Linear Algebra I Final Examination 1997

Write both of your ID number and name on each of the answer sheets. Do not forget to write the problem number as well.

1. True or false? Write ◯ for true and $\times$ for false in your answer sheet.

   For $(a) - (c)$ consider the following system of linear equations.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases}$$

   (a) If m, the number of equations, is less than or equal to n, the number of unknowns, the system of linear equations always has a solution.

   (b) Suppose m, the number of equations, is strictly less than n, the number of unknowns. Such a system of linear equations cannot have a unique solution.

   (c) Suppose $m < n$ and $b_i = a_{i1}$ for $i = 1, 2, \ldots, m$. Then there are infinitely many solutions.

   Let A be the following matrix for $(d) - (f)$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

   (d) Both AA^t and A^tA are square matrices.

   (e) If $m < n$ then the matrix AA^t is always invertible.

   (f) If A is a square matrix (, that is $m = n$), then the following holds.

$$\det(AA^t) = \det(A)^2.$$

2. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be as follows.

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 3 \\ 0 & -1 & 2 & 4 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   (a) Find rank$A$, the rank of the matrix A.

   (b) Determine whether the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ always has a solution for any b_1, b_2, b_3. Write your reason. If some condition for b_1, b_2, b_2 is required for the equation to have a solution, find the condition as well.

   (c) Find all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ when $b_1 = b_2 = b_3 = 0$.

   (d) Find all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ when $b_1 = b_2 = b_3 = 1$.

3. Find the following.

   (a) Let $\rho = (3, 7, 2, 1, 4, 6, 5)$ be a permutation of the set of integers $\{1, 2, 3, \ldots, 7\}$. Find $\ell(\rho)$, the total number of inversions and the signature $\mathrm{sgn}(\rho)$.

(b) $\begin{vmatrix} 1 & 3 & 5 \\ 3 & 7 & 11 \\ 2 & 4 & 6 \end{vmatrix}$

(c) $\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix}$

(d) $\begin{vmatrix} 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & e & 0 & 0 & 0 \end{vmatrix}$

4. For each pair $i, j = 1, 2, 3$, and a real number c, let $P(i, j; c) = I + c \cdot E_{i,j}$ be an elementary matrix of size 3. Here I denotes the identity matrix of size 3 and $E_{i,j}$ denotes the matrix unit, whose entries are zero except the (i, j) entry which is one. Express the following matrix as a product of certain number of $P(i, j; c)$'s.

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

5. Let $f(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ be a polynomial in t of degree at most 3 satisfying the following:

$$f(-1) = 2,\ f(1) = 5,\ f(3) = -1,\ f(5) = -3.$$

   (a) In order to determine the coefficients of the polynomial $f(t)$, consider a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. Determine the coefficient matrix A, and the vectors $\boldsymbol{x}$ and $\boldsymbol{b}$. You do not need to solve the equation.

   (b) Find $|A|$, the determinant of the matrix A in the previous problem. If you apply a formula, write the formula itself as well.

   (c) Express $|A| \cdot c_2$ by a 4×4 determinant. You do not need to determine the value of the determinant.

   (d) Write the inverse of the matrix A using the adjoint matrix of A. You do not need to determine the values of the determinants which appear in the entries of the adjoint matrix.

6. For a square matrix $X = [x_{i,j}]$ of size 3, let trace(X) denote the sum of the diagonal entries of X, that is

$$\mathrm{trace}(X) = \sum_{i=1}^{3} x_{i,i} = x_{1,1} + x_{2,2} + x_{3,3}.$$

For square matrices A, B of size 3, let $\ll A, B \gg = \mathrm{trace}(AB^t)$.

   (a) Show that $\ll A, B \gg = \ll B, A \gg$.

   (b) If $\ll A, A \gg = 0$ then A is a zero matrix, that is a square matrix of size 3 with all entries 0. (Here we assume that all entries of A are real numbers.)

Hiroshi Suzuki @ Department of Mathematics