

The Terwilliger Algebra of a Directed Hamming Graph

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Association Schemes

Definition 1

Let X be a nonempty finite set, and $X \times X = X_0 \cup X_1 \cup \cdots \cup X_d$ be a partition of $X \times X$. Let A_i be matrices rows and columns indexed by X such that $A_i[x, y] = 1$ if $(x, y) \in X_i$ and 0 otherwise. If A_0, A_1, \dots, A_d satisfy the following conditions, then X with its partition becomes an *association schemes*.

- ① $A_0 = I$, the identity matrix.
- ② $A_i^\top = A_{i'}$ for some $i' \in \{0, 1, \dots, d\}$.
- ③ For each $i, j \in \{0, 1, \dots, d\}$, there exist constants $p_{i,j}^0, \dots, p_{i,j}^d$ such that $A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k$.

The algebra $\mathcal{B} = \langle A_0, A_1, \dots, A_d \rangle \subseteq \text{Mat}_X(\mathbb{C})$ is called the *Bose-Mesner algebra* of the association scheme. If \mathcal{B} is commutative, it is called a *commutative association scheme*.

Examples of Association Schemes

$\mathcal{H}(H \backslash G / H)$

Let H be a subgroup of a finite group G . Let $X = G/H$ be the set of left cosets. The orbits of G on $X \times X$ define an association scheme with respect to the following action.

$$g : X \times X \rightarrow X \times X : (g_1H, g_2H) \mapsto (gg_1H, gg_2H) \quad \text{for } g, g_1, g_2 \in G.$$

$\mathcal{H}(G_x \backslash G / G_x)$

Let G be a transitive permutation group on a finite set X . Then, the orbits of G on $X \times X$ (called orbitals) define an association scheme.

Commutativity

The association scheme defined above is commutative if the permutation character of the permutation representation is multiplicity-free.

Primitive Idempotents

Let $\mathcal{B} = \langle A_0, A_1, \dots, A_d \rangle \subseteq \text{Mat}_X(\mathbb{C})$ be the Bose-Mesner algebra of a commutative association scheme with $n = |X|$. Then there are primitive idempotents E_0, E_1, \dots, E_d satisfying the following.

- (i) $E_0 = \frac{1}{n}J$, where J is the all one's matrix of size n , and $E_i E_j = \delta_{i,j} E_i$ for all $i, j \in \{0, 1, \dots, d\}$.
- (ii) $E_j^\top = E_{\hat{j}}$ for some $\hat{j} \in \{0, 1, \dots, d\}$.
- (iii) For $i, j \in \{0, 1, \dots, d\}$, there exists $q_{i,j}^k \in \mathbb{R}$ such that $E_i \circ E_j = \frac{1}{n} \sum_{k=0}^n q_{i,j}^k E_k$, where \circ is the entry-wise product, called the Hadamard product.
- (iv) For $i, j \in \{0, 1, \dots, d\}$, there exists $p_i(j), q_j(i) \in \mathbb{R}$ such that

$$A_i = \sum_{j=0}^d p_i(j) E_j, \quad E_j = \frac{1}{n} \sum_{i=0}^d q_j(i) A_i.$$

Definition 2 (DRGs and WDRDGs)

- Let $\Gamma = (X, \tilde{E})$, where \tilde{E} is a set of pairs of X . Γ is a *distance-regular graph* if Γ is connected of diameter d , and the following partition defines an association scheme. Here, $\partial(x, y)$ is the path distance.

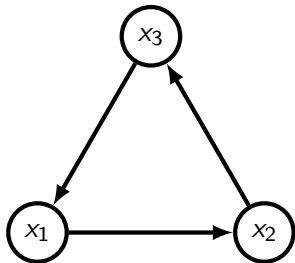
$$X \times X = \bigcup_{i=0}^d \Delta_i, \quad \Delta_i = \{(x, y) \mid \partial(x, y) = i\}.$$

- Let $\Gamma = (X, E)$, where E is a set of ordered pairs of X . Γ is a *weakly distance-regular digraph* if Γ is strongly connected, and the following partition defines a nonsymmetric association scheme.

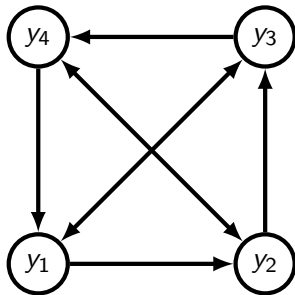
$$X \times X = \bigcup_{(i,j) \in \Delta} \Delta_{i,j}, \quad \Delta_{i,j} = \{(x, y) \mid \tilde{\partial}(x, y) = (i, j)\},$$

where $\tilde{\partial}(x, y)$ denotes the two-way distance of x and y , and $\Delta(\Gamma) = \{\tilde{\partial}(x, y) \mid x, y \in X\}$.

WDRDGs over K_3 and K_4



$$\Delta = \{(0, 0), (1, 2), (2, 1)\}$$



$$\Delta = \{(0, 0), (1, 1), (1, 2), (2, 1)\}$$

WDRDGs over a Clique (Semicomplete WDRDGs)

Proposition 1 (Y. Yang, Q. Zeng and K. Wang [4, Proposition 3.5])

Let $\Gamma = (X, E)$ be a weakly distance-regular digraph (wdrdg) of diameter d and girth g whose underlying graph is a complete graph. Then $d = 2$, $g \leq 3$ and

$$\Delta(\Gamma) = \{(0, 0), (1, 2), (2, 1)\}, \text{ or } \Delta(\Gamma) = \{(0, 0), (1, 1), (1, 2), (2, 1)\}.$$

Proof.

Suppose $x, y \in X$ with $\tilde{\partial}(x, y) = (1, p)$ and $p \geq 3$. Note that $X = \{x\} \cup \Gamma_{1,*}(x) \cup \Gamma_{*,1}(x)$. If $z \in \Gamma_{1,*}(y) \setminus (\{x\} \cup \Gamma_{1,*}(x))$, then

$$3 \leq p = \partial(y, x) \leq \partial(y, z) + \partial(z, x) = 2,$$

which is absurd. Hence, $\{y\} \cup \Gamma_{1,*}(y) \subseteq \Gamma_{1,*}(x)$. A contradiction. □

Two Types of Semicomplete WDRDGs

Type I: $d = 2$, $g = 3$, and $\Delta(\Gamma) = \{(0,0), (1,2), (2,1)\}$

- The existence is equivalent to that of a nonsymmetric association scheme with two classes with $4m - 1$ vertices.
- It exists if and only if a skew Hadamard matrix of size $4m$ exists.

Type II: $d = 2$, $g = 2$, and $\Delta(\Gamma) = \{(0,0), (1,1), (1,2), (2,1)\}$

Let $\mathcal{B} = \langle I, A, A^\top, B = B^\top \rangle$ be the Bose-Mesner Algebra of a nonsymmetric association scheme of class three.

- If the digraph defined by A is connected, it is semicomplete of type II.
- If the digraph defined by A is not connected, each connected component is a semicomplete wdrdg of type I.

Every association scheme of class at most four is commutative.

Weakly Distance-Regular Digraphs of Hamming Type

Theorem 2 (Y. Yang, Q. Zeng and K. Wang [4])

Let Γ be a commutative weakly distance-regular digraph. Then Γ has a Hamming graph as its underlying graph if and only if Γ is isomorphic to one of the following diagrams:

- (i) $\text{Cay}(\mathbb{Z}_4, \{1\}) \text{ --- } [H(2, 2)];$
- (ii) $\text{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1, 0), (0, 1)\}) \text{ --- } [H(3, 2)];$
- (iii) $\Delta^1 \text{ or } \Delta^1 \square \Delta^2 \text{ --- } [H(1, q) \text{ or } H(2, q)];$
- (iv) $\Gamma^1 \square \Gamma^2 \square \dots \square \Gamma^d \text{ --- } [H(d, q) \text{ with } q \equiv 3 \pmod{4}].$

Here, Δ^i (resp Γ^i) is a semicomplete weakly distance-regular digraph of girth 2 (resp 3) with the same intersection numbers for each i .

The cases when the underlying graph is a folded cube and a Doob graph are determined. $\text{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_4, \{(1, 0), (0, 1), (-1, -1)\})$ is the only nontrivial case coming from the Shrikhande graph.

Definition 3 ($H(d, q)$)

Let $N = \{a_1, a_2, \dots, a_n\}$. The Hamming graph $H(d, n)$ or more specifically, the Hamming graph of diameter d on the set N , $H(d, N)$ is defined by X , the set of vertices, and \tilde{E} , the set of edges.

$$X = \{(x_1, \dots, x_d) \mid \text{for all } i, x_i \in N\},$$

$$\tilde{E} = \{xy \mid \text{exactly 1 coordinate } i, x_i \neq y_i\} \subseteq X \times X.$$

Definition 4 ($H^*(d, 3)$)

The directed graph $H^*(d, 3)$ is defined by X , the set of vertices, and E , the set of arcs.

$$X = \{(x_1, \dots, x_d) \mid \text{for all } i, x_i \in \mathbb{F}_3\},$$

$$E = \{xy \mid \text{exactly 1 coordinate } i, x_i + 1 = y_i\} \subseteq X \times X.$$

(We assign the trivial direction on each coordinate, $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$.)

Adjacency Matrix

Definition 5

The adjacency matrix of $H^*(d, 3)$, $A = A^{(d)}$ is defined by the following.

$$(A)_{ij} = \begin{cases} 1 & \text{if there exists an arc from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

For x, y in X , $\partial(x, y)$ denotes the distance between x and y , i.e., the smallest number of arcs connecting from x to y . We also define the two-way distance $\tilde{\partial}(x, y) = (\partial(x, y), \partial(y, x))$, and Δ , the set of all two-way distances;

$$\Delta = \{\tilde{\partial}(x, y) \mid x, y \in X\}.$$

Vector Space V

Let $\Gamma = (X, E)$ be $H^*(d, 3)$. Fix a base vertex $x = (0, 0, \dots, 0) \in X$.

- $V = \mathbb{C}^{|X|}$: the vector space over the complex number field \mathbb{C} whose coordinates are indexed by the elements of X .
- $X_{i,j} = \{y \mid \partial(x, y) = i, \partial(y, x) = j\}$.
- If $y \in X$ has s ones, t twos, and $r = d - s - t$ zeros, then $\partial(x, y) = s + 2t, \partial(y, x) = 2s + t$.
- If $\tilde{\partial}(x, y) = (i, j)$, then $s = (2j - i)/3$, and $t = (2i - j)/3$.
- We also write

$$X_{[s,t]} = \{y \mid \text{there are } s \text{ ones and } t \text{ twos}\} = X_{s+2t, 2s+t}.$$

$E_{i,j}^*$

For $(i, j) \in \Delta$, $E_{i,j}^* = E_{i,j}^{(d)*}$ denotes a diagonal matrix such that

$$E_{i,j}^*(z, z) = \begin{cases} 1, & \text{if } \partial(x, z) = (i, j), \\ 0 & \text{otherwise,} \end{cases}$$

and the zero matrix of the same size if $(i, j) \notin \Delta$. Then,

$$E_{i,j}^* \mathbf{1} = \{ \sum \hat{y} \mid y \in X, \partial(x, y) = i, \partial(y, x) = j \}.$$

We set

$$E_{i,j}^* = E_{[(2j-i)/3, (2i-j)/3]}^*, \text{ and } E_{[s,t]}^* = E_{s+2j, 2s+j}^*.$$

If a vector $\mathbf{v} \in V$ satisfies $E_{[s,t]}^* \mathbf{v} = \mathbf{v}$, then we write

$$\text{type}(\mathbf{v}) = [r(\mathbf{v}), s(\mathbf{v}), t(\mathbf{v})]: \text{ the type of } \mathbf{v}.$$

Note that \mathbf{v} can be written as a linear combination of \hat{y} with $y \in X_{[s,t]}$.

The Terwilliger Algebra $\mathcal{T}(x)$

Definition 6

The Terwilliger algebra $\mathcal{T}(x)$ of $H^*(d, 3)$ with respect to a base vertex x is a \mathbb{C} -algebra generated by A , A^\top and $E_{i,j}^*$ for $(i, j) \in \Delta$. We also write $\mathcal{T}(H^*(d, 3))$ to specify the digraph.

Note

- ① $\Gamma = H^*(d, 3)$ is a Cayley digraph defined on an abelian group \mathbb{Z}_3^d . Hence, it is commutative.
- ② It is easy to calculate eigenvalues of $A = A^{(d)}$.
- ③ $H^*(1, 3)$ is a directed triangle, and the orientation of arcs of $H^*(d, 3)$ is uniquely determined.
- ④ $A^{(d)} = A^{(1)} \otimes I \otimes \cdots \otimes I + I \otimes A^{(1)} \otimes \cdots \otimes I + I \otimes I \otimes \cdots \otimes A^{(1)}$.

$\mathcal{T}(H^*(1, 3))$

Lemma 7

Let \mathcal{T}_1 be the Terwilliger algebra of $H^*(1, 3)$. Then

$$\mathcal{T}_1 = \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle \simeq \text{Mat}_3(\mathbb{C}).$$

Proof.

This is trivial. If $A^{(1)}$ is the adjacent matrix of $H^*(1, 3)$, then $A^{(1)2} = A^{(1)\top}$, and $E^{(1)*}_{0,0}$, $E^{(1)*}_{1,2}$ and $E^{(1)*}_{2,1}$ are diagonal matrix units $e_{1,1}$, $e_{2,2}$ and $e_{3,3}$. Hence, every matrix unit is in \mathcal{T}_1 . \square

Note that if the principal module of dimension n is the only irreducible module of the algebra, it is isomorphic to $\text{Mat}_n(\mathbb{C})$.

The Algebraic Structure of $\mathcal{T}(x)$

Theorem 3 (T.Miezaki, H.Suzuki, K.Uchida [5])

Let $\mathcal{T}(x)$ be the Terwilliger algebra of $H^*(d, 3)$ with respect to a base vertex x . Then,

$$\mathcal{T}(x) \simeq \text{Sym}^d(\text{Mat}_3(\mathbb{C})).$$

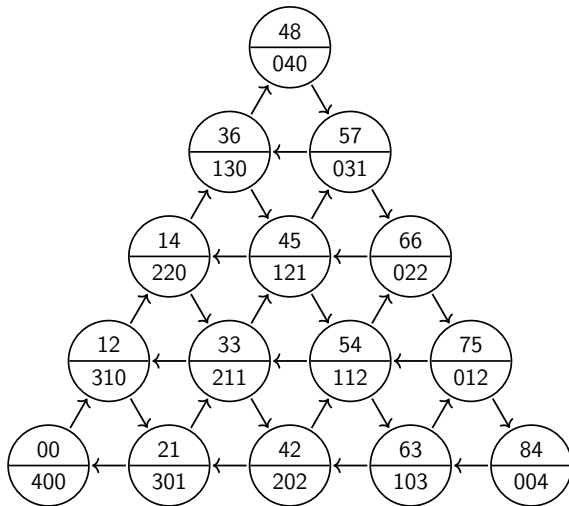
Moreover,

$$\mathcal{T}(x) \simeq \text{Sym}^d(\text{Mat}_3(\mathbb{C})) \simeq \bigoplus_{n \in \Lambda} \text{Mat}_n(\mathbb{C}), \text{ where}$$

$$\Lambda = \left\{ \frac{1}{2}(d - 3\ell - 2m + 1)(m + 1)(d - 3\ell - m + 2) \mid \right. \\ \left. 0 \leq \ell \leq \left\lceil \frac{d}{3} \right\rceil, 0 \leq m \leq \left\lceil \frac{d - 3\ell}{2} \right\rceil \right\}.$$

Example: $H^*(4, 3)$

Two-way distance: $\tilde{d}(x, y) = (i, j)$, and $\text{type}(\mathbf{v}) = [r(\mathbf{v}), s(\mathbf{v}), t(\mathbf{v})]$



Elements in $\mathcal{T}(x)$

Let $r = d - s - t$.

$$H_1 = \sum_{s,t} (r - s) E_{[s,t]}^*, \quad H_2 = \sum_{s,t} (s - t) E_{[s,t]}^*,$$

$$H_3 = H_1 + H_2,$$

$$R_1 = \sum_{s,t} E_{[s+1,t]}^* A E_{[s,t]}^*, \quad R_2 = \sum_{s,t} E_{[s-1,t+1]}^* A E_{[s,t]}^*,$$

$$R_3 = \sum_{s,t} E_{[s,t+1]}^* A^\top E_{[s,t]}^*,$$

$$L_1 = \sum_{s,t} E_{[s-1,t]}^* A^\top E_{[s,t]}^*, \quad L_2 = \sum_{s,t} E_{[s+1,t-1]}^* A^\top E_{[s,t]}^*,$$

$$L_3 = \sum_{s,t} E_{[s,t-1]}^* A E_{[s,t]}^*.$$

Note that $H_1, H_2, H_3, R_1, R_2, R_3, L_1, L_2, L_3$ are all in $\mathcal{T}(x)$.

Lemma 8

We have the following.

- (i) $A = R_1 + R_2 + L_3$ and $A^\top = L_1 + L_2 + R_3$.
- (ii) Let $\tilde{A} = A + A^\top$. Then \tilde{A} is the adjacency matrix of $H(d, 3)$.
- (iii) $\tilde{R} = R_1 + R_3$ is the raising operator, $\tilde{F} = R_2 + L_2$ the flat operator, and $\tilde{L} = L_1 + L_3$ the lowering operator of $H(d, 3)$.

For matrices M_1, M_2 , $[M_1, M_2] = M_1 M_2 - M_2 M_1$. Then $sl_n(\mathbb{C}) = \{M \in \text{Mat}_n(\mathbb{C}), \text{tr}(M) = 0\}$ becomes a simple Lie algebra.

Lemma 9

The following hold.

- (i) $[L_1, R_1] = H_1$, $[H_1, L_1] = 2L_1$, and $[H_1, R_1] = -2R_1$.
- (ii) $[L_2, R_2] = H_2$, $[H_2, L_2] = 2L_2$, and $[H_2, R_2] = -2R_2$.
- (iii) $[L_3, R_3] = H_3$, $[H_3, L_3] = 2L_3$, and $[H_3, R_3] = -2R_3$.

The Relation Between $\mathcal{T}(x)$ and $sl_3(\mathbb{C})$

Proposition 4

The following hold.

- (i) The Lie algebra generated by $H_1, H_2, L_1, L_2, R_1, R_2$ is isomorphic to $sl_3(\mathbb{C})$ with a Cartan subalgebra generated by H_1 and H_2 , and a Borel subalgebra generated by H_1, H_2, L_1, L_2, L_3 .
- (ii) The Lie algebra generated by the triple H_1, L_1, R_1 (resp. the triple H_2, L_2, R_2 and H_3, L_3, R_3) is isomorphic to $sl_2(\mathbb{C})$ with a Cartan subalgebra generated by H_1 (resp. H_2, H_3) and a Borel subalgebra generated by H_1, L_1 (resp. H_2, L_2 , and H_3, L_3).

Associative algebra $\mathcal{T}(x)$ and the associated Lie algebra

$$\mathcal{T}(x) = \langle A, A^\top, E_{i,j}^* \mid (i,j) \in \Delta(\Gamma) \rangle = \langle L_1, L_2, R_1, R_2 \rangle \supseteq \langle L_1, L_2, R_1, R_2 \rangle_{\text{Lie}}.$$

$\mathcal{T}(x)$ -modules and $sl_3(\mathbb{C})$ -modules

Proposition 5

The following hold. $\mathbb{C}[A] = \mathbb{C}[A^\top] = \mathbb{C}[A, A^\top]$ is the Bose-Mesner algebra of $H^*(d, 3)$.

Proposition 6

The following hold.

- (i) $\mathcal{T} = \mathcal{T}(x) = \langle A, E_{i,j}^* \mid (i, j) \in \Delta \rangle$.
- (ii) $\mathcal{T}(x) = \langle L_1, L_2, R_1, R_2 \rangle \supseteq \langle L_1, L_2, R_1, R_2 \rangle_{\text{Lie}}$.
- (iii) Let \mathcal{L} be the Lie algebra generated by L_1, L_2, R_1, R_2 . Then $\mathcal{L} \simeq sl_3(\mathbb{C})$, and \mathcal{L} submodules of V are \mathcal{T} submodules and vice versa. Moreover, an \mathcal{L} -submodule W of V is irreducible if and only if a \mathcal{T} -module W of V is irreducible.

Poincaré–Birkhoff–Witt Theorem and $\mathcal{U}(\mathcal{L})$

Proposition 7 ([1, Theorem on page 108])

Let $\mathcal{L} = \langle L_1, L_2, L_2, R_1, R_2, R_3 \rangle_{Lie}$, and W an irreducible submodule of V . Then there is a highest weight vector $\mathbf{v} \in W$ satisfying $L_1 \mathbf{v} = L_2 \mathbf{v} = L_3 \mathbf{v} = \mathbf{0} \neq \mathbf{v}$, determined up to nonzero scalar multiple, and W is spanned by the set of vectors

$$\{R_3^k R_1^j R_2^i \mathbf{v} \mid i, j, k \geq 0\}.$$

Highest weight vectors

$$H_1 = \sum_{s,t} (r-s) E_{[s,t]}^*, \quad H_2 = \sum_{s,t} (s-t) E_{[s,t]}^*.$$

Highest weight vector and the weight of $\mathbf{v} = E_{[s,t]}^* \mathbf{v}$

Definition 10

Let $\mathbf{v} \in V$ be a nonzero vector of V satisfying $H_1 \mathbf{v} = m_1 \mathbf{v}$ and $H_2 \mathbf{v} = m_2 \mathbf{v}$. Then we call $\lambda = (m_1, m_2)$ the weight of \mathbf{v} .

Lemma 11

Let W be an irreducible \mathcal{L} submodule of V with the highest weight vector $\mathbf{v} = E_{[s,t]}^* \mathbf{v}$ with weight $\lambda = (m_1, m_2)$. Set $r = d - s - t$, and $\mathbf{v}_{i_1, i_2, i_3} = R_3^{i_3} R_1^{i_1} R_2^{i_2} \mathbf{v}$. Then, the following hold.

- (i) The weight of $\mathbf{v}_{i_1, i_2, i_3}$ is $(r - s - 2i_1 + i_2 - i_3, s - t + i_1 - 2i_2 - i_3)$. In particular, $m_1 = r - s$ and $m_2 = s - t$.
- (ii) $\mathbf{v}_{i_1, i_2, i_3} = R_3^{i_3} R_1^{i_1} R_2^{i_2} \mathbf{v} \neq \mathbf{0}$ if and only if $0 \leq i_2 \leq s - t$, $0 \leq i_1 \leq r - s + i_2$ and $0 \leq i_3 \leq r - t - i_1 - i_2$.

Highest weights

Lemma 12

Suppose $\lambda = m_1\lambda_1 + m_2\lambda_2$ be the highest weight of an irreducible \mathcal{L} submodule W of V , and $\mathbf{v} = E_{[s,t]}^*\mathbf{v}$ a highest weight vector. Set $r = d - s - t$. Then, the following hold.

- (i) $(r, s, t) = (m_1 + m_2 + t, m_2 + t, t)$, and $r \geq s \geq t$.
- (ii) $t \in \{0, 1, \dots, \lfloor \frac{d}{3} \rfloor\}$.
- (iii) $m_1 = r - s = d - 3t - 2m_2$.
- (iv) $m_2 = s - t \in \{0, 1, \dots, \lfloor \frac{d-3t}{2} \rfloor\}$.

The structure of an irreducible module

Lemma 13

Let \mathbf{v} be a highest weight vector with weight $\lambda = (m_1, m_2)$ of an irreducible \mathcal{L} submodule of V . Let $\mathbf{v}_{i,j,k} = R_3^k R_1^j R_2^i \mathbf{v}$ for $0 \leq i \leq m_2$, $0 \leq j \leq m_1 + i$, $0 \leq k \leq m_1 + m_2 - i - j$. Let

$$W_{i,j} = \text{Span}(\mathbf{v}_{i,j,k} \mid 0 \leq k \leq m_1 + m_2 - i - j),$$

and

$$U_i = \text{Span}(\mathbf{v}_{i,j,k} \mid 0 \leq j \leq m_1, 0 \leq k \leq m_1 + m_2 - i - j).$$

Then $W_{i,j}$ is an irreducible $\mathcal{L}_3 = \langle L_3, R_3 \rangle \simeq sl_2(\mathbb{C})$ module of highest weight $m_1 + m_2 - i - j$ with a highest weight vector $\mathbf{v}_{i,j,0}$ of dimension $m_1 + m_2 - i - j + 1$. Moreover, $W_{i,j}$ and $W_{i',j'}$ are orthogonal if $i + j \neq i' + j'$.

We also have that $R_1 \mathbf{v}_{i,m_1,0} \in U_{i-1} + \cdots + U_0$.

The highest weight and the dimension of an irreducible module

Lemma 14

For each (r, s, t) with $r \geq s \geq t$, there is a highest weight vector \mathbf{v} with $\lambda = (m_1, m_2) = (r - s, s - t)$ such that $(r(\mathbf{v}), s(\mathbf{v}), t(\mathbf{v})) = (r, s, t)$. Every highest-weight vector is of this type.

Proposition 8

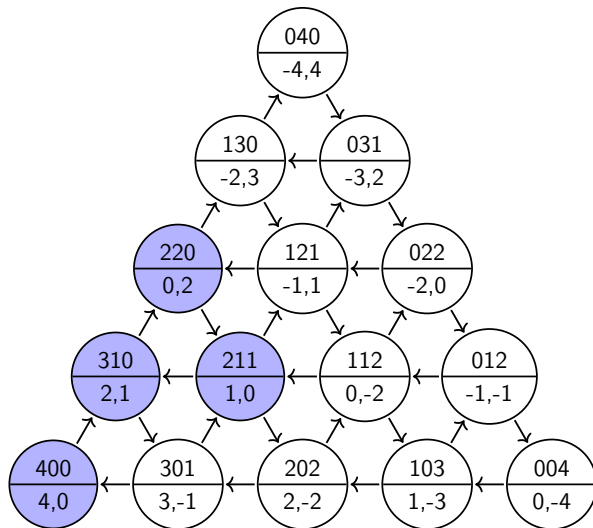
Let \mathbf{v} be a highest weight vector with weight $\lambda = (m_1, m_2)$ of an irreducible \mathcal{L} submodule W of V . Let $\mathbf{v}_{i,j,k} = R_3^k R_1^j R_2^i \mathbf{v}$. Then the set

$$\{\mathbf{v}_{i,j,k} \mid 0 \leq i \leq m_2, 0 \leq j \leq m_1, 0 \leq k \leq m_1 + m_2 - i - j\}$$

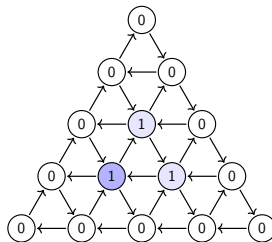
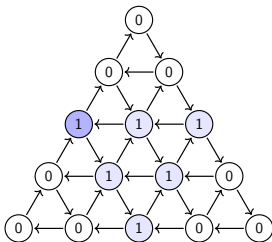
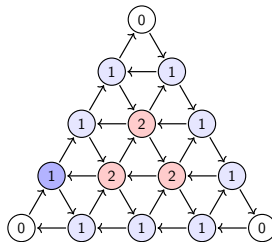
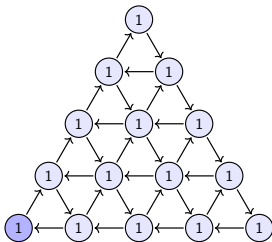
forms a basis of W . Moreover, $\dim W = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)$.

Example: $H^*(4, 3)$

$\text{type}(\mathbf{v}) = [r(\mathbf{v}), s(\mathbf{v}), t(\mathbf{v})]$, and $\text{weight}(\mathbf{v}) = (\ell, m) = (r(\mathbf{v}) - s(\mathbf{v}), s(\mathbf{v}) - t(\mathbf{v}))$



Example: $H^*(4, 3)$ - Irreducible Modules



Examples for small d

Example 15

For a small d , we list (r, s, t) of the highest weight vector and the highest weight $\lambda = (m_1, m_2)$ as $[r, s, t], (m_1, m_2) : \dim, m$, where \dim denotes the dimension of the corresponding irreducible module, and m the multiplicity in the standard module in the following.

- $d = 1$: $[0, 0, 0], (1, 0) : 3, 1$.
- $d = 2$: $[2, 0, 0], (2, 0) : 6, 1, [1, 1, 0], (0, 1) : 3, 1$.
- $d = 3$: $[3, 0, 0], (3, 0) : 10, 1, [2, 1, 0], (1, 1) : 8, 2, [1, 1, 1], (0, 0) : 1, 1$
- $d = 4$: $[4, 0, 0], (4, 0) : 15, 1, [3, 1, 0], (2, 1) : 15, 3, [2, 2, 0], (0, 2) : 6, 2, [2, 1, 1], (1, 0) : 3, 3$.
- $d = 5$: $[5, 0, 0], (5, 0) : 21, 1, [4, 1, 0], (3, 1) : 24, 4, [3, 2, 0], (1, 2) : 15, 5, [3, 1, 1], (2, 0) : 6, 6, [2, 2, 1], (0, 1) : 3, 5$.

Algebra generated by two matrices

Theorem 9

A and A^* Let A be the adjacency matrix of $H^*(d, 3)$. Let $\omega \in \mathbb{C}$ such that $\omega^2 + \omega + 1 = 0$ and

$$A^* = \sum_{s,t} ((d - s - t) + s\omega + t\omega^2) E_{[s,t]}^*.$$

Then, the following hold.

$$\begin{aligned} [A, [A, [A, A^*]]] &= (1 - \omega)^3 A^* \\ [A^*, [A^*, [A^*, A]]] &= -(1 - \omega)^3 A. \end{aligned}$$

The Algebra defined by generators A , B , and relations

$$\mathcal{L} = \langle A, B \mid [A, [A, [A, B]]] = B, [B, [B, [B, A]]] = A \rangle$$

Special type of distance-regular graphs

Q -polynomial distance-regular graphs

Most of the distance regular graphs of large diameter satisfy the following. There is a primitive idempotent E with the following property by setting

$$A^* = \frac{1}{|X|} \text{diagonal}(E_{0,0}, E_{0,1}, \dots, E_{0,d}).$$

$$\begin{aligned} [A, [A, [A, A^*]]] &= b^2[A, A^*] \\ [A^*, [A^*, [A^*, A]]] &= b^{*2}[A^*, A]. \end{aligned}$$

Onsager Algebra

Let \mathcal{O} be a Lie algebra generated by A and A^* with the relation above. If A and A^* act on a finite dimensional algebra as linear transformations. Then A, A^* acts on V as a tridiagonal pair.






Problems

- ① Find the multiplicity formula of the standard module of $H^*(d, 3)$.
- ② Study the structure of the irreducible modules of the Terwilliger algebra of $H^*(d, q)$, ($q \equiv 3 \pmod{4}$).
- ③ Study finite dimensional irreducible module of the following algebra generated by two elements and two relations.

$$\mathcal{L} = \langle A, B \mid [A, [A, [A, B]]] = B, [B, [B, [B, A]]] = A \rangle$$

- ④ Develop an algebraic theory of weakly distance-regular digraphs, and find a good class. Or, find a good class of commutative but not symmetric association scheme.
- ⑤ Are there noncommutative weakly distance-regular digraphs?

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THANK YOU!!

Matrices, Part I

Let $r = d - s - t$.

$$H_1 = \sum_{s,t} (r - s) E_{[s,t]}^* \sim h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$H_2 = \sum_{s,t} (s - t) E_{[s,t]}^* \sim h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$H_3 = H_1 + H_2, \text{ where } r = d - s - t,$$

$$R_1 = \sum_{s,t} E_{[s+1,t]}^* A E_{[s,t]}^* \sim e_{-s} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$R_2 = \sum_{s,t} E_{[s-1,t+1]}^* A E_{[s,t]}^* \sim e_{-t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

Matrices - Part II

$$R_3 = \sum_{s,t} E_{[s,t+1]}^* A^\top E_{[s,t]}^* \sim e_{-s-t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$L_1 = \sum_{s,t} E_{[s-1,t]}^* A^\top E_{[s,t]}^* \sim e_s = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L_2 = \sum_{s,t} E_{[s+1,t-1]}^* A^\top E_{[s,t]}^* \sim e_t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L_3 = \sum_{s,t} E_{[s,t-1]}^* A E_{[s,t]}^* \sim e_{-s-t} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $H_1, H_2, H_3, R_1, R_2, R_3, L_1, L_2, L_3$ are all in $\mathcal{T}(x)$.

$$[A, [A, [A, B]]] = B, [B, [B, [B, A]]] = A$$

Recall $[M, N] = MN - NM$. Hence,

$$[A, [A, [A, B]]] = A^3B - 3A^2BA + 3ABA^2 - BA^3.$$

Suppose A, B are diagonalizable linear transformations on a finite dimensional vector space V , and E_i (resp. F_j) are projections onto the eigenspaces written as a polynomial in A (resp. B). satisfying

$$AE_i = E_iA = \phi_i E_i \text{ and } BF_j = F_jB = \psi_j F_j.$$

Then, $[A, [A, [A, B]]] = B$ implies

$$\begin{aligned} E_i B E_j &= E_i [A, [A, [A, B]]] E_j = (\phi_i^3 - 3\phi_i^2\phi_j + 3\phi_i\phi_j^2 - \phi_j^3) E_i B E_j \\ &= (\phi_i - \phi_j)^3 E_i B E_j. \end{aligned}$$

Therefore, if $[A, [A, [A, B]]] = B$, and $[B, [B, [B, A]]] = A$,

$$E_i B E_j \neq 0 \Rightarrow \phi_i - \phi_j \in \{1, \omega, \omega^2\}, \text{ and}$$

$$F_i A F_j \neq 0 \Rightarrow \psi_i - \psi_j \in \{1, \omega, \omega^2\},$$

where $1 + \omega + \omega^2 = 0$.

Hexagonal Pairs

Let A and B be linear transformations on a finite dimensional vector space $V \neq 0$ satisfying the following four conditions.

- (i) A and B are diagonalizable on V .
- (ii) There is an indexing $V_{i,j}, (i,j) \in \Phi$ of the eigenspaces of A such that

$$BU_{i,j} \subseteq U_{i+1,j} + U_{i-1,j+1} + U_{i,j-1}, \quad (i,j) \in \Phi.$$

- (iii) There is an indexing $V_{i,j}, (i,j) \in \Psi$ of the eigenspaces of A such that

$$BV_{i,j} \subseteq V_{i+1,j} + V_{i-1,j+1} + V_{i,j-1}, \quad (i,j) \in \Psi.$$

- (iv) There is no subspace W of V such that $AW \subseteq W$, $BW \subseteq W$, other than $W = 0$ and $W = V$.

Examples with small number of vertices

Hanaki-Miyamoto, Classification of association schemes with a small number of vertices at <http://math.shinshu-u.ac.jp/~hanaki/as/>.

<http://math.shinshu-u.ac.jp/~hanaki/as/data/as07> No. 2

An example of a wdrdg with seven vertices:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 0 & 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 0 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 & 0 \end{bmatrix}$$