Shortest-Path Algorithms II

Shortest Path with Negative Costs

Given a weighted directed graph (weights can be negative), a start node **s**, and end node **e**, find the cost of the shortest path from **s** to **e** (or report that it does not exist)

Using Dijkstra’s Algorithm

Does Dijkstra’s Algorithm work for this problem?

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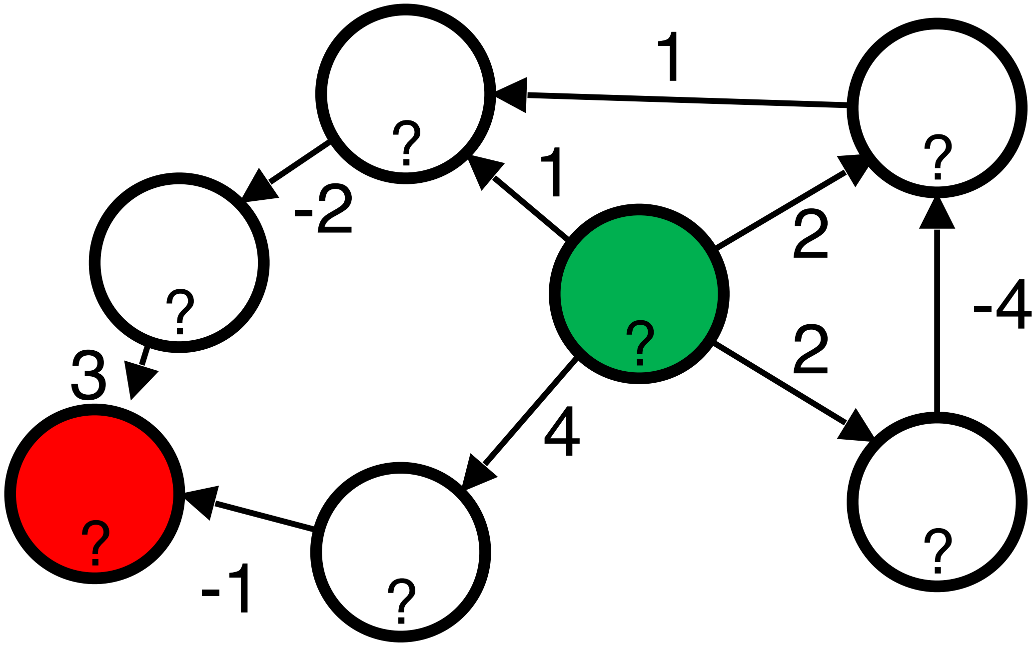
Breaks invariants

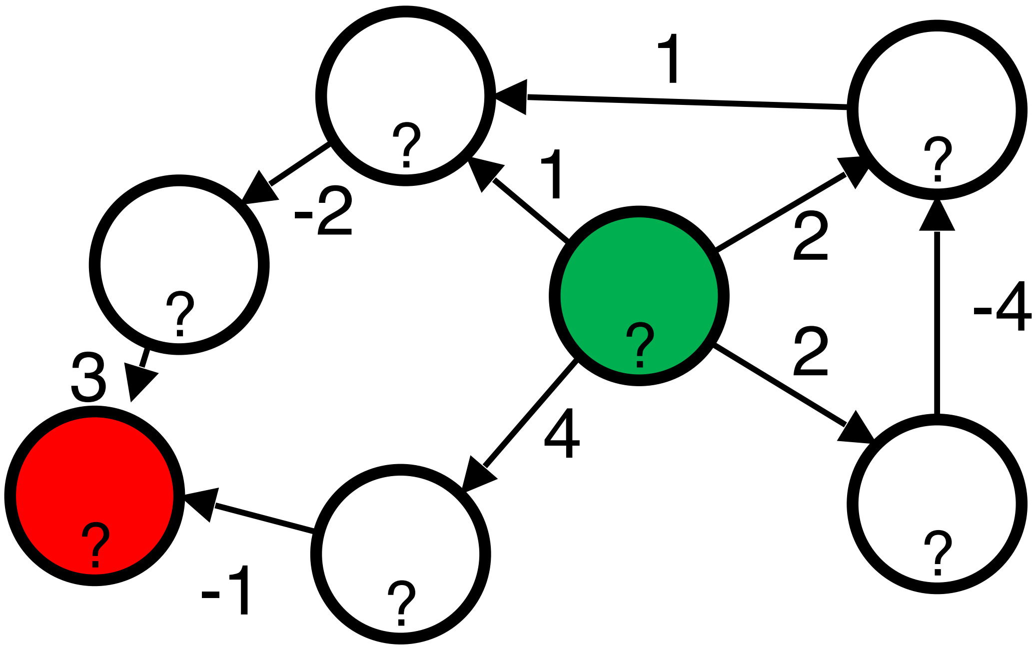
Negative Costs

start state
goal states



start state
goal states

Negative Costs



Is there anything we **know**?

Adjustment: Adding Constant

Proposed Idea: What if we add a fixed constant to each edge such that all negative edges become positive?

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Doesn’t work because this penalizes paths with more edges

Edge Relaxation

```
for each edge e:
  if(e.dest.cost > e.src.cost + e.cost)
    e.dest.cost = e.src.cost + e.cost;
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Questions:

1. Will this process terminate?
2. How many iterations are needed?

Key Observations

If the shortest path to a node involves L edges, it will be found in L iterations.

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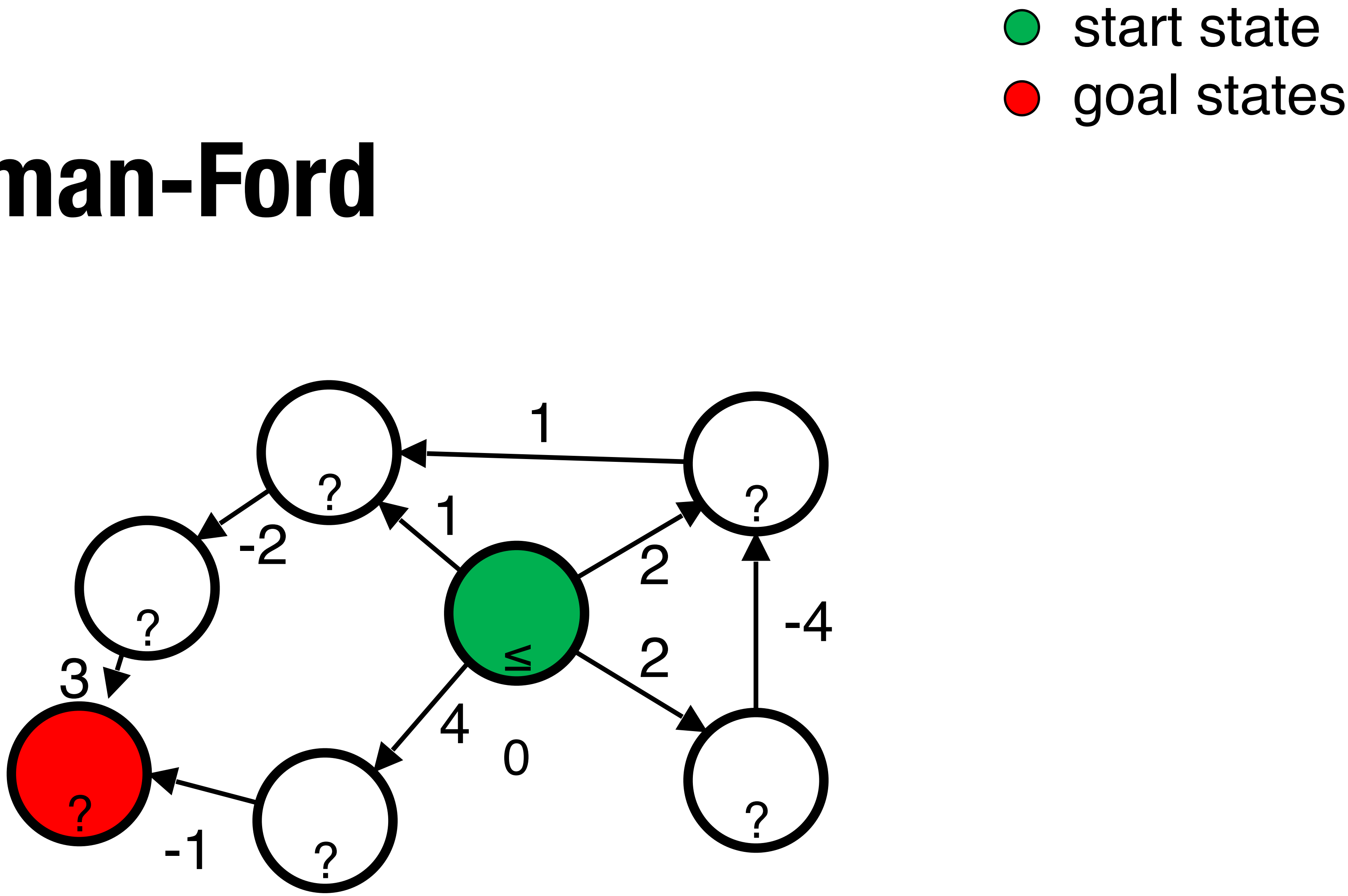
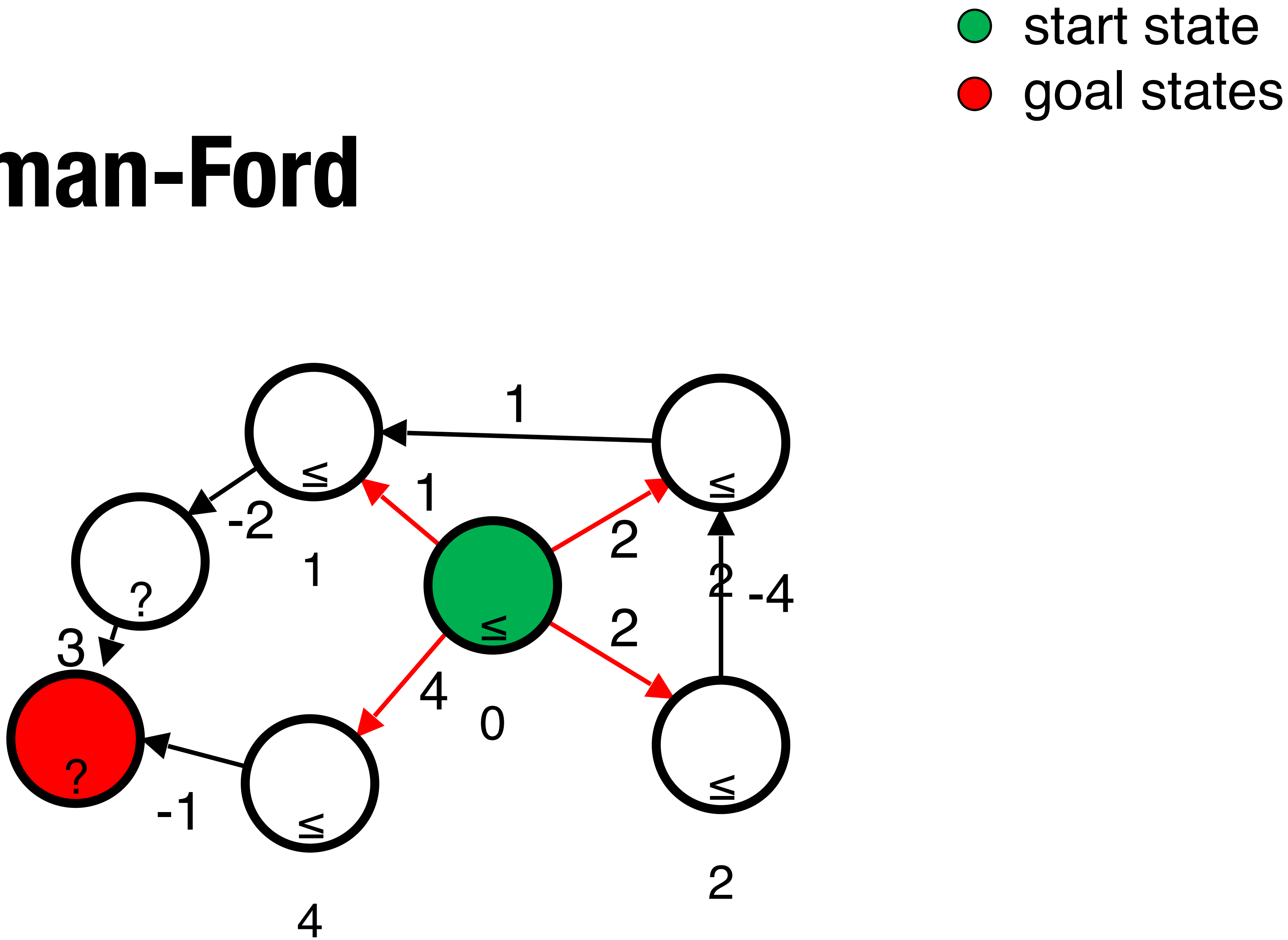
The shortest path to any node is either:

1) <= |V|
2) infinite

Bellman-Ford

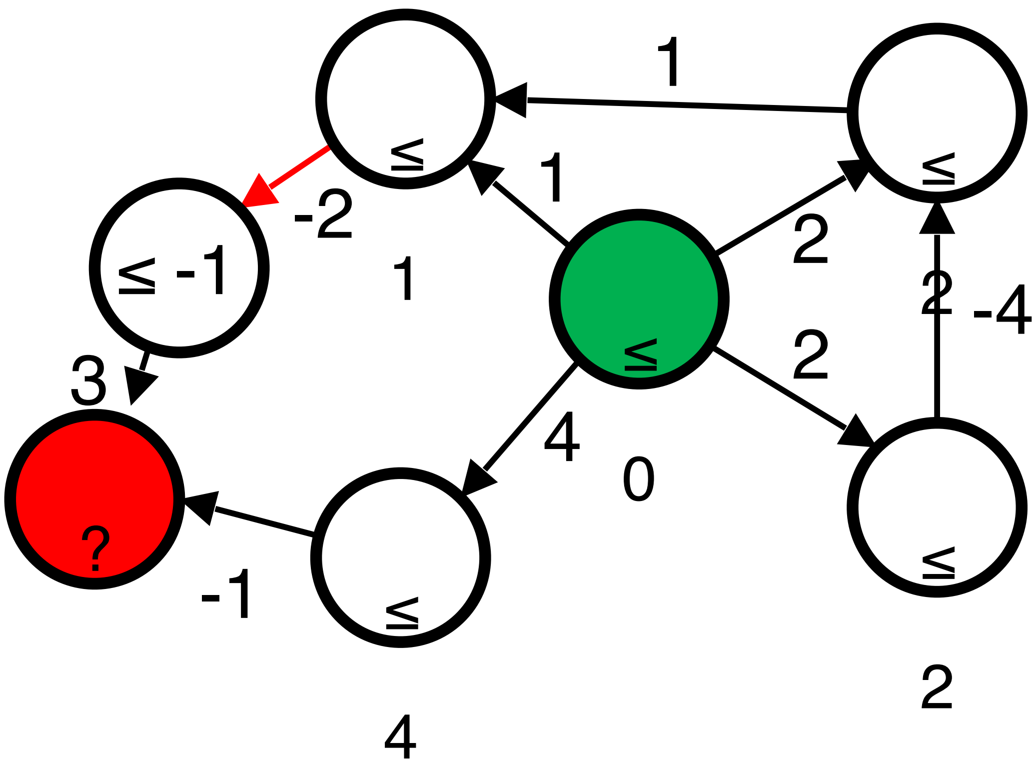


Bellman-Ford



start state
goal states

Bellman-Ford



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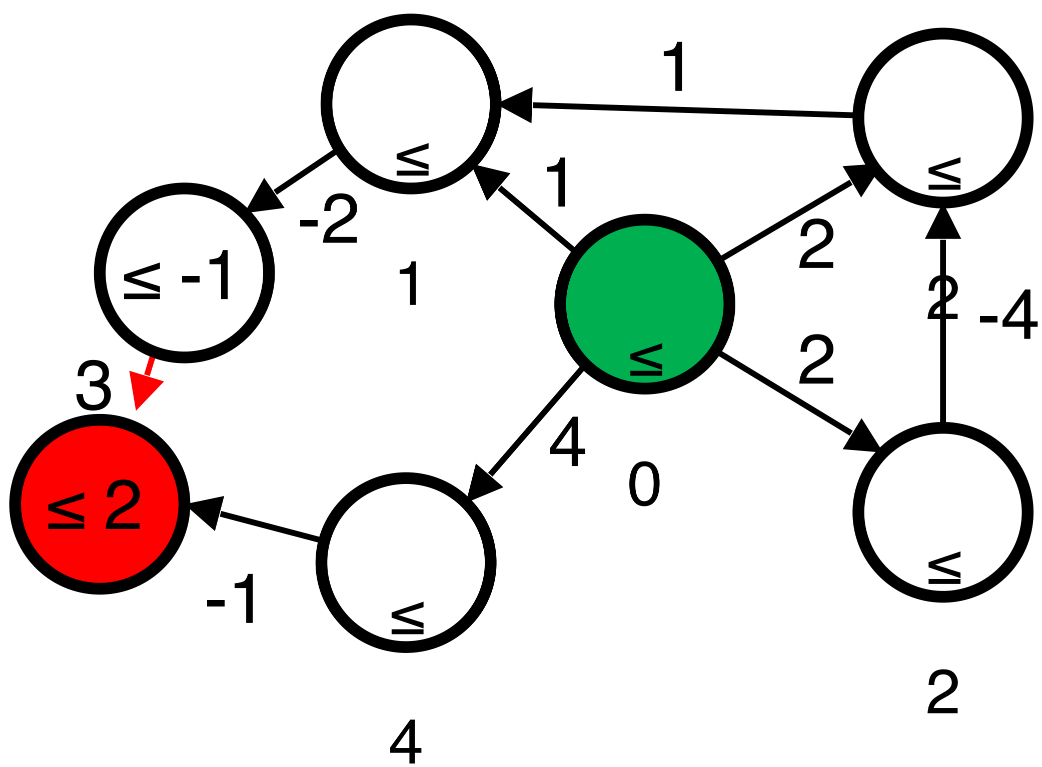

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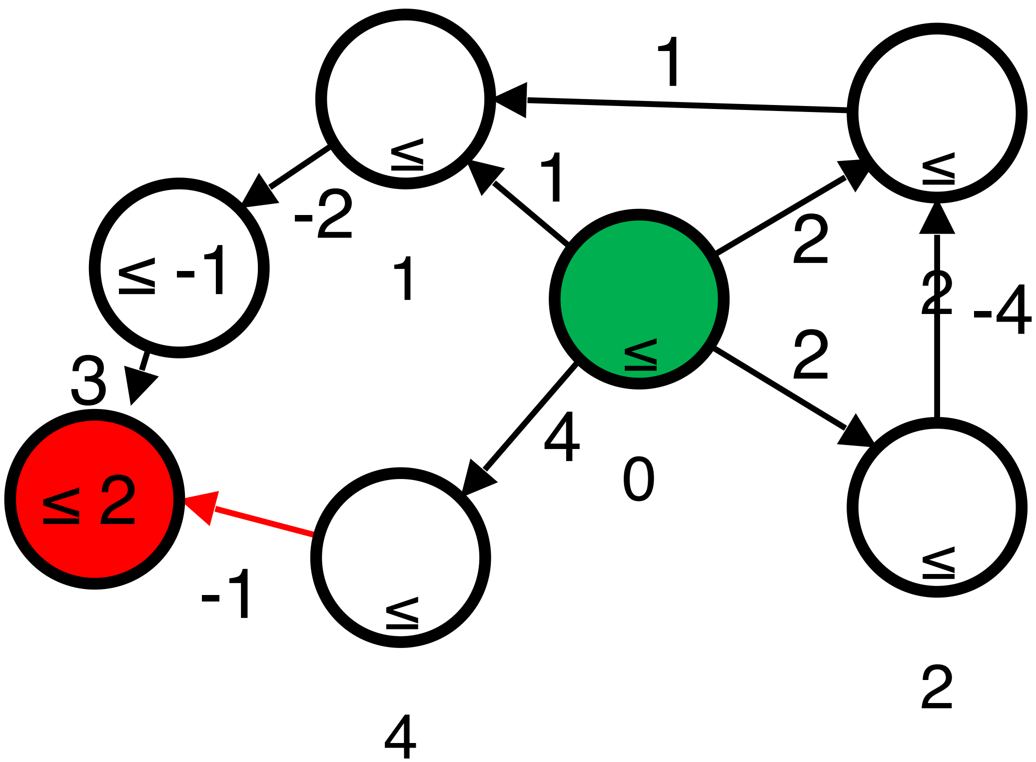

start state
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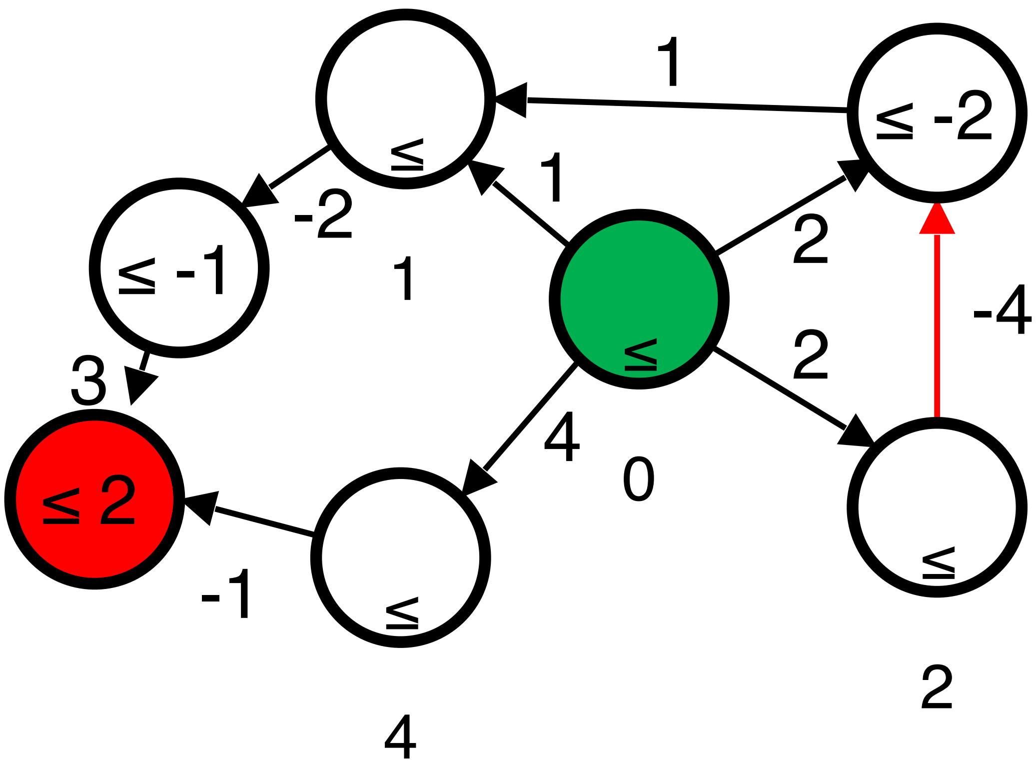

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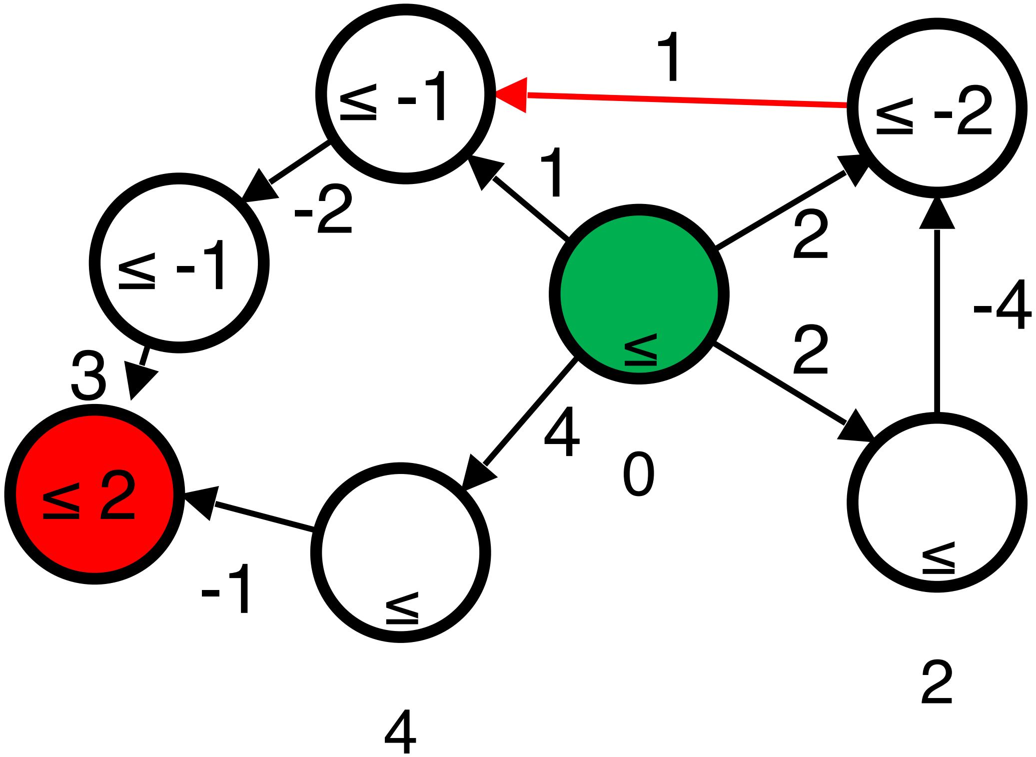

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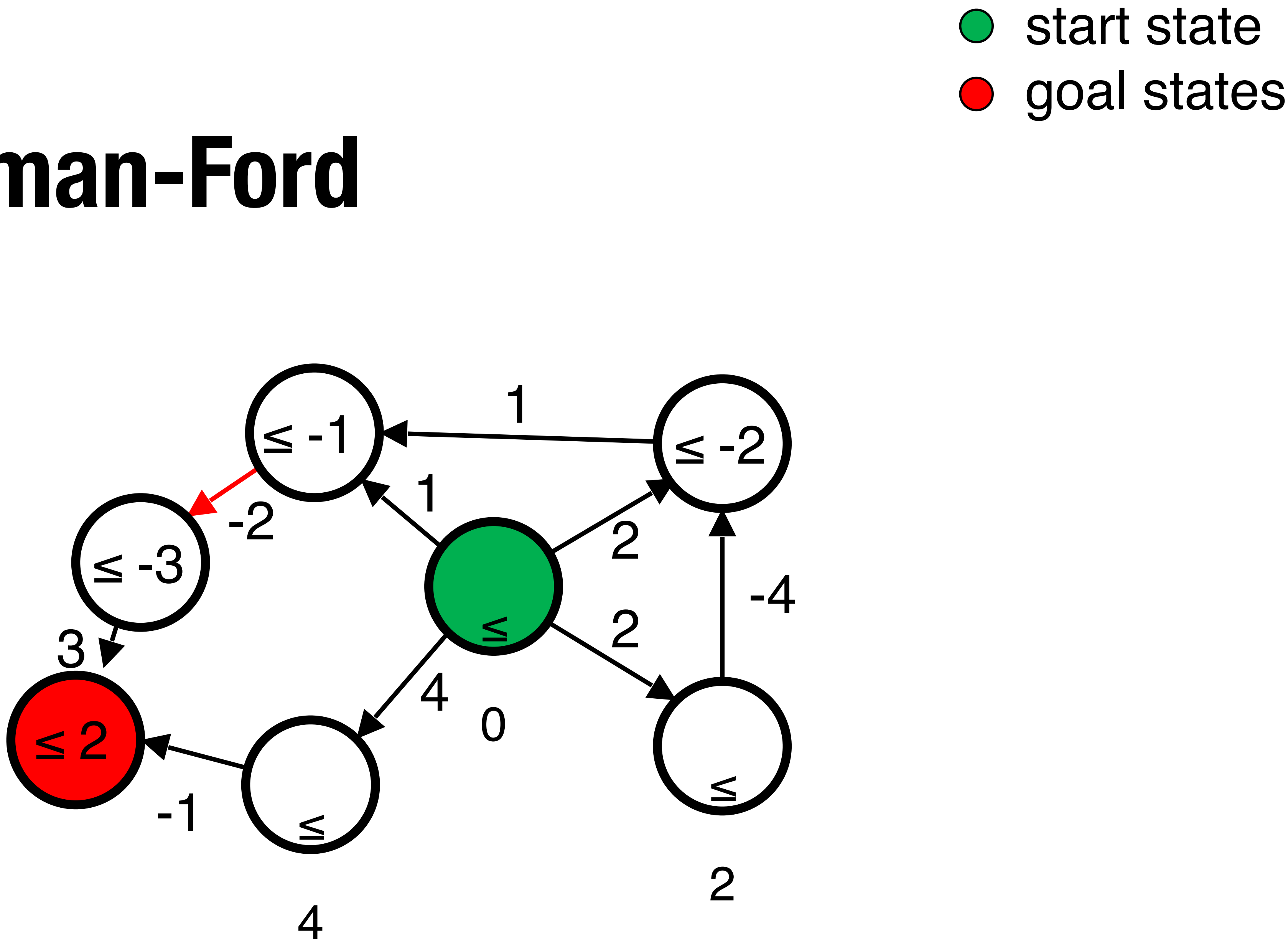

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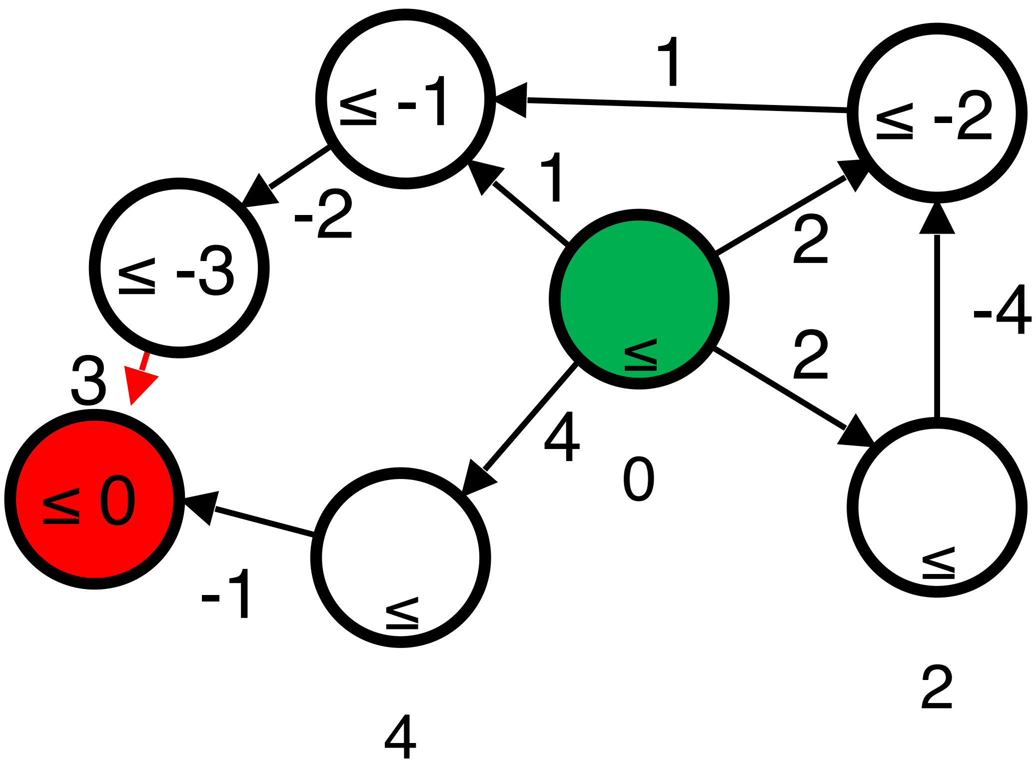
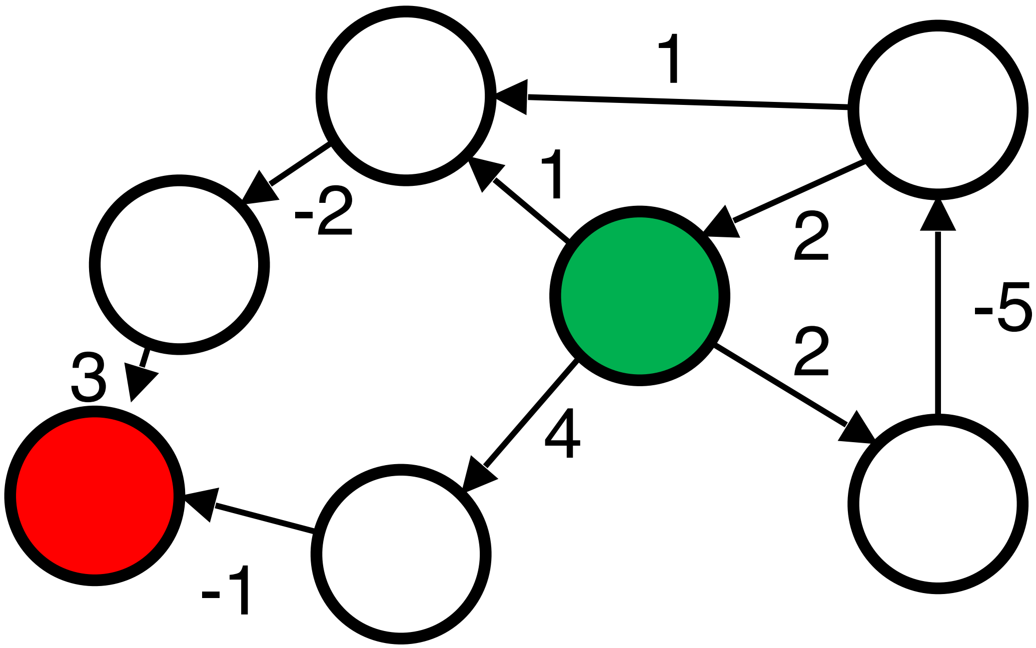
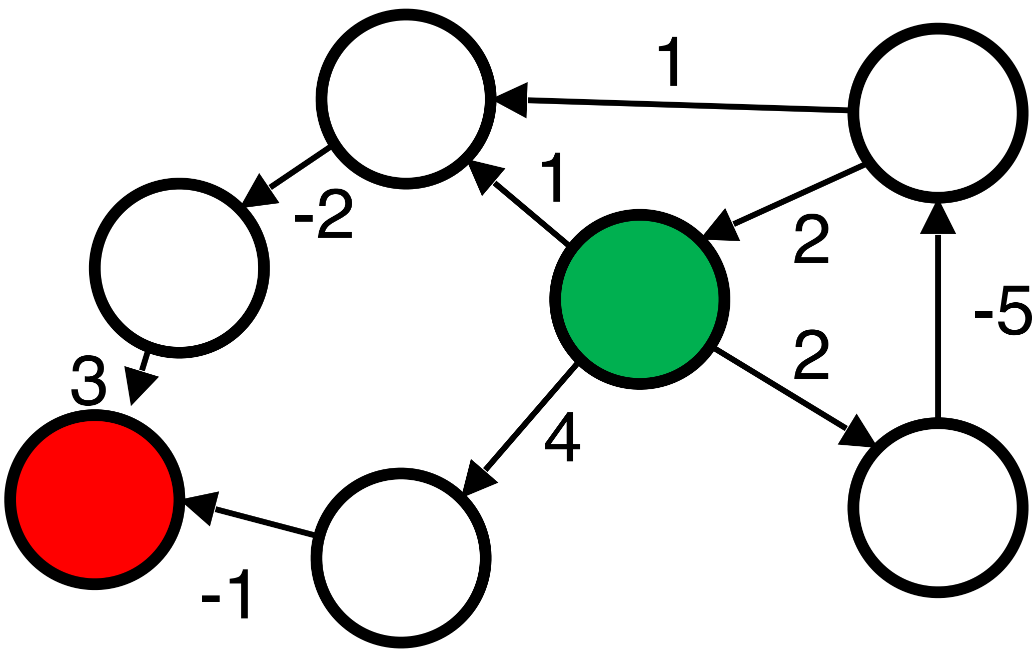


start state
goal states

Termination?

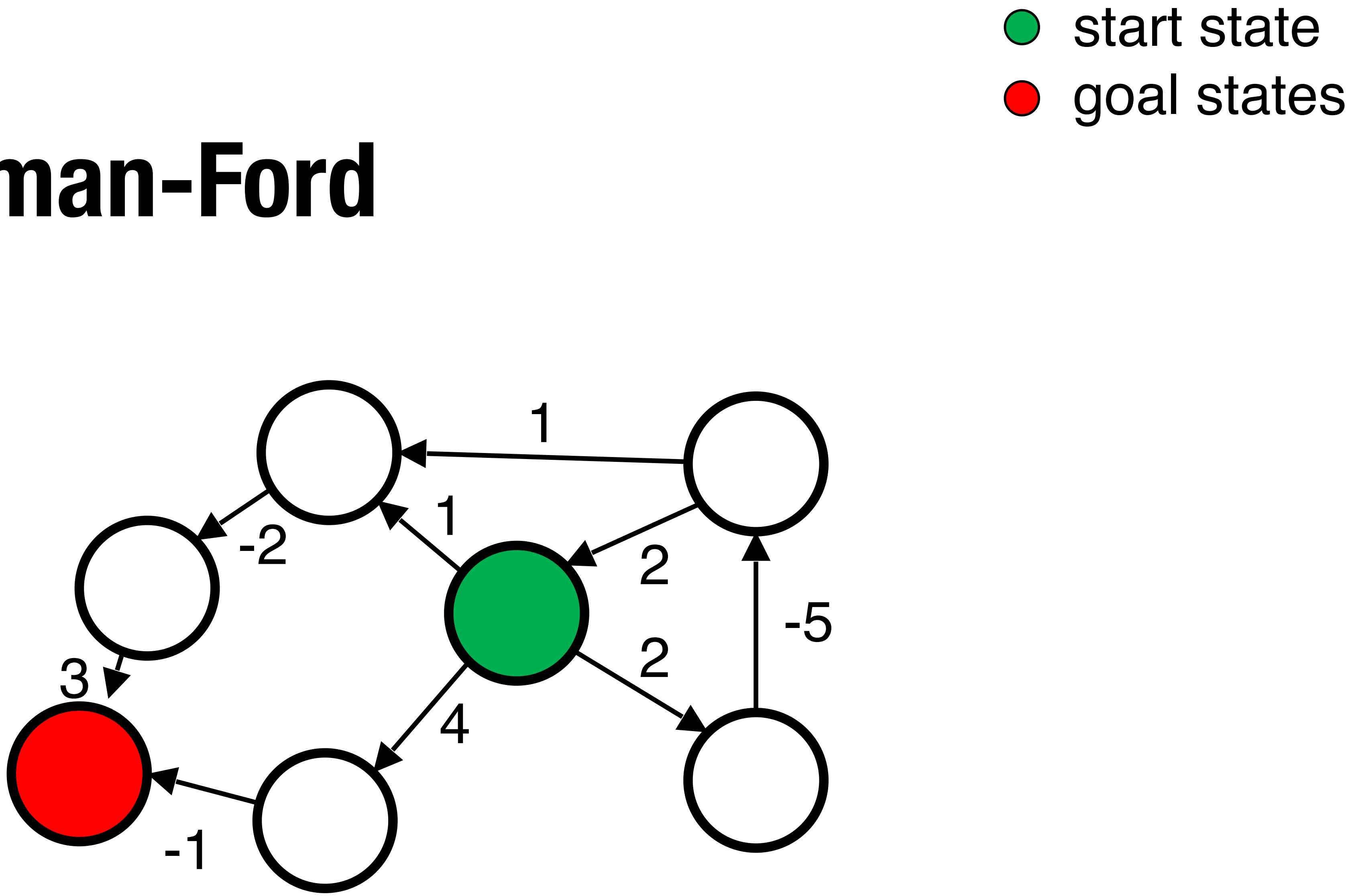


Termination?



changes on the V+1st iteration => negative cycle

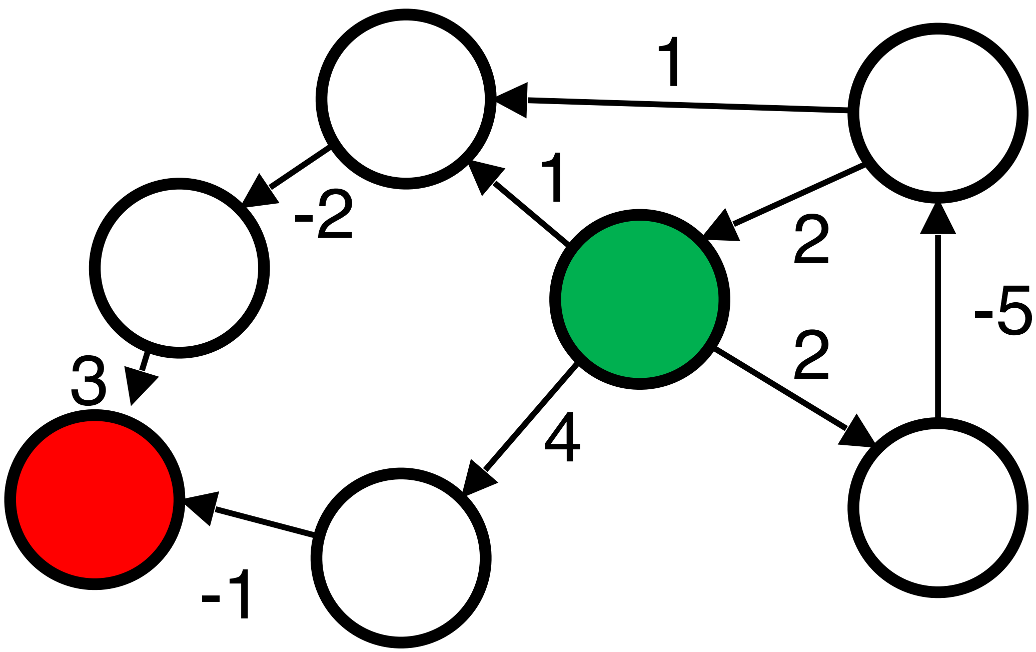
Bellman-Ford



What’s the run time?

start state
goal states

Bellman-Ford



Changes past iter $|V| \Rightarrow$ cycle

What's the run time? O(|V||E|)

Bellman-Ford Pseudocode

for **i** from 1 to |V| - 1:

    for edge (**u, v**) with weight **w**:

        if(dist(**v**) > dist(**u**) + **w**)

            dist(**v**) = dist(**u**) + **w**;

Time complexity O(|V||E|)

All-Pairs Shortest Path

Given a weighted, directed graph (negative weights possible). For each pairs of vertices (**v**, **w**), return length of shortest path from **v** to **w**.

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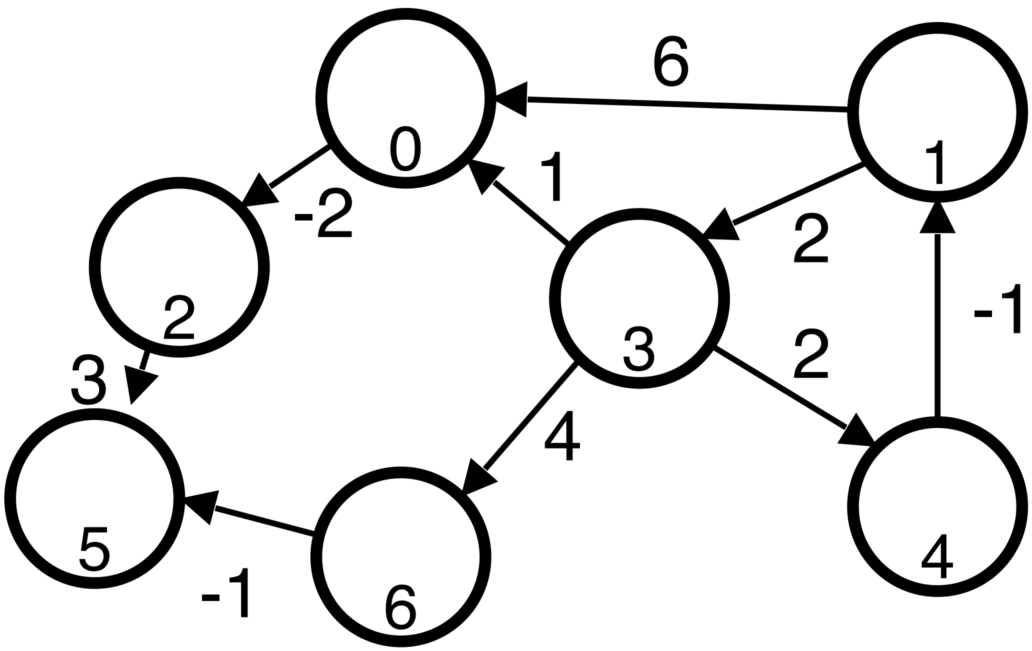
Run Bellman-Ford for each **v**: $O(|V|^2|E|)$

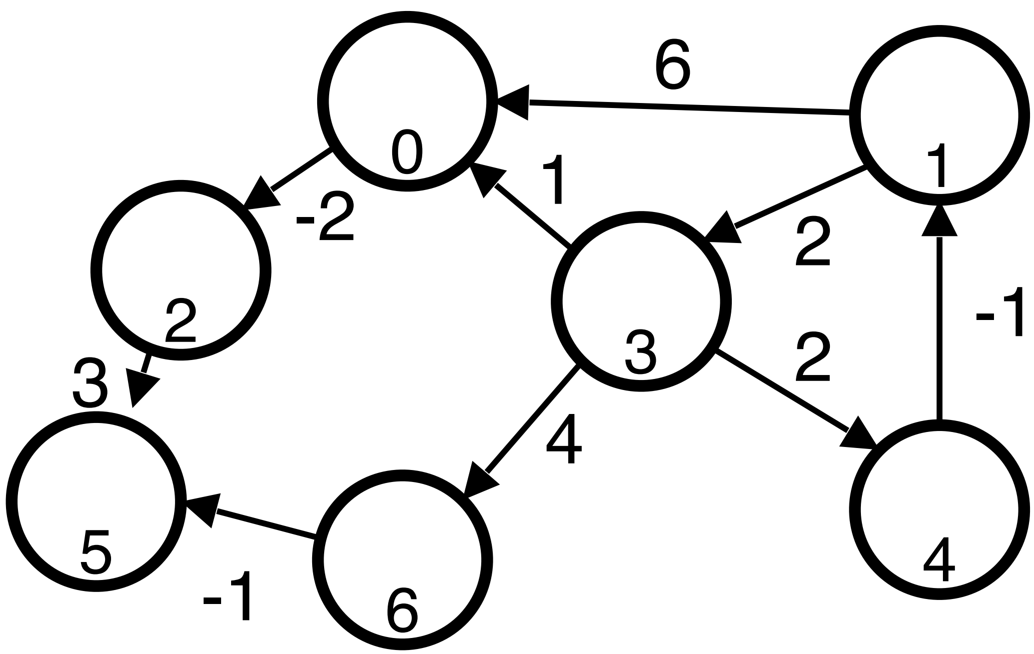
All-Pairs Shortest Path



$$\begin{bmatrix} 0 & ? & ? & ? & ? & ? & ? \\ ? & 0 & ? & ? & ? & ? & ? \\ ? & ? & 0 & ? & ? & ? & ? \\ ? & ? & ? & 0 & ? & ? & ? \\ ? & ? & ? & ? & 0 & ? & ? \\ ? & ? & ? & ? & ? & 0 & ? \\ ? & ? & ? & ? & ? & ? & 0 \end{bmatrix}$$

Vertex costs no longer useful; create **distance matrix**

All-Pairs Shortest Path



$$\begin{bmatrix} 0 & ? & -2 & ? & ? & ? & ? \\ 6 & 0 & ? & 2 & ? & ? & ? \\ ? & ? & 0 & ? & ? & 3 & ? \\ 1 & ? & ? & 0 & 2 & ? & 4 \\ ? & -1 & ? & ? & 0 & ? & ? \\ ? & ? & ? & ? & ? & 0 & ? \\ ? & ? & ? & ? & ? & -1 & 0 \end{bmatrix}$$

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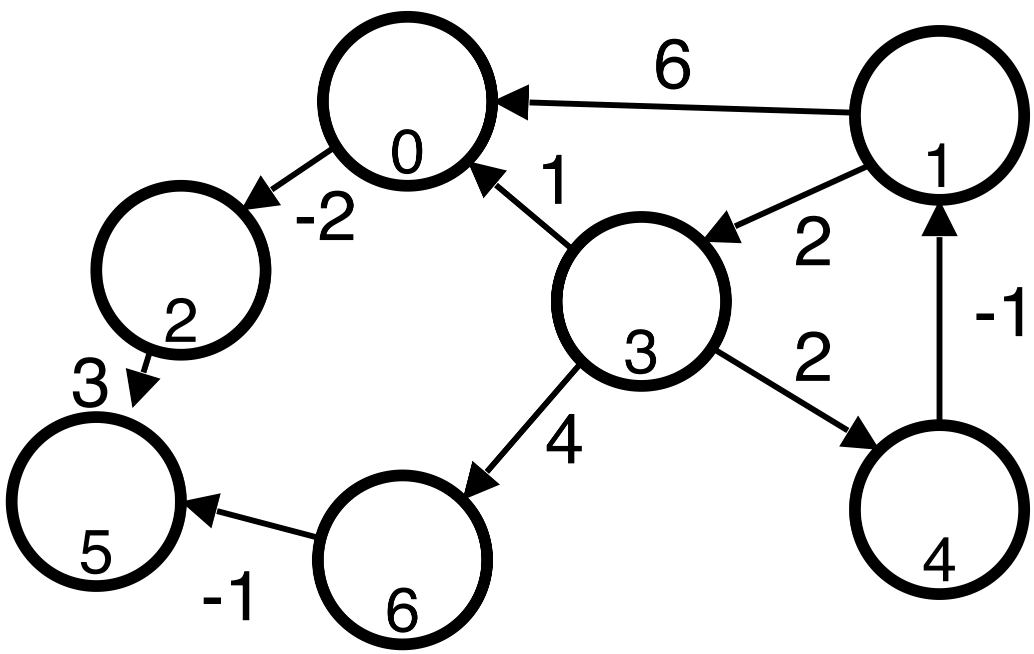

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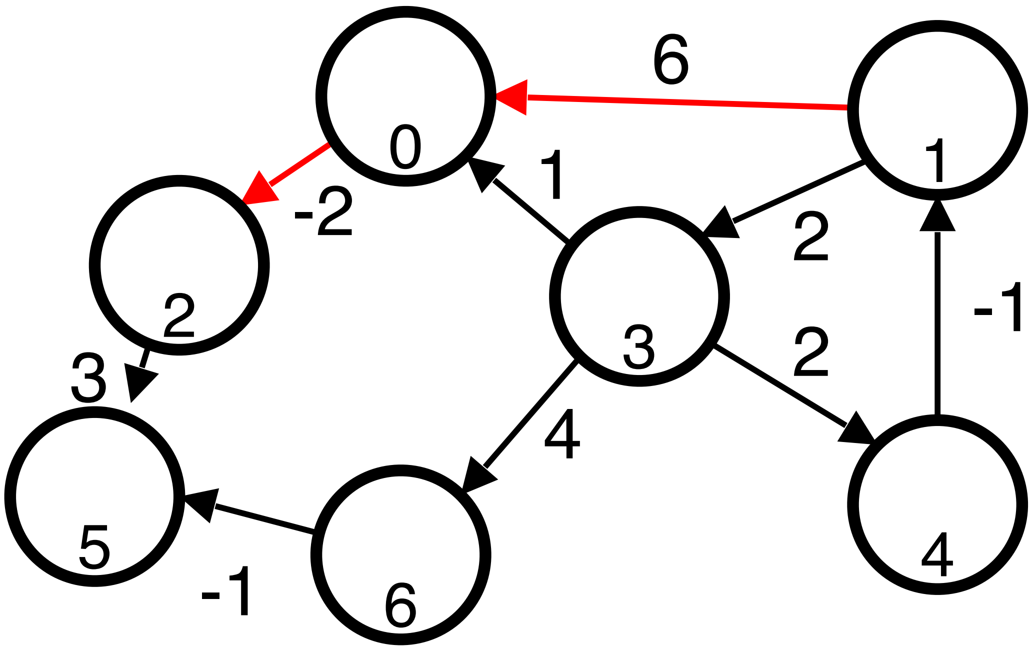
Which paths can improve by passing through node 0?

All-Pairs Shortest Path



$$\begin{bmatrix} 0 & ? & \boxed{-2} & ? & ? & ? & ? \\ \boxed{6} & 0 & \boxed{?} & 2 & ? & ? & ? \\ ? & ? & 0 & ? & ? & 3 & ? \\ 1 & ? & ? & 0 & 2 & ? & 4 \\ ? & -1 & ? & ? & 0 & ? & ? \\ ? & ? & ? & ? & ? & 0 & ? \\ ? & ? & ? & ? & ? & -1 & 0 \end{bmatrix}$$

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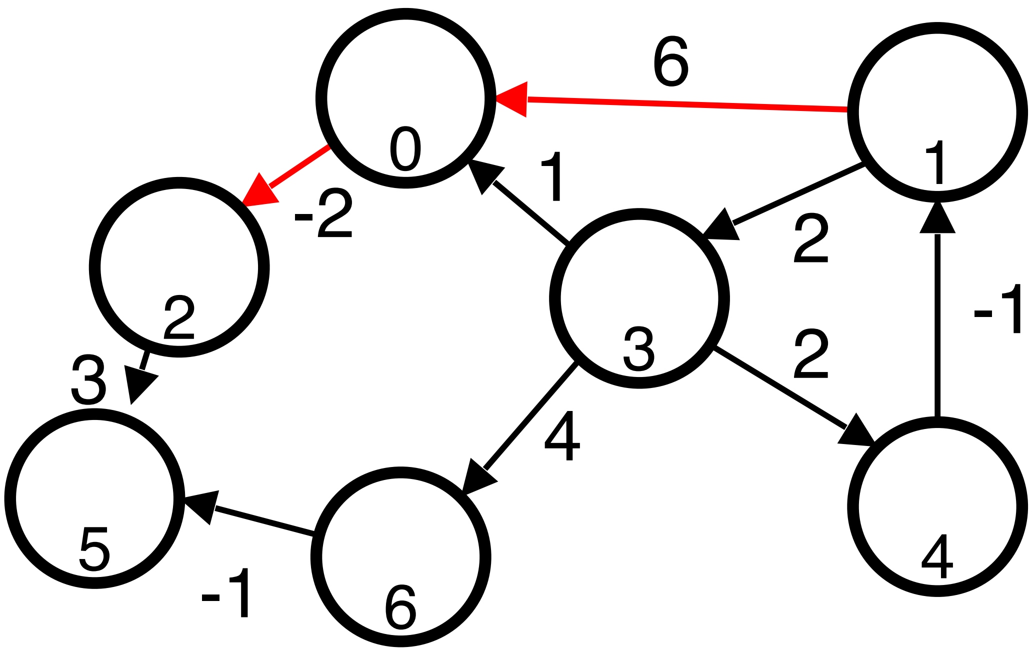

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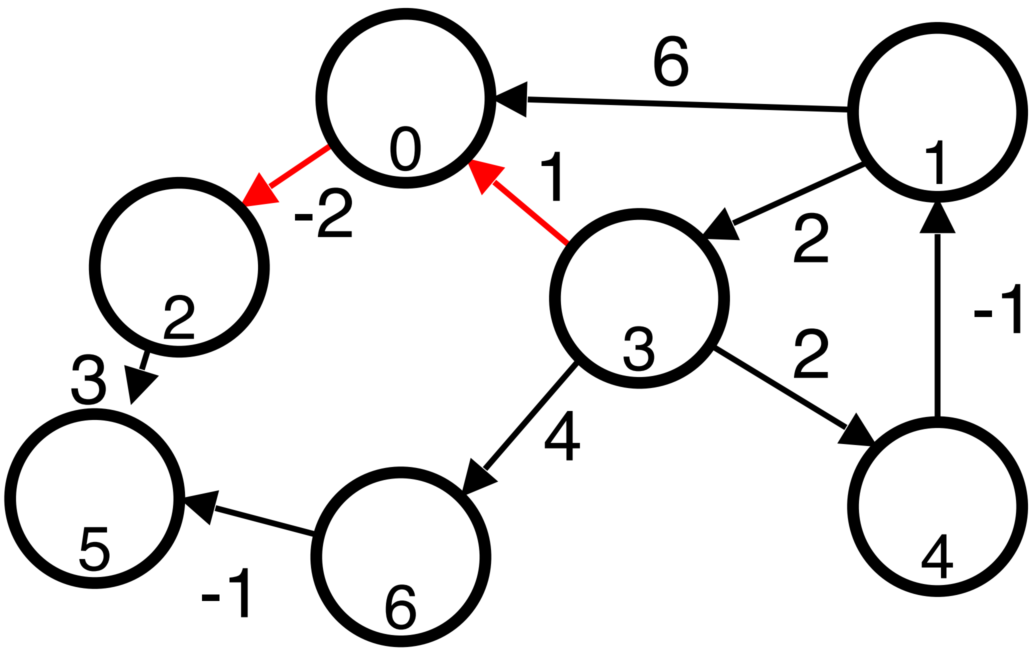

$$\begin{bmatrix} 0 & ? & \boxed{-2} & ? & ? & ? & ? \\ 6 & 0 & 4 & 2 & ? & ? & ? \\ ? & ? & 0 & ? & ? & 3 & ? \\ \boxed{1} & ? & \boxed{?} & 0 & 2 & ? & 4 \\ ? & -1 & ? & ? & 0 & ? & ? \\ ? & ? & ? & ? & ? & 0 & ? \\ ? & ? & ? & ? & ? & -1 & 0 \end{bmatrix}$$

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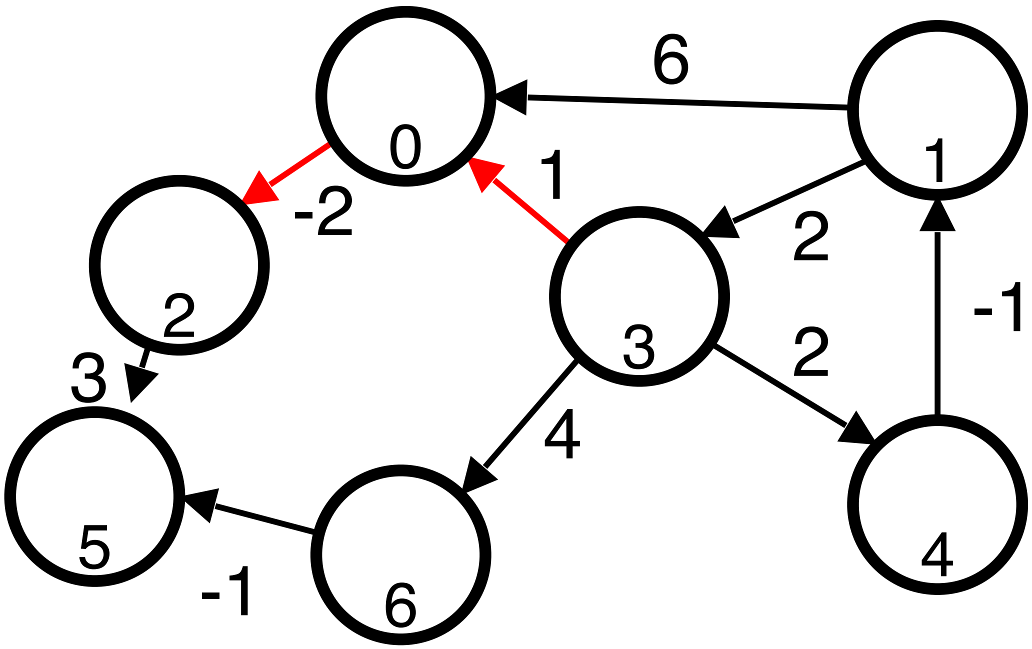

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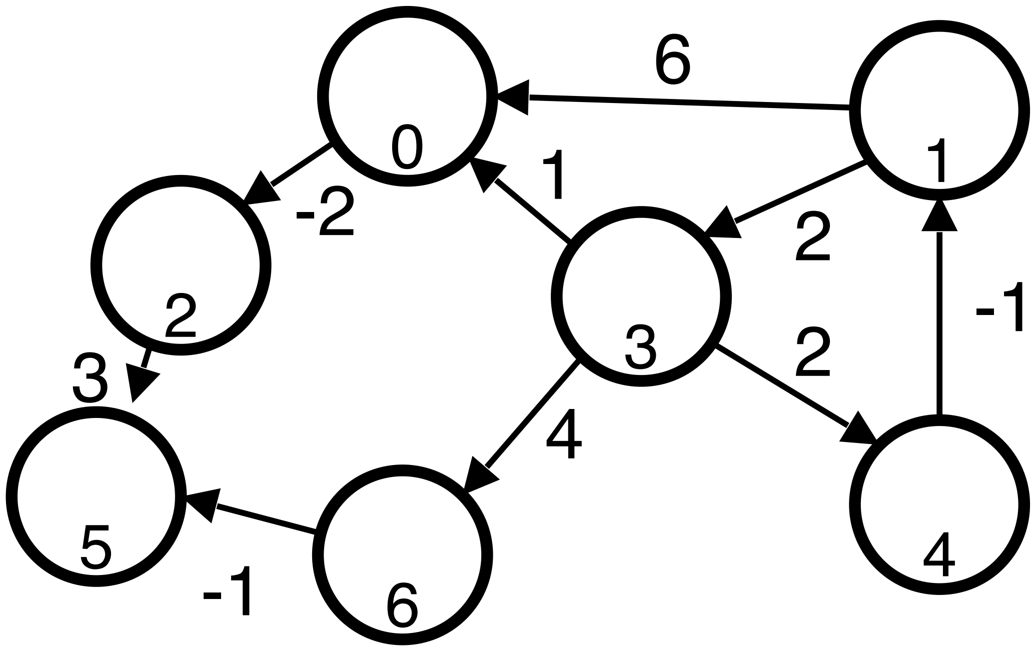

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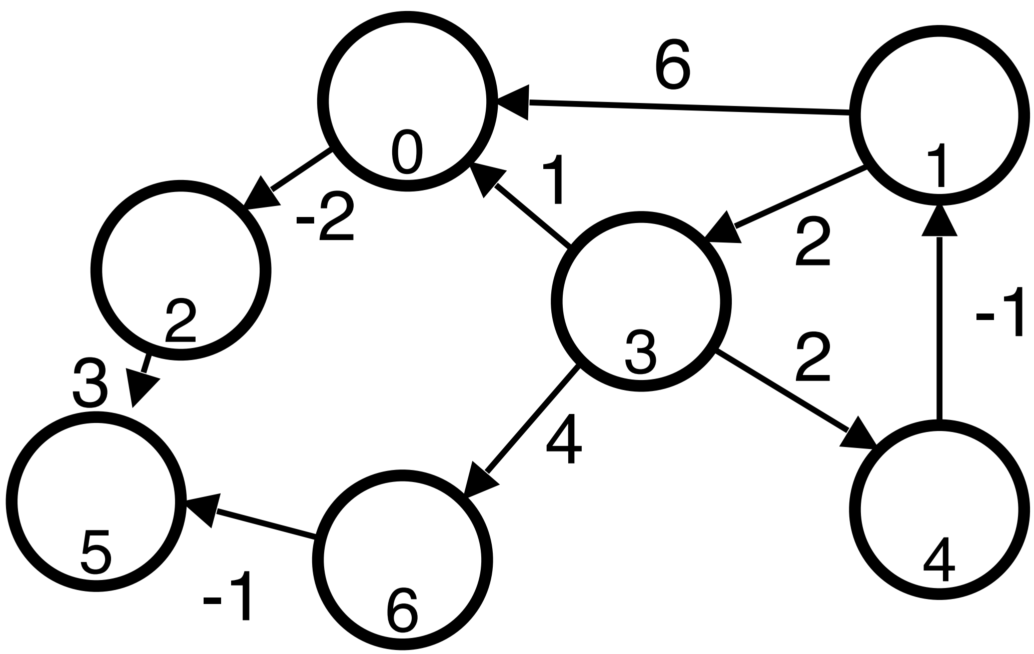
Which paths can improve by passing through node 1?

All-Pairs Shortest Path



$$\begin{bmatrix} 0 & ? & -2 & ? & ? & ? & ? \\ 6 & 0 & 4 & 2 & ? & ? & ? \\ ? & ? & 0 & ? & ? & 3 & ? \\ 1 & ? & -1 & 0 & 2 & ? & 4 \\ 5 & -1 & 3 & 1 & 0 & ? & ? \\ ? & ? & ? & ? & ? & 0 & ? \\ ? & ? & ? & ? & ? & -1 & 0 \end{bmatrix}$$

Which paths can improve by passing through node 1?

Floyd-Warshall

```
for each vertex k:
  for each vertex v:
    for each vertex w:
      if( dist(v,w) > dist(v,k) + dist(k,w) )
        dist(v,w) = dist(v,k) + dist(k,w);
```

Floyd-Warshall

for each vertex **k**:
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Claim: if no negative cycles, will find all shortest paths

Floyd-Warshall

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How can we tell if there are negative cycles?

Floyd-Warshall

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How can we tell if there are negative cycles?

Check for dist(**v,v**) < 0

Floyd-Warshall

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Danger: watch for overflow is using **MAXINT** to denote missing edges

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```

Time complexity $O(|V|^3)$