

# Hall's marriage theorem

In [mathematics](#), **Hall's marriage theorem**, proved by [Philip Hall](#) (1935), is a theorem with two equivalent formulations:

- The [combinatorial](#) formulation deals with a collection of [finite sets](#). It gives a necessary and sufficient condition for being able to select a distinct element from each set.
- The [graph theoretic](#) formulation deals with a [bipartite graph](#). It gives a necessary and sufficient condition for finding a [matching](#) that covers at least one side of the graph.

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## Combinatorial formulation

Let  $\mathcal{S}$  be a (possibly infinite) [family](#) of finite [subsets](#) of  $X$ , where the members of  $\mathcal{S}$  are [counted with multiplicity](#) (That is,  $\mathcal{S}$  may contain the same set several times).<sup>[1]</sup>

A **transversal** for  $\mathcal{S}$  is the [image](#) of an [injective function](#)  $f$  from  $\mathcal{S}$  to  $X$  such that  $f(s)$  is an element of the set  $s$  for every  $s$  in the family  $\mathcal{S}$ . In other words,  $f$  selects one [representative](#) from each set in  $\mathcal{S}$  in such a way that no two of these representatives are equal. An alternative term for *transversal* is *system of distinct representatives*.

The collection  $\mathcal{S}$  satisfies the **marriage condition** when for each subfamily  $\mathcal{W} \subseteq \mathcal{S}$ ,

$$|\mathcal{W}| \leq \left| \bigcup_{A \in \mathcal{W}} A \right|.$$

Restated in words, the marriage condition asserts that every subfamily of  $\mathcal{S}$  contains at least as many distinct members as the number of sets in the subfamily.

If the marriage condition fails then there cannot be a transversal  $f$  of  $\mathcal{S}$ .

### Proof

Hall's theorem states that the converse is also true:

**Hall's Marriage Theorem** — A family  $\mathcal{S}$  of finite sets has a transversal if and only if  $\mathcal{S}$  satisfies the marriage condition.

## Examples

Example 1: Consider  $\mathcal{S} = \{A_1, A_2, A_3\}$  with

$$A_1 = \{1, 2, 3\}$$

$$A_2 = \{1, 4, 5\}$$

$$A_3 = \{3, 5\}.$$

A valid transversal would be (1, 4, 5). (Note this is not unique: (2, 1, 3) works equally well, for example.)

Example 2: Consider  $\mathcal{S} = \{A_1, A_2, A_3, A_4\}$  with

$$A_1 = \{2, 3, 4, 5\}$$

$$A_2 = \{4, 5\}$$

$$A_3 = \{5\}$$

$$A_4 = \{4\}.$$

No valid transversal exists; the marriage condition is violated as is shown by the subfamily  $\mathcal{W} = \{A_2, A_3, A_4\}$ . Here the number of sets in the subfamily is  $|\mathcal{W}| = 3$ , while the union of the three sets  $A_2 \cup A_3 \cup A_4$  contains only 2 elements of  $X$ .

Example 3: Consider  $\mathcal{S} = \{A_1, A_2, A_3, A_4\}$  with

$$A_1 = \{a, b, c\}$$

$$A_2 = \{b, d\}$$

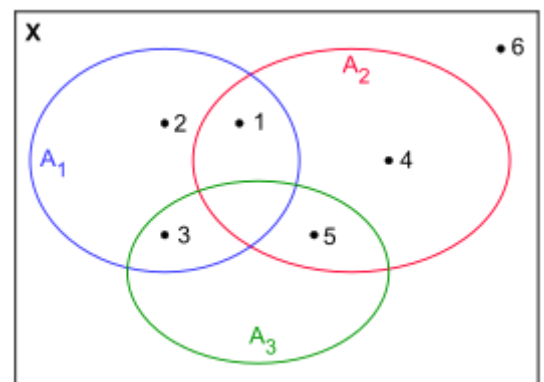
$$A_3 = \{a, b, d\}$$

$$A_4 = \{b, d\}.$$

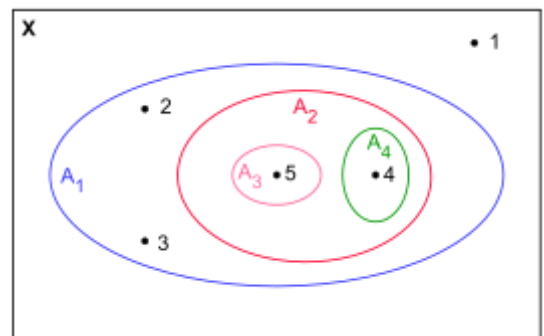
The only valid transversals are  $(c, b, a, d)$  and  $(c, d, a, b)$ .

## Application to marriage

The standard example of an application of the marriage theorem is to imagine two groups; one of  $n$  men, and one of  $n$  women. For each woman, there is a subset of the men, any one of which she would happily marry; and any man would be happy to marry a woman who wants to marry him. Consider whether it is possible to pair up (in marriage) the men and women so that every person is happy.



example 1, marriage condition met



example 2, marriage condition violated

If we let  $A_i$  be the set of men that the  $i$ -th woman would be happy to marry, then the marriage theorem states that each woman can happily marry a unique man if and only if for any subset  $I$  of the women, the number of men whom at least one of these women would be happy to marry,  $|\bigcup_{i \in I} A_i|$ , is at least as big as the number of women in that subset,  $|I|$ .

This condition is *necessary*, as if it does not hold, there are not enough men to share among the  $|I|$  women. What is interesting is that it is also a *sufficient* condition.

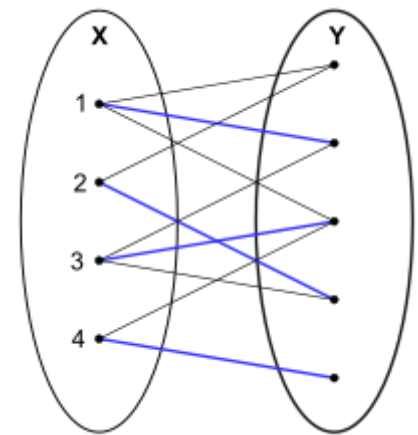
## Graph theoretic formulation

Let  $G$  be a finite bipartite graph with bipartite sets  $X$  and  $Y$  (i.e.  $G := (X + Y, E)$ ). An  *$X$ -perfect matching* (also called:  *$X$ -saturating matching*) is a matching which covers every vertex in  $X$ .

For a subset  $W$  of  $X$ , let  $N_G(W)$  denote the neighborhood of  $W$  in  $G$ , i.e., the set of all vertices in  $Y$  adjacent to some element of  $W$ . The marriage theorem in this formulation states that there is an  $X$ -perfect matching if and only if for every subset  $W$  of  $X$ :

$$|W| \leq |N_G(W)|.$$

In other words: every subset  $W$  of  $X$  has sufficiently many adjacent vertices in  $Y$ .



blue edges represent a matching

## Proof of the graph theoretic version

**Easy direction:** we assume that some matching  $M$  saturates every vertex of  $X$ , and prove that Hall's condition is satisfied for all  $W \subseteq X$ . Let  $M(W)$  denote the set of all vertices in  $Y$  matched by  $M$  to a given  $W$ . By definition of a matching,  $|M(W)| = |W|$ . But  $M(W) \subseteq N_G(W)$ , since all elements of  $M(W)$  are neighbours of  $W$ . So,  $|N_G(W)| \geq |M(W)|$  and hence,  $|N_G(W)| \geq |W|$ .

**Hard direction:** we assume that there is no  $X$ -perfect matching and prove that Hall's condition is violated for at least one  $W \subseteq X$ . Let  $M$  be a maximum matching, and let  $u$  be a vertex of  $X$  not saturated by  $M$ . Consider all *alternating paths* (i.e., paths in  $G$  alternately using edges outside and inside  $M$ ) starting from  $u$ . Let the set of all points in  $Y$  connected to  $u$  by these alternating paths be  $Z$ , and the set of all points in  $X$  connected to  $u$  by these alternating paths (including  $u$  itself) be  $W$ . No maximal alternating path can end in a vertex in  $Y$ , lest it would be an *augmenting path*, so that we could augment  $M$  to a strictly larger matching by toggling the status (belongs to  $M$  or not) of all the edges of the path. Thus every vertex in  $Z$  is matched by  $M$  to a vertex in  $W \setminus \{u\}$ . Conversely, every vertex  $v$  in  $W \setminus \{u\}$  is matched by  $M$  to a vertex in  $Z$  (namely, the vertex preceding  $v$  on an alternating path ending at  $v$ ). Thus,  $M$  provides a bijection of  $W \setminus \{u\}$  and  $Z$ , which implies  $|W| = |Z| + 1$ . On the other hand,  $N_G(W) \subseteq Z$ . Indeed, let  $v$  in  $N_G(W)$  be connected to a vertex  $w$  in  $W$ . If the edge  $(w, v)$  is in  $M$ , then  $v$  precedes  $w$  in any alternating path that goes from  $u$  to  $w$ . If the edge  $(w, v)$  is not in  $M$ , then one can use it to extend any alternating path that goes from  $u$  to  $w$ . In either case,  $v$  is visited by some alternating path issued from  $u$ . Hence  $N_G(W) \subseteq Z$ , and so  $|N_G(W)| \leq |Z| = |W| - 1 < |W|$ .

## Constructive proof of the hard direction

Define a Hall violator as a subset  $W$  of  $X$  for which  $|N_G(W)| < |W|$ . If  $W$  is a Hall violator, then there is no matching that saturates all vertices of  $W$ . Therefore, there is also no matching that saturates  $X$ . Hall's marriage theorem says that a graph contains an  $X$ -perfect matching if and only if it contains no Hall violators.

The following algorithm proves the hard direction of the theorem: it finds either an  $X$ -perfect matching or a Hall violator. It uses, as a subroutine, an algorithm that, given a matching  $M$  and an unmatched vertex  $x_0$ , either finds an  $M$ -augmenting path or a Hall violator containing  $x_0$ .

It uses

1. Initialize  $M := \{\}$ . // Empty matching.
2. Assert:  $M$  is a matching in  $G$ .
3. If  $M$  saturates all vertices of  $X$ , then **return the  $X$ -perfect matching  $M$** .
4. Let  $x_0$  be an unmatched vertex (a vertex in  $X \setminus V(M)$ ).
5. Using the Hall violator algorithm, find either a Hall violator or an  $M$ -augmenting path.
6. In the first case, **return the Hall violator**.
7. In the second case, use the  $M$ -augmenting path to increase the size of  $M$  (by one edge), and **go back to step 2**.

At each iteration,  $M$  grows by one edge. Hence, this algorithm must end after at most  $|E|$  iterations. Each iteration takes at most  $|X|$  time. The total runtime complexity is similar to the Ford-Fulkerson method for finding a maximum cardinality matching.

## Equivalence of the combinatorial formulation and the graph-theoretic formulation

Let  $S = (A_1, A_2, \dots, A_n)$  where the  $A_i$  are finite sets which need not be distinct. Let the set  $X = \{A_1, A_2, \dots, A_n\}$  (that is, the set of names of the elements of  $S$ ) and the set  $Y$  be the union of all the elements in all the  $A_i$ .

We form a finite bipartite graph  $G := (X + Y, E)$ , with bipartite sets  $X$  and  $Y$  by joining any element in  $Y$  to each  $A_i$  which it is a member of. A transversal of  $S$  is an  $X$ -perfect matching (a matching which covers every vertex in  $X$ ) of the bipartite graph  $G$ . Thus a problem in the combinatorial formulation can be easily translated to a problem in the graph-theoretic formulation.

## Topological proof

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Hall's theorem can be proved (non-constructively) based on Sperner's lemma.<sup>[2]</sup>:Thm.4.1,4.2

## Applications

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The theorem has many other interesting "non-marital" applications. For example, take a standard deck of cards, and deal them out into 13 piles of 4 cards each. Then, using the marriage theorem, we can show that it is always possible to select exactly 1 card from each pile, such that the 13 selected cards contain exactly one card of each rank (Ace, 2, 3, ..., Queen, King).

More abstractly, let  $G$  be a group, and  $H$  be a finite subgroup of  $G$ . Then the marriage theorem can be used to show that there is a set  $T$  such that  $T$  is a transversal for both the set of left cosets and right cosets of  $H$  in  $G$ .

The marriage theorem is used in the usual proofs of the fact that an  $(r \times n)$  Latin rectangle can always be extended to an  $((r+1) \times n)$  Latin rectangle when  $r < n$ , and so, ultimately to a Latin square.

## Logical equivalences

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This theorem is part of a collection of remarkably powerful theorems in combinatorics, all of which are related to each other in an informal sense in that it is more straightforward to prove one of these theorems from another of them than from first principles. These include:

- The König–Egerváry theorem (1931) (Dénes Kőnig, Jenő Egerváry)
- König's theorem<sup>[3]</sup>
- Menger's theorem (1927)
- The max-flow min-cut theorem (Ford–Fulkerson algorithm)
- The Birkhoff–Von Neumann theorem (1946)
- Dilworth's theorem.

In particular,<sup>[4][5]</sup> there are simple proofs of the implications Dilworth's theorem  $\Leftrightarrow$  Hall's theorem  $\Leftrightarrow$  König–Egerváry theorem  $\Leftrightarrow$  König's theorem.

## Infinite families

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### Marshall Hall Jr. variant

By examining Philip Hall's original proof carefully, Marshall Hall, Jr. (no relation to Philip Hall) was able to tweak the result in a way that permitted the proof to work for infinite  $S$ .<sup>[6]</sup> This variant refines the marriage theorem and provides a lower bound on the number of transversals that a given  $S$  may have. This variant is:<sup>[7]</sup>

Suppose that  $(A_1, A_2, \dots, A_n)$ , where the  $A_i$  are finite sets that need not be distinct, is a family of sets satisfying the marriage condition, and suppose that  $|A_i| \geq r$  for  $i = 1, \dots, n$ . Then the number of different transversals for the family is at least  $r!$  if  $r \leq n$  and  $r(r-1) \dots (r-n+1)$  if  $r > n$ .

Recall that a transversal for a family  $S$  is an ordered sequence, so two different transversals could have exactly the same elements. For instance, the family  $A_1 = \{1, 2, 3\}$ ,  $A_2 = \{1, 2, 5\}$  has both  $(1, 2)$  and  $(2, 1)$  as distinct transversals.

### Marriage condition does not extend

The following example, due to Marshall Hall, Jr., shows that the marriage condition will not guarantee the existence of a transversal in an infinite family in which infinite sets are allowed.

Let  $S$  be the family,  $A_0 = \{1, 2, 3, \dots\}$ ,  $A_1 = \{1\}$ ,  $A_2 = \{2\}$ ,  $\dots$ ,  $A_i = \{i\}$ ,  $\dots$

The marriage condition holds for this infinite family, but no transversal can be constructed.<sup>[8]</sup>

The more general problem of selecting a (not necessarily distinct) element from each of a collection of non-empty sets (without restriction as to the number of sets or the size of the sets) is permitted in general only if the axiom of choice is accepted.

The marriage theorem does extend to the infinite case if stated properly. Given a bipartite graph with sides  $A$  and  $B$ , we say that a subset  $C$  of  $B$  is smaller than or equal in size to a subset  $D$  of  $A$  *in the graph* if there exists an injection in the graph (namely, using only edges of the graph) from  $C$  to  $D$ , and that it is strictly smaller in the graph if in addition there is no injection in the graph in the other direction. Note that omitting *in the graph* yields the ordinary notion of comparing cardinalities. The infinite marriage theorem states that there exists an injection from  $A$  to  $B$  in the graph, if and only if there is no subset  $C$  of  $A$  such that  $N(C)$  is strictly smaller than  $C$  in the graph.<sup>[9]</sup>

## Fractional matching variant

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A *fractional matching* in a graph is an assignment of non-negative weights to each edge, such that the sum of weights adjacent to each vertex is at most 1. A fractional matching is *X*-perfect if the sum of weights adjacent to each vertex is exactly 1. The following are equivalent for a bipartite graph  $G = (X+Y, E)$ :<sup>[10]</sup>

- $G$  admits an  $X$ -perfect matching.
- $G$  admits an  $X$ -perfect fractional matching. The implication follows directly from the fact that  $X$ -perfect matching is a special case of an  $X$ -perfect fractional matching, in which each weight is either 1 (if the edge is in the matching) or 0 (if it is not).
- $G$  satisfies Hall's marriage condition. The implication holds because, for each subset  $W$  of  $X$ , the sum of weights near vertices of  $W$  is  $|W|$ , so the edges adjacent to them are necessarily adjacent to at least  $|W|$  vertices of  $Y$ .

## Quantitative variant

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When Hall's condition does not hold, the original theorem tells us only that a perfect matching does not exist, but does not tell what is the largest matching that does exist. To learn this information, we need the notion of deficiency of a graph. Given a bipartite graph  $G = (X+Y, E)$ , the *deficiency of  $G$  w.r.t.  $X$*  is the maximum, over all subsets  $W$  of  $X$ , of the difference  $|W| - |N_G(W)|$ . The larger is the deficiency, the farther is the graph from satisfying Hall's condition.

Using Hall's marriage theorem, it can be proved that, if the deficiency of a bipartite graph  $G$  is  $d$ , then  $G$  admits a matching of size at least  $|X|-d$ .

## Generalizations

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- A generalization of Hall's theorem to general graphs (that are not necessarily bipartite) is provided by the Tutte theorem.
- A generalization of Hall's theorem to bipartite hypergraphs is provided by various Hall-type theorems for hypergraphs.