

LECTURE 4: FUNDAMENTALS OF GENERAL RELATIVITY (II)

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In this lecture, we follow the convention of Ma & Bertschinger (hereafter M & B 95) [1] to summarise the Friedmann equations for the background and for the perturbations, which is the foundation of the CAMB code.¹

1. BACKGROUND

In M & B 95, the conformal time τ was used to denote time, instead of the physical time t , so let us make it clear that we shall use τ from now on, and use the overdot to denote the derivative wrt τ and $'$ for derivative wrt t .

Note that,

$$(1) \quad \tau \equiv \frac{t}{a}; \quad \mathcal{H} \equiv \frac{da}{d\tau}/a = \frac{\dot{a}}{a}; \quad H \equiv \frac{da}{dt}/a = \frac{a'}{a}; \quad \mathcal{H} = aH$$

$$(2) \quad a' = \frac{\dot{a}}{a}, \quad a'' = \frac{1}{a} \left(\frac{\ddot{a}}{a} - \mathcal{H}^2 \right)$$

The Friedmann equation in terms of conformal time,

$$(3) \quad \mathcal{H}^2 = \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} a^2 \rho$$

$$(4) \quad \frac{d}{d\tau} \left(\frac{\dot{a}}{a} \right) = -\frac{4\pi G}{3} a^2 (\rho + 3P)$$

2. PERTURBATION

The general metric including perturbations,

$$(5) \quad ds^2 = a(\tau)^2 \{ -(1 + 2\psi)d\tau^2 + 2w_i d\tau dx^i + [(1 - 2\phi)\gamma_{ij} + 2h_{ij}] dx^i dx^j \}$$

where

$$(6) \quad \gamma^{ij}\gamma_{jk} = \delta_k^i, \quad \gamma^{ij}h_{ij} = 0$$

So there are 1 (ψ) + 1 (ϕ) + 3 (w_i) + 6 (h_{ij}) - 1 ($\gamma^{ij}h_{ij} = 0$) = 10 degrees of freedom, which coincides with the number of independent entries of a 4×4 symmetric metric $g_{\mu\nu}$.

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¹Available at <http://camb.info>

If we focus on scalar perturbations, which is the most important and relevant part for large scale structure of the Universe, only ψ and ϕ are left, in the so-called Conformal-Newton gauge.

In this gauge, different components of the Einstein equation yield,

$$(7) \quad k^2\phi + 3\mathcal{H}(\dot{\phi} + \mathcal{H}\psi) = -4\pi G\rho\delta$$

$$(8) \quad k^2(\dot{\phi} + \mathcal{H}\psi) = 4\pi Ga^2(\rho + P)\theta$$

$$(9) \quad \psi = \phi$$

where $\delta \equiv \delta\rho/\rho$, and we have assumed that the fluids are perfect (thus there is no shear perturbations).

On sub-horizon scales, where the density perturbation is much more important than the velocity perturbation, we have the Poisson equation, by combining the first two equations,

$$(10) \quad \ddot{\delta} + \mathcal{H}\dot{\delta} = 4\pi G\rho a^2\delta$$

In a matter-dominated Universe, where,

$$(11) \quad H^2 = H_0^2(\Omega_m a^{-3})$$

so that

$$(12) \quad \mathcal{H}^2 \propto a^{-1}$$

and let us assume,

$$(13) \quad \delta \propto a^n$$

and substitute Eqs (13,12) into Eq (10), we find that,

$$(14) \quad n = 1$$

This means that in the matter-dominated era, $\delta \propto a$, which is an important conclusion to bear in mind!

An important implication of the Poisson equation Eq (10) is that one can perform consistency tests using it, as \mathcal{H} and $f \equiv \frac{d\ln \delta}{d\ln a}$ can be independently measured using SNe and RSD, respectively. If Eq (10) does not hold when the observables are substituted in, it might be a smoking gun of modified gravity!

There are other interesting implications of the Poisson equation. See [2] for an exercise of using it to reconstruct f from \mathcal{H} .

REFERENCES

- [1] C. P. Ma and E. Bertschinger, *Astrophys. J.* **455**, 7 (1995) doi:10.1086/176550 [astro-ph/9506072].
- [2] U. Alam, V. Sahni and A. A. Starobinsky, *Astrophys. J.* **704**, 1086 (2009) doi:10.1088/0004-637X/704/2/1086 [arXiv:0812.2846 [astro-ph]].
- [3] Dodelson, S. 2003, *Modern cosmology* / Scott Dodelson. Amsterdam (Netherlands): Academic Press. ISBN 0-12-219141-2, 2003, XIII + 440 p.

- [4] Peacock, J. A. 1999, *Cosmological Physics*, by John A. Peacock, pp. 704. ISBN 052141072X. Cambridge, UK: Cambridge University Press, January 1999., 704