Backstepping as a pole-shifting method
Dual optimization as a certification method

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Séminaire du L2S

Summary

Framework

Framework: **linear time-invariant control systems** with a **scalar control** (in state space representation)

$$\dot{x} = Ax + Bu(t).$$

A is a linear operator, B the *control operator* is linear, $u \in L^2(0,T)$.

Goal: ***exponential*** **stabilization** around 0 with a **linear feedback** (*i.e.*,stationary closed-loop control).

$$u(t) = Kx(t), \quad \dot{x} = Ax + BKx = (A + BK)x,$$

$$\|x(t)\| \leq Ce^{-\lambda t}\|x_0\|, \quad \lambda > 0,\ t \geq 0.$$

- In finite dimension, make $A + BK$ Hurwitz.
- In infinite dimension, make the system dissipative (Lyapunov function...).

Linearizations of important nonlinear models.

Summary

Examples in infinite dimension

- **Boundary control of the temperature in a rod**

$$\begin{cases} y_t - \Delta y = 0 \text{ on } [0, L], \\ y(0) = u(t), \quad y(L) = 0. \end{cases}$$

- **Control of a watergate**

$$\begin{cases} H_t + (HV)_x = 0, \\ V_t + \left(gH + \dfrac{V^2}{2}\right)_x = 0, \\ V(0,t) = u_0(t)\gamma\sqrt{H^0_{\mathrm{up}}(t) - H^0_{\mathrm{down}}(t)}, \\ V(L,t) = u_L(t)\gamma\sqrt{H^L_{\mathrm{up}}(t) - H^L_{\mathrm{down}}(t)} \end{cases}$$

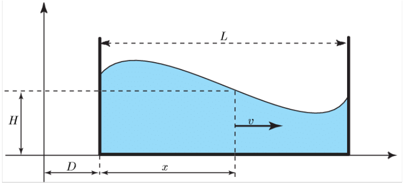
Distributed control in infinite dimension

- **Controlling a particle in an EM field**

$$\begin{cases} \mathrm{i}\, y_t = \Delta y + u(t)\mu(t,x)y \text{ on } \Omega, \\ y = 0 \text{ on } \partial\Omega. \end{cases}$$

- **Moving a water tank**

$$\begin{cases} H_t + (HV)_x = 0, \\ V_t + \left(gH + \dfrac{V^2}{2}\right)_x = \underbrace{-u(t)}_{\text{acceleration}}, \\ V(t,0) = V(t,L) = 0, \quad \forall t \geq 0. \end{cases}$$



Summary

Integrator backstepping

$$\dot{x_1} = ax_1 + x_2$$
$$\dot{x_2} = u(t)$$

Feedback for the first equation: $x_2 = -(\lambda + a)x_1$.
How do you "backstep" that through the integrator?

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$$V = x_1^2 + \underbrace{(x_2 + (\lambda + a)x_1)^2}_{\text{Penalization term}}$$

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Backstepping change of variable:

$$z_1 = x_1 \qquad \dot{z_1} = -\lambda z_1 + z_2$$
$$z_2 = x_2 + (\lambda + a)x_1 \qquad \dot{z_2} = u - \lambda(\lambda + a)z_1 + (\lambda + a)z_2$$

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Backstepping change of variable:

$$\begin{aligned} z_1 &= x_1 \quad \text{lower triangular} \\ z_2 &= x_2 + (\lambda + a)x_1 \end{aligned} \qquad \begin{aligned} \dot{z_1} &= -\lambda z_1 + z_2 \\ \dot{z_2} &= u - \lambda(\lambda + a)z_1 + (\lambda + a)z_2 \end{aligned}$$

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$$V = x_1^2 + (x_2 + (\lambda + a)x_1)^2$$

Feedback: $u(t) = \lambda(\lambda + a)z_1 - (2\lambda + a)z_2$

$$z_1 = x_1 \quad \text{lower triangular} \qquad \dot{z_1} = -\lambda z_1 + z_2$$
$$z_2 = x_2 + (\lambda + a)x_1 \qquad \dot{z_2} = -\lambda z_2$$

Difference between ODE and PDE backstepping

Boskovic, Balogh, Krstic, *Backstepping in infinite dimension for a class of parabolic distributed parameter systems*, MCSS 2003.

Krstic's backstepping comes from:

- *Modified* backstepping on *space discretization* of the heat equation
- *Continuum limit* of the resulting change of variables and feedback.

Modification? Compare the free systems.

ODE:

Free system:

$$\dot{x_1} = ax_1 + x_2$$
$$\dot{x_2} = 0$$

Stabilized system (with new variables)

$$\dot{z_1} = -\lambda z_1 + z_2$$
$$\dot{z_2} = -\lambda z_2.$$

The structure of the spectrum is changed.

In PDE backstepping, we want to *translate* the spectrum to the left.

PDE backstepping: a historical example

Unstable heat equation (Boskovic, Balogh, Krstic, MCSS 2003) :

$$\begin{cases} y_t - y_{xx} = \lambda y, \\ y(0) = 0, \quad y(1) = U(t). \end{cases}$$

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Transformation (Volterra, **always invertible**):

$$w(t,x) = y(t,x) - \int_0^x k(x,s)y(t,s)ds.$$

Target system:

$$\begin{cases} w_t - w_{xx} = 0, \\ w(0) = 0, \\ w(1) = U(t) - \int_0^1 k(1,s)y(t,s)ds. \end{cases}$$

The target system has to be *reasonably related* to the original system.
Its stability is *easier to read.*

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Feedback control design: $U(t) = \int_0^1 k(1,y) y(t,s) ds.$

Kernel equations

$w(t,x) = y(t,x) - \int_0^x k(x,s)y(t,s)ds$ has to solve the stable heat equation.

After formal computations (IBP...): PDE for $k(x,y)$.

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Kernel equations on $\mathcal{T} := \{0 \leq y \leq x \leq 1\}$:

$$\begin{cases} k_{xx} - k_{yy} = \lambda k, \\ k(x,0) = 0, \\ k(x,x) = -\lambda \dfrac{x}{2}. \end{cases}$$

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Explicit solution

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}.$$

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$$U(t) = \int_0^1 k(1,s)y(t,s)ds.$$

Summary

The development of backstepping

- Developed for **boundary stabilization**, with **Volterra transformations** (Krstic, Liu, Balogh, Smyshlaev, Bastin, Coron, Di Meglio, Auriol...)
- **More general transformations?** (Coron & Lü) Volterra structure $\leftrightarrow$ position of the control
- **Distributed control?** (Coron, Nguyen, Gagnon, Morancey, CZ, Hayat, Marx, Xiang...)

Results

Theorem (Gagnon, Hayat, Xiang, CZ, 2022)

*Let A be an anti-adjoint operator with domain $D(A) \subset L^2(0,L)$, with simple eigenvalues behaving in n^α, $\alpha > 1$. Let $B : \mathbb{C} \to D(A)'$ be such that (A,B) is **controllable**.*
*Then, for $\lambda > 0$ there exists a feedback K_λ such that the control $u(t) := K_\lambda x(t)$ **stabilizes the system exponentially** with decay rate λ:*

$$\|x(t)\|_{H^s} \leq C e^{-\lambda t} \|x(0)\|_{H^s}$$

- Covers many physical systems.
- K_λ is relatively explicit
- New spectrum: $\{\lambda_n - \lambda\}$.
- $D(A + BK_\lambda) \neq D(A)$! But...

Results

Some specific results for $\alpha = 1$ (**critical case...**).

Theorem (CZ, 2019)

Let φ be a piecewise H^1 function such that $|n\hat{\varphi}(n)|$ is bounded away from 0 for $n \in \mathbb{Z}$. Then, the transport equation

$$\begin{cases} y_t + y_x = u(t)\varphi, \\ y(t,0) = y(t,L), \end{cases}$$

*is controllable, and can be **stabilized exponentially**, as well as in **finite time**, with **explicit** feedbacks.*

Explicit: feedback given by its Fourier coefficients.
Spectrum: $\{\lambda_n - \lambda\}$.

Results

Theorem (Coron, Hayat, Xiang, CZ, 2020)

The linearized water tank system

$$\begin{cases} h_t + h^\gamma (v)_x = 0, \\ v_t + g\,(h)_x = -u(t), \\ v(t,0) = v(t,L) = 0, \quad \forall t \geq 0. \end{cases}$$

around the stationary state $(h_\gamma, 0, \gamma)$ *with constant acceleration* $\gamma > 0$ *can be stabilized exponentially with decay rate* $\lambda < -C \ln(\gamma)$.

- Bound on λ due to controllability criterion (technical or profound?)
- Feedback is actually Proportional Integral (to account for conservation of mass inside the tank).
- Spectrum: $\left\{ \lambda_n - \lambda + o\left(\frac{1}{n}\right) \right\}$

Summary

Generalization of backstepping

We keep the spirit of a *transformation*: map

$$\dot{x} = Ax + B(Kx + v(t))$$

into the stable system

$$\dot{(Tx)} = \tilde{A}(Tx) + Bv(t).$$

The mapping T should be **invertible** and satisfy

$$\begin{gathered} T(A + BK) = \tilde{A}T, \\ TB = B. \end{gathered}$$

$$\begin{gathered} Ty = y - \int_0^x k(x,s)y(t,s)ds; \\ k_{xx} - k_{yy} = \lambda k, \\ k(x,0) = 0, \\ k(x,x) = -\lambda \frac{x}{2}. \end{gathered}$$

“Backstepping equations”: linear/affine operator equations of two unknowns T and K.

Generalization of backstepping

We keep the spirit of a *transformation*: map

$$\dot{x} = Ax + B(Kx + v(t))$$

into the stable system

$$\dot{(Tx)} = (A - \lambda I)(Tx) + Bv(t).$$

The mapping T should be **invertible** and satisfy

$$\begin{gathered} TA + BK = (A - \lambda I)T, \\ TB = B. \end{gathered}$$

$$\begin{gathered} Ty = y - \int_0^x k(x,s)y(t,s)ds; \\ k_{xx} - k_{yy} = \lambda k, \\ k(x,0) = 0, \\ k(x,x) = -\lambda \frac{x}{2}. \end{gathered}$$

"Backstepping equations": linear/affine operator equations of two unknowns T and K.

From Volterra to Fredholm backstepping

General transformations on functions: integral operators, aka *Fredholm transforms*:

$$y(t,x) \mapsto \int_0^L K(x,s)y(t,x)ds.$$

Volterra transform	Fredholm transform
Specific form	General form
Solving the kernel equations is sufficient	Additional proof of **invertibility**
No additional assumption	

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Specific form	General form
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No additional assumption	**Controllability assumption**

Solving the backstepping equations

1. Start with the operator equation

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Solving the backstepping equations

1. Start with the operator equation

$$TAf_n + BKf_n = (A - \lambda I)Tf_n$$

   Take the linear version.
   Projection on f_n: ODE in (Tf_n).

$$Tf_n = (Kf_n)(A - (\lambda_n + \lambda)I)^{-1}B.$$

   We get T as a function of K.

2. Find K with second equation

$$TB = B.$$

   Infinite system of linear equations. Solutions requires fine estimates on some infinite sums.

Solving the backstepping equations

We have a candidate (T, K).
Controllability assumption $\rightarrow$ the Tf_n form a basis of the state space (and many more spaces) *i.e.*,T is invertible.
Main technical difficulty:

- Technical estimates $\rightarrow$ perturbative approach
- Fredholm alternative: invertible + compact = invertible?

$$Tf_n = (Kf_n)\left[-\frac{b_n}{\lambda}f_n + (A - (\lambda_n + \lambda)I)^{-1}(B - b_n f_n)\right]$$

- Specifying the spaces is important.

Perspectives

Some perspectives:

- Critical case: no perturbative approach. Perturbative approach in a much weaker space?
- Higher dimensions?
- Observer design?
- Link with Gramian stabilization and poleshifting: always the same feedback, obtained in different ways!
- Gramian, Riccati...Optimal control?

Papers:

- ✓ CZ, *Internal rapid stabilization of a 1-D linear transport equation with a scalar feedback*, MCRF
- ✓ CZ, *Finite-time internal stabilization of a 1-D linear transport equation*, Systems & Control Letters, Volume 133, 2018.
- ✓ Water tank: Jean-Michel Coron, Amaury Hayat, Shengquan Xiang & CZ. *Stabilization of the linearized water tank system.* ARMA, 2021.
- ✓ General result: Ludovick Gagnon, Amaury Hayat, Shengquan Xiang & CZ. *Fredholm backstepping for critical operators and application to rapid stabilization for the linearized water waves*, accepted in Ann. Inst. Fourier, 2023.

Summary

Framework

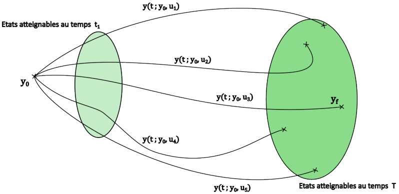
LTI systems with *convex constraints*

$$\dot{x} = Ax + Bu, \quad x(t) \in H, \quad u \in C \subset L^2(0,T;U).$$

Assumption: $x(0) = 0$. Denote the *input to state map* (Duhamel formula):

$$L_T := \int_0^T e^{(T-t)A} u(t) dt \quad \text{linear operator } U \to H.$$

Goal: what are the *reachable states* from 0, **in time** $T > 0$?



PhD work of Ivan Hasenohr (cosupervised with Sébastien Martin, Camille Pouchol and Yannick Privat)

Summary

Constraints in control problems

Constraints are natural

- Chemical reaction: concentration of a reactant

$$0 \leq u(t), \quad \int_0^T u(t)dt \leq C_{tot}.$$

- Heating a room with radiators: electrical power

$$0 \leq u(t) \leq P_{\max}$$

- Introducing infected mosquitoes to control the spread of the dengue virus

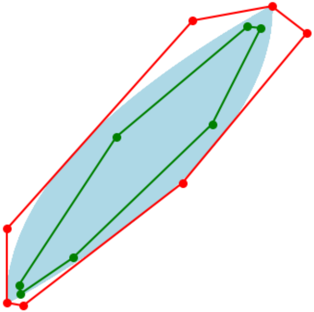
$$0 \leq u(t,x) \leq m, \quad \int_0^T \int_\Omega u(t,x)dtdx \leq M.$$

Summary

A priori certification of (non)-reachability

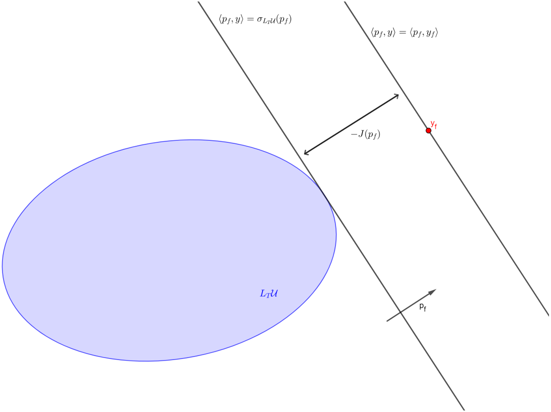
Baier, R., Büskens, C., Chahma, I. A., & Gerdts, M. (2007). *Approximation of reachable sets by direct solution methods for optimal control problems.* Optimisation Methods and Software, 22(3), 433-452.

Approximation of a convex set by a collection of *support hyperplanes*. Polyhedral approximation.



Separating hyperplanes

Focus on **nonreachability**: separating hyperplane between y_f and $L_T C$



Using optimization to be constructive

Is y_f reachable in a fixed time T?
Rewrite as an optimization problem:

$$\inf_{u \in C} \quad \underbrace{F(u)}_{\substack{\text{cost}\\ \text{constraints}}} \quad + \quad \underbrace{G(L_T u)}_{\text{reaching the target}}$$

$$\boxed{\pi := \inf_{u \in U} \delta_C(u) + \delta_{y_f}(L_T u).}$$

$$\delta_K(x) = \begin{cases} 0 \text{ if } x \in K, \\ +\infty \text{ otherwise} \end{cases}$$

Convex optimization problem (only made of penalization terms, very singular).

$$\pi \in \{0, +\infty\}, \quad y_f \text{ reachable} \iff \pi = 0.$$

Fenchel-Rockafellar duality

Idea: there is an auxiliary problem with a strong connexion to our optimal control problem.
Dual problem:

$$d := \inf_{p \in H} \sup_{v \in C} \langle L_T^* p, v \rangle - \langle p, y_f \rangle = \inf_{p \in H} J_{y_f}^T(p).$$

This problem is *nicer*.

Weak duality

$$\pi \geq -d$$

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Weak duality

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Even strong duality

$$\pi = -d$$

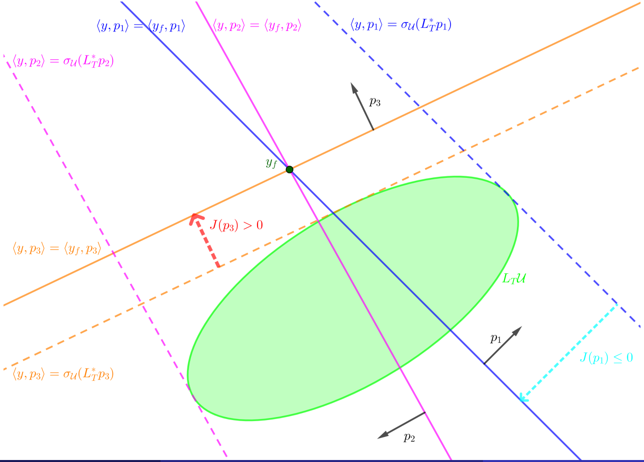
Idea! Prove that $d < 0$ so $\pi > 0$ *i.e.*, $\pi = +\infty$.

Certify **numerically** that J takes *negative values*.

Certification by the dual problem

Theorem

The state y_f is non reachable if and only if there exists $p \in H$ such that $J^T_{y_f}(p) < 0$.



Numerically

- Descent algorithm
  → **Chambolle-Pock algorithm**: primal-dual method. Search of min-max points of the Lagrangian:

$$\mathcal{L}(u,p)\delta_C(u) - \langle L_T^* p, u\rangle + \langle p, y_f\rangle$$

- Descend until you can certify that the current value is negative
  → INTLAB package (interval arithmetic to give precise error bounds).
- If the algorithm stops moving (or max iterations reached), abort.

Perspectives

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  - Streetcar
  - Rendez-vous problem
- Next benchmark: discretized heat equation
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  Upcoming preprints:
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  - ✓ Hasenohr, I., Pouchol, C., Privat, Y., & CZ, *Computer-assisted proof of non-controllability for linear finite-dimensional control systems*

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THANK YOU!