Breaking indecision in multi-agent multi-option dynamics

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Thank you

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Thanks!

A Franci, M Golubitsky, I Stewart, A Bizyaeva, NE Leonard, Breaking indecision in multi-agent, multi-option dynamics, To appear in SIAM J Applied Dynamical Systems

Indecision-breaking in nature, society, and technology

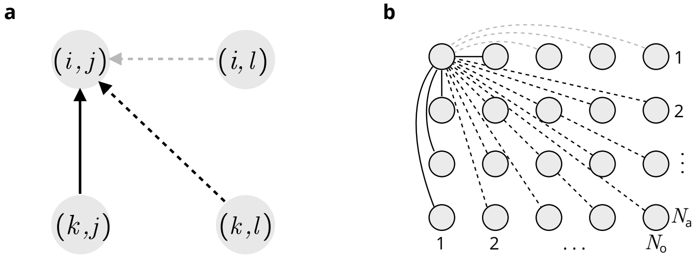
(Keynote)

Contents

Setting-up the mathematical framework

Influence networks

- We consider a set of $N_\mathrm{a} \geq 2$ identical agents who form valued preferences about a set of $N_\mathrm{o} \geq 2$ identical options.
- They do so through an **influence network**, whose connections describe mutual influences between valuations made by all agents.
- The network has $N_\mathrm{a} \times N_\mathrm{o}$ nodes (i, j), $i = 1, \ldots, N_\mathrm{a}$, $j = 1, \ldots, N_\mathrm{o}$, representing the evaluation of agent i about option j



Decision states and value patterns

- The **value** that agent i assigns to option j is represented by a real number z_{ij} which may be positive, negative or zero.

- A **state** of the influence network is a rectangular array of values $Z = (z_{ij})$ where $1 \leq i \leq N_\mathrm{a}$ and $1 \leq j \leq N_\mathrm{o}$.

- $z_{ij} \geq z_{kl}$ means that agent i values option j more than or equal to how agent k values option l.

- In the following, values are **color**-**coded** in the network representation.

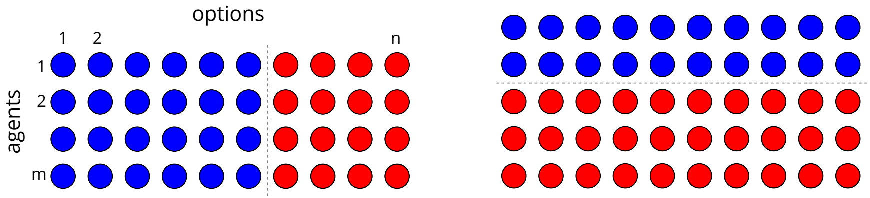
- Nodes z_{ij} and z_{kl} of the array Z have the same color if and only if $z_{ij} = z_{kl}$. Nodes with the same value, hence same color, are **synchronous**.

- A decision state is graphically represented by a **value pattern**.

Decision states and value patterns

Decision states:

- A state where each row consists of equal values is a *deadlock* state. Agent i is **deadlocked** if row i consists of equal values.
- A state where each column consists of equal values is a **consensus** state. There is *consensus about option j* if column j consists of equal values.



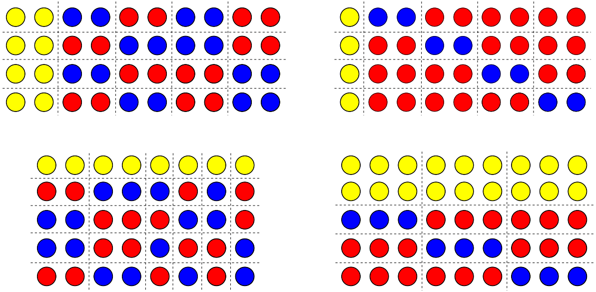
Deadlock

Consensus

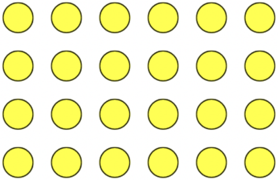
Decision states and value patterns

Decision states:

- A state that is neither deadlock nor consensus is **dissensus**.
- A state that is both deadlock and consensus is a state of **indecision**. An indecision state is **fully synchronous**, that is, in such a state all values are equal.



Dissensus



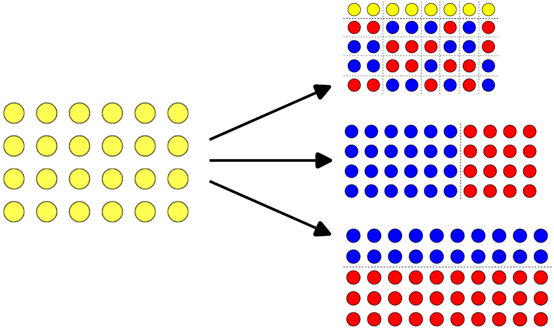
Indecision

Opinion formation as spontaneous synchrony breaking of symmetric influence networks

Opinion formation as symmetric synchrony breaking

We ask:

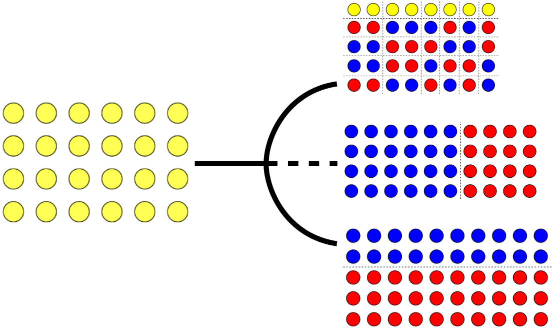
- How are value pattern formed from full indecision?



Opinion formation as symmetric synchrony breaking

We ask:

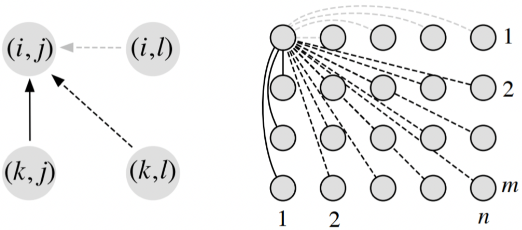
- How are value pattern formed from full indecision?
- We propose **spontaneous synchrony breaking** as a universal mechanism for valued opinion formation.



The value influence network

- 1 cell type (nodes are of the same kind)
- 4 arrow types
- All-to-all interconnection for all arrow types

- (A) intra-agent, inter-option
- (O) intra-option, inter-agent
- (E) inter-agent, inter-option
- internal (not shown)



Symmetries of the influence network

- The influence network $\mathcal{N}_{mn}$ is symmetric: its **automorphism group**, the set of permutations of the set of nodes $\mathcal{C}$ that preserve node and arrow types and incidence relations between nodes and arrows, is non-trivial.

- It is straightforward to prove that $\mathcal{N}_{mn}$ has symmetry group

$$\Gamma = \mathbf{S}_{N_\mathrm{a}} \times \mathbf{S}_{N_\mathrm{o}}$$

  where $\mathbf{S}_{N_\mathrm{a}}$ swaps rows (agents) and $\mathbf{S}_{N_\mathrm{o}}$ swaps columns (options).

- More precisely, $\sigma \in \mathbf{S}_{N_\mathrm{a}}$ and $\tau \in \mathbf{S}_{N_\mathrm{o}}$ act on $\mathcal{C}$ by

$$(\sigma, \tau)(i, j) = (\sigma(i), \tau(j)).$$

State space and vector fields over influence networks

- A point in the **state space** (or **phase space**) $\mathbb{P}$ is an $N_\mathrm{a} \times N_\mathrm{o}$ rectangular array of real numbers $z = (z_{ij})$.
- The action of Γ on points $z = (z_{ij}) \in \mathbb{P}$ is defined by

$$(\sigma, \tau)z = (z_{\sigma^{-1}(i)\tau^{-1}(j)}).$$

- Let

$$\boldsymbol{G} : \mathbb{P} \to \mathbb{P}$$

  be a smooth map $\boldsymbol{G} = (G_{ij})$ on $\mathbb{P}$, so that each component G_{ij} is smooth and real-valued.
- We assume that the value z_{ij} assigned by agent i to option j evolves according to the *value dynamics*

$$\dot{z}_{ij} = G_{ij}(z). \tag{1}$$

Network admissible vector field

- The map $\boldsymbol{G}$ for the influence network is assumed to be **admissible** [7, 13] for $\mathcal{N}_{mn}$, that is, it respects the network structure and its symmetries.

- E.g., if two nodes $(k_1, l_1), (k_2, l_2)$ input a third one (i, j) through the same arrow type, then $\boldsymbol{G}_{ij}(z)$ depends identically on $z_{k_1 l_1}$ and $z_{k_2 l_2}$.
  Meaning: the value of $\boldsymbol{G}_{ij}(z)$ does not change if $z_{k_1 l_1}$ and $z_{k_2 l_2}$ are swapped in the vector z.

- The components of the admissible map $\boldsymbol{G}$ in (1) satisfy

$$G_{ij}(z) = G(z_{ij}, \overline{z_{\mathsf{A_{ij}}}}, \overline{z_{\mathsf{O_{ij}}}}, \overline{z_{\mathsf{E_{ij}}}})$$

  where the function G is independent of i, j and the notation $\overline{z_{\mathsf{A_{ij}}}}$, $\overline{z_{\mathsf{O_{ij}}}}$, and $\overline{z_{\mathsf{E_{ij}}}}$ means that G is invariant under all permutations of the arguments appearing under each overbar.

- That is, each arrow type leads to identical interactions.

Symmetries of the admissible vector field

- It is straightforward to see that the admissible vector field is Γ-**equivariant**:
  Given $\gamma = (\sigma, \tau) \in \Gamma$, with $\sigma \in \mathbf{S}_{N_\mathrm{a}}$ and $\tau \in \mathbf{S}_{N_\mathrm{o}}$,

$$
\begin{aligned}
((\sigma,\tau)\boldsymbol{G})_{ij}(z) &= G_{\sigma^{-1}(i)\,\tau^{-1}(j)}(z) \\
&= G(z_{\sigma^{-1}(i)\,\tau^{-1}(j)}, \overline{z_{\mathrm{A}_{\sigma^{-1}(\mathrm{i})\,\tau^{-1}(\mathrm{j})}}}, \overline{z_{\mathrm{O}_{\sigma^{-1}(\mathrm{i})\,\tau^{-1}(\mathrm{j})}}}, \overline{z_{\mathrm{E}_{\sigma^{-1}(\mathrm{i})\,\tau^{-1}(\mathrm{j})}}}) \\
&= G_{ij}((\sigma,\tau)z).
\end{aligned}
$$

- This is a well-known general result [1]: all admissible maps are equivariant (with respect to the network symmetries).

- The converse is not true [12]: not all equivariant maps are admissible. That is, **network structure constraints dynamics more than symmetry**.

Equivariant vs admissible maps

- Indeed [12, Theorem 3]:

Theorem

The equivariant maps for $\mathbf{S}_m \times \mathbf{S}_n$ are the same as the admissible maps for $\mathcal{N}_{mn}$ if and only if $(m, n) = (m, 1), (1, n), (2, 2), (2, 3), (3, 2)$.

- However (proof in a few minutes):

Theorem

The **linear** *equivariant maps for $\mathbf{S}_m \times \mathbf{S}_n$ are the same as the linear admissible maps for $\mathcal{N}_{mn}$ for all m, n.*

Γ-irreducible representations

- A representation (linear subspace) $V \subset \mathbb{P}$ is Γ-**irreducible** if the only group-invariant subspaces of V are V itself and 0.

- The four irreducible representation of Γ on $\mathbb{P}$ are

  - V_{s} = all entries equal (*fully synchronous subspace*)
  - V_{c} = all rows identical with sum 0 (*consensus subspace*)
  - V_{dl} = all columns identical with sum 0 (*deadlock subspace*)
  - V_{d} = all rows and all columns have sum 0 (*dissensus subspace*)

  with dimensions $1, (N_{\mathrm{o}}-1)(N_{\mathrm{a}}-1), N_{\mathrm{o}}-1, N_{\mathrm{a}}-1$, respectively.

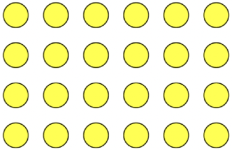
- The kernels of these representations are $\Gamma, \mathbf{S}_{N_{\mathrm{a}}} \times \mathbf{1}, \mathbf{1} \times \mathbf{S}_{N_{\mathrm{o}}}, \mathbf{1}$, respectively. Since the kernels are unequal the representations are non-isomorphic and (by counting dimensions)

$$\mathbb{R}^{N_{\mathrm{a}} N_{\mathrm{o}}} = V_{\mathrm{s}} \oplus V_{\mathrm{c}} \oplus V_{\mathrm{dl}} \oplus V_{\mathrm{d}}.$$

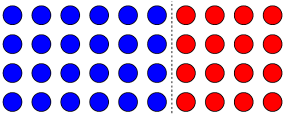
  is the **isotopic decomposition** of $\mathbb{P}$ with respect to Γ.

Γ-irreducible representations

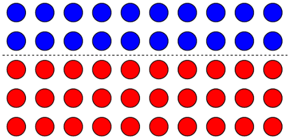
V_{s}



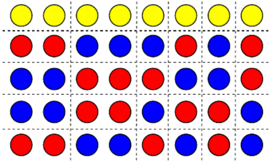
V_{c}



V_{dl}



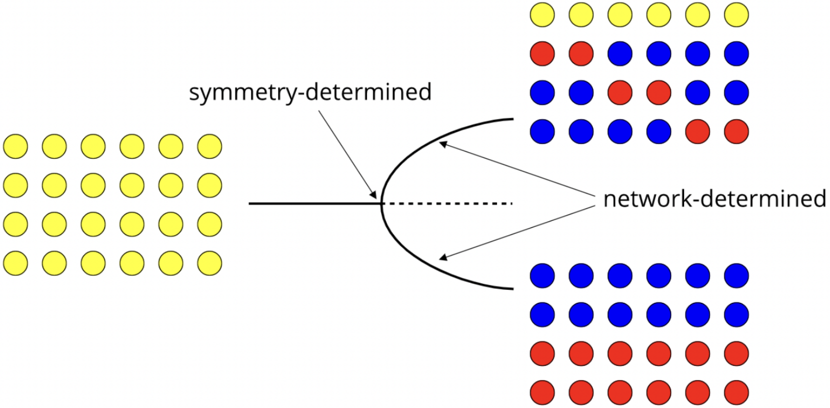
V_{d}



Proofs of linear equivariant of $\mathcal{N}_{nm}$ = linear admissible of $\mathcal{N}_{nm}$

- Hence, the dimension of the space of linear equivariant of $\mathcal{N}_{nm}$ is four because linear equivariant restricted to each irrep are multiple of the identity in that irrep.
- The dimension of the space of the linear admissible is also four because there are four types of arrows.
- Hence the two spaces coincides.

Consequences for opinion formation I



Consequences for opinion formation II

- Because the Jacobian J of any admissible vector field $\boldsymbol{G}$ is a linear admissible,

$$\begin{aligned} J|_{V_\mathrm{d}} &= c_\mathrm{d} I_{(N_\mathrm{a}-1)(N_\mathrm{o}-1)} \\ J|_{V_\mathrm{c}} &= c_\mathrm{c} I_{N_\mathrm{o}-1} \\ J|_{V_\mathrm{dl}} &= c_\mathrm{dl} I_{N_\mathrm{a}-1} \\ J|_{V_\mathrm{s}} &= c_\mathrm{s} \end{aligned}$$

where the four eigenvalues $c_\mathrm{d}, c_\mathrm{c}, c_\mathrm{dl}, c_\mathrm{s}$ are expressed as

$$\begin{aligned} c_\mathrm{d} &= \alpha - \beta - \gamma + \delta \\ c_\mathrm{c} &= \alpha - \beta + (N_\mathrm{a}-1)(\gamma - \delta) \\ c_\mathrm{dl} &= \alpha - \gamma + (N_\mathrm{o}-1)(\beta - \delta) \\ c_\mathrm{s} &= \alpha + (N_\mathrm{o}-1)\beta + (N_\mathrm{a}-1)\gamma + (N_\mathrm{a}-1)(N_\mathrm{o}-1)\delta\,. \end{aligned}$$

with

$$\alpha = \frac{\partial G_{ij}}{\partial z_{ij}} \ \text{(same-agent, same-option)}, \quad \beta = \frac{\partial G_{ij}}{\partial z_{il}} \ \text{(same-agent, different-option)},$$

$$\gamma = \frac{\partial G_{ij}}{\partial z_{kj}} \ \text{(different-agent, same-option)}, \quad \delta = \frac{\partial G_{ij}}{\partial z_{kl}} \ \text{(different-agent, different-option)}.$$

- Thus, the four eigenvalues can be set independently by changing the network arrow weights $\alpha, \beta, \gamma, \delta$.

Consequences for opinion formation III

- When an eigenvalue is zero a bifurcation along the associated irreducible representation happens:
  - When $c_{\mathrm{d}} = 0$, a dissensus bifurcation happens along V_{d}.
  - When $c_{\mathrm{c}} = 0$, a consensus bifurcation happens along V_{c}.
  - When $c_{\mathrm{dl}} = 0$, a deadlock bifurcation happens along V_{dl}.
  - When $c_{\mathrm{s}} = 0$, a bifurcation happens along V_{s}, which does not lead to value patterning (not studied further).
- Because the four eigenvalues can be set independently, any bifurcation can happen.

The parameterized opinion dynamics

- To study opinion formation as bifurcation, we introduce a **bifurcation parameter** $\lambda \in \mathbb{R}$ to define the parametrized dynamics

$$\dot{\boldsymbol{Z}} = \boldsymbol{G}(\boldsymbol{Z}, \lambda)$$
$$G_{ij}(\boldsymbol{Z}, \lambda) = G(z_{ij}, \overline{z_{\mathsf{A_{ij}}}}, \overline{z_{\mathsf{O_{ij}}}}, \overline{z_{\mathsf{E_{ij}}}}, \lambda)\,.$$

- By admissibility, and hence symmetry, for all $\gamma \in \Gamma$ and all $\lambda \in \mathbb{R}$,

$$\gamma \boldsymbol{G}(\boldsymbol{Z}, \lambda) = \boldsymbol{G}(\gamma \boldsymbol{Z}, \lambda)\,.$$

- Interpretations of λ: strength of agent interactions, attention, time or environmental pressure, urgency.

Bifurcations of symmetric opinion networks I: consensus and deadlock

Consensus and deadlock bifurcation

- Because the faithful actions of Γ on consensus and deadlock space are that of $\mathbf{S}_n$ and $\mathbf{S}_m$ respectively, the structure of consensus and deadlock bifurcations follow from well-known **equivariant branching lemma** [8] and are well-known.
- Consensus bifurcations are $\mathbf{S}_n$-equivariant bifurcations.

Theorem

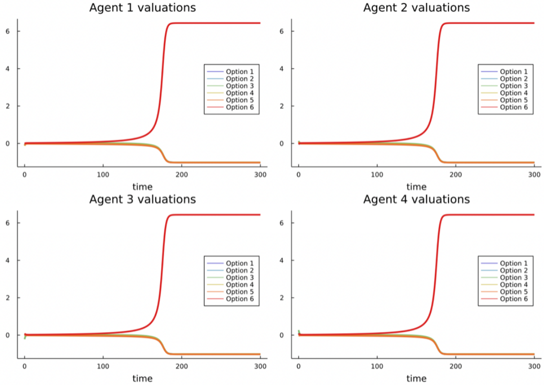
Suppose $c_{\mathrm{c}} = 0$ *for* $\lambda = \lambda_{\mathrm{c}}^*$*. Generically, there is a branch of equilibria corresponding to the axial subgroup* $\mathbf{1} \times (\mathbf{S}_k \times \mathbf{S}_{N_{\mathrm{o}}-k})$ *for all* $1 \leq k \leq N_{\mathrm{o}} - 1$*. These solution branches are tangent to* V_{c} *at* $\lambda = \lambda_{\mathrm{c}}^*$*, lie in the subspace* $\mathrm{Fix}(\mathbf{S}_{N_{\mathrm{a}}} \times \mathbf{1}) = V_{\mathrm{c}} \oplus V_{\mathrm{s}}$*, and are consensus solutions.*

- Deadlock bifurcations are $\mathbf{S}_m$-equivariant bifurcations.

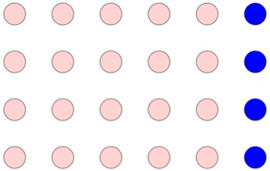
Theorem

Suppose $c_{\mathrm{dl}} = 0$ *for* $\lambda = \lambda_{\mathrm{dl}}^*$*. Generically, there is a branch of equilibria corresponding to the axial subgroup* $(\mathbf{S}_k \times \mathbf{S}_{N_{\mathrm{a}}-k}) \times \mathbf{1}$ *for* $1 \leq k \leq N_{\mathrm{a}} - 1$*. These solution branches are tangent to* V_{dl} *at* $\lambda = \lambda_{\mathrm{dl}}^*$*, lie in the subspace* $\mathrm{Fix}(\mathbf{1} \times \mathbf{S}_n) = V_{\mathrm{dl}} \oplus V_{\mathrm{s}}$*, and are deadlock solutions.*

Simulation of consensus opinion formation

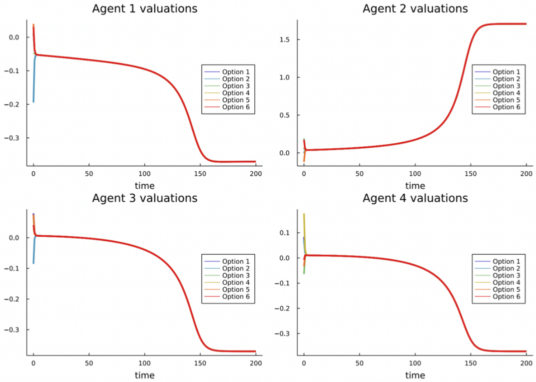


(a) Evolution of valuations by 4 agents on 6 options at consensus symmetry-breaking. Simulation of (10.1) with parameters (11.1).

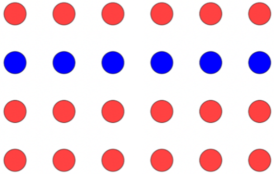


(b) Final simulation pattern of valuations of 4 agents on 6 options at consensus symmetry-breaking. Option 6 is chosen.

Simulation of deadlock opinion formation



(a) Evolution of valuation by 4 agents on 6 options at deadlock symmetry-breaking. Simulation of (10.1) with parameters (11.2).



(b) Final simulated value pattern by 4 agents on 6 options at deadlock symmetry-breaking. Agent 2 values all the options positively. Other agents value all the options negatively.

Bifurcations of symmetric opinion networks II: dissensus

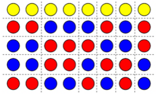
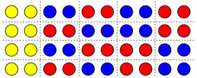
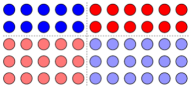
The synchrony-breaking branching lemma

- Because on dissensus space V_{d} the full symmetry group Γ acts faithfully and equivariant vector fields of Γ-symmetric networks are not the same as admissible vector field, the equivariant branching lemma cannot be applied to study dissensus opinion formation.
- Golubitsky and Stewart thus developed a network version of the equivariant branching lemma called the **Synchrony-breaking branching lemma** [7].
- Using this lemma, we were able to show the following.

Dissensus bifurcations

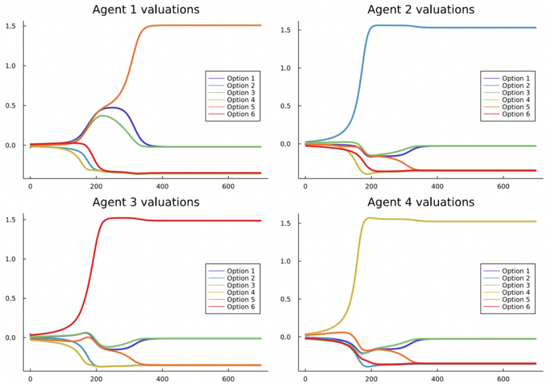
Theorem

Suppose $c_{\mathrm{d}} = 0$ *for* $\lambda = \lambda_{\mathrm{d}}^*$. *Generically, there is a branch of equilibria tangent to* V_{s} *at* $\lambda = \lambda_{\mathrm{d}}^*$ *made of dissensus solutions. Furthermore, modulo reordering columns or rows, only the following kinds of dissensus solutions can emerge at a dissensus bifurcation:*

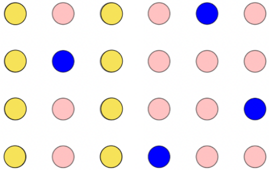
-  The yellow block might be empty
-  The yellow block might be empty
- 

positive value   negative value   ~neutral value

Simulation of dissensus opinion formation 1

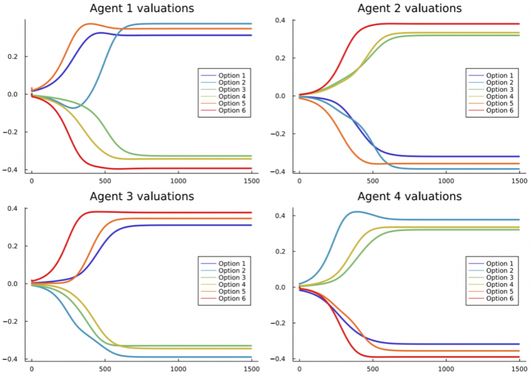


(a) Evolution of 4 agents' valuations about 6 options at dissensus symmetry-breaking with parameter set (11.3).

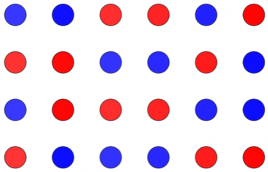


(b) Final value pattern of 4 agents' valuations about 6 options at dissensus symmetry-breaking with parameter set (11.3). All agents are neutral about Options 1 and 3. Agent 1 prefers Option 5. Agent 2 prefers Option 2. Agent 3 prefers Option 6. Agent 4 prefers Option 4.

Simulation of dissensus opinion formation 2



(a) Evolution of 4 agents' valuations about 6 options at dissensus symmetry-breaking with parameter set (11.4).



(b) Final value pattern of 4 agents' valuations about 6 options at dissensus symmetry-breaking with parameter set (11.4). Agent 1 prefers Options 1,2,5. Agent 2 prefers Options 3,4,6. Agent 3 prefers Options 1,5,6. Agent 4 prefers Options 2,3,4.

A general decision-making model and applications

A general model of decision-making

- All the studied phenomena can computationally be studied and generalized in the model [2]

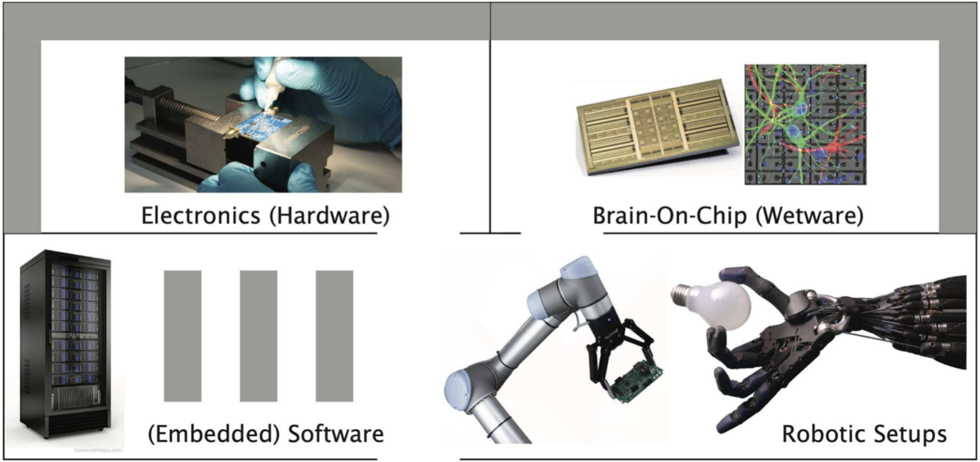
$$\dot{z}_{ij} = -d_i z_{ij} + S\left(u_i \sum_{kl} A_{ik}^{jl} z_{kl}\right) + b_i$$

- General interconnection topologies.
- Time-varying and feedback-controlled parameters.
- Rich behavior:
  - Opinion cascades and changes of mind [4, 6]
  - Tunable sensitivity to inputs
  - Tunable decision speed
  - Oscillations [3]
- Suitable for real-time embodied (robotics, neuromorphic) applications.

Applications

- Sensorimotor control for autonomous agents [10, 5]
- Political polarization [11]
- Bio-inspired control [9]
- Neuroscience (in progress)
- Neuromorphic Engineering (in progress)

A little advertisement: Neuromorphic Engineering Lab @ULiège



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