Fredholm 二择一定理 (矩阵语言)

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Theorem. 设 $A \in \mathbb{F}^{m\times n}$，$l : \mathbb{R}^n \to \mathbb{R}^m$。对于给定的 $b \in \mathbb{R}^m$，方程 $Ax = b$ 是否有解？

(1)$b \in \operatorname{Im}(f) \Leftrightarrow b \in (\,Ker(A^T))^{\perp}$;

(2)$x \in \operatorname{Ker}(A) \Leftrightarrow x \in (\,Im(A^T))^{\perp}$.

Proof. 只证 (1).

$b \in \operatorname{Im}(A) \iff \exists x \in \mathbb{R}^n, Ax = b$.

$\Rightarrow \forall \nu \in \operatorname{Ker}(A^T), b^T\nu = (Ax)^T\nu = x^TA^T\nu = 0(A^T\nu = 0)$.

另一方面, 要证 $(\operatorname{Ker}(A^T))^{\perp} \subset \operatorname{Im}(A)$,

只需证 $\operatorname{Ker}(A^T) \supseteq (\operatorname{Im}(A))^{\perp}$.

$\nu \in (\,Im(A))^{\perp} \Rightarrow \forall \omega \in \mathbb{R}^n, \nu^T(A\omega) = \omega^TA^T\nu = 0$.

不妨设 $\omega = A^T\nu$, 则 $(A^T\nu)^TA^T\nu = 0 \Rightarrow A^T\nu = 0 \Rightarrow \nu \in \operatorname{Ker}(A^T)$. $\square$

例. $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$,$Ax = b$ 在 b 为何值时有解？

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 3b_1 \end{pmatrix}$$

$$\Leftrightarrow b \in \operatorname{Im}(A)$$

$$\Leftrightarrow b \in (\operatorname{Ker}(A^T))^{\perp}.$$

$A^T = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$，令

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$，得 $A^T x = b$ 的通解 $\omega = t(-3, 1)^T$，即

$$\mathrm{Ker}(A^T) = t \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

$$\Leftrightarrow (\mathrm{Ker}(A^T))^{\perp} = t(1.3).$$

$$\Rightarrow (b_1, b_2) = t(1.3).$$