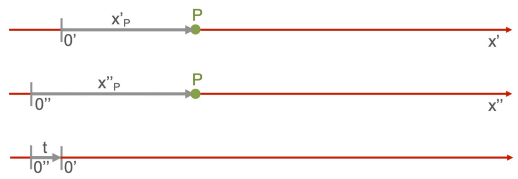
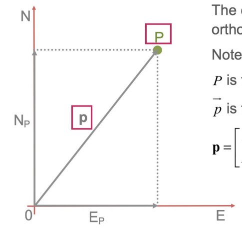
Reference Frames

RF in 1, 2 and 3 dimensions.



$t = x''_P - x'_P = O' - O''$

Case 1D



The coordinates of a point are its orthogonal projections on the axes.

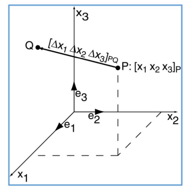
Note

P is the point,

$\vec{p}$ is the vector between the origin and P,

$\mathbf{p} = \begin{bmatrix} E_P \\ N_P \end{bmatrix}$ is P position wrt the axes.

Case 2D



3D Reference Systems

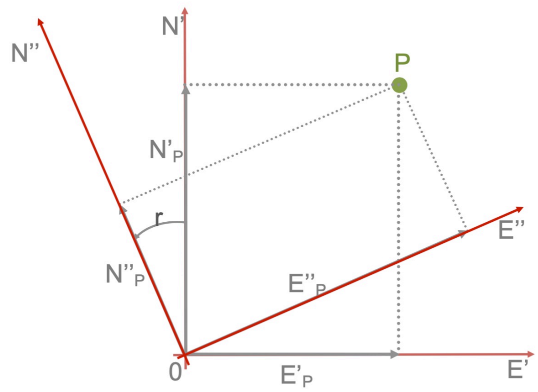
Three axes $\vec{x}_1, \vec{x}_2, \vec{x}_3$ with common origin, reciprocally orthogonal, and the unit of lengths (unitary vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$).

The coordinates of a point are the lengths of its orthogonal projections on the three axes. The vector between two points is the oriented difference between their coordinates.

Case 3D

- ***1D***
  - To compute the position of P (Xp), the **origin** and the **unit of length** is needed.
    - given a RS, the position of P (Xp) could be its distance (oriented) from the origin.
  - height RS: the origin height should be imposed, for example, the mean sea level
- ***2D***: **two orthogonal axes** (for example, East and North, origin is defined) + **one unit of length**
- ***3D***: **three reciprocally orthogonal axes with common origin** and **unit of length**

Rototranslation in 2D.



Effect of a rotation of the axes

$$P'' = \begin{vmatrix} E'' \\ N'' \end{vmatrix} \qquad P' = \begin{vmatrix} E' \\ N' \end{vmatrix}$$

$$P'' = \underbrace{t}_{\text{Origin change}} + \underbrace{\boxed{\lambda R p'}}_{\text{Rotation}}$$

λ: rotation between the units of lengths of 2 RS's

$$R = \begin{bmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{bmatrix}$$

property: $RR^T = I \rightarrow R^T = R^{-1}$

Helmert transformation in 3D: the decomposition of 3D rotations in 3 planar rotations.

- degrees of freedom (**DOF**) of a 3D RS: 3 DOF for origin translation and 3 DOF for axes rotation
- in geodetic framework, **R** is implemented by composition of 3 planar rotations around 3 axes.

$$x'_P = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}_{P,I} \qquad x''_P = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}_{P,II} \qquad t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \leftarrow \text{coordinates of I origin wrt II}$$

Rotate x_2-x_3 plane around x_1

$$R_1(r_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos r_1 & \sin r_1 \\ 0 & -\sin r_1 & \cos r_1 \end{bmatrix}$$

→

Rotate x_1-x_3 plane around x_2

$$R_2(r_2) = \begin{bmatrix} \cos r_2 & 0 & -\sin r_2 \\ 0 & 1 & 0 \\ \sin r_2 & 0 & \cos r_2 \end{bmatrix}$$

→

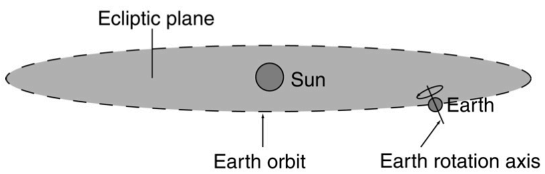
Rotate x_1-x_2 plane around x_3

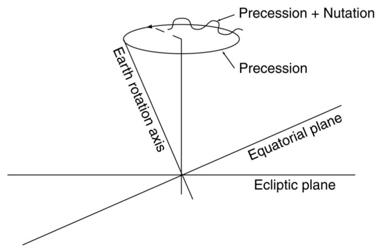
$$R_3(r_3) = \begin{bmatrix} \cos r_3 & \sin r_3 & 0 \\ -\sin r_3 & \cos r_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x''_P = \underbrace{t}_{\text{origin change}} + \underbrace{\boxed{\lambda R x'_P}}_{\text{Rotation}}$$

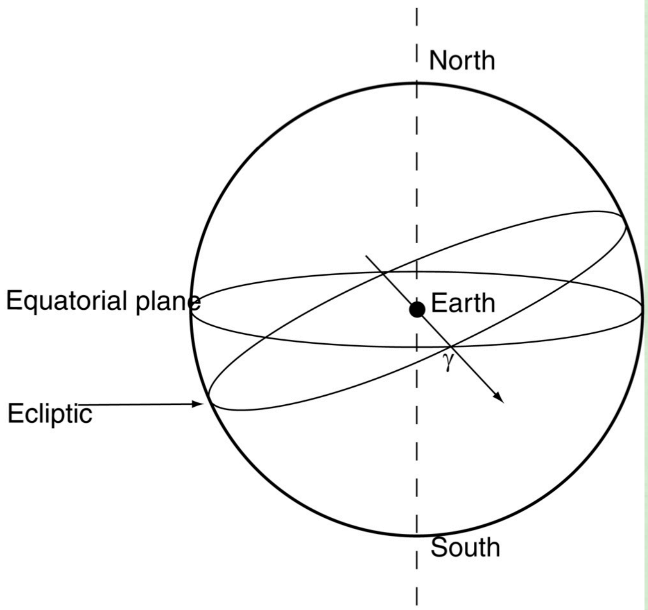
$$R = R(r_1, r_2, r_3) = R_3(r_3)\,R_2(r_2)\,R_1(r_1)$$

Earth dynamics in space.





- ***Earth revolution (地球公转)***: the Earth orbit around the sun is **elliptic**, **counter-clockwise**, with a period of 365.25 polar days.
  - **ecliptic plane**: the orbit plane.
- ***Earth rotation (地球自转)***: the Earth rotates **counter-clockwise** around its axis with a period of 1 solar day
  - the angle between the rotation axis and the perpendicular to the ecliptic plane is of about 23°27'
  - the rotation axis is moving in the space with two seperate conic motions (圆锥运动)
    - **precession** (岁差): a motion with an angle of 23°27' and period of 25800 years
    - **nutation** (章动): a fluctuation around the precession motion, with angle of about 9.2'' and period of 18.6 years



- **Celestial Sphere**: sphere with unitary radius, Earth centered, on which the stars positions are projected
  - for an Earth fixed observer, the sphere rotates westwards, around the Earth rotation axis
- **Celestial Equator**: projection of the equatorial plane on the Celestial Sphere
- **Ecliptic**: projection of the apparent sun orbit on Celestial Sphere
- **Equinoxes** (昼夜平分点): intersection of the ecliptic with Celestial Equator
  - γ: Spring Equinox; γ': Autumn Equinox
- Equator and Equinoxes fluctuate in time. For a given epoch:
  - **Mean Celestial Equator and Mean Equinoxes**: computed by subtracting the effects of nutation to the actual equator and equinoxes
  - Reference Celestial equator and equinox γ: mean celestial equator and spring equinox for 12.00 of 2000.01.01
- **Astronomical Reference Frame**: origin in the center of the celestial bodies, axes directions by astronomical directions

Chandler wobble (钱德勒摆动).

The planet fluctuates around its rotation axis. This motion is approximately conic, with amplitude of 0.1''-0.2'' and period of 435 days

- **Earth Orientation Parameters** (EOP，地球定向参数)
  - used to describe irregularities in the rotation of planet Earth
    - note: Earth rotation axis and the angular velocity of rotation are not constant.
  - includes 3 time variant parameters
    - **LOD (Length of Day)**: difference between the length of the solar day and the atomic day
    - two direction angles of the instantaneous rotation axis with respect to a conventional earth-fixed axis: **coordinates of the pole** and celestial pole offsets
      - **conventional position of the pole for a given period**: the mean position (wrt to the Chandler Wobble) of the pole in the period
        - **Conventional International Origin (CIO)**: mean in the 1900-1905 period
        - **Conventioanl Terrestrial Pole (CTP)**: Bureau Interational dell'Heure (BIH) definition in 1984

The Earth Dynamic

- The earth internal displacement and external shape are continously changing
  - **period phenomena**, like the solid tides
    - a mean position can be defined
  - **irreversible phenomena**, from the global (geodynamics) to the local scale (for example, subsidence)
    - each point of the earth occupies a specific position just for an infinitesimal instant: functions of time
      - the hypothesis of **linear motion** is usually adopted to model in time the geodynamics

$$X(t) = X_0 + (t-t_0)\underbrace{\dot{X}}_{\text{geodynamic velocity}}$$

      - **Discontinuities**: the linear hypothesis doesn't hold anymore in presence of earthquakes, strutural deformations or breaks, seasonal landslides.

discontinuity epoch

$$\begin{cases} x(t) = x_0 + (t-t_0)\dot{x}, & t < t_D \\ x(t) = x_0 + \underbrace{\delta x}_{\text{displacement caused by discontinuity}} + (t-t_0)\dot{x} + (t-t_D)\underbrace{\delta\dot{x}}_{\text{change of geodynamic velocity due to the discontinuity}}, & t \geq t_D \end{cases}$$

Terrestrial reference frames: ITRF and ETRF.

- ***Reference Systems (RSs) and Reference Frames (RFs)***
  - RSs
    - realized by (networks of) benchmarks or continuously operating stations, whose coordinates are
      - estimated (and continuously monitored
      - published in catalogues
    - then, other points can be estimated by differential positioning techniques wrt the reference networks.
  - RSs V.S RFs
    - RSs is the geometric definition (几何定义)
    - RFs is realization. A frame is **a set of points** whose coordinates has been estimated so that users can used them to estimate the position of other points.
  - two types of RFs
    - **continuously monitord RFs**: realized by networks of continously observing stations, whose coordinates are **continuously estimated and updated** → permanent stations and networks
    - **static RFs**: realized by networks of benchmarks that have been surveyed and estimated in a single compaign, whose coordinates are **statically published**.
- ***Global Reference systems and frames***
  - Global RS's are realized by global networks of fundamental stations that apply specific (satellite geodesy) techniques to estimate **the Earth goecenter** and **rotation axis** together with the **stations coordinates**.
    - Geocentric Cartesian and Geodesic
  - ITRS and ITRF
    - **International Terrestial Reference System (ITRS)**
      - geocentric
      - unit of length: meter, consistent with the TCG scale
      - initial axis orientation: BIH 1984 orientation
        - time evolution of the axes orientation: no net rotation wrt the Earth tectonics
    - ITRS permanent network
      - for spatial geodesy techniques
        - **VLBI**: Very Long Baseline Interferometry ([ˌintəfə'rɔmitri]), by space telescopes
        - **SLR**: Satellite Laser Ranging, by laser (['leɪzə(r)]) telescope
        - **DORIS**: Doppler Orbit determinination and Radiopositioning Integrated on Satellite
        - **GNSS**: Global Navigation Satellite System. (GPS + GLONASS + GALILEO)
    - **ITRF (ITRF Frame)**: A ITRF consists of **a catalogue of coordinates and velocities** of the PS's contributing to the solution.
      - Estimation and distribution of ITRF
        - **VLBI**: estimates the Z axis of ITRF
        - **SLR**: estimate the origin of ITRF
        - **IGS PN**: linked to VLBI and SLR and distributed ITRF coordinates to the GNSS user community
      - ITRF updates: one update about every 5 years. The current is **ITRF2014**

- ***Local Reference systems and frames***
    - at the scale of the continents / nations
        - GNSS networks adjusted in the global network, horizontal and vertical national networks
        - example: ETRS
    - at the local scale
        - Local Level (gravity field): used for terrestial survey
        - GNSS/topological networks for surveying, monitoring
            - Local Cartesian (ellipsoid): used for GPS survey
        - mobile cells points for geolocalization
        - Wi-Fi, RFID for indoor positioning
    - ETRS and ETRF
        - **European Terrestial Reference System (ETRS)**
            - ETRS89: coincides with ITRS in 1989, but is moving and rotates with the mean part of Europe. It is realized by the IAG European Reference Frame (EUREF) Commission by EPN (European Permanent Network)
        - Present Realization: **ETRF2014**
            - howto
                - ITRF coordinates and velocities of EPN stations are estimated.
                - The transformation parameters between ITRF and ETRF are published and distributed.
                - ETRF2014 coordinates and velocities of the stations are computed by applying the transformations

$$\underline{X}(t)_{P,E} = \underline{t}(t) + (1+\mu(t))\underline{X}(t)_{P,I} + SR(t)\underline{X}(t)_{P,I}$$

$$SR(t) = [\underline{r}_0 \times] + (t - \underline{2000.0})[\underline{\dot{r}} \times]$$

$$\underline{t}(t) = t_0 + (t - \underline{2000.0})\underline{\dot{t}}$$

$$\mu(t) = \mu_0 + (t - \underbrace{2000.0}_{\text{from 2000}})\dot{\mu}$$

            - ETRF2014 provides a RF with minimum motion wrt Europe which is useful for all the cartographic applications.
            - **IGM95**: Italian realization of ETRF
                - It is the national geodetic network, established and surveyed by IGM
                - IGM95 is composed of about 1250 benchmarks which densifies ETRF

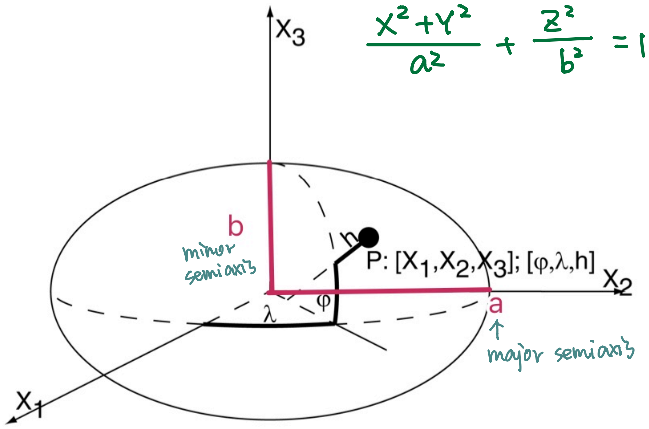
The IGS and EPN GNSS permanent networks.

- ***Permanent Station (PS)***: a continuously operating instrument which publishes raw data, continuously be monitored and provides its coordinates estimates.
    - Distributed estimates for each PS
        - known
            - initial geocentric position $X(t_0)$ and related covariance C_{00}
            - velocity $\dot{x}$ and related covariance C_{vv} for the reference epoch t_0
        - the coordinates can be propagated at each epoch $t \neq t_0$ by

$$x(t) = x(t_0) + \dot{x}(t - t_0), \quad C_{xx} = C_{00} + C_{vv} \cdot (t - t_0)^2$$

            - accuracies of the last ITRFs are at mm level.
- ***Permanent Networks (PNs)***
    - A PN cosists of
        - a set of permanent stations with a homogeneous ([ˌhɒməˈdʒiːniəs] 均匀的) spatial distribution in the interested area
        - one or more control centers that manage the PSs, check their data quility, process the network data, estimate PSs coordinates and if needed, distribute network data and products to the users.
- ***Global GNSS Network: IGS (International GNSS Service)***
    - It is a service of the International Association of Geodesy (IAG), established in 1993. Its goals are
        - contribute to the realization and distribution of ITRS
        - distribute GNSS products (ephemerides, EOP, ...)
        - define the standard for GNSS PNs and support GNSS research
    - consists of about 500 PSs, serveral Analysis centers, Working groups, Pilot projects, Services, and a Central Bureau
- ***EPN (European Permanent Network)***
    - about 330 GNSS PSs

Definition of Cartesian and geodetic coordinates: transformation between them.



$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1$$

Geodetic coordinates in a 3D RF

A rotaional ellipsoid in 3D is defined as the surface whose points satisfy the condition (⬅)

eccentricity : $e=\sqrt{1-\frac{b^2}{a^2}}$

flattening : $f=\frac{a-b}{a}$

$\Big\}\; e^2=f(2-f)$

Given a RF and the related ellipsoid, the geodetic coordinates of a point P are defined as:

- **geodetic latitude (φ)**: angle between the norml to ellipsoid and the equatorial plane
- **geodetic longitude (λ)**: angle between the meridian plane passing per P and the origin meridian plane
- **geodetic height (h)**: distance between P and the ellipsoid surface

Geodetic coordinates $[\varphi,\lambda,h]$ ⟶ Cartesian coordinates $[x,y,z]$

$$X_P=(R_N+h_P)\cos\varphi\cos\lambda$$
$$Y_P=(R_N+h_P)\cos\varphi\sin\lambda$$
$$Z_P=[R_N(1-e^2)]\sin\varphi$$

$$R_N=\frac{a}{\sqrt{(1-e^2\sin^2\varphi)}}$$

Cartesian coordinates $[x,y,z]$ ⟶ Geodetic coordinates $[\varphi,\lambda,h]$

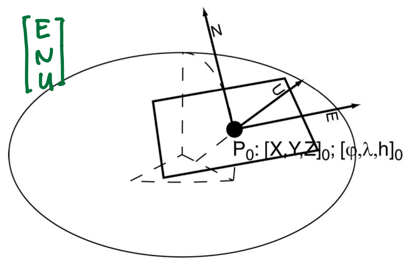
$$e_b^2=\frac{a^2-b^2}{b^2}$$
$$r=\sqrt{x^2+y^2}$$
$$\psi=\arctan\left(\frac{z}{r\sqrt{1-e^2}}\right)$$

⟶

$$\lambda=\arctan\frac{Y}{X}$$
$$\varphi=\arctan\frac{(z+e_b^2\,b\sin^3\psi)}{(r-e^2a\cos^3\psi)}$$
$$h=\frac{r}{\cos\varphi}-R_N$$

Definition of Local Cartesian: transformations between Geocentric Cartesian and LC.

Local Cartesian coordinates (East, North, Up): origin in a point P0; X and Y orthogonal to the normal to the **ellipsoid** in P0; X points to East, Y points to North, Z points to the **normal** to the ellipsoid.



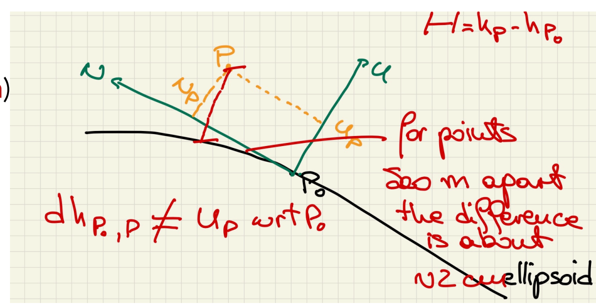
Geocentric baseline from P_0 to P

$$P_0=\begin{bmatrix}X_0\\Y_0\\Z_0\end{bmatrix},\quad P=\begin{bmatrix}X_P\\Y_P\\Z_P\end{bmatrix},\quad \Delta X_{P_0,P}=\begin{bmatrix}X_P-X_0\\Y_P-Y_0\\Z_P-Z_0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}E\\N\\U\end{bmatrix}_P=R_0\,\Delta X_{P_0,P}\qquad R_0=\begin{bmatrix}-\sin\lambda_0 & \cos\lambda_0 & 0\\-\sin\varphi_0\cos\lambda_0 & -\sin\varphi_0\sin\lambda_0 & \cos\varphi_0\\\cos\varphi_0\cos\lambda_0 & \cos\varphi_0\sin\lambda_0 & \sin\varphi_0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}X_P\\Y_P\\Z_P\end{bmatrix}=\begin{bmatrix}X_0\\Y_0\\Z_0\end{bmatrix}+R_0^T\begin{bmatrix}E\\N\\U\end{bmatrix}_P$$

- ***Note***: UP component is not the ellipsoidal height difference (△U ≠ △h)
  - for points 500m apart, the difference is aboutn 2cm
- Two applications for Local Cartesian coordinates:
  - to estimate coordinates of a set of points wrt a local origin
  - to monitor time series of coordinates



Definition of Local Level: differences between LC and LL.

Local Level coordinates (X, Y, Z): given an origin point P0 for the interest area, Z axis along the **gravity vertical** in P0, X axis contain the projection of another point on X, Y plane

LC → LL

vertical deflection components

P_i coordinates in LC

$$X_{LL}(P_i) = R\,x_{LC} = R_z(\alpha)\,R_y(\eta)\,R_x(-\xi)\,\boxed{x_{LC}(P_i)}$$

$$R_x(-\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\xi) & -\sin(\xi) \\ 0 & \sin(\xi) & \cos(\xi) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\xi \\ 0 & \xi & 1 \end{bmatrix}, \quad R_y(\eta) = \begin{bmatrix} \cos(\eta) & 0 & -\sin\eta \\ 0 & 1 & 0 \\ \sin\eta & 0 & \cos\eta \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\eta \\ 0 & 1 & 0 \\ \eta & 0 & 1 \end{bmatrix}$$

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Computation Flow (LL → LC)

$\xi,\ \eta,\ \boxed{x_{P_1,LC}}$ ← P_1 coordinates in LC

1) find α

$$x_{P_1,Tmp} = \begin{bmatrix} x_{P_1,tmp} \\ y_{P_1,tmp} \\ z_{P_1,tmp} \end{bmatrix} = R_y(\eta)\,R_x(-\xi)\,x_{P_1,LC} \longrightarrow \alpha = \operatorname{atan}\left(\frac{y_{P_1,tmp}}{x_{P_1,tmp}}\right)$$

2) transform.

$$x_{LC}(P_i) = R^T x_{LL}(P_i)$$

Diagram flux of the transformation between local systems (LC, LL, body frame) and geocentric systems.

- In many applications, the position of a set of points are determined in a local level reference system (LL), but are needed in a global one: LL → ETRF
- To transform from LL to GC (LL → GC)
  A. P0 and P1 must estimated in GC $x_{P_0,GC},\ x_{P_1,GC}$
  B. GC coordinates of P1 are transformed to LC wrt P0 $x_{P_1,GC} \to x_{P_1,LC}$
  C. local cartesian coordinates of P1 are used to estimated α $x_{P_1,LC} \to \alpha$
  D. given α, all the other points can be transformed from LL → LC → GC

The covariance propagation in RF transformations.

origin in P_0 ↳ LL

$$P_j = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{LL}, \quad C_{LLj} = \begin{bmatrix} \sigma_{x_j}^2 & 0 & 0 \\ 0 & \sigma_{y_j}^2 & 0 \\ 0 & 0 & \sigma_{z_j}^2 \end{bmatrix}$$

↓ $R_{(LL\to LC)}$

Same origin in P_0 ↳ LC

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} \qquad C_{LCj} = R_{(LL\to LC)}\,C_{LLj}\,R^T_{(LL\to LC)}$$

↓ $R_{(LC\to GC)}$

GC, ITRF

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{GC} \qquad C_{GCj} = R_{(LC\to GC)}\,C_{LCj}\,R^T_{(LC\to GC)}$$

↓

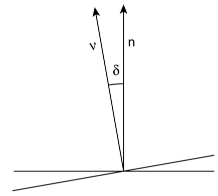
GC, Geodetic

$$\begin{bmatrix} \varphi \\ \lambda \\ h \end{bmatrix} \leftrightarrow \begin{bmatrix} E \\ N \\ U \end{bmatrix} \qquad C_{\left[\begin{smallmatrix} E \\ N \\ U \end{smallmatrix}\right]_j} = R_{(GC\to LC)}\,C_{GCj}\,R^T_{(GC\to LC)}$$

$$\longrightarrow \begin{cases} \sigma_{\lambda_j} = \sqrt{C_{\left[\begin{smallmatrix} E \\ N \\ U \end{smallmatrix}\right]_j}(1,1)} \\ \sigma_{\varphi_j} = \sqrt{C_{\left[\begin{smallmatrix} E \\ N \\ U \end{smallmatrix}\right]_j}(2,2)} \\ \sigma_{h_j} = \sqrt{C_{\left[\begin{smallmatrix} E \\ N \\ U \end{smallmatrix}\right]_j}(3,3)} \end{cases}$$

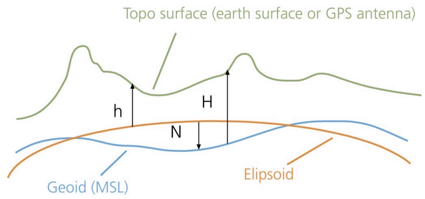
Geoid undulation, vertical deflection, orthometric height and heights transformations.

- ***Geoid undulation N(φ, λ)*** in a point φ, λ: the height of the geoid with respect to the reference ellipsoid
    - positive if geoid outside, negative if geoid inside
- ***vertical deflection***: angle between the normal vectors to the ellipsoid and the geoid, decomposed in local North ξ and East η components (smaller than 1')
- ***orthometric height H***: height of P wrt to geoid
- Note: by GNSS, h is estimated. For 99% of applications, H are needed ➡️ **H = h - N**
- The differences between orthometric height H / ellipsoidal heights h / height differences are never negligible. The problem can easily solved if geoid undulation is known.



Vertical deflection

h=H+N



h=elipsoid height
H=orthometric height
N=geoid height

height transformations

Hights conversion : $H(\varphi,\lambda) = h(\varphi,\lambda) - N(\varphi,\lambda)$

Conversion of heights differences

$$\begin{aligned}\Delta H_{ij} &= H(\varphi,\lambda)_j - H(\varphi,\lambda)_i \\ &= h(\varphi,\lambda)_j - N(\varphi,\lambda)_j - [h(\varphi,\lambda)_i - N(\varphi,\lambda)_i] \\ &= \Delta h_{ij} - \Delta N_{ij}\end{aligned}$$

Time Series analysis

Given a permanent stations

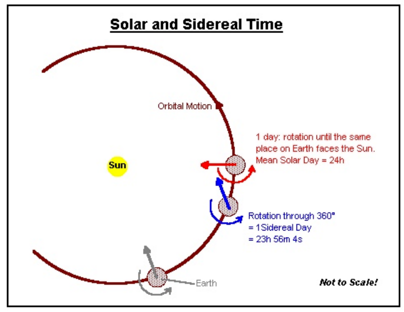
$\begin{vmatrix}X\\Y\\Z\end{vmatrix}(t_i)$ $i = \text{epoch } \phi \to T$

P_0 is the initial position of the position → the "origin" of a local cartesian reference system → $\begin{vmatrix}X\\Y\\Z\end{vmatrix}(t_0) = P_0$

All the coordinates $\begin{vmatrix}X\\Y\\Z\end{vmatrix}(t_i)$ are transformed as $\begin{vmatrix}E\\N\\U\end{vmatrix}(t_i)$ wrt P_0

Time Systems

Three main groups of time scales (时标)



- ***sidereal ([saɪ'dɪəriəl], 恒星的) and solar scales***
  - Driven by terrestrial rotation motion
    - sidereal wrt equinox direction: angle btw a reference meridian and γ
    - solar wrt Sun direction: angle btw a reference meridian and the Sun
  - Actual solar/sidereal time scale: angle wrt "**true**" sun/γ direction
  - Mean solar/sidereal time scale: angle wrt "**mean**" sun/γ direction
    - mean = true direction - effects of nutation
  - Hourly angle:
    - Global: reference meridian in Greenwich
      - **GAST (Greenwich Apparent Sidereal Time)**: Hourly angle between Greenwich meridian and the true equinox + 12hours
      - **GMST (Greenwich Mean Sidereal Time)**: Hourly angle between Greenwich meridian and mean equinox (nutation subtracted) + 12 hours
      - **UT1 (Universal Time 1)**: Hourly angle between Greenwich meridian and mean Sun direction + 12 hours
    - Local: the angle is cmoputed in a local meridian
  - In one year: 365.2422 solar days, 366.2422 sidereal days
    - the solar day is longer than the sidereal day
- ***dynamic scales***: observations of the positions of Sun system planets
  - positions of planets can be very accurately modeled in time
    - the inverse problem: observe position of planeets and determine time!!
  - two main scales: Ephemeris Time and Terrestrial Time
- ***atomic scales***
  - International Atomic Time scale TAI: driven by a set of worldwide atomic clocks
  - **UTC (Universal Time Coordinated)**: an atomic time scale that, periodically is shifted of 1 second in order to take it aligned with UT1, to make midnight of UTC = midnight of UT1
  - **GPST**: GPS time = TAI-19s

MEO: Medium Earth Oribts, earth-centered. All the orbits with a distance from Earth btw 2000km and 36000km
Geosynchronous: orbit period is 24 hours
Geostationary: geosyncronous and orbital plane in the equatorial plane

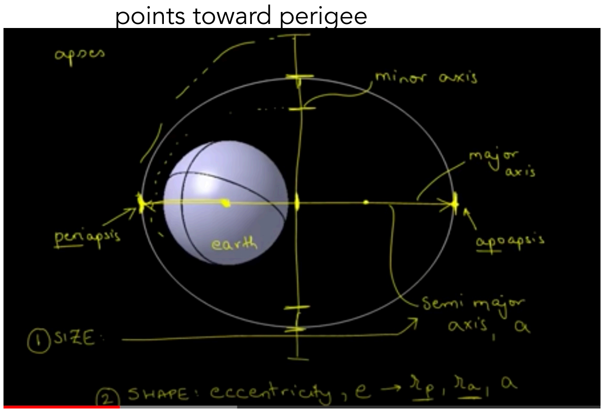
GNSS Point positioning

GPS, GLONASS, Compass / Beidou and Galileo: general description.

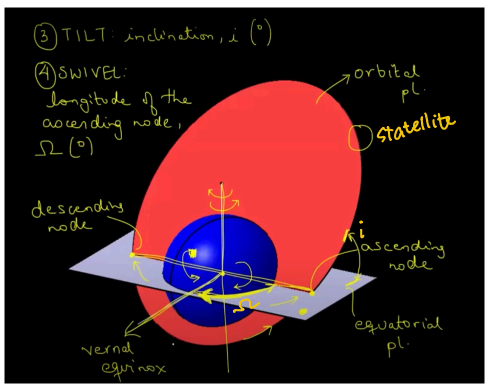
- ***GNSS (Global Navigation Satellite System)***: the set of all the satellites system aimed at positioning of users.
    - it includes:
        - several constellations of artificial satellites around the Earth
            - global constellation: a constellation that allows positioning everywhere at every time on the Earth
            - regional constellation: a constellation that allows positioning in limited region of the Earth
        - a receiver observes signal travel times from all the in-view satellites.
    - how to
        - the satellites positions are known from ephemerides ([,efi'meridi:z])
            - ephemerides: numerical data that allow to compute satellites position
        - the travel times are converted to distances ($d=cT$)
        - the observations are used to estimate the receiver postion
- ***GPS (Global Positioning Service)***: United States
    - development started in 1970s, global fully operating from 1995 and completely renewed after 2000
    - consists of 31 operational satellites in 6 different orbital plane (i=55°). (now Block III)
        - the system has been designed to always guarantee the visibility of at least 4 satellites everywhere on the Earth surface. Each satellite moves at about 4km/sec
    - the world's most utilized, accurate and reliable satellite navigation system
- ***GLONASS***: Russia
    - first project and development in 1970s, global fully operating from 2011, but still few operational problems
    - consists of 24 satellites in 3 orbital plane
- ***Compass/Beidou***: China
    - first project in 1980s, first testing in 2000
    - Beidou 1 includes 4 geostationary satellites, limited coverage to China and part of Assia
    - Beidou 2 includes 5 geostationary and 30 MEO satellites
    - Beidou 3 includes 3 geostaionary satellites, 3 geosynchronous and 24 MEO satellites
- ***Galileo***: European
    - first project in 1990s, first operational satellite in 2011
    - now consists of 22 operational satellites in 3 orbital plane

GPS: constellation, ephemerides and relevant errors.

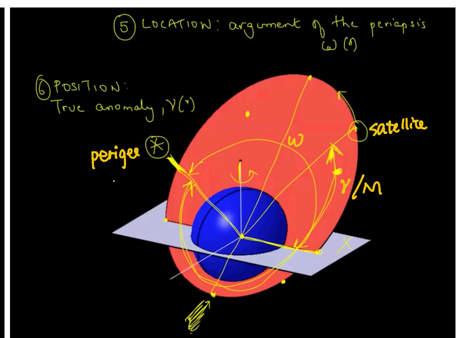
- ***constellation***
    - requirements for satellites: communicate their position and GPS time with the highest possible accuracy
    - GPS satellites position and orbits
        - the satellite transmits a set of parameters which alow us to compute its position (ephemerides)
        - In an ideal central force field, artificial satellites move along elliptic orbits predictable in time by Kepler laws (ref: https://youtu.be/AReKBoiph6g)
            - two stationary points wrt an inertial Earth Geocentric reference frame
                - Apogee: the furthest point of the orbit from the Earth
                - Perigee: the nearest point to the Earth
            - the position vector spans equal areas in equal times
    - **Orbit Reference System**: origin in the orbit focus; X-Y plane contains the oribital plane, Z of orbit equal to zero and X points toward perigee



Keplerian orbit: a and e



Keplerian orbit: Inclination and Ω



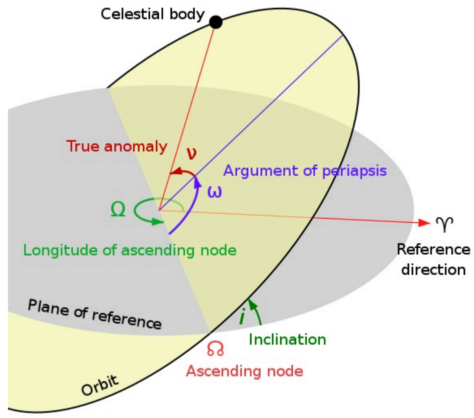
Kepleerian orbit: ω and M

- **Keplerian orbit**: origin is in the Earth center of mass, X axis along major semiaxis of orbit, Y orthogonal to X in the orbit plane, Z orthogonal to the orbit plane
  - A Keplerian orbit is stationary and defined by 6 parameters:

Orientation
    - orbit inclination **i** : angle between the orbital plane and the reference equatorial plane
    - right ascension of the ascending node **Ω** : angle on the reference equatorial plane, btw the equinox node and the ascending node

Position
    - perigee argument **ω** : angle btw the intersection of the orbital plane with the reference equatorial plane (vertical equinox) and the orbit perigee direction (counter-clockwise)
    - mean anomaly **M0**: angle between satellite mean motion and perigee at epoch zero

Size & shape
    - **a** and **e**: major semiaxis and orbit accentricity



GPS case

$$\begin{cases} \Omega(t) = \Omega_0 + \dot{\Omega}(t - t_0) \\ \omega(t) = \omega_0 + \dot{\omega}(t - t_0) \\ i(t) = i_0 + \dot{i}(t - t_0) \end{cases}$$ # in 2h.

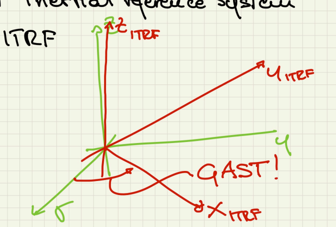
Keplerian orbit computation

1. $a, e, M_0 \longrightarrow x(t), y(t)$ of satellite in ORS

Mean motion $n = \sqrt{\dfrac{GM_E}{a^3}} \longrightarrow M(t) = M_0 + n(t - t_0)$ (Mean anomaly; Eccentric Anomaly) $\longrightarrow \eta(t) = M(t) + e\sin\eta(t)$ (interative method, $\eta_0(t) = M(t)$) $\longrightarrow \psi(t) = \tan^{-1}\left(\dfrac{\sin(\eta(t))\sqrt{1 - e^2}}{\cos\eta(t) - e}\right)$ (true anomaly)

$\longrightarrow r(t) = \dfrac{a(1 - e^2)}{1 + e\cos(\psi(t))}$ (distance of satellite from Earth) $\longrightarrow$ $\begin{cases} x(t) = r(t)\cos(\psi(t)) \\ y(t) = r(t)\sin(\psi(t)) \end{cases}$ $\begin{vmatrix} x(t) \\ y(t) \\ 0 \end{vmatrix}_{ORS}$

from Inertial reference system to ITRF



2. $i, \Omega, \omega \longrightarrow$ ITRF

ORS $\longrightarrow$ ICRS $\longrightarrow$ ITRF

$$\begin{vmatrix} x(t) \\ y(t) \\ z(t) \end{vmatrix}_{ICRS} = R_3(-\Omega)R_1(-i)R_3(-\omega) \begin{vmatrix} x(t) \\ y(t) \\ 0 \end{vmatrix}_{ORS}, \quad \begin{vmatrix} x(t) \\ y(t) \\ z(t) \end{vmatrix}_{ITRF} = R_1(y_p)R_2(-x_p)R_3(GAST) \begin{vmatrix} x(t) \\ y(t) \\ z(t) \end{vmatrix}_{ICRS}$$

angles in x, y direction btw convention pole and actual pole (EOP)

Final Transformations for GPS

$\Omega_E = 7.292115167 \times 10^5$ rad/sec

$\Omega(t) = \Omega_0 + (\dot{\Omega} - \dot{\Omega}_E)(t - t_0)$

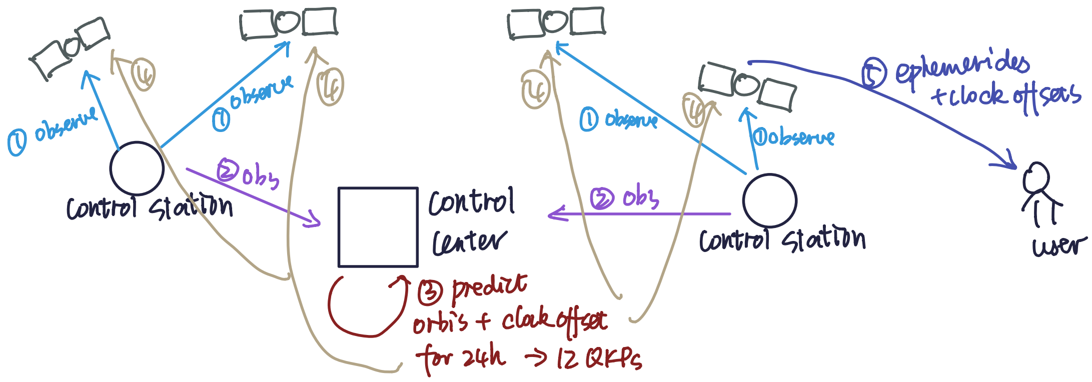
$\omega(t) = \omega_0 + \dot{\omega}(t - t_0)$

$i(t) = i_0 + \dot{i}(t - t_0)$

$$\longrightarrow \begin{vmatrix} x(t) \\ y(t) \\ z(t) \end{vmatrix}_{ITRF} = R_3(-\Omega(t))R_1(-i(t))R_3(-\omega(t)) \begin{vmatrix} x(t) \\ y(t) \\ 0 \end{vmatrix}_{ORS}$$

note: to model the satellite orbit in a sufficiently accurate (better than 10m) way, a slightly more complex parameterization: 16 **quasi-Keplerian parameters (Q-KP)** can be used with sufficient accuracy over interval of 2 hours.

- ***satellite ephemerides***: the set of 16 parameters necesasry and sufficient to compute the satellite position in time
  - each set lasts 2 hours.
  - Broadcast ephemerides: predicted by US National Geospatial Intelligence Agency(NGA)
    - A network of GPS PSs of US Army, also called **Control Segment** of the GPS system
      - the observation of each control station to each satellite are sent to the Control center
      - by complex modeling, the control center predicts the orbit and clock offset of each satellite for the next 24h
      - the 24h orbit is split into 12 quasi-keplerian arcs of two hours
      - the predicted ephemerides and clock offsets are transmitted to the satellite, which distribute them to the users
    - the broadcast ephemerides are in WGS84 realization of ITRF and have an accuracy of about 1m
  - Precsie ephemerides: a posteriori estimated ephemerides by IGS, delivered with a latency of 14 days via web in a tabular format
    - comparison: rapid ephemerides are available within 1 days while final (precise) ephemerides are available within 2 weeks.
    - Final ephemerides have an accuracy better than 2.5 cm.



The oscillator: frequency, phase and time.

- **satellite elevation**: vertical angle between the horizontal plane in the observer and the satellite direction of veiw
  - horizontal: orthogonal to the normal to the ellipsoid in P
- For an ideal oscillator
  - carrier phases

$$A(t) = \underbrace{A_0}_{\text{signal amplitude}} \sin(\underbrace{\omega}_{\text{Angular velocity (rad/s)}} t + \underbrace{\varphi_0}_{\text{initial phase}}) = A_0 \sin(\varphi(t)) \quad \varphi(t)\text{, phase (rad)}$$

$$\underbrace{\phi(t)}_{\text{in cycles}} = \frac{\varphi(t)}{2\pi} \leftarrow \text{in rad}$$

period: $T = \frac{2\pi}{\omega}$

frequency: $\underbrace{f}_{\text{constant}} = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{\Delta\phi}{\Delta t} = \frac{d\phi}{dt}$ (first derivative of phase in time.)

Propagation Law

$$A(\underbrace{t}_{\text{time}}, \underbrace{x}_{\text{position in the propagation direction}}) = A_0 \sin(\omega(t - \frac{x}{\underbrace{c}_{\text{propagation velocity}}}) + \varphi_0) = A_0 \sin[2\pi(\frac{t}{T} - \frac{x}{\underbrace{\lambda}_{\text{wave length}}}) + \varphi_0]$$

$\lambda = cT = c/f$

  - binary codes for an ideal chip generator
    - binary code: a sequence of pulses equal to +1 and -1. The pulses transmission sequence, according to a proper decoding key, represnets the signal content.
      - code duration: time needed to transmit the whole sequence
      - code frequency: the inverse of one pulse lasting
      - code wavelength: length of one pulse
- For a real oscillator

Frequency

$$f(t) = \frac{d\phi(t)}{dt} = \underbrace{f_0}_{\text{Nominal frequency}} + \underbrace{\delta f(t)}_{\text{fluctuation in time}} \longrightarrow f_i(t) = \frac{d\phi(t)}{dt} = \underbrace{f_{0i}}_{\text{Nominal frequency of } i} + \underbrace{\delta f_i(t)}_{\text{fluctuation in time}}$$

i = oscillator

phase

$$\phi_i(t) = \underbrace{\phi_0}_{\text{initial phase}} + \int_{t_0}^{t} f_i(\tau)d\tau = \phi_0 + \int_{t_0}^{t} f_{0i}\,d\tau + \int_{t_0}^{t} f_i(\tau)d\tau = \phi_0 + f_0 t_i(t) = \phi_0 + f_0 t + f_0 dt_i(t)$$

time

$$t_i(t) = \frac{\phi_i(t) - \boxed{\phi_i(t_0)}}{f_0} = \frac{1}{f_0}\int_{t_0}^{t} f_i(\tau)d\tau = \underbrace{\boxed{\frac{1}{f_0}\int_{t_0}^{t} f_0\,d\tau}}_{t\text{: true}} + \underbrace{\boxed{\frac{1}{f_0}\int_{t_0}^{t} f_i(\tau)d\tau}}_{dt_i(t)\text{: clock offset of } i \text{ (error)}} = t + dt_i(t)$$

$\phi_i(t_0) = \phi_0$

- GPS time and satellites clock

satellites transit broadcast clock parameters

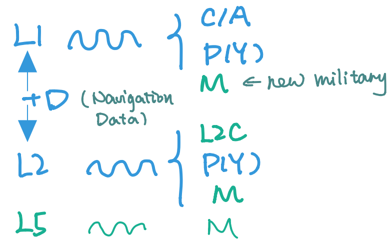
$$\underbrace{t^s}_{\text{time of a satellite clock}} = \underbrace{t_{GPS}}_{\text{ideal time scale of GPS (true)}} + \underbrace{dt^s(t_{GPS})}_{\text{satellite offset}} \xrightarrow{\text{describ by}} dt^s(t) = dt_0^s + a^s t_{GPS} + b^s t_{GPS}^2$$

(dt_0^s, a^s, b^s: communicated by the satellite by D message)

$$\underbrace{\Delta t}_{\text{Measured Travel Time}} = \underbrace{t_R}_{\text{Receiver clock; Measured arrival Time of the signal}} - \underbrace{t^s}_{\text{satellite clock; measured emission time of the signal}}$$

GPS binary codes and phase carriers.

The on-board oscillators produce a signal, whose nominal frequency f0 is equal to 10.23MHZ (fundamental signal) and stable in time. From it



- 3 sinusoidal ([,saɪnə'sɔɪdəl]) carrier phases (sinusoidal signal propagating in space) $\sin(\varphi(t))$
  - II { ○ **L1**: f=154f0, λ≈19cm
  - ○ **L2**: f=120f0, λ≈24cm
  - III ○ (from Block II-F)**L5**: f=115f0, λ≈26cm
    - to improve ambiguity fixing
- 5 binary codes $\pm 1(t)$
  - used for point positioning
    - ***Pseudo Random C/A: Coasrse acquisition code***
      - f=0.1f0, λ≈293m, pulses number=1023, T=1ms
      - one different C/A for each satellite, used to **identify** the satellite and for pseudo-range observations
      - Code Divition Multiple Access (CDMA)
      - only on L1
    - ***P(Y): P(Precise) or Y(Encrypted P) military code***
      - f=f0, λ≈29.3m, pulses number=3.2703264+e16, T=37weeks
      - common to all the satellites, guarantees better pseudo-range precisions than C/A code.
      - on both L1 and L2
  - ***Navigational Message D***: contains ephemerides and other data
    - f=50Hz, λ≈6e+6m, pulses number=37500, T=12.5min
  - From Block-IIR
    - Pseudo Rando **L2C**
      - f=0.1f0, λ≈293m, pulses number=10230, T=10ms
      - new civilian code, counterpart of C/A code on L2 carrier
    - Pseudo Rando ***M*** { L1 L2 L5
      - f=f0, λ≈29.3m
      - new military code
- combination of the signals S(t): carrier phase A(t) modulation by a binary code C(t) ➡️ **S(t) = A(t)C(t)**
  - result: a signal equal to the carrier phase but for slips equal to 180° corresponding to code state transitions
- ***Observation Equations***
  - Binary (PR) codes C/A, P(Y), L2C, M → **code (or pseudo-range) observation**
    - The satellite is identified by its own C/A code. After identification, the receiver performs a **correlation** between its internally generated C/A code and the one recevied from the satellite. This observations can be done with both C/A and P(Y) codes. The electronic correlator provides the observation of the **delay** btw the received signal the the internal one

Measurement into the receiver correlator of the time shift btw received signal and internal replica.

$$\Delta T_R^S(\underset{\text{observation epoch}}{t}) = \underbrace{t_R(t)}_{\text{time of the reception of the signal in the clock of the receiver}} - \underbrace{t^S(\overset{\text{true time of emission}}{t} - \overset{\text{travel time}}{\tau_R^S})}_{\text{time of emission of the signal in the clock of satellite}} = [t + dt_R(t)] - [(t-\tau_R^S) + dt^S(t-\tau_R^S)]$$

$$= \tau_R^S + dt_R(t) - dt^S(t-\overset{\simeq 66ms}{\tau_R^S}) \simeq \tau_R^S + dt_R(t) - dt^S(t)$$

obs → $$P_R^S(t) = c\Delta T_R^S(t) = c\tau_R^S + c[dt_R(t) - dt_S(t)]$$

#Noise { C/A & L2C : 20-40m; P(Y) : 10-20m

- howto
  - For each in view satellite (multichannel processing):
    - C/A code and navigation Data acquisition (C/A → satellite identification; D → satellite position and clock)
    - C/A → correlator → pseudorange observation in C/A
  - with all the observed C/A pseudoranges observation processing → C/A point (absolute) positioning
  - In case P on L1 available: P(Y) on L1 acquisition on all the in view satellites → pesudo ranges on P(Y) L1 → P(Y) on L1 point positioning
  - In case P on L2 available: P(Y) on L2 acquisition on all the in view satellites → pseudo ranges on P(Y) L1+L2 → Ionospheric free P1/P2 point positioning (no ionospheric disturbance)

- L1, L2 L5 → **Phase carriers observations**
  - The difference in **phase** btw the received carrier (L1 or L2) and a **sinusoid** with the same frequency internally generated by the receiver can be observed

argument of sin(φ(t)) in fractions of cycles

$$\phi_R^S(t) = \phi_R(t) - \phi^S(t-\tau_R^S) = [\phi_{0R} + f_0 t + f_0 dt_R(t)] - [\phi_0^S + f_0(t-\tau_R^S) + f_0 dt^S(t-\tau_R^S)]$$

reception time; phase of R at epoch t; travel time; phase of S at epoch $t-\tau_R^S$

$$= \phi_{0R} - \phi_0^S + f_0\tau_R^S + f_0 dt_R(t) - f_0 dt^S(t-\tau_R^S) \simeq \phi_{0R} - \phi_0^S + f_0\tau_R^S + f_0 dt_R(t) - f_0 dt^S(t) + N_R^S(t)$$

~66ms

fractions of cycles, constant in time

obs →

$$L_R^S(t) = \lambda\phi_R^S(t) = \overset{\simeq c}{\boxed{\lambda f_0}}[\tau_R^S + dt_R(t) - dt^S(t)] + \lambda[\phi_{0R} - \phi_0^S + N_R^S(t)]$$

$$= c[\tau_R^S + dt_R(t) - dt^S(t)] + \lambda[\phi_{0R} - \phi_0^S + N_R^S(t)]$$

$$= \boxed{c\tau_R^S} + c[dt_R(t) - dt^S(t)] + \lambda\eta_R^S(t)$$

Integer ambiguity: cannot be measured. It represents an unknown in the observation. **N(t)**: integer number of cycles passed from the satellite signal phase at the emission epoch and the satellite signal phase at the receiving epoch. It depends on time (distance btw S&R changes continuously, relative velocity of S wrt R is of about 1km/s)

Propagation in the atmosphere, initial ambiguities, cycle slips, multipath.

- In **vaccumn**, the signal speed c is constant ≈ 3e+08 m/s

$$c\tau_R^S = \sqrt{(x^S - x_R)^2 + (y^S - y_R)^2 + (z^S - z_R)^2} = \rho_R^S$$

distance btw S&R

- ***atmospheric disturbance***: In the atmosphere, the signal propagation velocity varies according to the physical state of the medium. It is the so called atmospheric effect/disturbance and ranges from 5m (very good conditions) to 200m (unfair conditions)

$$\tau_R^S = \int_{\rho_R^S} \frac{1}{\boxed{v(x)}} dx$$

path from S to R; varying velocity along the trajectory

→ $$c\tau_R^S = \int_{\rho_R^S} \boxed{\frac{c}{v(x)}} dx = \int_{\rho_R^S} n(x)dx = \int_{\rho_R^S}[n(x)+1-1]dx = \int_{\rho_R^S} dx + \int_{\rho_R^S}[n(x)-1]dx = \rho_R^S + \boxed{\Delta_R^S}$$

refraction index in x

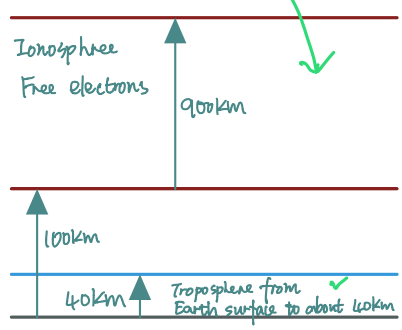
Atmospheric Affect in length units

- In a dispersive medium, group (codes) and phase (carriers) velocities are different → **Ionosphere**

$$n(x) = 1 \pm \frac{A\,\boxed{N_e(x)}}{f^2}$$

constant, positive; density of free electrons in x; frequency of the carrier

"1+" for codes → "delay"
"1−" for carries → "shortened"

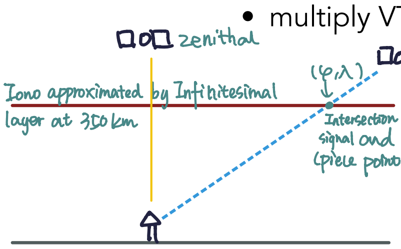


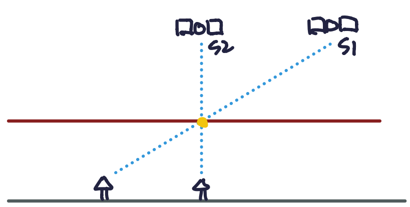
Ionospheric Delay

$$I_R^S(t) = \int_{IONO_R^S}[n(x)-1]dx = \pm\frac{A}{f^2}\int_{IONO_R^S} N_e(x)dx = \pm\frac{A}{f^2}TEC_R^S$$

Total Electron Content (TEC): superficial density of free electrons along the trajectory of the signal

- In GPS data processing, **Klobuchar ionospheric model** is adopted
  - satellite transmit the 8 coefficients through Navigation Data
  - compute the intersection point btw signal and ionosphere → φ, λ
  - compute VTEC in φ, λ (harmonics+8 coefficient) at epoch t
  - multiply VTEC with a function of elevation of satellite → delay





Intersection is the same, but S1 signal travelled more Iono. So VTEC must be multiplied to take into account satellite elevation

- **VETC**: TEC in (φ,λ) for a zenithal satellite. It is modeled as harmonic in φ,λ of pierce point
- Ionospheric delay on L1 and L2

$$\underbrace{I2_R^S(t)}_{\uparrow \text{ on L2}} = A\frac{TEC_R^S(t)}{f_2^2} = A\frac{f_1^2}{f_1^2}\frac{TEC_R^S(t)}{f_2^2} = A\frac{f_1^2}{f_2^2}\frac{TEC_R^S(t)}{f_1^2} = \frac{f_1^2}{f_2^2}\underbrace{I1_R^S(t)}_{\uparrow \text{ On L1}}$$

- In a not dispersive medium, group (codes) and phase (carriers) vevlocities are equal → **Troposphere**
  - disurbance is due to air and water vapor content of the atmosphere

$$n(x) = 1 + \underbrace{k_1}_{\text{constant}}\frac{\overset{\text{Air Pressure}}{P(x)}}{\underbrace{T(x)}_{\text{temperature}}} + \underbrace{k_2}_{\text{constant}}\frac{\overset{\text{Vapor Pressure}}{e(x)}}{T^2(x)}$$

Tropospheric Delay → $T_R^S(t) = \int_{Tropo_R^S}[n(x)-1]\,dx$

$$= \int_{Tropo_R^S}\left[k_1\frac{P(x)}{T(x)} + k_2\frac{e(x)}{T(x)^2}\right]dx = \underbrace{\boxed{\int_{Tropo_R^S}k_1\frac{P(x)}{T(x)}dx}}_{\text{Dry component, 90\%}} + \underbrace{\boxed{\int_{Tropo_R^S}k_2\frac{e(x)}{T(x)^2}dx}}_{\text{Wet Component : 10\%}}$$

  - To compute T by **Saastamoinen model**: approximate **h** of receiver is needed

P_R, T_R, e_R, are modeled by propagation in height (Berg formulas)

$\boxed{P_R} = P_0(1-0.0000226h_R)^{5.225}$, $P_0 = 1013.25\ mBar$

$\boxed{T_R} = T_0 - 0.0065h_R$, $T_0 = 291.15°K$

$\boxed{e_R} = H_R e^{c_1+c_2T+c_3T^2}$

$H_R = H_0 e^{-0.0006396h_R}$, $H_0 = 50\%$

H_R: relative humidity in the receiver site,

c_1, c_2, c_3: constants.

Step 1

$$T_R^S = \frac{0.002277}{\sin\eta_R^S}\left[P_R + \left(\frac{1255}{T_R}+0.05\right)e_R - \tan^{-2}\eta_R^S\right]$$

1

η_R^S: satellite elevation

P_R, T_R, e_R: pressure, temperature, ~~humidity~~ water vapor pressure in the receiver site.

Zenith Tropospheric Delay (ZTD): tropospheric delay at the zenith of a receiver

$\eta_R^S = 90°$, $T_{90°}(P_R, e_R, T_R)$

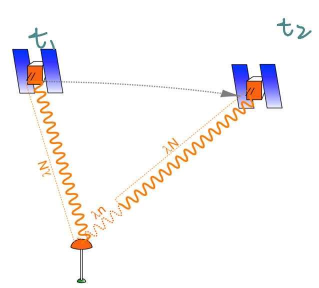
Step 2

  - note: Any modeling of T leaves a residual error, whose order of magnitude is about 5% of the total delay

- Sum

$$CT_R^S(t) = \underbrace{P_R^S(t)}_{\sqrt{(x^S-x_R)^2+(y^S-y_R)^2+(z^S-z_R)^2}} \pm I_R^S(t) + T_R^S(t)$$

- ***Initial ambiguities***



Initial ambiguities

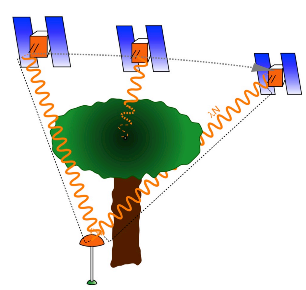
One R-S couple, two consecutive epoch t_1, t_2

$N_R^S(t_2) = N_R^S(t_1) + \underline{n_R^S(t_1,t_2)}$ → can be counted and recorded by receiver if no loss of clock

$$\eta_R^S(t_2) = N_R^S(t_2) + \phi_{0R} - \phi_{0S}$$

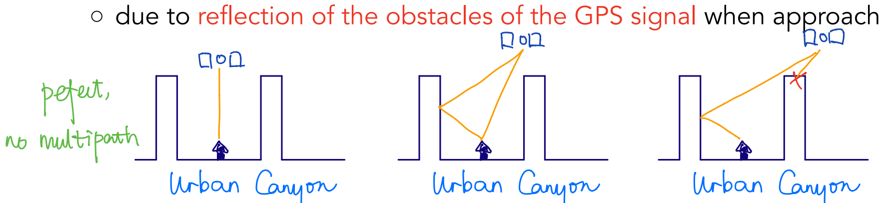
$$= N_R^S(t_1) + n_R^S(t_1,t_2) + \phi_{0R} - \phi_0^S = \underbrace{\boxed{\eta_R^S(t_1)}}_{\text{Initial Ambiguity} \downarrow \text{ unkown, but constant in time}} + n_R^S(t_1,t_2)$$

- ***Cycle slip (CS)***
  - usually, a cycle slip occurs when an obstacle (such as a building) blocks the signal going from S to R
  - during a CS, the counting is interrupted. When the signal is tracked again, a new unknown ambiguity has be introduced
  - *identification*: in the processing of phase time series, cycle slips need to be identified
  - repairing: a cycle slip is interger, once identified
    - a test to estimate the right integer
    - in positive case, the integer is subtracted to all the following observations



Cycle Slip

- ***Multipath effect***
  - due to reflection of the obstacles of the GPS signal when approaching the receiver antenna.



Final codes and phases observation equations. Ionospheric free combination.

- ***Final observation equations***

$$P_R^S(t) = c\tau_R^S + c[dt_R(t) - dt^S(t)]$$
$$= \rho_R^S + I_R^S(t) + T_R^S(t) + c[dt_R(t) - dt^S(t)]$$
$$= \sqrt{(X^S - X_R)^2 + (Y^S - Y_R)^2 + (Z^S - Z_R)^2} + I_R^S(t) + T_R^S(t) + c[dt_R(t) - dt^S(t)]$$

— unknown (4)

— known terms from models / ephemerides (7)

$$L_R^S(t) = c\tau_R^S + c[dt_R(t) - dt^S(t)] + \lambda[\phi_{0_R} - \phi_{0_S} + N_R^S(t)]$$
$$= \rho_R^S - I_R^S(t) + T_R^S(t) + c[dt_R(t) - dt^S(t)] + \lambda\eta_R^S(t)$$

satellite dependent

  - In single epoch with m satellites in view
    - by code observations → m observations in 4 unknowns → canbe solved if m ≥ 4
    - by phase observations → m observations in 4+m unknowns → the system cannot be solved
  - phase observations with a constant configuration of satellites, static receiver: m satellites in view, K epochs
    - mxK phase observations
    - unknown: 3 receiver coordinates + K clock offsets of the receiver
      - if the tracking is continuous, m ambiguities → solved if mxK > 3+K+m
- ***Combination of observations***: New observations can be obtained by linear combinations of code and phase observations $O_R^S(t) = \alpha_1 P1_R^S(t) + \alpha_2 P2_R^S(t) + \beta_1 L1_R^S(t) + \beta_2 L2_R^S(t)$
  - Ionospheric free combination

    \# No Ionospheric Effect !
    - Codes

$$P3_R^S(t) = \frac{f_1^2}{f_1^2 - f_2^2} P1_R^S(t) - \frac{f_2^2}{f_1^2 - f_2^2} P2_R^S(t)$$
$$= \rho_R^S(t) + c[dt_R(t) - dt^S(t)] + T_R^S(t)$$

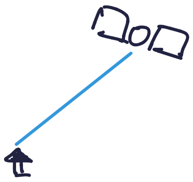
    - Phases

$$L3_R^S(t) = \frac{f_1^2}{f_1^2 - f_2^2} L1_R^S(t) - \frac{f_2^2}{f_1^2 - f_2^2} L2_R^S(t)$$
$$= \rho_R^S(t) + c[dt_R(t) - dt^S(t)] + T_R^S(t) + \frac{f_1^2 \lambda_1}{f_1^2 - f_2^2}\eta1_R^S(t) - \frac{f_2^2 \lambda_2}{f_1^2 - f_2^2}\eta2_R^S(t)$$

    - Ionospheric Free is used in
      - high accuracy point positioning by code observations
      - geodetic surveys by phase observations

Real time point positioning by Least Squares and final error budget.

- ***linearization of distance***



$$\underline{X}_R = \begin{bmatrix} x_R \\ y_R \\ z_R \end{bmatrix}, \quad \underline{\tilde{X}}_R = \begin{bmatrix} \tilde{x}_R \\ \tilde{y}_R \\ \tilde{z}_R \end{bmatrix}, \quad \underline{e}^S_R = \frac{1}{\rho^S_R}\begin{vmatrix} x_R - x^S \\ y_R - y^S \\ z_R - z^S \end{vmatrix}$$

← unitary vector btw S&R

$$\longrightarrow \quad \rho^S_R = \tilde{\rho}^S_R + \tilde{\underline{e}}^{S\,T}_R \cdot (\underline{X}_R - \tilde{\underline{X}}_R)$$

- ***Case: one epoch, one receiver***
  - all the in view satellites
    - single frequencey: P1
    - double frequency: P1& P2 → P3

$$P^S_R(t) = \underline{\tilde{\rho}^S_R} + \tilde{\underline{e}}^{S\,T}_R(\underline{X}_R - \tilde{\underline{X}}_R) + c\,dt_R(t) \underline{- c\,dt^S(t) + I^S_R(t) + T^S_R(t)}$$

$$= \tilde{\underline{e}}^{S\,T}_R(\underline{X}_R - \tilde{\underline{X}}_R) + c\,dt_R(t) + \underbrace{b^S_R(t)}_{\text{contains all the terms in O.E that can be modeled}} + \underbrace{\varepsilon^S_R(t)}_{\text{model error}}$$

Known term

$$\longrightarrow \quad \delta P^S_R(t) = P^S_R(t) - \boxed{b^S_R(t)} = \tilde{\underline{e}}^{S\,T}_R(\underline{X}_R - \tilde{\underline{X}}_R) + c\,dt_R(t) + \varepsilon^S_R(t)$$

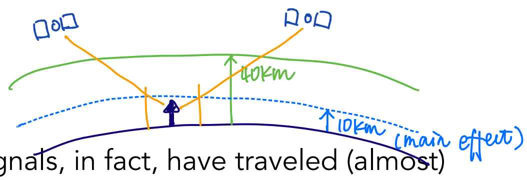
$$\longrightarrow \quad \begin{vmatrix} \delta P^1_R \\ \delta P^2_R \\ \cdots \\ \delta P^m_R \end{vmatrix} = \begin{bmatrix} \tilde{e}^1_{x_R} & \tilde{e}^1_{y_R} & \tilde{e}^1_{z_R} & c \\ \tilde{e}^2_{x_R} & \tilde{e}^2_{y_R} & \tilde{e}^2_{z_R} & c \\ & \cdots & & \\ \tilde{e}^m_{x_R} & \tilde{e}^m_{y_R} & \tilde{e}^m_{z_R} & c \end{bmatrix} \begin{bmatrix} x_R - \tilde{x}_R \\ y_R - \tilde{y}_R \\ z_R - \tilde{z}_R \\ dt_R \end{bmatrix} + \begin{vmatrix} \varepsilon^1 \\ \varepsilon^2 \\ \vdots \\ \varepsilon^m \end{vmatrix}, \quad \begin{cases} \text{Simplified: } C_{\delta P_0} = \sigma_0^2 I \\ \text{More Realistic: } Q^S_R \propto \dfrac{1}{\sin \eta^S_R} \end{cases}$$

$$\underline{\delta P} = A \qquad \underline{\xi} \qquad + \varepsilon$$

$$\text{if } m > 4 \longrightarrow \quad \hat{x}_R = \tilde{x}_R + \hat{\xi}_1, \quad \hat{y}_R = \tilde{y}_R + \hat{\xi}_2, \quad \hat{z}_R = \tilde{z}_R + \hat{\xi}_3$$

  - **model errors** (atmospheric and satellite related mis-modelling) effect

satellites
    - Ephemerides error: error in the S satellite position $\quad \rho^S_R = \tilde{\rho}^S_R + \tilde{e}^S_R \cdot \varepsilon(\underline{x}^S)$
      - typical error is about 1m
      - Slowly varying in time and uncorrelated between satellites
    - satellite clock offset error: error in the broadcast satellite clock offset $\quad c\,dt^S = c\,\tilde{dt}^S + c\,\varepsilon(dt^S)$
      - Typical error is about 1m
      - Slowly varying in time and uncorrelated between satellites
    - error in the atmospheric effect model
      - highly correlated in time
      - $\varepsilon(T^S_R)$ of the different in view satellites are similar: their signals, in fact, have traveled (almost) through the same troposphere
      - $\varepsilon(I^S_R)$ of the differenet in view satellites is similar only if their view directions are the same.
    - multipath (see above)



  - the observation geometry effect
    - the receiver positioning accuracy depends not only on the signal errors but also on the satellites geometry wrt the receiver.
  - final covariance matrix of the estimated coordinates and clock offset.

geometric part, Q_{xx}

$$C_{\xi\xi} = \sigma_0^2 (A^T Q^{-1} A)^{-1} \xrightarrow{Q = I} C_{\xi\xi} = \sigma_0^2 \boxed{(A^T A)^{-1}} \qquad Q_{\xi\xi} = (A^T A)^{-1} = \begin{bmatrix} q_{x_R} & & & \\ & q_{y_R} & & \\ & & q_{z_R} & \\ & & & q_{dt_R} \end{bmatrix}$$

  - final accuracy
    - with a good satellites goemetry: accuracy up to 5-10m, better accuracies in the horizontal than in vertical positions

PDOP and HDOP: definition and extraction.

- ***GDOP (Global Dilution of Precision)***

$$GDOP = \sqrt{q_{x_R} + q_{y_R} + q_{z_R} + q_{dt_R}}$$

- ***PDOP (Positional Dilution of Precision)***
  - suppose R is the rotation matrix to convert geocentric coordinates to local E,N,U wrt our point P

$$Q_{ll} = RQ_{xx}R^T = \begin{bmatrix} q_E & & \\ & q_N & \\ & & q_U \end{bmatrix} \rightarrow PDOP = \sqrt{q_E + q_N + q_U}$$

typical DOP value:

- 1-5: good
- 5-10: moderate
- 10-20: fair
- >20: poor

  - ***HDOP (Horizontal Dilution of Precision)*** $HDOP = \sqrt{q_E + q_N}$
  - ***VDOP (Vertical Dilution of Precision)*** $VDOP = \sqrt{q_U}$

Definition of integrity (accuracy+reliability → integrity)

- Integrity is a complex drived measure of the reliability of the information supplied by the system itself. It includes the ability of the system to provide timely warnings to users when the system could not be used for navigation
  - 例, for civil aviation(民用航空), differente integrity levels are sued, that range btw 99.999% to 99.9999999%
- ***Integrity V.S. Accuracy***: integrity requirements involve alarms being raised when system's performance is bad enough to become risky, while accuracy requirements do not.

Assisted GNSS and SBAS systems.

- ***Assisted GNSS***
  - A GNSS augmentation system that often significantly improves the startup performance of a GNSS.
  - In order to improve the accuracy, local organizations have been set up in different regions of the world to estimate in real time local atmosphereic delays and broadcast ephemerides errors. The 'corrections' are provided to GPS users via satellite, radio, GMS messages and can be used correct the observations.
- ***Satellite-based Augmentation System (SBAS)***: a system that supports wide-area (typically continental) augmentation of existing GNSS constellation(s).
  - *goal*
    - provide integrity assurance
    - increase the point positioning accuracy at 1meter level or better
    - provice pseudo range signal
  - based on
    - a ground infrastructure that monitors integrity
      - A network of GNSS permanent stations: acquire data from GNSS satellites
      - some processing facilities for computing (CC):
        - estimate the unknown parameters of the error models
      - on or more transmission facilities that upload to the SBAS satellites
    - a constellation of satellites that transmit integrity.
      - broadcast augmentation information and PR signals to the users
        - augmentation information is relevant to satellite ephemerides and healthy, satellites clock/time offsets and ionospheric distrubances
          - computed on a regional basis and should be more accurate than global estimates broadcast by GNSS constellations
        - PR signals: similar to GPS binary pseudorandom codes in structure and provide additional pseudo-range observations to increase the redundancy in point positioning.
  - when the users receive broadcast signals
    a. apply the SBAS corrections and integrity data to improve the GNSS pseudo-range observations
    b. point positioning is performed by corrected pseudo-range observations, both from GNSS constellations and SBAS satellites.

The EGNOS example.

- EGNOS: European Geostationary Navigation Overlay Service
- ***constellation***: 3 geostationary satellites
- ***ground segment***
  - 34 Ranging and Integrity Monitoring Stations (RIMS): acquire observations from the inview EGNOS and GPS satellites, then send the data to the Master Control Centers
  - 4 Master Control Centers: compute integrity and corrections to the observations, then send to the Up-link stations
    - at any time, only one Master is "the master"
  - 6 transmission (Up-link) stations: send the augmentation information to the 3 EGNOS satellites, which then transmit it to the GPS users with an EGNOS enabled receiver.
    - only 3 is needed, one for one EGNOS satellite

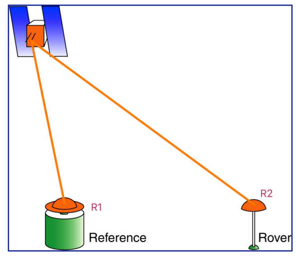
GNSS relative positioning

Baselines concept and estimation.

- ***relative positioning***: a rover is estimated wrt a reference whose coordinates are known and fixed in the processing.
- ***on codes (pseudorange observations)***: single epoch, real time positioning and navigation
- ***on carriers (phase observations)***
    - static processing
    - batch (more epochs needed) kinematic processing
        - post-processing (office) of the data
        - near real time processing of the data: real time data exchange from the reference to rover needed

Single differences and double differences: the geometric content for short baselines.

- ***Single differences (SD)***: differences between two simultaneous ([ˌsɪmlˈteɪniəs]) observations from two receivers R1 and R2 to the same satellite S.



SD

- codes

$$P^S_{R_1,R_2}(t) = P^S_{R_1}(t) - P^S_{R_2}(t) = \{\rho^S_{R_1}(t) + c[dt_{R_1}(t) - \cancel{dt^S(t)}] + I^S_{R_1}(t) + T^S_{R_1}(t)\} - \{\rho^S_{R_2}(t) + c[dt_{R_2}(t) - \cancel{dt^S(t)}] + I^S_{R_2} + T^S_{R_2}\}$$

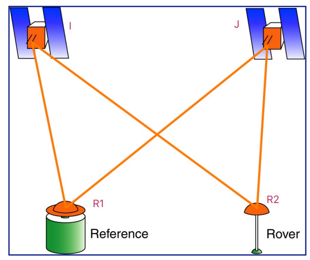
$$= \rho^S_{R_1} - \rho^S_{R_2} + c[dt_{R_1}(t) - dt_{R_2}(t)] + I^S_{R_1}(t) - I^S_{R_2}(t) + T^S_{R_1}(t) - T^S_{R_2}(t)$$

$$= \rho^S_{R_1R_2}(t) + c[dt_{R_1}(t) - dt_{R_2}(t)] + \underbrace{\boxed{I^S_{R_1R_2}(t) + T^S_{R_1R_2}(t)}}_{\text{diminish}}$$

- Phases

$$L^S_{R_1,R_2}(t) = L^S_{R_1}(t) - L^S_{R_2}(t) = \rho^S_{R_1R_2}(t) + c[dt_{R_1}(t) - dt_{R_2}(t)] \boxed{- I^S_{R_1R_2}(t) + T^S_{R_1R_2}(t)} + \lambda[N^S_{R_1R_2}(t) + \phi_{0R_1} - \phi_{0R_2}]$$

- ***Double differences (DD)***: differences of two simultaneous [single differences] to two satellites I and J.



DD

- codes

$$P^{S_1S_2}_{R_1R_2}(t) = P^{S_1}_{t_1t_2} - P^{S_2}_{t_1t_2} = \underbrace{\boxed{\rho^{S_1S_2}_{R_1R_2}(t)}}_{\text{geometric}} + \underbrace{\boxed{I^{S_1S_2}_{R_1R_2}(t) + T^{S_1S_2}_{R_1R_2}(t)}}_{\text{Reduced Atmospheric Effect}}$$

- phases

$$L^{S_1S_2}_{R_1R_2}(t) = L^{S_1}_{R_1R_2} - L^{S_2}_{R_1R_2} = \boxed{\rho^{S_1S_2}_{R_1R_2}(t)} \boxed{- I^{S_1S_2}_{R_1R_2}(t) + T^{S_1S_2}_{R_1R_2}(t)} + \underbrace{\boxed{\lambda N^{S_1S_2}_{R_1R_2}(t)}}_{\text{integer ambiguities parameters}}$$

- ***geometric content of SD/DD***
    - SD

Range from S_1, S_2 to R are very similar

$$P^S_{R_1R_2} = P^S_{R_1} - P^S_{R_2} = \hat{\rho}^S_{R_1} + \hat{e}^S_{R_1} \cdot \delta x_{R_1} - (\hat{\rho}^S_{R_2} + \hat{e}^S_{R_2} \cdot \delta x_{R_2}) = \hat{\rho}^S_{R_1R_2} - \hat{e}^S_{R_2}(\delta x_{R_2} - \delta x_{R_1}) + (\hat{e}^S_{R_1} - \hat{e}^S_{R_2}) \cdot \delta x_{R_1}$$

$$\hat{e}^S_{R_1} - \hat{e}^S_{R_2} \simeq \frac{\hat{x}_{R_1} - \hat{x}_{R_2}}{\bar{\rho}^S} = \frac{-\Delta \hat{x}_{R_1R_2}}{\underbrace{\boxed{\bar{\rho}^S}}_{\frac{1}{2}(\hat{\rho}^S_{R_1} + \hat{\rho}^S_{R_2})}}$$

$$\simeq \hat{\rho}^S_{R_1R_2} - \underbrace{\boxed{\hat{e}^S_{R_2}}}_{\simeq \frac{1}{\sqrt{3}} = 0.57} \cdot \delta \Delta x_{R_1R_2} \gg \underbrace{\boxed{\frac{\overbrace{\Delta x_{R_1R_2}}^{\text{baseline btw } R_1 \,\&\, R_2}}{\bar{\rho}^S}}}_{\simeq 2.5 \times 10^{-3}} \delta x_{R_1} \simeq \hat{\rho}^S_{R_1R_2} - \hat{e}^S_{R_2} \cdot \delta \Delta x_{R_1R_2}$$

- DD

$$P^{S_1S_2}_{R_1R_2} = P^{S_1}_{R_1R_2} - P^{S_2}_{R_1R_2} \simeq \hat{\rho}^{S_1}_{R_1R_2} - \hat{\rho}^{S_2}_{R_1R_2} - \hat{e}^{S_1}_{R_2} \cdot \delta\Delta x_{R_1R_2} + \hat{e}^{S_2}_{R_2} \cdot \delta\Delta x_{R_1R_2} = \hat{\rho}^{S_1S_2}_{R_1R_2} + (\hat{e}^{S_2}_{R_2} - \hat{e}^{S_1}_{R_2})\delta\Delta x_{R_1R_2}$$

- ***For short baselines***

"+" for code, "-" for phase

$$b^{S_1S_2}_{R_1R_2} = \hat{\rho}^{S_1S_2}_{R_1R_2} + I^{S_1S_2}_{R_1R_2} \boxed{\pm} T^{S_1S_2}_{R_1R_2}$$

contain unknow baseline component

$$\delta P^{S_1S_2}_{R_1R_2} = P^{S_1S_2}_{R_1R_2} - b^{S_1S_2}_{R_1R_2} = (\hat{e}^{S_2}_{R_2} - \hat{e}^{S_1}_{R_2}) \cdot \boxed{\delta\Delta x_{R_1R_2}} + \boxed{\varepsilon}$$

$$\delta L^{S_1S_2}_{R_1R_2} = L^{S_1S_2}_{R_1R_2} - b^{S_1S_2}_{R_1R_2} = (\hat{e}^{S_2}_{R_2} - \hat{e}^{S_1}_{R_2}) \cdot \boxed{\delta\Delta x_{R_1R_2}} + \boxed{\lambda N^{S_1S_2}_{R_1R_2}} + \boxed{\varepsilon}$$

model error

contains unknown integer ambiguties (check cycle slips)

Order of magnitudes of the error terms in DD in single epoch.

$$\underline{\varepsilon} = \underbrace{\varepsilon(X_{R_1}, X^{S_1}, X^{S_2}, T^{S_1S_2}_{R_1R_2}, I^{S_1S_2}_{R_1R_2})}_{\text{model error } \varepsilon^{S_1S_2}_{R_1R_2}} + \underbrace{\nu_{DD}}_{\text{electronic noise of DD}}$$

- ***model error*** $\varepsilon^{S_1S_2}_{R_1R_2} = f(\varepsilon(\hat{X}^{S_1}), \varepsilon(\hat{X}^{S_2}), \varepsilon(I), \varepsilon(T)) = \varepsilon^{S_1}_{R_1} - \varepsilon^{S_2}_{R_1} - (\varepsilon^{S_1}_{R_2} - \varepsilon^{S_2}_{R_2})$

  typically, the residual error is expressed in part per baseline

  $$\varepsilon = \underbrace{10^{-6}}_{1ppm} l \begin{cases} l = 1km \rightarrow \varepsilon = 1mm \\ l = 10km \rightarrow \varepsilon = 1cm \\ l = 100km \rightarrow \varepsilon = 10cm = 0.1m \end{cases}$$

  - satellite ephemerides error: propagation of the errors in coordinates of satellite/reference stations

    $$\varepsilon(X^s) = \varepsilon^s \cdot \frac{\Delta X_{R_1R_2}}{\rho^s}$$

  - Atmospheric effect error

    $$\varepsilon(I^s_R) = 0 \sim 8\times10^{-6} \quad , \quad \varepsilon(T^s_R) = 0 \sim 2\times10^{-6}$$

  - Multipath effect

In single epoch
- ***the phase DD error ranges btw 0.5cm+0.5ppm and 4cm+10ppm pf base line***
- ***for codes, the noise and multipath contributions increase up to meters***

DD error changes very slowly in time.

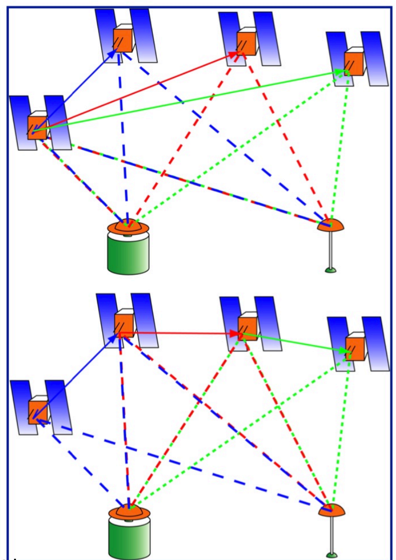
The single epoch estimation for code DD.

one epoch, 2 reciever, m satellites → m-1 indepent DD

$$\underline{\delta P_0}(t) = \underline{\delta P}(t) + \underline{\varepsilon}(t)$$

$$\underline{\delta P} = \begin{bmatrix} P^{12}_{12}(t) \\ P^{13}_{12}(t) \\ \cdots \\ P^{1m}_{12}(t) \end{bmatrix} - \begin{bmatrix} \hat{b}^{12}_{12}(t) \\ \hat{b}^{13}_{12}(t) \\ \cdots \\ \hat{b}^{1m}_{12}(t) \end{bmatrix} = \begin{bmatrix} \hat{e}^{21}_{2x}(t) & \hat{e}^{21}_{2y}(t) & \hat{e}^{21}_{2z}(t) \\ \hat{e}^{31}_{3x}(t) & \hat{e}^{31}_{3y}(t) & \hat{e}^{31}_{3z}(t) \\ \cdots & \cdots & \cdots \\ \hat{e}^{m1}_{mx}(t) & \hat{e}^{m1}_{my}(t) & \hat{e}^{m1}_{mz}(t) \end{bmatrix} \delta\Delta\underline{X}_{R_1R_2}$$

$$\underline{\delta P} = \underline{P}(t) - \underline{b}(t) = \underline{E}(t) \cdot \delta\Delta\underline{X}_{R_1R_2}$$


DD estimation

In principle, 3 unknowns (baseline) → 3 observations needed → 4 satellites at least

- ***error budget***
  - up to 10 ppm for single frequency: 10mm/km → 1m/100km
  - if P3 combination is used, maximum error reduces to 2ppm: 2mm/km → 0.2m/100km
  - + electronic noise and multipath: in typical condtions 0.2m ≤ v ≤ 10m

Real time navitgation for code DD

Differential GPS (DGPS): for each epoch
- a reference station transmit its coordinates and observations to the rover in real time
- the rover has a modem that acquires the data
- the rover builds and processes the code DD to estimate the baseline and its position

Single epoch for phase DD

a single epoch system of phse DD **can't be solved** with respect to the satellite because both baseline and ambiguities are unknown (always 3+(m-1) unknowns > m-1 observations)

The processing steps of phase DD (static/kinematic surveys)

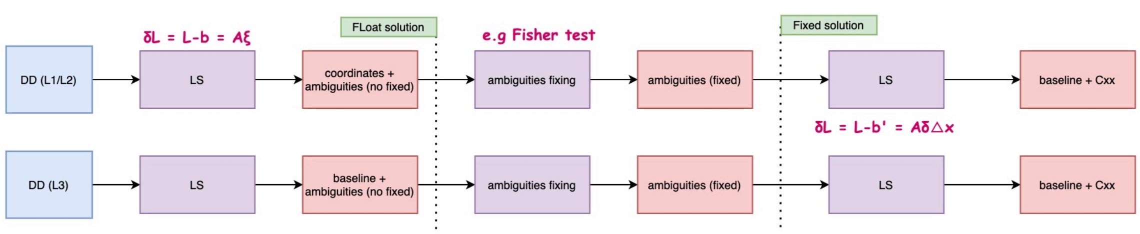
- ***two main cases***
  - static solution: both reference and rover receivers occupy the same points for the whole session → one baseline is estimated
  - kinematic solution: the reference is static, the rover is moving → one baseline per epoch (the rover trajectory) is estimated
- ***static processing of short baselines***
  - **input data**: double frequency observations of codes and phases
  - **known terms**
    - satellites coordinates (from ephemerides)
    - reference station coordinates (previsouly estimated)
    - tropospheric effect (model)
  - **steps**
    - ① preprocessing: data preparation and cleaning
      - DD code observables are adjusted by LS to obtain an approximate estimate of the baseline
      - the results are used to identify and repair possible cycle slips in phase observation.(fundamental)
        - ()Given a time series of DDs **L(t)**
        - 1 compute residual DDs: **δL(t) = L(t) - b(t) + ε**
        - 2 for each couple of consecutive residual DDS
          - compute their differences in time: **△δL(t1, t2)= δL(t2)- δL(t1)**
          - check |△**δL(t1, t2)**| against a threshhold
            - if **|△δL(t1, t2)| ≤ ε** → no cycle slip
            - if **|△δL(t1, t2)| > ε** → a cycle slip is identified, try to repair
              - **x = △δL(t1, t2) / λ** → **n** is the nearest integer to x
              - if **λ|n-x| > ε** → cycle slip identified, but no repairing is possible
              - if **λ|n-x| ≤ ε** → n is the true value of the cycle slip → all the data from t1 to T are correlated by λn:
    - ② processing: parameters estimate
      - prerequisite
        - short baseline (≤20km), short session (≤20min)
        - constant constellation of satellites and no cycle slips/all cycle slips repaired
      - 1 ***Float solution***: A LS system on all the epochs is built by phase DD and solved wrt static baseline trajectory and ambiguities
        - input: n epochs, m satellites, 2 frequencies → (m-1)*n*2 DD observations
        - unknown: 3 coordinates + (m-1)*2 ambiguities → solve
        - output: estimated baselines and ambiguities(no fixed)
      - 2 ***Ambiguities fixing***: real valued, estiamted ambiguities are fixed to integer values
        - fixing by Fisher test

$$\hat{N} = \begin{bmatrix} N_1 \\ N_2 \\ \cdots \\ N_m \\ N_1 \\ \cdots \\ N_m \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} N_1 \\ N_2 \\ \cdots \\ N_m \end{matrix}}\right\} L1 \\ \left.\vphantom{\begin{matrix} N_1 \\ \cdots \\ N_m \end{matrix}}\right\} L2 \end{matrix}, \; C_{\hat{N}\hat{N}} \longrightarrow (\underset{\uparrow \text{True}}{\underline{N}} - \hat{\underline{N}})^T C_{\hat{N}\hat{N}}^{-1} (\underline{N} - \hat{\underline{N}}) \sim F$$

$$\longrightarrow \text{Test } (\underline{N} - \hat{\underline{N}})^T C_{\hat{N}\hat{N}} (\underline{N} - \hat{\underline{N}}) < F_t$$

- In case in the confidence interval only one integer vector N′ exists: N is 'fixed' to N′.
- In case two integer vectors exist, a further choice criterion has to be applied
- In case no integer vector is in the α−confidence region, ambiguities cannot be fixed, or another choice criterion has to be applied

      - 3 ***Fixed solution***: A LS system is solved wrt static ba
delay for long sessions.



- ***kinematic case of short baselines***
    - input: n epochs and m satellites ➝ 2n(m-1) observations
        - if m>5, a good float solution usually is achieved
    - unknown: 3 coordinates + 2(m-1) ambiguities
    - steps
        - 1️⃣ ***Float solution***
            - if m > 5, a good float solution usually is achieved
        - 2️⃣ ***Ambiguities fixing***
        - 3️⃣ ***Fixed solution***
- ***real time processing of short baselines***
    - suppose the rover can communicate with the reference station and reference station transmit its observations and related metadata to rover
    - process
        a. the rover builds and processes the double frequency DD to estimate the float vector of ambiguities
        b. fix the float ambiguities
        c. for each epoch, the fixed solution is computed: baseline btw reference and rover ➝ rover position
    - *note*: LS allows you to compute the increase to the solution, so the solution could be updated.
- ***longer sessions, longer baselines***
    - long sessions ➝ tropospheric effect is estimated
    - longer baselines
        - local positioning services and SBAS networks ➝ estimate and distribute all the spatially correlated errors
        - regional permanent networks ➝ individual positions of the stations are estimated
        - global networks ➝ orbits of the satellites are estimated

Order of magnitudes of final errors in baseline estimates as function of baseline length and survey time.

- remarks
    - given a survey time, the accuracy decreases with the baseline length
    - given a baseline, the accuracy increases with the survey time

Typical final accuracies in baselines estimates

Time\ Baseline	1 km	10 km	50 km	1000 km
1 epoch	2.5 cm	4.0 cm	ne	ne
1-2 minutes	2.0 cm	2.5 cm	?	ne
10 minutes	1.5 cm	2.5 cm	5 cm	ne
1 hour	1.0 cm	1.5 cm	2 cm	ne
6 hours	0.5 cm	1.0 cm	1.5 cm	ne
24 hours	0.3 cm	0.5 cm	0.5 cm	1.0 cm (*)
1 week	0.1 cm	0.1 cm	0.3 cm	<1 cm (*)

Kinematic. Fast static. Static. Long static (permanent stations)

ne: not estimable; (*) multibase networks (not single baseline) processing

Indoor positioning

// No GNSS!!

Introduction to applications and user requirements.

- applications
  - **private homes**: ambient assistant living (AAL) systems (环境辅助生活系统)
  - **hospitals**
    - location of medical personnel or patients
    - navigation of unmanned vehicles (carrying medicines, ...)
    - robotic assistance during surgery
  - **security** (police or firefighters)
    - instantaneous detection of theft/burglary
    - location of stolen products
    - location of injured people
    - location of firmen
  - **industries**
    - robotic guidance
    - robot cooperation
    - tagged tools finding
  - **museum**: piece finding
  - **navigation to parking slots**
  - **surveying and monitoring of buildings**
- user requirements
  - **accuracy**: confidence region at 95%
    - from mm to decameter
  - **availability**: percentage of time in which the system is available
    - low < 95%; regular > 99%; high > 99.9%
  - **coverage area**: from mm level (robot arm), rom level (navigation in building) to global level (blind navigation)
    - **scalable system**: can increase the coverage area (in case by adding new components) without decreasing accuracy/availability
  - **output positions**
    - 3D: complete goereferencing
    - 2D: floor location
    - symbolic: proximity to an object (e.g. artwork in an museum)
  - **update rate**
    - on event: sensor passes a door
    - on request: where am I?
    - periodic: periodic monitoring
  - **latency**
    - real time: no perceivable delay
    - sooner or later: as soon as possible
    - **STB with an upper limit:** a threshold is set
    - proprecessing: no specific requirements
  - **privacy**
    - active techniques: user sends signals
    - passive techniques: user receives signals
    - **server side computation** (estimate of the positions are computed by a central server) or **user side computation** (each user autonomously estimates its position)
  - **number of users**
    - single user: for example, surveying
    - restricted number of users: active / passive mobile sensors
    - unlimited users: passive mobile sensors
  - **costs**: robotic industry can afford expensive systems, individual citizens do not
    - system set up, users devices, maintenance

Measuring principles (observables): RSSI, ToA, TDoA, RTT, AoA.

- **RSSI (Received Signal Strength Indicator)**: Received Signal Strength (RSS) averaged over a certain period. Usually speicified as power in Decibels (['desɪbelz]) (dB)
  - **RSS**: instantanous observed power in decibels
  - Physical attenuation general model

$$dB = 10\log_{10}\frac{P}{P_0}$$

P ← observed power
P_0 ← reference power

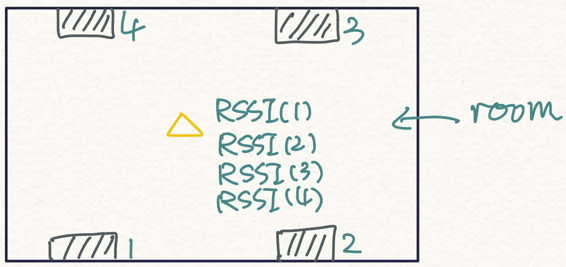
$$P_R = kP_T \frac{G_T G_R}{4\pi d_{TR}^p}$$

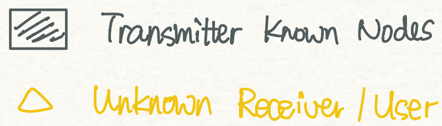
P_R, P_T : received and transmitted signal strengths
G_T, G_R : transmitter, received antenna gains
d_{TR} : distance between transmitter and receiver
p : path loss exponent

In open space, $p = 2$. In indoor, typically $4 \leq p \leq 6$ accordingly to the geometry, but in narrow corridors, $p < 2$

$$\frac{P_T}{P_R} = K d_{TR}^{p}$$

p ← depent on local conditions
K ↑ proportionality constant





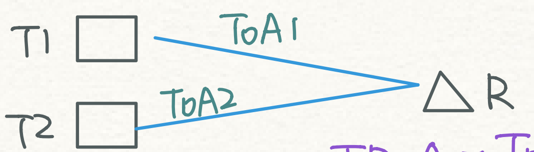
RSSI(4) > RSSI(3)
RSSI(4) >> RSSI(2)
RSSI(4) > RSSI(1)

- **ToA (Time of Arrival)**: travel time between transmitter and receiver
  - technical issue: synchronization between clocks
  - observation errors
    - multipath and not-LoS
    - slowering of signal passing through walls and obstacles

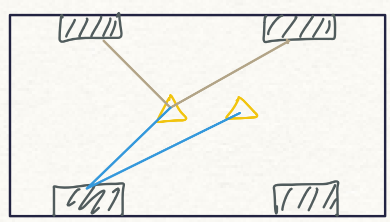
$$c\tau_{TR} \begin{cases} \text{in vaccum } d_{TR} \\ \text{in indoor} \simeq d_{TR} \begin{cases} \text{multipath} \\ \text{attenuation} \\ \text{(walls)} \end{cases} \end{cases}$$

Shared characteristics of RSSI, ToA, TDoA: **transmitter / receiver sensors** can be either on known nodes or unknown mobiles
- transmitter on known nodes, receiver on unknown mobiles: best suited for **distributed computation**
- transmitter on unknown mobiles, reciever on known nodes: best suired for **central computation**

- **TDoA (Time Difference of Arrival)**: difference in arrival time from transmitter to receiver
  - no problem of receiver clock, same error sources of ToA

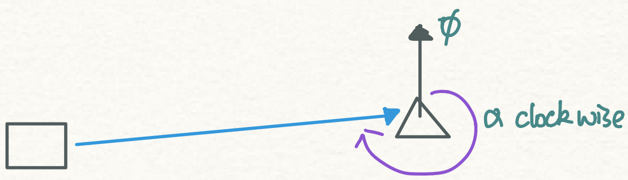


$$TDoA = ToA1 - ToA2$$



- **RTT (Round Trip Time)** or two ways ranging time: time needed by the signal to travel from transmitter to receiver and back

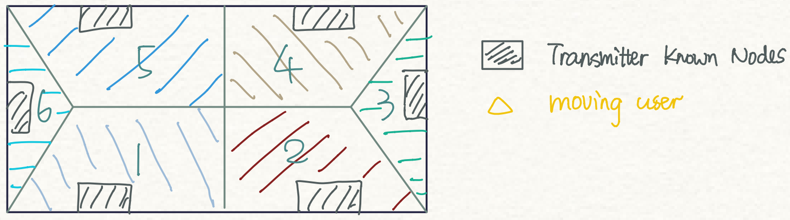
- **AoA (Angle of Arrival)**: the angular direction of incoming signals can be measured by a directional antenna in the receiver



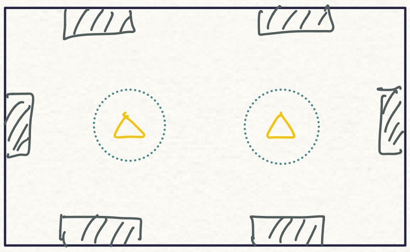
Processing algorithms: CoO, Centroid determination, Multilateration, Polar points, Fingerprinting.

None parametric approach

- **CoO (Cell of Origin)**: the user is located in the position of the **nearest** known node
    - nearest: highest RSSI, smallest ToA



- **Centroid determination**: the user is located in the **centroid (weight mean)** of the sensed known nodes.
    - weights: proportional to RSSI observations, inversely proportional to ToA observations



coordinates of known nodes

$$\underline{X}_u = \sum_{i=1}^{N} w_i X_i$$

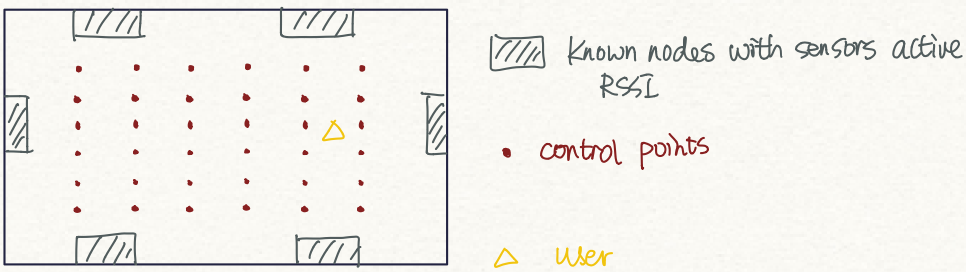
coordinates of user

weight of known nodes

$\sum_{i=1}^{N} w_i = 1$

Solution by Least Squares

- **Multilateration**: multiple distances derived from RSSI or ToA or DToA are adjusted by **Least Squares** to estimate positions / paths
- **Polar points**: Distances and angles are adjusted by **Least Squares** to estimate positions.
- **Fingerprinting**
    - 1 Calibration (suppose N known nodes serve the interest area with either transmitters or receivers)
        a. choose **M control points** in interest area and estimate their coordinates accurately
        b. for each control points, all the signals (RSSI, ToA, DToA) received by all the known nodes are registered



$$\underline{y}_{CP_i} = \begin{vmatrix} X_{CP_i} \\ Y_{CP_i} \end{vmatrix}, \quad \underline{r}_{CP_i} = \begin{vmatrix} \text{register from } 1 \\ 2 \\ \vdots \\ N \end{vmatrix}$$

    - 2 Usage: at one epoch in an unknown position, the mobile is registered and compared with all the contorl points in database. The coordinates of the mobile is the coordinates of the control point whose vector of observation is the "**nearest**" to the vector of observation of the mobile.

$$\underline{S}_u = \begin{vmatrix} S_{KN1} \\ S_{KN2} \\ \vdots \\ S_{KNN} \end{vmatrix} \rightarrow$$

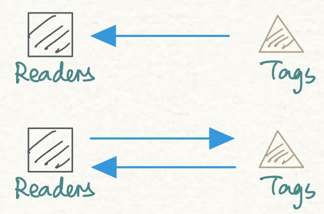
compared with all the r_{cpi}

$$\forall CP_i : \sqrt{(\underline{S}_u - r_{CP_i})^T (\underline{S}_u - r_{CP_i})}$$

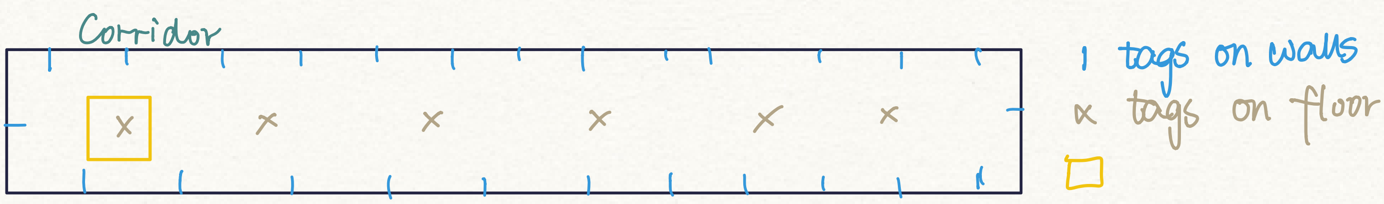
find the minimum

Technologies: WLAN / WI-FI, Radio Frequency Identification (RFID).

- **WLAN (Wireless Local Area Network) / WIFI**: active beacons that create network through wireless techniques, at frequencies of 2-5 GHz
  - main purpose: let users access to Internet connections and the access points are available in many indoor environments
  - range: up to 50-100m
  - most popular techniques: **RSSI**
    - the users need to known the positions of the beacons
  - processing strategy: **CoO**, fingerprinting (promising, more expensive)
- **Radio Frequency Identification (RFID, 射频识别)**: a cheap technique to identify objects carrying RFID sensors
  - accuracy: depend on the density of known nodes
  - measurement and processing: typically **RSSI** by **CoO**
  - two kinds of sensors
    - **readers**: read signal from tags (antenna / scanner)
    - **tags**: send their identification codes
      - active: **emit** a signal (identification code)
      - passive: **reflect** a signal (identification code)
        - signal emit by readers

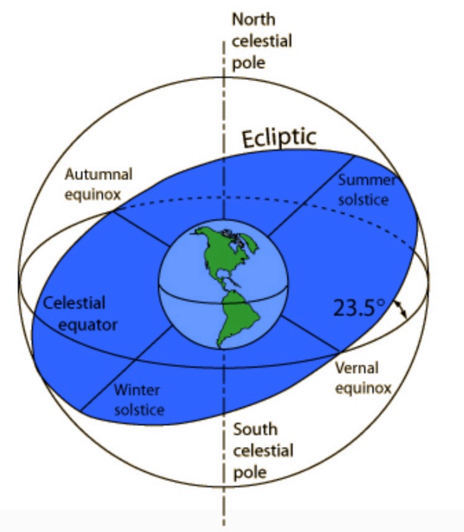


  - two possible configurations
    - readers on known nodes, tag on user ➡️ centralized process
    - reader on user, tags (usually passive) on known nodes ➡️ typically distributed processing



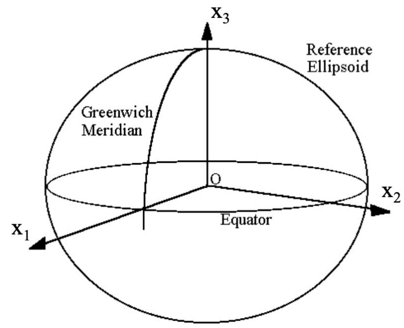
Inertial positioning

Connections to reference frames.



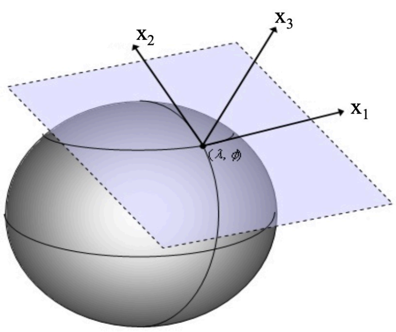
Pseudo inertial system

- **origin**: in the earth center
- **x3**: orientd to the North celestial pole
- **x1**: intersection between ecliptic and celestial equatorial plane
- **x2**: complete the right-handed model



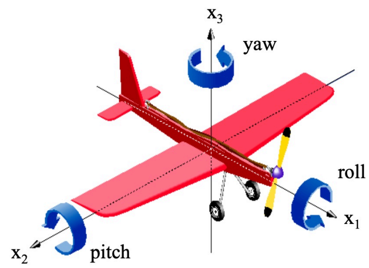
Earth-fixed system

- **origin**: in the earth center
- **x3**: orientd towards a conventional North pole
- **x1**: intersection between prime meridian and terrestrial equatorial plane
- **x2**: complete the right-handed model



Navigation System (Local Cartesian)

- **origin**: in the given point P
- **x3**: orientd to the normal to the ecllipsoid passing per P
- **x1**: oriented to the East direction
- **x2**: oriented to the North



Body System

- **origin**: in the a point P (typically inside the body)
- **x3**: orthogonal to vehicle plane and in UP direction
- **x1**: typical direction of motion
- **x2**: to complete the right-handed model

The Inertial Measurement Units and their observations: accelerometers and gyroscopes.

- Inertial Measurement Units (IMU) = 3 **accelerometers (a)** + 3 **gyroscopes (ω)**
- **accelerometer**: measure **acceleration (a)** in one specific direction
  - basic principle: measure the forces acting on a proof mass.
  - two types
    - open loop: measure the **displacement** of the proof mass resulting from the **external forces** acting on the sensor
      - example: spring based accelerometers
    - closed loop: keep the proof mass in a state of **equilibrium** by generating a force that is opposite to the applied force
      - example: pendulous or electrostatics (accuracy order: 10^-10 m/s2) accelerometers

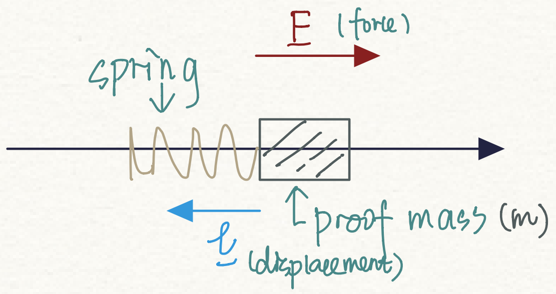
$$\underline{a} = \begin{vmatrix} a_1 \\ a_2 \\ a_3 \end{vmatrix}$$

- **gyroscope**: measure **angle velocity (ω)** along one axis
  - two types
    - mechanical
      - more expensive
    - optical
      - cheaper, miniaturizable
      - example: Fiber optical gyroscopes

$$\underline{\omega} = \begin{vmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{vmatrix}$$

The spring accelerometer and the optical gyroscope.

- spring accelerometer



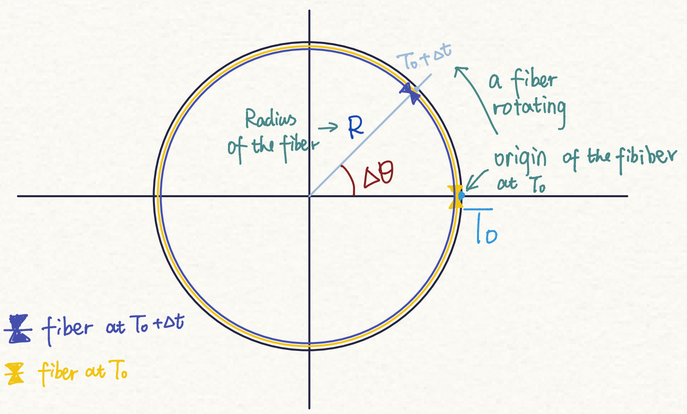
$$m\ddot{l} + k_v\dot{l} - k_e l = F$$

$F = \text{const}$ → $-k_e l = F$

$F = ma$ → $a = -\dfrac{k_e l}{m}$

↑ external acceleration along spring axis

- optical gyroscope: based on **"Sagnac" effect**
  - An observer moving with fiber sees the light to cover exactly one revolution: **2π**
  - An inertial observer sees the light to cover an angle: **2π + △θ**
    - For a light emitted on the opposite direction of the rotation: **2π - △θ**



$$\Delta\theta = \omega \cdot \frac{2\pi R}{c}$$

→ $\begin{cases} \Delta_+ L = \Delta\theta R = 2\pi R^2 \dfrac{\omega}{c} = 2\dfrac{\omega A}{c} \\ \Delta_- L = -2\dfrac{\omega A}{c} \end{cases}$

→ $\Delta = \Delta_+ L - \Delta_- L = 4\dfrac{\omega A}{c}$

→ $\omega = \dfrac{\Delta \times c}{4A}$

Typical error budget of IMU and Inertial Navigation Systems.

- accelerometer error

注. $1\ \text{Gal} = 1\ \text{cm/s}^2$, $1\text{mGal} = 10^{-3}\ \text{Gal}$

scale factor, $\simeq 5\times10^{-5}$; temperature

$$\delta a = b + \lambda a + C_T(T - T_0) + \sigma_v$$

bias, $\simeq 25\ \text{mGal}$; thermal constant, $\simeq 0.5\ \text{mGal}/°\text{C}$; measurement noise, $\simeq 40\ \text{mGal}/\sqrt{\text{Hz}}$

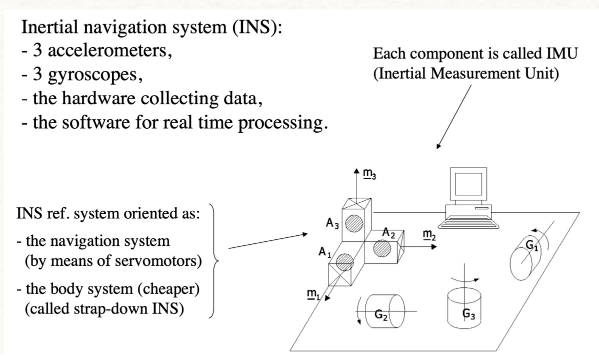
$\begin{cases} f = 1\text{Hz}, & \sigma_v = 40\text{mGal} \\ f = 10\text{Hz}, & \sigma_v \simeq 13\text{mGal} \\ f = 100\text{Hz}, & \sigma_v = 4\text{mGal} \end{cases}$

- gyroscope error

$$\delta\omega = b + \lambda\omega + C_T(T - T_0) + \sigma_\omega$$

bias, $\simeq 10^{-3}\ °/\text{h}$; λ: 2×10^{-6} ; C_T: $5\times10^{-5}\ °/\text{h}°\text{C}$; σ_ω: $6\times10^{-7}\ \text{rad}/\text{s}\sqrt{\text{Hz}}$

- **Inertial Navigation System (INS)**
  - A navigation device that uses a **computer**, **accelerometers** and **gyroscopes** to continuously calculate the **position**, the **orientation**, and the **velocity** (direction and speed of movement) of a moving object without the need for external references.
  - in our applications, INS measures in the body system of the vehicle



The basic rules of rotations.

R_A^B : rotation from R.S. A to R.S. B

2D: $\underline{x}' = R_A^B \underline{x}$, $R_A^B = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

3D: $\dot{R}_A^B(t_0) = -\Omega_A^B R_A^B(t_0)$, $\Omega_A^B = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$

Matrix of angular velocity

time derivative of Rotation Matrix

rotatation from A to B at epoch t_0 $(R_A^B R_B^A = I)$

$\longrightarrow \quad \dot{R}_B^A = R_B^A \Omega_A^B$

The navigation equations in IRF and ERF.

The link between the **unknowns** (namely position, velocity and attitude) and the **observations** of the accelerometers and gyroscopes.

Body frame ➡️ Inertial system ➡️ earth-fixed system

- ***navigation equations in an inertial reference system (IRF)***

Dynamics of IRS

acceleration sensed in IRS

$$\boxed{F^{i\,T}} = m\boxed{\ddot{x}^i} = F^i + m\boxed{g^i}$$ ← acceleration produced by local gravitational field.

$$\longrightarrow \boxed{\begin{cases} \dfrac{d\dot{x}^i}{dt} = \ddot{x}^i = \boxed{\dfrac{F^i}{m}} + g^i = \boxed{f^i} + g^i \\[2mm] \dfrac{dx^i}{dt} = \dot{x}^i \end{cases}}$$

\# i stays for inertial reference system

\# m is known

$\longrightarrow \quad f^i = R_b^i \boxed{a^b}$ ← acceleration in Body System, measured by accelerometers

$\dot{R}_b^i = -\boxed{\Omega_b^i} R_b^i$

observation of gyroscopes

$$\dot{R}_b^i, \Omega_b^i \longrightarrow R_b^i \xrightarrow{a^b} f^i \xrightarrow{g^i} \ddot{\underline{x}}^i \longrightarrow \dot{\underline{x}}^i \longrightarrow \underline{x}^i$$

- ***navigation equations in Earth-fixed reference system (ERF)***

$$\boxed{\begin{cases} \dfrac{d\underline{x}^e}{dt} = \dot{\underline{x}}^e \\[2mm] \dfrac{d\dot{\underline{x}}^e}{dt} = \underbrace{\boxed{R_i^e R_b^i f^b}}^{f^e} + g^e - \Omega_i^e \Omega_i^e \underline{x}^e - 2\Omega_i^e \dot{\underline{x}}^e \\[2mm] \dot{R}_b^i = R_b^i \Omega_i^b \end{cases}}$$

Location Based Services

The general problem and typical application.

- ***What is LBS***: fully location aware applications delivering geolocated to contextualized services through client-service or peer to peer architectures by wireless, cellular and satellite communication networks
  - **DEF 1**. geographically oriented data and information services to users across mobile telecommunication ntwork
  - **DEF 2.** any service that extends spatial information process (GIS capabilities) to end users via Internet and/or wireless networks
- ***general problem***
  - **user** (who is): central to LBS and must be designed on a user centerd view
    - user modeling
      - who is the users?
      - what they need?
      - when and where they need?
    - 例如： people who need rescye by location of emergence calls just need
      - rescuers known their position
      - to be rescued as soon as possible
  - **location** (what is it and 'how' 'user' use)
    - mathematically: 3D coordinates estimate
    - typically: (addresses+) directions
  - **data**: data modeling
    - geographic (referenced) model: locations and objects are points, lines, areas and volumes consistently goereferenced in a given reference system
    - symbolic (conceptual) model: locations are classes with a set of attributes.
- ***typical application***
  - location of emergencies calls in USA (911) or Europe (112) for rescue operations

The database of routes and connections: topological rules.

- ***nodes***: intersections between roads and terminal points of roads
- ***road***: a sequence of connections between couples of intersections
  - for each couple of connections: connecting node
  - for each turn: rights allowed yes/no, costs
- ***stored data***
  - general attributes: regulatory / physical / geometrical descriptive
    - name, kind, surface, slope, width, max speed ...
  - topological attributes: connections
    - for each way
      - rights: allowed yes/no
      - cost: at least 3 possible definition → time, fee, length
    - *note 1*: in the same (same name) way, two or more different geometrical/topological conditions exist: split the road into two or more connections
    - *note 2*: two roads geographically overlap but without any topological intersections (cross), i.e. bridges/ tunnels: no nodes between them.

ID	X	Y	N
1 univocal identifier	X coordinate	Y coordinate	Number of connected connections
2	…		

database of nodes

Nodes turn costs

ID	Name	Node_S	Node_E	S_E	E_S	C1_S_E	C1_T_F	C2_S_E
1 univocal identifier, nothing to do with official name!	Official name of the road, can be shared by several connections	Starting node of the road	Ending node of the road	Allowed way (Y/N)	Allowed way (Y/N)	Cost 1 (for example time) from S to E	Cost 1 from E to S	Cost 2…
2		…						

Database of connections

Node ID	Road_1	Road_2	1_2	2_1	C1_1_2	C1_2_1	C2_1_2
1 Node identifier	First connected road	Second connected road	Allowed turn (Y/N)	Allowed turn (Y/N)	Cost 1 (for example time) from 1 to 2	Cost 1 from 2 to 1	Cost 2…
1 for each node, many records as the couples of connections …	…						

Rods turn cost

Database of turns of nodes

Path optimization: the Dijkstra algorithm.

- zero iteration
  - current explored (CE) node: origin
  - from CE, all the costs of the connected nodes are estimated
- in each following iteration
  A. the CE node is **chosen** among the already estimated nodes, in particular, the nearest (less expensive node) to the origin and not yet already explored
     - the ID of all the already explored nodes is stored
  A. all the connected nodes to the CE node are **estimated**:
     a. for each one, the sum of the cost of CE and the cost from CE to it is computed.
     b. the updated cost of the node is the minimum between the old (if exists) and the new estimate.
        - note: the path of minimum cost to all the ready estiamted nodes is stored
- iterations stop when the destination has been reached, i.e destimation = CE !