The Terwilliger Algebra of a Hamming Digraph

Seminar at Beijing Normal University

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March 20, 2024

Table of Contents

My Journey in Mathematics

- Undergraduate University Student
  - J. P. Serre, Representation of finite groups [R]
  - J. E. Humphreys, Introduction to Lie algebra and representation theory [S & R]
  - R. W. Carter, Simple groups of Lie type [S & R]
  - D. Gorenstein, Finte Groups [S]
- Master Course Student
  - R. Steinberg, Lecture Note on Chevalley Groups [S & R]
  - A. Gagen, Topics of Finite Groups [S]
  - M. Suzuki, Group Theory I, II. [S]
  - Papers of Goldschmidt, Bender, Aschbacher, etc. [S]
- Doctor Course Student
  - Groups with a Standard Component of $L_4(3)$ [S]
    - A part of the classification theorem of finite simple groups:
    - $\mathbb{Z}_p$, A_n, Lie type groups, 26 sporadic groups; $M_{11}, \ldots,$ Monster

[R]: algebraic methods, representations, [S]: structure theory

Algebraic Combinatorics: My Journey in Mathematics

- Post classification
  - Automorphism groups of multilinear maps - Norton algebra
  - Designs over $GF(q)$ - designs in the Grassmann scheme
- Group Theory without Groups
  - E. Bannai and T. Ito, Algebraic Combinatorics, I
  - Brouwer, Cohen, Neumaier, Distance-Regular Graphs

Distance-Regular Graphs

Let $\Gamma = (X, \tilde{E})$ be a graph, where $\tilde{E}$ is a set of pairs of X. Suppose Γ is connected, and $\partial(x, y)$ denotes the path distance between x and y. Γ is said to be distance-regular, if for $x, y \in X$,

$$p^k_{i,j}(x, y) = |\{(z) \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

depends only on the distance $\partial(x, y) = k$ and does not depend on the choice of x, y.

Association Schemes

Definition 1

Let X be a nonempty finite set, and $X \times X = X_0 \cup X_1 \cup \cdots \cup X_d$ be a partition of $X \times X$. Let A_i be matrices rows and columns indexed by X such that $A_i[x, y] = 1$ if $(x, y) \in X_i$ and 0 otherwise. If $A_0, A_1, \ldots, A_d$ satisfy the following conditions, then X with its partition becomes an *association schemes*.

1. $A_0 = I$, the identity matrix.
2. $A_i^\top = A_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$.
3. For each $i, j \in \{0, 1, \ldots, d\}$, there exist constants $p^0_{i,j}, \ldots, p^d_{i,j}$ such that $A_i A_j = \sum_{k=0}^d p^k_{i,j} A_k$.

The algebra $\mathcal{B} = \langle A_0, A_1, \ldots, A_d\rangle \subseteq \mathrm{Mat}_X(\mathbb{C})$ is called the *Bose-Mesner algebra* of the association schme. If $\mathcal{B}$ is commutative, it is called a *commutative* association scheme.

Primitive Idempotents

Let $\mathcal{B} = \langle A_0, A_1, \ldots, A_d \rangle \subseteq \mathrm{Mat}_X(\mathbb{C})$ be the Bose-Mesner algebra of a commutative association scheme with $n = |X|$. Then there are primitive idempotents $E_0, E_1, \ldots, E_d$ satisfying the following.

- (i) $E_0 = \frac{1}{n}J$, where J is the all one's matrix of size n, and $E_iE_j = \delta_{i,j}E_i$ for all $i, j \in \{0, 1, \ldots, d\}$.
- (ii) $E_j^{\top} = E_{\hat{j}}$ for some $\hat{j} \in \{0, 1, \ldots, d\}$.
- (iii) For $i, j \in \{0, 1, \ldots, d\}$, there exists $q_{i,j}^k \in \mathbb{R}$ such that $E_i \circ E_j = \frac{1}{n}\sum_{k=0}^{n} q_{i,j}^k E_k$, where $\circ$ is the entry-wise product, called the Hadamard product.
- (iv) For $i, j \in \{0, 1, \ldots, d\}$, there exists $p_i(j), q_j(i) \in \mathbb{R}$ such that

$$A_i = \sum_{j=0}^{d} p_i(j)E_j, \quad E_j = \frac{1}{n}\sum_{i=0}^{d} q_j(i)A_i.$$

Basic Examples of Association Schemes

$\mathcal{H}(H\backslash G/H)$

Let H be a subgroup of a finite group G. Let $X = G/H$ be the set of left cosets. The orbits of G on $X \times X$ define an association scheme with respect to the following action.

$$g : X \times X \to X \times X : \ (g_1 H, g_2 H) \mapsto (gg_1 H, gg_2 H) \quad \text{for } g, g_1, g_2 \in G.$$

$\mathcal{H}(G_x\backslash G/G_x)$

Let G be a permutation group on a finite set X. Then, the orbits of G on $X \times X$ (called orbitals) define an association scheme with respect to the following action.

Commutativity

The association scheme defined above is commutative if the permutation character of the permutation representation is multiplicity-free.

Definition 2

- Let $\Gamma = (X, \tilde{E})$, where $\tilde{E}$ is a set of pairs of X. Γ is a *distance-regular graph* if Γ is connected of diamter d, and the following partition defines an association scheme. Here, $\partial(x, y)$ is the path distance.

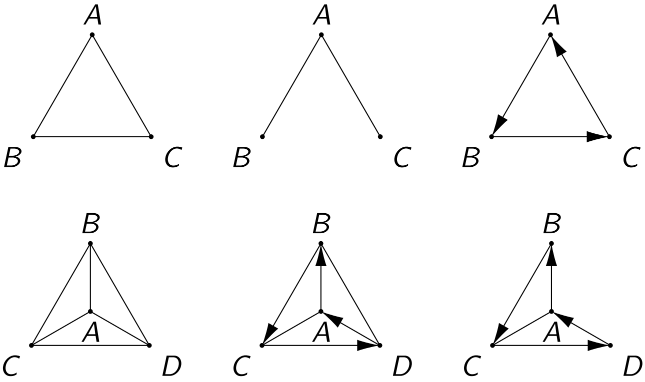
$$X \times X = \bigcup_{i=0}^{d} \Delta_i,\ \Delta_i = \{(x, y) \mid \partial(x, y) = i\}.$$

- Let $\Gamma = (X, E)$, where E is a set of ordered pairs of X. Γ is a *weakly distance-regular digraph* if Γ is strongly connected, and the following partition defines an association scheme.

$$X \times X = \bigcup_{(i,j)\in\Delta} \Delta_{i,j},\ \Delta_{i,j} = \{(x, y) \mid \widetilde{\partial}(x, y) = (i, j)\},$$

where $\widetilde{\partial}(x, y)$ denotes the two-way distance of x and y, and $\Delta = \{\widetilde{\partial}(x, y) \mid x, y \in X\}$.

Examples and non-examples with three and four vertices



drg, drg wrt each vertex, wdrdg

drg, wdrdg, wdrdg wrt each vertex

Examples with small number of vertices

Hanaki-Miyamoto, Classification of association schemes with a small number of vertices at `http://math.shinshu-u.ac.jp/~hanaki/as/`.

`http://math.shinshu-u.ac.jp/~hanaki/as/data/as07` No. 2

An example of a wdrdg with seven vertices:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 0 & 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 0 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 & 0 \end{bmatrix}$$

Examples of Distance-Regular Graphs

A Result on Completely Regular Clique Graphs by hs in 2018 [5]

Let Γ be one of the distance-regular graphs listed below:

> *The polygons, the Johnson graphs, the folded Johnson graphs, the Odd graphs, the doubled Odd graphs, the Grassmann graphs, the doubled Grassmann graphs, the Hamming graphs, the halved hypercubes, the folded hypercubes, the halved folded hypercubes, the dual polar graphs, the half dual polar graphs of type D, the Ustimenko graphs, the Hemmeter graphs, the bilinear forms graphs, the alternating forms graphs, and the quadratic forms graphs.*

Then Γ is a completely regular clique graph.

The Doob graphs, the twisted Grassmann graphs, and the Hermitian forms graphs of diameter of at least three are not completely regular clique graphs with respect to any collection of cliques.

A Brief Summary of the Study of Algebraic Combinatorics

Algebraic Combinatorics is a study of combinatorial objects with algebra.

Methods

- Combinatorial analysis of structures
- Algebraic study of the parameters
- Computer search

Study everything by yourself, or study with collaboration.

Weakly Distance-Regular Digraphs of Hamming Type

Theorem 1 (Y. Yang, Q. Zeng and K. Wang [4])

Let Γ be a commutative weakly distance-regular digraph. Then Γ has a Hamming graph as its underlying graph if and only if Γ is isomorphic to one of the following diagrams:

(i) $\mathrm{Cayley}(\mathbb{Z}_4, \{1\})$ — $[H(2,2)]$;

(ii) $\mathrm{Cayley}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1,0),(0,1)\})$ — $[H(3,2)]$;

(iii) Δ^1 *or* $\Delta^1 \square \Delta^2$ — $[H(1,q)$ *or* $H(2,q)]$;

(iv) $\Gamma^1 \square \Gamma^2 \square \cdots \square \Gamma^d$ — $[H(d,q)]$.

Here, Δ^i (resp Γ^i) is a semicomplete weakly distance-regular digraph of girth 2 (resp 3) with the same intersection numbers for each i.

The cases when the underlying graph is a folded cube and a Doob graph are determined.

Hamming Graphs and Digraphs

There are few studies of algebraic properties of association schemes associated with weakly distance-regular digraphs. We will study the Terwilliger algebra of $H^*(d,3)$, the smallest case with unbounded diameter in connection with $H(d,2)$ and $H(d,3)$, hoping to find the good class of weakly distance-regular digraphs or related algebraic structures.

T-Algebras Associated with the Hamming Graphs and Digraphs

1. $H(d,q)$.
2. $H^*(d,q)$ with $q \equiv 3 \pmod 4$.
3. $H(d,2)$, $H(d,3)$.
4. $H^*(d,3)$.

$H(d, q)$

Definition 3

Let $N = \{a_1, a_2, \ldots, a_n\}$. The Hamming graph $H(d, n)$ or more specifically, the Hamming graph of diameter d on the set N, $H(d, N)$ is defined by X, the set of vertices, and $\tilde{E}$, the set of edges.

$$\begin{aligned} X &= \{(x_1, \ldots, x_d) \mid \text{ for all } i,\ x_i \in N\}, \\ \tilde{E} &= \{xy \mid \text{exactly 1 coordinate } i,\ x_i \neq y_i\} \subseteq X \times X. \end{aligned}$$

$H^*(d,3)$

We use a trivial direction on each coordinate, $0 \to 1 \to 2 \to 0$.

Definition 4

The directed graph $H^*(d,3)$ is defined by X, the set of vertices, and E, the set of arcs.

$$X = \{(x_1, \ldots, x_d) \mid \text{ for all } i,\ x_i \in \mathbb{F}_3\},$$
$$E = \{xy \mid \text{exactly 1 coordinate } i,\ x_i + 1 = y_i\} \subseteq X \times X.$$

Let $\Gamma = (X, E)$ be $H^*(d,3)$.

Adjacecy Matrix

Definition 5

The adjacency matrix of $H^*(d,3)$, $A = A^{(d)}$ is defined by the following.

$$(A)_{ij} = \begin{cases} 1 & \text{if there exits an arc from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

For x, y in X, $\partial(x,y)$ denotes the distance between x and y, i.e., the smallest number of arcs connecting from x to y. We also define the two-way distance $\tilde{\partial}(x,y) = (\partial(x,y), \partial(y,x))$, and Δ, the set of all two-way distances;

$$\Delta = \{\tilde{\partial}(x,y) \mid x, y \in X\}.$$

Vector Space V

Fix a base vertex $x = (0, 0, \ldots, 0) \in X$.
$V = \mathbb{C}^{|X|}$: the vector space over the complex number field $\mathbb{C}$ whose coordinates are indexed by the elements of X.
$X_{i,j} = \{y \mid \partial(x, y) = i, \partial(y, x) = j\}$.
If $y \in X$ has s ones, t twos, and $r = d - s - t$ zeros, then $\partial(x, y) = s + 2t, \partial(y, x) = 2s + t$. If $\tilde{\partial}(x, y) = (i, j)$, then $s = (2j - i)/3$, and $t = (2i - j)/3$.
We also write

$$X_{[s,t]} = \{y \mid \text{there are } s \text{ ones and } t \text{ twos}\} = X_{s+2t,2s+t}.$$

$E^*_{i,j}$

For $(i,j) \in \Delta$, $E^*_{i,j} = E^{(d)*}_{i,j}$ denotes a diagonal matrix such that

$$E^*_{i,j}(z,z) = \begin{cases} 1, & \text{if } \partial(x,z) = (i,j), \\ 0 & \text{othersise}, \end{cases}$$

and the zero matrix of the same size if $(i,j) \not\in \Delta$. Then,

$$E^*_{i,j}\mathbf{1} = \{\sum \hat{y} \mid y \in X, \partial(x,y) = i, \partial(y,x) = j\}.$$

We set

$$E^*_{i,j} = E^*_{[(2j-i)/3,(2i-j)/3]}, \text{ and } E^*_{[s,t]} = E^*_{s+2j,2s+j}.$$

If a vector $\boldsymbol{v} \in V$ satisfies $E^*_{[s,t]}\boldsymbol{v} = \boldsymbol{v}$, then we write

$$\mathrm{type}(\boldsymbol{v}) = (r(\boldsymbol{v}), s(\boldsymbol{v}), t(\boldsymbol{v})) \text{: the type of } \boldsymbol{v}.$$

Note that $\boldsymbol{v}$ can be written as a linear combination of $\hat{y}$ with $y \in X_{[s,t]}$.

The Terwilliger Algebra $\mathcal{T}(x)$

Definition 6

The Terwilliger algebra $\mathcal{T}(x)$ of $H^*(d,3)$ with respect to a base vertex x is a $\mathbb{C}$-algebra generated by A, $A^\top$ and $E^*_{i,j}$ for $(i,j) \in \Delta$. We also write $\mathcal{T}(H^*(d,3))$ to specify the digraph.

The Algebraic Structure of $\mathcal{T}(x)$

Theorem 2

Let $\mathcal{T}(x)$ be the Terwilliger algebra of $H^*(d,3)$ with respect to a base vertex x. Then,

$$\mathcal{T}(x) \simeq \mathrm{Sym}^d(\mathrm{Mat}_3(\mathbb{C})).$$

Moreover,

$$\mathcal{T}(x) \simeq \mathrm{Sym}^d(\mathrm{Mat}_3(\mathbb{C})) \simeq \bigoplus_{n\in\Lambda} \mathrm{Mat}_n(\mathbb{C}), \text{ where}$$

$$\Lambda = \left\{ \frac{1}{2}(d-3\ell-2m+1)(m+1)(d-3\ell-m+2) \mid 0 \le \ell \le \left[\frac{d}{3}\right], 0 \le m \le \left[\frac{d-3\ell}{2}\right] \right\}.$$

$\mathcal{T}(H^*(1,d))$

Lemma 7

Let $\mathcal{T}_1$ be the Terwilliger algebra of $H^*(1,3)$. Then

$$\mathcal{T}_1 = \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle \simeq \mathrm{Mat}_3(\mathbb{C}).$$

Proof.

This is trivial. If $A^{(1)}$ is the adjacenct matrix of $H^*(1,3)$, then ${A^{(1)}}^2 = {A^{(1)}}^\top$, and ${E^{(1)}}^*_{0,0}$, ${E^{(1)}}^*_{1,2}$ and ${E^{(1)}}^*_{2,1}$ are diagonal matrix units $e_{1,1}$, $e_{2,2}$ and $e_{3,3}$. Hence, every matrix unit is in $\mathcal{T}_1$. $\square$

Note that if the principal module of dimension n is the only irreducible module of the algebra, it is isomorphic to $\mathrm{Mat}_n(\mathbb{C})$.

Matrices, Part I

Let $r = d - s - t$.

$$H_1 = \sum_{s,t} (r-s) E^*_{[s,t]} \sim h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$H_2 = \sum_{s,t} (s-t) E^*_{[s,t]} \sim h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$H_3 = H_1 + H_2, \text{ where } r = d - s - t,$$

$$R_1 = \sum_{s,t} E^*_{[s+1,t]} A E^*_{[s,t]} \sim e_{-s} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$R_2 = \sum_{s,t} E^*_{[s-1,t+1]} A E^*_{[s,t]} \sim e_{-t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

Matrices - Part II

$$R_3 = \sum_{s,t} E^*_{[s,t+1]} A^\top E^*_{[s,t]} \sim e_{-s-t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$L_1 = \sum_{s,t} E^*_{[s-1,t]} A^\top E^*_{[s,t]} \sim e_s = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L_2 = \sum_{s,t} E^*_{[s+1,t-1]} A^\top E^*_{[s,t]} \sim e_t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L_3 = \sum_{s,t} E^*_{[s,t-1]} A E^*_{[s,t]} \sim e_{-s-t} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Lemma 8

We have the following.

(i) $A = R_1 + R_2 + L_3$ *and* $A^\top = L_1 + L_2 + R_3$.

(ii) *Let* $\tilde{A} = A + A^\top$. *Then* $\tilde{A}$ *is the adjacency matrix of* $H(d,3)$.

(iii) $\tilde{R} = R_1 + R_3$ *is the raising operator,* $\tilde{F} = R_2 + L_2$ *the flat operator, and* $\tilde{L} = L_1 + L_3$ *the lowering operator of* $H(d,3)$.

For matrices M_1, M_2, $[M_1, M_2] = M_1 M_2 - M_2 M_1$. Then $sl_n(\mathbb{C}) = \{M \in \mathrm{Mat}_n(\mathbb{C}), \mathrm{tr}(M) = 0\}$ becomes a simple Lie algebra.

Lemma 9

The following hold.

(i) $[L_1, R_1] = H_1$, $[H_1, L_1] = 2L_1$, *and* $[H_1, R_1] = -2R_1$.

(ii) $[L_2, R_2] = H_2$, $[H_2, L_2] = 2L_2$, *and* $[H_2, R_2] = -2R_2$.

(iii) $[L_3, R_3] = H_3$, $[H_3, L_3] = 2L_3$, *and* $[H_3, R_3] = -2R_3$.

Proposition 3

The following hold.

(i) *The Lie algebra generated by* $H_1, H_2, L_1, L_2, R_1, R_2$ *is isomorphic to* $sl_3(\mathbb{C})$ *with a Cartan subalgebra generated by* H_1 *and* H_2*, and a Borel subalgebra generated by* H_1, H_2, L_1, L_2, L_3.

(ii) *The Lie algebra generated by the triple* H_1, L_1, R_1 *(resp. the triple* H_2, L_2, R_2 *and* H_3, L_3, R_3*) is isomorphic to* $sl_2(\mathbb{C})$ *with a Cartan subalgebra generated by* H_1 *(resp.* H_2*,* H_3*) and a Borel subalgebra generated by* H_1, L_1 *(resp.* H_2, L_2*, and* H_3, L_3*).*

Proposition 4

The following hold. $\mathbb{C}[A] = \mathbb{C}[A^\top] = \mathbb{C}[A, A^\top]$ *is the Bose-Mesner algebra of* $H^*(d, 3)$.

Proposition 5

The following hold.

(i) $\mathcal{T} = \mathcal{T}(x) = \langle A, E^*_{i,j} \mid (i, j) \in \Delta \rangle$.

(ii) $\mathcal{T}(x) = \langle L_1, L_2, R_1, R_2 \rangle$.

(iii) *Let* $\mathcal{L}$ *be the Lie algebra generated by* L_1, L_2, R_1, R_2. *Then* $\mathcal{L} \simeq sl_3(\mathbb{C})$, *and* $\mathcal{L}$ *submodules of* V *are* $\mathcal{T}$ *submodules and vice versa. Moreover, an* $\mathcal{L}$*-submodule* W *of* V *is irreducible if and only if a* $\mathcal{T}$*-module* W *of* V *is irreducible.*

Proposition 6 ([1, Theorem on page 108])

Let $\mathcal{L} = \langle L_1, L_2, L_2, R_1, R_2, R_3 \rangle$, and W an irreducible submodule of V. Then there is a highest weight vector $\boldsymbol{v} \in W$ satisfying $L_1\boldsymbol{v} = L_2\boldsymbol{v} = L_3\boldsymbol{v} = \mathbf{0} \neq \boldsymbol{v}$, determined up to nonzero scalar multiple, and W is spanned by the set of vectors

$$\{R_3^k R_1^j R_2^i \boldsymbol{v} \mid i, j, k \geq 0\}.$$

Definition 10

Let $\boldsymbol{v} \in V$ be a nonzero vector of V satisfying $H_1 \boldsymbol{v} = m_1 \boldsymbol{v}$ and $H_2 \boldsymbol{v} = m_2 \boldsymbol{v}$. Then we call $\lambda = (m_1, m_2)$ the weight of $\boldsymbol{v}$.

Lemma 11

Let W be an irreducible $\mathcal{L}$ submodule of V with the heighest weight vector $\boldsymbol{v} = E^*_{[s,t]} \boldsymbol{v}$ with weight $\lambda = (m_1, m_2)$. Set $r = d - s - t$, and $\boldsymbol{v}_{i_1,i_2,i_3} = R_3^{i_3} R_1^{i_1} R_2^{i_2} \boldsymbol{v}$. Then, the following hold.

(i) *The weight of $\boldsymbol{v}_{i_1,i_2,i_3}$ is $(r - s - 2i_1 + i_2 - i_3, s - t + i_1 - 2i_2 - i_3)$. In particular, $m_1 = r - s$ and $m_2 = s - t$.*

(ii) *$\boldsymbol{v}_{i_1,i_2,i_3} = R_3^{i_3} R_1^{i_1} R_2^{i_2} \boldsymbol{v} \neq \boldsymbol{0}$ if and only if $0 \leq i_2 \leq s - t$, $0 \leq i_1 \leq r - s + i_2$ and $0 \leq i_3 \leq r - t - i_1 - i_2$.*

Lemma 12

Suppose $\lambda = m_1\lambda_1 + m_2\lambda_2$ *be the heighest weight of an irreducible* $\mathcal{L}$ *submodule* W *of* V, *and* $\boldsymbol{v} = E^*_{[s,t]}\boldsymbol{v}$ *a highest weight vector. Set* $r = d - s - t$. *Then, the following hold.*

(i) $(r, s, t) = (m_1 + m_2 + t, m_2 + t, t)$, *and* $r \geq s \geq t$.

(ii) $t \in \{0, 1, \ldots, [\frac{d}{3}]\}$.

(iii) $m_1 = r - s = d - 3t - 2m_2$.

(iv) $m_2 = s - t \in \{0, 1, \ldots, [\frac{d-3t}{2}]\}$.

Lemma 13

Let $\boldsymbol{v}$ *be a heigest weight vector with weight* $\lambda = (m_1, m_2)$ *of an irreducible* $\mathcal{L}$ *submodule of* V*. Let* $\boldsymbol{v}_{i,j,k} = R_3^k R_1^j R_2^i \boldsymbol{v}$ *for* $0 \leq i \leq m_2,\ 0 \leq j \leq m_1 + i,\ 0 \leq k \leq m_1 + m_2 - i - j$*. Let*

$$W_{i,j} = \mathrm{Span}(\boldsymbol{v}_{i,j,k} \mid 0 \leq k \leq m_1 + m_2 - i - j),$$

and

$$U_i = \mathrm{Span}(\boldsymbol{v}_{i,j,k} \mid 0 \leq j \leq m_1, 0 \leq k \leq m_1 + m_2 - i - j).$$

Then $W_{i,j}$ *is an irreducible* $\mathcal{L}_3 = \langle L_3, R_3 \rangle \simeq sl_2(\mathbb{C})$ *module of heigest weight* $m_1 + m_2 - i - j$ *with a heighest weight vector* $\boldsymbol{v}_{i,j,0}$ *of dimenstion* $m_1 + m_2 - i - j + 1$*. Moreover,* $W_{i,j}$ *and* $W_{i',j'}$ *are orthogonal if* $i + j \neq i' + j'$*.*
We also have that $R_1 \boldsymbol{v}_{i,m_1,0} \in U_{i-1} + \cdots + U_0$*.*

Lemma 14

For each (r, s, t) with $r \geq s \geq t$, there is a highest weight vector $\boldsymbol{v}$ with $\lambda = (m_1, m_2) = (r - s, s - t)$ such that $(r(\boldsymbol{v}), s(\boldsymbol{v}), t(\boldsymbol{v})) = (r, s, t)$. Every highest-weight vector is of this type.

Proposition 7

Let $\boldsymbol{v}$ be a heigest weight vector with weight $\lambda = (m_1, m_2)$ of an irreducible $\mathcal{L}$ submodule W of V. Let $\boldsymbol{v}_{i,j,k} = R_3^k R_1^j R_2^i \boldsymbol{v}$. Then the set

$$\{\boldsymbol{v}_{i,j,k} \mid 0 \leq i \leq m_2, 0 \leq j \leq m_1, 0 \leq k \leq m_1 + m_2 - i - j\}$$

forms a basis of W. Moreover, $\dim W = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)$.

Examples

Example 15

For a small d, we list (r, s, t) of the highest weight vector and the highest weight $\lambda = (m_1, m_2)$ as $[r, s, t], (m_1, m_2) : \dim, m$, where dim denotes the dimension of the corresponding irreducible module, and m the multiplicity in the standard module in the following.

- $d = 1$: $[0, 0, 0], (1, 0) : 3, 1$.
- $d = 2$: $[2, 0, 0], (2, 0) : 6, 1$, $[1, 1, 0], (0, 1) : 3, 1$.
- $d = 3$: $[3, 0, 0], (3, 0) : 10, 1$, $[2, 1, 0], (1, 1) : 8, 2$, $[1, 1, 1], (0, 0) : 1, 1$
- $d = 4$: $[4, 0, 0], (4, 0) : 15, 1$, $[3, 1, 0], (2, 1) : 15, 3$, $[2, 2, 0], (0, 2) : 6, 2$, $[2, 1, 1], (1, 0) : 3, 3$.
- $d = 5$: $[5, 0, 0], (5, 0) : 21, 1$, $[4, 1, 0], (3, 1) : 24, 4$, $[3, 2, 0], (1, 2) : 15, 5$, $[3, 1, 1], (2, 0) : 6, 6$, $[2, 2, 1], (0, 1) : 3, 5$.

Algebra generated by two matrices

Theorem 8

A and A^* Let A be the adjacency matrix of $H^*(d,3)$. Let $\omega \in \mathbb{C}$ such that $\omega^2+\omega+1=0$ and

$$A^* = \sum_{s,t}((d-s-t)+s\omega+t\omega^2)E^*_{[s,t]}.$$

Then, the following hold.

$$[A,[A,[A,A^*]]] = (1-\omega)^3 A^*$$
$$[A^*,[A^*,[A^*,A]]] = -(1-\omega)^3 A.$$

Special type of distance-regular graphs

Q-polynomial distance-regular graphs

Most of the distance regular graphs of large diameter satisfy the following. There is a primitive idempotent E with the following property by setting

$$A^* = \frac{1}{|X|}\text{diagonal}(E_{0,0}, E_{0,1}, \ldots, E_{0,d}).$$

$$[A,[A,[A,A^*]]] = b^2[A,A^*]$$
$$[A^*,[A^*,[A^*,A]]] = b^{*2}[A^*,A].$$

Onsagar Algebra

Let $\mathcal{O}$ be a Lie algebra generated by A and A^* with the relation above. If A and A^* act on a finite dimensional algebra as linear transformations. Then A, A^* acts on V as a tridiagonal pair.

Problems

1. Find the multiplicity formula of the standard module of $H^*(d, 3)$.
2. Study the relation between the Terwilliger algebra of a wdrdg $H^*(d, 3)$ and the underlying distance-regular graph $H(d, 3)$.
3. Study the structure of the irreducible modules of the Terwilliger algebra of a weakly distance-regular digraph of Hamming type determined in [4].
4. Determine the structure of a weakly distance regular digraph with respect to every vertex.
5. Study finite dimensional irreducible module of a Lie algebra generated by A and A^* with the following relations.

$$[A, [A, [A, A^*]]] = A^*, \quad [A^*, [A^*, [A^*, A]]] = A.$$

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