>> Postulates of QM

P1: Associated to any isolated (closed) physical system there is a Hilbert space known as state space of the system. The system is completely determined by its state vector which is a unit vector in $\mathcal{H}$.

[normalization condition].

P2: Evolution of $\boxed{\text{closed}}$ quantum system is described by a unitarry transformation.

$$|\psi(t_2)\rangle = U(t_2, t_1)\,|\psi(t_1)\rangle .$$

[Discuss with class: Measurement of qubits is non-unitary].

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle \qquad \text{[Schrödinger equation]}.$$

Hermitian op.

$$H = \sum_E E\, |E\rangle\langle E| \qquad \text{eigen decomposition}$$

$\{|E\rangle\}$ are called stationary states, lowest E is called ground state energy, & corresponding $|E\rangle$ is ground state.

if $|\psi(t=0)\rangle = |E\rangle$, then $|\psi(t)\rangle = e^{-iEt/\hbar}|E\rangle$.

[from 1st order ODE theory].

— as E is ~~finite matrix~~ scaler, $e^{-iEt/\hbar}$ is well defined. series converges.

$$|\psi(t)\rangle = e^{-\frac{it}{\hbar}H}\,|\psi(t=0)\rangle .$$

← also well defined as H is a finite dim matrix.

[Discuss this notation of matrix exponential].

$$|\psi(t_2)\rangle = \exp\left[\frac{-iH_1(t_2-t_1)}{\hbar}\right]|\psi(t_1)\rangle$$

so $$U(t_2,t_1) = \underbrace{\exp\left[\frac{-iH(t_2-t_1)}{\hbar}\right]}_{\text{unitary op.}}.$$

$$U^\dagger U = UU^\dagger = I.$$

- Need to use fact
  $$e^A e^B = e^{A+B} \quad \text{iff } A, B \text{ commute}$$
  - definitely true for Hermitian matrices
  - how to show in general? [Density arguments].

Because of unitary evolution, final state also is a normalized state of Hilbert space.

<u>P3</u>: (Quantum measurement). $\{M_m\}$ are a collection of measurement operators, $M_m : \mathcal{H} \to \mathcal{H}$, m refers to measurement outcomes.

- M_m satisfy completeness relation
  $$\sum_m M_m^\dagger M_m = I$$
- if $|\psi\rangle$ is state before measurement, then after measurement, prob. of getting m is
  $$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle,$$ and state after measurement is $$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$
- completeness relation expresses $\sum_m P_m = 1$.

[Ex: $\sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = 1 \;\forall\; |\psi\rangle$ iff $\sum_m M_m^\dagger M_m = I$]
(Hint: $M_m^\dagger M_m$ are Hermitian).

Ex: measurement in computational basis.

$M_0 = |0\rangle\langle 0|, \quad M_1 = |1\rangle\langle 1|$

$I = M_0 + M_1$
$= M_0^\dagger M_0 + M_1^\dagger M_1.$

$|\psi\rangle = a|0\rangle + b|1\rangle.$

$p(0) = |a|^2; \quad p(1) = |b|^2$

if 0 : final state $\dfrac{M_0|\psi\rangle}{|a|} = \dfrac{a}{|a|}|0\rangle \equiv |0\rangle$

if 1 : final state $\dfrac{M_1|\psi\rangle}{|b|} = \dfrac{b}{|b|}|1\rangle \equiv |1\rangle$

Orthogonal states can be distinguished with appropriate measurement.

[Do Bell states in class].

Thm: Non-orthogonal states cannot be reliably distinguished.

Pf $|\psi_1\rangle, |\psi_2\rangle$ non orthogonal.

$\{M_i\}$ are measurement ops.

$$E_1 \equiv \sum_{j : f(j)=1} M_j^\dagger M_j, \qquad E_2 = \sum_{j : f(j)=2} M_j^\dagger M_j.$$

$$\langle\psi_1|E_1|\psi_1\rangle = 1, \qquad \langle\psi_2|E_2|\psi_2\rangle = 1.$$

$$E_1 + E_2 = I \Rightarrow \langle\psi_1|E_2|\psi_1\rangle = 0 \Rightarrow \sqrt{E_2}|\psi_1\rangle = 0.$$

$$|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\phi\rangle, \qquad |\beta| < 1, \text{ so } \alpha \neq 0.$$

$$\sqrt{E_2}|\psi_2\rangle = \beta\sqrt{E_2}|\phi\rangle \Rightarrow \langle\psi_2|E_2|\psi_2\rangle = |\beta|^2\langle\phi|E_2|\phi\rangle$$

$\langle\psi_1|\phi\rangle = 0$

$$\leq |\beta|^2\langle\phi|\phi\rangle^2\|E_2\| < 1.$$

■

Projective Measurements

- Described by an observable M (Hermitian op on state space of system being observed).
- Observations are eigenvals of M, post measurement state is the eigenstate.

$M = \sum_m m P_m$, P_m are projectors on an eigenspace with eigenval m.
[m distinct]

$P_m^2 = P_m$.

$M^r = \sum_m m^r P_m$.

$\mathbb{P}[\text{result } m] = p(m) = \langle\psi|P_m|\psi\rangle$.

post measurement state: $\dfrac{P_m|\psi\rangle}{\langle\psi|P_m|\psi\rangle}$.

[Postulate 3 if additionally required to satisfy the condition that M_m are orthogonal projectors, then we get projective measurements]

- M_m are Hermitian
- $M_m M_{m'} = \delta_{m,m'} M_m$.

<u>Properties of proj. measurements</u>

$\langle M\rangle = \mathbb{E}(M) = \sum_m m\,p(m) = \langle\psi|M|\psi\rangle$.

$\langle M^r\rangle = \mathbb{E}(M^r) = \sum_m m^r p(m) = \langle\psi|M^r|\psi\rangle$.

$[\Delta(M)]^2 = \mathbb{E}\left((M-\langle M\rangle)^2\right) = \langle (M-\langle M\rangle)^2\rangle = \langle M^2\rangle - \langle M\rangle^2$.

$\parallel$

$\sum_m (m - \mathbb{E}(M))^2 p(m)$

[Ex: Suppose $|\psi\rangle$ is an eigenstate of an observable M with eval m. What is $\langle M\rangle$, $\Delta(M)$?] Ans: m, 0.

<u>Heisenberg Uncertainty Principle</u>

C, D observables, $|\psi\rangle$ quantum state.

<u>Thm</u>: $\Delta(C)\,\Delta(D) \geq \dfrac{|\langle\psi|[C,D]|\psi\rangle|}{2}$.

Interpretation: If we prepare a large number of identical copies of $|\psi\rangle$, & do measurements of observables C, D on disjoint subsets of these copies, then the std dev of these measurements satisfy the Thm..

Pf. Let A, B Hermitian op., $|\psi\rangle$ quantum state.

$\langle\psi|AB|\psi\rangle = x+iy$, $\langle\psi|AB|\psi\rangle^* = x-iy$

$x, y \in \mathbb{R}$.

$$\langle\psi|(AB)\psi\rangle^* = \langle(AB)\psi|\psi\rangle = \langle\psi|BA|\psi\rangle.$$

$\langle\psi|[A,B]|\psi\rangle = 2iy$, $\langle\psi|\{A,B\}|\psi\rangle = 2x$.

By C-S. $|\langle\psi|AB|\psi\rangle|^2 = |\langle A\psi|B\psi\rangle|^2 \leq |\langle\psi|A^2|\psi\rangle||\langle\psi|B^2|\psi\rangle|$

$= \langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle$.

Also.

$$|\langle\psi|[A,B]|\psi\rangle|^2 + |\langle\psi|\{A,B\}|\psi\rangle|^2 = 4|\langle\psi|AB|\psi\rangle|^2$$

$$\Rightarrow |\langle\psi|[A,B]|\psi\rangle| \leq 2|\langle\psi|AB|\psi\rangle| \leq 2\sqrt{\langle\psi|A^2|\psi\rangle}\sqrt{\langle\psi|B^2|\psi\rangle}$$

Let $A = C - \langle C\rangle$, $B = D - \langle D\rangle$

then $[A,B] = [C,D]$

$$|\langle\psi|[C,D]|\psi\rangle| \leq 2\,\Delta(C)\,\Delta(D). \quad \blacksquare$$

e.g. $[X,Y] = 2iZ$, $|\psi\rangle = |0\rangle$ 1-qubit system.

$$\Delta(X)\Delta(Y) \geq \frac{1}{2}|\langle 0|2iZ|0\rangle| = |\langle 0|Z|0\rangle| = 1.$$

$\Rightarrow$ both $\Delta(X)$ must be strictly +ve.

e.g. projective measurement of observable Z

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, Let $|\psi\rangle = a|0\rangle + b|1\rangle$.

$p(+1) = \langle\psi|\,|0\rangle\langle 0|\,|\psi\rangle = |a|^2$

$p(-1) = \langle\psi|\,|1\rangle\langle 1|\,|\psi\rangle = |b|^2$

e.g. More generally

$\vec{v}\cdot\vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$ | $\vec{v}$: real unit vector in 3-d.

↑ Hermitian op. $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices. X, Y, Z

[Ex: $\vec{v}\cdot\vec{\sigma}$ has eval ± 1, $P_{\pm} = (I \pm \vec{v}\cdot\vec{\sigma})/2$].

↑ "measurement of spin along $\vec{v}$-axis".

POVM measurements

- For many applications, post-measurement state is of little significance, but mainly interested in measurement outcome statistics.

Suppose $\{M_m\}$ are measurements.

Define: $E_m = M_m^\dagger M_m \Rightarrow \sum_m E_m = I$.

$p(m) = \langle\psi|E_m|\psi\rangle$.

$\{E_m\}$ are called "POVM elements" associated with measurement.

Thm: Any measurement where measurement ops & POVM elements coincide is a projective measurement.

Pf $M_m^\dagger M_m = M_m \Rightarrow M_m$ is Hermitian, positive.
(so $M_m^\dagger = M_m \Rightarrow M_m^2 = M_m$).

$\therefore$ eval of $M_m \in \{0,1\}$. $\Rightarrow M_m$ are projectors.

$\sum_m M_m = I \Rightarrow M_m$ are orthogonal projectors b/c $\mathrm{Ran}(M_m) \in \mathrm{Ker}(M_{m'})$ $\forall\, m' \neq m$. ∎

Convenient to define POVM as $\{E_m\}$

(i) each E_m positive
(ii) $\sum_m E_m = I$

} can define corresponding measurement ops $M_m = \sqrt{E_m}$.

An important application.

[Ex 2.64] Linearly indep states $|\psi_1\rangle, \ldots, |\psi_m\rangle$.
Construct POVM $\{E_1, \ldots, E_{m+1}\}$ s.t if outcome E_i occurs $1 \le i \le m$, then Bob knows with certainty that input state was $|\psi_i\rangle$.

[Prove in class].

Define $V_i = \text{span}\{|\psi_1\rangle, \ldots, \widehat{|\psi_i\rangle}, \ldots, |\psi_m\rangle\}$
excluded

$V_i^{\perp}$: orthogonal complement of V_i

$|\tilde{\psi}_i\rangle = \text{proj}_{V_i^{\perp}}(|\psi_i\rangle) \neq 0$ (by linear indep).
normalized.

Take POVM to be

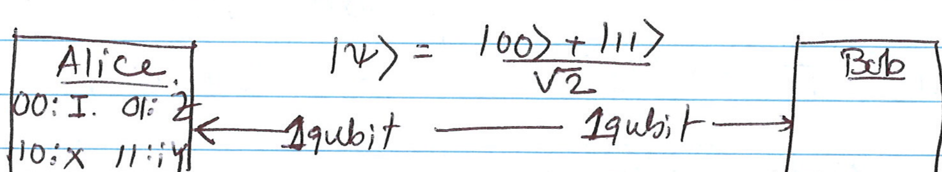
$\{\frac{1}{m}|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1|, \ldots, \frac{1}{m}|\tilde{\psi}_m\rangle\langle\tilde{\psi}_m|, I - \sum_{i=1}^{m} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|\}$.

Postulate 4 (P4): State space of a composite physical system is the tensor prod of state spaces of the component physical systems. Moreover if we have systems numbered 1 to n, & system i is prepared in state $|\psi_i\rangle$, then joint state of total system is $|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$.

[Ex. 2-qubit system, $|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. Show that $\langle X_1 Z_2 \rangle = 0$.] [Do in class].

$X_1 Z_2 |\psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \Rightarrow \langle\psi| X_1 Z_2 |\psi\rangle = 0$.

>>> Superdense Coding



- Alice needs to send 2 classical bits of info to Bob.
- Can be achieved by sending a single qubit if you have entanglement.

Classical bits	Op.	result state
0 0	I	$(\|00\rangle + \|11\rangle)/\sqrt{2}$
0 1	Z	$(\|00\rangle - \|11\rangle)/\sqrt{2}$
1 0	X	$(\|10\rangle + \|01\rangle)/\sqrt{2}$
1 1	iY	$(\|01\rangle - \|10\rangle)/\sqrt{2}$

} Bell states

Can be distinguished by Bob!

[Ex: if some malevolent party intercepts Alice's qubit before reaching Bob. Can the person get any info about what Alice is trying to send?]. [Ans. No]

get $|0\rangle, |1\rangle$ with prob. $1/2$.

>>> Density Operator

— powerful way to capture situations when quantum state is not completely known

— $|\psi_i\rangle$, i is an index, p_i (prob. that state is in $|\psi_i\rangle$).

($|\psi_i\rangle$ are distinct).

"ensemble of pure states".

Def: density op. for system is defined as

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|. \qquad \text{[Density matrix]}.$$

[All 4 postulates can be re-expressed in density op. language].

- Evolution: $|\psi_i\rangle \xrightarrow{U} U|\psi_i\rangle$ (U unitary)

  prob. p_i

$$\rho_f = \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho_i U^\dagger.$$

- Measurement: $\{M_m\}$.

$$p(m|i) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = tr\,(M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|)$$

$$\boxed{tr\,(xy^*) = y^*x}$$

$$p(m) = \sum_i p(m|i)\, p(i) = \sum_i p(m|i) p_i$$

$$= \sum_i p_i \, tr(M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|) = tr(M_m^\dagger M_m \rho).$$

- Post-measurement state.

$$|\psi_i\rangle \xrightarrow{\text{if result } m} |\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}}$$

with prob $p(i|m)$.

$$\rho_m = \sum_i p(i|m) |\psi_i^m\rangle\langle\psi_i^m| = \sum_i p(i|m) \frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}.$$

$$p(i|m) = p(m,i)/p(m) = p(m|i)p_i/p_m$$

$$\rho_m = \sum_i \frac{tr(\cancel{M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|})\, p_i\, M_m|\psi_i\rangle\langle\psi_i| M_m^\dagger}{tr(M_m^\dagger M_m \rho)\; tr(\cancel{M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|})}$$

$$= \frac{M_m \rho M_m^\dagger}{tr(M_m^\dagger M_m \rho)}$$

[Note: if result m obtained, then $tr(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) \neq 0$.]

Pure state / Mixed state

- if quantum state known exactly, then it is pure stat $|\psi\rangle$, density op is $|\psi\rangle\langle\psi|$.

- otherwise mixed state, which is an ensemble of pure states with some probabilities.

Q. If a system is prepared in state ρ_i with prob p_i ($\sum_i p_i = 1$). What is the resulting density matrix?

A. $\sum_i p_i \rho_i$ $\qquad \rho_i = \sum_j p_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|$

$$\rho = \sum_i p_i \left(\sum_j p_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|\right) = \sum_i p_i \rho_i.$$

Q. Start with ρ. $\{M_m\}$ measurement op. Don't observe result. What is final density op after measurement?

A. $P(m) = tr(M_m^\dagger M_m \rho)$, $\rho_m = \dfrac{M_m \rho M_m^\dagger}{tr(M_m^\dagger M_m \rho)}$

$$\rho_f = \sum_m P(m)\, \rho_m = \sum_m M_m \rho M_m^\dagger .$$

Note. $tr(\rho_f) = tr\left(\sum_m M_m^\dagger M_m \rho\right) = tr(\rho)$.

General prop. of density operator

Thm: An op ρ is density op associated with some ensemble $\{p_i, |\psi_i\rangle\}$ iff it satisfies

(i) $tr(\rho) = 1$, (ii) ρ is positive.

Pf. ($\Rightarrow$) $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \Rightarrow tr(\rho) = \sum_i p_i = 1$.

$\rho \geq 0$.

($\Leftarrow$) $\rho = \sum_i \lambda_i |i\rangle\langle i|$, as ρ positive $\Rightarrow \rho$ Hermitian.

& $tr(\rho) = \sum_i \lambda_i = 1$, $\lambda_i \geq 0$. $\forall i$.

$\{\sqrt{\lambda_i}, |i\rangle\}$ is an ensemble.

[Density op is a positive op with trace 1].

QM postulates rewritten in terms of density op

P1: Associated to physical system (isolated) is a Hilbert space (over $\mathbb{C}$), called state space. System completely described by density op acting on state space. If system in state ρ_i with prob p_i then state is $\sum p_i \rho_i$.

P2 Evolution of closed quantum system described by unitary transformation $\rho' = U\rho U^\dagger$.

P3 Measurement $\{M_m\}$. $p(m) = tr(M_m^\dagger M_m \rho)$ is prob of outcome m, with post-measurement state $\dfrac{M_m \rho M_m^\dagger}{tr(M_m^\dagger M_m \rho)}$.

Ops satisfy $\sum_m M_m^\dagger M_m = I$.

P4 Composite systems: State space is tensor prod of state spaces of component physical systems. If systems 1 to n, and i^{th} system is prepared in state ρ_i, then state of composite system is $\rho_1 \otimes \cdots \otimes \rho_n$.

Thm: ρ is density op. Then $tr(\rho^2) \leq 1$ with equality iff ρ is pure state.

Pf. $\rho = \sum_i \lambda_i |i\rangle\langle i|$, $\Rightarrow \sum_i \lambda_i = 1$, $\lambda_i \in [0,1]\ \forall i$.

$\therefore tr(\rho^2) = \sum_i \lambda_i^2 \leq \sum_i \lambda_i = 1$.

if ρ is pure state, $\rho = |\psi\rangle\langle\psi| \Rightarrow tr(\rho) = 1$.

if $tr(\rho^2) = 1$, $\sum_i \lambda_i^2 = 1 \Rightarrow \lambda_i = 1$ for exactly one index i, rest 0. ∎.

[Note: different ensembles can result in same density matrix.

e.g. $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$

Let $|a\rangle \equiv \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$,

$|b\rangle = \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle$.

Then $\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|$.].