm: The sets $\{|\tilde{\psi}_i\rangle\}$, $\{|\tilde{\phi}_j\rangle\}$ generate the same density matrix iff $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\phi}_j\rangle$ where u_{ij} is a unitary matrix where we pad the smaller set with zeros (zero vectors).

Pf. ($\Rightarrow$) $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\phi}_j\rangle$

$$\sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_{ijk} u_{ij} u_{ik}^* |\tilde{\phi}_j\rangle\langle\tilde{\phi}_k|$$

$$= \sum_{jk} \left(\sum_i u_{ij} u_{ik}^*\right) |\tilde{\phi}_j\rangle\langle\tilde{\phi}_k| = \sum_{jk} \delta_{kj} |\tilde{\phi}_j\rangle\langle\tilde{\phi}_k|$$

$$= \sum_j |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j| .$$

$$\left[\ \tilde{\Psi} = \tilde{\Phi} U \qquad \tilde{\Psi}\tilde{\Psi}^* = \tilde{\Phi} \underbrace{U U^*}_{I} \tilde{\Phi}^* = \tilde{\Phi}\tilde{\Phi}^*\ \right].$$

[$\Leftarrow$ HW].

[Sketch basic idea in class]

>>> Reduced density operator

A, B composite system, ρ^{AB} is a state.

$$\rho^A \equiv tr_B(\rho^{AB}),\quad \rho^B = tr_A(\rho^{AB}).$$

| V, W Hilbert spaces of A, B resp.

"partial trace map"

linear map from $L(V\otimes W, V\otimes W) \rightarrow L(V,V)$

Def: $\forall$ pure states $|a_1\rangle, |a_2\rangle \in V$, $|b_1\rangle, |b_2\rangle \in W$,

$$tr_B\left(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right) \equiv |a_1\rangle\langle a_2|\, tr\left(|b_1\rangle\langle b_2|\right)$$
$$\equiv \left(|a_1\rangle\langle a_2|\right) \cdot \langle b_2|b_1\rangle.$$

Extend using linearity.

[Why this definition? ρ^A or ρ^B provides correct measurement statistics for systems A, B].

e.g. $\rho \otimes \sigma = \rho^{AB}$

$$\left.\begin{aligned} \rho^A &= tr_B(\rho^{AB}) = tr_B(\rho \otimes \sigma) = \rho \\ \rho^B &= tr_A(\rho^{AB}) = \sigma. \end{aligned}\right\} \quad \begin{aligned} &\rho: \text{density op of } A \\ &\sigma: \text{density op of } B. \end{aligned}$$

e.g. Bell state $|\psi\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$.

$$\rho = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right).$$

$$\rho' = tr_2(\rho) = \frac{1}{2} tr_2\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$
$$= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = I/2.$$

$$tr\left(\left(\frac{I}{2}\right)^2\right) = \frac{1}{4} \cdot 2 = \frac{1}{2} < 1 \quad \text{(mixed state)}.$$

[Joint state of system is pure, yet a subsystem is in a mixed state $\Rightarrow$ quantum entanglement].

$\uparrow$

This property is a signature for entangled states.

[Note: Partial trace is the unique op. that gives rise to correct description of observable quantities for subsystems of a composite system].

$M = \sum_m m P_m$ observable on A. $M|m\rangle = m|m\rangle$.

$\tilde{M}$ same observable on $A+B$.

$\tilde{M} = M \otimes I_B$ [as $\tilde{M}|m\rangle|x\rangle = m|m\rangle|x\rangle$ $\forall |x\rangle$ of B, $|m\rangle$ eigenstate of A with eval m].

Measurement stat.

$\langle M\rangle = tr(M\rho^A) = \langle\tilde{M}\rangle = tr(\tilde{M}\rho^{AB})$ for all observables M.

Q: $\rho^A = f(\rho^{AB})$ → What f satisfies the above?

A. $f(\rho^{AB}) = tr_B(\rho^{AB})$ is the unique function.

[$tr(M f(\rho^{AB})) = tr((M\otimes I_B)\rho^{AB})$

$f(\rho^{AB}) = \sum_i M_i \alpha_i$ where M_i are orthonormal (H-S) Hermitian matrices which is a basis for Hermitian matrices

$\alpha_i = tr(M_i f(\rho^{AB}))$

So, $f(\rho^{AB}) = \sum_i M_i \, tr(M_i f(\rho^{AB}))$

$= \sum_i M_i \, tr((M_i \otimes I_B)\rho^{AB})$].

So, f is uniquely determined. [Check does not depend on choice of basis]

Schmidt Decomposition

Thm: Suppose $|\psi\rangle$ is a pure state of composite system AB. Then $\exists$ orthonormal states $|i_A\rangle$ for A, $|i_B\rangle$ for B s.t $|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$, where $\lambda_i \geq 0$ $\forall i$, $\sum \lambda_i^2 = 1$.

The multiset $\{\lambda_i\}$ are uniquely determined by $|\psi\rangle$.

[Prove in class if time].

[Note: if λ_i are distinct, the vectors $\{|i_A\rangle\}$, $\{|i_B\rangle\}$ are uniquely determined upto phase freedom].

$$\rho^{AB} = |\psi\rangle\langle\psi| = \left(\sum_i \lambda_i |i_A\rangle |i_B\rangle\right)\left(\sum_j \lambda_j \langle j_A| \langle j_B|\right)$$
$$= \sum_{i,j} \lambda_i \lambda_j \, |i_A\rangle\langle j_A| \otimes |i_B\rangle\langle j_B|$$

$\rho^A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$, $\rho^B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$, so eigenvalues of ρ^A, ρ^B are identical [also gives uniqueness of coeff].

[Fact: A state $|\psi\rangle$ of a composite system AB is a product state iff $|\psi\rangle$ has Schmidth # = 1. Also ρ is a product state iff ρ^A (& thus ρ^B) are pure states.]
[Ask class to prove].

Purifications

- ρ^A is a state of A. Can we introduce another system R, & define a pure state $|AR\rangle$ s.t $\rho^A = \mathrm{tr}_R (|AR\rangle\langle AR|)$?

YES → always possible (this process is called purification)

Pf. $\rho^A = \sum_i p_i |i^A\rangle\langle i^A|$ orthonormal decomp.

Define: $|AR\rangle = \sum_i \sqrt{p_i} \, |i^A\rangle |i^R\rangle$

$$\Rightarrow \mathrm{tr}_R (|AR\rangle\langle AR|) = \sum_i p_i |i^A\rangle\langle i^A| = \rho^A. \quad \blacksquare$$

>>> Quantum Fourier Transform (QFT)

Let $N = 2^n$.

Discrete Fourier transform (DFT)

$$\{x_0, \ldots, x_{N-1}\} \to \{y_0, \ldots, y_{N-1}\}$$

$$y_k \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i j k/N} \quad \to \text{[unitary operator]}.$$

Quantum Fourier transform (QFT)

$$\{|0\rangle, \ldots, |N-1\rangle\}$$

$$|j\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i j k/N} |k\rangle$$

$$\sum_{j=0}^{N-1} x_j |j\rangle \to \sum_{j=0}^{N-1} x_j \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i j k/N} |k\rangle \right)$$

$$= \sum_{k=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i j k/N} \right) |k\rangle$$

$$= \sum_{k=0}^{N-1} y_k |k\rangle.$$

So, coefficients $\{y_0, \ldots, y_{N-1}\}$ are DFT of $\{x_0, \ldots, x_{N-1}\}$.

Thm: QFT is a unitary map.

Pf. The transformation is: $\begin{pmatrix} y_0 \\ \vdots \\ y_{N-1} \end{pmatrix} = F \begin{pmatrix} x_0 \\ \vdots \\ x_{N-1} \end{pmatrix}$, and F is exactly the DFT matrix, which we know is unitary. ∎.

[Ex: What is QFT of $|0\rangle$?].

$$|0\rangle \xrightarrow{QFT} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle \quad \text{[i.e. uniform superposition]}.$$

We will use equivalent notations.

- $|j\rangle = |j_1, \dots, j_n\rangle$ where $j_1 j_2 \cdots j_n = j$ is the binary representation of j, i.e. $j = j_1 2^{n-1} + \cdots + j_n 2^0$.

- $0.j_\ell j_{\ell+1} \cdots j_m = j_\ell/2 + j_{\ell+1}/4 + \cdots + j_m/2^{m-\ell+1}$ (binary fraction).

Thm: QFT can also be written as the following map.

$$|j_1 \cdots j_n\rangle \to \frac{1}{\sqrt{N}}\left(|0\rangle + e^{2\pi i\, 0.j_n}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i\, 0.j_{n-1} j_n}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2\pi i\, 0.j_1 \cdots j_n}|1\rangle\right).$$

Pf.

$$|j\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i j \sum_{l=1}^{n} k_l 2^{-l}} |k\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{k_1,\dots,k_n=0}^{1} e^{2\pi i j \sum_{l=1}^{n} k_l 2^{-l}} |k_1 \cdots k_n\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{k_1,\dots,k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} |k_l\rangle$$

$$k = k_1 2^{n-1} + \cdots + k_n 2^0 = \sum_{l=1}^{n} k_l 2^{n-l}$$

$$j 2^{-l} = j_1 2^{n-1-l} + \cdots + j_n 2^{-l} = \sum_{k=1}^{n-l} j_k 2^{n-l-k} + \sum_{k=n-l+1}^{n} j_k 2^{n-l-k}$$

$$= \frac{1}{\sqrt{2^n}} \bigotimes_{l=1}^{n} \left[\sum_{k_l=0}^{1} e^{2\pi i j k_l 2^{-l}} |k_l\rangle\right] = \frac{1}{\sqrt{2^n}} \bigotimes_{l=1}^{n} \left(|0\rangle + e^{2\pi i j 2^{-l}} |1\rangle\right).$$

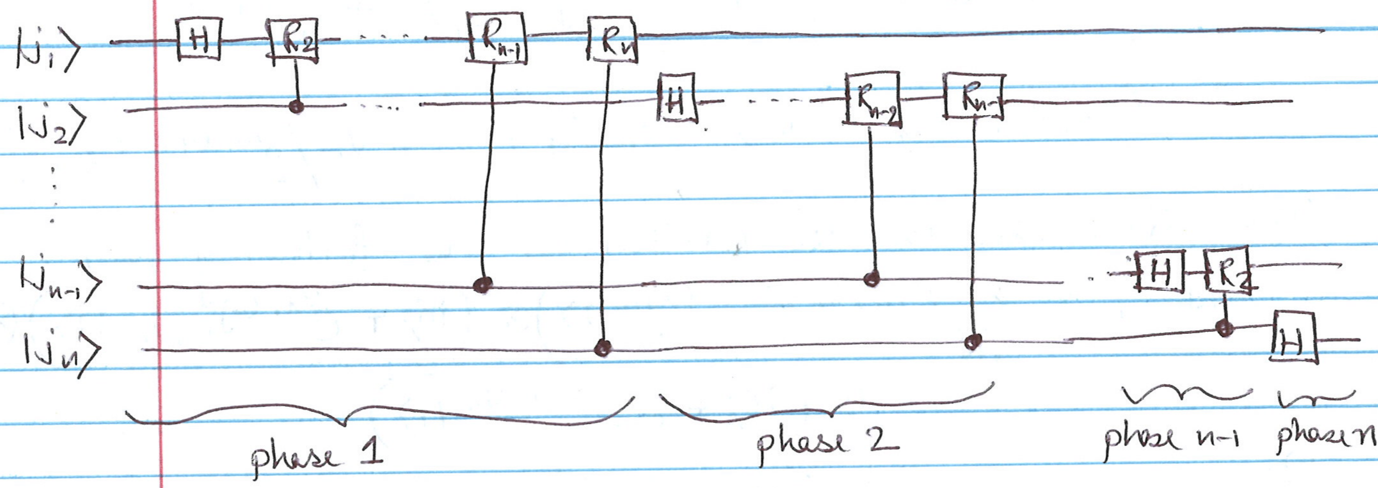
$$= \frac{1}{\sqrt{2^n}} \bigotimes_{l=1}^{n} \left(|0\rangle + e^{2\pi i \sum_{k'=1}^{l} j_{k'+n-l}/2^{k'}} |1\rangle\right)$$

$$k + l - n = k'$$

$$= \frac{1}{\sqrt{2^n}} \left(|0\rangle + e^{2\pi i\, 0.j_n}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i\, 0.j_{n-1} j_n}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2\pi i\, 0.j_1 \cdots j_n}|1\rangle\right). \quad \blacksquare$$

Denote the 1-qubit gate

$$R_k \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}$$



Lets track the state in phase 1

- after H : $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_1}|1\rangle\right)|j_2 \cdots j_n\rangle$

- after controlled R_2 :

 if $j_2 = 0$, nothing happens : $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_1}|1\rangle\right)|j_2 \cdots j_n\rangle$

 $j_2 = 1$, : $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (j_1/2 + 1/2^2)}|1\rangle\right)|j_2 \cdots j_n\rangle$

 $\Rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_1 j_2}|1\rangle\right)|j_2 \cdots j_n\rangle$.

- after controlled R_n : $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_1 j_2 \cdots j_n}|1\rangle\right)|j_2 \cdots j_n\rangle$

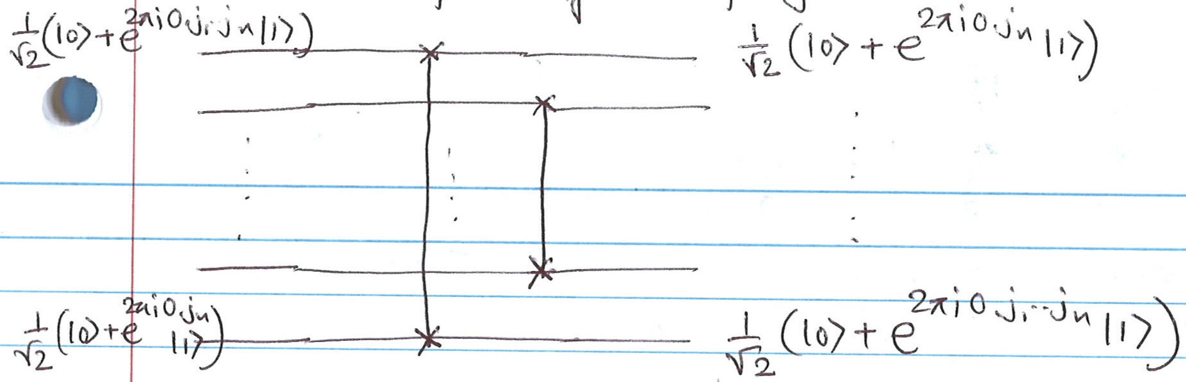
Similarly state after phase 2

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_1 \cdots j_n}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_2 \cdots j_n}|1\rangle\right)|j_3 \cdots j_n\rangle$$

After phase n

$$\frac{1}{\sqrt{2^n}}\left(|0\rangle + e^{2\pi i\, 0.j_1 \cdots j_n}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2\pi i\, 0.j_n}|1\rangle\right)$$

One can finish up using swap gates.



[Q: Design a circuit to implement IQFT].

A. one solution is to run the circuit in reverse.
[Find another solution].

How many quantum gates did we use?

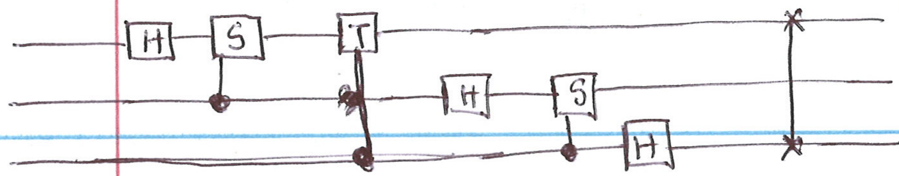
- each phase (k) uses 1 Hadamard + ($n-k$) controlled gates. so a total of $(n-k+1)$ gates.
- total gates: $\sum_{k=1}^{n}(n-k+1) = n+(n-1)+\cdots+1 = \frac{n(n+1)}{2}$.
- swap gates: $\frac{n}{2}$ if n even, $\frac{n-1}{2}$ if n odd.
  (each swap gate $\equiv$ 3 controlled NOT).
- $\Theta(n^2)$ quantum gates.

Contrast: Classical FFT $\Theta(2^n \log(2^n)) = \Theta(n 2^n)$.

Q. Can we use this circuit to compute classical DFT?

A. No, even reading coeff. out will need atleast 2^n measurements. Even worse, no easy way to prepare the input state!

e.g. Circuit for QFT on 3 qubits.



(*)

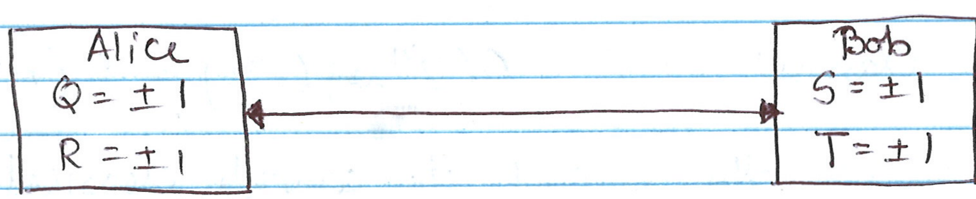
[Ex: Implement controlled S, controlled T using CNOT & single qubit gates].

>>> Di-Vincenzo's Criteria

1. A scalable physical system with well characterized qubit.
2. Ability to initialize the state of the qubits to a simple fiducial state ($|0\cdots0\rangle$).
3. Long relevant decoherence times.
4. A "universal set" of quantum gates.
5. A qubit specific measurement capability.

[Ex: Read about Di-Vincenzo's criteria on wikipedia].

>>> Bell's Inequality



— Charlie prepares two particles, & sends one particle to Alice, & other to Bob. (Charlie should be able to ~~prepare identical copies)~~. repeat experimental procedure).