一道高等代数习题的解答

icnow

April 2024

$Problem$: 设 $W_1, W_2, \ldots, W_s$ 是 n 维向量空间 V 的真子空间, 则存在 V 的一个基, 使得基中的每一个向量均不在 $W_1, W_2, \ldots, W_s$ 中.

Proof. 不妨设 $dim(W_i) = n-1, i = 1, 2, \ldots, s$. (此时若 $\exists \alpha \in V \setminus W_i$, 则 $\alpha \notin W, \forall W$满足$dim(W) \leqslant dim(W_i)$, 符合题意.)

构造线性同构 $\phi : V \to \mathbb{R}$, 则 $\phi(W_i) \subset \mathbb{R}$.(设 $W_i = Span\{e_1, e_2, \ldots, e_{n-1}\}$, 则 $\phi(W_i) = Span\{\phi(e_1), \phi(e_2), \ldots, \phi(e_{n-1})\}$. 若 $\mathbb{R}^n = Span\{V_1, V_2, \ldots, V_n\}, V_i \notin \phi(W_j)$, 则 $\phi^{-1}(V_i) \notin W_j, Span\{\phi^{-1}(V_1), \phi^{-1}(V_2), \ldots, \phi^{-1}(V_n)\} \neq V$.)

$Step1$: 由于 $W_1 \subset \mathbb{R}^n, dim(W_1) = n-1$, 则 $\exists \xi_1 \in \mathbb{R}$,使得$W_1 = \{x \in \mathbb{R}^n | < x, \xi >= 0\}$. 上述引理的证明:$dim(W_1) = n-1, W_1 \subset \mathbb{R}^n$, 则 $\exists y \in \mathbb{R}^n \setminus W_1$.

设 $W_1 = Span\{e_1, e_2, \ldots, e_{n-1}\}, \{e_1, e_2, \ldots, e_{n-1}\}$ 是单位正交基. 构造

$$\xi = y - \langle y, e_1 \rangle e_1 - \langle y, e_2 \rangle e_2 - \cdots - \langle y, e_{n-1} \rangle e_{n-1},$$

则 $\langle \xi, e \rangle = 0, \forall 1 \leqslant i \leqslant n-1 \Rightarrow \xi_1 \perp W$, 即 $\forall \nu \in W \Rightarrow \xi \perp \nu$.

记

$W_1 = \{x \in \mathbb{R}^n | a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0\}$(令$\xi = (a_{11}, a_{12}, \cdots, a_{1n})$),
$W_2 = \{x \in \mathbb{R}^n | a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0\}$,
$\cdots$
$W_n = \{x \in \mathbb{R}^n | a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0\}$.

构造 $f_i(\lambda) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n, 1 \leqslant i \leqslant s$, 该多项式共有 $s * n = ns$ 个根. 在 $\mathbb{R}$ 内把这个 ns 个根去掉, 再任取 $\mathbb{R}$ 上的数 $y_1, y_2, \cdots, y_n$, 构造 $\nu_1 = (1, y_1, y_1^2, \cdots, y_1^{n-1} \in \mathbb{R}$, 则 $\nu_1 \notin W_1 (a_{11} + a_{12}y_1 + \cdots + a_{1n}y_1^{n-1} \neq 0)$. 同理构造 $\nu_2, \cdots, \nu_n$, 则

$$\nu_i \notin W_j, \forall i, j,$$

$$\Rightarrow \nu_1, \nu_2, \cdots, \nu_n \text{是} \mathbb{R}^n \text{的一组基}.$$

验证:$det(\nu_1, \nu_2, \cdots, \nu_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_n \\ y_1^2 & y_2^2 & \cdots & y_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{vmatrix}$

$$= \prod_{1 \leqslant i < j \leqslant n} (y_j - y_i) \neq 0,$$

即该向量组线性无关. □