Graphs, Digraphs and Association Schemes

Seminar at Kanazawa University

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Distance-Regular Graphs

Let $\Gamma = (X, \tilde{E})$ be a graph, where $\tilde{E}$ is a set of pairs of X. Suppose Γ is connected, and $\partial(x, y)$ denotes the path distance between x and y. Γ is said to be distance-regular, if for $x, y \in X$,

$$p_{i,j}^{k}(x, y) = |\{(z) \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

depends only on the distance $\partial(x, y) = k$ and does not depend on the choice of x, y.

Examples of Distance-Regular Graphs

1. The polygons (n-cycles: $\mathrm{Cay}(\mathbb{Z}_n, \{1\})$)
2. (The skeleton of) the Platonic solids: Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron.
3. Graphs related to classical geometries: the Johnson graphs, the folded Johnson graphs, the Odd graphs, the doubled Odd graphs, the Grassmann graphs, the doubled Grassmann graphs, the Hamming graphs, the halved hypercubes, the folded hypercubes, the halved folded hypercubes, the dual polar graphs, the half dual polar graphs of type D, the Ustimenko graphs, the Hemmeter graphs, the bilinear forms graphs, the alternating forms graphs, the quadratic forms graphs, and the Hermitian forms graphs.
4. Graphs with the same parameters as one of the above: The Doob graphs, the twisted Grassmann graphs.
5. Sporadic graphs with relatively small diameter.

Association Schemes

Definition 1

Let X be a nonempty finite set, and $X \times X = X_0 \cup X_1 \cup \cdots \cup X_d$ be a partition of $X \times X$. Let A_i be matrices rows and columns indexed by X such that $A_i[x, y] = 1$ if $(x, y) \in X_i$ and 0 otherwise. If $A_0, A_1, \ldots, A_d$ satisfy the following conditions, then X with its partition becomes an *association schemes*.

1. $A_0 = I$, the identity matrix.
2. $A_i^\top = A_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$.
3. For each $i, j \in \{0, 1, \ldots, d\}$, there exist constants $p^0_{i,j}, \ldots, p^d_{i,j}$ such that $A_i A_j = \sum_{k=0}^d p^k_{i,j} A_k$.

The algebra $\mathcal{B} = \langle A_0, A_1, \ldots, A_d \rangle \subseteq \mathrm{Mat}_X(\mathbb{C})$ is called the *Bose-Mesner algebra* of the association schme. If $\mathcal{B}$ is commutative, it is called a *commutative* association scheme. If $i = i'$ for every i, $\mathcal{B}$ consists of symmetric matrices, and it is commutative.

Examples of Association Schemes

$\mathcal{H}(H\backslash G/H)$

Let H be a subgroup of a finite group G. Let $X = G/H$ be the set of left cosets. The orbits of G on $X \times X$ define an association scheme with respect to the following action.
$g : X \times X \to X \times X : \ (g_1H, g_2H) \mapsto (gg_1H, gg_2H) \quad \text{for } g, g_1, g_2 \in G.$

$\mathcal{H}(G_x\backslash G/G_x)$

Let G be a transitive permutation group on a finite set X. Then, the orbits of G on $X \times X$ (called orbitals) define an association scheme.

Commutativity

The association scheme defined above is commutative if the permutation character of the permutation representation is multiplicity-free. [a]

[a] When G is a classical group and H a subgroup, (G, H) is called a Gelfand pair if $\mathcal{H}(H\backslash G/H)$ is commutative.

Graphs and Digraphs Defined by Association Schemes

Let $\mathcal{X} = (X, \{X_i \mid i \in \{0, 1, \ldots, d\})$ be an association scheme on a set X, $A_0, A_1, \ldots, A_d$ adjacency matrices, $S \subseteq \{1, \ldots, d\}$, and $E = \bigcup_{s \in S} X_s$.

1. If the relations in E are symmetric, $\Gamma_S = (X, E)$ defines a graph and $A = \sum_{s \in S} A_s$ is its adjacency matrix.
2. If a relation in E is not symmetric, $\Gamma_S = (X, E)$ defines a digraph (directed graph) and $A = \sum_{s \in S} A_s$ is its adjacency matrix.

Connectivity and Primitivity

1. The graph $\Gamma_S = (X, E)$ is connected if and only if $J \in \langle A \rangle$.
2. The digraph $\Gamma_S = (X, E)$ is strongly connected if and only if $J \in \langle A \rangle$.
3. If $\Gamma_S = (X, E)$ is strongly connected for every $\emptyset \neq S \subseteq \{1, \ldots, d\}$, then $\mathcal{X}$ is called primitive, otherwise imprimitive.[a]

[a] If $\mathcal{X}$ is imprimitive, there exist factor schemes and subschemes.

Definition 2 (The Second Definition of DRGs)

Let $\Gamma = (X, E)$, where E is a set of pairs of X. Γ is a *distance-regular graph* if Γ is connected of diamter d, and the following partition defines an association scheme. Here, $\partial(x, y)$ is the path distance.

$$X \times X = \bigcup_{i=0}^{d} \Delta_i,\ \Delta_i = \{(x, y) \mid \partial(x, y) = i\}.$$

Definition 3 (The Third Definition of DRGs)

$\mathcal{X} = (X, \{X_i \mid i \in \{0, 1, \ldots, d\})$ be an association scheme on a set X, $A_0, A_1, \ldots, A_d$ adjacency matrices, and $E = X_1$. Then, $\Gamma = (X, E)$, is a distance-regular graph (with respect to the ordering), if there are polynomials $p_0, p_1, \ldots, p_d \in \mathbb{C}[x]$ such that $\deg p_i = i$[a], satisfying

$$A_i = p_i(A_1),\ i \in \{0, 1, \ldots, d\}.$$

[a] Let $A = A_1$. Then $p_0(x) = 1$, $I = p_0(A)$, $p_1(x) = x$, $p_1(A) = A$, $\mathcal{B} = \langle A \rangle = \mathbb{C}[A]$. In this case, $\mathcal{X}$ is called a P-polynomial scheme. Since $J \in \mathcal{B}$, Γ is connected.

Definition 4 (The Fourth Definition of DRGs)

Let $\Gamma = (X, E)$, where E is a set of pairs of X, and A_i the i-th adjacency matrix, i.e., $(A_i)_{x,y} = 1$ if $\partial(x, y) = i$ and $(A_i)_{x,y} = 0$ otherwise. Γ is a *distance-regular graph* if Γ is connected of diamter d, and there exist constants b_i, a_i, c_i, $(i = 0, 1, \ldots, d)$ such that

$$A_i A = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}, \quad (i = 0, 1, \ldots, d-1)$$
$$A_d A = b_{d-1} A_{d-1} + a_d A_d. \text{ where } A_{-1} = O, A_0 = I, A = A_1, b_{-1} = 0.$$

Intersection array

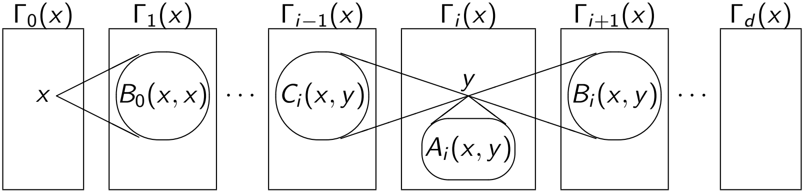
- $\iota(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called the intersection array.
- Let $\Gamma_i(x) = \{z \mid \partial(x, z) = i\}$, and $\Gamma(x) = \Gamma_1(x)$.

$$(A_i A)_{x,y} = |\Gamma_i(x) \cap \Gamma_1(y)| = \begin{cases} b_{i-1} = p^{i-1}_{i,1}, & \text{if } \partial(x, y) = i-1 \\ a_i = p^i_{i,1} & \text{if } \partial(x, y) = i \\ c_{i+1} = p^{i+1}_{i,1} & \text{if } \partial(x, y) = i+1 \end{cases}$$

Intersection array: $\iota(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$

$$(A_iA)_{x,y} = |\Gamma_i(x) \cap \Gamma_1(y)| = \begin{cases} b_{i-1} = p^{i-1}_{i,1}, & \text{if } \partial(x,y) = i-1 \\ a_i = p^i_{i,1} & \text{if } \partial(x,y) = i \\ c_{i+1} = p^{i+1}_{i,1} & \text{if } \partial(x,y) = i+1 \end{cases}$$

- $b_0 = |\Gamma_1(x)|$ and Γ is regular of valency $k = b_0$.
- $b_i + a_i + c_i = k$ and a_i is determined by the intersection array



$$B_i(u,v) = \Gamma_{i+1}(x) \cap \Gamma(y),\ A_i(u,v) = \Gamma_i(x) \cap \Gamma(y),$$
$$C_i(u,v) = \Gamma_{i-1}(x) \cap \Gamma(y).$$

Intersection array: $\iota(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$

$$\iota(\Gamma) = \left\{ \begin{array}{ccccccc} * & c_1 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_i & \cdots & b_{d-1} & * \end{array} \right\}.$$

The intersection array of the skeleton of the Platonic Solids

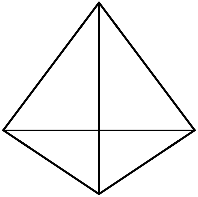
$$\text{Tetrahedron} : \left\{ \begin{array}{cc} * & 1 \\ 0 & 2 \\ 3 & * \end{array} \right\}. \quad \text{Cube} : \left\{ \begin{array}{cccc} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{array} \right\}.$$

$$\text{Octahedron} : \left\{ \begin{array}{ccc} * & 1 & 4 \\ 0 & 2 & 0 \\ 4 & 1 & * \end{array} \right\}. \quad \text{Icosahedron} : \left\{ \begin{array}{cccc} * & 1 & 2 & 5 \\ 0 & 2 & 2 & 0 \\ 5 & 2 & 1 & * \end{array} \right\}.$$

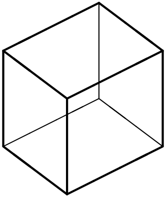
$$\text{Dodecahedron} : \left\{ \begin{array}{cccccc} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{array} \right\}.$$

Is it possible to determine imprimitivity by the intersection array? Yes.

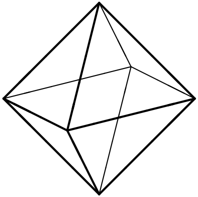
Platonic Solids



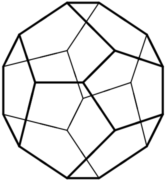
Tetrahedron



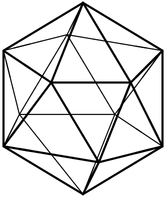
Cube



Octahedron



Dodecahedron



Icosahedron

Johnson graph $J(n, d)$:

Let V be a set of size n. Let

$$X = \{\alpha \subset V \mid |\alpha| = d\},\ E = \{(\alpha, \beta) \in X \times X \mid |\alpha \cap \beta| = d - 1\}.$$

Then the graph $\Gamma = (X, E)$ is a distance-regular graph and is called the *Johnson graph* $J(n, d)$[a]

[a] $b_i = (d - i)(n - d - i),\ c_i = i^2$. $J(4, 2)$ is the skeleton of the Octahedron.

Hamming graph $H(d, q)$:

Let Q be a set of size $q > 1$. Let

$$X = Q^d,\ E = \{(\alpha, \beta) \mid |\{i \mid \alpha_i \neq \beta_i,\ 1 \leq i \leq d\}| = 1\}.$$

Then the graph $\Gamma = (X, E)$ is a distance-regular graph and is called the *Hamming graph* $H(d, q)$. When $q = 2$ this graph is called a *hypercube*.[a]

[a] $b_i = (d - i)(q - 1),\ c_i = i$. $H(3, 2)$ is the skeleton of the Cube.

Primitive Idempotents

Let $\mathcal{B} = \langle A_0, A_1, \ldots, A_d \rangle \subseteq \mathrm{Mat}_X(\mathbb{C})$ be the Bose-Mesner algebra of a commutative association scheme[1] with $n = |X|$. Then there are primitive idempotents $E_0, E_1, \ldots, E_d$ satisfying the following.

(i) $E_0 = \frac{1}{n}J$, where J is the all one's matrix of size n, and $E_iE_j = \delta_{i,j}E_i$ for all $i, j \in \{0, 1, \ldots, d\}$.

(ii) $E_j^\top = E_{\hat{j}}$ for some $\hat{j} \in \{0, 1, \ldots, d\}$.

(iii) For $i, j \in \{0, 1, \ldots, d\}$, there exists $q_{i,j}^k \in \mathbb{R}$ such that $E_i \circ E_j = \frac{1}{n}\sum_{k=0}^n q_{i,j}^k E_k$, where $\circ$ is the entry-wise product, called the Hadamard product.

(iv) For $i, j \in \{0, 1, \ldots, d\}$, there exists $p_i(j), q_j(i) \in \mathbb{R}$ such that

$$A_i = \sum_{j=0}^{d} p_i(j)E_j, \quad E_j = \frac{1}{n}\sum_{i=0}^{d} q_j(i)A_i.$$

[1] The P-polynomial scheme associated with a distance-regular graph is symmetric, hence commutative.

Definition 5 (The Third Definition of DRGs - P-Polynomial Scheme)

$\mathcal{X} = (X, \{X_i \mid i \in \{0, 1, \ldots, d\})$ be an association scheme on a set X, $A_0, A_1, \ldots, A_d$ adjacency matrices, and $E = X_1$. Then, $\Gamma = (X, E)$, is a distance-regular graph (with respect to the ordering), if there are polynomials $p_0, p_1, \ldots, p_d \in \mathbb{C}[x]$ such that $\deg p_i = i$, satisfying

$$A_i = p_i(A_1),\ i \in \{0, 1, \ldots, d\}.$$

Definition 6 (Q-Polynomial Scheme)

$\mathcal{X} = (X, \{X_i \mid i \in \{0, 1, \ldots, d\})$ be an association scheme on a set X, and $E_0, E_1, \ldots, E_d$ primitive idempotents. Then, $\mathcal{X}$, is a Q-polynomial schme graph (with respect to the ordering), if there are polynomials $q_0, q_1, \ldots, q_d \in \mathbb{C}[x]$ such that $\deg q_i = i$ satisfying

$$E_i = q_i(E_1)/|X|,\ i \in \{0, 1, \ldots, d\}$$

with respect to the entry-wise multiplication $\circ$.

Definition 7 (Q-Polynomial Scheme)

$\mathcal{X} = (X, \{X_i \mid i \in \{0, 1, \ldots, d\})$ be an association scheme on a set X, and $E_0, E_1, \ldots, E_d$ primitive idempotents. Then, $\mathcal{X}$, is a Q-polynomial schme (with respect to the ordering), if there are polynomials $q_0, q_1, \ldots, q_d \in \mathbb{C}[x]$ such that $\deg q_i = i$ satisfying

$$E_i = q_i(E_1)/|X|, \; i \in \{0, 1, \ldots, d\}$$

with respect to the entry-wise multiplication $\circ$.

Definition 8 (The Second Definition of Q-Polynomial Scheme)

$\mathcal{X} = (X, \{X_i \mid i \in \{0, 1, \ldots, d\})$ be an association scheme on a set X, and $E_0, E_1, \ldots, E_d$ primitive idempotents. Fix $x \in X$ and let $A_i^* = \frac{1}{|X|}\mathrm{diag}((E_i)_{x,y} \mid y \in X)$. Then, $\mathcal{X}$, is a Q-polynomial schme (with respect to the ordering), if there are polynomials $q_0, q_1, \ldots, q_d \in \mathbb{C}[x]$ such that $\deg q_i = i$ satisfying

$$A_i^* = q_i(A_1^*), \; i \in \{0, 1, \ldots, d\}.$$

Terwilliger Algebra of a Q-polyomial DRG

Definition 9

Let $\Gamma = (X, E)$ be a distance-regular graph, and A its adjacecy matrix. Suppose its association scheme is Q-polynomial with respect to the ordering of its primitive idempotents $E_0, E_1, \ldots, E_d$. Fix $x \in X$ and let $A^* = \mathrm{diag}((E_1)_{x,y} \mid y \in X)$. Then, the Terwilliger algebra is defined by the following.

$$\mathcal{T} = \mathcal{T}(x) = \langle A, A^* \rangle \subseteq \mathrm{Mat}_X(\mathbb{C}).$$

Theorem 1 (Terwilliger)

Let $\Gamma = (X, E)$ be a Q-polynomial distance-regular graph. Let A and A^* be matrices defined above. Then every irreducible $\mathcal{T}$-module is a $\mathcal{U}$ module defined by the following.

$$\mathcal{U} = \langle A, A^* \mid [A, [A, [A, A^*]]] = b^2[A, A^*],\ [A^*, [A^*, [A^*, A]]] = b^{*2}[A^*, A] \rangle,$$

where $[M, N] = MN - NM$, and b and b^* are constants.

DSRGs and CRCGs

Definition 10 (Distance-Semiregular Graphs)

Let $\mathcal{P}$ and $\mathcal{L}$ be finite sets. Let $\tilde{\Gamma} = (\mathcal{P} \cup \mathcal{L}, \tilde{E})$ be a connected bipartite graph with parts $\mathcal{P}$ and $\mathcal{L}$. $\tilde{\Gamma}$ is called a distance-semiregular graph if for every vertex $x \in \mathcal{P}$, there are constants b_i's and c_i's satisfying

$$|\tilde{\Gamma}_i(x) \cap \tilde{\Gamma}(y)| = \begin{cases} \tilde{b}_{i-1}, & \text{if } \partial_{\tilde{\Gamma}}(x,y) = i-1, \\ \tilde{c}_{i+1}, & \text{if } \partial_{\tilde{\Gamma}}(x,y) = i+1. \end{cases}$$

Definition 11 (Completely Regular Clique Graphs)

Let $\tilde{\Gamma} = (\mathcal{P} \cup \mathcal{L}, \tilde{E})$ be a distance semiregular graph. Then the distance-2-graph $\Gamma = (\mathcal{P}, E)$ on $\mathcal{P}$, where $E = \{(u,v) \mid \partial_{\tilde{\Gamma}}(u,v) = 2\}$, is a distance-regular graph and it is called a completely regular clique graph.[a]

[a] For each $\ell \in \mathcal{L}$, $C_\ell = \tilde{\Gamma}(\ell) \subset \mathcal{P}$ is a clique and a complete regular code in Γ.

A Result on Completely Regular Clique Graphs by hs in 2018 [4]

Let Γ be one of the distance-regular graphs listed below:

> *The polygons, the Johnson graphs, the folded Johnson graphs, the Odd graphs, the doubled Odd graphs, the Grassmann graphs, the doubled Grassmann graphs, the Hamming graphs, the halved hypercubes, the folded hypercubes, the halved folded hypercubes, the dual polar graphs, the half dual polar graphs of type D, the Ustimenko graphs, the Hemmeter graphs, the bilinear forms graphs, the alternating forms graphs, and the quadratic forms graphs.*

Then Γ is a completely regular clique graph.

The Doob graphs, the twisted Grassmann graphs, and the Hermitian forms graphs of diameter of at least three are not completely regular clique graphs with respect to any collection of cliques.

[6] H. Suzuki, Distance-regular graphs of large diameter that are completely regular clique graphs, J. Algebr. Comb. 48 (2018), 369-404.

Summary of DRGs

- DRGs are graphs with high regularity.
- The distance partition defines a symmetric association scheme, and it is easy to use algebraic (eigenvalue) properties.
- Good examples related to classical geomety, and interesting sporadics.
- Most of the primitive DRGs with large diameter are Q-polynomial and have good parameter set.
- Most of the DRGs with large diameter are completely regular clique graphs and associated with a good geometry.
- *Classification is still very challenging even the class of prmitive DRGs of large diameter.*

Excellent Textbooks and Monographs

Bannai-Ito [1], Brouwer-Cohen-Neumair [2], van Dam-Koolen-Tanaka [3], Terwilliger [5], Lecturenote of Terwilliger edited by HS, etc. [6]

Definition 12 (Distance-Regular Digraphs (DRDGs))

Let $\Gamma = (X, E)$, where E is a set of ordered pairs of X. Γ is a *distance-regular digraph* if Γ is strongly connected of diameter d, and the following partition defines a nonsymmetric association scheme.

$$X \times X = \bigcup_{i=0}^{d} \Delta_i, \ \Delta_i = \{(x, y) \mid \partial(x, y) = i\}.$$

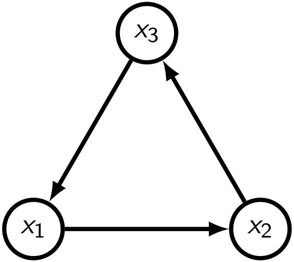
Definition 13 (Weakly Distance-Regular Digaphs (WDRDGs))

Let $\Gamma = (X, E)$, where E is a set of ordered pairs of X. Γ is a *weakly distance-regular digraph* if Γ is strongly connected, and the following partition defines a nonsymmetric association scheme.

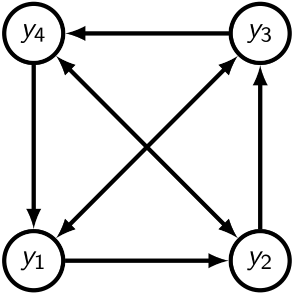
$$X \times X = \bigcup_{(i,j)\in\Delta} \Delta_{i,j}, \ \Delta_{i,j} = \{(x, y) \mid \widetilde{\partial}(x, y) = (i, j)\},$$

where $\widetilde{\partial}(x, y)$ is the two-way distance, and $\Delta(\Gamma) = \{\widetilde{\partial}(x, y) \mid x, y \in X\}$.

Digraphs over K_3 and K_4



$$\Delta = \{(0,0),(1,2),(2,1)\}$$



$$\Delta = \{(0,0),(1,1),(1,2),(2,1)\}$$

The digraph on the left is both distance-regular and weakly distance-regular. The digraph on the right is not distance-regular but is weakly distance-regular, and is imprimitive.

WDRDGs over a Clique (Semicomplete WDRDGs)

Proposition 2 (Y. Yang, Q. Zeng and K. Wang [3, Proposition 3.5])

Let $\Gamma = (X, E)$ be a weakly distance-regular digraph (wdrdg) of diameter d and girth g whose underlying graph is a complete graph. Then $d = 2$, $g \leq 3$ and

$$\Delta(\Gamma) = \{(0,0), (1,2), (2,1)\}, \text{ or } \Delta(\Gamma) = \{(0,0), (1,1), (1,2), (2,1)\}.$$

Proof.

Suppose $x, y \in X$ with $\tilde{\partial}(x, y) = (1, p)$ and $p \geq 3$. Note that $X = \{x\} \cup \Gamma_{1,*}(x) \cup \Gamma_{*,1}(x)$. If $z \in \Gamma_{1,*}(y) \setminus (\{x\} \cup \Gamma_{1,*}(x))$, then

$$3 \leq p = \partial(y, x) \leq \partial(y, z) + \partial(z, x) = 2,$$

which is absurd. Hence, $\{y\} \cup \Gamma_{1,*}(y) \subseteq \Gamma_{1,*}(x)$. A contradiction. $\square$

Two Types of Semicomplete WDRDGs

Type I: $d = 2$, $g = 3$, and $\Delta(\Gamma) = \{(0,0),(1,2),(2,1)\}$

- The existence is equivalent to that of a nonsymmetirc association scheme with two classes with $4m-1$ vertices.
- It exists if and only if a skew Hadamard matrix of size $4m$ exists.

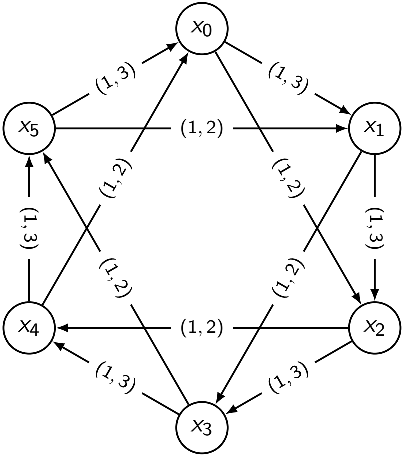
Type II: $d = 2$, $g = 2$, and $\Delta(\Gamma) = \{(0,0),(1,1),(1,2),(2,1)\}$

Let $\mathcal{B} = \langle I, A, A^\top, B = B^\top \rangle$ be the Bose-Mesner Algebra of a nonsymmetric association scheme of class three.

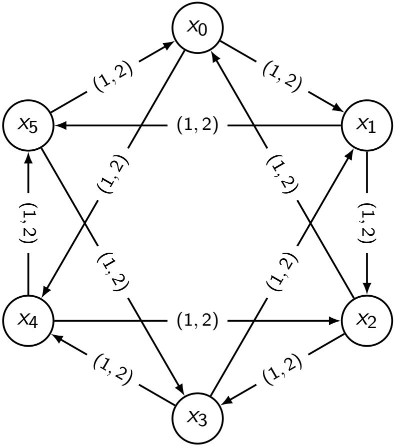
- If the digraph defined by A is connected, it is semicomplete of type II.
- If the digraph defined by A is not connected, each connected component is a semicomplete wdrdg of type I.

Every association scheme of class at most four is commutative.

$\mathrm{Cay}(\mathbb{Z}_6, \{1,2\})$



$\mathrm{Cay}(\mathbb{Z}_6, \{1,4\})$



The edge label shows the arc-type, i.e., the two-way distance of the nodes connected by the arc. The underlying graph is $J(4,2)$.

A Brief Summary of DRDGs and WDRDG

- Leonard and Nomura proved that except directed cycles, the diameter of distance-regular digraph is at most eight, and the girth is at most seven, [1].[a]
- WDRGs are digraphs with high regularity. The two-way distance partition defines an association scheme, but it is not known if it is commutative. No algebraic theory is developed yet.
- There are many examples of WDRDGs with unbounded diameter. What are the good class of WDRDGs? Classical type? Good class with a good algebraic condition?[b]
- Only papers. No good reference books.

[a] Infinitely many examples of girth 4 are known. There are studies of the cases with girth 5,6,7.

[b] Valency 2 and 3 and other special classes are classified. DRGs as underlying graphs case are now developed. The Terwilliger algebra of $H^*(d,3)$ is determined.

Problems 1: Association Schemes, P-and/or Q-polynomial Schemes

1. Determine the parameter set of distance-regular graphs. Five parameters to four parameters.
2. Characterize good class of distance-regular graphs.
3. Develop design and coding theory in each distance-regular graphs of classical type.
4. Develop the theory of Q-polynomial schemes without assuming the P-polynomial property.

Problems 2: Weakly Distance-Regular Digraphs

1. Develop the theory of weakly distance-regular digraphs.
2. Develop the algebraic theory of weakly distance-regular digraphs, and find a good class. Or, find a good class of commutative but not symmetric assocation scheme.
3. Are there noncommutative weakly distance-regular digraphs?

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- P.H. Zieschang, Theory of Association Schemes, Springer Monograph in Mathematics, Springer, Berlin, 2005.[3]

[2] See the refeerences in the paper on WDRDGs.

[3] A theory of noncommutative association schemes as a generalization of groups and buildings is developed.

THANK YOU!!

$[A,[A,[A,B]]]=B,\ [B,[B,[B,A]]]=A$

Recall $[M,N]=MN-NM$. Hence,

$$[A,[A,[A,B]]]=A^3B-3A^2BA+3ABA^2-BA^3.$$

Suppose A,B are diagonalizable linear transformations on a finite dimensional vector space V, and E_i (resp. F_j) are projections onto the eigenspaces written as a polynomial in A (resp. B). satisfying $AE_i=E_iA=\phi_iE_i$ and $BF_j=F_jB=\psi_jF_j$.
Then, $[A,[A,[A,B]]]=B$ implies

$$\begin{aligned} E_iBE_j &= E_i[A,[A,[A,B]]]E_j = (\phi_i^3-3\phi_i^2\phi_j+3\phi_i\phi_j^2-\phi_j^3)E_iA^*E_j \\ &= (\phi_i-\phi_j)^3E_iA^*E_j. \end{aligned}$$

Therefore, if $[A,[A,[A,B]]]=B$, and $[B,[B,[B,A]]]=A$,

$$E_iBE_j\neq O\Rightarrow \phi_i-\phi_j\in\{1,\omega,\omega^2\},\ \text{and}$$
$$F_iAF_j\neq O\Rightarrow \psi_i-\psi_j\in\{1,\omega,\omega^2\},$$

where $1+\omega+\omega^2=0$.

Hexagonal Pairs

Let A and B be linear transformations on a finite dimensional vector space $V \neq 0$ satisfying the following four conditions.

(i) A and B are diagonazable on V.

(ii) There is an indexing $V_{i,j}, (i,j) \in \Phi$ of the eigenspaces of A such that

$$BU_{i,j} \subseteq U_{i+1,j} + U_{i-1,j+1} + U_{i,j-1}, \quad (i,j) \in \Phi.$$

(iii) There is an indexing $V_{i,j}, (i,j) \in \Psi$ of the eigenspaces of A such that

$$BV_{i,j} \subseteq V_{i+1,j} + V_{i-1,j+1} + V_{i,j-1}, \quad (i,j) \in \Psi.$$

(iv) There is no subspace W of V such that $AW \subseteq W$, $BW \subseteq W$, other than $W = 0$ and $W = V$.