A System of Linear Equations:

$$\left\{\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \cdots\cdots\cdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}\right., \quad B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

where $x_1, x_2, \ldots, x_n$ are the unknowns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad B = [A, \boldsymbol{b}].$$

Then the equation can be written as $A\boldsymbol{x} = \boldsymbol{b}$. When $\boldsymbol{b} = \boldsymbol{0}$, the equation is called *homogeneous*. If the equation is homogeneous, $\boldsymbol{x} = \boldsymbol{0}$ is always a solution and is called the *trivial* solution; if there are other solutions, they are called *nontrivial solutions*.

Theorem 10.2 *Every matrix can be transformed to a reduced echelon form by performing a series of elementary row operations, and the reduced row echelon form of a matrix is unique.*

We obtained the reduced echelon form C of B by performing a sequence of elementary row operations. Let $P_1, P_2, \ldots, P_\ell$ be the corresponding elementary matrices, and $P = P_\ell P_{\ell-1} \cdots P_1$. Then

$$PB = C, \; P[A, \boldsymbol{b}] = [PA, P\boldsymbol{b}] \text{ and } PA\boldsymbol{x} = P\boldsymbol{b}.$$

$$\boldsymbol{x} \text{ is a solution to } PA\boldsymbol{x} = P\boldsymbol{b} \Leftrightarrow \boldsymbol{x} \text{ is a solution to } A\boldsymbol{x} = \boldsymbol{b}.$$

We call the number of leading 1's in the reduced echelon form of a matrix the *rank* of the matrix.

Theorem 10.3 *Let A be an $m \times n$ matrix and $A\boldsymbol{x} = \boldsymbol{b}$ is a system of linear equation expressed as a matrix equation. Let $B = [A, \boldsymbol{b}]$ be the augmented matrix and $C = [A_1, \boldsymbol{b}_1]$ is the reduced echelon form of B. Then*

$$\boldsymbol{x} \textit{ is a solution to } A\boldsymbol{x} = \boldsymbol{b} \Leftrightarrow \boldsymbol{x} \textit{ is a solution to } A_1\boldsymbol{x} = \boldsymbol{b}_1.$$

Moreover,

(i) *If* $\operatorname{rank}(A) \neq \operatorname{rank}(B)$, *the equation $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent.*

(ii) *If* $\operatorname{rank}(A) = \operatorname{rank}(B) = n$, *the equation $A\boldsymbol{x} = \boldsymbol{b}$ has the only one solution.*

(iii) *If* $\operatorname{rank}(A) = \operatorname{rank}(B) < n$, *the equation $A\boldsymbol{x} = \boldsymbol{b}$ has solutions with $n - \operatorname{rank}(A)$ parameters.*

Theorem 10.4 *Let A be the augmented matrix of a system of linear equations in n unknowns $x_1, x_2, \ldots, x_n$ and B the reduced echelon form of A. Suppose the leading 1's of B are at columns $i_1, i_2, \ldots, i_r$ and rows $j_1, j_2, \ldots j_{n-r}$ are those colums without leading 1's. Then the following hold.*

(i) *The system of linear equations is inconsistent if and only if $i_r = n+1$, i.e., a leading 1 is in the last column.*

(ii) *The system of linear equation has a unique solution if and only if it is consistent and $r = n$, i.e., $i_1 = 1, i_2 = 2, \ldots, i_r = i_n = n$.*

(iii) *If the system is consistent, the solutions are expressed with $n-r$ parameters $t_1 = x_{j_1}, t_2 = x_{j_2}, \ldots, t_{n-r} = x_{j_{n-r}}$.*

Theorem 10.5 (1.2.1) *A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions. In particular, if A is an $m \times n$ matrix, then a system of linear equation $A\boldsymbol{x} = \mathbf{0}$ has a nonzero solution.*

Example 10.1 The right hand side matrix is the reduced row echelon form of the left hand side matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & -4 \end{bmatrix} \to\to \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \begin{array}{rcc} x_1 & = & 1 - s - 4u, \\ x_2 & = & 3 + 2s - 3t + u, \\ x_3 & = & s, \\ x_4 & = & t, \\ x_5 & = & -2 + u, \\ x_6 & = & u. \end{array} \right. \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ -2 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \cdot \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

s, t and u are parameters.

Theorem 10.6 *Let $A\boldsymbol{x} = \boldsymbol{b}$ be an equation. Suppose $A\boldsymbol{x}_0 = \boldsymbol{b}$. Then*

$$\boldsymbol{x} \text{ is a solution to } A\boldsymbol{x} = \boldsymbol{b} \Leftrightarrow \boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{y} \text{ where } \boldsymbol{y} \text{ is a solution to } A\boldsymbol{x} = \mathbf{0}.$$

$\boldsymbol{x}_0$ is called a particular solution.

Proof. $A(\boldsymbol{x} - \boldsymbol{x}_0) = \boldsymbol{b} - \boldsymbol{b} = \mathbf{0}$. Hence $\boldsymbol{x} - \boldsymbol{x}_0$ is a solution $\boldsymbol{y}$ to $A\boldsymbol{x} = \mathbf{0}$. Hence $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{y}$. ∎

Exercise 10.1

1. If m, the number of equations, is less than or equal to n, the number of unknowns, the system of linear equations always has a solution.

2. Suppose m, the number of equations, is strictly less than n, the number of unknowns. Such a system of linear equations cannot have a unique solution.

3. Suppose $m < n$ and $b_i = a_{i1}$ for $i = 1, 2, \ldots, m$. Then there are infinitely many solutions.