ICMT Division A Sample Problems

1. Evaluate
$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+4)!}.$$

2. A monoid M is a set S equipped with a binary operation $\cdot : S \times S \to S$ having the properties of

   - Associativity: For all $a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
   - Identity: There exists $1_M \in S$ such that $a \cdot 1_M = 1_M \cdot a = a$ for all $a \in S$.

   Let M be a monoid with the 4 (initially free) elements $a, b, c, d \in M$ and the 12 relations $xyx = 1_M$ for all $x, y \in \{a, b, c, d\}$ where $x \neq y$. What is the minimum number of relations that need to be added to M to reduce the number of distinct elements of M to 1?

3. Let $\mathcal{M} = \mathbb{F}_4^{4\times 4}$ be the set of 4×4 matrices with elements in $\mathbb{F}_4$. Let $\mathcal{N}$ be the nilpotent subgroup of $\mathcal{M}$; i.e.
$$\mathcal{N} = \{M \in \mathcal{M} : \exists n \in \mathbb{N},\, M^n = 0\}.$$
   Define the equivalence relation $A \sim B$ for matrices in $\mathcal{M}$ if there exists invertible $V \in \mathcal{M}$ such that $AV = VB$. Compute $|\mathcal{N}/\sim|$.

4. Define S_N to be the number of integral solutions (x, y, z) to the equation $x + y + z = 0$ such that $-N \leq x, y, z \leq N$ and $\gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1$. Compute
$$\lim_{N\to\infty} \frac{S_N}{N^2}.$$

5. Let S be the set of all differentiable functions with continuous derivatives $f : [0, 1] \to \mathbb{R}$ such that
$$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx = 1.$$
   Compute
$$\inf_{f\in S} \left(\int_0^1 (f'(x))^2\,dx \right).$$

6. Compute the number of ordered pairs (g, h) of elements of S_5, the group of permutations of 5 elements, such that $gh = hg$.