GEN024 Final Exam 2017/8

Write your name and student ID number, and all your answers in the places provided on the separate answer sheets. (5pts× 20)

Part I.

1. Complete the truth tables of $p \Rightarrow (q \vee r)$ and $(\neg(p \wedge \neg q)) \vee r$, and determine whether they are logically equivalent.

2. Write $\boldsymbol{T}$ for true and $\boldsymbol{F}$ for false in the answer sheets.

$$\left\{ \begin{array}{lcl} x - y & = & 3 \\ 2x - y + z & = & 0 \\ 6x - 2y + 3z & = & 1 \end{array} \right., \ A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 6 & -2 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -3 & 1 \\ -2 & 4 & -1 \end{bmatrix}.$$

   (a) The matrix A is the augmented matrix of the system of linear equations above.

   (b) The matrix B is the inverse matrix of A.

   (c) The rank of A is 3.

   (d) The rank of B is 3.

   (e) There are infinitely many solutions to the system of linear equations above.

Part II. Write the answers of the following in the places provided in answer sheets. Show work! If you apply a proposition, state the number or the statement clearly.

3. Let $p(x)$ be a polynomial satisfying $p(0) = 0$, $p(5) = 3$, $p(10) = 0$ and $p(15) = 1$. Write two such polynomials $p(x)$, one with degree at most three and the other with degree exactly four.

4. Let $f(x) = 3x^3 - 26x^2 + 75x - 54 = q(x)(x-3) + r = c_3(x-3)^3 + c_2(x-3)^2 + c_1(x-3) + c_0$. Find a polynomial $q(x)$, constants r and c_3, c_2, c_1, c_0.

5. Let $f(x)$ be a polynomial in Problem 4. (a) Show that there is a zero between 0 and 3, i.e., there is c with $0 < c < 3$ such that $f(c) = 0$. (b) Determine whether $c \geqq 2$ or $c < 2$.

6. Let $f(x)$ be a polynomial in Problem 4. Determine whether $f(x)$ is increasing, decreasing at $x = 3$ or $f(3)$ is a local maximum or a local minimum. Why?

7. Find the limit $\displaystyle\lim_{x \to 3} \frac{2x^3 - 13x^2 + 24x - 9}{x^3 - 11x^2 + 39x - 45}$.

8. Find the limit $\displaystyle\lim_{x \to 0} \frac{\log(x+1) - x}{x^2}$. Note that $\log 1 = 0$.

9. Find the derivative of $\dfrac{1}{(2x-3)^7}$.

10. Find the derivative of $x^2 e^{x^3}$.

11. Find the indefinite integral $\int \left(\frac{1}{2x^2} + 1 - 3\sqrt{x} \right) dx$.

12. Find the indefinite integral $\int \frac{1}{(2x-3)^8} dx$.

13. Find the definite integral $\int_0^1 e^{-3x} dx$.

14. Find the derivative of $F(x)$, where $F(x) = \int_0^x e^{-t^4} dt$.

Part III. Write your answers on the answer sheets.

$$B = \begin{bmatrix} 2 & 0 & -6 & 10 & -3 & 3 & -10 \\ 0 & 1 & 2 & 0 & -1 & -2 & -8 \\ 0 & -2 & -4 & 1 & 2 & 4 & 17 \\ 1 & 0 & -3 & 5 & -2 & 2 & -9 \end{bmatrix} \to\to\to C = \begin{bmatrix} 1 & 0 & -3 & 5 & -2 & 2 & -9 \\ 0 & 1 & 2 & 0 & -1 & -2 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 8 \end{bmatrix}$$

15. The matrix C is obtained from the matrix B by performing elementary row operations three times. Write them in order using notation $[i, j; c]$ (add c times row j to row i), $[i, j]$ (interchange row i and row j), $[i; c]$ (multiply every entry in row i by c).

16. Find a 4×4 matrix T satisfying $TB = C$.

17. Find the inverse of the matrix T in Problem 16 above.

18. Suppose the matrix B above is an augmented matrix of a system of linear equations with unknowns $x_1, x_2, x_3, x_4, x_5, x_6$. Find (a) the reduced row echelon form of B, and (b) the solutions of the system.

19. Let $f(x) = (x+1)e^x$. Find the equation of the tangent line to $y = f(x)$ at $x = 0$.

20. Apply Proposition 7.4, and solve the differential equation below. (a) identify $h(x)$, $g(y)$, and (b) find $G(y)$, $H(x)$ and write the equation $G(y) = H(x) + C$. Finally (c) solve it with an initial condition below for $y = f(x)$.

$$y' = \frac{dy}{dx} = 8x^3\sqrt{y}, \quad y(0) = f(0) = 1.$$

$$\boxed{y' = \frac{h(x)}{g(y)}, H'(x) = h(x), G'(y) = g(y) \Rightarrow G(y) = H(x) + C.}$$

GEN024 FINAL 2017/8 Answer Sheets

ID#: **Name:**

Part I-1.

p	q	r	$p \Rightarrow (q \vee r)$	$(\neg(p \wedge \neg q)) \vee r$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Are these logically equivalent?

2.

(a)	(b)	(c)	(d)	(e)

No.	PTS.
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	
11.	
12.	
13.	
14.	
15.	
16.	
17.	
18.	
19.	
20.	
Total	

Message: Did you enjoy mathematics, or did you suffer a lot? I appreciate your feedbacks on the following. 数学少しは楽しめましたか。苦しんだ人もいるかな。以下のことについて書いて下さい。(If you don't want your message to be posted, write "Do Not Post."「HP 掲載不可」は明記の事。)

(A) About this class, especially on improvements. この授業について。改善点など何でもどうぞ。

(B) About the education at ICU, especially on improvements. Any comments concerning ICU are welcome. ICU の教育一般について。改善点など、ICU に関すること何でもどうぞ。

Part II.

3. degree at most three:

   degree four:

4. Let $f(x) = 3x^3 - 26x^2 + 75x - 54 = q(x)(x-3) + r = c_3(x-3)^3 + c_2(x-3)^2 + c_1(x-3) + c_0$. Find a polynomial $q(x)$, constants r and c_3, c_2, c_1, c_0.

5. Let $f(x)$ be a polynomial in Problem 4.
   (a) Show that there is a zero between 0 and 3, i.e., there is c with $0 < c < 3$ such that $f(c) = 0$.

   (b) Determine whether $c \geqq 2$ or $c < 2$.

6. Let $f(x)$ be a polynomial in Problem 4. Determine whether $f(x)$ is increasing, decreasing at $x = 3$ or $f(3)$ is a local maximum or a local minimum. Why?

7. Find the limit $\lim_{x \to 3} \frac{2x^3 - 13x^2 + 24x - 9}{x^3 - 11x^2 + 39x - 45}$.

8. Find the limit $\lim_{x \to 0} \frac{\log(x+1) - x}{x^2}$. Note that $\log 1 = 0$.

9. Find the derivative of $\frac{1}{(2x-3)^7}$.

10. Find the derivative of $x^2 e^{x^3}$.

11. Find the indefinite integral $\int \left(\frac{1}{2x^2} + 1 - 3\sqrt{x} \right) dx$.

12. Find the indefinite integral $\int \frac{1}{(2x-3)^8} dx$.

13. Find the definite integral $\int_0^1 e^{-3x} dx$.

14. Find the derivative of $F(x) = \int_0^x e^{-t^4} dt$.

Part III.

15. The matrix C is obtained from the matrix B by performing elementary row operations three times. Write them in order using notation $[i, j; c]$ (add c times row j to row i), $[i, j]$ (interchange row i and row j), $[i; c]$ (multiply every entry in row i by c).

16. Find a 4×4 matrix T satisfying $TB = C$.

17. Find the inverse of the matrix T in Problem 16 above.

18. Suppose the matrix B above is an augmented matrix of a system of linear equations with unknowns $x_1, x_2, x_3, x_4, x_5, x_6$. Find (a) the reduced row echelon form of B, and (b) the solutions of the system.

(a)

(b)

19. Let $f(x) = (x + 1)e^x$. Find the equation of the tangent line to $y = f(x)$ at $x = 0$.

20. Apply Proposition 7.4, and solve the differential equation below. (a) identify $h(x)$, $g(y)$, and (b) find $G(y)$, $H(x)$ and write the equation $G(y) = H(x) + C$. Finally (c) solve it with an initial condition below for $y = f(x)$.

$$y' = \frac{dy}{dx} = 8x^3\sqrt{y}, \quad y(0) = f(0) = 1.$$

(a)

(b)

(c)

Solutions to GEN024 FINAL 2017/8

Part I.

1.

p	q	r	p	$\Rightarrow$	$(q$	$\vee$	$r)$	$(\neg$	$(p$	$\wedge$	$\neg q))$	$\vee$	r
T	T	T	T	$\boldsymbol{T}$		T		T		F		$\boldsymbol{T}$	T
T	T	F	T	$\boldsymbol{T}$		T		T		F		$\boldsymbol{T}$	F
T	F	T	T	$\boldsymbol{T}$		T		F		T		$\boldsymbol{T}$	T
T	F	F	T	$\boldsymbol{F}$		F		F		T		$\boldsymbol{F}$	F
F	T	T	T	$\boldsymbol{T}$		T		T		F		$\boldsymbol{T}$	T
F	T	F	T	$\boldsymbol{T}$		T		T		F		$\boldsymbol{T}$	F
F	F	T	T	$\boldsymbol{T}$		T		T		F		$\boldsymbol{T}$	T
F	F	F	T	$\boldsymbol{T}$		F		T		F		$\boldsymbol{T}$	F

Are these logically equivalent? : **YES**

2.

(a)	(b)	(c)	(d)	(e)
$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$

(a) A is the coefficient matrix. (b) $AB = BA = I$. (c), (d) By Invertible Matrix Theorem (IMT) and (b), rank A = rank $B = 3$. (e) By IMT, the system has a unique solution.

Part II.

3. degree at most three:

$$p(x) = 3\frac{x(x-10)(x-15)}{(5)(5-10)(5-15)} + \frac{x(x-5)(x-10)}{15(15-5)(15-10)} = \frac{x(x-10)(2x-25)}{375}.$$

degree four: Let $p(x)$ be the one with degree at most three above. Then the following is a polynomial of degree four satisfying the conditions. See Proposition 4.2.

$$p(x) + x(x-5)(x-10)(x-15).$$

4. Let $f(x) = 3x^3 - 26x^2 + 75x - 54 = q(x)(x-3) + r = c_3(x-3)^3 + c_2(x-3)^2 + c_1(x-3) + c_0$. Find a polynomial $q(x)$, constants r and c_4, c_3, c_2, c_1, c_0.

Soln. $f(x) = 3x^3 - 26x^2 + 75x - 54 = (3x^2 - 17x + 24)(x-3) + 18 = 3(x-3)^3 + (x-3)^2 + 18$. Hence $q(x) = 3x^3 - 17x + 24$, $r = c_0 = 18$, $c_1 = 0$, $c_2 = 1$, $c_3 = 3$. Use synthetic division.

5. Let $f(x)$ be a polynomial in Problem 4. (a) Show that there is a zero between 0 and 3, i.e., there is c with $0 < c < 3$ such that $f(c) = 0$. (b) Determine whether $c \geqq 2$ or $c < 2$.

Soln. Since $f(x)$ is a polynomial, it is continuous in a closed interval $[0, 3]$. Since $f(0) = -49 < 0$ and $f(3) = 18 > 0$, by Intermediate Value Theorem, there is $c \in [0, 3]$ such that $f(c) = 0$. Since $f(2) = 16 > 0$, there is a zero between 0 and 2. Hence $c < 2$.

6. Let $f(x)$ be a polynomial in Problem 4. Determine whether $f(x)$ is increasing, decreasing at $x = 3$ or $f(3)$ is a local maximum or a local minimum. Why?

   Soln. $f'(x) = 9(x-3)^2 + 2(x-3)$, $f''(x) = 18(x-3) + 2$. Hence, $f'(3) = 0$, $f''(3) = 2$. Thus $f(3)$ is a local minimum by Second Derivative Test.

7. Find the limit $\lim\limits_{x\to 3} \dfrac{2x^3 - 13x^2 + 24x - 9}{x^3 - 11x^2 + 39x - 45}$.

   Soln. Since $2x^3-13x^2+24x-9 = (x-3)^2(2x-1)$ and $x^3-11x^2+39x-45 = (x-3)^2(x-5)$ by synthetic division,

   $$= \lim_{x\to 3} \frac{(x-3)^2(2x-1)}{(x-3)^2(x-5)} = \lim_{x\to 3} \frac{2x-1}{x-5} = \frac{5}{-2} = -\frac{5}{2}.$$

8. Find the limit $\lim\limits_{x\to 0} \dfrac{\log(x+1) - x}{x^2}$. Note that $\log 1 = 0$.

   Soln. Use l'Hôpital's rule.

   $$\lim_{x\to 0} \frac{\log(x+1) - x}{x^2} = \lim_{x\to 0} \frac{\frac{1}{x+1} - 1}{2x} = \lim_{x\to 0} \frac{-(x+1)^{-2}}{2} = -\frac{1}{2}.$$

9. Find the derivative of $\dfrac{1}{(2x-3)^7}$.

   Soln.

   $$\left(\frac{1}{(2x-3)^7}\right)' = ((2x-3)^{-7})' = -7(2x-3)^{-8}(2x-3)' = -14(2x-3)^{-8} = -\frac{14}{(2x-3)^8}.$$

10. Find the derivative of $x^2 e^{x^3}$.

    Soln. Since $(e^{x^3})' = e^{x^3}(x^3)' = 3x^2e^{x^3}$ by the Chain Rule, using the Product Rule we have

    $$(x^2e^{x^3})' = (x^2)'e^{x^3} + x^2(e^{x^3})' = 2xe^{x^3} + 3x^4e^{x^3} = x(2+3x^3)e^{x^3}.$$

11. Find the indefinite integral $\displaystyle\int \left(\frac{1}{2x^2} + 1 - 3\sqrt{x}\right) dx$.

    Soln. Since $\sqrt{x} = x^{\frac{1}{2}}$,

    $$\int \left(\frac{1}{2x^2} + 1 - 3\sqrt{x}\right) dx = \int \left(\frac{1}{2}x^{-2} + 1 - 3x^{\frac{1}{2}}\right) dx = -\frac{1}{2x} + x - 2x^{\frac{3}{2}} + C = -\frac{1}{2x} + x - 2\sqrt{x^3} + C.$$

12. Find the indefinite integral $\displaystyle\int \frac{1}{(2x-3)^8} dx$.

    Soln. By Problem 9, $-\frac{1}{14(2x-3)^7}$ is an antiderivative of $\frac{1}{(2x-3)^8}$. Hence

    $$\int \frac{1}{(2x-3)^8} dx = -\frac{1}{14(2x-3)^7} + C.$$

13. Find the definite integral $\displaystyle\int_0^1 e^{-3x} dx$.

    Soln.

    $$\int_0^1 e^{-3x} dx = \left[-\frac{1}{3}e^{-3x}\right]_0^1 = \frac{1}{3}(1 - e^{-3}) = \frac{1}{3}(1 - \frac{1}{e^3}).$$

14. Find the derivative of $F(x)$, where $F(x) = \int_0^x e^{-t^4}\,dt$.

Soln. Since $F(x)$ is an antiderivative of e^{-x^4}, $F'(x) = e^{-x^4}$ by the Fundamental Theorem of Calculus.

Part III.

15. The matrix C is obtained from the matrix B by performing elementary row operations three times. Write these operations in order using notation $[i, j; c]$, $[i, j]$, and $[i; c]$.

Soln.

$$B = \begin{bmatrix} 2 & 0 & -6 & 10 & -3 & 3 & -10 \\ 0 & 1 & 2 & 0 & -1 & -2 & -8 \\ 0 & -2 & -4 & 1 & 2 & 4 & 17 \\ 1 & 0 & -3 & 5 & -2 & 2 & -9 \end{bmatrix} \xrightarrow{[1,4]} \begin{bmatrix} 1 & 0 & -3 & 5 & -2 & 2 & -9 \\ 0 & 1 & 2 & 0 & -1 & -2 & -8 \\ 0 & -2 & -4 & 1 & 2 & 4 & 17 \\ 2 & 0 & -6 & 10 & -3 & 3 & -10 \end{bmatrix} \xrightarrow{[4,1:-2]}$$

$$\begin{bmatrix} 1 & 0 & -3 & 5 & -2 & 2 & -9 \\ 0 & 1 & 2 & 0 & -1 & -2 & -8 \\ 0 & -2 & -4 & 1 & 2 & 4 & 17 \\ 0 & 0 & 0 & 0 & 1 & -1 & 8 \end{bmatrix} \xrightarrow{[3,2;2]} C = \begin{bmatrix} 1 & 0 & -3 & 5 & -2 & 2 & -9 \\ 0 & 1 & 2 & 0 & -1 & -2 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 8 \end{bmatrix}$$

Hence the operations are $[1, 4], [4, 1; -2], [3, 2; 2]$ in this order. There are other solutions.

16. Find a 4×4 matrix T satisfying $TB = C$.

Soln. We obtain the matrix T by applying $[1, 4], [4, 1; -2], [3, 2; 2]$ to I in this order.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{[1,4]} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{[4,1:-2]} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{[3,2;2]} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} = T,$$

or

$$T = E(3, 2; 2)E(4, 1; 2)E(1, 4).$$

17. Find the inverse of the matrix T in 16 above.

Soln.

$$[T, I] = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{[3,2;-2]} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{[4,1:2]}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{[1,4]} \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \ T^{-1} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

18. Suppose the matrix B above is an augmented matrix of a system of linear equations with unknowns $x_1, x_2, x_3, x_4, x_5, x_6$. Find (a) the reduced row echelon form of B, and (b) the solutions of the system.

Soln. (a) Using $B \to\to\to C$, we start from C.

$$C = \begin{bmatrix} 1 & 0 & -3 & 5 & -2 & 2 & -9 \\ 0 & 1 & 2 & 0 & -1 & -2 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 8 \end{bmatrix} \xrightarrow{[1,3;-5]} \begin{bmatrix} 1 & 0 & -3 & 0 & -2 & 2 & -14 \\ 0 & 1 & 2 & 0 & -1 & -2 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 8 \end{bmatrix} \xrightarrow{[1,4;2]}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 & -2 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 8 \end{bmatrix} \xrightarrow{[2,4;1]} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 8 \end{bmatrix}, \quad \left\{ \begin{array}{rcl} x_1 & = & 3s+2, \\ x_2 & = & -2s+3t, \\ x_3 & = & s\text{: free}, \\ x_4 & = & 1, \\ x_5 & = & t+8, \\ x_6 & = & t\text{: free}. \end{array} \right.$$

19. Let $f(x) = (x+1)e^x$. Find the equation of the tangent line to $y = f(x)$ at $x = 0$.

Soln. Since $f'(x) = e^x + (x+1)e^x = (x+2)e^x$, $f(0) = 1$,

$$y = f(0) + f'(0)(x-0) = 2x+1.$$

20. Apply Proposition 7.4 and solve the differential equation below. (a) identify $h(x)$, $g(y)$, and (b) find $G(y)$, $H(x)$ and write the equation $G(y) = H(x) + C$. Finally (c) solve it with an initial condition below for $y = f(x)$.

$$y' = \frac{dy}{dx} = 8x^3\sqrt{y}, \quad y(0) = f(0) = 1.$$

Soln.

$$\frac{dy}{dx} = y' = 8x^3\sqrt{y} = \frac{8x^3}{y^{-1/2}}.$$

So one of the choices is $h(x) = 8x^3$ and $g(y) = y^{-1/2}$. Hence $H(x) = 2x^4$ and $G(y) = 2y^{1/2}$.

$$2y^{1/2} = 2x^4 + C, \; C = 2y(0)^{1/2} = 2.$$

Therefore, $y = (x^4+1)^2$.

$$y' = 2(x^4+1)(4x^3) = 8x^3(x^4+1) = 8x^3\sqrt{y}, \text{ and } y(0) = 1.$$