Covariance propagation

The mean theorem

Linear case: $\mathbf{x} = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mu_x = \mathbf{A}\mu_y + \mathbf{b}$

Not linear case: $\mathbf{x} = \mathbf{g}(\mathbf{y})$, $\mu_x \cong \mathbf{g}(\mu_y)$

The **law of covariance propagation** is a kind of corollary of the mean theorem, and can be formally written as follows

Linear case: $\mathbf{x} = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mathbf{C}_{xx} = \mathbf{A}\mathbf{C}_{yy}\mathbf{A}^T$

Not linear case: $\mathbf{x} = \mathbf{g}(\mathbf{y})$, $\mathbf{C}_{xx} = \mathbf{J}\mathbf{C}_{yy}\mathbf{J}^T$

$\mathbf{y}$ random m-dimensional variable

$\mathbf{x}$ random n-dimensional variable

$\mathbf{x} = \mathbf{g}(\mathbf{y})$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ .. \\ y_i \\ .. \\ y_m \end{bmatrix} \quad \mu_y = \begin{bmatrix} \mu_{yl} \\ \ldots \\ \mu_{yi} \\ \ldots \\ \mu_{ym} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ .. \\ x_i \\ .. \\ x_n \end{bmatrix}$$

J is the **Jacobi or Jacobian matrix** of g: it contains the partial derivatives of x with respect to y.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \ldots & \frac{\partial x_1}{\partial y_m} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \ldots & \frac{\partial x_n}{\partial y_m} \end{bmatrix}_{\mu_y}$$

By these formulas, the global precision information can be propagated from observations to computed quantities.

注: *lecture in 20200925*

Principles of Least squares adjustment

Estimate a posterior value for redundant systems of observations

General properties

- not dependent on the probability distribution of the observations
- minimal variance estimates (more precise) within the class of unbiased and linear ones
- parameters and their accuracy can be computed in a closed form
- by simulations final accuracies of the parameters can be predicted from the a priori precision of the observations

Two cases

1. Linear Case:

- A is full rank -> all the columns are independent from each other -> N is regular
- A is not full rank -> N is not invertible

The Deterministic or Functional Model: $\underline{y} = A\underline{x} + \underline{b}$

Unknown parameters : $\underline{x} = \begin{vmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{vmatrix} \quad (\underline{x} \in \mathbb{R}^n, n \leqslant m)$

Design matrix : A

Known term : $\underline{b}$

The Model: $\underline{y_0} = \underline{y} + \underline{\varepsilon} = A\underline{x} + \underline{b} + \underline{\varepsilon}$

Unknown errors : $\underline{\varepsilon}$

An observation vector : $\underline{y_0} = \begin{vmatrix} y_{10} \\ y_{20} \\ \dots \\ y_{m0} \end{vmatrix} \quad (\underline{y_0}, \underline{y} \in \mathbb{R}^m \quad \underline{y} \in \mathbb{V}^n \leqslant \mathbb{R}^m \quad \underline{y_0} \notin \mathbb{V}^n, n \leqslant m)$

The covariance matrix of observations : $\mathbf{C_{y_0 y_0}} = \mathbf{C_{\varepsilon\varepsilon}} = \sigma_0^2 \mathbf{Q}$

A prior variance : σ_0^2 (in principle could be unknown)

Cofactor matrix : Q

Solution for $\underline{x}$ and $\underline{y}$ such that :

$$\begin{cases} \hat{\underline{y}} = A\hat{\underline{x}} + \underline{b} \\ \hat{\underline{y}} \text{ at minimal distance from } \underline{y_0} \Rightarrow (\hat{\underline{y}} - \underline{y_0})^T Q^{-1} (\hat{\underline{y}} - \underline{y_0}) : \textit{minimum} \quad \text{(stochastic distance)} \end{cases}$$

Least Square Solution

1st. Normal Matrix : $N = A^T Q^{-1} A$

2nd. The Normal Known Term : $\underline{t} = A^T Q^{-1} (\underline{y}_0 - \underline{b})$

3rd. The Estimated Solution for Unknown Parameters :

$$N\hat{\underline{x}} = A^T Q^{-1} A \hat{\underline{x}} = A^T Q^{-1} (\underline{y}_0 - \underline{b}) = \underline{t} \quad \Rightarrow \quad \hat{\underline{x}} = N^{-1} \underline{t} = N^{-1} A^T Q^{-1} (\underline{y}_0 - \underline{b}) \quad \text{(the normal system)}$$

4th. The Estimated Observations : $\hat{\underline{y}} = A\hat{\underline{x}} + \underline{b}$

5th. The Estimated Residuals/Errors : $\hat{\underline{\varepsilon}} = \underline{y}_0 - \hat{\underline{y}}$

6th. Redundancy or degrees of freedom : $R = m - n$

7th. A posteriori variance : $\hat{\sigma}^2 = \frac{\hat{\underline{\varepsilon}}^T Q^{-1} \hat{\underline{\varepsilon}}}{m-n}$

8th. the covariance matrix of estimated unknown : $C_{\hat{x}\hat{x}} = \hat{\sigma}^2 N^{-1}$

9th. the covariance matrix of estimated observables : $C_{\hat{y}\hat{y}} = A C_{\hat{x}\hat{x}} A^T$

10th. the covariance matrix of estimated residuals : $C_{\hat{\varepsilon}\hat{\varepsilon}} = \hat{\sigma}^2 (Q - A N^{-1} A^T)$

2. Non-Linear Case:

$$\mathbf{y_0} = \mathbf{y} + \varepsilon = \mathbf{f(x)} + \varepsilon \quad \textit{where} \quad f(x) = \begin{cases} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \dots \\ f_m(x_1, x_2, \dots, x_n) \end{cases}$$

Linearize the problem first

(1) approximate the values of unknown parameters (by solve the obs equations): $\tilde{\underline{x}}$

(2) the Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{\tilde{\underline{x}}}$$

(3) the design matrix (independently observed): $A = J$

(4) the known term: $b = \tilde{y} = f(\tilde{x})$

(5) the model: $y_0 = f(\tilde{x}) + J(\tilde{x})(x - \tilde{x})$

Least squares solution

(1) the covariance matrix of the observations $Cy_0y_0 = \sigma_0^2 Q$

(2) the normal matrix $N = A^T Q^{-1} A$

(3) the normal known term $\underline{t} = A^T Q^{-1} (y_0 - \tilde{y})$

(4) the estimated solution $\hat{\underline{x}} = N^{-1} \underline{t}$ $+\tilde{x}$

(5) the estimated observations $\hat{y} = A\hat{\underline{x}} + \tilde{y}$

(6) the estimated residuals $\hat{\underline{\varepsilon}} = \hat{y} - \underline{y_0}$

(7) degree of freedom $R = m - n$

(8) A posteriori variance : $\hat{\sigma}^2 = \frac{\hat{\underline{\varepsilon}}^T Q^{-1} \hat{\underline{\varepsilon}}}{m-n}$

(9)the covariance matrix of estimated unknown : $C_{\hat{x}\hat{x}} = \hat{\sigma}^2 N^{-1}$

(10) the covariance matrix of estimated observables : $C_{\hat{y}\hat{y}} = A C_{\hat{x}\hat{x}} A^T$

(11) the covariance matrix of estimated residuals : $C_{\hat{\varepsilon}\hat{\varepsilon}} = \hat{\sigma}^2 (Q - A N^{-1} A^T)$

Note: Not rank deficient problems!

Note1: Redundancy V.S Rank Deficiency

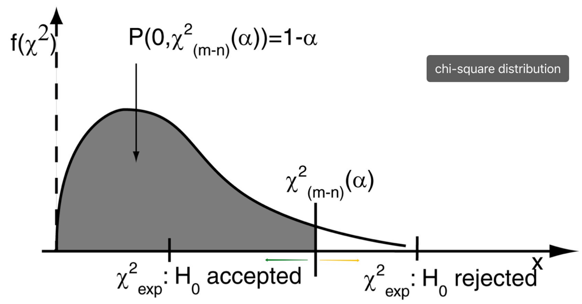
- redundancy is related to the cross check of the observations
- rank deficiency is related to the estimability of a given set of parameters wrt a given set of observables
    - Rank deficiency can be solved by fixing the minimum number of parameters to obtain a proper solution (minimal constraints adjustment)

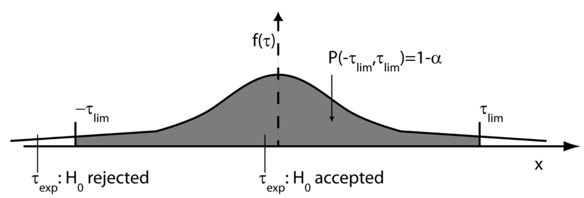
However not robust estimates: the estimates can be distorted if the functional and/or the stochastic models do not properly describe the observations: outliers

注: *lecture in 20201001 for linear problem*

Global test and outlier identification by local test

Need to identify and remove outliers. Use normality hypothesis on the not blundered observations.





11th. Global Test:

(1) Hypothesis $H_0 : \hat{\sigma}^2 = \sigma^2$

(2) Compute $\chi^2_{exp} = \frac{\hat{\sigma}^2}{\sigma^2}(m-n)$

(3) Look for in Chi-Square Distribution table $\chi^2_{lim} = \chi^2_{(m-n)}(\alpha)$

(4) $\begin{cases} \chi^2_{exp} \leqslant \chi^2_{lim} & \text{Accept } H_0 \\ \chi^2_{exp} > \chi^2_{lim} & \text{Reject } H_0 \end{cases}$

Note: The test on the global model has been originally designed to identify errors in the deterministic model, but it can fail just because of errors in the stochastic model.

12th. Local Test (Outlier identification):

(1) Hypothesis $H_0 : \frac{\hat{\varepsilon}_i}{\sigma_{\varepsilon_i}} = \tau_{exp} \sim \tau_{(m-n)}$

(2) Compute $\tau_{exp} = \frac{\hat{\varepsilon}_i}{\sigma_{\varepsilon_i}}$ *where* $\sigma_{\varepsilon_i} = \hat{\sigma}\sqrt{C_{\hat{\varepsilon}\hat{\varepsilon}}(i,i)}$

(3) Look for in Thomson Distribution table $\tau_{(m-n)}(\alpha) \Rightarrow \tau_{lim} = \tau_{(m-n)}(\frac{\alpha}{2})$ (T distribution table)

(4) $\begin{cases} |\tau_{exp}| \leqslant \tau^2_{lim} & \text{Accept } H_0 \\ |\tau_{exp}| > \tau^2_{lim} & \text{Reject } H_0 \end{cases}$

An iterative process is needed to identify outliers (Data Snooping)

注: *lecture in 20201016*

The Earth gravity field

The earth:

- Significant heterogeneities: the gravity field of the Earth is not spherical
- Significant inner (convective) displacements. The crust plates float on the mantle and move: geodynamic motion

gravity potential of the Earth = gravitational potential + rotational potential

$$W(X,Y,Z) = V(X,Y,Z) + \Phi(X,Y,Z) = G \iiint_{Earth} \frac{\rho(Q) dv_Q}{r_{PQ}} + \frac{w^2}{2}(X^2 + Y^2)$$

$\rho(Q)$: the inner density of the Earth

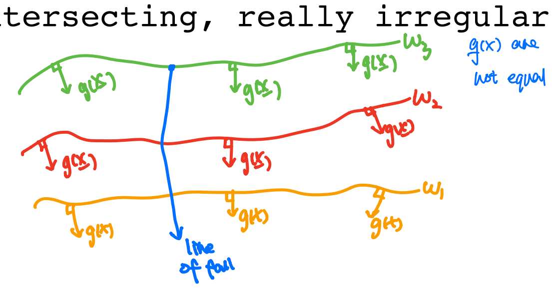
w : the Earth rotation velocity, a constant angular velocity

The gravity of Earth, denoted by g, is the net acceleration that is imparted to objects due to the combined effect of gravitation (from mass distribution within Earth) and the centrifugal force (from the Earth's rotation)

Equipotential surface of the gravity field:

$$\because W(X,Y,Z) = W_0 \text{ const}$$
$$\therefore dW(\delta X_{W_0}) = g \cdot \delta X = 0$$
$$\therefore g \perp \delta X_{W_0} \Rightarrow g \perp W_0(X,Y,Z)$$

A gravity equipotential surface is orthogonal to the gravity force. For the earth, they are closed, not intersecting, really irregular.



注: *lecture in 20201008*

the geoid

The surface that passes for the so called mean sea level is called geoid.

It is the best surface to represent a physical meaningful earth surface.

The Earth is a heterogeneous body, only approximately symmetric. The geoid is an irregular surface: it is quite smooth but cannot be modeled with simple analytical functions.

注: *lecture in 20201008, 20201014*

The ellipsoid reference surface

A proper rotational ellipsoid provides a first order approximation of the geoid.

A rotational ellipsoid in 3D is defined as the surface whose points satisfy the following condition

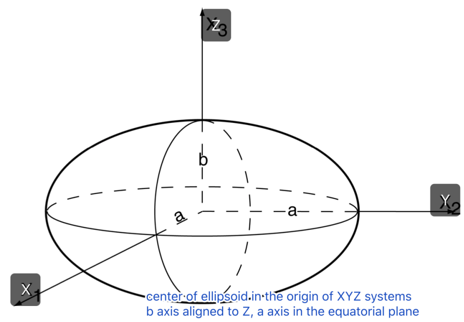
$$\frac{X^2+Y^2}{a^2} + \frac{Z^2}{b^2} = 1$$

a : major semiaxis b : minor semiaxis

the eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$

the flattening: $f = \frac{a-b}{a}$

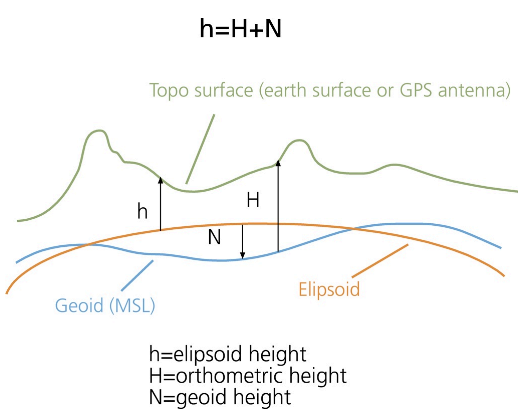
the relation: $e^2 = f(2-f)$



注：*lecture in 20201008, 20201014*

The geoid undulation

The geoid undulation N(φ,λ) in a point P(φ,λ) is defined as the height of the geoid with respect to a reference ellipsoid.



- H: height of P along the gravity field, with respect to geoid
- h: height (distance) of P with respect to ellipsoid
- N: the distance between ellipsoid and the geoid (geoid undulation)
  - N is positive when the geoid is external to the ellipsoid

注：*lecture in 20201014*

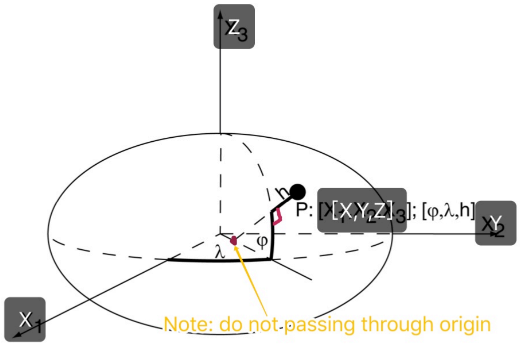
Geodetic coordinates and transformation to cartesian

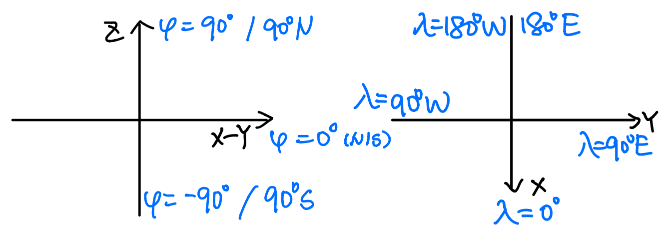
Given a XYZ and the related ellipsoid, the geodetic coordinates of a point P are defined as (φ, λ, h) :

geodetic latitude φ : angle between the normal to ellipsoid passing per P and the equatorial plane [X,Y]

geodetic longitude λ : anglebetweenthemeridianplanepassing per P and the origin meridian plane [X,~~Y~~Z]

geodetic height h : distancebetweenPandtheellipsoidsurface





The relation between cartesian and geodetic coordinates:

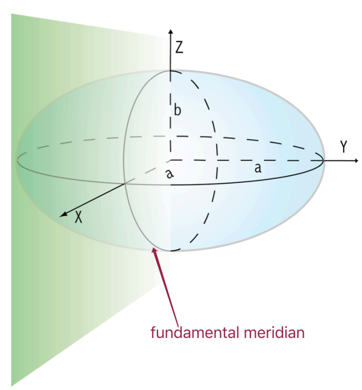
$$\begin{cases} X_P = (R_N + h_P)cos\varphi_P cos\lambda_P \\ Y_P = (R_N + h_P)cos\varphi_P sin\lambda_P \\ Z_P = [(R_N(1-e^2) + h_P)sin\varphi_P \end{cases}$$

$$where \quad R_N = \frac{a}{\sqrt{(1-e^2 sin^2\varphi_P)}}$$ (the Grand Normal curvature radius)

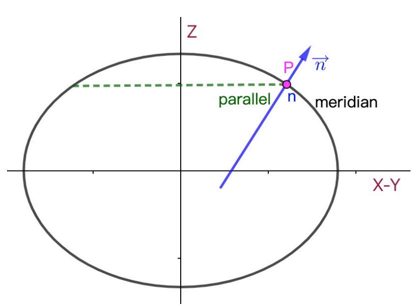
注: *lecture in 20201014*

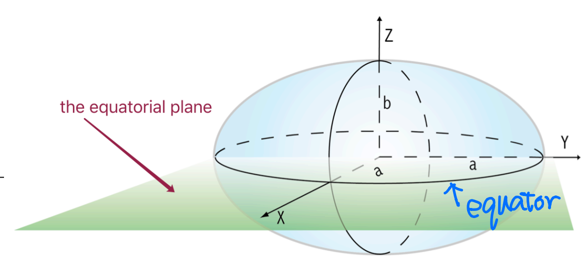
Meridians, parallels, normal sections

- **Meridians(子午线, 经线):** take a plane that contains the Z axis, and the intersection of that plane with the ellipsoid surface is a meridian.
  - definitely, you have infinite meridians.



- **Parallels:** take a plane parallel to the equator, the intersection with the ellipsoid is a parallel.

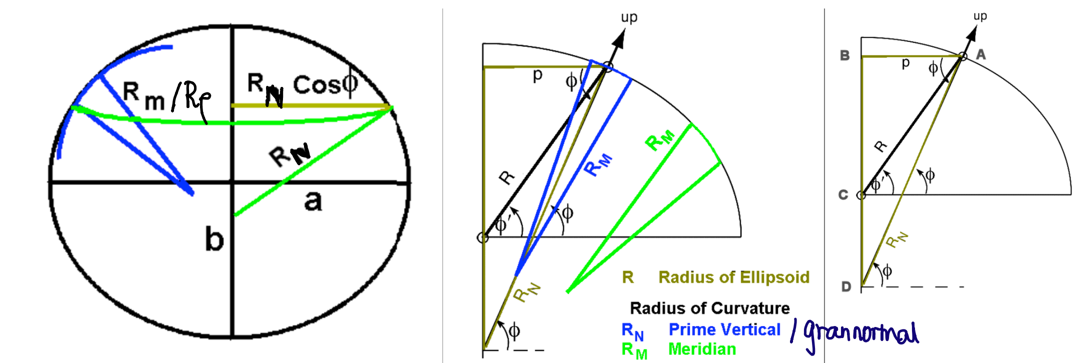




- A normal section in a point P is the intersection between the ellipsoid and a plane that contains the normal to ellipsoid in P
    - infinite normal sections
    - The parallel is not the normal section!
    - Two principal normal sections:
        - Meridian section
            - Generated by the normal and Z axis
            - minimum curvature radius
        - Grannormal section
            - The normal section in P orthogonal to the meridian
            - maximum curvature radius
            - Note: it is not the parallel!

注: *lecture in 20201014, 20201015*

Curvature radii of meridian and grand normal sections



Radii of Curvature

- meridian section

$$R_\rho = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\phi)^3}}$$

- grannormal section

$$R_N = \frac{a}{\sqrt{(1-e^2\sin^2\phi)}}$$

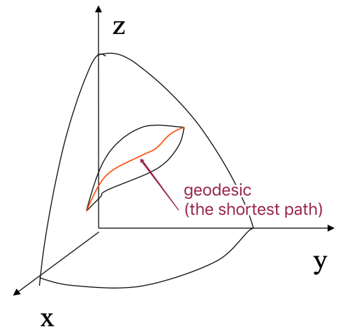
Radius	Formula
Radius of Curvature: in Prime Vertical, terminated by minor axis — grannormal section	$R_N = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$
Radius of Curvature: in Meridian — meridian section	R_M Rρ $= \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}}$ $= R_N \frac{1-e^2}{1-e^2\sin^2\phi}$
Radius of ellipsoid	$R = \frac{a\sqrt{[(1-e^2)^2\sin^2\phi+\cos^2\phi]}}{\sqrt{1-e^2\sin^2\phi}}$
Radius of Curvature: (Euler Theorem) At azimuth α	$\frac{1}{R_\alpha} = \frac{\cos^2\alpha}{R_M} + \frac{\sin^2\alpha}{R_N}$

注: *lecture in 20201015*

geodesics

Given P and Q, the minimum path curve on the ellipsoid connecting P and Q is called geodesic(测地线).

- It's unique
- It is between the two normal sections (P to Q and Q to P)
    - The normal section in P that passes per Q is not a normal section in Q
    - The normal section in Q that passes per P is not a normal section in P



注: *lecture in 20201015*

The theorems of geodesy and the planar approximation

Geodesy theorems:

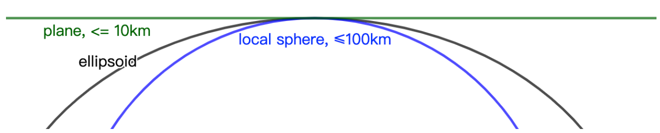
1. For distances up to 200 Km, the angles between geodesics can be approximated by angles between the normal sections.
2. For distances up to 200 Km the length of a geodesic can be approximated by the length of one of the normal section.

Geodesy operating rules:

1. For distances up to 100 Km, the horizontal computations on the ellipsoid can be approximated by the horizontal computations on a local sphere (significant simplifications) with radius
    - The error in distances / positions computation is no more than 3cm at 100km

$$R = \sqrt{R_\rho R_N}$$

2. For distances up to 10 Km, the horizontal computations on the local sphere can be reduced to computations on the local plane: intuitive simplifications!
    - With errors of the order of 1mm at 10km.



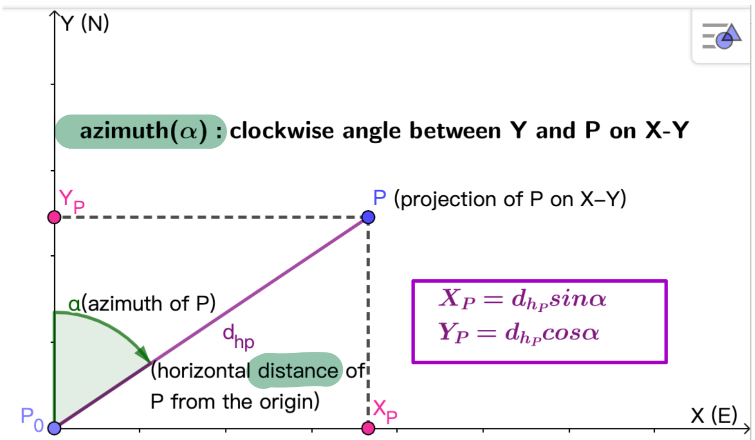
注: *lecture in 20201015*

Azimuth, distance and zenithal angle in the local plane, mathematical relation with cartesian coordinates

Azimuth(方位角): In a point P, the azimuth of a direction on the ellipsoid surface, is the angle between the north and that

direction.
• Azimuth of grannormal: 90 deg

For point P



the mathematical relation:

$d_{hp} = d_P sin\zeta$

$\Rightarrow$

$x_P = d_{hp} sin\alpha = d_P sin\zeta sin\alpha$

$y_P = d_{hp} cos\alpha = d_P sin\zeta cos\alpha$

$z_P = d_P cos\zeta$

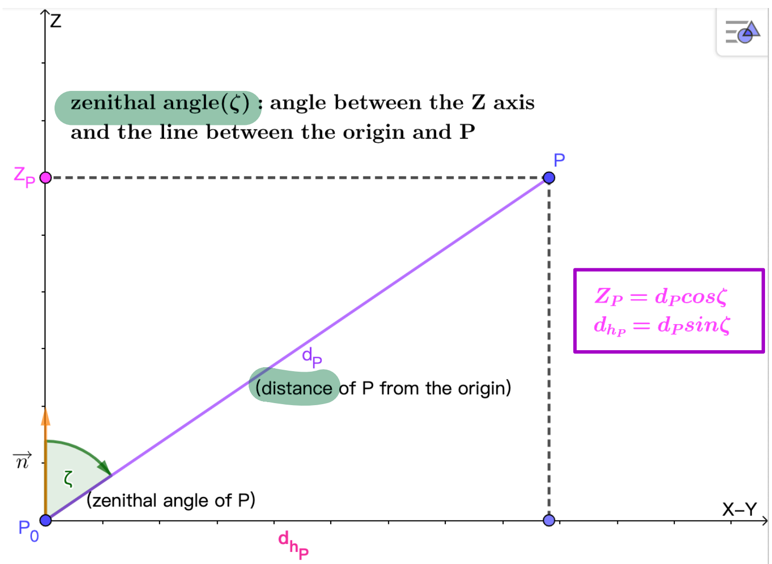
the inverse:

$\alpha = atan\frac{x_P}{y_P}$

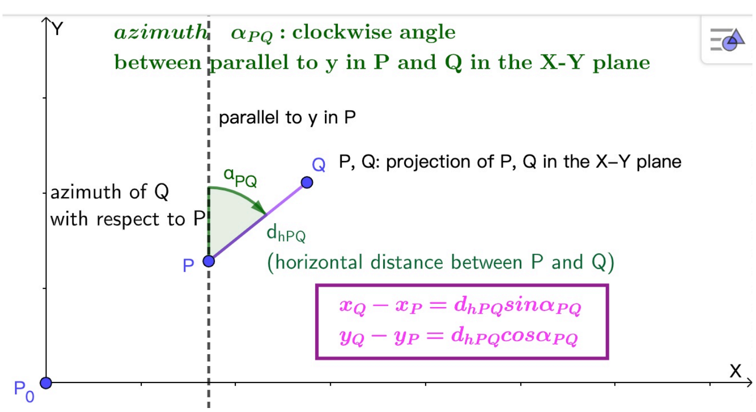
$d_{hp} = \sqrt{x_P^2 + y_P^2}$

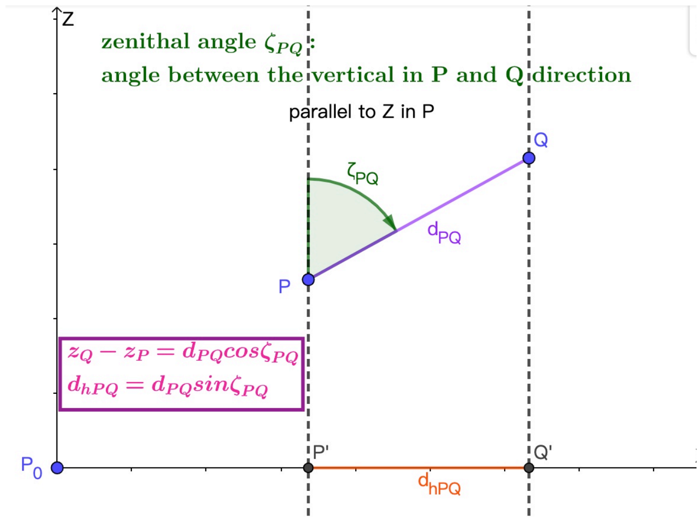
$d_P = \sqrt{x_P^2 + y_P^2 + z_P^2}$

$\zeta = acos\frac{z_P}{d_P}$



For point P and Q





the mathematical relation:

$d_{hPQ} = d_{PQ}sin\zeta_{PQ}$

$\Rightarrow$

$x_Q - x_P = d_{hPQ}sin\alpha_{PQ} = d_{PQ}sin\zeta_{PQ}sin\alpha_{PQ}$

$y_Q - y_P = d_{hPQ}cos\alpha_{PQ} = d_{PQ}sin\zeta_{PQ}cos\alpha_{PQ}$

$z_Q - z_P = d_{PQ}cos\zeta_{PQ}$

the inverse:

$\alpha_{PQ} = atan\frac{x_Q - x_P}{y_Q - y_P}$

$d_{hPQ} = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$

$d_{PQ} = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}$

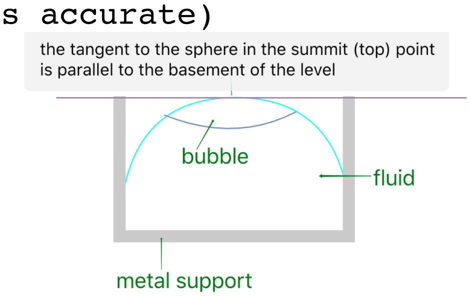
$\zeta_{PQ} = acos\frac{z_Q - z_P}{d_{PQ}}$

注: *lecture in 20201022*

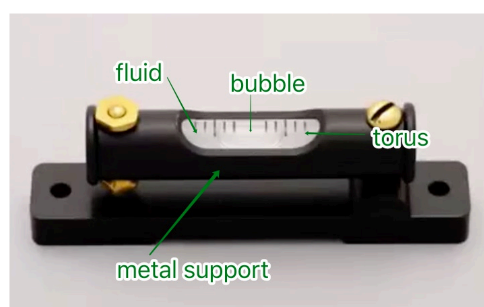
Spherical and toroidal bubbles

Bubble Level: A metal support that contains a glass phial that contains a fluid not viscous (for example, alcohol) with a small bubble of vapour, used to put horizontal:

- Spherical bubble (simpler, but less accurate)
  - When the bubble is in the center of the phial (top), the basement of the level is horizontal.
  - Typically sensitive (error) to 4’ (4 minutes)



- Torical / Linear bubble (more complicate, but more accurate)
  - The phial is a section of a torus with a graduation on the top.
  - The tangent to the torus in the middle (top) is parallel to the basement of the level.
  - When the bubble is in the top of the torus, the basement of the level is horizontal.
  - The sensitive (error) is about 10 seconds to 2 seconds

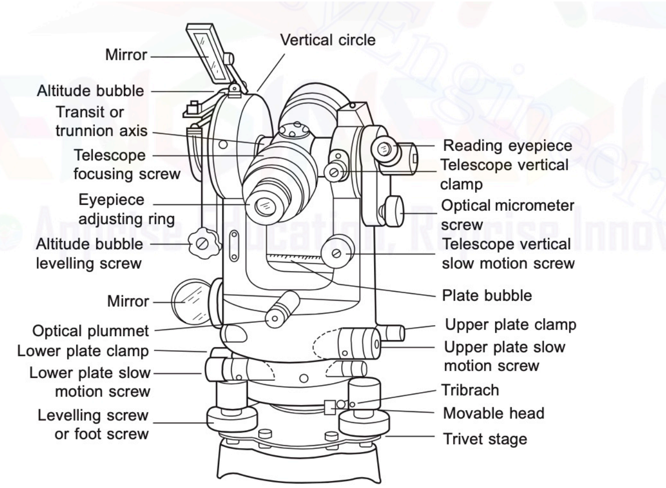


注: *lecture in 20201022*

Theodolite components and requirements

Theodolite: The instrument used in the measurement of angles

- A basement with screws that allows to put in horizontal the "lower plate"
- The lower plate that contains the so called "horizontal circle", that is a goniometer from 0 gon to 400 gon
  - Fixed
- An upper plate that rotates concentrically and horizontally to the lower plate.
- Mounted exactly in the center of the upper plate and orthogonal to it is a vertical support (axis) finished by the so called "alidade (照准仪)"
  - Free to rotate
- a telescope mounted on the trunnion axis (耳轴).
- The vertical circle is a goniometer, orthogonal to the upper plate with the center in the trunnion axis.



The requirements:

- vertical axis:
  - In the center of the upper plate
  - Orthogonal to the upper plate
- trunnion axis
  - Orthogonal to the vertical axis (horizontal)
  - Orthogonal to the vertical circle
  - Passes in the center

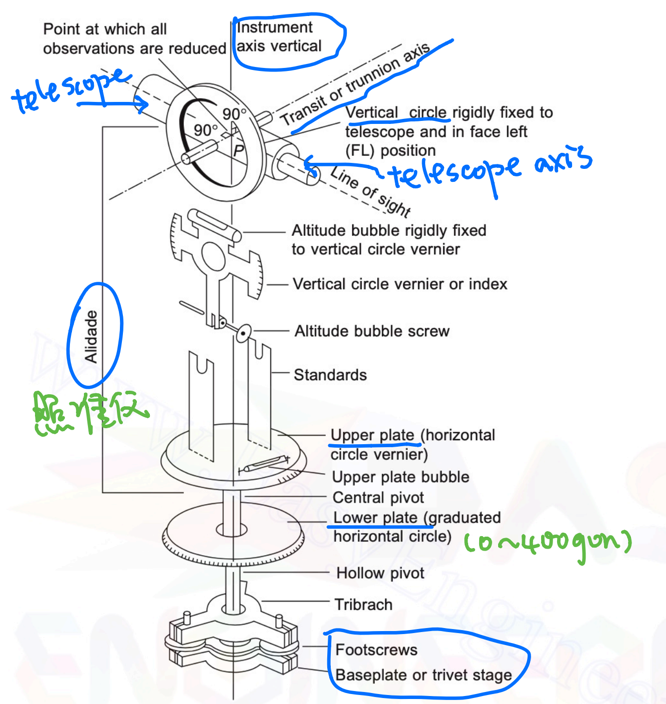
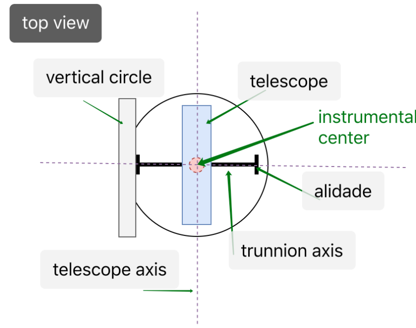
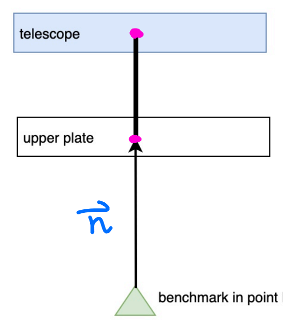


Fig. 4.3 *Simplified vernier theodolite*

    of the vertical circle
- Telescope axis
    - Orthogonal to the trunnion axis
- Vertical axis, trunnion axis and telescope axis intersect in one point that is called instrumental center or primary point of the theodolite.



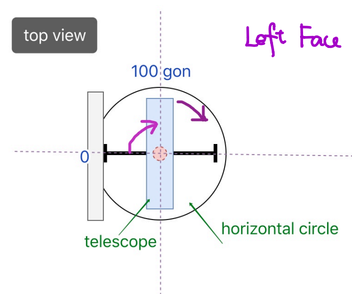
The ideal case: perfectly adjusted theodolite
1. the center of the upper plate is exactly on the vertical of P
2. The upper plate is perfectly horizontal.

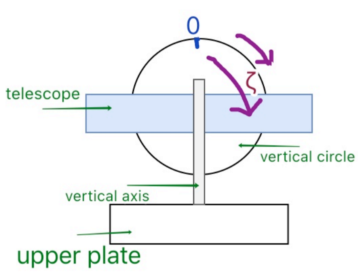


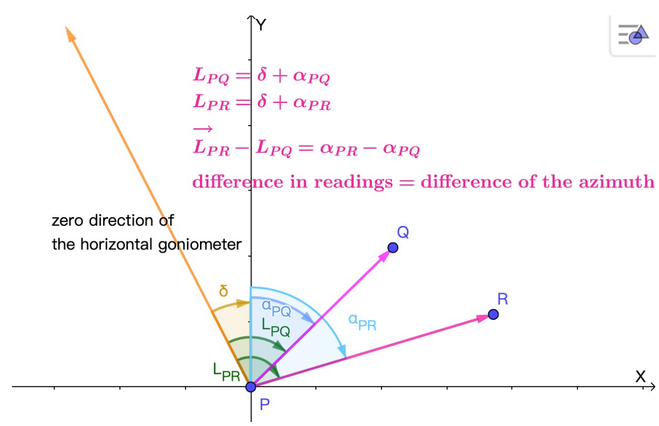
注: *lecture in 20201022*

Theodolite observations and their accuracies

Readings:
- horizontal reading: angle on the horizontal goniometer (horizontal circle) between 0 and the telescope.
- Vertical reading: zenithal angle (less accuracy)







The observation equations:

$$L_{PR} - L_{PQ} = \alpha_{QR} = \alpha_{PR} - \alpha_{PQ} = \mathrm{atan}(\frac{x_R - x_P}{y_R - y_P}) - \mathrm{atan}(\frac{x_Q - x_P}{y_Q - y_P})$$

Instrumental errors: costs range from 400€ ~ 5000€ according to
1. Accuracy
    - 400 € → σ ≈ 20cc
    - 50000 € → σ ≈ 1.5cc
2. Automation of the instrument

Even the most accurate instrument has errors. To eliminate the effect of instrumental error from the measurements, there are many measurement technics can be applied. The main error can be simply eliminated by making two readings (A double face observation).

Remember: Every time you collimation a point to make azimuthal reading, a proper average of left/right reading has to be done.

注: *lecture in 20201022*

Derivation of azimuthal angles from horizontal readings

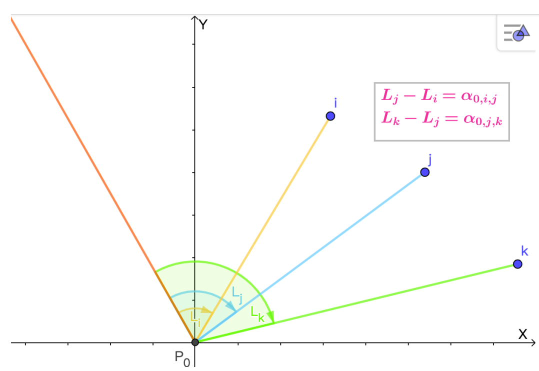
Suppose L_i, L_j, L_k are indepent with the same accuracy:
$L_i, L_j, L_k \leftrightarrow \sigma$(instrumental error)
$\therefore$
$\alpha_{ij} = L_j - L_i$
$\alpha_{jk} = L_k - L_j$
that is

$$\underline{\alpha} = \begin{vmatrix} \alpha_{ij} \\ \alpha_{jk} \end{vmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{vmatrix} L_i \\ L_j \\ L_k \end{vmatrix}$$

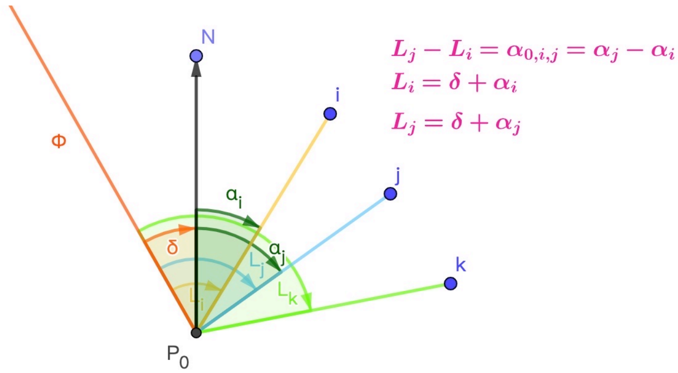
and the covariance matrix of $\underline{L}$: $C_{LL} = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$

According to covariance propagation:

$$C_{\alpha\alpha} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} C_{LL} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^T = \sigma^2 \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$



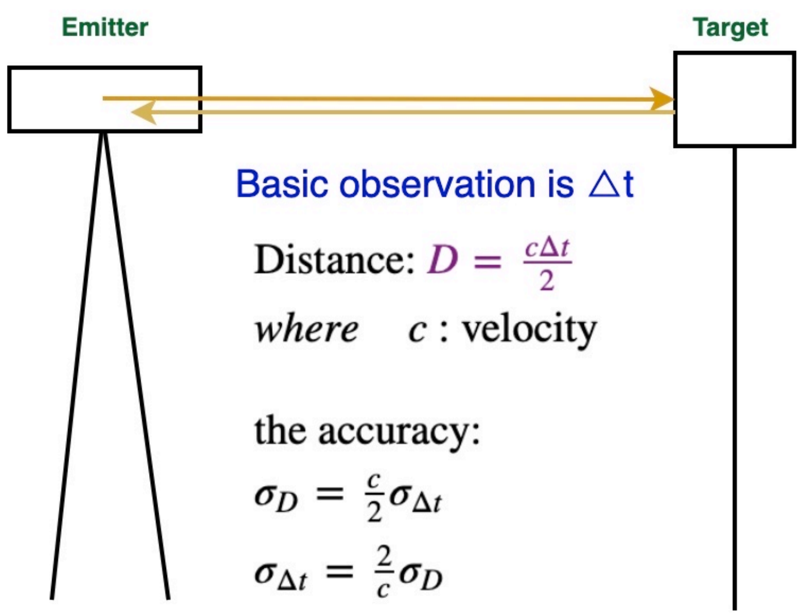
Therefore, if the azimuthal readings L have a certain σ
1) derived azimuthal angles have standard deviation = sqrt(2)*σ
2) consecutive azimuthal angles derived by the same station are correlated.



注: *lecture in 20201023*

EDM observation principles

Electronic distance measurement (EDM) is a method of determining the length between two points using electromagnetic waves. EDM is commonly carried out with digital instruments called theodolites.



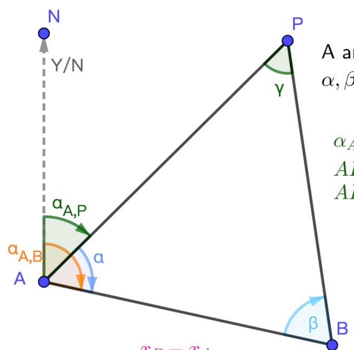
// pulse method

注: *lecture in 20201023*

Direct intersections

Direct intersection is used when the surveying is done from one or two known points to an unknown point. (Opposite side: inverse intersection)

by angles measurement



A and B: known coordinates x_A, y_A, x_B, y_B
α, β have been measured

$\alpha_{A,B}$: azimuth of B wrt A
AP : horizontal distance from A to P
AB : horizontal distance from A to B

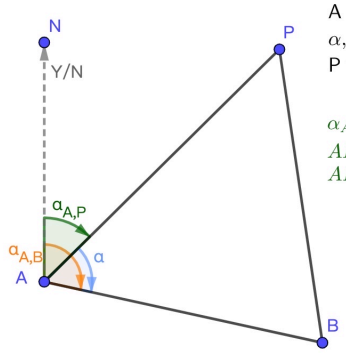
```
// fix A and B
```

$$\alpha_{A,B} = atan\frac{x_B - x_A}{y_B - y_A}$$
$$\gamma = \pi - \alpha - \beta$$
$$\frac{AP}{sin\beta} = \frac{AB}{sin\gamma}$$
$$\therefore \frac{AP}{sin\beta} = \frac{AB}{sin(\pi - \alpha - \beta)} = \frac{AB}{sin(\alpha + \beta)}$$
$$\therefore AP = AB\frac{sin\beta}{sin(\alpha + \beta)}$$

$$\because \boldsymbol{\alpha_{A,P} = \alpha_{A,B} - \alpha}$$
$$\therefore$$
$$\begin{cases} \boldsymbol{x_P = x_A + APsin(\alpha_{A,P})} \\ \boldsymbol{y_P = y_A + APcos(\alpha_{A,P})} \end{cases}$$

By angles and distances

//Simpler than previous one~

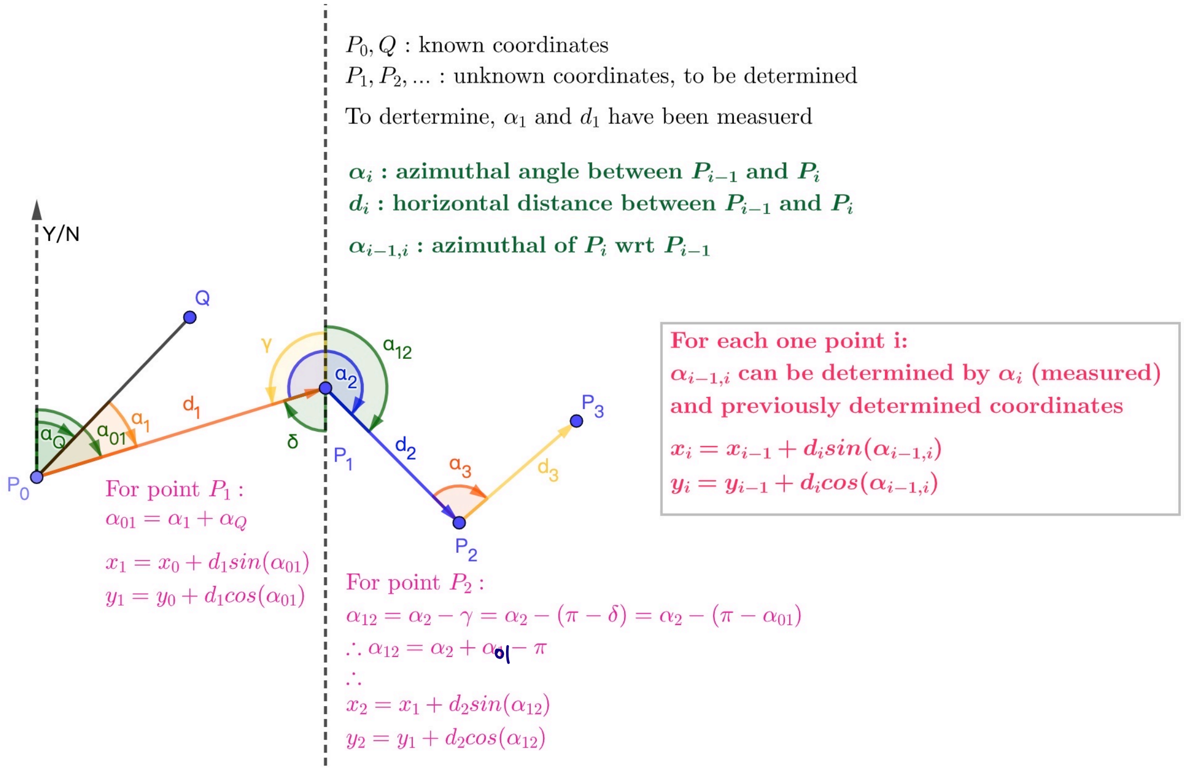


A and B: known coordinates x_A, y_A, x_B, y_B
α, AP have been measured
P to be determined

$\alpha_{A,B}$: azimuth of B wrt A
AP : horizontal distance from A to P
AB : horizontal distance from A to B

$$\because \boldsymbol{\alpha_{A,B} = atan\frac{x_B - x_A}{y_B - y_A}}$$
$$\therefore \boldsymbol{\alpha_{A,P} = \alpha_{A,B} - \alpha}$$
$$\therefore$$
$$\boldsymbol{x_P = x_A + APsin(\alpha_{A,P})}$$
$$\boldsymbol{y_P = y_A + APcos(\alpha_{A,P})}$$

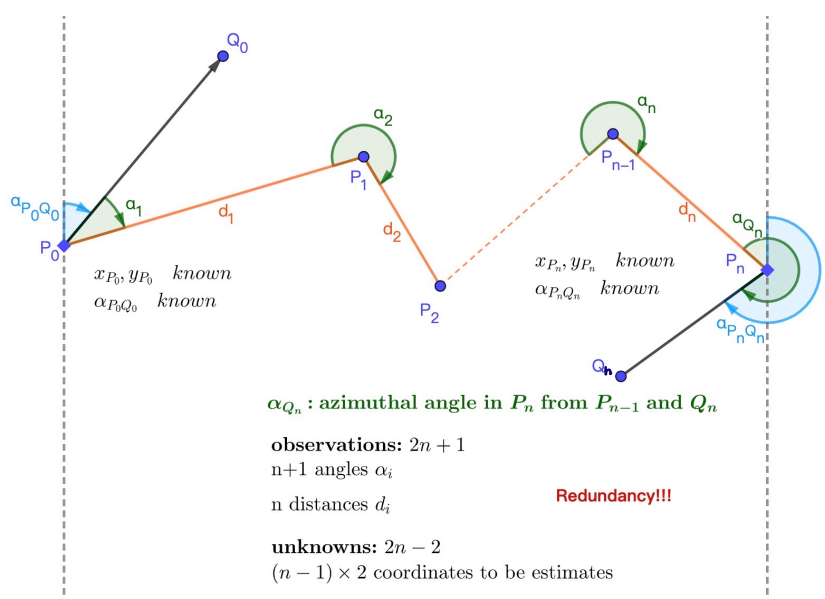
Open network



注：*lecture in 20201023*

Open and closed traverses

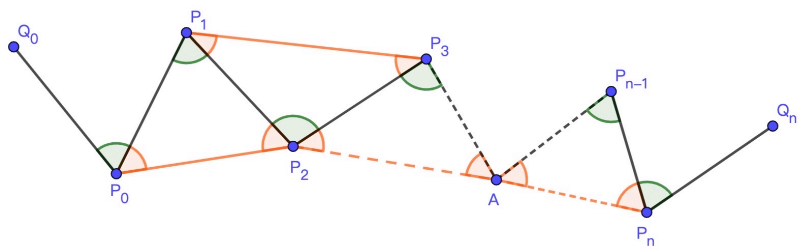
Open traverse



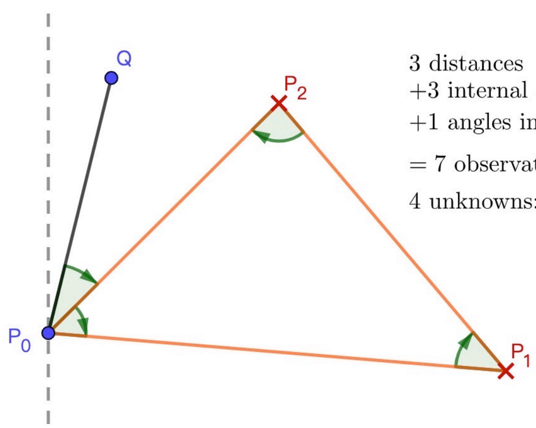
Condition on the coordinates: $\begin{cases} \sum_{i=1}^{n-1}(x_i - x_{i-1}) = x_n - x_0 \\ \sum_{i=1}^{n-1}(y_i - y_{i-1}) = y_n - y_0 \end{cases}$

Condition on the angles: $\sum_{i=1}^{n} \alpha_i = \alpha_{P_0Q_0} + \alpha_{P_nQ_n} + K\pi$ *where* K is an integer

How to improve redundancy?



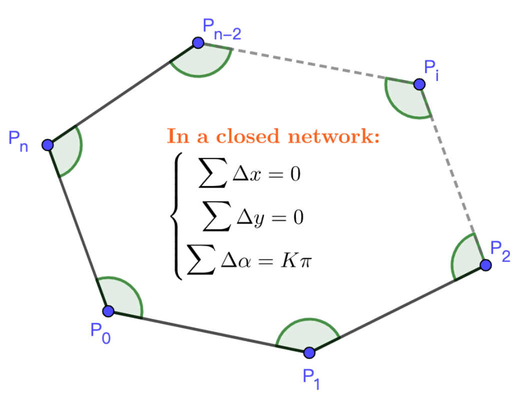
Close network



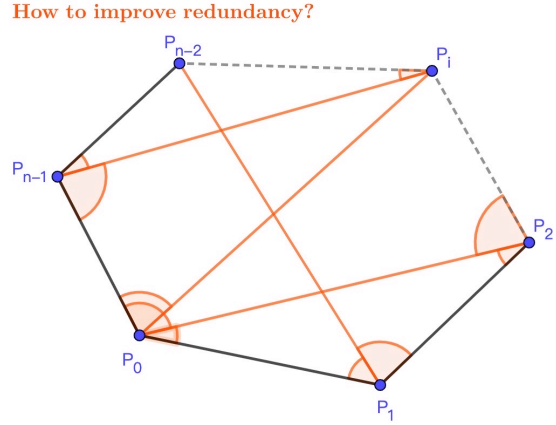
3 distances
+3 internal angles
+1 angles in P_0 from Q_0 to P_2
= 7 observations
4 unknowns: $x_{P_1}, y_{P_1}, x_{P_2}, y_{P_2}$

In the triangle: $\begin{cases} \sum \Delta x = 0 \\ \sum \Delta y = 0 \\ \sum \Delta \alpha = \pi \end{cases}$

Generalizing to any closing network:



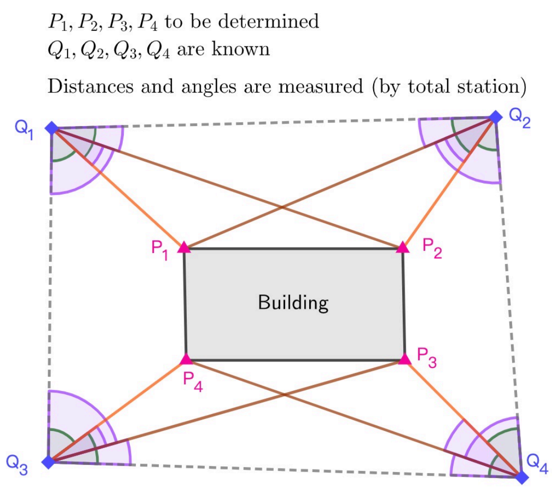
How to improve redundancy?



注：*lecture in 20201028*

Other redundant schemes for horizontal networks (be able to present and discuss examples)

Surveying of the building



How to improve redundancy?

Trilateration (三边测量术) + Triangulation (三角测量)：To measure as much as possible azimuthal angles and distances in a network.

How to find the less expensive configuration that provides a given accuracy?

the basic idea is,

1. Simulate the network with few redundancy,
2. Compute A, compute N.
   A. The elements of A contain partial derivative of Yj with respect to Xi
3. Check if the diagonal elements satisfy the requirement
   A. If no, the scheme is needed to improve.
      a. Add redundancy
      b. Then repeat step 2 and step 3
   B. if yes, stop.

In a least squares problem

the accuracy of the final parameters is: $\mathbf{C}_{\hat{\mathbf{x}}\hat{\mathbf{x}}} = \hat{\sigma}^2 \mathbf{N}^{-1}$

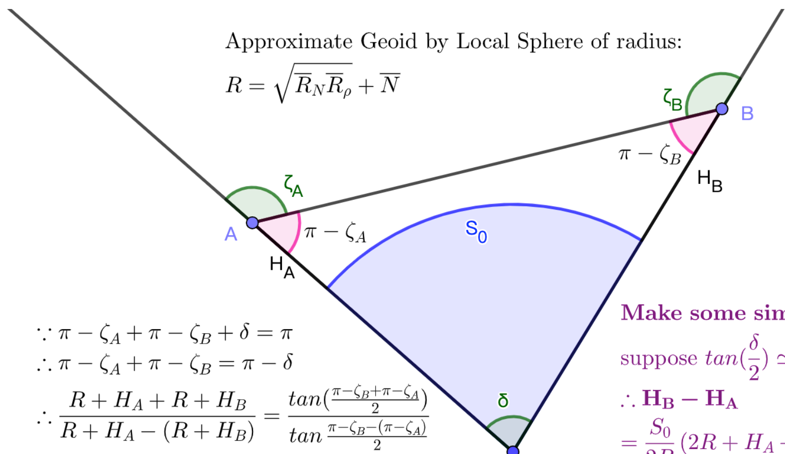
In case the knowledge of the instrumental accuracy is good: $\hat{\sigma}^2 \simeq \sigma_0^2$

and $\mathbf{N} = \mathbf{A}^{\mathrm{T}} \mathbf{Q}^{-1} \mathbf{A}$ (Q is cofactor matrix, a prior given; A is design matrix

注: *lecture in 20201028*

Trigonometric levelling (both the extremes / one extreme)

`// by theodolite`



$$\because \pi - \zeta_A + \pi - \zeta_B + \delta = \pi$$

$$\therefore \pi - \zeta_A + \pi - \zeta_B = \pi - \delta$$

$$\therefore \frac{R + H_A + R + H_B}{R + H_A - (R + H_B)} = \frac{tan(\frac{\pi - \zeta_B + \pi - \zeta_A}{2})}{tan\,\frac{\pi - \zeta_B - (\pi - \zeta_A)}{2}}$$

$$\therefore \frac{2R + H_A + H_B}{H_A - H_B} = \frac{tan(\frac{\pi - \delta}{2})}{tan(\frac{\zeta_A - \zeta_B}{2})} = \frac{cotg(\frac{\delta}{2})}{-tan(\frac{\zeta_B - \zeta_A}{2})}$$

$$\therefore H_B - H_A = (2R + H_A + H_B) tan(\frac{\zeta_B - \zeta_A}{2}) tan(\frac{\delta}{2})$$

Make some simplifications

suppose $tan(\frac{\delta}{2}) \simeq \frac{\delta}{2} \simeq \frac{S_0}{2R}$

$$\therefore \mathbf{H_B - H_A}$$

$$= \frac{S_0}{2R}(2R + H_A + H_B) tan(\frac{\zeta_B - \zeta_A}{2})$$

$$= S_0(1 + \frac{H_A + H_B}{2R}) tan(\frac{\zeta_B - \zeta_A}{2})$$

$$\mathbf{= S_0(1 + \frac{H_m}{R}) tan(\frac{\zeta_B - \zeta_A}{2})}$$

$$(where \quad H_m = \frac{H_A + H_B}{2})$$

`How to compute?`

At each one iteration

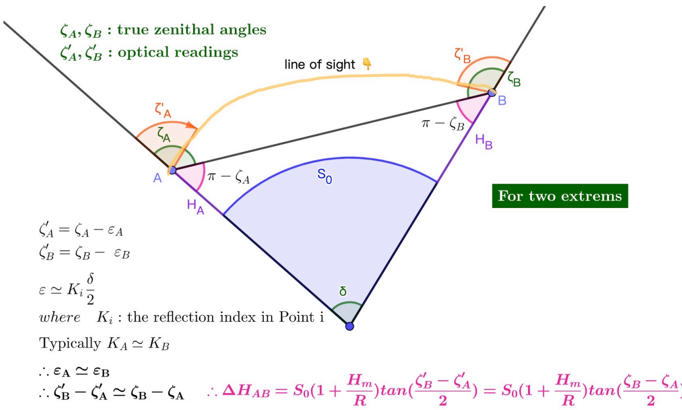
(1) $H_{m_i} = \frac{H_A + H_{B_{i-1}}}{2}$ $\quad$ ($when \quad i = 1, \quad H_{m_1} = H_A$)

(2) Compute $\Delta H_{AB_i} = S_0(1 + \frac{H_{m_i}}{R}) tan(\frac{\zeta_B - \zeta_A}{2})$

(3) Then $H_{B_i} = H_A + \Delta H_{AB_i}$

Iterations stop when $|\Delta H_{AB_i} - \Delta H_{AB_{i-1}}| < \varepsilon$ $\quad$ (ε : consistent with the required accuracy)

Errors



reflection effect is eliminated

one extreme

// determine by only measurement of one point

$$\because \pi - \zeta_A + \pi - \zeta_B = \pi - \delta$$
$$\therefore \zeta_B = \pi - \zeta_A + \delta$$
$$\therefore \Delta H_{AB} = S_0(1 + \frac{H_m}{R})tan(\frac{\zeta_B - \zeta_A}{2})$$
$$= S_0(1 + \frac{H_m}{R})tan(\frac{\pi - \zeta_A + \delta - \zeta_A}{2})$$
$$= S_0(1 + \frac{H_m}{R})tan[\frac{\pi}{2} - (\zeta_A - \frac{\delta}{2})]$$
$$= S_0(1 + \frac{H_m}{R})cotg(\zeta_A - \frac{\delta}{2})$$
$$= \mathbf{S_0(1 + \frac{H_m}{R})[cotg(\zeta_A) + \frac{1}{sin^2(\zeta_A)}(\frac{S_0}{2R})]}$$

Again, to be solved in an iterative way.

In planar approximation, $R \to \infty$
$\therefore \Delta H_{AB} = d_0 cotg(\zeta_A) \quad (where \quad d_0$ is the distance between A and B)

For one extreme

$$\Delta H_{AB} = S_0(1 + \frac{H_m}{R})[cotg(\zeta'_A) + \frac{1}{sin^2(\zeta'_A)}(\frac{S_0}{2R})]$$

Therefore, reflection effect completely propagates into the result.

注: *lecture in 20201029*

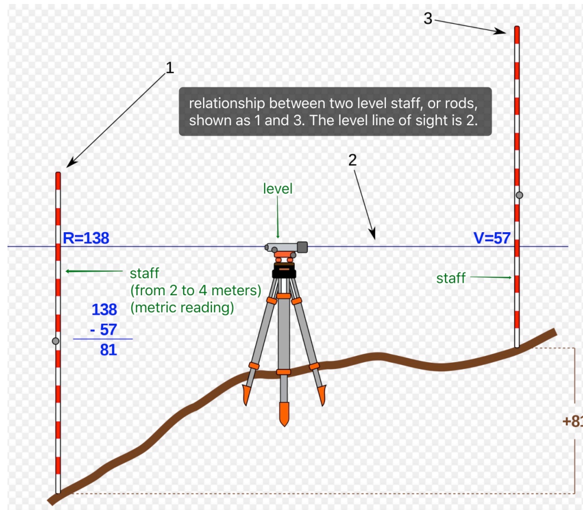
Optical levels and graduated staffs

Components

- Level: a telescope perfectly horizontal that allows to collimate the staff and to make the vertical reading.
- Staff: a pole that could be wood, metal or glass fibre

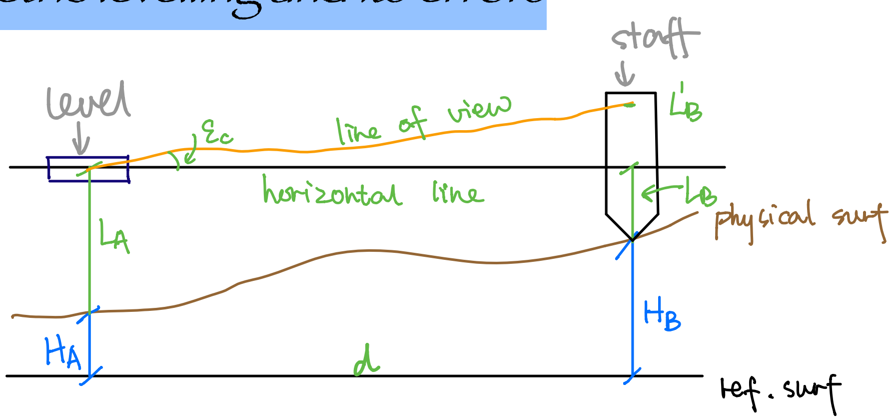
 with a graduation in meters

 with a bubble that allows to put the staff perfectly vertical



注: `lecture in 20201029`

Geometric levelling and its errors



$$\Delta H_{AB} = H_B - H_A = L_A - L_B$$

Error #1: Collimation error

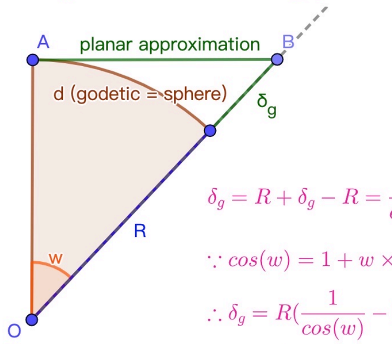
the level is perfectly horizontal but the collimation axis is not parallel to the telescope.

$$L'_B = L_B + d \times tan(\varepsilon_C) \quad where \quad d: \text{horizontal distance btw A and B}$$

for small ε_C : $\mathbf{L'_B = L_B + d \times \varepsilon_C}$

Error #2 Curvature of the geoid

δ_g : error introduced by planar approximation



$$\delta_g = R + \delta_g - R = \frac{R}{cos(w)} - R = R(\frac{1}{cos(w)} - 1)$$

$$\because cos(w) = 1 + w \times sin(0) - \frac{w^2}{2} \times cos(0) = 1 - \frac{w^2}{2} \text{ (Tylor expansion)}$$

$$\therefore \delta_g = R(\frac{1}{cos(w)} - 1)$$

$$= R(\frac{1}{1-\frac{w^2}{2}} - 1)$$

$$= R(1 + \frac{w^2}{2} - 1)$$

$$= R \times \frac{w^2}{2}$$

$\because \frac{w^2}{2}$ is very small

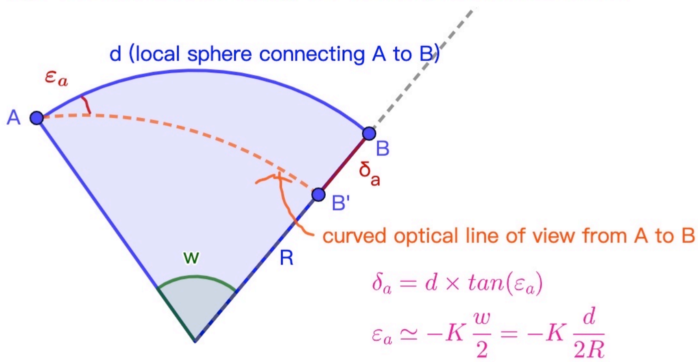
$\therefore \frac{1}{1-\frac{w^2}{2}} \simeq 1 + \frac{w^2}{2}$

$$\boxed{\begin{aligned} &\because \boldsymbol{w = \frac{d}{R}} \\ &\therefore \boldsymbol{\delta_g = R \times \frac{w^2}{2}} \\ &\boldsymbol{= R \times \frac{(\frac{d}{R})^2}{2}} \\ &\boldsymbol{= \frac{d^2}{2R}} \end{aligned}}$$

① $d = 100m$
$\delta g = 0.8mm$
② $d = 50m$
$\delta g = 0.1mm$

Error #3 atmospheric refraction

δ_a : error in the reading for atmospheric refraction



$$\delta_a = d \times tan(\varepsilon_a)$$

$$\varepsilon_a \simeq -K\frac{w}{2} = -K\frac{d}{2R}$$

$where \quad K$: refraction index $(0.1 \leq K \leq 0.2)$

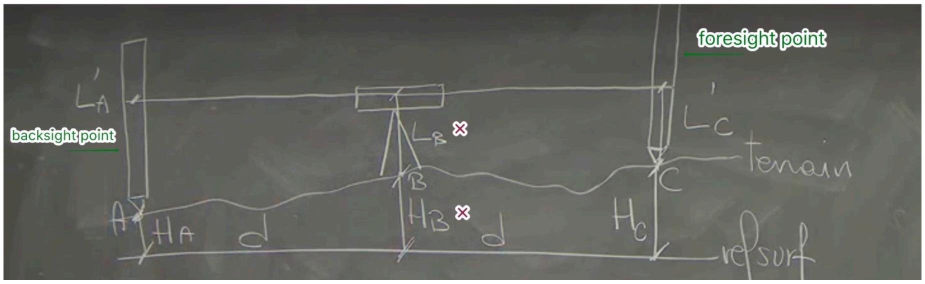
$$\boxed{\begin{aligned} &\therefore \delta_a = d \times tan(-K\frac{d}{2R}) \\ &= d(-K\frac{d}{2R}) \\ &= -K\frac{d^2}{2R} \end{aligned}}$$

Summarize the error

$$\mathbf{L'} = L + d \times \varepsilon_C + \frac{d^2}{2R} - K\frac{d^2}{2R}$$
$$= \mathbf{L} + \mathbf{d} \times \varepsilon_C + \frac{\mathbf{d}^2}{2\mathbf{R}}(\mathbf{1} - \mathbf{K})$$

ε

How to solve the error?
Levelling in from the middle!



$$\because \begin{cases} L'_A + H_A = L_B + H_B + \varepsilon_C \times d + (1-K)\frac{d^2}{2R} \\ L'_C + H_C = L_B + H_B + \varepsilon_C \times d + (1-K)\frac{d^2}{2R} \end{cases}$$

$$\therefore L'_A - L'_C = [-H_A + L_B + H_B + \varepsilon_C \times d + (1-K)\frac{d^2}{2R}] - [-H_C + L_B + H_B + \varepsilon_C \times d + (1-K)\frac{d^2}{2R}]$$

$$\therefore L'_A - L'_C = H_C - H_A = \Delta H_{AC}$$

Conclusion:

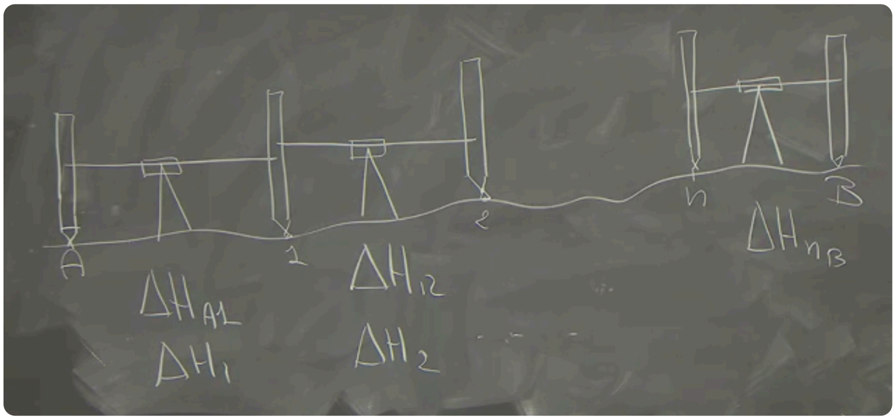
1. Exactly clean all the error sources we have seen.
2. Do not need anymore to make the instrumental reading in the telescope (L_B)

注：*lecture in 20201029, 20201030*

Covariance propagation in levelling chains

Chains of levelling
// used to connect points that are very far

$$\Delta H_{AB} = \sum_{i=1}^{n} \Delta H_i + \Delta H_{nB}$$



Covariance propagation

Suppose that to connect A and B, n ΔH have been measured

Suppose that each one reading L has a standard deviation σ_L

$$Then \quad \Delta H_k = L_{k-1} - L_k = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} L_{k-1} \\ L_k \end{bmatrix} \quad and \quad C_{LL} = \begin{bmatrix} \sigma_L^2 & 0 \\ 0 & \sigma_L^2 \end{bmatrix}$$

$$\therefore \sigma^2_{\Delta H_k} = 2\sigma_L^2$$

After chains of levelling done

$$\Delta H_{AB} = \sum_{i=1}^{n} \Delta H_i = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} \Delta H_1 \\ \Delta H_2 \\ \dots \\ \Delta H_n \end{bmatrix}$$

$\therefore$ (By covariance propagation) $\quad \sigma^2_{\Delta H_{AB}} = 2n \times \sigma_L^2$

$where \quad n = \frac{d_{AB}}{d_L}$ (d_L is distance in levelling from staff to staff)

$and \quad \sigma_L^2 = \alpha d_L$

$\therefore \sigma^2_{\Delta H_{AB}} = 2\frac{d_{AB}}{d_L} \times \alpha d_L = 2\alpha d_{AB}$ (2α = variance per unit of distance $= \sigma_0^2$)

$\therefore \sigma^2_{\Delta H_{AB}} = \sigma_0^2 d_{AB}$

$\because$ Conventionally, the variance per unit of length is given per Kilometer

$\therefore \sigma^2_{AB} = \sigma^2_{0(Km)} \times d_{AB(\text{in Km})}$

How to improve the precision of the result of levelling?

1. Improve the quality of the instrumentation
2. Take the points very near

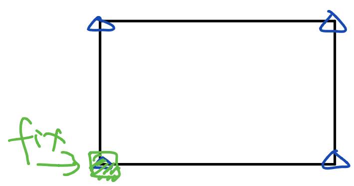
A. For example, for hight precise levelling, the distance between two points is no more than 40m

3. Reduce the redundancy of the network by adjusting the network

注：*lecture in 20201030*

Vertical networks (be able to present and discuss examples)

#1 Minimal redundant scheme



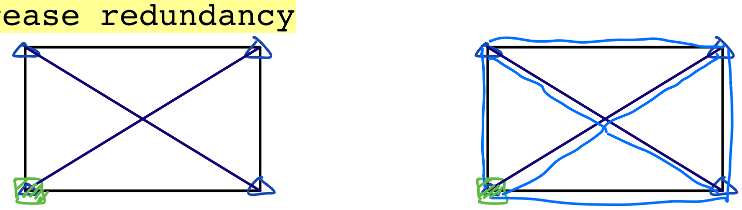
$\Sigma \Delta H = \phi$

Redundancy = 1

Closed Configuration: A network of 4 points (3 unknowns), closed with 4 observations

From the outlier check point of view, what you can do to against outliers?
Practically, nothing. Let's suppose there are two of outliers with same value but opposite sign, that produce a closure error that is 0, but they are two outliers.

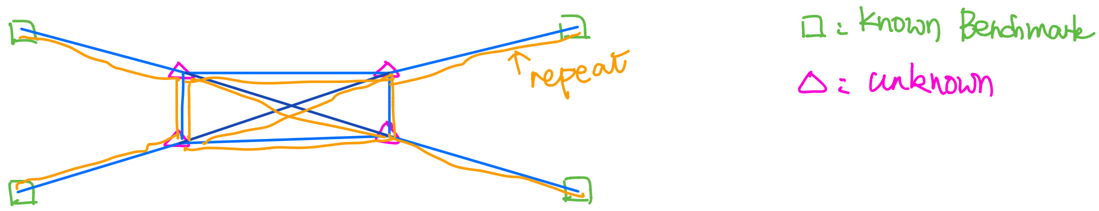
#2 Increase redundancy



A network of 4 points (3 unknowns)
- with all connection: 6 observations
- With 2 repetition: 12 observations

From the outlier check point of view (closure errors), more redundancy, more outliers identified.

#3



Introduced the redundancy by

1. Putting the network more than one known benchmark
2. Closing the polygons (triangles) exactly as for horizontal network and for levelling network, in order to choose the optimal surveying scheme

注：*lecture in 20201030*