Linear Algebra I *November 15, 2018*

Final Exam 2018

(Total: 100 pts, 50% of the grade)

ID#: **Name:**

1. Let $\boldsymbol{u} = [1, -1, -2]^\top$, $\boldsymbol{v} = [-1, -2, 1]^\top$, $\boldsymbol{w} = [2, 1, -2]^\top$, $\boldsymbol{e}_1 = [1, 0, 0]^\top$, $\boldsymbol{e}_2 = [0, 1, 0]^\top$ and $\boldsymbol{e}_3 = [0, 0, 1]^\top$ be vectors in $\mathbb{R}^3$. (15 pts)

   (a) Find (i) $\boldsymbol{u} \times \boldsymbol{v}$ and (ii) the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Show work.

   (b) Find the standard matrix A of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{u}) = \boldsymbol{e}_1$, $T(\boldsymbol{v}) = \boldsymbol{e}_2$ and $T(\boldsymbol{w}) = \boldsymbol{e}_3$. Show work.

Points:

1.(a)	$(b)\ast$	2.$(a)\ast$	(b)	(c)	(d)	(e)	(f)	$\ast$	**Total**
								10 pts	
3.$(a)\ast$	$(b)\ast$	4.$(a)\ast$	(b)	(c)	$(d)\ast$			*none*	
								5 pts	

メッセージ欄：この授業について、特に改善点について、その他何でもどうぞ。

2. Consider the system of linear equations with augmented matrix $C = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7]$, where $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7$ are the columns of C. Let $A = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_6]$ be its coefficient matrix. We obtained a row echelon form G after applying a sequence of elementary row operations to the matrix C. (35 pts)

$$C = \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 1 & -4 \\ 2 & -3 & 12 & -4 & 3 & 3 & 0 \\ 0 & 1 & -2 & 0 & -1 & -1 & -6 \\ -3 & 0 & -9 & 3 & 1 & 5 & 9 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 1 & -4 \\ 0 & 1 & -2 & 0 & -1 & -1 & -6 \\ 0 & 0 & 0 & -2 & 0 & -2 & -10 \\ 0 & 0 & 0 & 0 & 1 & 8 & -3 \end{bmatrix}.$$

(a) Describe each step of a sequence of elementary row operations to obtain G from C by $[i, j], [i, j; c], [i; c]$ notation. Show work.

(b) Find an invertible matrix P of size 4 such that $G = PC$ and express P as a product of <u>four</u> elementary matrices. Do not forget writing P. Show work.

(c) Explain that the invertible matrix P of size 4 such that $G = PC$ is unique.

(d) Find the reduced row echelon form of the matrix C. Show work.

(e) Find all solutions of the system of linear equations.

(f) Explain that the linear transformation defined by $T : \mathbb{R}^6 \to \mathbb{R}^4$ $(\boldsymbol{x} \mapsto A\boldsymbol{x})$, i.e., $T(\boldsymbol{x}) = A\boldsymbol{x}$ is NOT one-to-one.

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. (20 pts)

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ -2 & -1 & -5 & 5 \\ 4 & 1 & 3 & -3 \\ -2 & 3 & 1 & 4 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 8 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Show work.

(b) Express x_4 as a quotient (*bun-su*) of determinants when $A\boldsymbol{x} = \boldsymbol{b}$, and write $\operatorname{adj}(A)$, the adjugate of A. Don't evaluate the determinants.

$x_4 =$, $\operatorname{adj}(A) =$

4. Let A, $\boldsymbol{v}$ and P be given below. (30 pts)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 & {x_5}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & {x_4}^3 & {x_5}^3 \\ {x_1}^4 & {x_2}^4 & {x_3}^4 & {x_4}^4 & {x_5}^4 \end{bmatrix}.$$

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + x^5$.

(a) Find the determinant of P. Show work. You may use the following: $\det(Q) = (x_3 - x_2)(x_4 - x_2)(x_5 - x_2)(x_4 - x_3)(x_5 - x_3)(x_5 - x_4)$, where Q is the matrix obtained from P by deleting the 5th row and the 1st column.

(b) Show the following: If $\boldsymbol{v}$ is an eigenvector of A corresponding to an eigenvalue α, then $\alpha = \lambda$ and $f(\alpha) = 0$.

(c) Suppose x_1, x_2, x_3, x_4, x_5 are distinct and $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5) = 0$. Show that A is diagonalizable.

(d) Show that $\det(xI - A) = f(x)$.