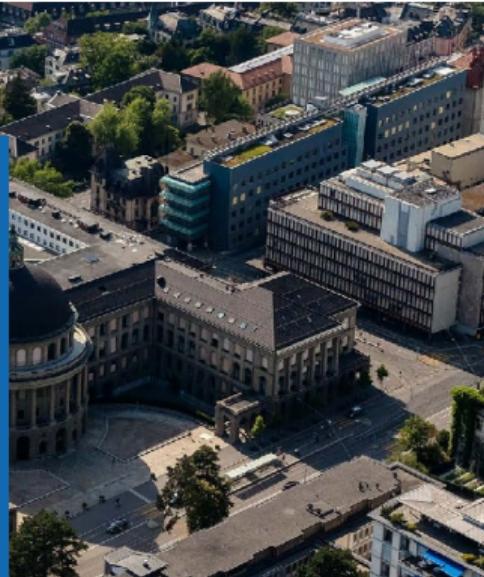




First-Order and Zeroth-Order Optimization Algorithms as Model-Free Feedback Controllers

Saverio Bolognani

Séminaire d'Automatique du plateau de Saclay



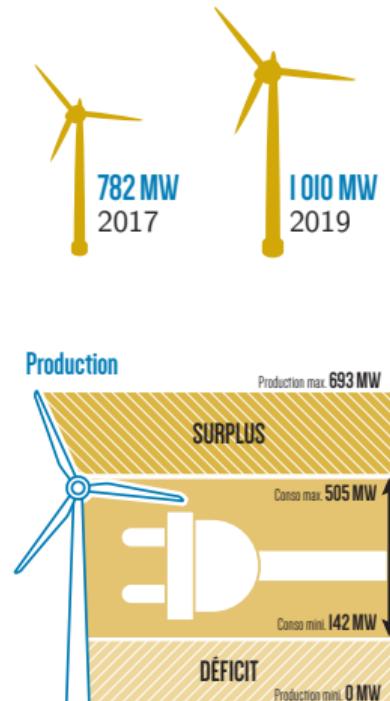
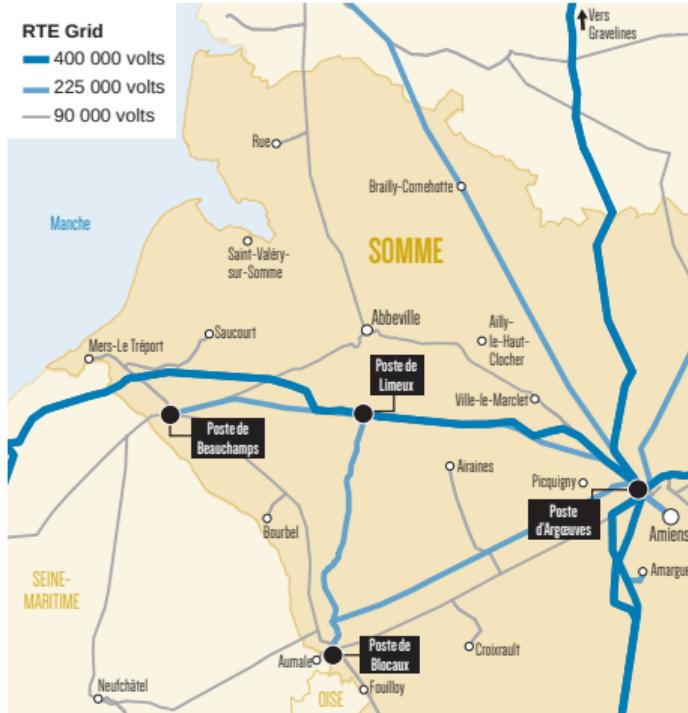
Optimal steady-state regulation



A typical *automation* task:

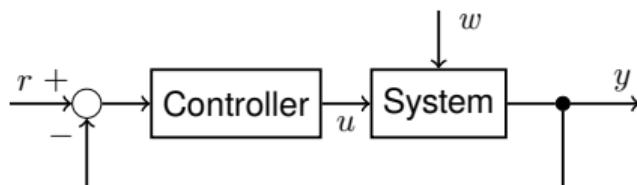
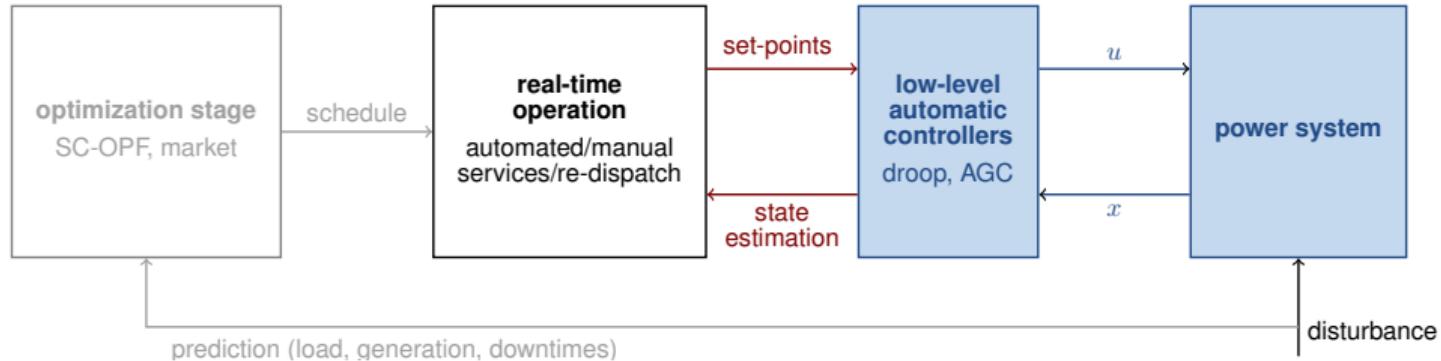
- **Stable or pre-stabilized nonlinear plant**
- **Steady-state specifications** in terms of optimality
 - yield
 - economic cost
 - CO₂ emissions
- Multiple **constrained inputs** to be set
- Multiple **steady-state constraints** to be satisfied
- **Exogenous factors** (disturbances) affect the plant steady state
 - disturbances
 - unknown parameters

Example: real-time power system operation



- Larger amount of wind
- Many small-scale distributed generators
- Reactive power control
- Voltage and flow constraints
- Goal: minimize curtailment

Example: real-time grid operation



- Steady-state power system dynamics
- Real-time sensing and actuation
- Goal: make the solution of an Optimal Power Flow problem an asymptotically stable equilibrium for the closed-loop system ← feedback optimization

Outline



Feedback optimization



First-order algorithms



Zeroth-order algorithms



Beyond optimization

Outline



Feedback optimization



First-order algorithms

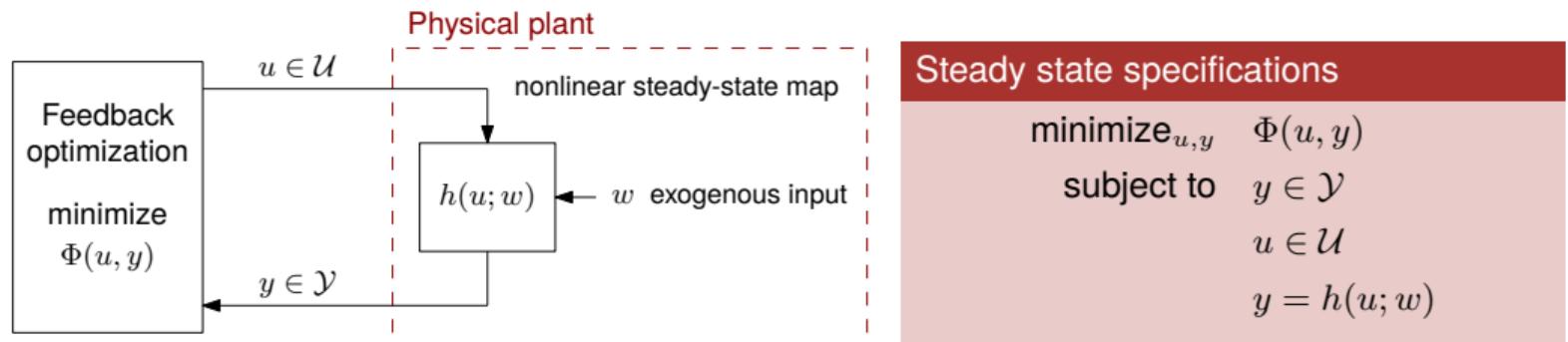


Zeroth-order algorithms



Beyond optimization

“Certainty-equivalence” design



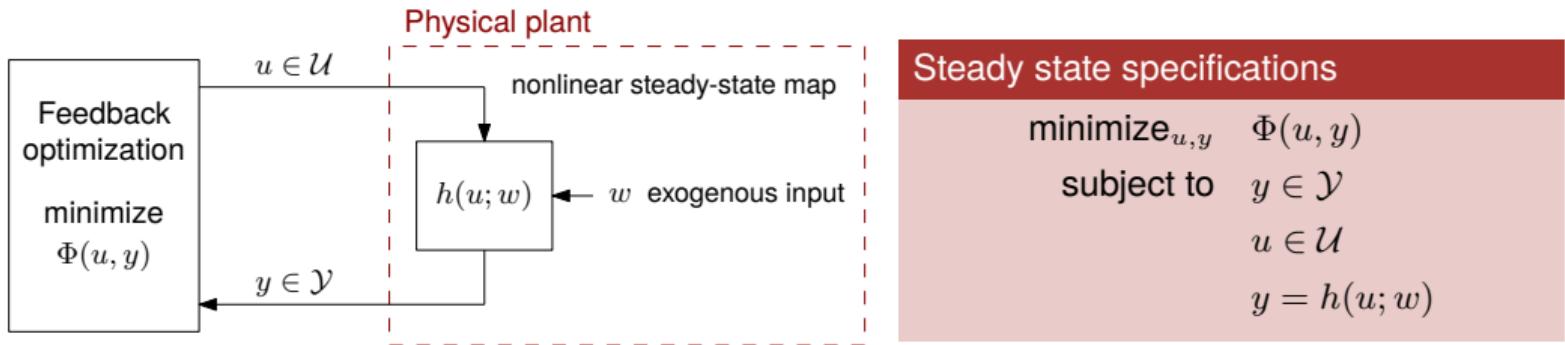
How to design **feedback controllers** that yield the desired **(autonomous) closed-loop behavior**?

Algebraic **input/output steady-state map** $y = h(u; w)$

+

Classical **iterative nonlinear optimization algorithms**

“Certainty-equivalence” design



Optimization perspective

Analysis and design of algorithms with the tools of dynamical systems

Difference

Part of the algorithm implemented by physics

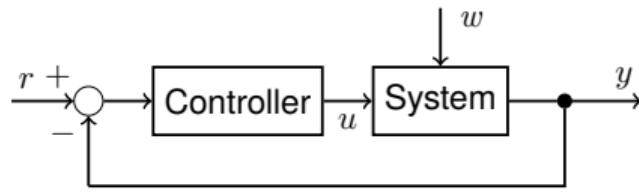
Control perspective

Feedback controllers interpreted as solvers of a specific optimization problem

Difference

Design for general objective and constraints

Toy example: re-inventing PI controllers



$$\begin{aligned} & \text{minimize}_{u,y} \quad \|u\|^2/2 \\ & \text{subject to} \quad y = r \quad (\text{tracking}) \\ & \qquad \qquad \qquad y = Hu + w \quad (\text{linear plant steady-state}) \end{aligned}$$

Dual ascent iteration

$$\mathcal{L}(u, \lambda) = \|u\|^2/2 + \lambda^\top (Hu + w - r)$$

- Primal minimization $u = -H^\top \lambda$
- Dual ascent $\dot{\lambda} = \alpha \frac{\partial \mathcal{L}}{\partial \lambda} = \alpha(Hu + w - r)$

I controller

$$\dot{u} = \alpha H^\top (r - y)$$

Augmented dual ascent iteration

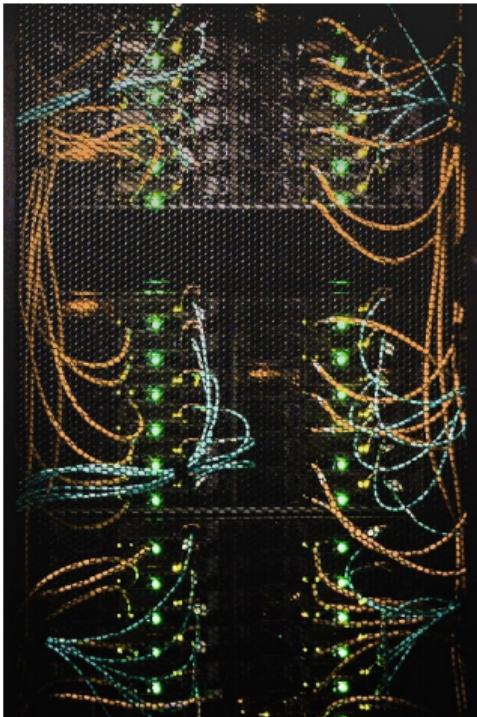
$$\begin{aligned} \mathcal{L}(u, \lambda) = & \|u\|^2/2 + \lambda^\top (Hu + w - r) \\ & + \frac{\rho}{2} \|Hu + w - r\|^2 \end{aligned}$$

PI controller

$$\dot{\lambda} = \alpha(r - y)$$

$$u = H^\top \lambda + \rho H^\top (r - y)$$

Historical note (1998)



TCP congestion control in data networks

- **Inputs:** source-destination data rates
- **Output:** data channel use
 - measured via delays, dropped packets, congestion signals
- **Goal:** network utility maximization

Rate control for communication networks: shadow prices, proportional fairness and stability

FP Kelly, AK Maulloo and DKH Tan
University of Cambridge, UK

Outline



Feedback optimization



First-order algorithms



Zeroth-order algorithms



Beyond optimization

Design of feedback optimizers

Borrow ideas from **first-order algorithms** for **nonlinear optimization** and interpret these algorithms as dynamical systems

- Gradient Flows
[Brockett, 1991], [Bloch, 1992], [Helmke & Moore, 1994], ...
- Interior-point methods
[Karmarkar, 1984], [Khachian, 1979], [Faybusovich, 1992], ...
- Acceleration & Momentum methods
[Su, 2014], [Wibisono, 2016], [Krichene, 2015], [Wilson, 2016], [Lessard, 2016], [Mühlebach & Jordan, 2020], ...
- Saddle-Point Flows
[Arrow, 1958], [Kose, 1956], [Feijer & Paganini, 2010], [Cherukuri, 2017], [Holding & Lestas, 2014], [Cortés & Niederländer, 2018], [Qu & Li, 2019], ...

Example: projected gradient descent

$$\text{minimize}_{u,y} \quad \Phi(u, y)$$

subject to $y \in \mathcal{Y}$ output constraints

$u \in \mathcal{U}$ input saturation

$y = h(u; w)$ steady state map

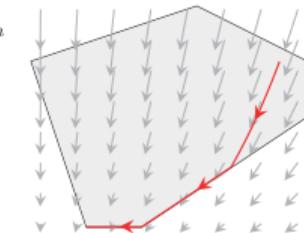
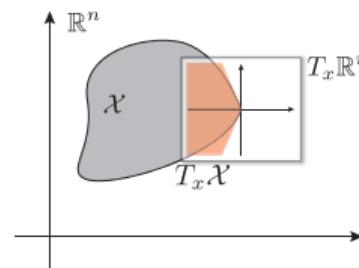
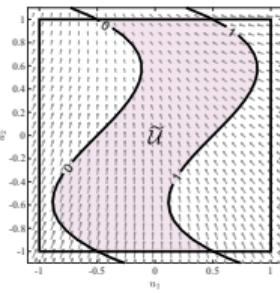
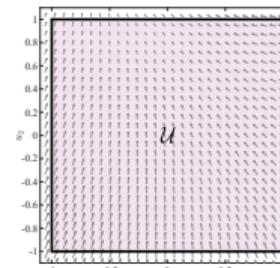
Projected gradient descent (Hauswirth 2016, Häberle 2020, ...)

Projection on the input and output constraints

$$\dot{u} = \Pi_{\tilde{\mathcal{U}}} \left[-\nabla_u \Phi(u, y) - \underbrace{\nabla h(u; w)' \nabla_y \Phi(u, y)}_{\text{model}} \right]$$

Also for time-varying constraints if special constraint qualification holds, see Hauswirth et al. IEEE CDC 2018 ↗

$$\tilde{\mathcal{U}} = \mathcal{U} \cap h^{-1}(\mathcal{Y})$$



Projected gradient flow via repeated Quadratic Programming

Continuous-time flow

$$\dot{u} = \Pi_{\bar{\mathcal{U}}} \left[-\nabla_u \Phi(u, y) - \underbrace{\nabla h(u; w)' \nabla_y \Phi(u, y)}_{\text{model}} \right]$$

Assumption

$$\mathcal{U} := \{u \in \mathbb{R}^p \mid Au \leq b\}$$

$$\mathcal{Y} := \{y \in \mathbb{R}^n \mid Cy \leq d\}$$

Discrete-time approximation

$$\begin{aligned} \delta u &:= \arg \min_v \quad \|v - (-\nabla_u \Phi(u, y) - \nabla h(u; w)' \nabla_y \Phi(u, y))\|^2 \\ &\text{subject to} \quad A(u + \alpha v) \leq b \\ &\quad C(y + \alpha \nabla h(u; w)' v) \leq d, \end{aligned}$$

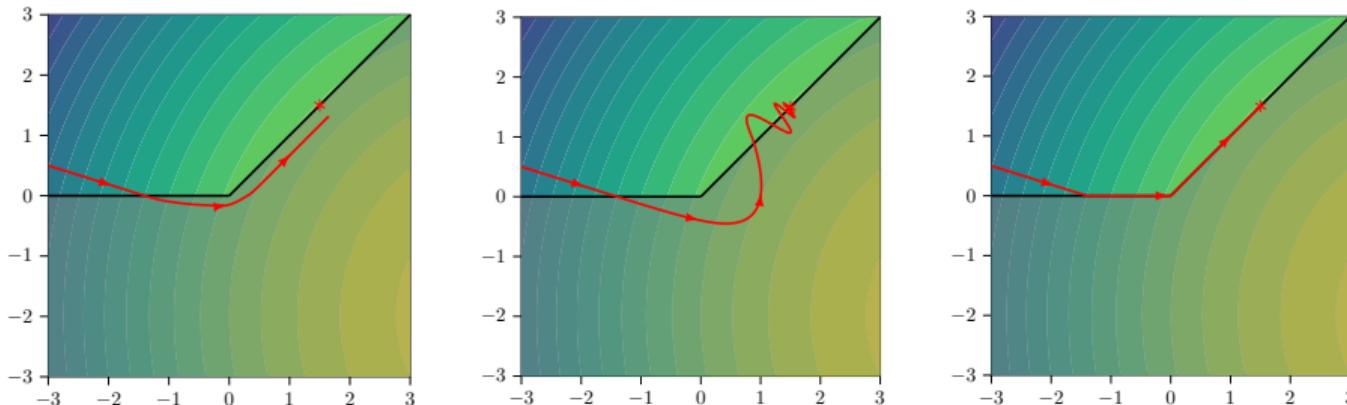
1st order approx of $h^{-1}(\mathcal{Y})$ centered at the measurement y

Theorem: (Häberle et al., IEEE Control Systems Letters, 2021) ↗

LICQ + Lipschitz + differentiability + small α → global convergence to the set of local minima

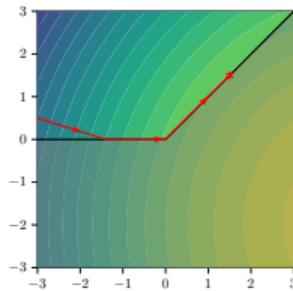
First-order methods

	Penalty	(Augmented) saddle-flow	Projected gradient
Feasibility	arbitrarily small violation	asymptotic feasibility	any-time feasibility
Controller	“proportional”	“(proportional)-integral”	quadratic programming
Stability	steep penalty limits speed	requires exp. stability	simple gain limit



All these methods require **gradient** information, and therefore **input output sensitivities** ∇h .

Note on discontinuous flows



The analysis of convergence of discontinuous flows is not trivial

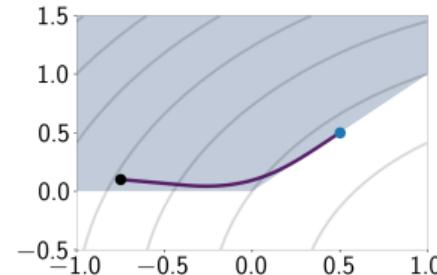
- existence of (Krasovskii)solutions
 - uniqueness of solutions (under prox-regularity conditions)
- Hauswirth et al., SIAM J. on Control and Optimization, 2021 ↗

Control Barrier Function-Based Design of Gradient Flows for Constrained Nonlinear Programming

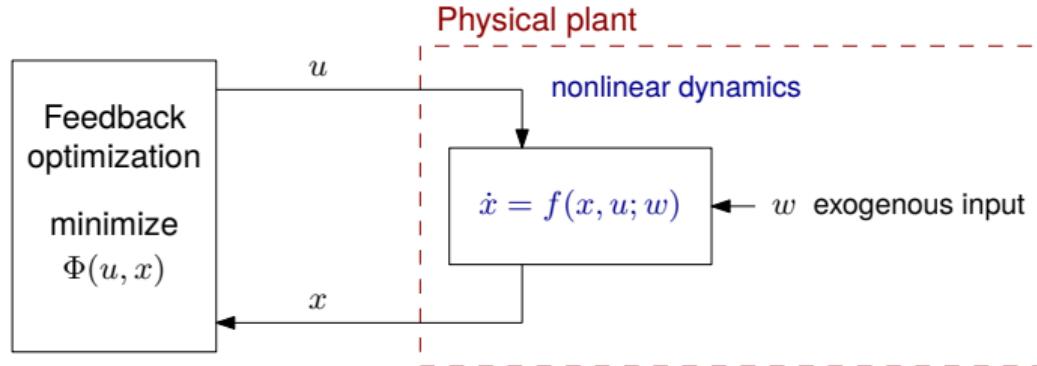
Ahmed Allibhoy Jorge Cortés

$$\begin{aligned}\mathcal{G}_\alpha(x) &= \underset{\xi \in \mathbb{R}^n}{\operatorname{argmin}} \quad \frac{1}{2} \|\xi + \nabla f(x)\|^2 \\ \text{subject to} \quad &\frac{\partial g(x)}{\partial x} \xi \leq -\alpha g(x) \\ &\frac{\partial h(x)}{\partial x} \xi = -\alpha h(x)\end{aligned}$$

arXiv:2204.01930v3 [math.OC] 20 Feb 2023



Closing the loop with a dynamical system



Optimization dynamics

Example: Generalized gradient descent

$$\dot{u} = -Q(u) \left(-\nabla_u \Phi(u, x) - \nabla h(u; w)' \nabla_y \Phi(u, x) \right)$$

with $Q(u) \succ 0$

Plant dynamics

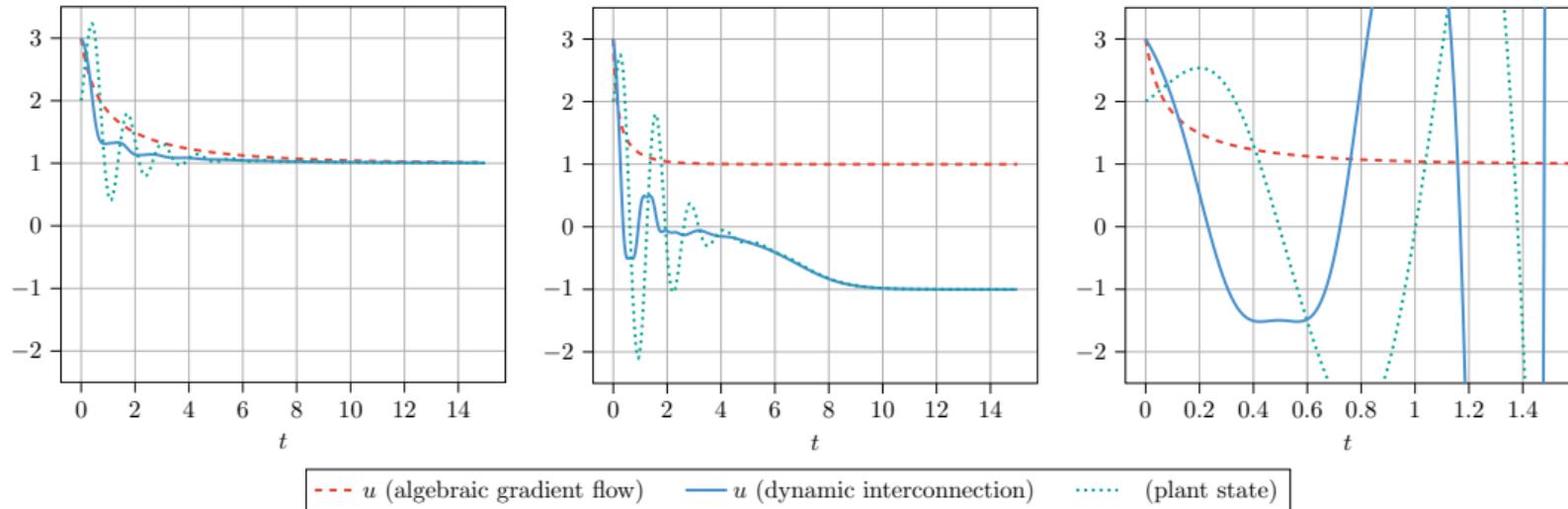
Exponentially stable system

$$\dot{x} = f(x, u; w)$$

with steady-state map $x = h(u; w)$

Closed loop instability

» Increasing gain $\|Q(u)\|$ »



Singular perturbation analysis

Theorem A. Hauswirth, S. Bolognani, G. Hug, F. Dörfler, IEEE TAC, 2021 ↗

Assume

- Physical system **exponentially stable** with Lyapunov function $W(u, x)$ s.t.

$$\dot{W}(u, x) \leq -\gamma \|x - h(u; w)\|^2 \quad \|\nabla_u W(u, x)\| \leq \zeta \|x - h(u; w)\| .$$

- Nonconvex cost $\Phi(u, x)$ has compact level sets and **L -Lipschitz** gradient.

Then, all trajectories converge to the set of KKT points whenever

$$\sup_u \|Q(u)\| < \frac{\gamma}{\zeta L} .$$

- Asymptotically stable equilibrium \Rightarrow strict local minimizer
- Strict local minimizer \Rightarrow stable equilibrium

Gradient-based Feedback Optimization

Projected Gradient Descent

Choose $Q(u) = \varepsilon I$.

Stability is guaranteed if

$$\varepsilon \leq \frac{\gamma}{\zeta L}$$

→ **prescription on global control gain**

Saddle-flow

Similar prescription on ε , but

- requires **exponential stability** of the saddle flow

[Qu & Li, IEEE Control Systems Letters, 2019]

Newton GD

Choose $Q(u) = (\nabla^2 \Phi(h(u; w), u))^{-1}$

(if Φ μ -strongly convex and twice differentiable)

Stability is guaranteed if

$$\frac{L}{\mu} \leq \frac{\gamma}{\zeta}$$

→ **invariant under scaling of Φ**

But not (in general)

- Subgradient methods
- Accelerated gradient method

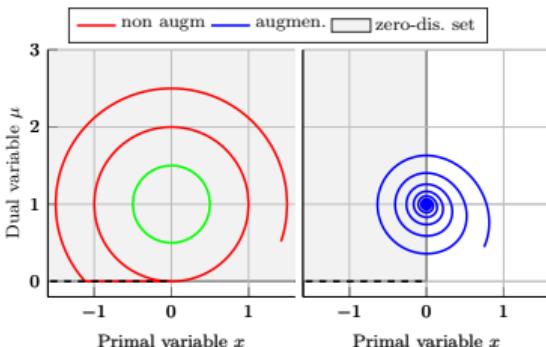
General feedback optimization controllers

General Slow Dynamics

$$\dot{u} = \varepsilon g(h(u; w), u, z)$$

$$\dot{z} = \varepsilon k(h(u; w), u, z)$$

- E.g. saddle flows
- requires exponential stability
(open problem!)



Theorem

- $(g(x, u, z), k(x, u, z))$ is L -Lipschitz in x
- $(g(h(u; w), u, z), k(h(u; w), u, z))$ is ℓ -Lipschitz
- \exists Lyapunov function $V(u, z)$ for the slow dynamics

$$\dot{V}(u, z) \leq -\mu \|e(u, z)\|^2 \quad \|\nabla V(u, z)\| \leq \kappa \|e(u, z)\|$$

- \exists Lyapunov function $W(x, u)$ for the plant

$$\dot{W}(x, u) \leq -\gamma \|x - h(u; w)\|^2, \quad \|\nabla_u W(x, u)\| \leq \zeta \|x - h(u; w)\|$$

Then, asymptotic stability is guaranteed if

$$\epsilon < \frac{\gamma}{\zeta L(1 + \frac{\kappa \ell}{\mu})}.$$

Take home message

If

- **first-order model** of the system is available (input-output sensitivities)
- the system is **fast-stable**

then

- gradient and saddle flows are **valid closed-loop trajectories**
- first-order optimization methods can be used as **optimizing controllers**
- controller design is **almost model-free***
- **stability** can be guaranteed with minimal model information

Adrian Hauswirth, Saverio Bolognani, Gabriela Hug, and Florian Dörfler
Optimization Algorithms as Robust Feedback Controllers
arXiv:2103.11329 [math.OC], 2021. ↗

Outline



Feedback optimization



First-order algorithms



Zeroth-order algorithms

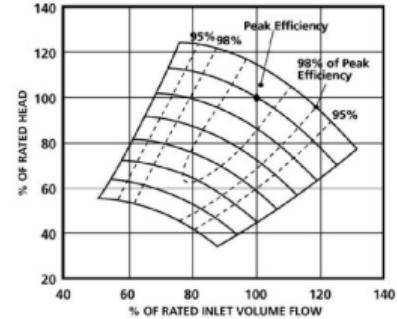


Beyond optimization

When the gradient (plant sensitivities) is not available

sample
bandit
oracle

...



- **Estimation of the plant input-output sensitivities**
 - Kalman filter (Picallo 2022)
 - shape-constrained Gaussian processes (Ospina 2021)
 - data-driven/behavioral systems theory (Bianchin 2021, Nonhoff 2021)
 - recursive least-square (A. D. Dominguez-Garcia 2023)
- **Optimization without a gradient**
 - extremum seeking, reinforcement learning, **zeroth-order optimization**

Descent without first-order information

How to determine a descent direction without first-order model information?

Extremum seeking

- Exploration via low dimensional perturbation signals
- Averaging / demodulation (time-scale separation)

$$\begin{aligned}\dot{u} &= \Phi(h(u + \sin(\omega t))) \sin(\omega t) \\ \frac{1}{T} \int_0^T \dot{u}(t) dt &\approx -\nabla \Phi(u)\end{aligned}$$

Reinforcement learning

- Policy gradient can be evaluated as an expectation $\nabla_{\theta} \rho = \sum_s d^{\pi}(s) \sum_a \nabla_{\theta} \pi(s, a) Q^{\pi}(s, a)$
- Sample-based approximations of that expectation $Q^{\pi}(s, a) \approx \dots$

Zero-th order optimization

- Gradient estimates based on {1, 2, many} function evaluations
- Descent iteration (typically: stochastic gradient)

Unbiased gradient estimators

Exploration parameter δ

Perturbation v sampled from the normal distribution

1-point
(Flaxmann 2005)

$$\widehat{\nabla \Phi}(u) = \frac{v}{\delta} \Phi(u + \delta v)$$

Large estimation variance
→ slow SGD convergence

2-point
(Nesterov Spokoiny 2017)

$$\widehat{\nabla \Phi}(u) = \frac{v}{\delta} (\Phi(u + \delta v) - \Phi(u))$$

Smaller estimation bias
Multiple oracle calls

1-point residual feedback
(Zhang 2022)

$$\widehat{\nabla \Phi}(u_k) = \frac{v_k}{\delta} (\Phi(u_k + \delta v_k) - \Phi(u_{k-1} + \delta v_{k-1}))$$

Evaluation on streaming data

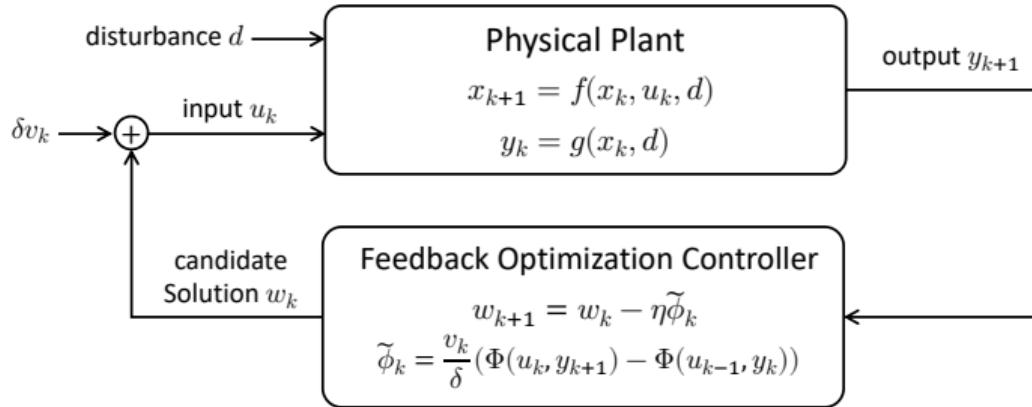
1-point residual feedback

- unbiased gradient estimator for the Gaussian-smoothed Φ_δ
- small est. variance under the SGD rule $u_{k+1} = u_k - \eta \widehat{\nabla \Phi}(u_k)$

ETH **ETH**

$$\Phi_\delta := \mathbb{E}_{v \sim \mathcal{N}} [\Phi(u + \delta v)]$$

Zero-th order feedback optimization



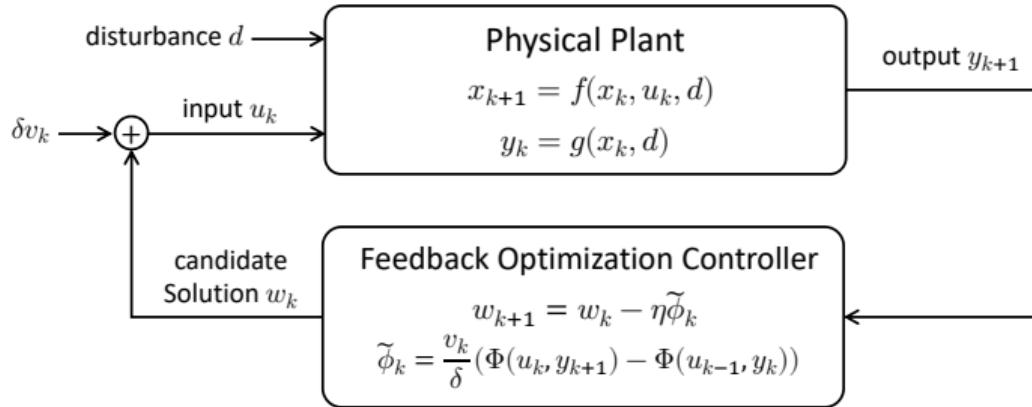
Non-convex optimization

$$\begin{aligned} & \text{minimize}_{u,y} \quad \Phi(u, y) \\ & \text{subject to} \quad y = h(u, d) \end{aligned}$$

Φ Lipschitz, bounded below

- **Exponentially stable** system (with constant input $u_k = u$)
- **Unique steady-state map** $x_{ss}(u, d) \rightarrow y = h(u, d)$
- $x_{ss}(u, d)$ and g are **Lipschitz continuous**

Zero-th order feedback optimization



Non-convex optimization

$$\begin{aligned} & \text{minimize}_{u,y} \quad \Phi(u, y) \\ & \text{subject to} \quad y = h(u, d) \end{aligned}$$

Φ Lipschitz, bounded below

- **Exploration signal** $\delta v_k \sim \mathcal{N}(0, 1)$
- **Measurement** y_{k+1}
- **Cost oracle** $\Phi(u_k, y_{k+1})$ (each sample used twice)
- Descent direction $\tilde{\phi}_k$ via **1-point residual-feedback gradient estimation**
- **Stochastic gradient descent** iteration

Closed-loop stability

Let M be the **Lipschitz constant** of $\tilde{\Phi}(u) := \Phi(u, h(u))$. Let p be the **input dimension**.

Let μ be the **rate of convergence** of the plant.

Let $\epsilon > 0$ be desired **precision** of the Gaussian approx. $\rightarrow \delta = \frac{\epsilon}{M\sqrt{p}}$ ensures $|\Phi_\delta(u) - \Phi(u)| \leq \epsilon$.

Theorem (He, Bolognani, He, Dörfler, Guan, 2022) 

If the SGD gain η satisfies

$$0 < \eta < \eta^*$$

$$\eta^* = \dots, \quad \rho = \dots$$

then

$$\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}_{v_{[T-1]}} [\|\nabla \Phi_\delta(w_k)\|^2] = \mathcal{O}\left(\frac{p^{\frac{3}{2}}}{\sqrt{\epsilon T}(1-\rho)}\right) + \mathcal{O}\left(\frac{p^2 \sqrt{\mu}}{1-\rho} \cdot \left(1 + \frac{1}{T\epsilon^2\alpha_2}\right)\right).$$

For **fast systems** ($\mu \rightarrow 0$)

$$0 < \eta < \frac{\epsilon}{p} \quad \Rightarrow \quad \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}_{v_{[T-1]}} [\|\nabla \Phi_\delta(w_k)\|^2] = \mathcal{O}\left(\frac{p^{\frac{3}{2}}}{\sqrt{\epsilon T}(1-\rho)}\right).$$

Sketch of the proof

$$\begin{aligned}\alpha_1 \|x - x_{\text{ss}}(u)\|^2 &\leq V(x, u) \leq \alpha_2 \|x - x_{\text{ss}}(u)\|^2 \\ V(f(x, u), u) - V(x, u) &\leq -\alpha_3 \|x - x_{\text{ss}}(u)\|^2\end{aligned}$$

$$\mu = \frac{2\alpha_2}{\alpha_1} \left(1 - \frac{\alpha_3}{\alpha_2}\right)$$

- Upper bound on the **error introduced by transient dynamics**

$$|\Phi(u_k, y_{k+1}) - \Phi(u_k, h(u_k))| \leq \mu \kappa_1 V(x_k, u_k)$$

- Interconnection of two recursive inequalities

- on the **expected Lyapunov function**

$$\mathbb{E}_{v[k]}[V(x_k, u_k)] \leq \mu \mathbb{E}_{v[k]}[V(x_{k-1}, u_{k-1})] + \kappa_2 \mathbb{E}_{v[k]}[\|\tilde{\phi}_{k-1}\|^2] + \kappa_3$$

- on the **second moment of the gradient estimate**

$$\mathbb{E}_{v[k]}[\|\tilde{\phi}_k\|^2] \leq \kappa_4 \mathbb{E}_{v[k]}[\|\tilde{\phi}_k\|^2] + \kappa_5 \mathbb{E}_{v[k]}[V(x_k, u_k)] + \kappa_6 \mathbb{E}_{v[k]}[V(x_{k-1}, u_{k-1})] + \kappa_7$$

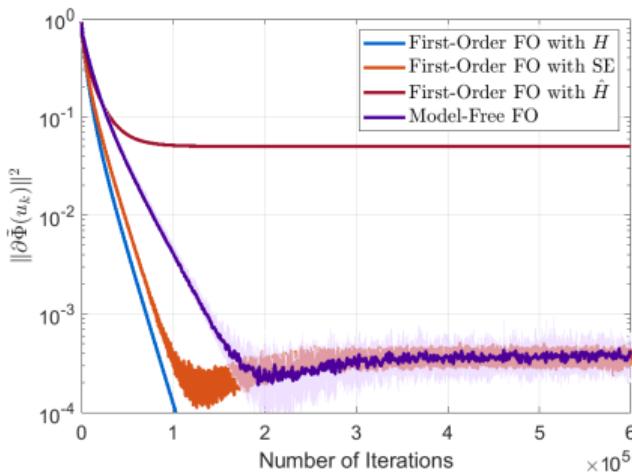
Numerical experiment

Nonlinear dynamics

$$x_{k+1} = Ax_k + Bu_k + Ed^{(x)} + F(x_k - x_{\text{ss}}(u_k, v))^{\otimes 2}$$

$$y_k = Cx_k + Dd^{(y)},$$

$$x \in \mathbb{R}^{10}, u \in \mathbb{R}^5, d \in \mathbb{R}^5, y \in \mathbb{R}^5$$



ℓ_1 -regularized QP

$$\min_{u,y} \underbrace{-\lambda \|u\|^3 + u' M_1 u + M_2' u + \|y\|^2}_{\Phi_1(u,y)} + \underbrace{\lambda' \|u\|_1}_{\Phi_2(u)},$$

First-order proximal gradient

$$u_{k+1} = \text{prox}_{\eta \Phi_2} \left(u_k - \eta \left(\nabla_u \Phi_1(u_k, y_{k+1}) + H' \nabla_y \Phi_1(u_k, y_{k+1}) \right) \right)$$

Zeroth-order proximal gradient

$$w_{k+1} = \text{prox}_{\eta \Phi_2} (w_k - \eta \tilde{\phi}_k),$$

$\text{prox}_{\eta \Phi_2}(u)$: shrinkage / soft thresholding

Take home message

If

- **function evaluations** are available (zeroth-order oracle)
- the system is **fast-stable**

then

- a **1-point gradient estimate** is possible (even for non-smooth cost, e.g. input saturation)
- the gradient estimate is unbiased and is a valid direction for **stochastic gradient descent**
- the resulting **model-free controller** achieves $\mathcal{O}(1/\sqrt{T})$ regret
- **stability** can be guaranteed with minimal model information

Zhiyu He, Saverio Bolognani, Jianping He, Florian Dörfler, and Xinping Guan
Model-Free Nonlinear Feedback Optimization
arXiv:2201.02395 [math.OC], 2022. ↗

Outline



Feedback optimization



First-order algorithms



Zeroth-order algorithms



Beyond optimization

Optimal steady-state regulation in a multi-agent world



In many engineering applications

- **multiple decision makers**
- different, possibly conflicting, **individual costs**
- **independent inputs**
- each cost depends on all agents' actions

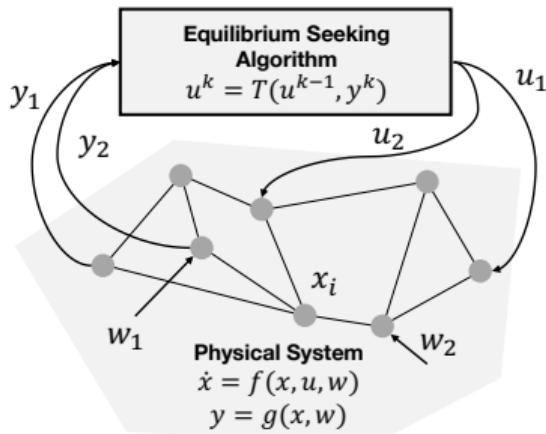
Solution concept

(Generalized) Nash equilibrium

$$\forall i : \quad \text{minimize}_{u_i \in \mathcal{U}_i} \{ \Phi_i(u_i, \bar{y}) \mid \bar{y} = h(u_i, \bar{u}_{-i}; w) \}$$

Feedback equilibrium seeking

For convex & differentiable J_i , NE = **variational inequality**



FES sampled-data closed loop

$$0 \in F(\bar{u}, h(\bar{u}, w)) + \mathcal{N}_{\mathcal{U}}(\bar{u}),$$

where F is the **pseudo-gradient mapping**

$$F_i(u_i, y) := \nabla_{u_i} J_i(u_i, y) + \nabla_{u_i} h_i(u_i, w_i)' \nabla_{y_i} J_i(u_i, y).$$

Example: best response iteration

$$\forall i : \quad T_i(w_i, y) = \arg \min_{\xi \in \mathcal{U}_i} \tilde{J}_i(\xi, y_{-i})$$

$$= \arg \min_{\xi \in \mathcal{U}_i} J_i\left(\xi, (h_i(\xi_i, w_i), y_{-i})\right)$$

Feedback equilibrium seeking

Assumptions:

- exponential stability of the plant
- linear convergence of the iteration
- strong monotonicity of the game map

Interconnected stability

- Sufficiently long sampling times
- (possibly) first-pass filtering of the iteration

⇒ the interconnected system is **input-to-state** stable w.r.t. w .

G. Belgioioso, D. Liao-McPherson, M. Hudoba de Badyn, S. Bolognani, J. Lygeros, F. Dörfler.

Sampled-data online feedback equilibrium seeking: Stability and tracking

arXiv:2103.13988 [math.OC], 2021. ↗



NCCR
Automation

<https://nccr-automation.ch>

TSO-DSO coordination

- Transmission grid operator
 - requires services from distribution grids
 - offers economic incentives
→ **feedback eq. seeking**
- Each distribution grid
 - optimizes its operation to satisfy constraints and respond to incentives
→ **feedback optimization**

Projected-gradient
feedback optimization has
been controlling a Swiss
distribution grid (AEW)
since December 2022!

