Convex liftings in control design. Connections with inverse optimality and path planning

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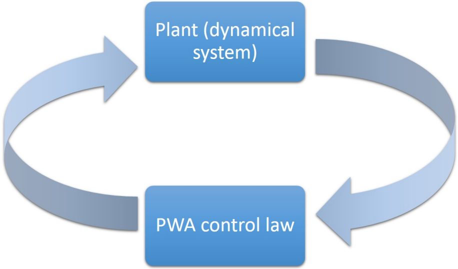
November 17, 2022

Outline

Motivation

Emergence of Piecewise Affine (PWA) formulations in control design



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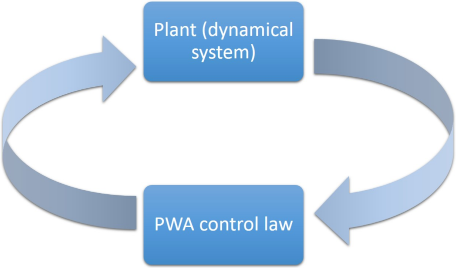

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- PWA controllers

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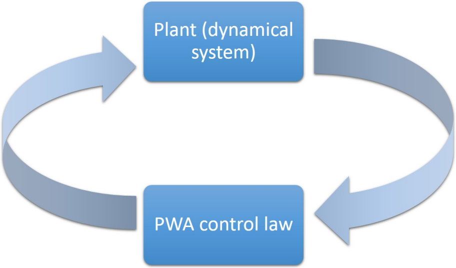

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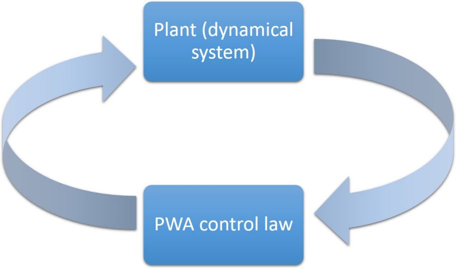

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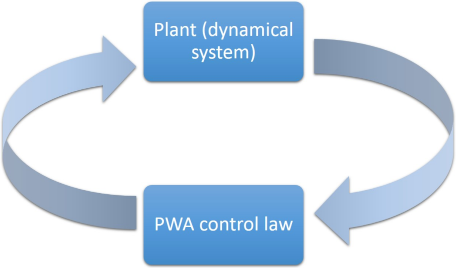

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Illustrative problem: constrained MPC

Minimization (or minimax in the robust case) of a finite-time optimal control performance index in the presence of input, state constraints and using a PWA prediction model:

$$J(\mathbf{u}_k, x_k) = \sum_{i=0}^{N-1} \left\{ \|Qx_{k+i|k}\|_{1/2/\infty} + \|Ru_{k+i|k}\|_{1/2/\infty} \right\} + \|Px_{k+N|k}\|_{1/2/\infty}$$

s.t.

$$\begin{array}{l} x_{k+1} = g_{pwa}(x_k, u_k, w_k) \\ x_{k+i|k} \in \mathbb{X},\ u_{k+i|k} \in \mathbb{U} \text{ for } i = 0 \ldots N-1 \\ x_{k+N|k} \in \mathbb{X}_T \end{array}$$

Attractive features:

- piece-wise convex formulation
- finite dimensional optimization over $\mathbf{u}_k = [u_k, \ldots, u_{k+N-1}]$
- piece-wise affine dependence of the optimum on parameters

Explicit constrained control formulations (MPC)

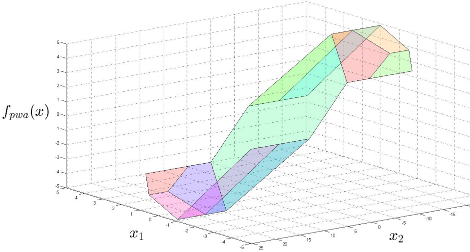
For polyhedral constraints, and linear prediction model, the MPC design leads to:

(Implicit) Parametric optimization:

$$\mathbf{u}^*(x) = \arg\min_{\mathbf{u}} \mathbf{u}^T H \mathbf{u} + (x^T F + C)\mathbf{u}$$
$$\text{s.t. } G\mathbf{u} \leq Ex + W$$

Explicit solution:

$$\mathbf{u}^*(x) = f_{pwa} : \bigcup_{i=1}^{N} \mathcal{X}_i \subset \mathbb{R}^{d_x} \longrightarrow \mathbb{R}^{d_u}$$
$$x \longmapsto f_i^T x + g_i \text{ for } x \in \mathcal{X}_i$$



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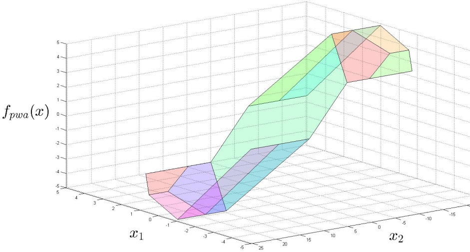
Explicit vs. implicit MPC formulation:

- avoid iterative search
- numerous regions to store
- difficult point-location problem



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Explicit vs. implicit MPC formulation:

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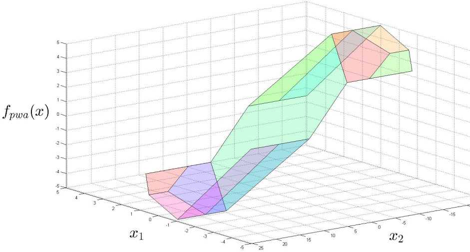
- difficult point-location problem

Our objective: revert to implicit MPC but use the minimal number of optimization arguments.



Further motivations: inverse optimality of PWA control

Recalling the inverse optimality studies:

R. E. Kalman. When is a linear control system optimal? Trans. ASME J., 1964.

When Is a Linear Control System Optimal?

R. E. KALMAN
Research Institute for Advanced Studies (RIAS), Baltimore, Md.

The purpose of this paper is to formulate, study, and (in certain cases) resolve the Inverse Problem of Optimal Control Theory, which is the following: Given a control law, find all performance indices for which this control law is optimal.

Under the assumptions of (a) linear constant plant, (b) linear constant control law, (c) measurable state variables, (d) quadratic loss functions with constant coefficients, (e) single control variable, we give a complete analysis of this problem and obtain various explicit conditions for the optimality of a given control law. An interesting feature of the analysis is the central role of frequency-domain concepts, which have been ignored in optimal control theory until very recently. The discussion is presented in rigorous mathematical form.

The central conclusion is the following (Theorem 6): A stable control law is optimal if and only if the absolute value of the corresponding return difference is at least equal to one at all frequencies. This provides a beautifully simple connecting link between modern control theory and the classical point of view which regards feedback as a means of reducing component variations.

Further motivations: inverse optimality of PWA control

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Our objective: answer (constructively) to the questions:

- "when is a PWA (control) function optimal?"
- "what (convex) optimal control formulation?"

The mathematical formalism - to fix the ideas

Polyhedral partition

A collection of $N \in \mathbb{N}_+$ full-dimensional polytopes $\mathcal{X}_i \subset \mathbb{R}^d$ is called a *polyhedral partition* if:

1. $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i$ is a compact set in $\mathbb{R}^d$.
2. $\mathrm{int}(\mathcal{X}_i) \bigcap \mathrm{int}(\mathcal{X}_j) = \emptyset$ with $i \neq j,\ (i,j) \in \mathcal{I}_N^2$,

Also, $(\mathcal{X}_i, \mathcal{X}_j)$ are called neighbours if $(i,j) \in \mathcal{I}_N^2$, $i \neq j$ and $\dim(\mathcal{X}_i \cap \mathcal{X}_j) = d - 1$.

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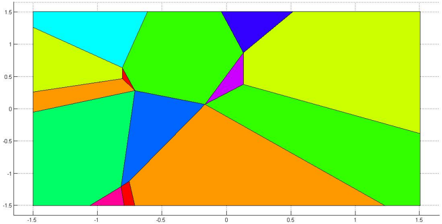


Problem statement – inverse optimal PWA control

Given a polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^{d_x}$ and a continuous piecewise affine function $f_{pwa} : \mathcal{X} \to \mathbb{R}^{d_u}$, find

- A convex cost function: $J(x, u, z)$,
- A set of convex constraints describing the feasible domain by the pair of matrices H_x, H_u, H_z, K

such that

$$f_{pwa}(x) = Proj_{\mathbb{R}^{d_u}} \arg \min_{[u^T \quad z]^T} J(x, u, z),$$
$$\text{subject to} \quad H_u u + H_x x + H_z z \leq K.$$

Several bad news and some hope

The following constructions are not valid:

- cost functions involving the PWA control:

$$\min_{u} \|u - f_{pwa}(x)\|$$

- convex cost and convex $h(x)$ coupled with the PWA constraints:

$$\min_{u,z} z \text{ subject to } \{z \geq h(x); u = f_{pwa}(x)\}$$

A good news (existence):

M. Baes, M. Diehl, and I. Necoara. "Every continuous nonlinear control system can be obtained by parametric convex programming." IEEE Transactions on Automatic Control (2008).

However:

- Both the cost and constraints choice are generic and do not pertain to a manageable form (linear/quadratic)
- The results are not constructive (how many constraints?, what structure?, dimension of the optimization?)

Outline

Main concept and formal definition

- Tackle the non-convexity of the PWA functions by lifting its partition to an extended space.
- Ensure the existence of an inverse operator in terms of projection that can retrieve the original partition.

Convex liftings

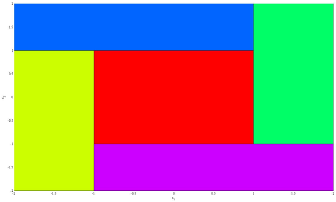
Given a polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^{d_x}$, a *piecewise affine lifting* is described by a function:

$$\begin{aligned} z_{pwa} &: \mathcal{X} \to \mathbb{R} \\ & x \mapsto z_{pwa}(x) = A_i^T x + a_i \quad \text{for} \quad x \in \mathcal{X}_i, \end{aligned}$$

where $A_i \in \mathbb{R}^{d_x}$ and $a_i \in \mathbb{R}$.

Basic pre-treatement

- The continuity at the frontier of the polyhedral regions induce geometrical constraints for the piece-wise affine function.



- The polyhedral partition impose regularity of the partition's frontiers for non-trivial solutions.

Non-restrictive assumption: polyhedral partition $\longrightarrow$ cell complex

A *cell complex* $\mathcal{C}$ is defined as a set of polytopes provided:

- every face of a member of $\mathcal{C}$ is itself a member of $\mathcal{C}$.
- the intersection of any two members of $\mathcal{C}$ is a face of each of them.

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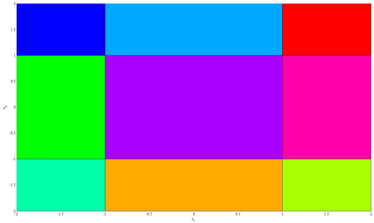

- Polyhedral partition defined as a cell complex.

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Regularization in view of convex lifting

Convex lifting on polyhedral partition

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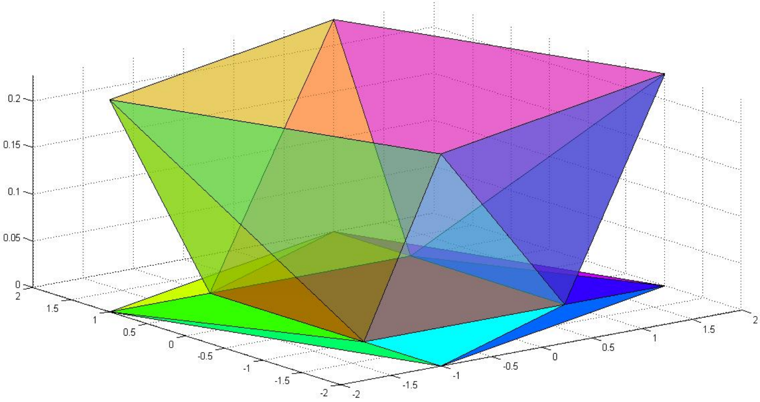
Convex liftings on cell complex

Given a cell complex $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^{d_x}$, a piecewise affine lifting $z(x) = A_i^T x + a_i \; \forall x \in \mathcal{X}_i$, is called *convex piecewise affine lifting* if the following conditions hold true:

- $z(x)$ is continuous over $\mathcal{X}$,
- for each $i \in \mathcal{I}_N$, $z(x) > A_j^T x + a_j$ for all $x \in \mathcal{X}_i \backslash \mathcal{X}_j$ and all $j \neq i$, $j \in \mathcal{I}_N$.

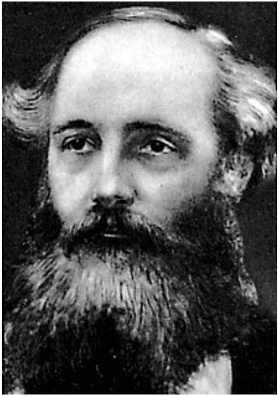
Convex liftings



- From the construction of the lifting (and its convexity) it follows that the projection of the epigraph on the original space $\mathbb{R}^{d_x}$ retrives the original partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i$.

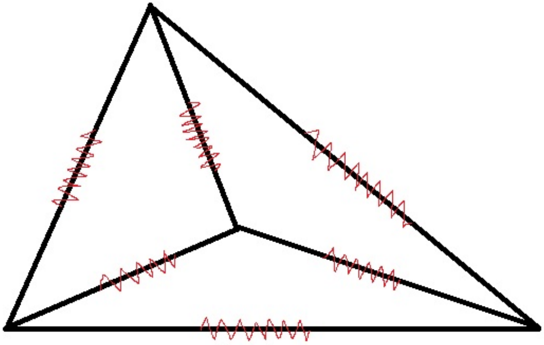
Some historical references

- James Clerk Maxwell(1831-1879), Scottish, mathematician, physicist
- He is the first one putting forward the notion of *reciprocal diagram* of a *cell complex* which is isomorphic to *convex lifting* in the plane $\mathbb{R}^2$.
- Links the geometrical problem to the notion of $k-$stress



Mechanical implication of convex lifting

What elastic coefficients generate steady positions with the same (x, y) coordinates independent on the vertical coordinates.



Historical remarks about $k-$stress

$n(F, C)$ inward unit normal vector to C at its facet F
$s(C) \in \mathbb{R}$: stress on the face C

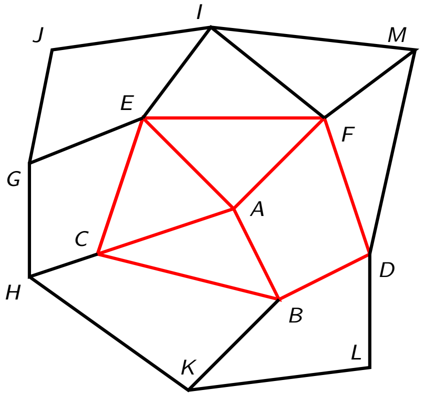
A real-valued function $s(\cdot)$ defined on the $(k-1)-$faces of a polyhedral cell complex $K \subset \mathbb{R}^k$ is called a $k-$stress if at each internal $(k-2)-$face F of K:

$$\sum_{C|F \subset C} s(C)n(F, C) = 0, \tag{1}$$

where this sum ranges over all $(k-1)-$faces in the star of F (the $(k-1)-$faces such that F is their common facet). The quantities $s(C)$ are the coefficients of the $k-$stresses, are called a *tension* if the sign is strictly positive, and a *compression* if the sign is strictly negative.

$k-$stress

$$s(AE)n_{AE} + s(AF)n_{AF} + s(AB)n_{AB} + s(AC)n_{AC} = 0,$$
$$n_{AE} = \frac{\overrightarrow{AE}}{AE}, n_{AF} = \frac{\overrightarrow{AF}}{AF}, n_{AB} = \frac{\overrightarrow{AB}}{AB}, n_{AC} = \frac{\overrightarrow{AC}}{AC}$$

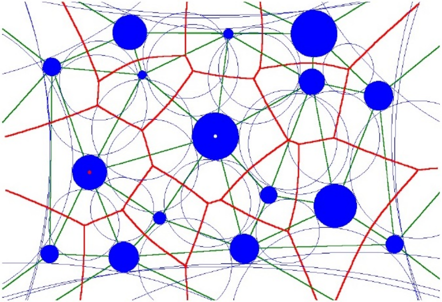


Convex lifting – existence conditions

Computational geometry

- it admits a strictly positive $d-$ stress,
- it is an additively weighted Dirichlet-Voronoi diagram,
- it is an additively weighted Delaunay diagram,
- it is the section of a $(d+1)-$dimensional Dirichlet-Voronoi diagram,
- it has a dual partition.

Gueorgui Feodossievitch Voronoi (1868-1908)



Constructive (effective numerical) solutions were missing for:

- test of convex liftability for general partitions in $\mathbb{R}^{d_x}$
- effective construction of convex liftings functions

Construction of convex liftings

An approach

Input: A given cell complex $\mathcal{X} = \bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^{d_x}$.
Output: $(A_i,\ a_i)$, $\forall i \in \mathcal{I}_N$.

1: Register all couples of neighboring regions in the cell complex $\mathcal{X}$.
2: For each couple $(i,j) \in \mathcal{I}_N^2$ such that $(\mathcal{X}_i, \mathcal{X}_j)$ are neighbors:

$$\text{(continuity)} A_i^T v + a_i = A_j^T v + a_j, \forall v \in \mathrm{vert}(\mathcal{X}_i \cap \mathcal{X}_j).$$
$$\text{(convexity)} A_i^T u + a_i > A_j^T u + a_j, \forall u \in \mathrm{vert}\mathcal{X}_i, u \notin \mathrm{vert}(\mathcal{X}_i \cap \mathcal{X}_j).$$

3: define a cost function: $f(A,a) = \sum_{i\in\mathcal{I}_N} \|A_i\| + |a_i|$
4: $\min_{A_i,a_i} f(A,a)$ subject to the constraints (continuity+convexity)

Construction of convex liftings

Important step forward:

$$\boxed{\textbf{Feasibility of a LP problem} \longleftrightarrow \textbf{Convex liftability}}$$

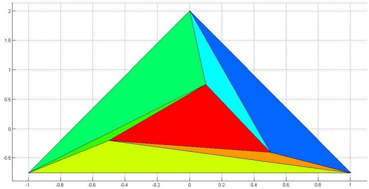
- With the convex lifting's epigraph describing the convex set:

$$\widetilde{\mathcal{X}} = conv \left\{ \begin{bmatrix} v \\ z(v) \end{bmatrix} \in \mathbb{R}^{d_x+1} \mid \begin{array}{l} v \in \bigcup_{i \in \mathcal{I}_N} \text{vert}\mathcal{X}_i, \\ z(v) = A_i^T v + a_i \quad \text{if} \quad v \in \mathcal{X}_i \end{array} \right\}.$$

- The projection of its faces are retriving the regions of the original partition.

Non-convexly liftable partitions

- Infeasibility of the construction is equivalent to non-convex liftability

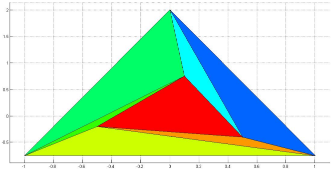


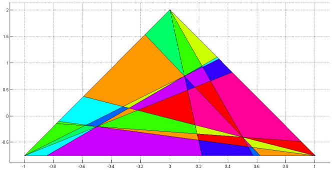
How to deal with non-convexly liftable partitions?

Nonconvexly liftable partitions

Main result

Given a non convexly liftable polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^d$, there exists at least one subdivision, preserving the internal boundaries of this partition, such that the new cell complex is convexly liftable.





Proof:

The **key point** of the proof is the notion of **hyperplane arrangement**. Interestingly this relates with Maxwell's $k-$stress existence conditions.

Nguyen, N.A., Olaru, S., Rodriguez-Ayerbe, P., Hovd, M. and Necoara, I., 2014, June. On the lifting problems and their connections with piecewise affine control law design. In European Control Conference (ECC), 2014.

Outline

Solution to Inverse Parametric L/Q Program

Given a polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^{d_x}$ and a continuous piecewise affine function $f_{pwa} : \mathcal{X} \to \mathbb{R}^{d_u}$, find

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such that

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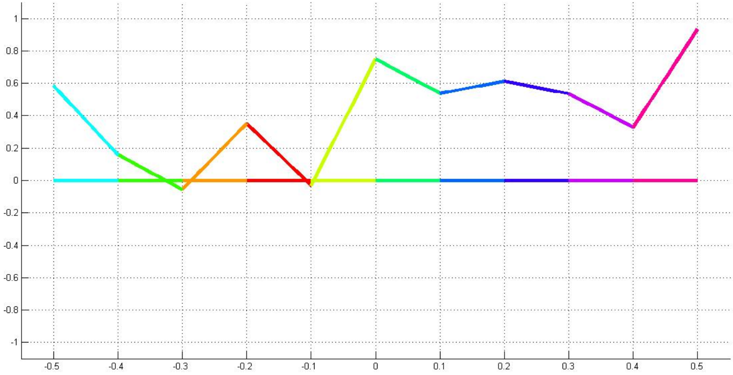
$$\text{subject to} \quad H_u u + H_x x + H_z z \leq K.$$

Assumptions

- The given partition $\mathcal{X}$ is convexly liftable (proved this can be obtained by pre-treatement)
- $\mathcal{X}$ is a partition of a polytope.

Constructive solution

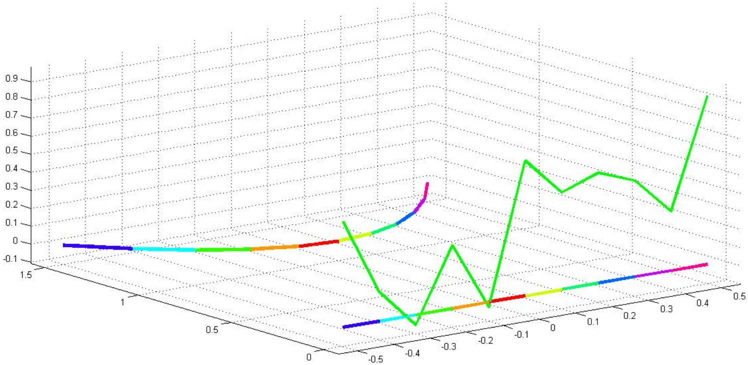
1. Construct a convex lifting $z_{pwa}(x)$ of the given cell complex $\mathcal{X} = \bigcup_{i=1}^{N} \mathcal{X}_i$



Constructive solution

1. Construct a convex lifting $z_{pwa}(x)$ of the given cell complex $\mathcal{X} = \bigcup_{i=1}^{N} \mathcal{X}_i$
2. Construct constraint set $\Pi_{[x^T\ z\ u^T]^T}$:

$$V_x = \bigcup_{i=1}^{N} \text{vert}(\mathcal{X}_i),\ \ V_{[x^T\ z]^T} = \left\{ \begin{bmatrix} x \\ z(x) \end{bmatrix} \mid x \in V_x \right\}$$



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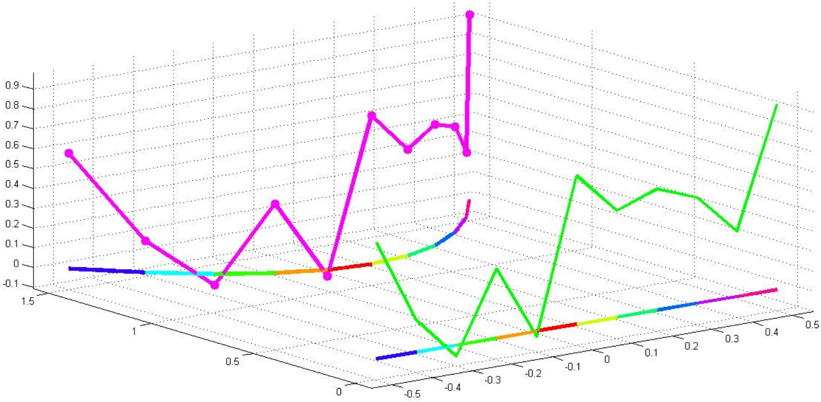
$$V_{[x^T \ z \ u^T]^T} = \left\{ \begin{bmatrix} x \\ z(x) \\ f_{pwa}(x) \end{bmatrix} \mid x \in V_x \right\}$$

$$\Pi_{[x^T \ z \ u^T]^T} = \mathrm{conv}\, V_{[x^T \ z \ u^T]^T}$$

Solution



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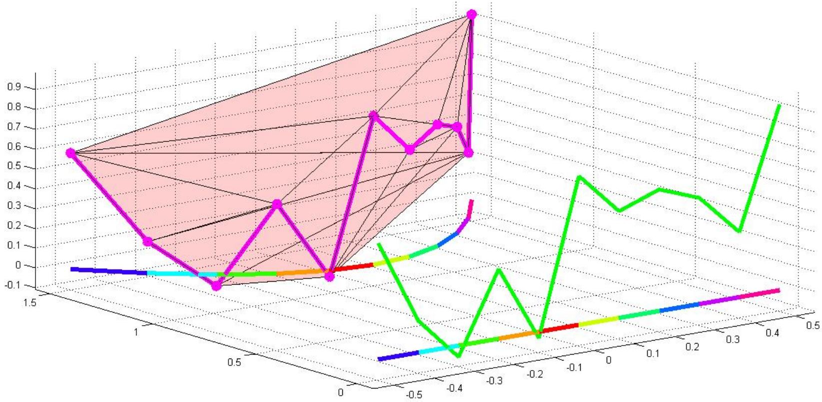


Constructive solution

3. Inverse optimal solution:

$$\begin{bmatrix} z^* \\ u^* \end{bmatrix}(x) = \arg \min_{[z \ \ u^T]^T} \quad z$$
$$\text{s.t.} \ \ \begin{bmatrix} x^T & z & u^T \end{bmatrix}^T \in \Pi_{[x^T \ z \ \ u^T]^T}.$$

4. The original PWA function is obtained by restricting to the appropriate subcomponent of the above optimal vector:

$$u^* = \mathrm{Proj}_{\mathbb{R}^{d_u}} \begin{bmatrix} z^* \\ u^* \end{bmatrix} = f_{pwa}(x).$$

Inverse Parametric L/QP

Complexity of Parametric linear/quadratic programming

Any continuous PWA function defined over a polyhedral partition can be obtained by a parametric linear/quadratic programming problem with at most one supplementary 1-dimensional variable.

Parametric linear programming

The parameter space partition associated with an optimal solution to a parametric linear programming problem admits affinely equivalent polyhedra

Linear MPC

Minimize:

$$J(U,x_k) = \left\{\sum_{i=0}^{N-1} x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k}\right\} + x_{k+N|k}^T P x_{k+N|k}$$

or

$$J(U,x_k) = \left\{\sum_{i=0}^{N-1} \|Qx_{k+i|k}\|_{1/\infty} + \|Ru_{k+i|k}\|_{1/\infty}\right\} + \|Px_{k+N|k}\|_{1/\infty}$$

$$\text{s.t.} \quad \begin{array}{l} x_{k+i+1} = Ax_{k+i} + Bu_{k+i} \\ x_{k+i|k} \in \mathbb{X},\ u_{k+i|k} \in \mathbb{U} \text{ for } i = 0 \ldots N-1 \\ x_{k+N|k} \in \mathbb{X}_T \end{array}$$

Explicit solution:

$$\begin{array}{rl} \mathbf{u}^*(x) = f_{pwa} : \bigcup_{i=1}^{N} \mathcal{X}_i \subset \mathbb{R}^{d_x} & \longrightarrow \mathbb{R}^{N \times d_u} \\ x & \longmapsto f_i x + g_i \text{ for } x \in \mathcal{X}_i \end{array}$$

Applications to linear MPC

The continuous explicit solution of a generic linear MPC problem with respect to a linear/quadratic cost function is equivalently obtained through a linear MPC problem with a linear or quadratic cost function and the control horizon at most equal to 2 prediction steps.

Hint: the lifting represents the only auxiliary variable needed to convexify the MPC solution:

$$\begin{cases} f_{pwa}(x_k) = Proj_{\mathbb{R}^{d_u}} \arg \min\limits_{\begin{bmatrix} u_k^T & u_{k+1}^T \end{bmatrix}^T} J(x_k, u_k, u_{k+1}), \\ \qquad \text{s.t.} \quad H_u u_k + H_z u_{k+1} + H_x x \leq K. \end{cases}$$

Double integrator 1/2

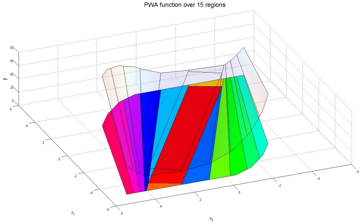
Double integrator model

$$x_{k+1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix} u_k \qquad Q = 10 * I_2, R = 0.5, N = 5$$

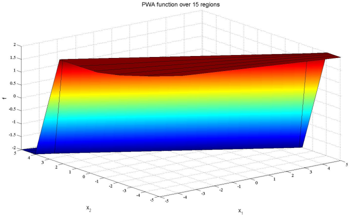
$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \qquad -2 \leq u_k \leq 2, -5 \leq y_k \leq 5$$

$$J = x_{k+5|k}^T P x_{k+5|k} + \sum_{i=0}^{4} (x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k})$$

Polyhedral partition



Associated PWA function



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Equivalent formulations

Standard MPC problem

$$\min_{\mathbf{u}} J$$

$$\text{s.t:} \quad -5 \leq \begin{bmatrix} 1 & 0 \end{bmatrix} x_{k+i|k} \leq 5$$
$$-2 \leq u_{k+i|k} \leq 2$$
$$0 \leq i \leq 4$$
$$x_{k+5|k} \in \mathbb{X}_f$$

IOPCP

$$\min_{\begin{bmatrix} z & u_k^T \end{bmatrix}^T} z$$

s.t:

$$H \begin{bmatrix} x_k \\ z \\ u_k \end{bmatrix} \leq K, \; H \in \mathbb{R}^{24\times 4}, K \in \mathbb{R}^{24}$$

Double integrator 1/2

Double integrator model

$$x_{k+1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix} u_k \qquad Q = 10 * I_2, R = 0.5, N = 5$$

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$$x_{k+5|k} \in \mathbb{X}_f$$

IOPCP

$$\min_{\begin{bmatrix} u_{k+1} & u_k^T \end{bmatrix}^T} u_{k+1}$$

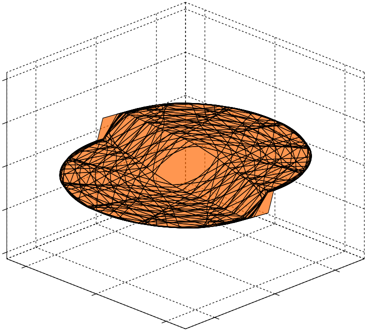
$$\text{s.t: } H \begin{bmatrix} x_k \\ u_{k+1} \\ u_k \end{bmatrix} \leq K, \; H \in \mathbb{R}^{24\times 4}, K \in \mathbb{R}^{24}$$

Control of a damped cantilever beam

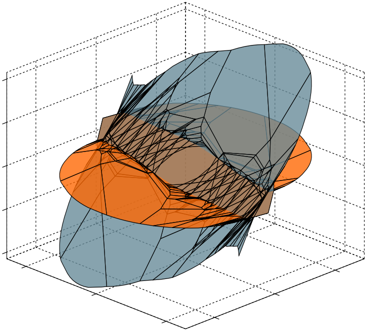
$$x_{k+1} = \begin{bmatrix} 0.867 & 1.119 \\ -0.214 & 0.870 \end{bmatrix} x_k + \begin{bmatrix} 9.336\text{E-}4 \\ 5.309\text{E-}4 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k$$

with $|u| \leq 120$, $Q_x = C^T C$, and $Q_u = 1\text{E-}4$



(a) original PWA control law ($N = 40$, 3397 polyhedra)



(b) PWA obtained via IpLP vith clipping ($N = 2$, 811 polyhedra)

Case-study example

		task execution time [s]				
Formulation	$N =$	10	20	30	40	50
Standard MPC problem						
mpQP		6.3	24.2	91.6	171.9	356.2
Extended lpLP problem						
convex lifting		0.1	0.3	0.5	0.6	1.8
facet enumeration		0.05	0.07	0.12	0.17	0.23
constraint removal		2.8	14.9	38.3	74.3	135.3
mpLP		1.7	7.3	24.5	90.8	150.5

The convex lifting as a point location mechanism

- Explicit MPC is based on the evaluation of a PWA function which stores the partition and performs a point location
- The convex lifting can perform a region-free point location through:
  - the selection of the unique active constraint of the optimization:

$$i^*(x) = \arg\max_{i \in \mathcal{I}_N} A_i^T x + a_i$$

  - evaluation of the corresponding PWA branch:

$$f_{pwa}(x) = f_{i*(x)}^T * x + g_{i^*(x)}$$

- aside the memory storage, the advantage of this solution resides in:
  - effective (direct) use of the MPC explicit solution
  - reduced pre-processing time due to the vertex enumeration and feasible domain construction through convex-hull operation

Region-free MPC: cross-platforms experiments

Implementaion based on the inverse optimal problem

Implementation based on region-free evaluation

Online implementation via Algorithm 4					using $\tilde{\ell}^{\mathrm{I}}(x)$, $\tilde{\kappa}^{\mathrm{I}}(x)$ obtained per Section II-C1												using $\tilde{\ell}^{\mathrm{II}}(x)$, $\tilde{\kappa}^{\mathrm{II}}(x)$ obtained per Section II-C2											
				Precision	double						single						double						single					
				Metrics†	ROM			RAM	TET		ROM			RAM	TET		ROM			RAM	TET		ROM			RAM	TET	
MCU type	ARM Cortex architecture	Clock frequency [MHz]	Available ROM [kB]	Available RAM [kB]	CODE [B]	DATA [B]	Occupied ROM [kB]	Occupied RAM [B]	TET [ms]	Norm. TET [µs/DMIPS]	CODE [B]	DATA [B]	Occupied ROM [kB]	Occupied RAM [B]	TET [ms]	Norm. TET [µs/DMIPS]	CODE [B]	DATA [B]	Occupied ROM [kB]	Occupied RAM [B]	TET [ms]	Norm. TET [µs/DMIPS]	CODE [B]	DATA [B]	Occupied ROM [kB]	Occupied RAM [B]	TET [ms]	Norm. TET [µs/DMIPS]
STM32F051R8T6‡	M0	48	64	8	22 836	37 180	58.6	552	18.43	485	13 124	18 604	31.0	536	9.99	263	15 140	21 788	36.1	552	10.75	283	9280	10 908	19.7	536	5.84	154
STM32F030R8T6	M0	48	64	8	22 820	37 180	58.6	552	18.48	458	13 108	18 604	31.0	536	10.02	249	15 124	21 788	36.0	552	10.79	268	9264	10 908	19.7	536	5.86	145
STM32F100RB	M3	24	128	8	23 014	37 182	58.8	552	19.31	644	13 198	18 606	31.1	536	10.81	360	15 318	21 790	36.2	552	11.28	376	9354	10 910	19.8	536	6.35	212
STM32L100RCT6	M3	32	256	16	23 594	37 182	59.4	552	17.64	441	13 778	18 606	31.6	536	10.68	267	15 898	21 790	36.8	552	10.25	256	9934	10 910	20.4	536	6.54	163
STM32L152RCT6	M3	32	256	32	23 590	37 182	59.3	552	16.07	402	13 774	18 606	31.6	536	9.87	247	15 894	21 790	36.8	552	9.38	235	9930	10 910	20.4	536	5.83	146
STM32F303VCT6	M4	64	256	40	14 100	46 464	59.1	552	9.95	124	8264	23 248	30.8	536	1.63	20	10 252	27 224	36.6	552	5.80	73	11 770	8197	19.5	536	0.95	12
STM32F401VCT6	M4	84	256	64	13 832	46 464	58.9	552	5.16	49	7996	23 248	30.5	536	0.67	6	9984	27 224	36.3	552	3.03	29	11 502	8198	19.2	536	0.40	4
STM32F407VGT	M4	168	1024	192	14 084	46 464	59.1	552	2.63	13	8248	23 248	30.8	536	0.35	2	10 236	27 224	36.6	552	1.54	7	11 754	8198	19.5	536	0.21	1
STM32F429ZI	M4	180	2048	256	14 204	46 464	59.2	552	2.41	11	8368	23 248	30.9	536	0.32	1	10 356	27 224	36.7	552	1.42	6	11 874	8198	19.6	536	0.19	1
STM32F746NGH6	M7	216	1024	340	14 504	46 464	59.5	552	5.52	42	8668	23 248	31.2	536	1.03	2	10 656	27 224	37.0	552	2.99	6	12 174	8198	19.9	536	0.60	1

† ROM (non-volatile, read-only memory) = CODE (program) + DATA (static variables); RAM – volatile, random-access memory; TET – task execution time (MPC only); Normalized TET = TET/(clock × DMIPS/MHz), DMIPS – Dhrystone MIPS (millions of instructions per second).
‡ The embedded target used in the experiments in Fig. 8.

Changing the formulation from complete inverse optimal solution to region-free implemntaion based on convex lifting

- will further reduce the non-volatile memory footprint by a factor of ≈ 1.6 for both numerical precision cases
- the TET will be reduced anywhere from a factor of ≈ 1.5–7.7 (averaging around 4) which depends on architecture and clock speed.

The fragility of Convex Lifting

Consider a linear system

$$\begin{aligned} x_{k+1} &= g_{pwa}(x_k, u_k) \\ u_k &= F_i^T x_k + g_i, x_k \in \mathcal{X}_i, i \in \{1, \cdots, 4\}, \end{aligned}$$

with $\mathcal{X} = \bigcup_{i=1}^{4} \mathcal{X}_i$

- Nominal closed loop fulfills the stability and performance criteria

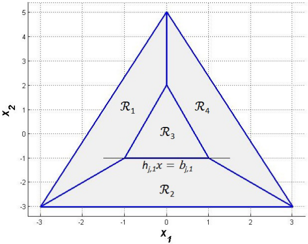


Figure: Original convex lifting

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- Nominal closed loop fulfills the stability and performance criteria
- Issue: fragility of the polyhedral partition.

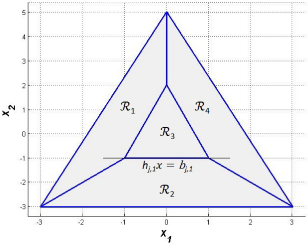

Figure: Original convex lifting

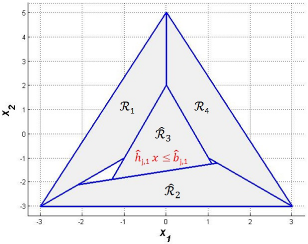

Figure: Disturbance within a half-plane representation.

The fragility of Convex Lifting

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- Nominal closed loop fulfills the stability and performance criteria
- Issue: fragility of the polyhedral partition.
- Difficulty: the reconstruction of explicit PWA control is costly.

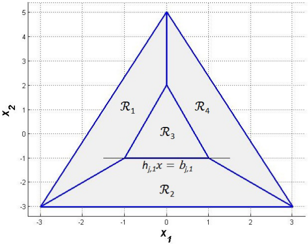
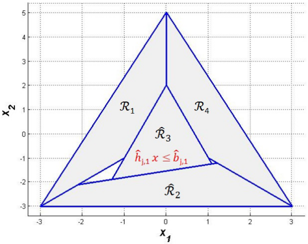


Figure: Original convex lifting



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- Nominal closed loop fulfills the stability and performance criteria
- Issue: fragility of the polyhedral partition.
- Difficulty: the reconstruction of explicit PWA control is costly.
- Objective: retuning of the controller $\mathcal{X}_i \to \tilde{\mathcal{X}}_i$, $x_{k+1} \in \mathcal{X}$ without reconstruction of the explicit PWA formulation.

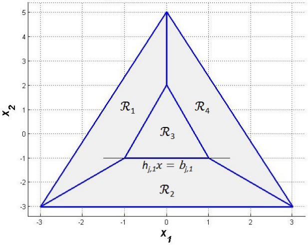

Figure: Original convex lifting

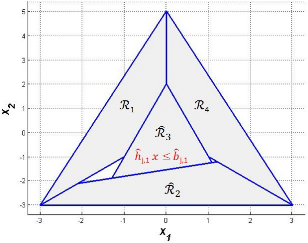

Figure: Disturbance within a half-plane representation.

The fragility of Convex Lifting

Region-free implementation of the PWA control via convex-lifting:

- Construct a relevant convex lifting:

$$l(x) = A_i^T x + a_i, i \in \mathcal{X}_i$$

- Properties of convex lifting:

$$\forall x \in \mathcal{X}_i \longrightarrow i : \max_r \tilde{a}_r^T x + \tilde{b}_r, \tag{2a}$$

$$\forall x \in \mathcal{X} \setminus \mathcal{X}_i \longrightarrow j : \max_r \tilde{a}_r^T x + \tilde{b}_r, j \neq i \tag{2b}$$

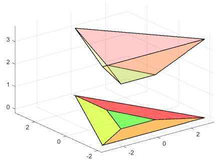

Figure: Initial convex lifting

*Undergoing work with Songlin Yang and Pedro Rodriguez; see ECC' 2023

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- Construct the covering of each cell $\overline{\mathcal{X}_i}$ which guarantees the invariance of $\mathcal{X}$ in closed loop.

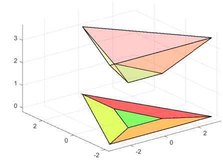

Figure: Initial convex lifting

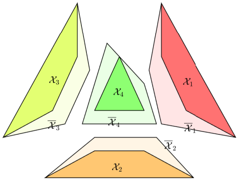

Figure: $\mathcal{X}_i$ and $\overline{\mathcal{X}}_i$

Undergoing work with Songlin Yang and Pedro Rodriguez: see ECC' 2023.

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- Construct the covering of each cell $\overline{\mathcal{X}_i}$ which guarantees the invariance of $\mathcal{X}$ in closed loop.
- Condition for partition retuning:

$$\forall x \in \mathcal{X} \setminus \overline{\mathcal{X}}_i \longrightarrow j : \max_r \tilde{a}_r^T x + \tilde{b}_r, j \neq i, \tag{3}$$

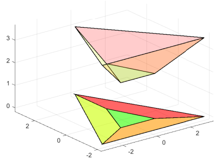

Figure: Initial convex lifting

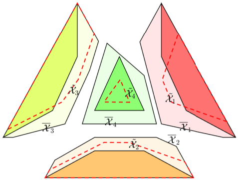

Figure: $\tilde{\mathcal{X}}_i$ and $\overline{\mathcal{X}}_i$

Undergoing work with Songlin Yang and Pedro Rodriguez: see ECC' 2023.

The fragility of Convex Lifting

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$$\forall x \in \mathcal{X} \setminus \overline{\mathcal{X}}_i \longrightarrow j : \max_r \tilde{a}_r^T x + \tilde{b}_r, j \neq i, \tag{3}$$

- Resulting re-tuned PWA control partition:

$$i : \max_r \tilde{a}_r^T x + \tilde{b}_r, \forall x \in \tilde{\mathcal{X}}_i \longrightarrow \tilde{\mathcal{X}}_i \subset \overline{\mathcal{X}}_i \tag{4}$$

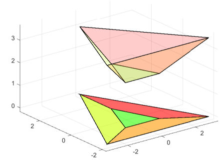


Figure: Initial convex lifting

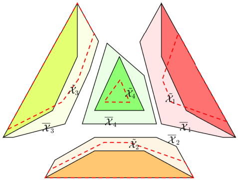


Figure: $\tilde{\mathcal{X}}_i$ and $\overline{\mathcal{X}}_i$

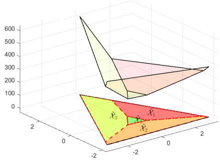
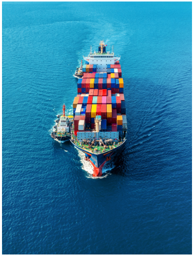


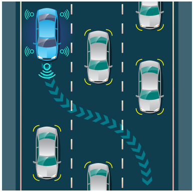
Figure: Alternative convex lifting

Undergoing work with Songlin Yang and Pedro Rodriguez: see ECC' 2023.

Outline

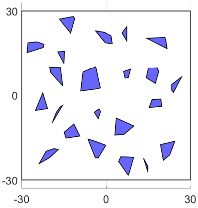
Navigation through multi-obstacle environment





- Topic of interest in several fields
  - Autonomous road vehicles
  - Unmanned aerial vehicles
  - Naval vehicles
- Motion planning is divided into three main tasks
  - *(i)* Path planning
  - *(ii)* Trajectory generation
  - *(iii)* Low-level feedback control
- Main difficulty for *i)* and *ii)*: non-convexity of the obstacle-free area
  - The search for collision avoiding paths is nontrivial
  - Needs for tools for the modelization of the obstacle-free area

The mathematical framework



- N_o obstacles lying in a finite dimensional space $\mathbb{X} \subset \mathbb{R}^d$
  - $P_i \in \text{Com}\left(\mathbb{R}^d\right)$: obstacles as compact (convex) sets in $\mathbb{R}^d$
  - $P_i \cap P_j = \emptyset$: non overlapping sets
  - $\mathbb{P} = \bigcup_{i=1}^{N_o} P_i$
  - $\mathcal{C}_{\mathbb{X}}(\mathbb{P})$: the (non-convex) obstacle-free area

Path definition

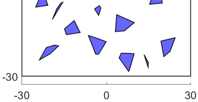
Given the obstacles $\mathbb{P}$, a corridor between two points $x_0, x_f \in \text{int}\left(\mathcal{C}_{\mathbb{X}}(\mathbb{P})\right)$ is characterized by the existence of two continuous functions

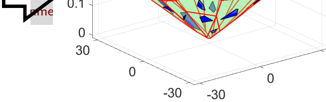
$$\gamma : [0,1] \to \mathcal{C}_{\mathbb{X}}(\mathbb{P}), \;\; \rho : [0,1] \to \mathbb{R}_{\geq 0} \tag{5}$$

satisfying $\gamma(0) = x_0$, $\gamma(1) = x_f$, and $\gamma(\theta) \oplus \mathbb{B}_{0,\rho(\theta)} \subset \mathcal{C}_{\mathbb{X}}(\mathbb{P})$, $\forall\, \theta \in [0,1]$. Then a corridor is defined as

$$\Pi = \{x \in \mathbb{R}^d \;:\; \exists \theta \in [0,1] \text{ s.t. } x \in \gamma(\theta) \oplus \mathbb{B}_{0,\rho(\theta)}\}$$

Overview on convex





A lift and project philosophy:

Step 1. Computation of a PWA convex lifting associated with each obstacle: $z(x) = a_i^T x + b_i$, with a_i, b_i s.t.

$$\min_{a_i, b_i} \sum_{i=1}^{N_o} |[a_i \;\; b_i]^T|_2^2 \text{subject to} \quad a_i^T v + b_i \geq a_j^T v + b_j + \epsilon, \; \forall v \in \mathcal{V}(P_i), \; \forall i \neq j,$$

$$a_i^T v + b_i \leq M, \; \forall v \in \mathcal{V}(P_i), \; \forall i,$$

Step 2. Construction of the $d+1$ dimensional polyhedron

$$\mathcal{P} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{d+1} \; : \; \begin{bmatrix} a_i^T & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq -b_i, \; i \in \mathcal{I} \right\}$$

Step 3. Projection of facets of $\mathcal{P}$ into $\mathbb{X}$ to obtain a collection of sets $\{X_i\}_{i=1}^{N_o}$

(1) $P_i \subset \text{int}\left(X_i\right), \forall \, i$

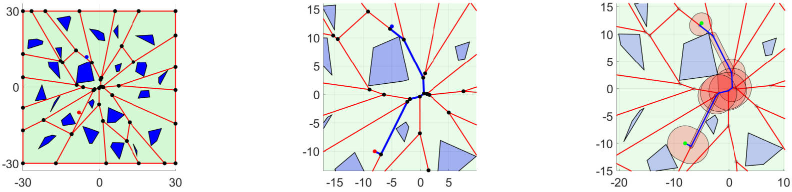
(2) $X_i \cap P_j = \emptyset, \forall \, j \neq i$

From partition to corridors

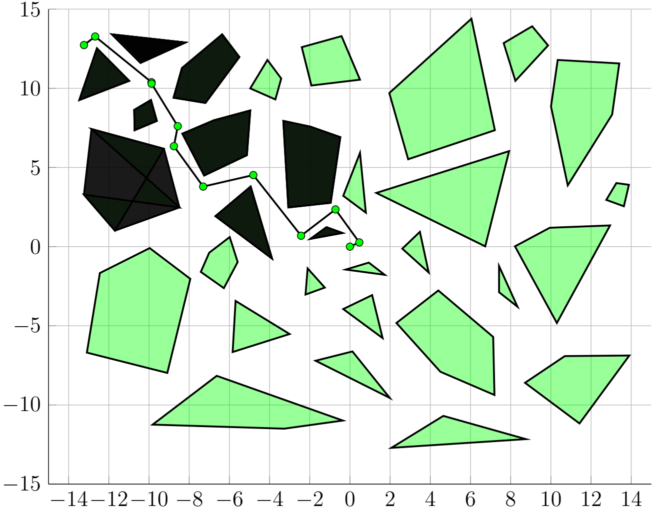
1. Define of a graph $\Gamma = (\mathcal{N}, \mathcal{E}, f)$, $f : \mathcal{E} \to \mathbb{R}$, based on the partition $\{X_i\}_{i=1}^{N_o}$
   - $\mathcal{N} = \bigcup_{i=1}^{N_o} \mathcal{V}(X_i)$
   - $\mathcal{E}$: the facets of the patition cells
2. Connect of x_0, x_f to Γ: extended graph $\tilde{\Gamma}$
3. Run a graph search algorithm (e.g. Dijkstra's algorithm): the result is $\gamma(\theta)$
4. Construct the corridor as the union of sub corridors

$$\Pi_i = \left\{ x \in \mathbb{R}^d \ : \ \exists \tilde{\theta} \in [0,1] \text{ s.t. } x \in \gamma_i(\tilde{\theta}) \oplus \mathbb{B}_{0,\rho_i(\tilde{\theta})} \right\}$$

where $\gamma_i(0) = x_i$ and $\gamma_i(1) = x_{i+1}$



Example for obstacle avoidance using the obtained path and MPC



Corridors enlargement problem

1. In order to compare the corridors one needs a performance index
   - Average width of the corridors associated to the partition

2. Using this criterion, the enlargement of the cells within the partition can be sought
   - Idea: different obstacles displacement lead to different partitions
     - Force the LP problem at the basis of convex lifting to return a different partition
     - The virtual reorganization of obstacles has to ensure the feasibility of the corridors with respect to the original arrangement
     - Repeat the computation of the new partition as long as it is possible to rearrange the obstacles
     - The existence of different partitions is ensured by the next result

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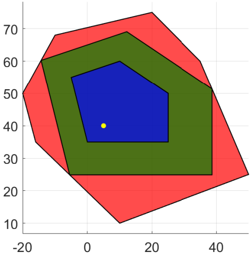
Monotonic improvement

Consider a sequence of collections of obstacles $\mathbb{P}^k$ inducing corridors that are feasible with respect to $\mathbb{P}^0 = \mathbb{P}\ \forall\, k\ \geq 0$, and such that $P_i^{k+1} \supset P_i^k$. Suppose that P_i^{k+1} "touches" X_i^k. As long as $\mathbb{P}^k$ is convex liftable, then the width of the corridors monotonically increases.

Virtual obstacles: scaling based on Farkas' Lemma



<u>Idea</u>: enlarge as much as possible the obstacles

- Problem formulation: consider $Q \subset H$, where

$$Q = \{x \in \mathbb{R}^d : A_q x \leq b_q\},\ A_q \in \mathbb{R}^{q \times d},\ b_q \in \mathbb{R}^q,$$
$$H = \{x \in \mathbb{R}^d : A_h x \leq b_h\},\ A_h \in \mathbb{R}^{h \times d},\ b_p \in \mathbb{R}^h$$

  <u>Find</u> the maximum scaling factor λ_M and the center of scaling c_q such that the enlarged $Q^\lambda \subseteq H$, where

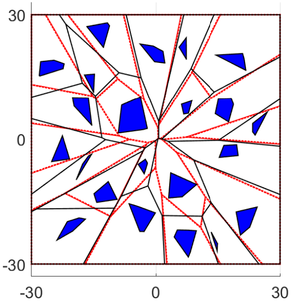
$$Q^\lambda = \{x \in \mathbb{R}^n : A_q x \leq \lambda b_q + (1-\lambda) A_q c_q\},$$

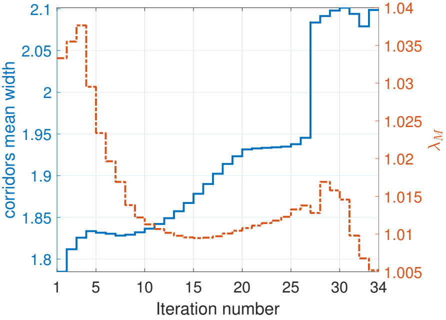
- Based on the Extended Farkas' Lemma, λ_M and c_q are computed as solution of

$$\min_{c_q, \mu_1, \mu_2, \tilde{U}} \mu_1$$
$$\text{s.t. } \tilde{U} A_q = \mu_1 A_h,\ \tilde{U} b_q - A_p c_q \leq \mu_2,\ b_h A_q c_q \leq b_q$$
$$\mu_1 - \mu_2 = 1,\ \mu_1 \geq 1,\ \mu_2 \geq 0$$
$$\tilde{U}_{i,j} \geq 0,\ i = 1, \ldots, h,\ j = 1, \ldots, q.$$

Then $\lambda_M = 1 + 1/\mu_2$

Numerical results





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Conclusions and perspectives

- In a general perspective
  - Convex lifting pertain to the class of lift and project methods
  - They allow to tackle in a convex optimization framework design problem that are notoriously complex
- Main results:
  - Provided a constructive inverse optimality result for any PWA control law
  - Allow a region-free MPC implementation
  - Open the way to robustification of PWA control (see PWA fragility)
  - Provided a path planning tool for cluttered environments
- Open problems:
  - Inverse optimal MPC control with minimal prediction horizon based on a nominal model
  - Commensurate the fragility of PWA controllers
  - Properly define and construct the optimal corridor with respect to width and constrained navigation margins.

Selected references

For the proofs and applications of the concepts:

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Other control problems dealing with convex liftings:

- Construction of Polyhedral Control Lyapunov functions
  Nguyen, Ngoc Anh, and Sorin Olaru. "A family of piecewise affine control Lyapunov functions." Automatica 90 (2018): 212-219.
- Model predictive control for hybrid systems via bi-level optimization
  Hempel, Andreas B., Paul J. Goulart, and John Lygeros. "Inverse parametric optimization with an application to hybrid system control." IEEE Transactions on automatic control 60.4 (2014): 1064-1069.