* An interesting feature of this process is that # of Grover iterations when increased causes prob to oscillate, sometimes becoming very small (in general).

* [Especially if θ is irrational multiple of 2π]

* What to do if M is not known?

Mon Oct 28, 2019 (Lec 8)

Outline:

1. QPE
2. HHL (Harrow, Hassidim, Lloyd)
3. QSVE + QLSP 2

QPE

Goal: Input $U \in \mathbb{C}^{2^n \times 2^n}$

Output eigenvalues λ_j

$U|u_j\rangle = \lambda_j |u_j\rangle$, $|\lambda_j| = 1$

$\lambda_j = e^{2\pi i \phi_j}$, $0 \leq \phi_j \leq 1$

Binary decimal notation

$$\phi = \sum_i \phi_i 2^{-i} = 0.\phi_1\phi_2 \cdots \phi_p$$ (p bits of precision)

$\phi_i \in \{0,1\}$. $p \in \mathbb{Z}$

$$\lambda = e^{2\pi i \, 0.\phi_1\phi_2 \cdots \phi_p}$$

B.D.	
0.1	1/2
0.11	3/4
0.11...	1

$$\lambda^{2^k} = \left(e^{2\pi i \sum_j 2^{-j}\phi_j}\right)^{2^k} = \left(e^{2\pi i}\right)^{\sum_j 2^{k-j}\phi_j}$$

$$= \left[e^{2\pi i\, 2^{k-1}\phi_1}\right]\left[e^{2\pi i\, 2^{k-2}\phi_2}\right]\cdots\left[e^{2\pi i\, 2^{k-k-1}\phi_{k+1}}\right]\cdots$$

$$= \quad (1) \qquad (1) \quad \cdots \; e^{2\pi i\, 0.\phi_{k+1}\phi_{k+2}\cdots\phi_p}.$$

$$U|u\rangle = \lambda|u\rangle = e^{2\pi i\, 0.\phi_1\cdots\phi_p}|u\rangle$$

$$U^2|u\rangle = \lambda^2|u\rangle = e^{2\pi i\, 0.\phi_2\cdots\phi_p}|u\rangle.$$

$$\vdots$$

$$U^{2^{p-1}}|u\rangle = \lambda^{2^{p-1}}|u\rangle = e^{2\pi i\, 0.\phi_p}|u\rangle.$$

Recap of QFT.

$$|\phi_1\rangle|\phi_2\rangle\cdots|\phi_p\rangle \xmapsto{QFT} \left(\frac{|0\rangle + e^{2\pi i\, 0.\phi_p}|1\rangle}{\sqrt{2}}\right) \otimes \left(|0\rangle + e^{2\pi i\, 0.\phi_{p-1}\phi_p}|1\rangle\right)$$

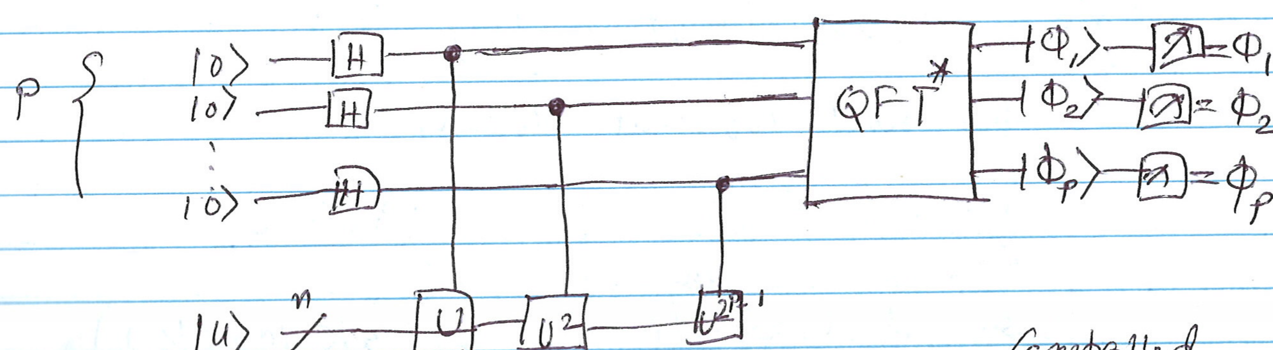
$$\cdots \otimes \left(\frac{|0\rangle + e^{2\pi i\, 0.\phi_1\cdots\phi_p}|1\rangle}{\sqrt{2}}\right)$$

$$\Big\downarrow QFT^*$$

$$|\phi_1\rangle|\phi_2\rangle\cdots|\phi_p\rangle.$$

"We have seen quantum circuit for QFT".

"We have seen QPE circuit".



$$1 + 2 \cdots + 2^{p-1} = 2^p - 1 \quad \text{(controlled unitaries)}.$$

$U = e^{-iHt}$

$U^m = e^{-iH(mt)}$ ← the form of unitary in HHL.

$|\psi\rangle = \sum_k c_k |u_k\rangle$, measure ϕ_k with prob $\propto |c_k|^2$.

$c_k = \langle \psi | u_k \rangle$.

Thm [Kitaev 1995]

Let $U \in \mathbb{C}^{2^n \times 2^n}$, $U|u_j\rangle = e^{2\pi i \phi_j}|u_j\rangle$. Then, $\exists$ quantum algorithm that maps

$$\sum_{j=1}^{2^n} c_j |u_j\rangle |0\rangle \mapsto \sum_j c_j |u_j\rangle |\tilde{\phi}_j\rangle \text{ where}$$

$|\tilde{\phi}_j - \phi_j| \leq \varepsilon$ with prob. $1 - 1/\mathrm{poly}(n)$

with runtime $O(T_U \log(N)/\varepsilon)$.

HHL

Refs.

"Primer": 1802.08227 ← recommend start with this paper.

Orig. Paper: 0811.3171

LSP (linear system problem)

Given $A \in \mathbb{C}^{M \times N}$, $\vec{b} \in \mathbb{C}^M$. Goal: Find $x \in \mathbb{C}^N$ s.t. $A\vec{x} = \vec{b}$, else output a flag $\nexists \vec{x}$.

We will look at $N = M$.

Classical Algs	Runtime
Gaussian Elimination	$O(N^3)$
Conjugate Gradient (CG)	$O(N s K \log(1/\varepsilon))$.

$s \equiv$ sparsity
A is s-sparse iff each row of A has at most s non-zero elements.

$\kappa \equiv$ condition number $= \dfrac{|\lambda_{max}|}{|\lambda_{min}|}$

Important Note:
$\nexists$ Q.A for LSP w/ $O(\log N)$ scaling.

QLSP

Input: $A, \vec{b}$ as before, $|b\rangle := \dfrac{\vec{b}}{\|b\|_2}$, $|x\rangle = \dfrac{\vec{x}}{\|x\|_2}$

Output: quantum state $|\tilde{x}\rangle$ w/ $\| |\tilde{x}\rangle - |x\rangle \| \leq \varepsilon$ $\quad (A\vec{x} = \vec{b})$
w/ prob. at least $1/2$.

(1) $\langle \tilde{x} | \hat{O} | \tilde{x} \rangle$, $\hat{O} = \hat{O}^\dagger$ ← Use case is where we only need to find such expectations.

Big idea for HHL

- A needs to be invertible
- $A^\dagger = A$

Q. If A is not ~~invertible~~ Hermitian? $A' = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \begin{pmatrix} \vec{x}' \\ \vec{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{b} \end{pmatrix}.$$

- Spectral decomp: $A = \sum \lambda_j |u_j\rangle\langle u_j|$

$$A|u_j\rangle = \lambda_j |u_j\rangle$$

$$A^{-1} = \sum_j \frac{1}{\lambda_j} |u_j\rangle\langle u_j|.$$

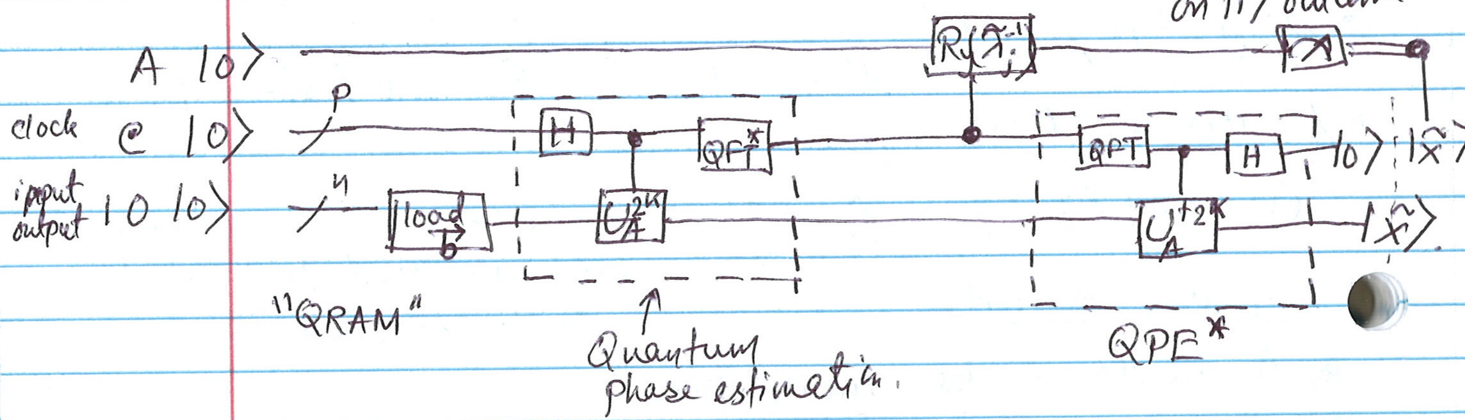
$$A^{-1}|b\rangle = \sum_j \frac{1}{\lambda_j} |u_j\rangle \underbrace{\langle u_j | b\rangle}_{\beta_j} = \sum_j \frac{\beta_j}{\lambda_j} |u_j\rangle = |x\rangle$$

Q: λ_j

▶ $U_A(t) = e^{-iAt}$ is an unitary as A is Hermitian.

$$Au = \lambda u \Rightarrow e^A u = e^\lambda u.$$

Steps of HHL
0. load $\vec{b}$
1. QPE w/ U_A
2. "Divide" by λ_j
3. Uncompute
4. Post selection on ancilla qubit



$$|0\rangle_A |0\rangle_c |0\rangle_{IO} \mapsto |0\rangle_A |0\rangle_c |b\rangle_{IO} = \sum_j |0\rangle_A |0\rangle_c \underbrace{\beta_j |v_j\rangle_{IO}}_{\text{after loading of } \vec{b}.}$$

$$\xrightarrow{QPE} \sum_j \beta_j |0\rangle_A |\tilde{\lambda}_j\rangle_c |v_j\rangle_{IO}$$

$c \in \mathbb{R}$ (hyperparameter)

$$\xrightarrow{\text{division}} \sum_j \beta_j \left(\sqrt{1-\frac{c^2}{\tilde{\lambda}_j^2}}\,|0\rangle + \frac{c}{\tilde{\lambda}_j}|1\rangle\right) |\lambda_j\rangle_c |v_j\rangle_{IO}$$

$\xrightarrow{\text{uncompute}}$ $\displaystyle\sum_j \beta_j \left(\sqrt{1 - \frac{c^2}{\tilde{\lambda}_j^2}}\, |0\rangle + \frac{c}{\tilde{\lambda}_j} |1\rangle \right) |0\rangle_c \, |v_j\rangle_{IO}$

$$\xrightarrow{\text{post selection.}} \boxed{\frac{1}{N} \sum_j \beta_j \frac{1}{\tilde{\lambda}_j} |v_j\rangle_{IO}} = |\hat{x}\rangle \propto A^{-1}|b\rangle.$$

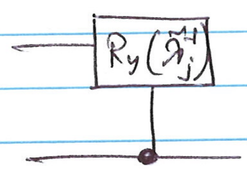
Complexity

$$\tilde{O}\left(\log(N)\, s^2 \kappa^2 / \varepsilon \right)$$

- This comes from QPE & Hamiltonian simulation
  - HS: $\tilde{O}\left(\log(N)\, s^2 t\right)$

Thm (HHL 08): Matrix inversion is BQP complete.

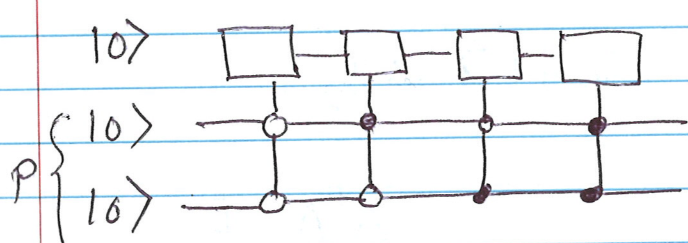
Division step.



$$R_y(2\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta = \cos^{-1}\frac{c}{\lambda}$$

$$\mapsto \begin{pmatrix} c/\lambda & -\sqrt{} \\ \sqrt{1 - c^2/\lambda^2} & c/\lambda \end{pmatrix}$$

$$R_y\left(\cos^{-1}\frac{c}{\lambda}\right)|c\rangle = \frac{c}{\lambda}|c\rangle + \sqrt{1 - c^2/\lambda^2}\,|1\rangle$$



$\phi = 0.\phi_1 \phi_2 \cdots$

2^P (large prefactor).

"Filter Functions"
↑
divide by bad eigenvalues.

QSVE

- SVD, $A \in \mathbb{R}^{M \times N}$, $A = U \Sigma V^* = \sum_{j=1}^{rk(A)} \sigma_j |u_j\rangle\langle v_j|$

  $U \in \mathbb{R}^{M \times M}$ unitary

  $\Sigma \in \mathbb{R}^{M \times N}$ non-neg. on diag.

  $V \in \mathbb{R}^{N \times N}$ unitary

q. Alg for Sing. value estimation

$\rightarrow$ QPE w/ a unitary, $W \in \mathbb{R}^{MN \times MN}$ determined from A.

Thm [Kerenidis, Prakash 17] For A as before, $\exists$ unitary W s.t. $\cos \theta_j^{\pm} = \frac{\sigma_j}{\|A\|_F}$ where $e^{i\theta_j^{\pm}}$ are evals of W, $\|A\|_F = \sqrt{\sum_{j=1}^{rk(A)} \sigma_j^2}$.

Pf Sketch

$$A = \begin{bmatrix} \text{---} A_1 \text{---} \\ \text{---} A_M \text{---} \end{bmatrix}, \text{ each } A_i \text{ is a row of } A$$

$$|A_i\rangle = \frac{A_i}{\|A_i\|_2}$$

$$|\overline{A}\rangle = [\|A_1\|_2, \ldots, \|A_M\|] / \|A\|_F.$$

Define isometry:

$$\Lambda|i\rangle = |\overline{A}\rangle \otimes |i\rangle, \quad \Lambda^+\Lambda = I$$

$$\Lambda \in \mathbb{R}^{MN \times N}$$

$\Omega|i\rangle = |i\rangle \otimes |A_i\rangle$

e.g. $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\Lambda = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Omega = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$

$W = (2\Omega\Omega^* - I)(2\Lambda\Lambda^* - I)$.

$W^*W = I$.

$\Omega^*\Lambda = \frac{A}{\|A\|_F}$

$W(\Lambda v_j) = \frac{2\sigma_j}{\|A\|_F}\Omega u_j - \Lambda v_j$

$W(\Omega u_j) \in \mathrm{span}\{\Omega u_j, \Lambda v_j\}$.

θ_j s.t. $\cos\frac{1}{2}\theta_j = \frac{\sigma_j}{\|A\|_F}$

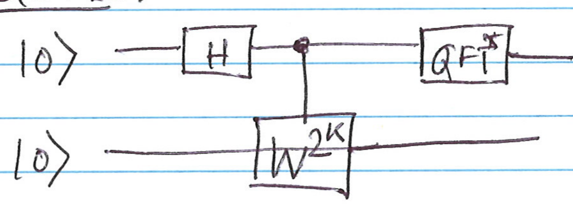
$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longrightarrow$ evals are $e^{\pm i\theta}$

$W|w^+\rangle = e^{i\theta}|w^+\rangle$

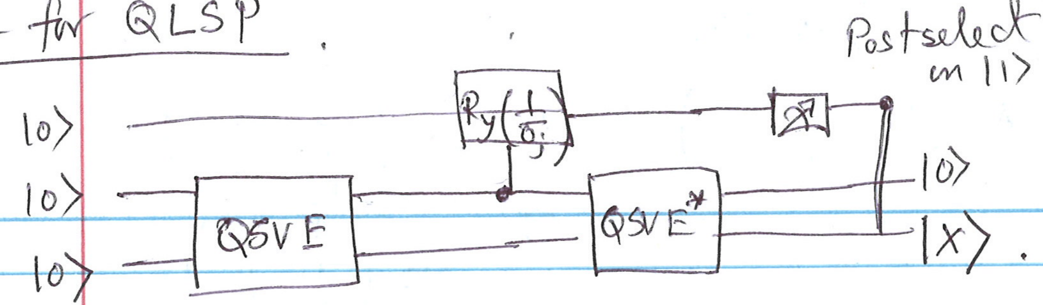
$W|w^-\rangle = e^{-i\theta}|w^-\rangle$ $\blacksquare$.

Do phase estimation with unitary W.

<u>QSVE</u>.



Use for QLSP



$O\left(\text{poly}\log(N)\,\kappa^2 \|A\|_F / \varepsilon\right)$ complexity for QLSP.