

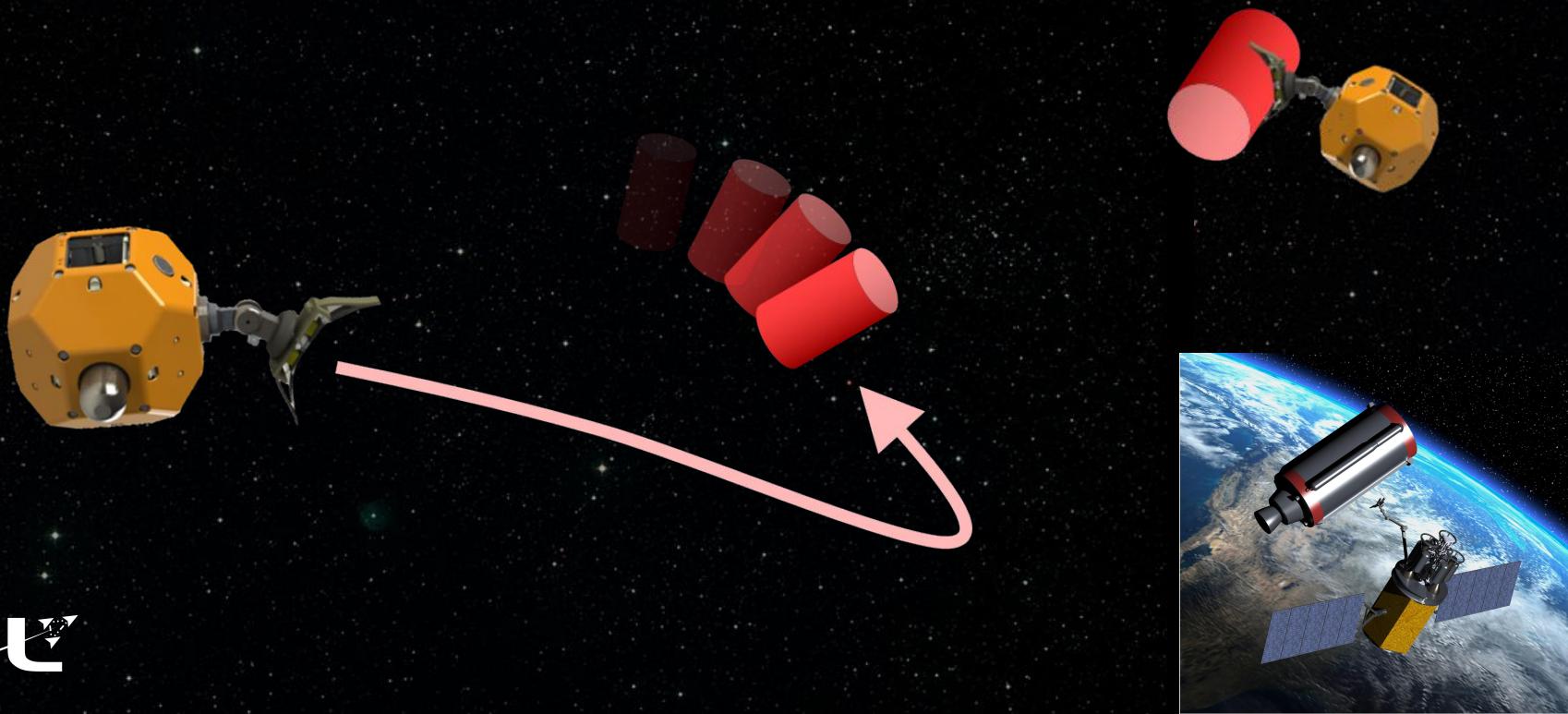
# Uncertainty-aware control strategies for safe learning-based robotic autonomy

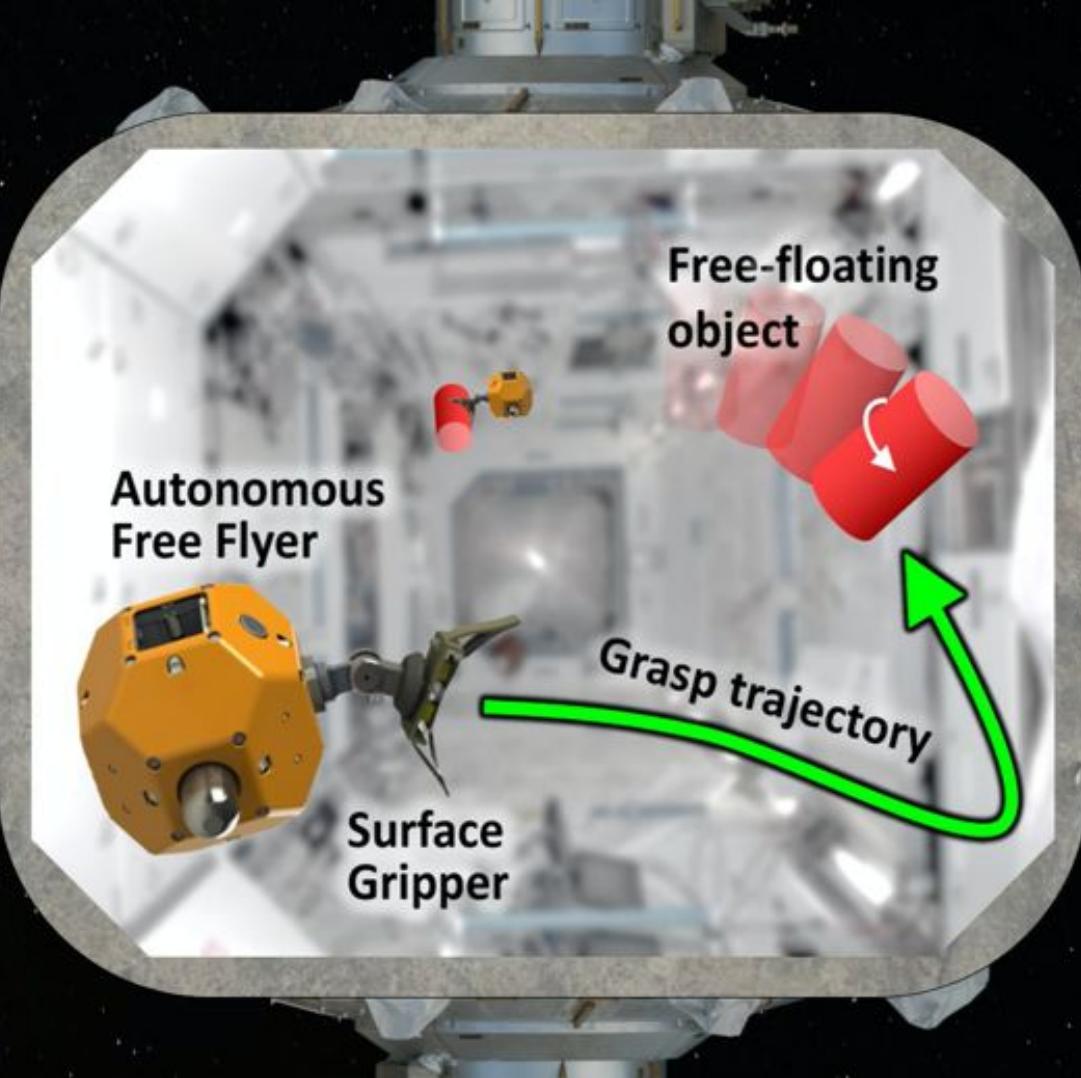
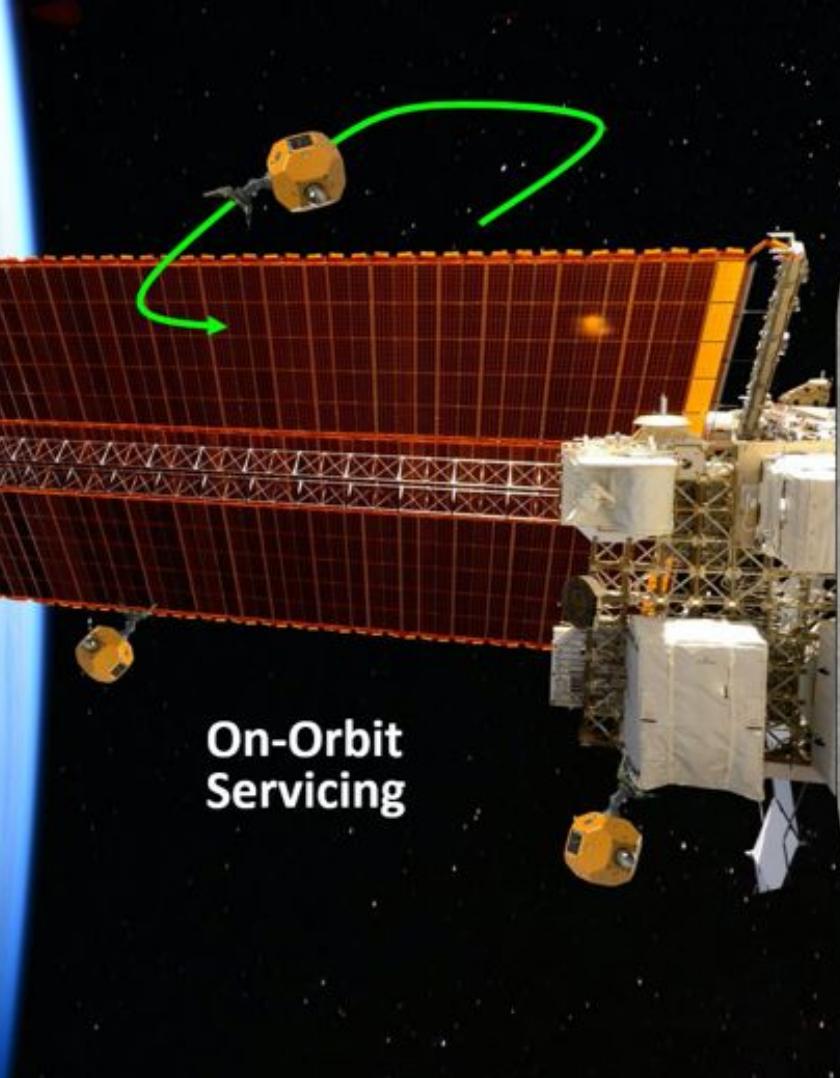


Thomas Lew

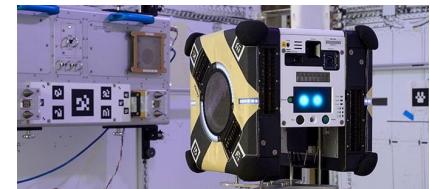


# Robotic autonomy in space

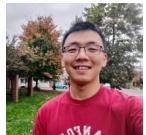




# NASA Astrobee free-flying robots on-board the ISS

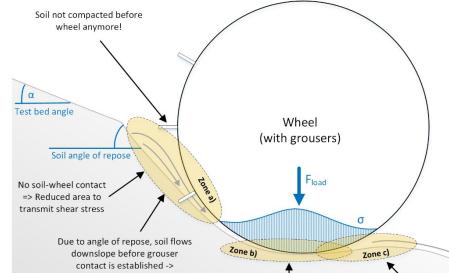
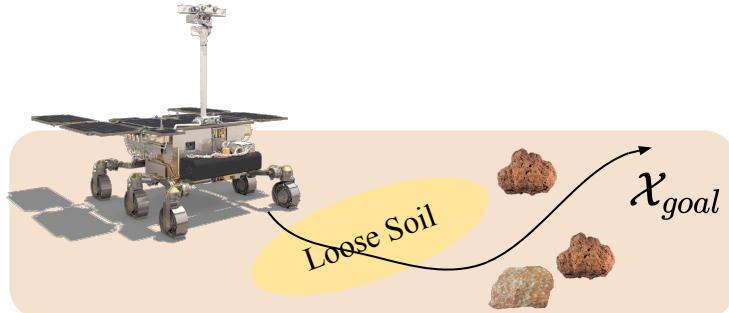
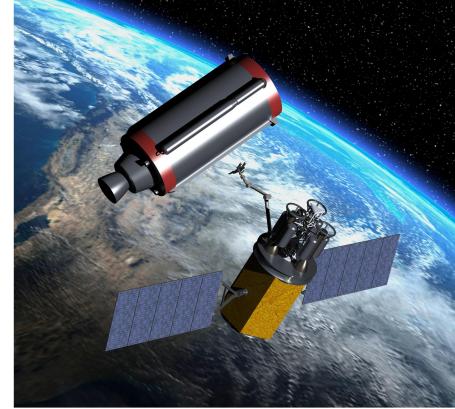


Astronaut McClain  
with Astrobee.

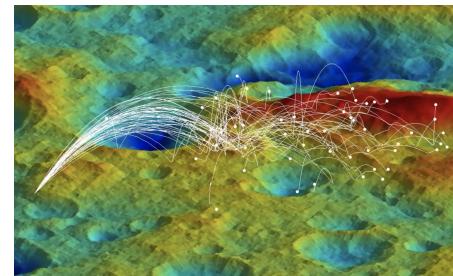
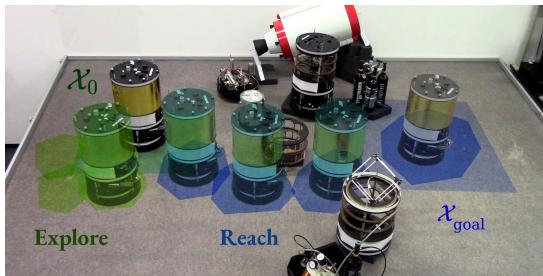
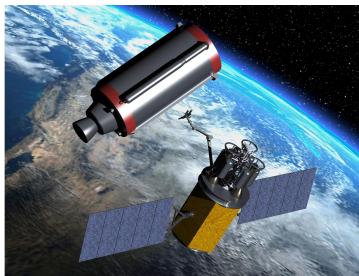


# Robotic autonomy: challenges

- Uncertain dynamics (unknown payload properties and flexible grasp)
- Safe use of learning-based components?
- Accounting for uncertainty?

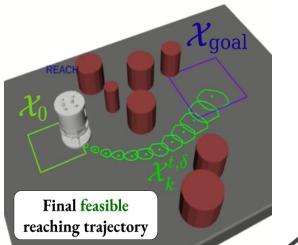
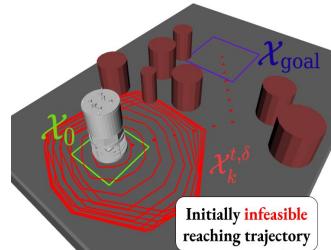


# Towards safe learning-based robotic autonomy



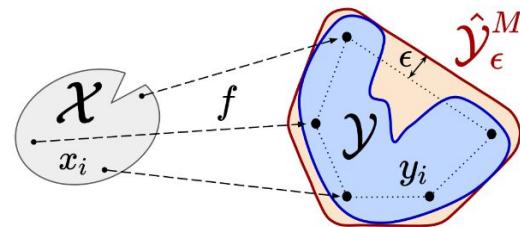
1

safe active dynamics  
learning and control

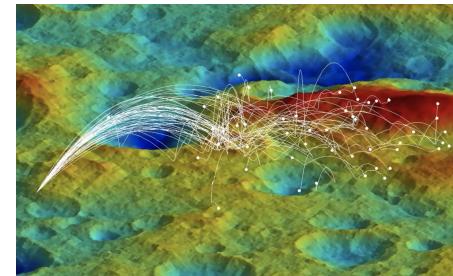
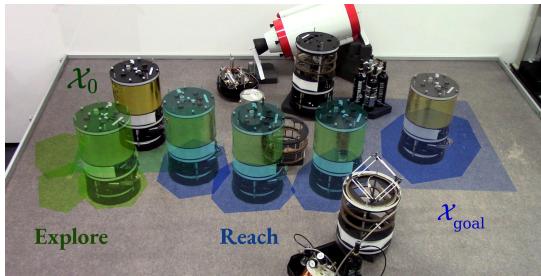
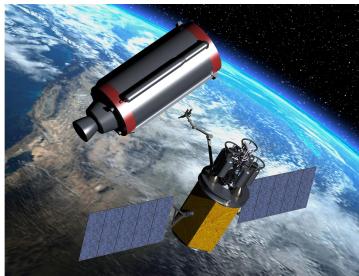


2

sampling-based  
reachability analysis

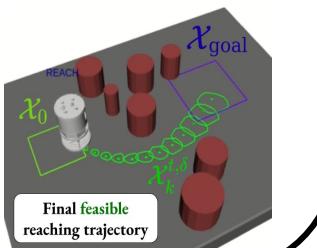
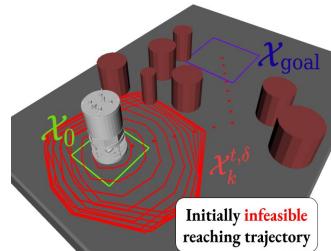


# Towards safe learning-based robotic autonomy



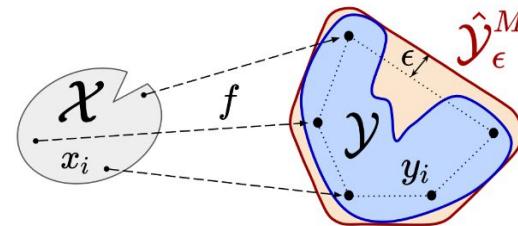
①

**safe active dynamics  
learning and control**

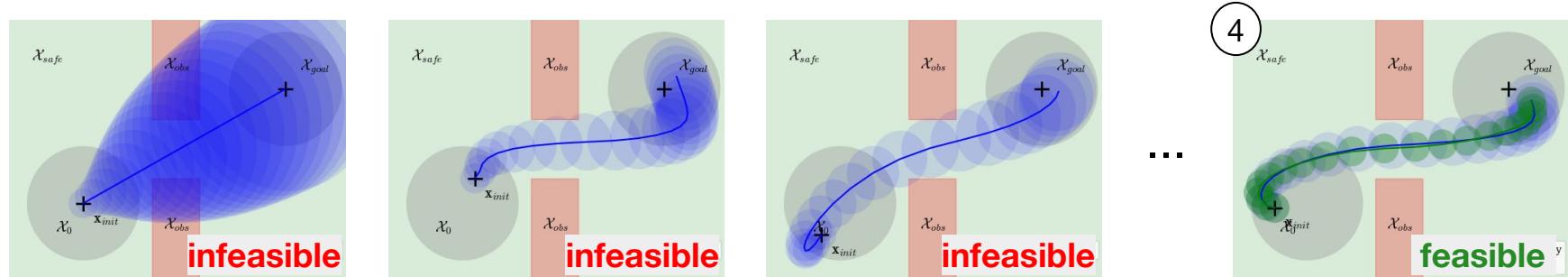


②

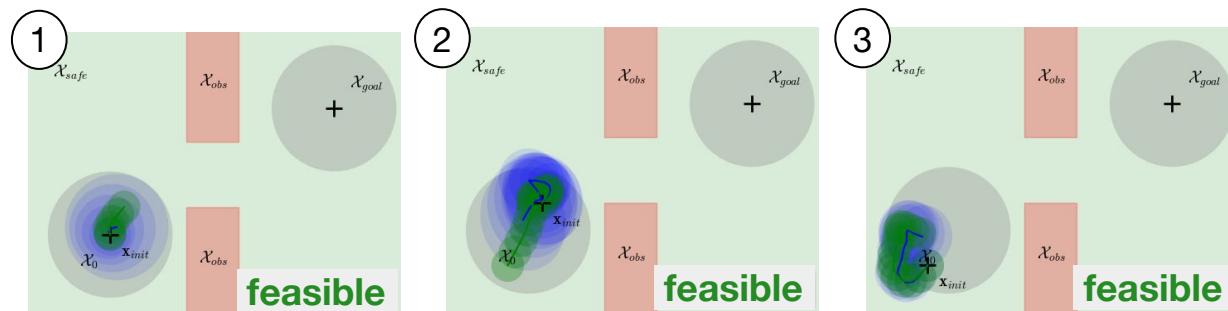
sampling-based  
reachability analysis



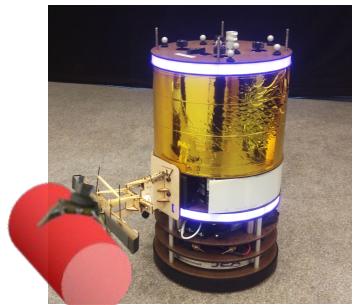
# Uncertainty may be too high and cause infeasibility



=> first, gather data to learn and reduce uncertainty



free-flying space robot  
with uncertain payload



# Problem formulation

- Uncertain dynamics

$$x_{t+1} = h(x_t, u_t) + g(x_t, u_t, \xi) + \epsilon_t$$

- Constraints  $(x, u) \in \mathcal{X} \times \mathcal{U}$

$x$  State  
 $u$  Control  
 $\xi$  Parameters  
 $\epsilon$  Disturbances  
(bounded)

$$\min_{x,u} \mathbb{E} \left[ \sum_{t=0}^{T-1} \ell(x_t, u_t) \right] \quad \text{s.t.} \quad x_0 = x(0)$$

$$x_{t+1} = \text{dynamics}(x_t, u_t), \quad t = 0, \dots, T-1,$$

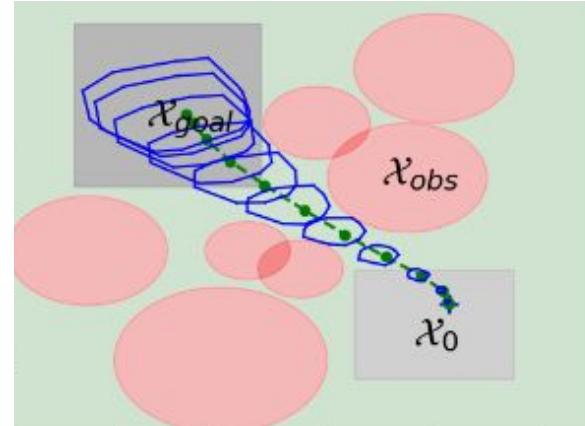
$$\mathbb{P} \left( \bigwedge_{t=1}^T x_t \in \mathcal{X} \wedge \bigwedge_{t=0}^{T-1} u_t \in \mathcal{U} \wedge x_T \in \mathcal{X}_{\text{goal}} \right) \geq 1 - \delta.$$

# Problem formulation

$$\begin{aligned} \min_{x,u} \quad & \mathbb{E} \left[ \sum_{t=0}^{T-1} \ell(x_t, u_t) \right] \quad \text{s.t.} \quad x_0 = x(0) \\ & x_{t+1} = h(x_t, u_t) + g(x_t, u_t, \xi) + \epsilon_t, \quad t = 0, \dots, T-1, \\ & \mathbb{P} \left( \bigwedge_{t=1}^T x_t \in \mathcal{X} \wedge \bigwedge_{t=0}^{T-1} u_t \in \mathcal{U} \wedge x_T \in \mathcal{X}_{\text{goal}} \right) \geq 1 - \delta. \end{aligned}$$

Additional knowledge:

- Control-invariant set  $\mathcal{X}_0$
- Prior data  $\mathcal{D}_t = \{(x_k^i, u_k^i, x_{k+1}^i)\}_{k=0 \dots t}^{i=0 \dots B}$



# Related work

- Approaches to learn a model / controller to improve performance
  - Reinforcement learning (e.g., Hwangbo et al, 2019)
  - Learning-based MPC (e.g, Deisenroth et al, 2015. Koller et al, 2018)
  - Learning-based control barrier functions (e.g., Khojasteh et al, 2020)
- Dual control typically does not consider constraints (e.g., Mania et al, 2020)
- We enforce a **joint** chance constraint

$$\mathbb{P} \left( (x_t \in \mathcal{X} \wedge u_t \in \mathcal{U} \ \forall t), (x_T \in \mathcal{X}_{\text{goal}}) \right) \geq (1 - \delta)$$

which gives stronger guarantees than *pointwise* chance constraints

$$\mathbb{P}(x_t \in \mathcal{X}) \geq 1 - \delta \quad \text{for all } t \geq 0$$

# Modeling: linear meta-learning

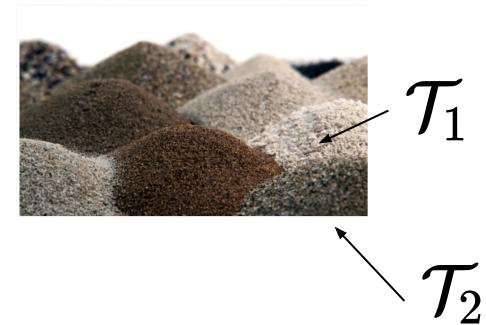
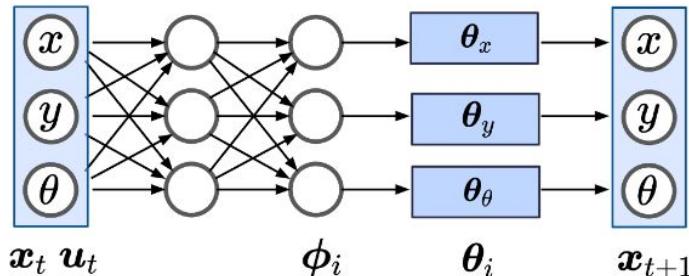
- Uncertain dynamics

$$x_{t+1} = h(x_t, u_t) + g(x_t, u_t, \xi) + \epsilon_t$$

- Approximate model

$$\hat{g}_i(x_t, u_t, \theta_i) = \theta_i^\top \phi_i(x_t, u_t)$$

train a model capable  
of *rapid adaptation*



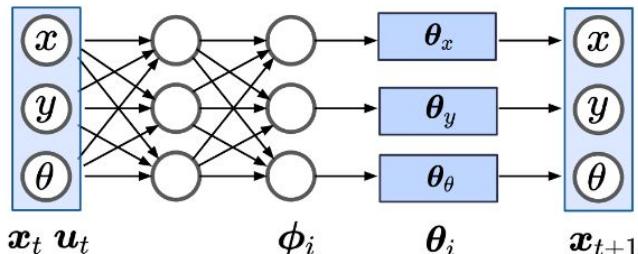
# Modeling: linear meta-learning

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- Approximate model

$$\hat{g}_i(x_t, u_t, \theta_i) = \theta_i^\top \phi_i(x_t, u_t)$$



- Train a prior  $(\bar{\theta}_{i,0}, \Lambda_{i,0})$
- Observe trajectory data  $(x_0, u_0, x_1, \dots, x_t, u_t, x_{t+1})$
- Update estimates  $(\bar{\theta}_{i,t}, \Lambda_{i,t})$ :

$$\Lambda_{i,t} = \Phi_{i,t-1}^\top \Phi_{i,t-1} + \Lambda_{i,0}$$

$$\bar{\theta}_{i,t} = \Lambda_{i,t}^{-1} (\Phi_{i,t-1}^\top G_{i,t} + \Lambda_{i,0} \bar{\theta}_{i,0})$$

# Self-normalizing martingales

- Uncertain dynamics  $x_{t+1} = h(x_t, u_t) + g(x_t, u_t, \xi) + \epsilon_t$
- Approximate model  $\hat{g}_i(x_t, u_t, \theta_i) = \theta_i^\top \phi_i(x_t, u_t)$
- Define the confidence sets

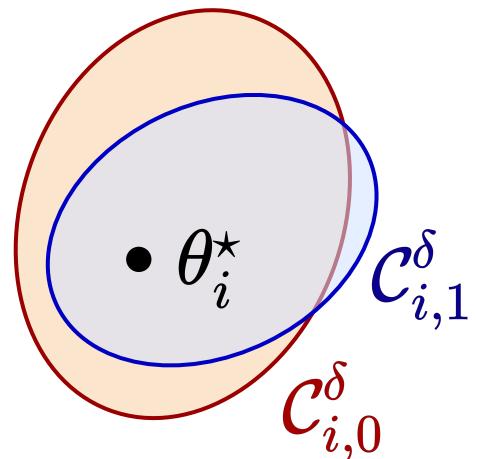
$$\mathcal{C}_{i,t}^\delta = \{\theta_i \in \mathbb{R}^d : \|\theta_i - \bar{\theta}_{i,t}\|_{\Lambda_{i,t}} \leq \beta_{i,t}^\delta\} \quad \text{with } \beta_{i,t}^\delta = \sigma_i \left( \sqrt{2 \log \left( \frac{1}{\delta_i} \frac{\det(\Lambda_{i,t})^{1/2}}{\det(\Lambda_{i,0})^{1/2}} \right)} + \sqrt{\frac{\bar{\lambda}(\Lambda_{i,0})}{\lambda(\Lambda_{i,t})}} B \right)$$

**Theorem\***: If the  $\epsilon_t$  are  $\sigma_i$ -subGaussian and

$$g \in \mathcal{H}_{\phi_i} = \left\{ \hat{g}_{i,\theta_i} = \theta_i^\top \phi_i : \|\theta_i - \bar{\theta}_{i,0}\|_{\Lambda_{i,0}}^2 \leq \sigma_i^2 B \right\}.$$

then

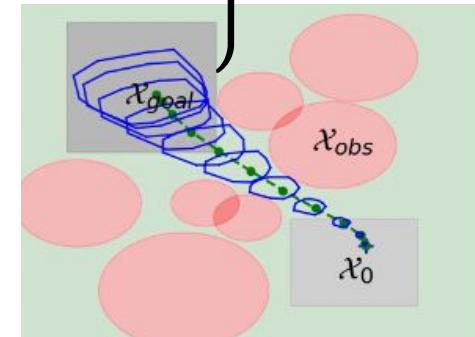
$$\mathbb{P} \left( \theta_i^* \in \mathcal{C}_{i,t}^\delta \ \forall t \geq 0 \right) \geq 1 - 2\delta_i$$



# Robust planning via reachability analysis

- Define the reachable sets

$$\mathcal{X}_k^{t,\delta}(u) = \left\{ \begin{array}{l} x_k = f(\cdot, u_{k-1}, \theta, \epsilon_{k-1}) \circ \cdots \circ f(x_0, u_0, \theta, \epsilon_0) \\ \text{with } x_0 = x(t), \theta_i \in \mathcal{C}_{i,t}^\delta, \epsilon_{i,s} \in \mathcal{E}_i, \\ s=1, \dots, k-1, i=1, \dots, n \end{array} \right\}$$



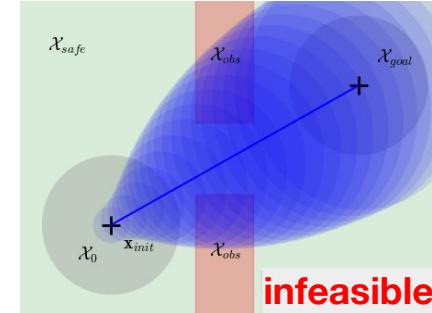
- Define **Reach-OCP(N)**:

$$\min_{\mu, u} \sum_{k=0}^{N-1} \ell(\mu_k, u_k) \text{ s.t. } \begin{aligned} \bigwedge_{k=1}^N \mathcal{X}_k^{t,\delta}(u) &\subseteq \mathcal{X}_{safe}, \quad \mathcal{X}_N^{t,\delta}(u) \subseteq \mathcal{X}_{goal}, \\ \bigwedge_{k=0}^{N-1} u_k &\in \mathcal{U}, \quad \mathcal{X}_0^{t,\delta} = \{x(t)\}, \end{aligned}$$

# If uncertainty is too high: robust exploration

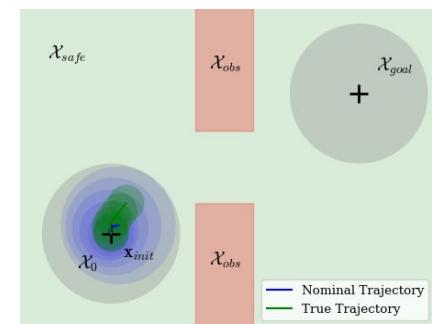
- **Reach-OCP(N):**

$$\min_{\mu, u} \sum_{k=0}^{N-1} \ell(\mu_k, u_k) \text{ s.t. } \begin{aligned} \bigwedge_{k=1}^N \mathcal{X}_k^{t,\delta}(u) &\subseteq \mathcal{X}_{safe}, \quad \mathcal{X}_N^{t,\delta}(u) \subseteq \mathcal{X}_{goal}, \\ \bigwedge_{k=0}^{N-1} u_k &\in \mathcal{U}, \quad \mathcal{X}_0^{t,\delta} = \{x(t)\}, \end{aligned}$$



- **Explore-OCP(N):**

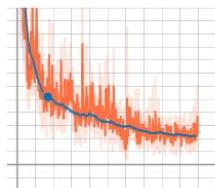
$$\min_{\mu, u} \sum_{k=0}^{N-1} (\ell + \ell_{info})(\mu_k, u_k) \text{ s.t. } \begin{aligned} \bigwedge_{k=1}^N \mathcal{X}_k^{t,\delta}(u) &\subseteq \mathcal{X}_{safe}, \quad \mathcal{X}_N^{t,\delta}(u) \subseteq \underline{\mathcal{X}_0}, \\ \bigwedge_{k=0}^{N-1} u_k &\in \mathcal{U}, \quad \mathcal{X}_0^{t,\delta} = \{x(t)\}, \end{aligned}$$



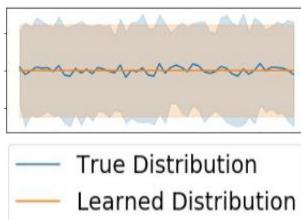
## Offline: meta-train

1. Collect data
2. Train model

$$\theta_i^\top \phi_i(x, u)$$



3. Verify accuracy



$$\theta_i^\top \phi_i(x, u)$$

## Online: exploration - exploitation

1. If **ReachOCP** is **feasible**:

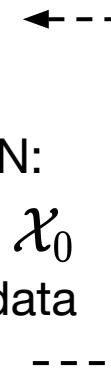
Reach  $\mathcal{X}_{\text{goal}}$

2. Else, for different horizons N:

Solve **ExploreOCP(N)** to  $\mathcal{X}_0$

Apply controls & gather data

Adapt parameters  $\theta_i$



while  
 $x \notin \mathcal{X}_{\text{goal}}$



This approach is a feasible solution to the original problem

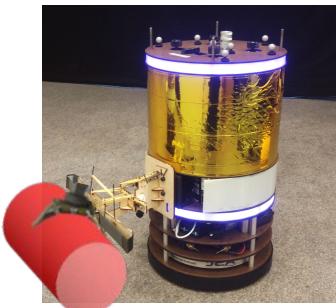
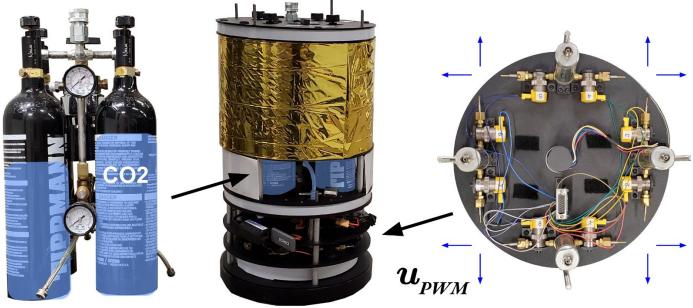
**Theorem:** assume that the  $\epsilon_t$  are  $\sigma_i$ -subGaussian and bounded and that  $g \in \mathcal{H}_{\phi_i} = \left\{ \hat{g}_{i,\theta_i} = \theta_i^\top \phi_i : \|\theta_i - \bar{\theta}_{i,0}\|_{\Lambda_{i,0}}^2 \leq \sigma_i^2 B \right\}$ .

Then, with probability at least  $(1 - \delta)$  :

- each **Explore-OCP** is feasible
- all constraints are always satisfied
- if **Reach-OCP** is eventually feasible, then

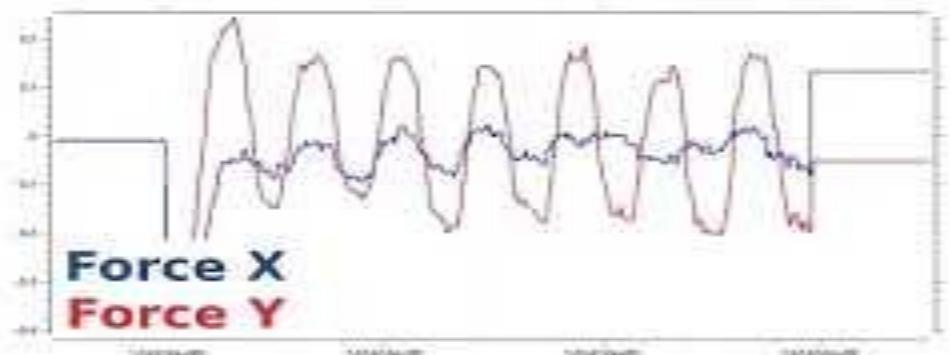
$$\mathbb{P} \left( (x_t \in \mathcal{X} \wedge u_t \in \mathcal{U} \ \forall t), (x_T \in \mathcal{X}_{\text{goal}}) \right) \geq (1 - \delta)$$

# Results on a free-flyer robotic testbed

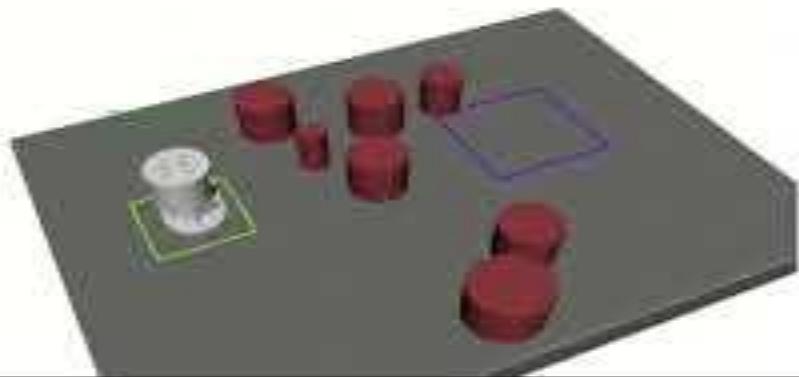


uncertainty:

- CoM offset
- Table tilt (constant force)



Explore & learn  
inertial properties

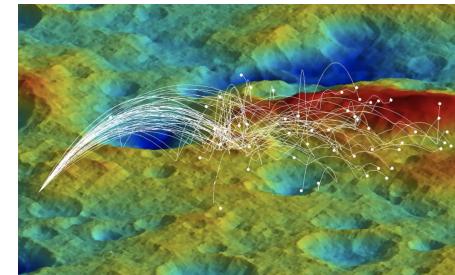


## Conclusion (1st part):

- Need for active learning to safely perform tasks under uncertainty
- Exploit linear model structure to quantify uncertainty
- Open questions:
  - Time to solve the problem? Regret bounds?
  - Efficient computation of the reachable sets accounting for parametric uncertainty?

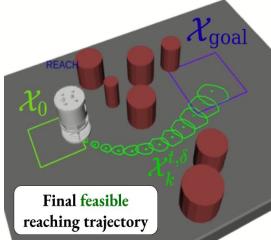
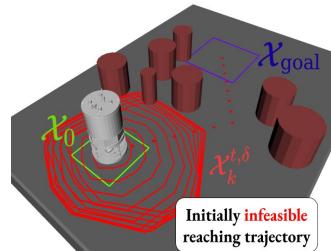
$$\mathcal{X}_k^{t,\delta}(u) = \left\{ \begin{array}{l} x_k = f(\cdot, u_{k-1}, \theta, \epsilon_{k-1}) \circ \cdots \circ f(x_0, u_0, \theta, \epsilon_0) \\ \text{with } x_0 = x(t), \theta_i \in \mathcal{C}_{i,t}^\delta, \epsilon_{i,s} \in \mathcal{E}_i, \\ s=1, \dots, k-1, i=1, \dots, n \end{array} \right\}$$

# Towards safe learning-based robotic autonomy



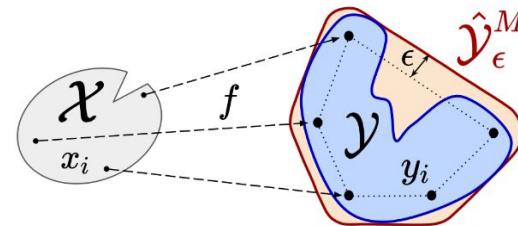
1

safe active dynamics  
learning and control

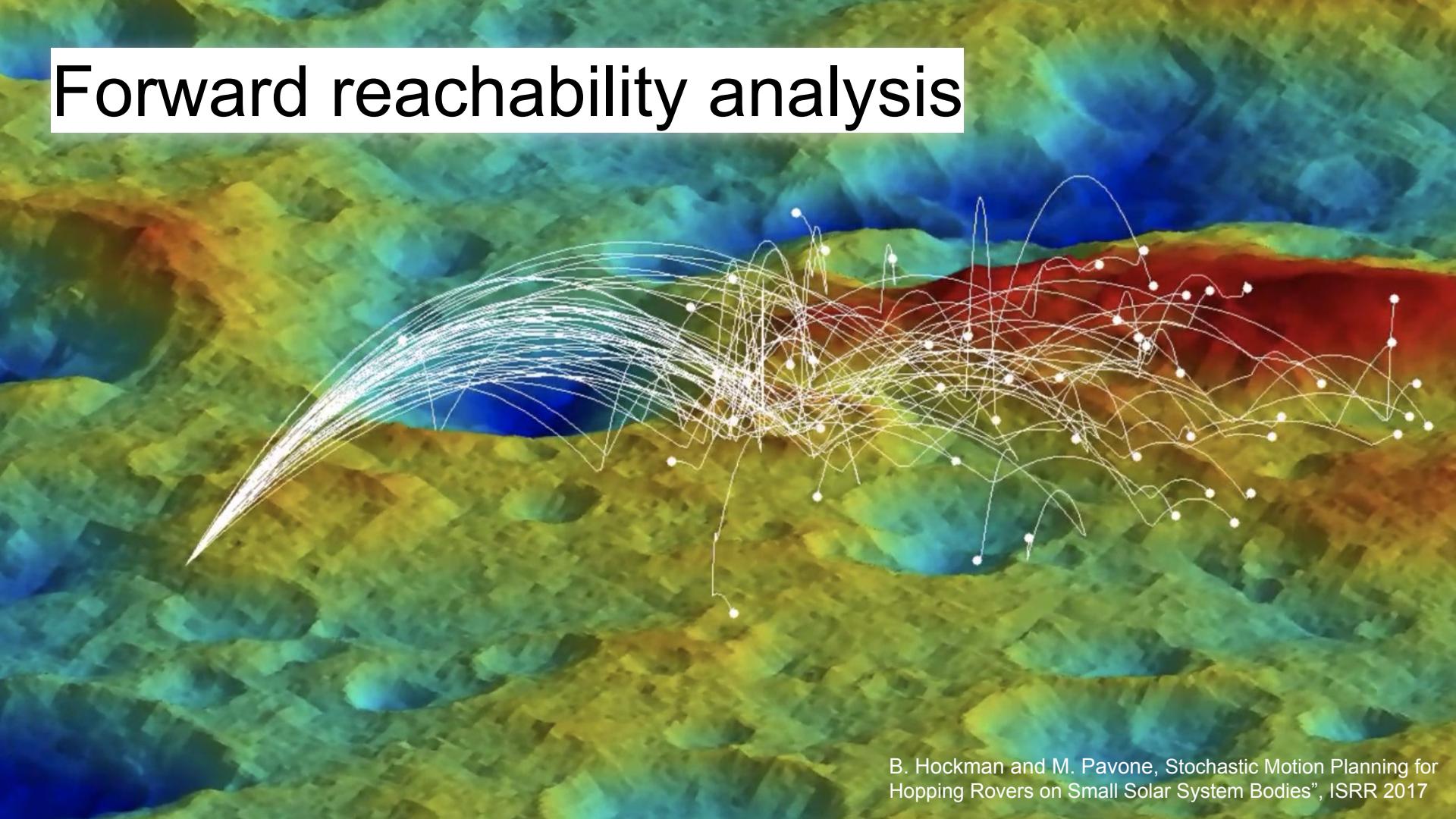


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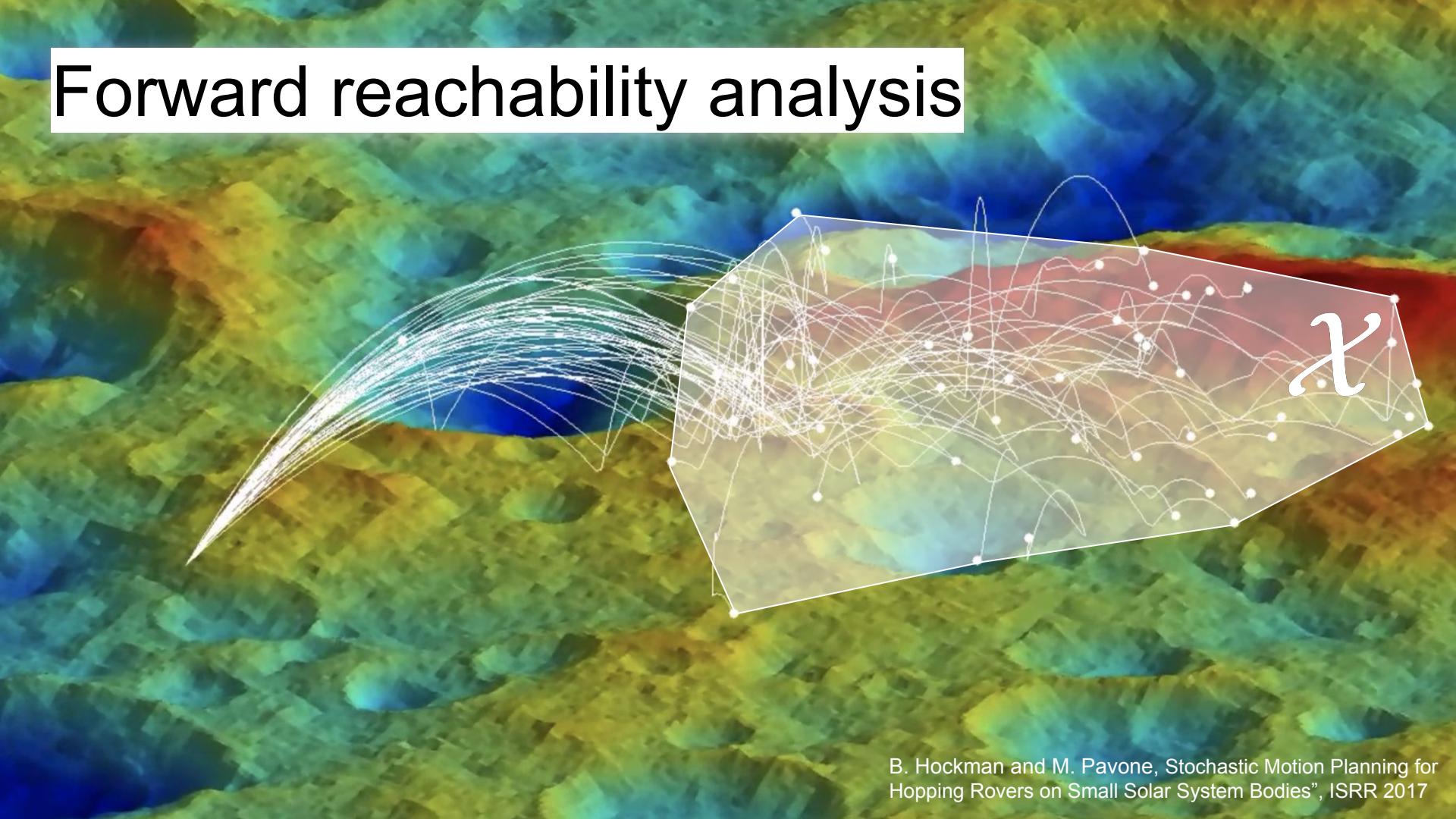
**sampling-based  
reachability analysis**



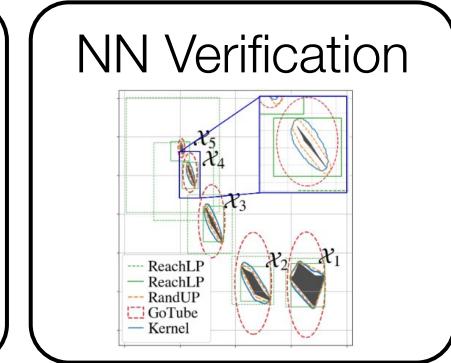
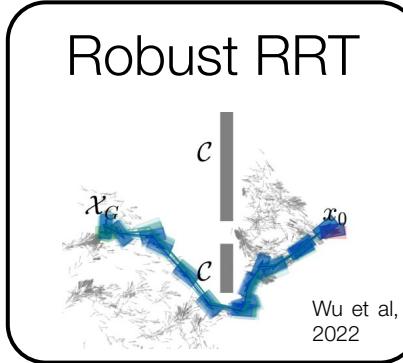
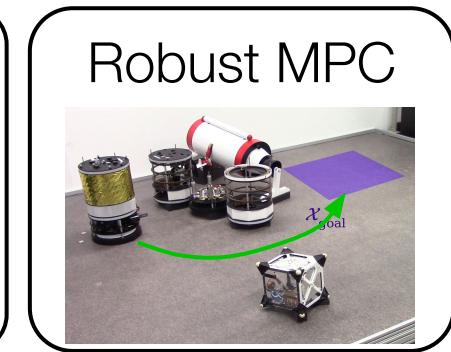
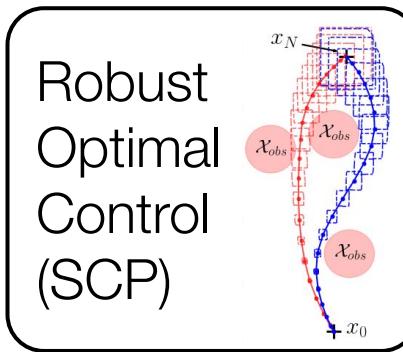
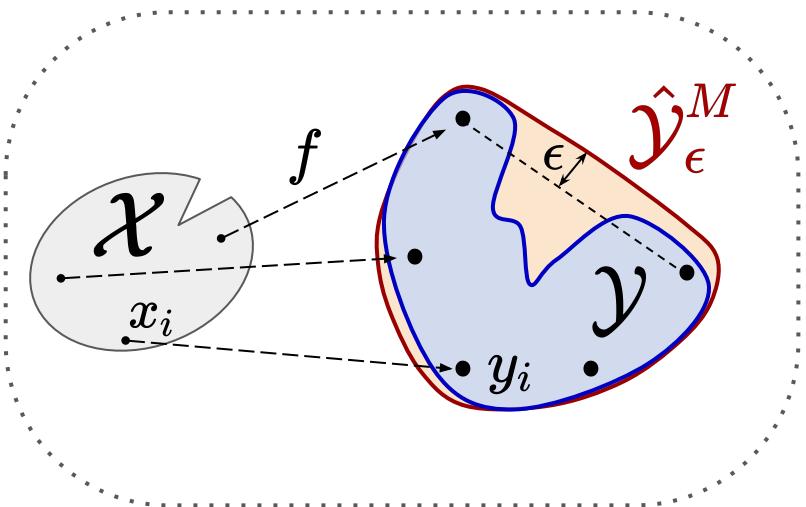
# Forward reachability analysis



# Forward reachability analysis



# Applications of reachability analysis

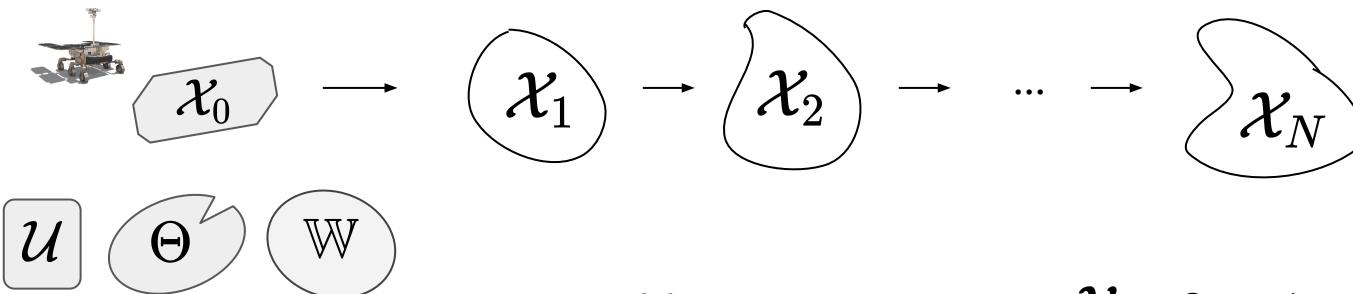


# Forward reachability analysis

$$x_{k+1} = f(x_k, u_k, \theta, w_k)$$

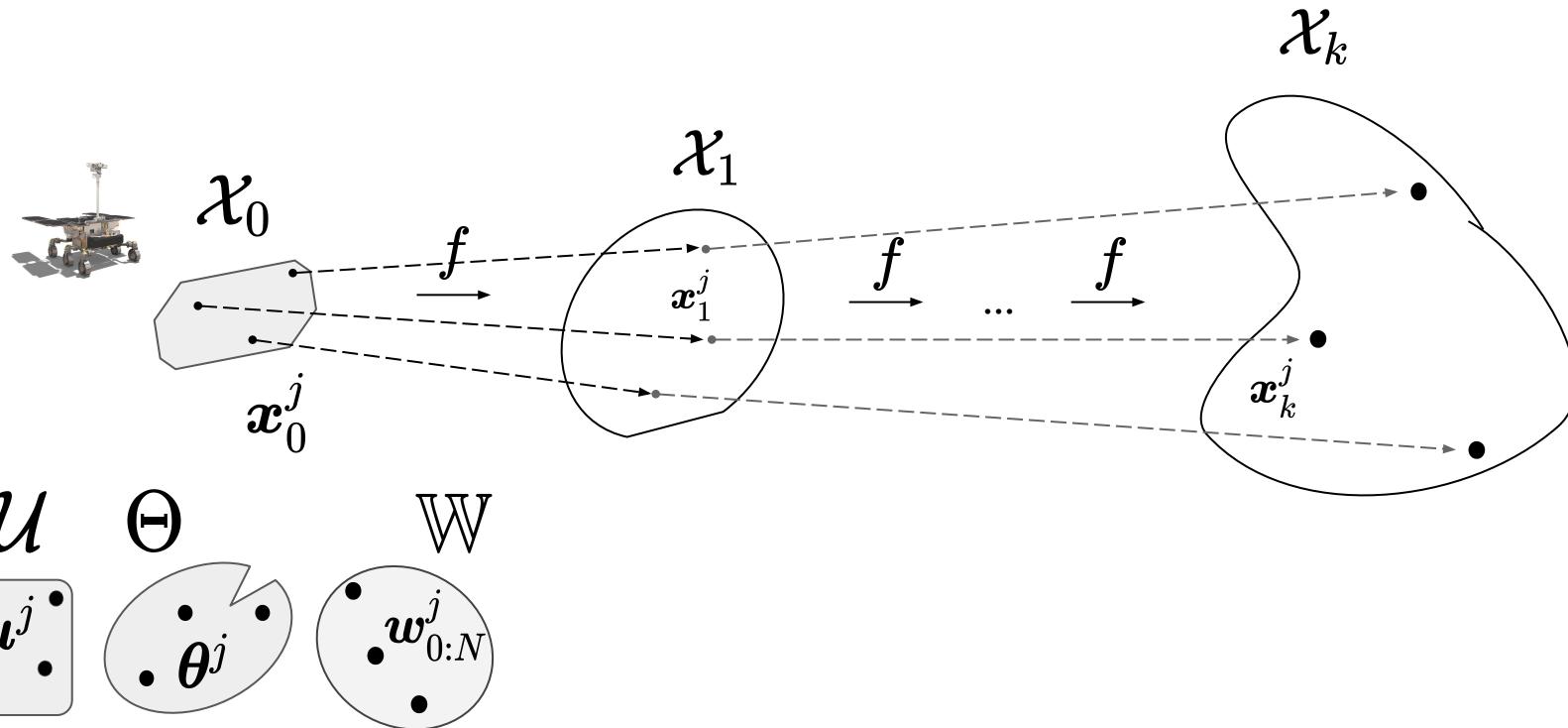
with  $(x_0, u, \theta, w) \in \mathcal{X}_0 \times \mathcal{U} \times \Theta \times \mathcal{W}$  compact

$x$	State
$u$	Control
$\theta$	Parameters
$w$	Disturbances

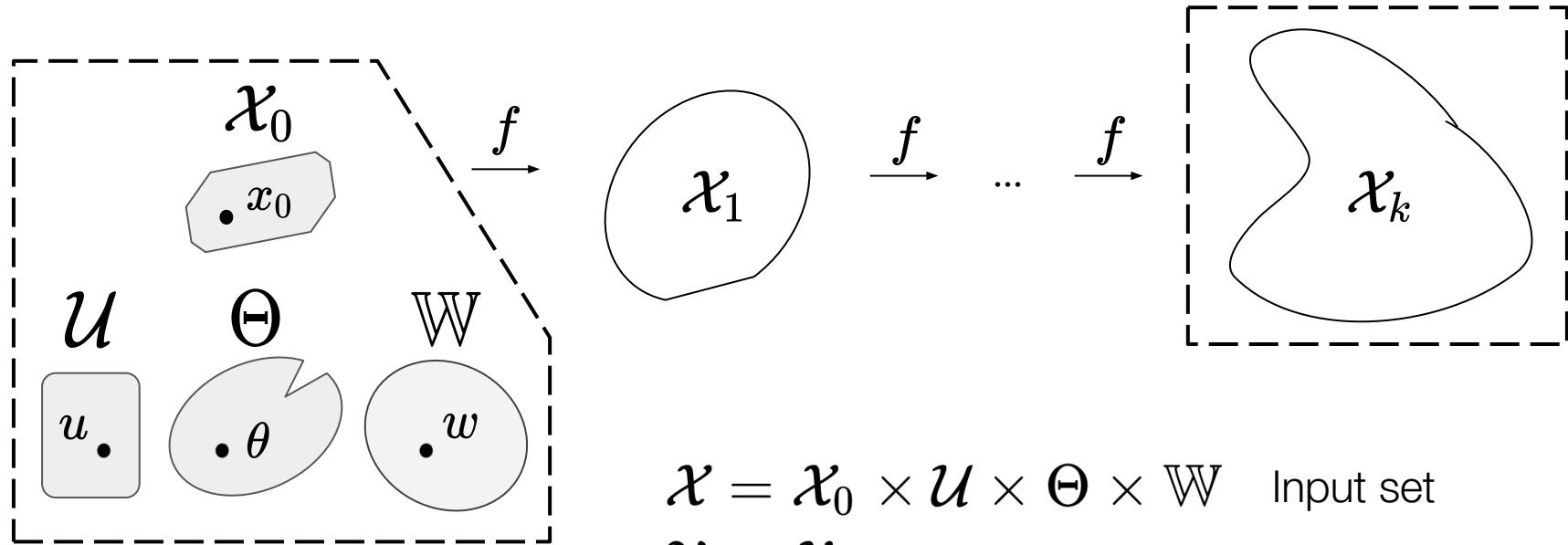


How to compute  $\mathcal{X}_k$  ?  $(k = 1, \dots, N)$

# Sampling-based forward reachability analysis



# General formulation of reachability analysis



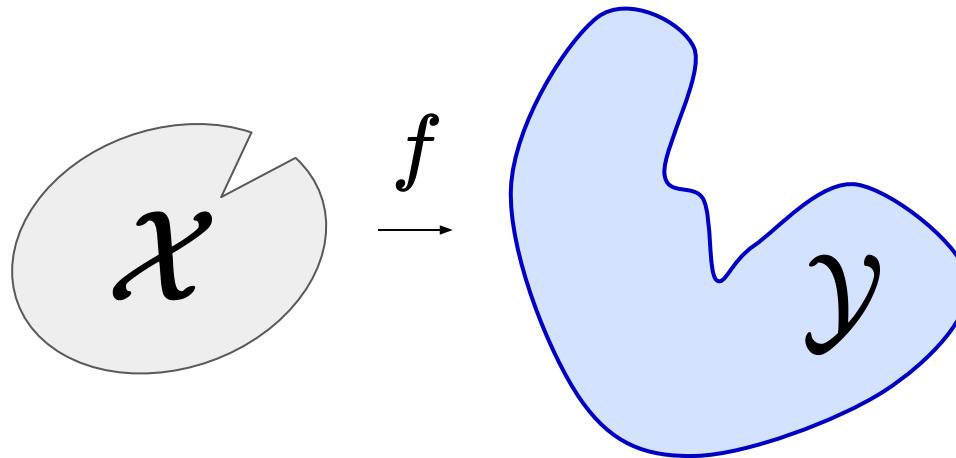
Reachability map:

$$f : \mathcal{X} \rightarrow \mathcal{Y}$$
$$(x_0, u, \theta, w) \mapsto f(\cdot, u_{k-1}, \theta, w_{k-1}) \circ \cdots \circ f(x_0, u_{k-1}, \theta, w_{k-1})$$

# General formulation of reachability analysis

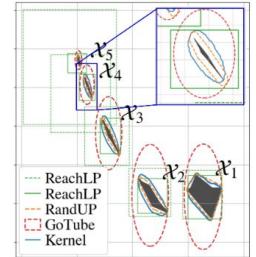
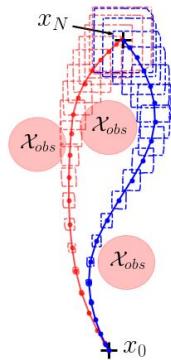
$\mathcal{X} \subset \mathbb{R}^p$  compact reachable set to estimate

$f : \mathbb{R}^p \rightarrow \mathbb{R}^n$  continuous  $\mathcal{Y} = f(\mathcal{X})$



# Related work

- Applications: robust trajectory optimization, robust MPC, ...
  - Convex over-approximations are often sufficient
- Polytopic & ellipsoidal reachable sets for systems with disturbances
  - Problem-specific: linear/polynomial dynamics (Girard, 2005), (Chen et al, 2013), (Althoff et al, 2021)
  - Nonlinear: techniques that use monotonicity or bound linearization error via Lipschitz constants (Koller et al., 2018)
  - Difficult to account for parametric uncertainty: time correlations along state trajectories
- Neural network verification community
  - (Chen et al, 2013), (Weimer et al, 2019), (Fazlyab et al, 2019), (Tran et al, 2019), (Ivanov, 2019), (Everett et al, 2021) developed libraries for NN verification (Flow\*, Verisig, LipSDP)
- How can we compute accurate approximations in milliseconds?
  - Sampling-based approaches (Devonport & Arcak, 2020, Thorpe et al, 2021)



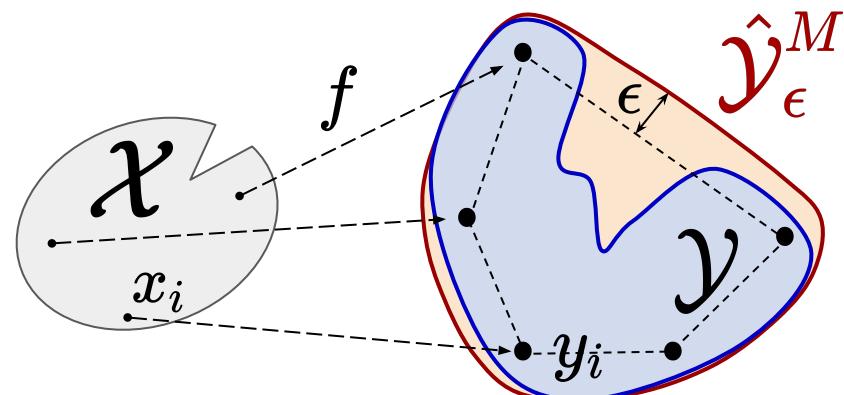
# Sampling-based reachability analysis

$\mathcal{X} \subset \mathbb{R}^p$  compact reachable set to estimate

$f : \mathbb{R}^p \rightarrow \mathbb{R}^n$  continuous  $\mathcal{Y} = f(\mathcal{X})$

3 parameters:

- 1) a number of samples  $M \in \mathbb{N}$
- 2) a padding constant  $\epsilon > 0$
- 3) a probability measure  $\mathbb{P}_{\mathcal{X}}$  on  $(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p))$



# Sampling-based reachability analysis

$$\mathcal{X} \subset \mathbb{R}^p \quad \text{compact}$$

reachable set to estimate

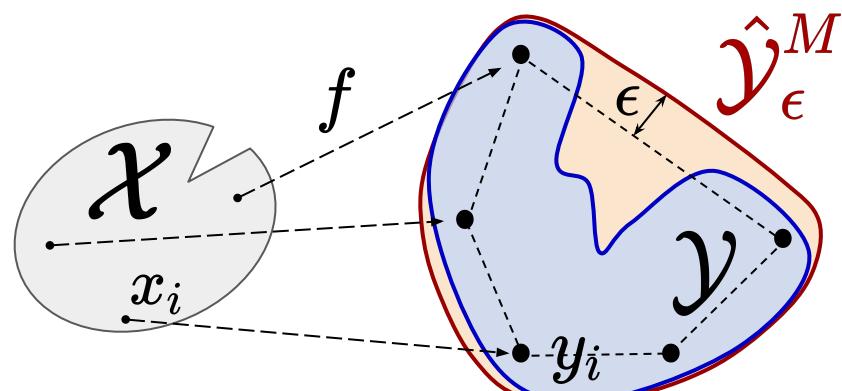
$$f : \mathbb{R}^p \rightarrow \mathbb{R}^n \quad \text{continuous}$$

$$\mathcal{Y} = f(\mathcal{X})$$

3 steps:

- 1) Sample  $M$  inputs  $x_i \sim \mathbb{P}_{\mathcal{X}}$
- 2) Evaluate outputs  $y_i = f(x_i)$
- 3) Evaluate estimator

$$\hat{\mathcal{Y}}_{\epsilon}^M = H(\{y_i\}_{i=1}^M) \oplus B(0, \epsilon)$$



# Sampling-based reachability analysis

$\mathcal{X} \subset \mathbb{R}^p$  compact

reachable set to estimate

$f : \mathbb{R}^p \rightarrow \mathbb{R}^n$  continuous

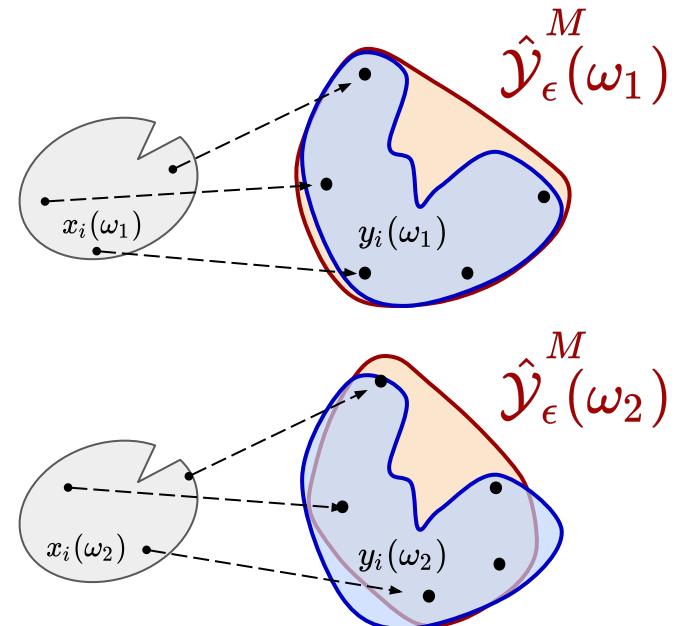
$$\mathcal{Y} = f(\mathcal{X})$$

Let  $(\Omega, \mathcal{G}, \mathbb{P})$  be a probability space

Let  $\mathcal{K}$  denote the family of nonempty compact subsets of  $\mathbb{R}^n$

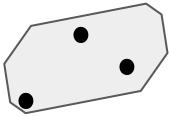
$\hat{\mathcal{Y}}_\epsilon^M : \Omega \rightarrow \mathcal{K}$  is a

**random compact set**

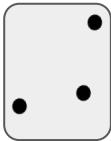


# Asymptotics

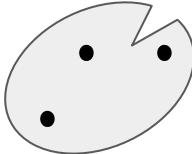
$x_0$



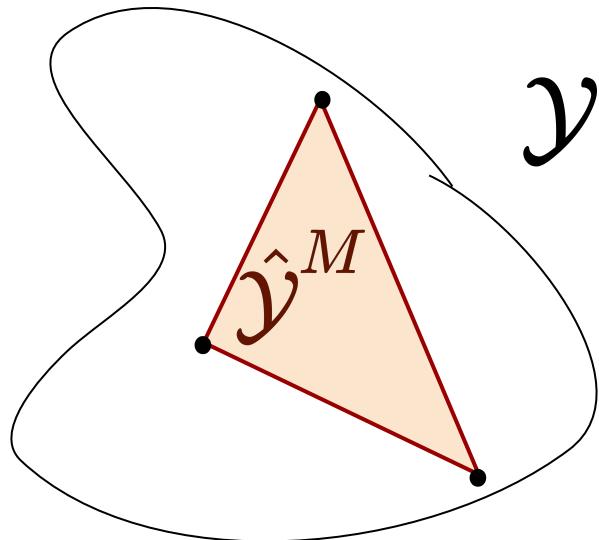
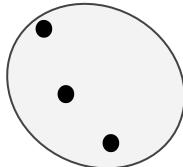
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$\Theta$

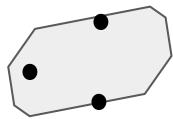


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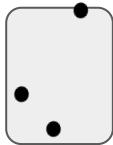


# Asymptotics

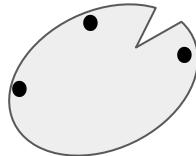
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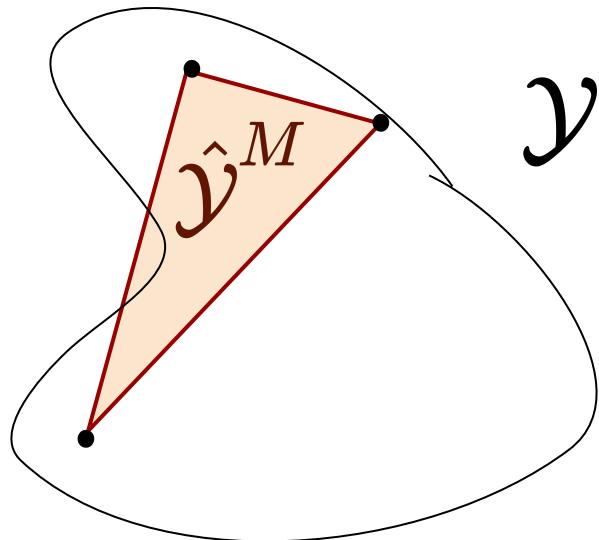
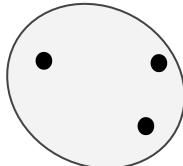
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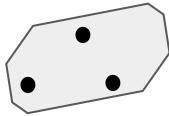


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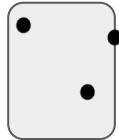


# Asymptotics

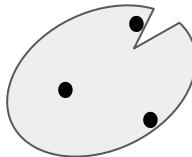
$x_0$



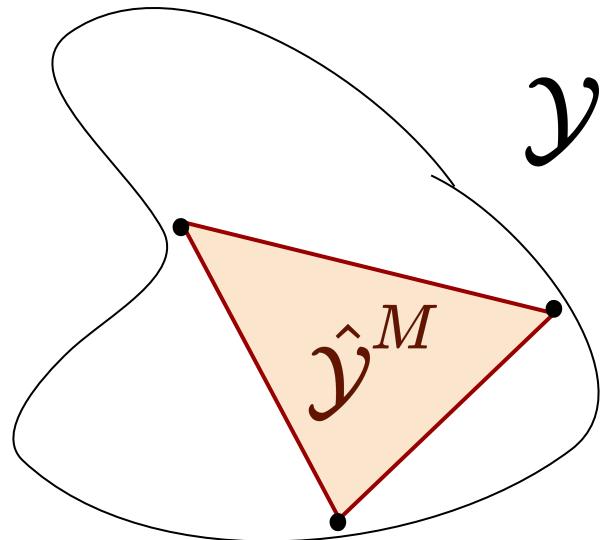
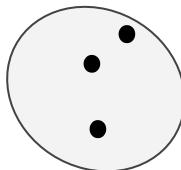
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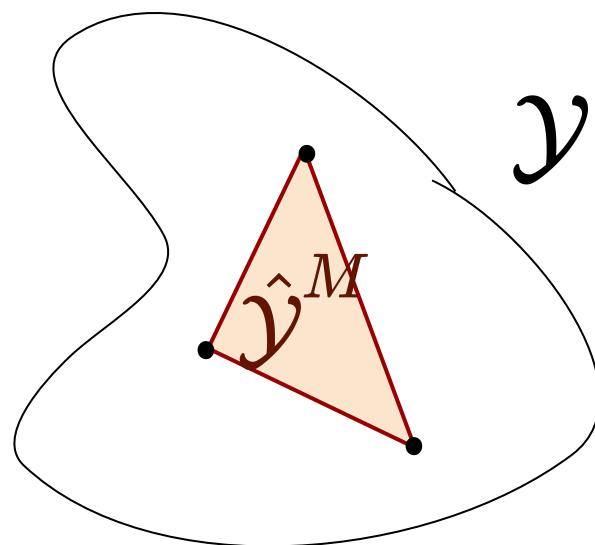
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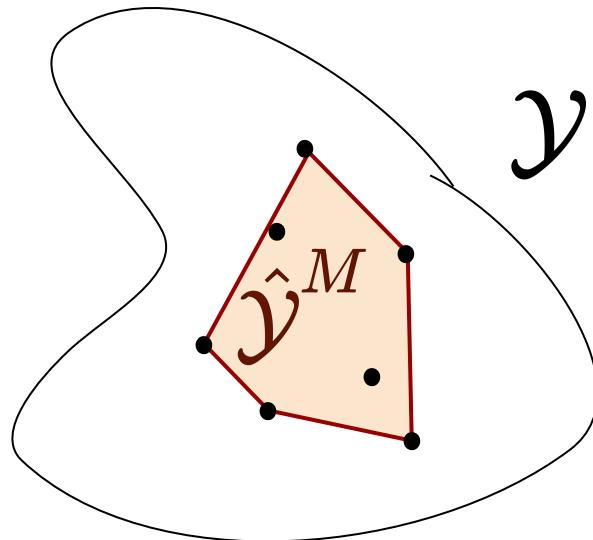
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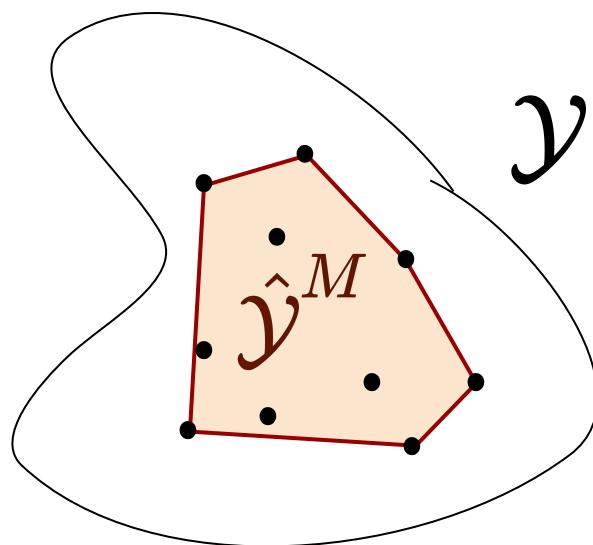
# Asymptotics



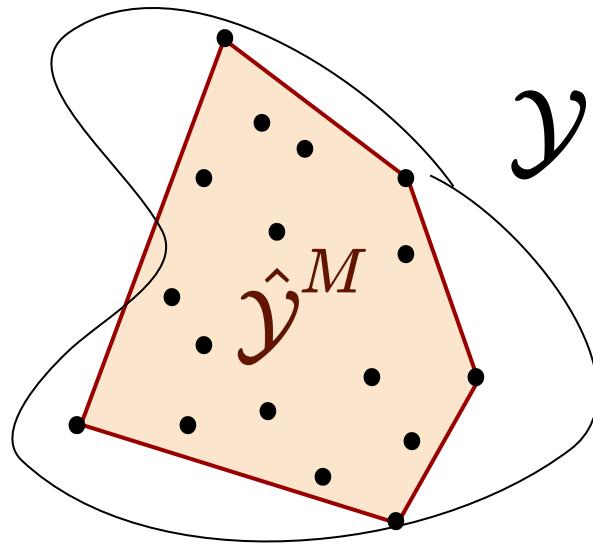
# Asymptotics



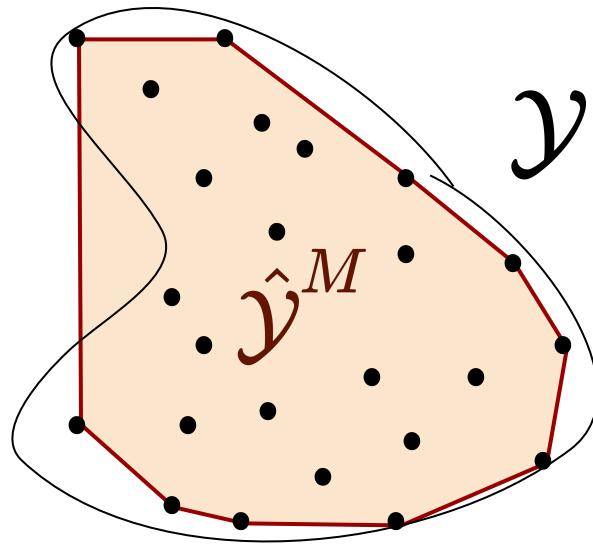
# Asymptotics



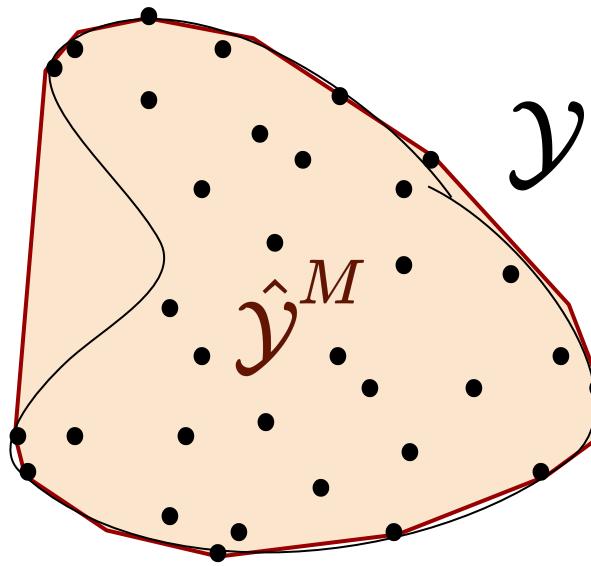
# Asymptotics



# Asymptotics



# Asymptotics

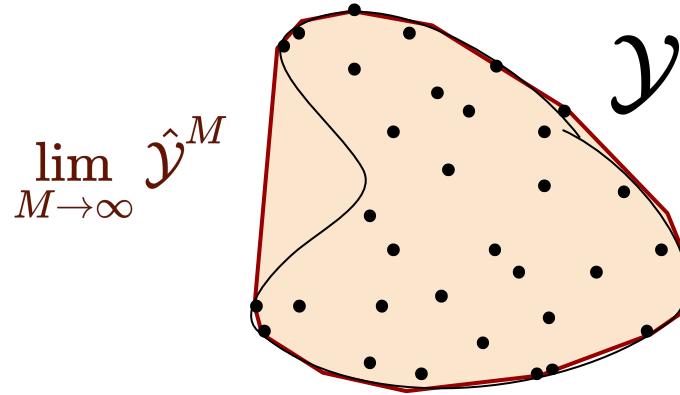


# Characterizing the accuracy of the random compact set estimator

## Asymptotic analysis

$\mathbb{P}$ -almost surely, as  $N \rightarrow \infty$ ,

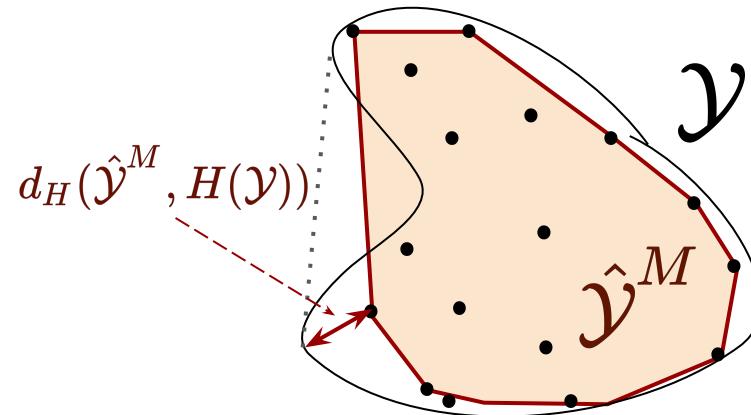
$$d_H(\hat{\mathcal{Y}}_\epsilon^M, H(\mathcal{Y}) \oplus B(0, \bar{\epsilon})) \rightarrow 0$$



## Finite-sample analysis

with  $\mathbb{P}$ -probability at least  $1 - \delta$ ,

$$d_H(\hat{\mathcal{Y}}^M, H(\mathcal{Y})) \leq \epsilon$$



**Theorem:** Let  $\mathcal{Y} \in \mathcal{K}$  and  $(\hat{\mathcal{Y}}^M)_{M \in \mathbb{N}}$  be a seq. of random compact sets.

Assume that

- For any  $K \in \mathcal{K}$  such that  $\mathcal{Y} \cap K = \emptyset$ :

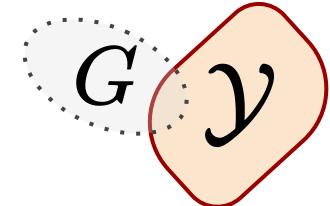
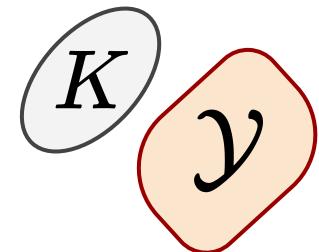
$$\mathbb{P}(\hat{\mathcal{Y}}^M \cap K \neq \emptyset \text{ i.o.}) = \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{M=N}^{\infty} \{\hat{\mathcal{Y}}^M \cap K \neq \emptyset\}\right) = 0.$$

- For any open  $G \subset \mathbb{R}^n$  such that  $\mathcal{Y} \cap G \neq \emptyset$ :

$$\mathbb{P}(\hat{\mathcal{Y}}^M \cap G = \emptyset \text{ i.o.}) = \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{M=N}^{\infty} \{\hat{\mathcal{Y}}^M \cap G = \emptyset\}\right) = 0.$$

Then,  $\mathbb{P}$ -almost surely,

$$d_H(\hat{\mathcal{Y}}^M, \mathcal{Y}) \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$



**Theorem:** Assume that  $f^{-1}(\partial\mathcal{Y}) \subseteq \text{supp}(\mathbb{P}_X)$ .

Then,

- For any  $K \in \mathcal{K}$  such that  $H(\mathcal{Y}) \cap K = \emptyset$ :

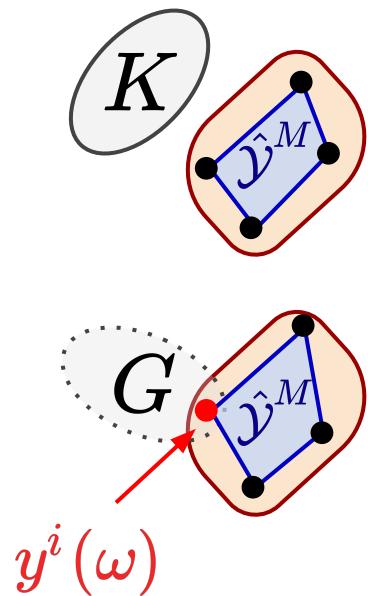
$$\mathbb{P}(\hat{\mathcal{Y}}^M \cap K \neq \emptyset \text{ i.o.}) = \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{M=N}^{\infty} \{\hat{\mathcal{Y}}^M \cap K \neq \emptyset\}\right) = 0.$$

- For any open  $G \subset \mathbb{R}^n$  such that  $H(\mathcal{Y}) \cap G \neq \emptyset$ :

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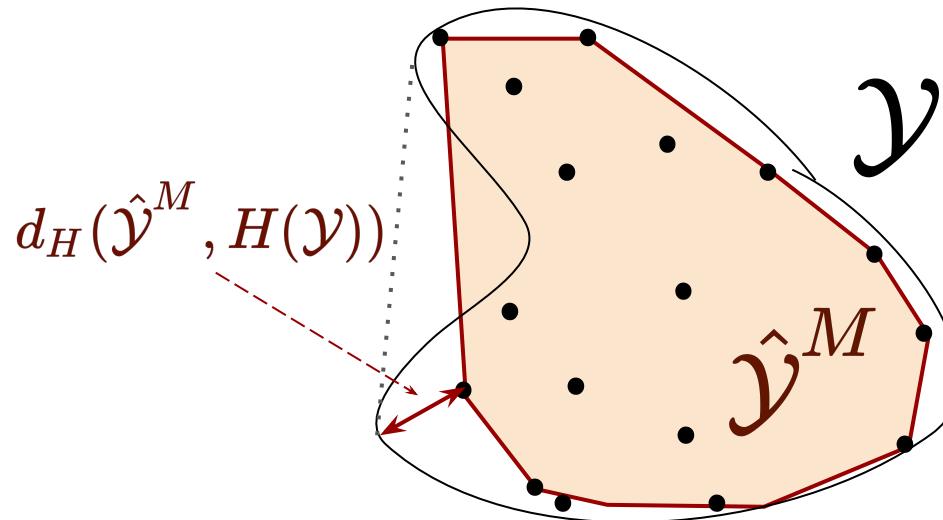
Thus,  $\mathbb{P}$ -almost surely,

$$d_H(\hat{\mathcal{Y}}^M, \mathcal{Y}) \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$



# Finite-sample error bounds

$$\mathbb{P} \left( d_H(\hat{\mathcal{Y}}^M, H(\mathcal{Y})) \leq \epsilon \right) \geq 1 - \delta$$



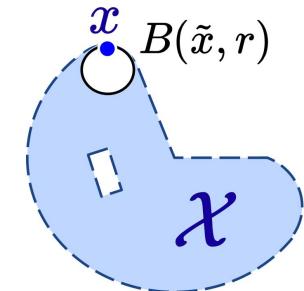
# Finite-sample error bounds

Assume:

$\mathcal{X}^c$  is  $r$ -convex

$f$  is  $L$ -Lipschitz

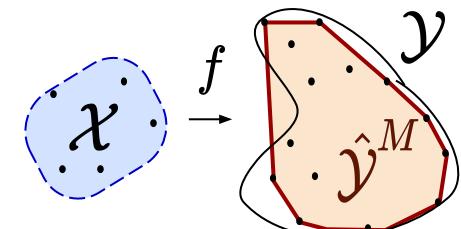
$\mathbb{P}_{\mathcal{X}}(A) \geq p_0 \lambda(A)$  for all  $A \subset \mathcal{X}$



Then,

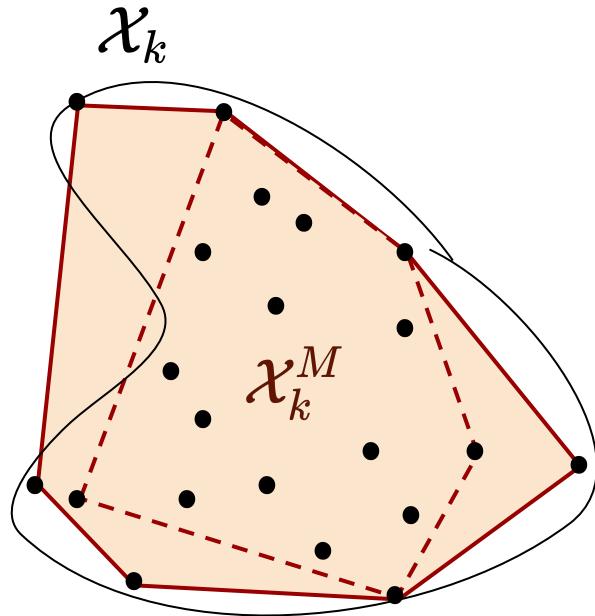
$$\mathbb{P} \left( d_H \left( \hat{\mathcal{Y}}^M, H(\mathcal{Y}) \right) \leq \epsilon \right) \geq 1 - \delta_M$$

with  $\delta_M = D(\mathcal{X}, \epsilon/2L)(1 - p_0 \Lambda_{\epsilon}^{r,L})^M$



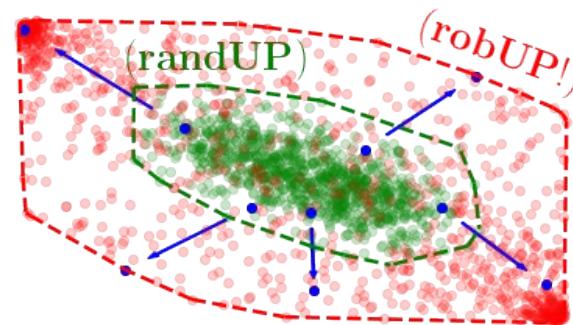
(proof via covering argument. Assumptions can be relaxed along the boundary of the input set.)

# Adversarial Sampling

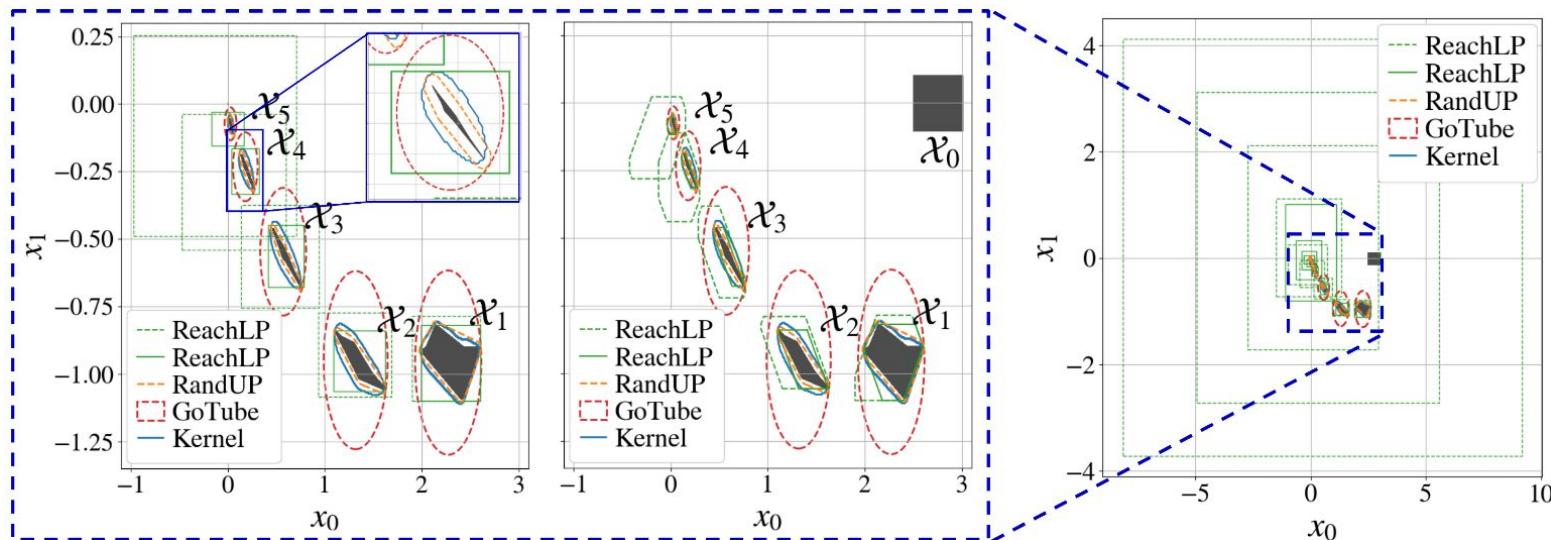
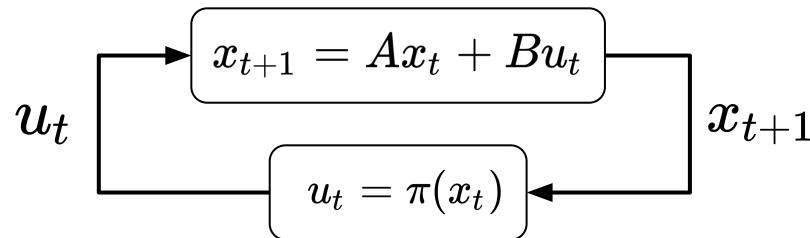


Accuracy depends on samples

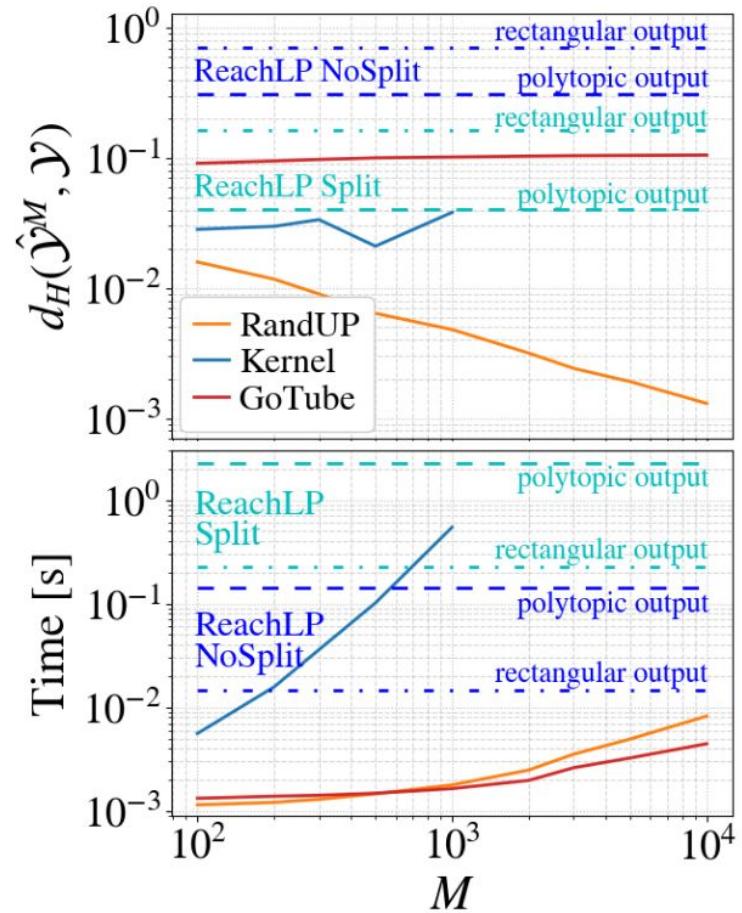
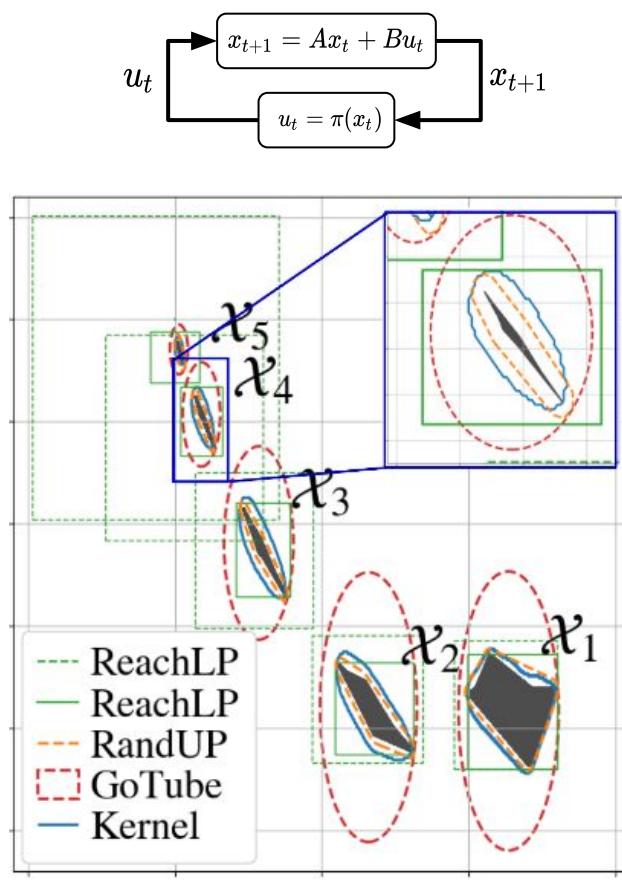
=> **Adversarial sampling**



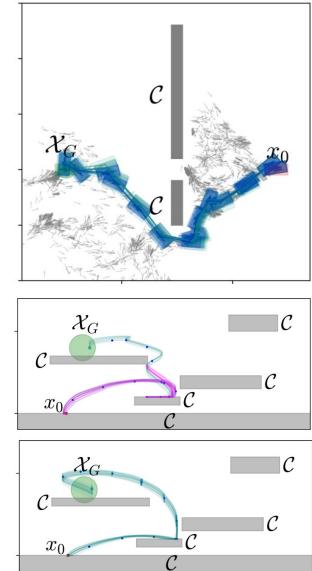
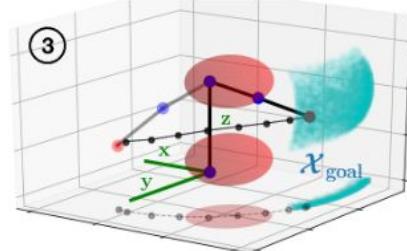
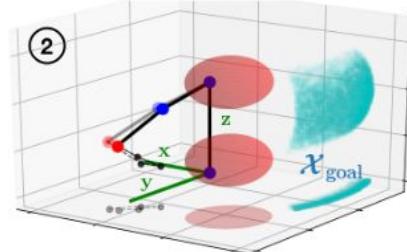
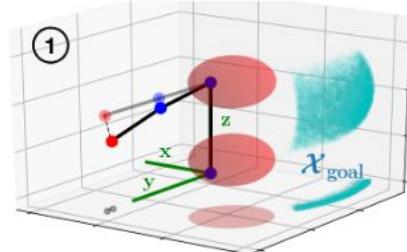
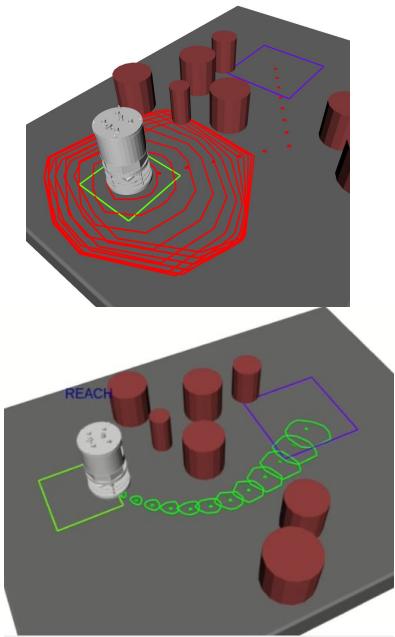
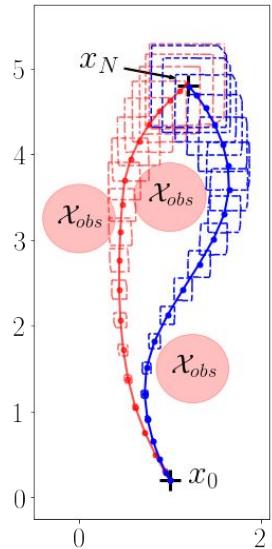
# Results for neural network verification



# Results for neural network verification

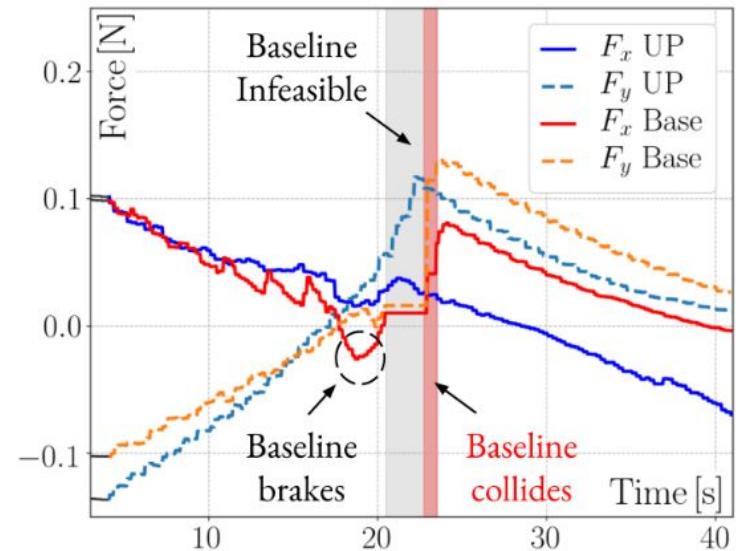
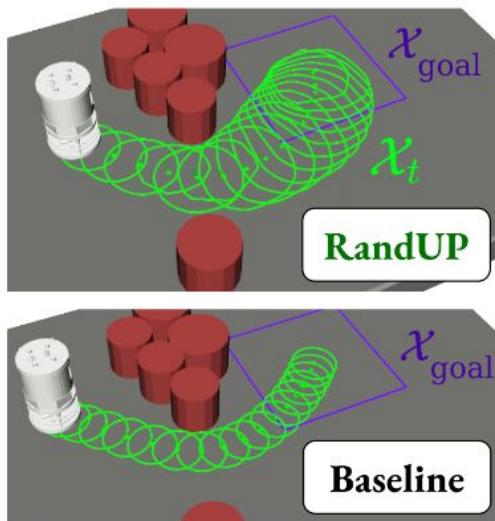
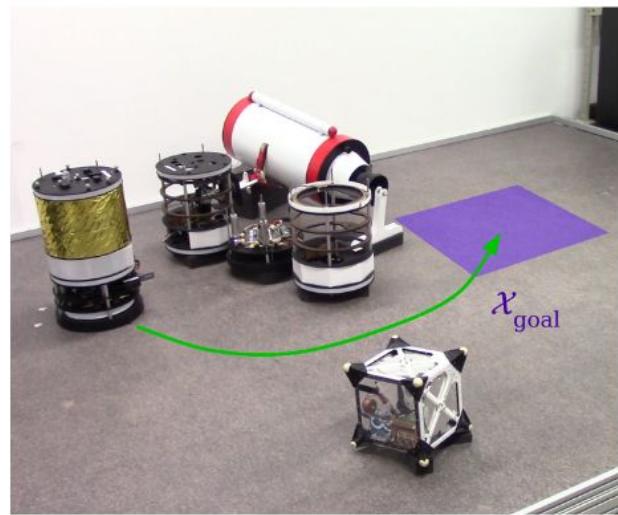


# Applications to robust planning & control

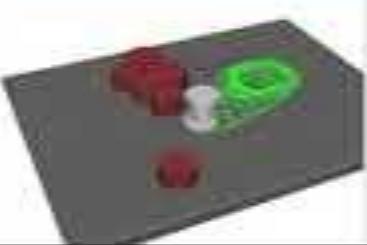


Wu, Lew, Solovey,  
Schmerling, Pavone,  
*Robust-RRT*, 2022  
(to appear)

# Results: robust MPC



**RandUP-MPC**

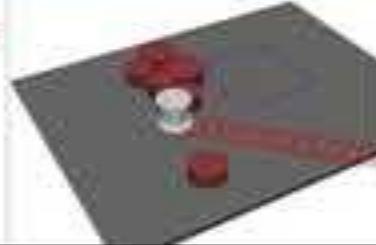


MPC feasible  
constraints  
satisfied

**Baseline-MPC**



MPC infeasible  
constraints  
violated  
obstacle  
collision



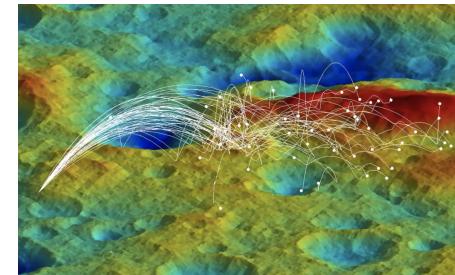
# Sampling-based reachability analysis

- simple + efficient approach for general reachability analysis
- analysis using random set theory
- can be used for uncertainty-aware planning and control
- hardware-tested

## **Future work:**

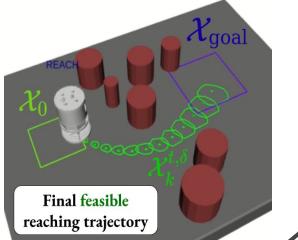
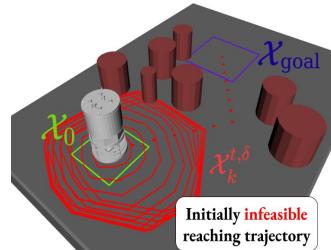
- Tighter rates of convergence
- Parallelization on GPUs

# Towards safe learning-based robotic autonomy



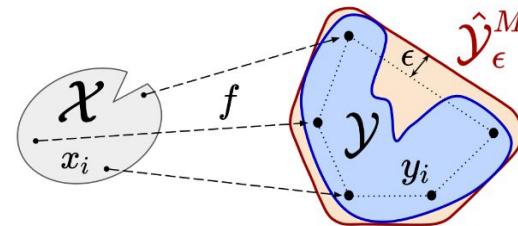
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safe active dynamics  
learning and control

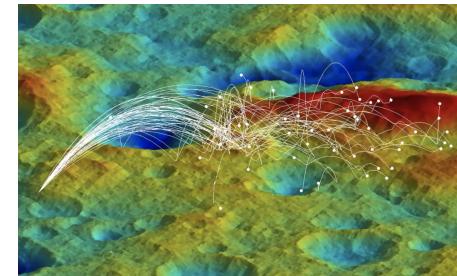


2

sampling-based  
reachability analysis



# Towards safe learning-based robotic autonomy



- Extensions: (1) continuous-time analysis for systems modeled as stochastic differential equations (2) tighter analysis to close the theory-to-practice gap
- New tools to guarantee the performance of complex autonomous systems with learning-in-the-loop components

# Acknowledgements



Marco Pavone



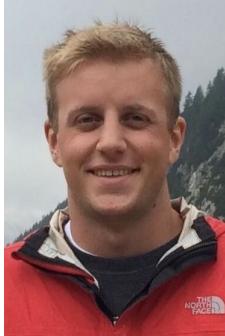
Riccardo Bonalli



Lucas Janson



Apoorva Sharma

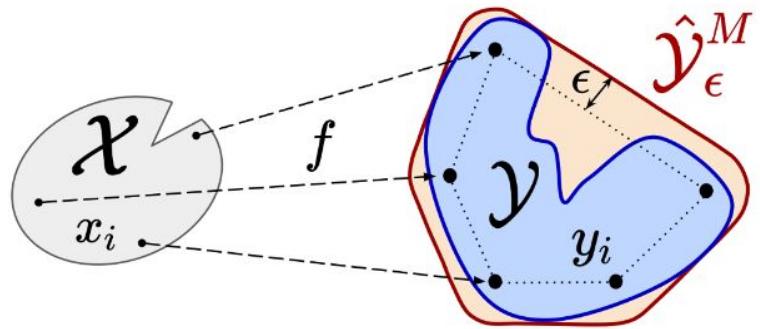


James Harrison



Andrew Bylard





Uncertainty-aware control strategies for  
safe learning-based robotic autonomy



Thomas Lew

