New results on asymptotic consensus formation in graphon dynamics

(in collaboration with N. Pouradier Duteil and M. Sigalotti)

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Outline of the talk

Multi-agent systems – From microscopic to macroscopic models

Review of consensus methods for microscopic cooperative systems

Consensus analysis in the context of graphon dynamics

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Multi-agent systems – *Finite dimensional setting*

Multi-agents dynamics can be described by **systems of ODEs**

$$\dot{x}_i(t) = \boldsymbol{v}_i(t, \boldsymbol{x}(t), x_i(t))$$

for $i \in \{1, \ldots, N\}$, where

- $\boldsymbol{x} = (x_1, ..., x_N) \in (\mathbb{R}^d)^N$ encodes the **states** of the agents,
- $\boldsymbol{v}_i : [0, T] \times (\mathbb{R}^d)^N \times \mathbb{R}^d \to \mathbb{R}^d$ are **non-local velocity fields**.

Breadcrumb trail example (Time-dependent cooperative dynamics)

$$\boldsymbol{v}_i(t, \boldsymbol{x}, x_i) = \frac{1}{N} \sum_{j=1}^{N} a_{ij}(t) \psi(x_i - x_j).$$

Central observation (pattern formation)

Simple **microscopic interactions** $\rightsquigarrow$ rich **macroscopic structures**.

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Example (Classical patterns arising in multi-agent systems)

- **Consensus** (everybody goes at the same place)
- **Flocking** (everybody goes in the same direction)
- **Synchronisation** (periodic motions arise in the system)

Macroscopic approximations (Main motivations)

- Interested in **global** patterns that involve **many agents**.
- Usually N is **very large** $\rightsquigarrow$ **scale-stability** of consensus ?

Today: **Consensus** for **micro** and **macro** cooperative systems.

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Multi-agent systems – *General cooperative dynamics*

We consider the **cooperative** dynamics

$$\dot{x}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a_{ij}(t)\phi(|x_i(t) - x_j(t)|)(x_j(t) - x_i(t)), \qquad \text{(CS)}$$

where

- $\phi \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$ encodes **distance-based** interactions,
- $a_{ij}(\cdot) \in L^\infty(\mathbb{R}_+, [0,1])$ represent **communication links**.

Definition (Asymptotic consensus formation)

A solution $x(\cdot)$ of (CS) converges to **consensus** if

$$\lim_{t \to +\infty} |x_i(t) - x^\infty| = 0,$$

for all $i \in \{1, \ldots, N\}$ and some $x^\infty \in \mathbb{R}^d$.

Question: **Stability**/dependence of consensus for $N \to +\infty$?

$\hookrightarrow$ Embed (CS) and analyse consensus in a **macroscopic model**!

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Multi-agent systems – *Reformulation with graphons (1)*

First idea: Study consensus for **mean-field dynamics**

$$\partial_t \mu_N(t) + \mathrm{div}_x\Big((\Phi(t) \star \mu_N(t))\mu_N(t)\Big) = 0.$$

[Ha&Liu'09], [Carrillo,Fornasier,Rosado&Toscani'10, [Piccoli,Rossi&Trélat'15].

System of ODEs on N agents $(x_1, ..., x_N) \in (\mathbb{R}^d)^N$

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$$

Single PDE on the density of agents $\mu_N : \mathbb{R}^d \to \mathbb{R}$.

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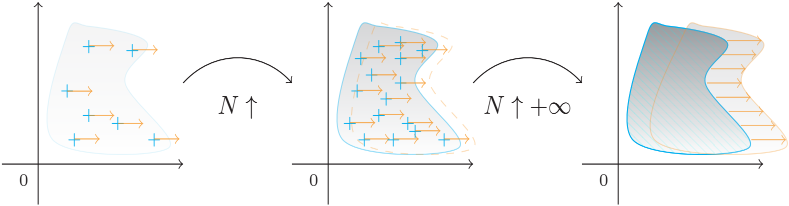

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Multi-agent systems – *Reformulation with graphons (2)*

Problem: Mean-field **needs** **indistinguishability**, i.e. $a_{ij}(t) = 1$.

$\hookrightarrow \mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ is spatial and "forgets" who is who.

Definition (Graph limit) [LS'07,M'14]

Given a solution $x(\cdot)$ of (CS), define the **piecewise constant** maps

$$i \in I \mapsto x_N(t,i) := \sum_{k=1}^N x_k(t)\, \mathbb{1}_{\left[\frac{k-1}{N},\frac{k}{N}\right)}(i)$$

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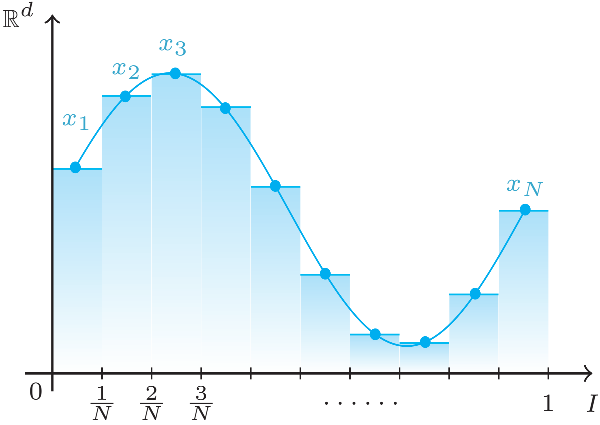

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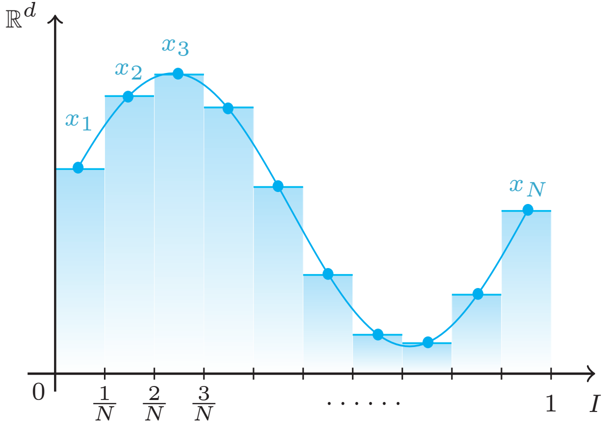

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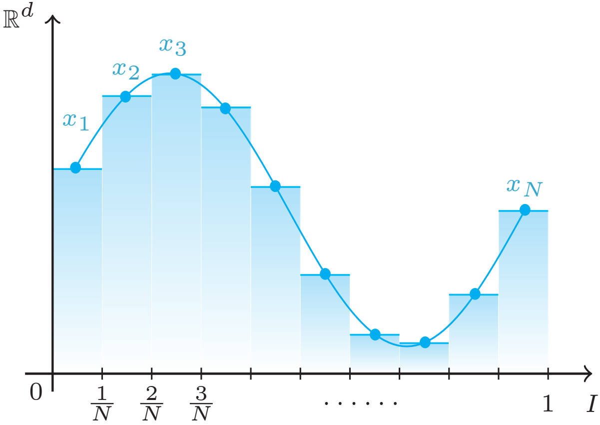

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Consensus analysis – *Reformulation and main ideas*

Definition (Adjacency and graph-Laplacian matrices)

Given an adjacency matrix $\boldsymbol{A}_N := (a_{ij})_{i,j=1}^N \in [0,1]^{N\times N}$ satisfying $a_{ii} = 1$, we define its **graph-Laplacian** by

$$\boldsymbol{L}_N : \boldsymbol{x} \in (\mathbb{R}^d)^N \mapsto \left(\tfrac{1}{N}\textstyle\sum_{j=1}^N a_{ij}(x_i - x_j)\right)_{1\leq i\leq N} \in (\mathbb{R}^d)^N.$$

$\hookrightarrow$ Reformulation of (CS) as $\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}_N(t)\boldsymbol{x}(t)$ when $\phi \equiv 1$.

Idea: **Quantitative** convergence results $\rightsquigarrow$ **Lyapunov** methods!

Definition (Candidate energy functionals)

We define the **variance functional**

$$\mathcal{V}(\boldsymbol{x}) := \frac{1}{2N^2}\sum_{i,j=1}^N |x_i - x_j|^2 \qquad (\ell_2\text{-convergence}),$$

and the **diameter**

$$\mathcal{D}(\boldsymbol{x}) := \max_{i,j\in\{1,\dots,N\}} |x_i - x_j| \qquad (\ell_\infty\text{-convergence}).$$

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$\hookrightarrow$ Reformulation of (CS) as $\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}_N(t)\boldsymbol{x}(t)$ when $\phi \equiv 1$.

Idea: Quantitative convergence results $\rightsquigarrow$ **Lyapunov** methods!

Definition (Candidate energy functionals)

We define the **variance functional**

$$\mathcal{V}(\boldsymbol{x}) := \frac{1}{2N^2}\sum_{i,j=1}^N |x_i - x_j|^2 \qquad (\ell_2\text{-convergence}),$$

and the **diameter**

$$\mathcal{D}(\boldsymbol{x}) := \max_{i,j\in\{1,\dots,N\}} |x_i - x_j| \qquad (\ell_\infty\text{-convergence}).$$

Consensus analysis – *Reformulation and main ideas*

Definition (Adjacency and graph-Laplacian matrices)

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Consensus – *Scrambling coefficient and diameter estimates*

Definition (Scrambling coefficient)[Seneta'79]

The **scrambling** of a graph $\boldsymbol{A}_N = (a_{ij})_{i,j=1}^N$ is defined by

$$\eta(\boldsymbol{A}_N) := \min_{1\leq i,j\leq N} \frac{1}{N}\left(\sum_{k=1,k\neq i,j}^{N} \min\left\{a_{ik}, a_{jk}\right\} + a_{ij} + a_{ji}\right)$$

$\hookrightarrow$ Positive if each (i,j) either **interact**, or **follow** the same k.

Theorem (Quantitative diameter decay)[Motsch&Tadmor'14]

For each $\boldsymbol{x}^0 \in (\mathbb{R}^d)^N$, it holds that

$$\mathcal{D}(\boldsymbol{x}(t)) \leq \mathcal{D}(\boldsymbol{x}^0) \exp\left(- \int_0^t \eta(\boldsymbol{A}_N(s))\mathrm{d}s\right).$$

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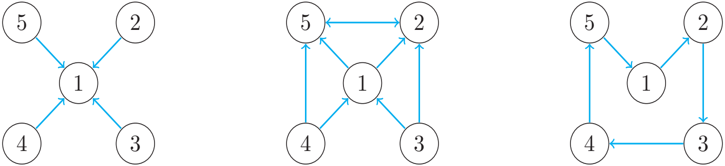

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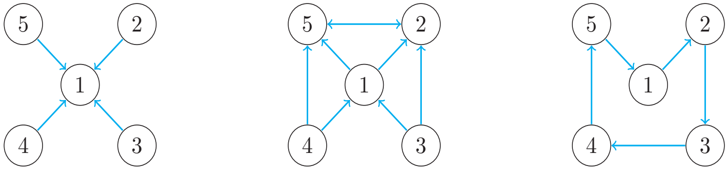

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Consensus – *Fiedler number and variance estimates*

Definition (Algebraic connectivity of a graph)[Fiedler'73, Mohar'91]

The **Fiedler number** of a **symmetric** graph $\boldsymbol{A}_N = (a_{ij})_{i=1}^N$ is

$$\lambda_2(\boldsymbol{A}_N) = \inf_{\boldsymbol{x} \in \mathscr{C}_N^\perp} \frac{\langle \boldsymbol{L}_N \boldsymbol{x}, \boldsymbol{x} \rangle_N}{|\boldsymbol{x}|_N^2},$$

where $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_N := \frac{1}{N} \sum_{i=1}^N \langle x_i, y_i \rangle$ and

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\boldsymbol{x}(t)) = \frac{1}{N}\sum_{i=1}^{N}\left\langle \dot{x}_i(t), x_i(t) - \bar{\boldsymbol{x}}\right\rangle$$

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Consensus – *About algebraic and graph connectivity (1)*

Theorem (Characterisation of graph connectivity)[Mohar'91]

A symmetric graph $\boldsymbol{A}_N = (a_{ij})_{i,j=1}^N$ is **strongly connected**, i.e.

for all i, j there exists $i = k_1, \ldots, k_m = j$ s.t. $a_{k_l k_{l+1}} > 0$

if and only if $\lambda_2(\boldsymbol{A}_N) > 0$.

Question: What happens when $\boldsymbol{A}_N$ is not symmetric ?

Theorem (Characterisation of graph connectivity)[Wu'05]

A graph $\boldsymbol{A}_N$ is a **disjoint union** of **str. connected components** ("DUSCC") **if and only if** there exists $(v_1, \ldots, v_N) \in (\mathbb{R}_+^*)^N$ s.t.

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Remark: $\boldsymbol{L}_N^*(1, \ldots, 1) = 0$ if $\boldsymbol{A}_N$ is symmetric $\rightsquigarrow$ always DUSCC!

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A symmetric graph $\boldsymbol{A}_N = (a_{ij})_{i,j=1}^N$ is **strongly connected**, i.e.

for all i, j there exists $i = k_1, \dots, k_m = j$ s.t. $a_{k_l k_{l+1}} > 0$

if and only if $\lambda_2(\boldsymbol{A}_N) > 0$.

Question: What happens when $\boldsymbol{A}_N$ is not symmetric ?

Theorem (Characterisation of graph connectivity)[Wu'05]

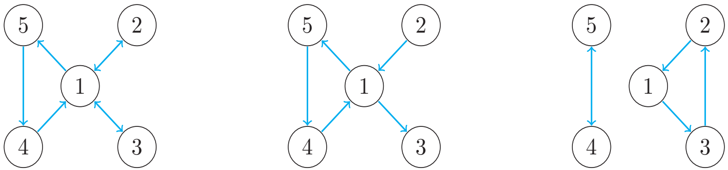
A graph $\boldsymbol{A}_N$ is a **disjoint union** of **str. connected components** ("DUSCC") **if and only if** there exists $(v_1, \dots, v_N) \in (\mathbb{R}_+^*)^N$ s.t.

$$\boldsymbol{L}_N^* \boldsymbol{v} = 0 \qquad \text{and} \qquad \frac{1}{N} \sum_{i=1}^N v_i = 1$$

where $\boldsymbol{v} := (v_1, \dots, v_1, \dots, v_N, \dots, v_N) \in (\mathbb{R}_+^*)^{dN}$.

Remark: $\boldsymbol{L}_N^*(1, \dots, 1) = 0$ if $\boldsymbol{A}_N$ is symmetric $\rightsquigarrow$ always DUSCC!

Consensus – *About algebraic and graph connectivity (2)*



Definition (Generalised algebraic connectivity for graphs)[Wu'05]

The **algebraic connectivity** of a DUSCC graph $\boldsymbol{A}_N$ is

$$\lambda_2(\boldsymbol{A}_N) := \inf_{\boldsymbol{x} \in \mathscr{C}_N^{\perp}} \frac{\langle \boldsymbol{L}_N^{\boldsymbol{v}} \boldsymbol{x}, \boldsymbol{x} \rangle_N}{|\boldsymbol{x}|_N^2}$$

where $\boldsymbol{L}_N^{\boldsymbol{v}} := \operatorname{diag}(\boldsymbol{v}) \boldsymbol{L}_N$ is the **renormalised graph-Laplacian**.

Question: Link with **graph connectivity** and variance estimates ?

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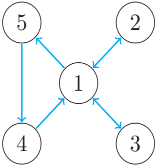

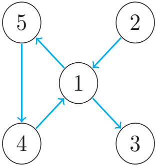

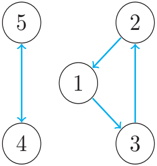

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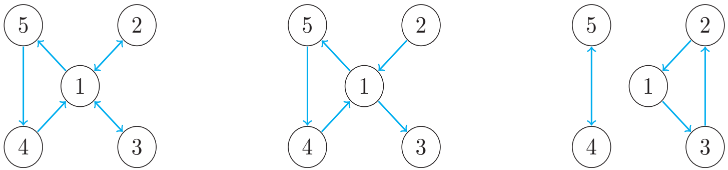

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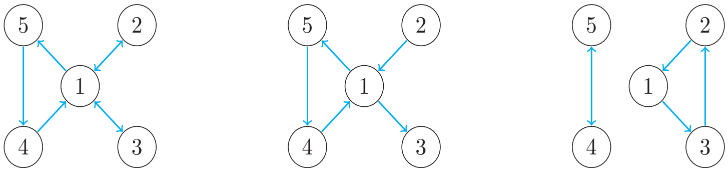

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Outline of the talk

Multi-agent systems – From microscopic to macroscopic models

Review of consensus methods for microscopic cooperative systems

Consensus analysis in the context of graphon dynamics

Graphon dynamics – *Adjacency and graph-Laplacian*

We consider the graphon dynamics

$$\partial_t x(t,i) = \int_I a(t,i,j)(x(t,j) - x(t,i))\mathrm{d}j$$

where $a(t) \in L^\infty(I \times I, [0,1])$ represents the **communications**.

Definition (Adjacency and graph-Laplacian operators)

We define the **adjacency** operator $\mathcal{A}(t) : L^2(I, \mathbb{R}^d) \to L^2(I, \mathbb{R}^d)$

$$\mathcal{A}(t)\, y : i \in I \mapsto \int_I a(t,i,j) y(j)\mathrm{d}j,$$

as well as the **graph-Laplacian** $\mathbb{L}(t) : L^2(I, \mathbb{R}^d) \to L^2(I, \mathbb{R}^d)$ by

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Graphon dynamics – *Scrambling and diameter estimate*

Definition (Scrambling coefficient and diameter)[BPDS'22]

We define the **diameter** of a map $x \in L^\infty(I, \mathbb{R}^d)$

$$\mathcal{D}(x) := \sup_{i,j \in I} |x(i) - x(j)|$$

as well as the **scrambling coefficient** of a graphon $\mathcal{A}$ by

$$\eta(\mathcal{A}) := \inf_{i,j \in I} \int_I \min\{a(i,k), a(j,k)\} \mathrm{d}k.$$

Theorem (Quantitative diameter decay)[BPDS'22]

For each $x^0 \in L^\infty(I, \mathbb{R}^d)$, it holds that

$$\mathcal{D}(x(t)) \leq \mathcal{D}(x^0) \exp\left(- \int_0^t \eta(\mathcal{A}(s)) \mathrm{d}s \right).$$

Two technical novelties

- No **stochastic normalisation** trick $\rightsquigarrow$ **Geometric** argument.
- $t \mapsto \mathcal{D}(x(t))$ not diff. $\rightsquigarrow$ approx. with **Scorza-Dragoni**.

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Graphon dynamics – *Strong connectivity for graphons*

Definition (Graphon connectivity)[Boudin,Salvarini&Trélat'21]

A graphon $\mathcal{A}$ is **strongly connected** if the following holds.

(*i*) (**Connectivity**) For $\mathscr{L}^1$-almost every $i, j \in I$, there exists $i = k_1, \ldots, k_m = j$ such that $k_{l+1} \in \mathrm{supp}(a(k_l, \cdot))$.

(*ii*) (**Degree lower-bound**) $\inf_{i\in I} \int_I a(i,j)\mathrm{d}j \geq \delta > 0$.

Theorem (Canonical kernel of $\mathbb{L}^*$)[Boudin,Salvarini&Trélat'21]

If $\mathcal{A}$ is strongly connected, there exists a unique $v \in L^2(I, \mathbb{R}_+^*)$ s.t.

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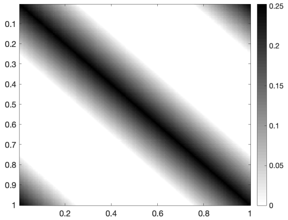

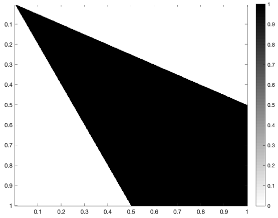

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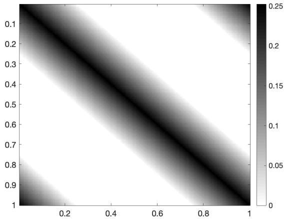

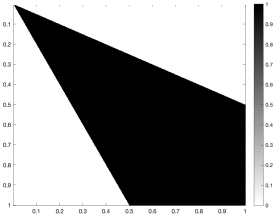

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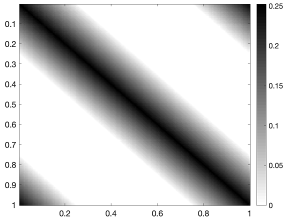

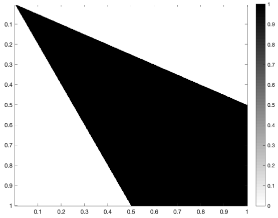

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For a graphon $\mathcal{A}$, the following connectivity characterisations hold.

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Graphon dynamics – *Variance decay and connectivity (1)*

Theorem (Variance decay for symmetric graphons) [BPDS'22,BF'21]

Suppose that $\mathcal{A}(t)$ is symmetric for a.e. $t \in \mathbb{R}_+$. Then for each $x^0 \in L^\infty(I, \mathbb{R}^d)$ and every $\tau > 0$, there exist $\alpha_\tau, \gamma_\tau > 0$ s.t.

$$\mathcal{V}(x(t)) \leq \alpha_\tau \mathcal{V}(x^0) \exp\left(-\gamma_\tau \int_0^t \lambda_2\Big(\tfrac{1}{\tau}\int_s^{s+\tau} \mathcal{A}(\sigma)\mathrm{d}\sigma\Big)\mathrm{d}s\right).$$

$$\mathcal{V}(x) := \frac{1}{2}\int_I |x(i) - \bar{x}|^2 \mathrm{d}i$$

$$\lambda_2\Big(\tfrac{1}{\tau}\int_s^{s+\tau} \mathcal{A}(\sigma)\mathrm{d}\sigma\Big) > 0$$

Idea: **Exponential** consensus with mere **average connectivity**.

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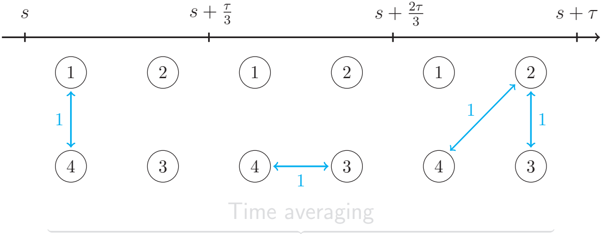

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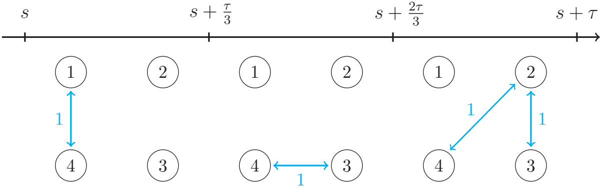

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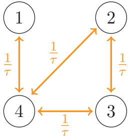

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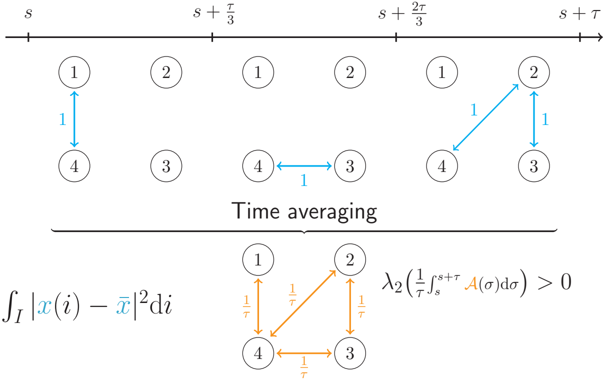

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Definition (Balanced interaction topology)

A graphon $\mathcal{A}$ is said to be **balanced** if $\mathbb{L}^* 1 = 0$, namely

$$\int_I a(i,j)\mathrm{d}j = \int_I a(j,i)\mathrm{d}j.$$

$\hookrightarrow$ Equality between the **in-degree** and **out-degree** at a.e. node.

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Suppose that $\mathcal{A}(t)$ is balanced for $\mathscr{L}^1$-almost every $t \in [0,T]$. Then for each $x^0 \in L^\infty(I, \mathbb{R}^d)$, it holds that

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Issue: If $\mathcal{A}(t)$ is DCUSCC $\rightsquigarrow$ $\mathcal{V}(\cdot)$ not **Lyapunov** anymore!

Definition (Weighted variance)

If a graphon $\mathcal{A}$ is DCUSCC, we define the **weighted variance** by

$$\mathcal{V}_v(x) := \frac{1}{2}\int_I \left|x(i) - \langle v, x\rangle_{L^2(I)}\right|^2 \mathrm{d}i.$$

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Observation: Under the **sufficient condition** for L^2-consensus

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Theorem (Equivalence between L^2- and L^∞-consensus)[BPDS'22]

Suppose that there exist constants $(\tau,\mu)\in\mathbb{R}_+^*\times(0,1]$ s.t.

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for $\mathscr{L}^1$-almost every $i\in I$. Then

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Conclusion – *That's all friends!*

1) Convergence to **consensus** in **micro** and **macro** dynamics.
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3) Still some interesting **open problems** to investigate!

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