Segment Trees

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Computing Sums

- **Problem:** Given an array a which has n integers.
- Given q queries of the form $[l, r]$ (1-indexed, both inclusive).
- For each query, return $a[l] + a[l+1] + \cdots + a[r]$.

- Example: Say $a = [3, 5, 2, 7]$.
- $(l, r) = (1, 4)$, then $sum = 17$.
- $(l, r) = (2, 3)$, then $sum = 7$.
- What's the overall runtime if we do this naïvely over all queries?
- $O(nq)$

Prefix sums

- Now we **precompute** some information that will help you answer queries faster.
- $a = [3, 5, 2, 7]$
- $p = [3, 8, 10, 17]$
- Given p, what is the formula for $\sum_{i=l}^{r} a[i]$?
- $p[r] - p[l-1]$ (assuming $l > 1$)

Visualizing prefix sums

$a[1]$

$a[1] + a[2]$

$a[1] + a[2] + a[3]$

$a[1] + a[2] + a[3] + a[4]$

$a[1]$	$a[2]$	$a[3]$	$a[4]$

Visualizing prefix sums

$a[1]$

$a[1] + a[2]$

$a[1] + a[2] + a[3]$

$a[1] + a[2] + a[3] + a[4]$

$a[1]$	$a[2]$	$a[3]$	$a[4]$

Prefix sum limitations

- What functions can we use prefix sums on?
- We know addition and multiplication work
- The function needs to have an *inverse*.

Handling Updates

$a[1]$			
$a[1] + a[2]$			
$a[1] + a[2] + a[3]$			
$a[1] + a[2] + a[3] + a[4]$			
$a[1]$	$a[2]$	$a[3]$	$a[4]$

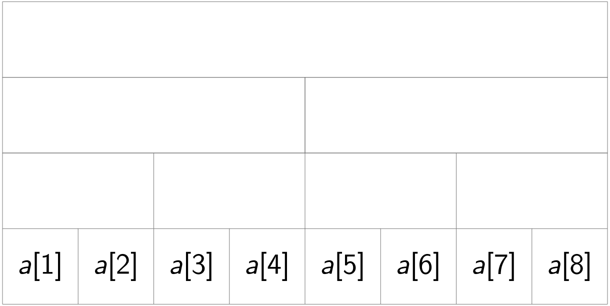
- Say now we also allow updates to specific elements.
- How many prefix sums do we need to update?
- Each update requires changing $O(n)$ prefix sums.
- Can we do better?

Single block system

<table>
<tr><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><td>$a[1]$</td><td>$a[2]$</td><td>$a[3]$</td><td>$a[4]$</td><td>$a[5]$</td><td>$a[6]$</td><td>$a[7]$</td><td>$a[8]$</td></tr>
</table>

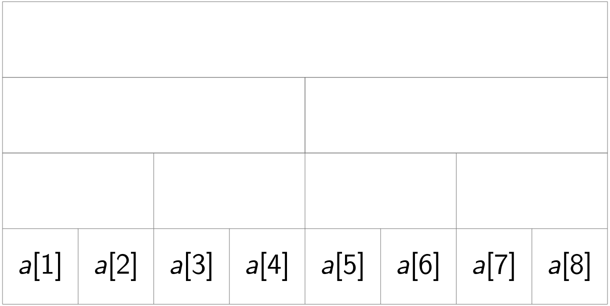
- Say each element can only belong to a single block
- What's the runtime for updating? $O(1)$
- What's the runtime to compute a sum? Let B be the size of a block $O(\max(B, N/B))$
- What value of B should we use to minimize the runtime? $O(\sqrt{N})$.
- This technique is called **Square Root decomposition**

$O(\log(N))$ block system



- Now let's allow each element to belong to $O(\log(N))$ blocks.
- Each block is of size 2^k.
- What's the runtime for updating? $O(\log(N))$
- What's the runtime to compute a sum? $O(\log(N))$

Segment Trees



- The above structure is a binary tree called a **Segment Tree**.

Implementation: Query

- Represent tree as an array; `tree[0]` is the root, and node i's children are $2 * i + 1$ and $2 * i + 2$.
- i is the node in the tree
- $[l, r]$ is the interval that node i contains the sum of.
- $[a, b]$ is the query interval.
- Returns the part of the interval sum contained in the subtree at node i

```
int[2 * N] tree;
int query(int i, int l, int r, int a, int b) {
    if (r < a || l > b) return 0;
    if (l >= a || r <= b) return tree[i];
    return query(2 * i + 1, l, (l + r) / 2, a, b) +
        query(2 * i + 2, (l + r) / 2 + 1, r, a, b);
}
```

- Why is the tree of size $2 * N$?
- What if N is not a power of 2?

Implementation: Update

- i is the node in the tree
- $[l, r]$ is the interval that node i contains the sum of.
- j is the index we want to change.
- v is the value we want to change $A[v]$ to.
- Recursive function returns value of `tree[i]` after applying `A[j] = v`

```
int update(int i, int l, int r, int v, int j) {
    if (j < l || j > r) return tree[i];
    if (l == r) {
        tree[i] = v
    } else {
        tree[i] =
            update(2 * i + 1, l, (l + r) / 2, j, v) +
            update(2 * i + 2, (l + r) / 2 + 1, r, j, v)
    };
    return tree[i];
}
```

Segment Tree Operations

- What kinds of operations can be supported on a segment tree?
- Any **associative** function will work
- Recall associative means $a + (b + c) = (a + b) + c$.
- What are some examples of associative functions?
- sum, product, min, max, gcd, matrix multiplication
- What are not associative functions?
- subtract, divide