Problems of Distance-Regular Graphs

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Chapter 1

Introduction

In this chapter we will introduce some basic definitions and notational convention of graphs.

1.1 Graphs

A *graph* Γ is a pair (X, R) consisting of a nonempty set X, referred to as the *vertex set* of Γ, and a set R of unordered pairs of distinct vertices of X, referred to as the *edge set* of Γ. We will use xy rather than $\{x, y\}$ to denote an edge. If xy is an edge, then we say that x and y are *adjacent*, denoted by $x \sim y$. Throughout this note, we always assume that all graphs $\Gamma = (X, R)$ are finite, i.e., $|X| < \infty$.

A graph Γ on n vertices is said to be a *complete graph* if $x \sim y$ for any two distinct vertices of Γ, denoted by K_n. A graph with no edges is called *empty*.

A *subgraph* of a graph $\Gamma = (X, R)$ is a graph $\Gamma' = (X', R')$ such that

$$X' \subseteq X, R' \subseteq R.$$

A subgraph $\Gamma' = (X', R')$ of Γ is an *induced subgraph* on X' if two vertices of Γ' are adjacent in Γ' if and only if they are adjacent in Γ.

Definition 1.1.1 Let $\Gamma = (X, R)$ be a graph, and let u and v be vertices of Γ.

(i) A *walk* of length l from u and v in Γ is a finite sequence of vertices $(u = w_0, w_1, \cdots, w_l = v)$ of Γ such that

$$u = w_0 \sim w_1 \sim \cdots \sim w_l = v.$$

(ii) A walk $(w_0, w_1, \cdots, w_l)$ is said to be a *path* of length l if $w_{i-1} \neq w_{i+1}$ for any $i = 1, 2, \ldots, l-1$. Moreover a path $(w_0, w_1, \cdots, w_l)$ is said to be *reduced* if $w_{i-1} \not\sim w_{i+1}$ for every $i = 1, \cdots, l-1$.

(iii) A path $(w_0, w_1, \cdots, w_l)$ is said to be a *cycle* or a *circuit* if $w_0 = w_l$. The length of a shortest cycle in graph Γ is the *girth* $g(\Gamma)$ of Γ.

(iv) The *distance* between u and v is the length of a shortest path connecting u and v in Γ, denoted by $\partial(u, v)$. By convention $\partial(u, v) = \infty$ if there is no path connecting u and v.

(v) The *diameter* of Γ is the maximal distance between any two vertices of Γ, denoted by $d = d(\Gamma)$. That is $d = \max\{\partial(x, y) \mid x, y \in \Gamma\}$.

(vi) A graph Γ is said to be *connected* if, for any two vertices x and y in Γ, there is a path connecting x and y, i.e., $d(\Gamma) < \infty$.

(vii) Let $x \in X$. The *valency (or degree)* $k(x)$ of x is the number of vertices adjacent to x in Γ. If $k := k(y)$ is a constant for all $y \in \Gamma$, then Γ is said to be *regular* of valency k.

(viii) $\Gamma = (X, R)$ is said to be *bipartite* if there is a nontrivial bipartition $X = X^+ \cup X^-$ of vertices such that both induced subgraphs on X^+ and X^- are empty.

Clearly, the distance function satisfies the following triangular inequality:

$$\partial(u, v) \leq \partial(u, w) + \partial(w, v), \text{ for all vertices } u,\ v,\ w \in X. \tag{1.1}$$

Let $\Gamma = (X, R)$ and $\Gamma' = (X', R')$ be two graphs. A bijection from X to X' is an *isomorphism* from Γ to Γ' if $xy \in R$ if and only if $\sigma(x)\sigma(y) \in R'$. An isomorphism from Γ to Γ is called an *automorphism* of Γ. The set of all automorphisms of Γ with the operation of composition is called the *automorphism group* of Γ, denoted by $\mathrm{Aut}(\Gamma)$. Let σ be an isomorphism from Γ to Γ', and let x be a vertex of Γ. Then the valency of x is equal to that of $\sigma(x)$. We do not distinguish between two isomorphic graphs.

1.2 Exercises

1. Show that the distance function satisfies the triangular inequality:

$$\partial(u, v) \leq \partial(u, w) + \partial(w, v), \text{ for all vertices } u,\ v,\ w \in X.$$

2. Show that every automorphism of a connected graph $\Gamma = (X, R)$ preserves distance, that is for any vertices u and $v \in X$, and an automorphism $\sigma \in \mathrm{Aut}(\Gamma)$,

$$\partial(u, v) = \partial(\sigma(u), \sigma(v)).$$

3. Let $u,\ v,\ x,\ y \in X$, and let $\sigma \in \mathrm{Aut}\Gamma$ such that $(\sigma(u), \sigma(v)) = (x, y)$. Show that

$$|\Gamma_i(u) \cap \Gamma_j(v)| = |\Gamma_i(x) \cap \Gamma_j(y)| \text{ for all } i,\ j.$$

4. A connected graph of diameter d is bipartite if and only if Γ has no circuits of odd length.

Chapter 2

Basic Theory of Distance-Regular Graphs

2.1 Distance-Transitive and Distance-Regular Graphs

Let $\Gamma = (X, R)$ be a connected graph of diameter d. For $i \in \{0, 1, \ldots, d\}$, let

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\} \text{ and } \Gamma(x) = \Gamma_1(x).$$

Let u, v, x, $y \in X$ and consider the following conditions:

(i) There exists $\sigma \in \mathrm{Aut}\Gamma$ such that $(\sigma(u), \sigma(v)) = (x, y)$.

(ii) $\partial(u, v) = \partial(x, y)$.

(iii) $|\Gamma_i(u) \cap \Gamma_j(v)| = |\Gamma_i(x) \cap \Gamma_j(y)|$ for all i, j.

The condition (i) always implies (ii) and (iii). If (ii) implies (i) for all vertices u, v, x, $y \in X$, then Γ is called a *distance-transitive graph* (dtg). Hence, if Γ is a distance-transitive graph, then condition (ii) implies (iii).

Definition 2.1.1 Let $\Gamma = (X, R)$ be a connected graph of diameter d. Then Γ is said to be *distance regular* whenever for all integers $h, i, j (0 \leq h, i, j \leq d)$ and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{i,j} = |\{x \in X \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

is independent of the choice of x and y.

By our observation above, distance-transitive graphs are distance regular, but the converse is not generally true.

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d. The numbers

$$p^h_{i,j}, 0 \leq i, j, h \leq d$$

are called the *intersection numbers* of Γ. For $i \in \{0, 1, \ldots, d\}$ let $k_i = p^0_{i,i}$. Then $k_i = |\Gamma_i(u)|$ for every $u \in X$. In particular, $k = k_1$ is the degree of each vertex $u \in X$ and Γ is k-regular.

Lemma 2.1.1 *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d. Then the following hold.*

(i) $p^h_{i,j} = p^h_{j,i}$ *for* $0 \leq h, i, j \leq d$.

(ii) $p^h_{i,j} = 0$ *if* $h > i + j$, $j > h + i$ *or* $i > h + j$.

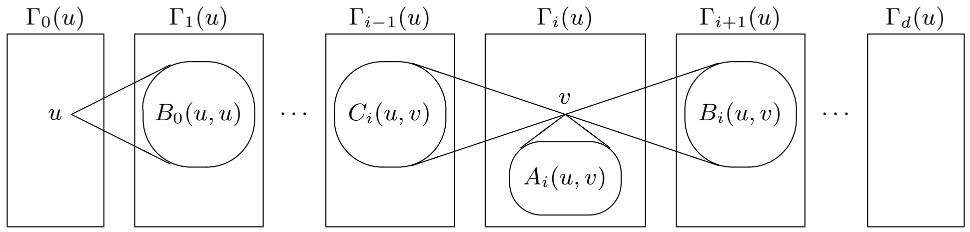
(iii) $p^h_{i,j} \neq 0$, *if* $h = i + j, j = h + i$ *or* $i = h + j$.

Proof. It is obvious from Definition 2.1.1 and (1.1).

Let $\Gamma = (X, R)$ be a distance-regular graph. For vertices u, v of Γ with $\partial(u, v) = i$, let

$$\begin{aligned} C(u,v) &= C_i(u,v) = \Gamma_{i-1}(u) \cap \Gamma(v), \\ A(u,v) &= A_i(u,v) = \Gamma_i(u) \cap \Gamma(v), \\ B(u,v) &= B_i(u,v) = \Gamma_{i+1}(u) \cap \Gamma(v). \end{aligned}$$

Then $\Gamma(v) = C_i(u, v) \cup A_i(u, v) \cup B_i(u, v)$ for u, v at distance i.



Let c_i, a_i and b_i denote the cardinalities of the sets $C(u, v)$, $A(u, v)$ and $B(u, v)$ as $\partial(u, v) = i$, respectively. Note that

$$c_i = p^i_{i-1,1}, \ a_i = p^i_{i,1}, \ \text{ and } b_i = p^i_{i+1,1},$$

and $k = |\Gamma(v)| = c_i + a_i + b_i$. It is easy to check that $c_0 = a_0 = b_d = 0$, $c_1 = 1$ and $b_0 = k$.

These numbers c_i, a_i and b_i play important roles in the study of distance-regular graphs. The following is called the *intersection array* of Γ, where $d = d(\Gamma)$.

$$\iota(\Gamma) = \left\{ \begin{array}{ccccccc} * & c_1 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_i & \cdots & b_{d-1} & * \end{array} \right\}.$$

A distance-regular graph of valency 2 is nothing but an ordinary polygon with at least 3 vertices. We always assume the valency k is at least 3 in the following.

2.2 Intersection Diagrams

We introduce intersection diagrams as a tool to investigate the structures of distance-regular graphs.

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d. Let x, y be vertices of Γ with $\partial(x, y) = h$. Set

$$D^i_j = D^i_j(x, y) = \Gamma_i(x) \cap \Gamma_j(y).0 \leq i, j \leq d.$$

An *intersection diagram* of rank h is the collection $\{D^i_j\}_{i,j}$ with lines between D^i_j's and D^s_t's. We draw a line

$$D^i_j \;\text{———}\; D^s_t$$

if there is possibility of existence of edges. We erase the line when we know there is no edge between D^i_j and D^s_t.

For sets of vertices A and B, let $e(A, B)$ denote the number of edges between A and B. If $A = \{u\}$, $e(u, B) := e(\{u\}, B)$. We sometimes write

$$D^i_j \overset{p}{\text{———}} D^s_t$$

in order to indicate that $e(x, D^s_t) = p$ for every $x \in D^i_j$,

$$\overset{q}{D^i_j} \text{———} D^s_t,$$

when $e(x, D^i_j) = q$ for every $x \in D^i_j$. But it does not hold for any distance-regular graph.

The following are straightforward and useful to determine the shape of intersection diagrams. (See Figure 2.1.)

(i) $p^h_{i,j} = |D^i_j|$ and $|D^i_j| = |D^j_i|$.

(ii) $D^i_j = \emptyset$, if $h > i + j$, $i > j + h$ or $j > h + i$.

(iii) $D^i_j \neq \emptyset$ if $h = i + j, i = j + h$ or $j = h + i$.

(iv) There is no edge between D^i_j and D^s_t if $|i - s| > 1$ or $|j - t| > 1$.

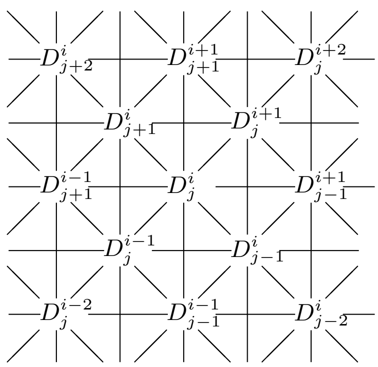


Figure 2.1: Intersection Diagram

Intersection diagrams of rank 1 is very important. Let x and y be two adjacent vertices of Γ, and let $D^i_j = \Gamma_i(x) \cap \Gamma_j(y)$. By (1.1), if $D^i_j \neq \emptyset$, then $1 \leq i + j, i \leq 1 + j$, or $j \leq 1 + i$. Hence, the intersection diagram with respect to adjacent vertices x, y is shown in Figure 2.2.

Lemma 2.2.1 *Let Γ be a distance-regular graph and let x, y be adjacent vertices of Γ. Set $D^i_j = \Gamma_i(x) \cap \Gamma_j(y)$. Let $u \in D^{i+1}_i$. Then the following hold.*

(i) $c_i = e(u, D^i_{i-1}) \leq e(u, D^i_{i-1} \cup D^i_i \cup D^i_{i+1}) = c_{i+1}$.
In particular, $c_i = c_{i+1}$ if and only if $e(u, D^i_i \cup D^i_{i+1}) = 0$.

(ii) $b_i = e(u, D^{i+2}_{i+1} \cup D^{i+1}_{i+1} \cup D^i_{i+1}) \geq e(u, D^{i+2}_{i+1}) = b_{i+1}$.
In particular, $b_i = b_{i+1}$ if and only if $e(u, D^{i+1}_{i+1} \cup D^i_{i+1}) = 0$.

Proof. It is easy to see from the diagram. ∎

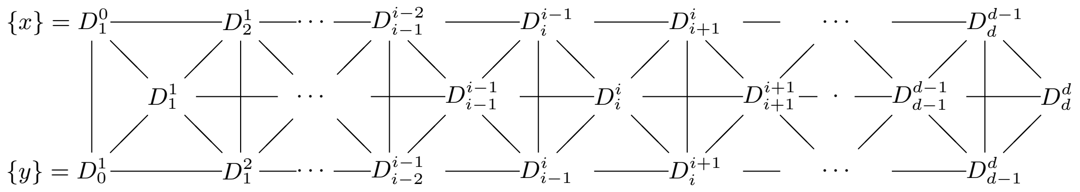


Figure 2.2: Intersection Diagram of Rank 1

2.3 A. A. Ivanov's Bound

As an application of intersection diagrams, we prove an important result first shown by A. A. Ivanov in 1983 [153]. Recall that by Lemma 2.2.1

$$1 = c_1 \le c_2 \le \cdots \le c_d \le k,\ k = b_0 > b_1 \ge \cdots \ge b_{d-1} \ge 1.$$

Let $\ell(c,a,b) = |\{i \mid (c_i, a_i, b_i) = (c,a,b)\}|$ and $r(\Gamma) = \ell(c_1, a_1, b_1)$.

Theorem 2.3.1 *Let Γ be a distance-regular graph. Suppose that $s \ge 1$ and $(c_s, a_s, b_s) \ne (c_{s+1}, a_{s+1}, b_{s+1})$. Then $\ell(c_{s+1}, a_{s+1}, b_{s+1}) \le s+1$. In particular, $d = d(\Gamma) < 2^{k-1}(r+1)$, where $r = r(\Gamma)$.*

Proof. Assume that $\ell(c_{s+1}, a_{s+1}, b_{s+1}) \ge s+2$, i.e.,

$$(c_{s+1}, a_{s+1}, b_{s+1}) = (c_{s+2}, a_{s+2}, b_{s+2}) = \cdots = (c_{2s+2}, a_{2s+2}, b_{2s+2}).$$

Let x and y be two adjacent vertices. Let $D_j^i = \Gamma_i(x) \cap \Gamma_j(y)$. By Lemma 2.2.1 the intersection diagram of rank 1 has the shape in Figure 2.3.

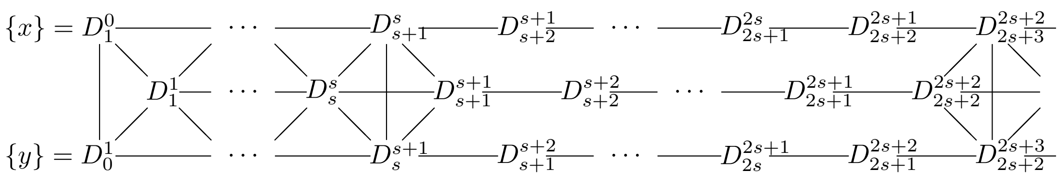


Figure 2.3: Rank 1 diagram with respect to x, y

Since $(c_s, a_s, b_s) \ne (c_{s+1}, a_{s+1}, b_{s+1})$, $e(D_s^{s+1}, D_s^s \cup D_{s+1}^s \cup D_{s+1}^{s+1}) \ne 0$. Let

$$u_i \in D_{s+i-1}^{s+i},\ i = 1, 2, \ldots, s+1,\ u_1 \sim u_2 \sim \cdots \sim u_{s+1}.$$

Case 1. $e(D_{s+1}^s, D_s^{s+1}) \ne 0$.
We may assume that there is $u_0 \in \Gamma(u_1) \cap D_{s+1}^s$. Then $\partial(u_0, u_{s+1}) = s+1$ and

$$(\Gamma(u_0) \cap D_s^{s-1}) \cup (\Gamma(u_0) \cap D_{s+2}^{s+1}) \subset \Gamma(u_0) \cap \Gamma_{s+2}(u_{s+1}).$$

Hence $c_s + b_{s+1} \le b_{s+1}$. This is a contradiction. Thus $e(D_{s+1}^s, D_s^{s+1}) = 0$.

Case 2. $e(D_s^{s+1}, D_{s+1}^{s+1}) \ne 0 = e(D_{s+1}^s, D_s^{s+1})$.

We may assume that there is $u_0 \in \Gamma(u_1) \cap D^{s+1}_{s+1}$. Since

$$e(u_0, D^{s+1}_s) + e(u_0, D^s_s) = c_{s+1} = e(u_0, D^s_{s+1}) + e(u_0, D^s_s)$$

and $u_1 \in \Gamma(u_0) \cap D^{s+1}_s$, $e(u_0, D^s_{s+1}) \neq 0$. Thus

$$(\Gamma(u_0) \cap D^s_{s+1}) \cup (\Gamma(u_0) \cap D^{s+2}_{s+2}) \subset \Gamma(u_0) \cap \Gamma_{s+2}(u_{s+1}).$$

We have $1 + b_{s+1} \leq b_{s+1}$, a contradiction.

Case 3. $e(D^{s+1}_s, D^s_s) \neq 0 = e(D^{s+1}_s, D^{s+1}_{s+1} \cup D^s_{s+1})$.

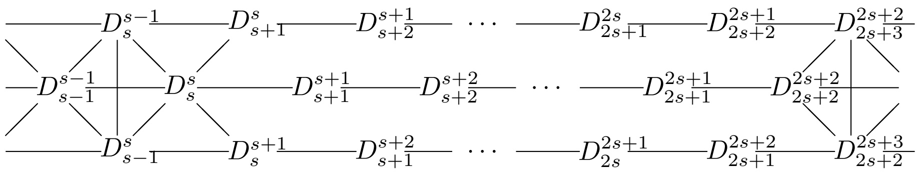


Figure 2.4: The case $b_s = b_{s+1}$

Let $u_0 \in \Gamma(u_1) \cap D^s_s$. Then

$$e(u_0, D^{s-1}_s \cup D^{s-1}_{s-1}) + e(u_0, D^s_{s+1} \cup D^{s+1}_{s+1}) \leq b_{s+1}.$$

Hence $c_s + b_s \leq b_{s+1}$, a contradiction.

Therefore, we have $\ell(c_{s+1}, a_{s+1}, b_{s+1}) \leq s + 1$ as desired.

Let $m = \min\{i \mid c_i > b_i\}$. Then by Exercise 9 in this chapter, $d < 2m$. On the other hand, since

$$k - 2 \geq b_1 - c_1 \geq b_2 - c_2 \geq \cdots \geq b_{m-1} - c_{m-1} \geq 0,$$

there are at most $k - 1$ different columns in the intersection array besides the 0-th column up to $(m-1)$-th column. Now by the result we just proved above, we have

$$m \leq r + (r+1) + 2(r+1) + \cdots + 2^{k-3}(r+1) + 1 = 2^{k-2}(r+1).$$

Therefore, we have $d < 2m \leq 2^{k-1}(r+1)$. ■

2.4 $J(n,d)$, $H(d,q)$ and O_k

Johnson graph $J(n,d)$**:** Let V be a set of size n. Let $X = \{\alpha \subset V \mid |\alpha| = d\}$. Two vertices α and $\beta \in X$ are adjacent if and only if

$$|\alpha \cap \beta| = d - 1.$$

Then this graph is a distance-regular graph and is called the *Johnson graph* $J(n,d)$. Note that $J(n,d) \simeq J(n, n-d)$, and so sometimes we assume that $n \geq 2d$.

Hamming graph $H(d,q)$**:** Let Q be a set of size $q > 1$. Let $X = Q^d$, i.e., the direct product of d copies of Q. Two vertices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in X$ are adjacent if and only if

$$|\{i \mid \alpha_i \neq \beta_i,\ 1 \leq i \leq d\}| = 1.$$

Then this graph is a distance-regular graph and is called the *Hamming graph* $H(d,q)$. When $q = 2$ this graph is often called a *d-cube* or a *hypercube*.

Odd graph O_k: Let V be a set of size $2k-1$. Let $X = \{\alpha \subset V \mid |\alpha| = k-1\}$. Two vertices α and $\beta \in X$ are adjacent if and only if

$$|\alpha \cap \beta| = 0.$$

Then this graph is a distance-regular graph and is called the *Odd graph O_k*.

2.5 Exercises

1. Write the intersection arrays of ordinary polygons C_n.
2. The vertices and edges (line segments) (called 1-skeleton) of five Platonic solids define distance-regular graphs. Check this fact and determine the intersection arrays of the Tetrahedron, the Octahedron, the Dodecahedron and the Icosahedron.

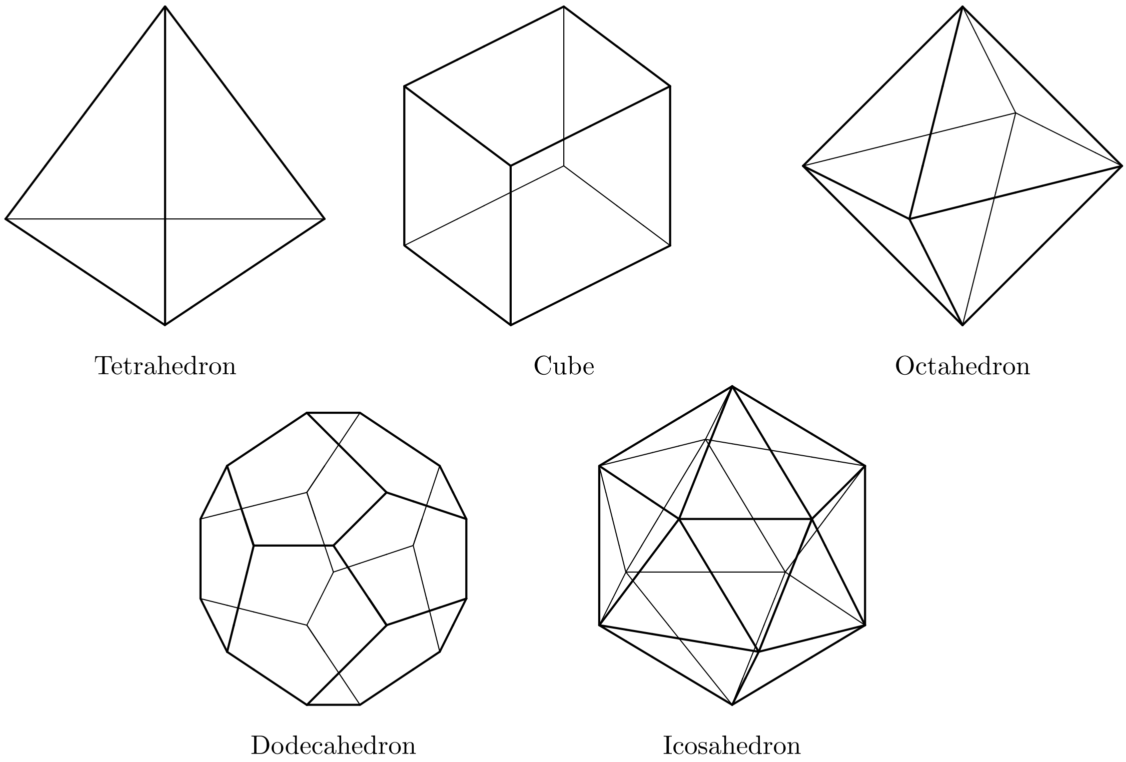


Figure 2.5: Platonic Solids

3. Let Γ be the Johnson graph $J(n, d)$. Prove the following.
   (a) For any $r \geq 1$, $\partial(\alpha, \beta) = r$ if and only if $|\alpha \cap \beta| = d - r$. In particular the diameter of $J(n, d)$ is d.
   (b) Γ is distance-transitive.
4. Write down the intersection array of the Johnson graph $J(n, d)$.
5. Let Γ be the Hamming graph $H(d, q)$. Prove the following.

(a) For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in X$, $\partial(\alpha, \beta) = r$ if and only if $|\{j \mid \alpha_j \neq \beta_j,\ 1 \leq j \leq d\}| = r$. In particular the diameter of $H(d, q)$ is d.

(b) Γ is distance-transitive.

6. Write down the intersection array of the Hamming graph $H(d, q)$.

7. Let Γ be the Odd graph O_k. Prove the following.

   (a) $\partial(\alpha, \beta) = 2r$ if and only if $|\alpha \cap \beta| = (k-1) - r$, moreover $\partial(\alpha, \beta) = 2r + 1$ if and only if $|\alpha \cap \beta| = r$. In particular the diameter of O_k is $k - 1$.

   (b) Γ is distance-transitive.

8. Write down the intersection array of the Odd graph O_k.

9. For intersection numbers show

$$k_i p^i_{j,h} = k_j p^j_{h,i} = k_h p^h_{i,j}$$

   for all $0 \leq h, i, j \leq d$.

10. For each $i = 0, 1, \ldots, d-1$, show

$$b_i k_i = c_{i+1} k_{i+1}$$

   in two ways.

11. Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d. Let x and y be vertices of Γ at distance h and let $D^i_j = \Gamma_i(x) \cap \Gamma_j(y)$. Show that for each vertex $u \in D^i_j$ the following hold.

   (a) $|D^i_j| = p^h_{i,j}$.

   (b) $D^i_{h+i} \neq \emptyset$ if $0 \leq i \leq d - h$ and $D^i_{h-i} \neq \emptyset$ if $0 \leq i \leq h$.

   (c) $c_i = e(u, D^{i-1}_{j+1}) + e(u, D^{i-1}_j) + e(u, D^{i-1}_{j-1})$. $c_j = e(u, D^{i+1}_{j-1}) + e(u, D^i_{j-1}) + e(u, D^{i-1}_{j-1})$.

   (d) $a_i = e(u, D^i_{j+1}) + e(u, D^i_j) + e(u, D^i_{j-1})$. $a_j = e(u, D^{i+1}_j) + e(u, D^i_j) + e(u, D^{i-1}_j)$.

   (e) $b_i = e(u, D^{i+1}_{j+1}) + e(u, D^{i+1}_j) + e(u, D^{i+1}_{j-1})$. $b_j = e(u, D^{i+1}_{j+1}) + e(u, D^i_{j+1}) + e(u, D^{i-1}_{j+1})$.

12. Show that $c_i \leq b_j$ if $i + j = h \leq d$.

13. Let Γ be a distance-regular graph and let x, y be adjacent vertices of Γ. Set $D^i_j = \Gamma_i(x) \cap \Gamma_j(y)$. Let $u \in D^{i+1}_i$ and $v \in D^i_i$. Show the following.

   (a) $a_i = e(v, D^i_{i-1}) + e(v, D^i_i) + e(v, D^i_{i+1}) = e(v, D^{i-1}_i) + e(v, D^i_i) + e(v, D^{i+1}_i)$, $c_i = e(v, D^{i-1}_i) + e(v, D^{i-1}_{i-1})$ and $b_i = e(v, D^i_{i+1}) + e(v, D^{i+1}_{i+1})$.

   (b) $a_i = 0$ if and only if $D^i_i = \emptyset$. In particular, if $a_i = 0 \neq a_{i+1}$ then we have $c_i \leq a_{i+1}$. Similarly, if $a_{i+1} = 0 \neq a_i$ then we have $b_i \leq a_i$.

14. Draw rank 1 diagrams of Five Platonic solids.

15. Draw rank 1 diagrams of the following Johnson graphs.

   (a) $J(n, 2)$, $J(n, 3)$, $J(n, 4)$ and $J(n, d)$ with $n \geq 2d$.

   Is anything different when $n = 2d$.

16. Draw rank 1 diagrams of the following Hamming graphs.

(a) $H(2,2)$, $H(3,2)$, $H(4,2)$ and $H(d,2)$.

(b) $H(2,q)$, $H(3,q)$, $H(4,q)$ and $H(d,q)$ with $q \geq 3$.

17. Draw rank 1 diagrams of the following Odd graphs.

    (a) O_2, O_3, O_4 and O_k.

18. Draw rank 1 diagrams of all cubic distance-regular graphs, i.e., those of valency three. (See [21, Theorem 7.5.1]).

19. Show that $J(n,d) \simeq J(n, n-d)$.

20. Let $\Gamma = (X,R)$ be a connected graph of diameter d. Let A_i be a matrix of size $|X|$, whose rows and columns are indexed by vertices in X such that (x,y)-entry is defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{otherwise.} \end{cases}$$

    A_i is called the *i-th adjacency matrix*, and $A = A_1$ the *adjacency matrix*. Show that the adjacency matrices of a distance-regular graph satisfy the following.

    (a) $A_0 = I$.

    (b) $A_0 + A_1 + \cdots + A_d = J$, where J is the all 1's matrix.

    (c) ${}^tA_i = A_i$ for $i = 0, 1, \ldots, d$.

    (d) $A_iA_j = \sum_{h=0}^{d} p^h_{i,j} A_h$, where $p^h_{i,j} = |\Gamma_i(x) \cap \Gamma_j(y)|$ with $h = \partial(x,y)$.

21. Let A_i's be the adjacency matrices of a distance-regular graph of diameter d. Prove the following.

    (a) $A_0, A_1, \ldots, A_d$ are linearly independent over the real number field $\boldsymbol{R}$.

    (b) $A_iA = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}$ for all i, where b_j and c_j with $j < 0$ or $j > d$ are indeterminate.

    (c) Let $v_0(t) = 1$, $v_1(t) = t$, and $v_{i+1}(t)$ is defined by

$$v_i(t)t = b_{i-1}v_{i-1}(t) + a_iv_i(t) + c_{i+1}v_{i+1}(t)$$

    for $i = 1, \ldots, d$ with $c_{d+1} = 1$. Then $v_i(A) = A_i$. In particular, $v_{d+1}(t)$ is a minimal polynomial of A.

    (d) $\boldsymbol{R}[A] \simeq \boldsymbol{R}[t]/(v_{d+1}(t))$ as an algebra over $\boldsymbol{R}$, where $\boldsymbol{R}[A] = \{p(A) \mid p(t) \in \boldsymbol{R}[t]\}$. Moreover, $\mathrm{Span}(A_0, A_1, \ldots, A_d) = \mathrm{Span}(I, A, A^2, \ldots, A^d) = \boldsymbol{R}[A]$.

22. Let $\Gamma = (X,R)$ be a distance-regular graph. Show the following.

    (a) $b_{i-1}p^h_{i-1,j} + a_jp^h_{i,j} + c_{i+1}p^h_{i+1,j} = p^{h-1}_{i,j}c_h + p^h_{i,j}a_h + p^{h+1}_{i,j}b_h$.

    (b) $p^h_{i,j+1} = \dfrac{1}{c_{j+1}}(p^h_{j,i-1}b_{i-1} + p^h_{j,i}(a_i - a_j) + p^h_{j,i+1}c_{i+1} - p^h_{j-1,i}b_{j-1})$.

23. Let $\Gamma = (X,R)$ be a connected graph of valency k and diameter d. Show that if the cardinalities $|C_i(u,v)|$ and $|B_i(u,v)|$ depend only on i for all $i \in \{0, 1, \ldots, d\}$, then Γ is distance regular.

Chapter 3

Antipodal Graphs

3.1 Basic Properties

Let $\Gamma = (X, R)$ be a graph of diameter d. For each $i \in \{0, 1, \ldots, d\}$, let $R_i = \{(x, y) \mid \partial(x, y) = i\}$, and $\Gamma^{(i)} = (X, R_i)$.

Definition 3.1.1 A graph Γ of diameter d is said to be *primitive* if $\Gamma^{(i)}$ is connected for any $i = 1, 2, \ldots, d$.

Definition 3.1.2 A distance-regular graph Γ of diameter $d \geq 2$ is said to be *antipodal*, if $\partial(x, y) = \partial(x, z) = d$ and $y \neq z$ implies $\partial(y, z) = d$, i.e., $\Gamma^{(d)} = (X, R_d)$ is a disjoint union of cliques.

Proposition 3.1.1 ([21]) *Let Γ be a distance-regular graph. Then Γ is imprimitive if and only if Γ is bipartite or antipodal.*

Proof. Suppose that Γ is imprimitive. Let i be the least integer such that $\Gamma^{(i)}$ is disconnected. Then for any $j = 1, 2, \ldots, i-1$, the configuration xyz satisfying

$$\partial(x, y) = \partial(x, z) = i, \partial(y, z) = j$$

is forbidden.

Case 1. $i = 2 < d$.

We claim that $a_1 = 0$. Suppose not. Let x and y be two vertices at distance 3, and let (x, z, w, y) be the path connecting x and y. Pick $u \in \Gamma(z) \cap \Gamma(w)$. Then $\partial(x, u) = 1$ or 2. If $\partial(x, u) = 1$, then $\partial(y, u) = 2$; and so the configuration uzy is forbidden. If $\partial(x, u) = 2$, the configuration uwx is forbidden. Therefore our claim is valid.

Suppose that there exists an integer $s \geq 2$ such that $a_s \neq 0$. For any two adjacent vertices x_0 and x_1, there exists a circuit of length $2s + 1$

$$(x_0, x_1, \ldots, x_{2s}).$$

Since $a_1 = 0$, $\partial(x_i, x_{i+2}) = 2$ for all i, where all subscripts taken modulo $2s + 1$. In $\Gamma^{(2)}$, there is a path

$$(x_0, x_2, \ldots, x_{2s}, x_1)$$

connecting x_0 and x_1. Therefore, $\Gamma^{(2)}$ is connected, a contradiction.

By above argument, $a_j = 0$ for any $j = 1, 2, \ldots, d$. Hence, Γ is bipartite.

Case 2. $2 < i < d$.

Let x_0 and x_d be two vertices at distance d, and let

$$(x_0, x_1, \ldots, x_d)$$

be a path connecting x_0 and x_d. Since $k > 2$, we may choose a vertex $w \in \Gamma(x_{i+1}) \setminus \{x_i, x_{i+2}\}$. Then $\partial(x_0, w) = i + l$, where $l = 0, 1, 2$. If $l = 0$, the configuration $x_0 w x_i$ is forbidden. If $l = 2$, $\partial(x_2, w) = i$; and so the configuration $x_2 w x_{i+2}$ is forbidden. If $l = 1$, $\partial(x_1, w) = i$ or $i + 1$. The configuration $x_1 w x_{i+1}$ is forbidden whenever $\partial(x_1, w) = i$; and the configuration $x_2 w x_{i+2}$ is forbidden whenever $\partial(x_1, w) = i + 1$. Hence Case 2 does not appear.

Case 3. $i = d$.

In this case, $\Gamma^{(d)}$ is a disjoint union of cliques. Therefore, Γ is antipodal.

The converse is obvious. ■

Remarks. Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d.

1. If Γ is bipartite, then $\Gamma^{(2)}$ is disconnected. The converse is not true. For example, let Γ be the Octahedron. Then $\Gamma^{(2)}$ is disconnected, but Γ is not bipartite.

2. If $\Gamma^{(i)}$ is connected for $i = 0, 1, \ldots, d-1$, and $\Gamma^{(d)}$ is not connected, then Γ is antipodal. The converse is not true. For example, let Γ be the Hamming graph $H(d, 2)$ with $d \geq 3$. Then Γ is antipodal and bipartite, but $\Gamma^{(2)}$ is disconnected.

Lemma 3.1.2 *Let Γ be a distance-regular graph of diameter d and valency k. Then the following hold.*

(i) *$c_j \leq b_i$ if $i + j \leq d$.*

(ii) *$k_i \leq k_j$ if $0 \leq i \leq j$ and $i + j \leq d$.*

(iii) *Suppose $k_i = k_j$ with $0 \leq i < j$ and $i + j \leq d$. Then*

$$b_i = c_j,\ b_{i+1} = c_{j-1}, \cdots, b_{j-1} = c_{i+1},$$

and in particular $k_{i+1} = k_{j-1}$. Moreover $k_i = k_{d-i}$.

(iv) *Prove that k_i's have unimodal property, i.e., there exist h, l with $1 \leq h \leq l \leq d$ such that*

$$1 = k_0 < k_1 < \cdots < k_h = \cdots = k_l > \cdots > k_d \geq 1.$$

In particular if $k_i = k_j$ with $0 \leq i < j$ and $i + j \leq d$, then $k_i = k_{d-i}$ as $k_i = k_j \leq k_{d-i}$ with $j \leq d - 1$. Moreover, if $j < d - i$ then $k_i = k_{i+1} = \cdots = k_{d-i}$.

(v) *$k_d = 1$ if and only if $b_i = c_{d-i}$ for all $i \in \{0, 1, \ldots, d\}$.*

(vi) *Γ is antipodal if and only if $b_i = c_{d-i}$ for all $i \neq [d/2]$.*

Proof. (i) follows from Exercise 12 in Chapter 1. By (i), we have

$$k_j = \frac{b_i b_{i+1} \cdots b_{j-1}}{c_{i+1} c_{i+2} \cdots c_j} k_i = \frac{b_i}{c_j} \frac{b_{i+1}}{c_{j-1}} \cdots \frac{b_{j-1}}{c_{i+1}} k_i \geq k_i,$$

as each quotient is at least 1. Therefore (ii) and (iii) hold. Moreover (iv) and (v) are obvious.

Suppose Γ is antipodal. If $b_i \neq c_{d-i,}$ i.e., $b_i > c_{d-i}$ by (ii). Let u, w, v be vertices of Γ such that $\partial(u, v) = d$ and $w \in \Gamma_i(u) \cap \Gamma_{d-i}(v)$. Then there exists a vertex $x \in B(u, w) \setminus C(v, w)$ such that

$$l := \partial(x, v) = d - i \text{ or } d - i + 1.$$

Since $|\Gamma_d(u) \cap \Gamma_{d-i-1}(x)| = p^{i+1}_{d,d-i-1} \neq 0$, there exists a vertex y such that $\partial(u, y) = d$ and $\partial(x, y) = d - i - 1$. Note that $y \neq v$. The fact Γ is antipodal implies that $\partial(y, v) = d$. By (1.1), we have

$$d = \partial(y, v) \leq \partial(y, w) + \partial(w, v) = 2(d - i).$$

Therefore, $i \leq [\frac{d}{2}]$.

Let $z \in X$ with $\partial(z, x) = d - l$ and $\partial(z, v) = d$. Then $\partial(z, x) = i$ or $i - 1$. By $z \neq u$, $\partial(u, z) = d$. By (1.1),

$$d = \partial(z, u) \leq \partial(z, x) + \partial(x, u) = i + 1 + d - l.$$

Then $d \leq 2i + 1$ or $d \leq 2i$; consequently, $i \geq [\frac{d}{2}]$.

Therefore, $i = [\frac{d}{2}]$.

Conversely, suppose that $b_i = c_{d-i}$ for all $i \neq [d/2]$. We only need to prove that $\partial(y, z) = d$ for any $z \in \Gamma_d(x) \setminus \{y\}$. Let $m = [\frac{d}{2}]$. Since $c_d = b_0 = k$, $\partial(z, w_1) = d - 1$ for any $w_1 \in \Gamma(x)$. Hence, $\Gamma(x) \subseteq \Gamma_{d-1}(z)$. By induction, $\Gamma_i(x) \subseteq \Gamma_{d-i}(z)$ for $i = 1, 2, \ldots, m$.

For any $w_j \in \Gamma_j(x)$ with $m + 1 \leq j \leq d - 1$, pick $u_j \in \Gamma_m(x) \cap \Gamma_{j-m}(w_j)$. Then

$$\partial(z, w_j) \leq \partial(z, u_j) + \partial(u_j, w_j) = (d - m) + (j - m).$$

If $d = 2m$, then $\partial(z, w_i) \leq j \leq d - 1$. Now suppose $d = 2m + 1$. Then $\partial(z, w_j) \leq j + 1$. We claim that $\partial(z, w_j) < j + 1$. Suppose not. Then there exists some j such that

$$\partial(z, w_j) = \partial(z, u_j) + \partial(u_j, w_j) = j + 1.$$

Pick $z_j \in C(w_j, u_j.)$ Then $z_j \in \Gamma_{m+1}(x) \cap B(z, u_j)$. Since $b_{m+1} = c_m$, we have $B(z, u_j) = C(x, u_j)$; and so $z_j \in C(x, u_j)$, a contradiction. Hence our claim is proved.

By above argument, for each vertex $z_i \in \Gamma_i(x)$ with $1 \leq i \leq d - 1$, we have $\partial(z, z_i) \leq d - 1$. Since $|\Gamma_d(x)| = |\Gamma_d(z)|$, $\partial(y, z) = d$. Therefore, Γ is antipodal. ■

3.2 Graphs with $k_i = k_{d-i}$

Problem 3.2.1 *Let Γ be a distance-regular graph of diameter d. If $k_i = k_{d-i}$ with $i < d - i$, then $k_d = 1$ and Γ is an antipodal 2-cover.*

Proposition 3.2.1 ([236]) *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d with $k_i = k_j$, $i < j$ and $i + j \leq d$. Then one of the following hold.*

(i) $k_d = 1$; *or*

(ii) $k_i = k_{i+1} = \cdots = k_j$. *Moreover if $k_j \neq k_{j+1}$ then $\Gamma_d(u)$ is a clique for any vertex $u \in X$.*

In the following we prove a special case:

> *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter 6 with $k_2 = k_4$. Then one of the following hold.*
>
> (i) $k_6 = 1$; *or*
>
> (ii) $k_2 = k_3 = k_4$. *Moreover if $k_4 \neq k_5$ then $\Gamma_6(u)$ is a clique for any vertex $u \in X$.*

Proof. Let x and y be two vertices of Γ at distance 6, and let $D^i_j = \Gamma_i(x) \cap \Gamma_j(y)$. Then the intersection diagram of rank 6 has the shape in Figure 3.1.

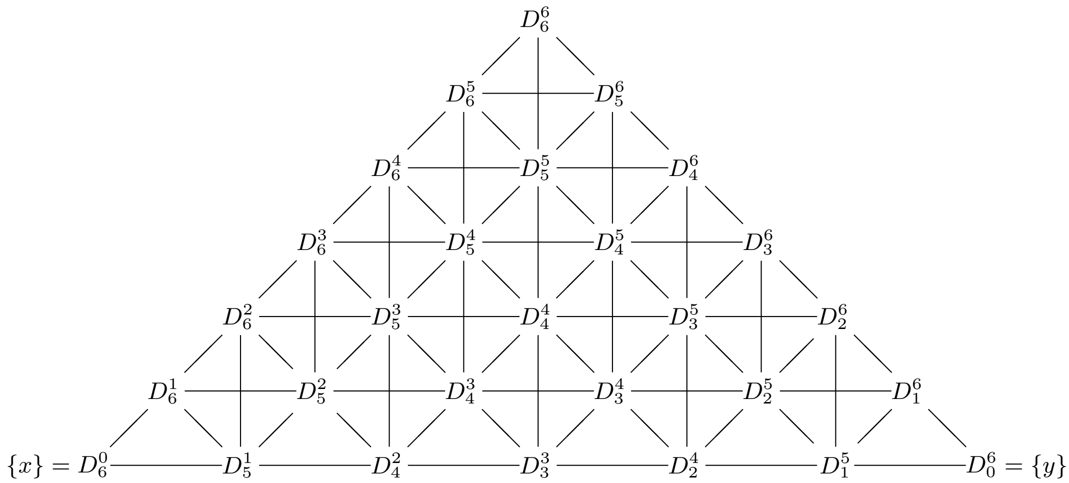


Figure 3.1: Intersection Diagram of Rank 6

Case 1. $k_4 = k_5$.
By Lemma 3.1.2 (iii), We get $k_2 = k_3 = k_4 = k_5$.

Case 2. $k_4 > k_5$.
By $k_5c_5 = k_4b_4$, we have $b_4 < c_5$. By Lemma 3.1.2, $k_2 = k_4$ implies that $b_2 = c_4$ and $b_3 = c_3$.

Step 1 $e(D^2_4 \cup D^3_3 \cup D^4_2, D^3_4 \cup D^3_5 \cup D^4_3 \cup D^4_4 \cup D^5_3) = 0$.
Pick a vertex $x_1 \in D^2_4$. Then $e(x_1, D^3_3 \cup D^3_4 \cup D^3_5) = b_2$ and $e(x_1, D^3_3) = c_4$. Since $b_2 = c_4$, we have $e(D^2_4, D^3_5 \cup D^3_4) = 0$. By the symmetry of the diagram, $e(D^4_2, D^5_3 \cup D^4_3) = 0$. Similarly $b_3 = c_3$ implies that $e(D^3_3, D^3_4 \cup D^4_4 \cup D^4_3) = 0$.

Step 2 $e(D^3_4, D^4_5 \cup D^3_5 \cup D^4_4) = e(D^4_3, D^5_4 \cup D^5_3 \cup D^4_4) = 0$.
If $D^3_4 = \emptyset$, the above equality holds. Now suppose $D^3_4 \neq \emptyset$. Pick a vertex $x_2 \in D^3_4$. Since $e(x_2, D^4_3 \cup D^4_4 \cup D^4_5) = b_3$ and $e(x_2, D^4_3) = c_4$, we have $b_3 \geq c_4 = b_2$; and so $b_3 = c_4$. Hence $e(D^3_4, D^4_4 \cup D^4_5) = 0$. Since $e(x_2, D^2_5) = c_3$ and $e(x_2, D^2_5 \cup D^3_5) = b_4$, we have $b_4 \geq c_3 = b_3$; and so $b_4 = c_3$. Hence, $e(D^3_4, D^3_5) = 0$. By the symmetry of the diagram, the equality is valid.
Hence the intersection diagram of rank 6 has the shape in Figure 3.2.

Step 3. $D^3_5 = D^4_4 = D^5_3 = \emptyset$.
Suppose $D^3_5 \neq \emptyset$. Pick a vertex $x_3 \in D^3_5$. Since $\Gamma(x_3) \cap \Gamma_4(y) = \Gamma(x_3) \cap D^4_4 \neq \emptyset$, we have $D^4_4 \neq \emptyset$. Note that

$$c_5 = e(x_3, D^4_4) \leq e(x_3, D^4_4 \cup D^4_5 \cup D^4_6) = b_3.$$

By $b_3 = c_3$, we obtain $c_5 = c_4$. Pick a vertex $x_4 \in D^4_4$. Then

$$c_5 = c_4 = e(x_4, D^5_3) \leq e(x_4, D^5_3 \cup D^5_4 \cup D^5_5) = b_4,$$

a contradiction. Hence $D^3_5 = D^5_3 = \emptyset$. Since $e(D^4_4, \Gamma_3(x)) = 0$, we get $D^4_4 = \emptyset$.

Step 4 $D^4_5 = D^5_4 = D^5_5 = \emptyset$.
Suppose $D^4_5 \neq \emptyset$. Pick a vertex $x_5 \in D^4_5$. Then

$$c_5 = e(x_5, D^5_4) \leq e(x_5, D^5_4 \cup D^5_5 \cup D^5_6) = b_4,$$

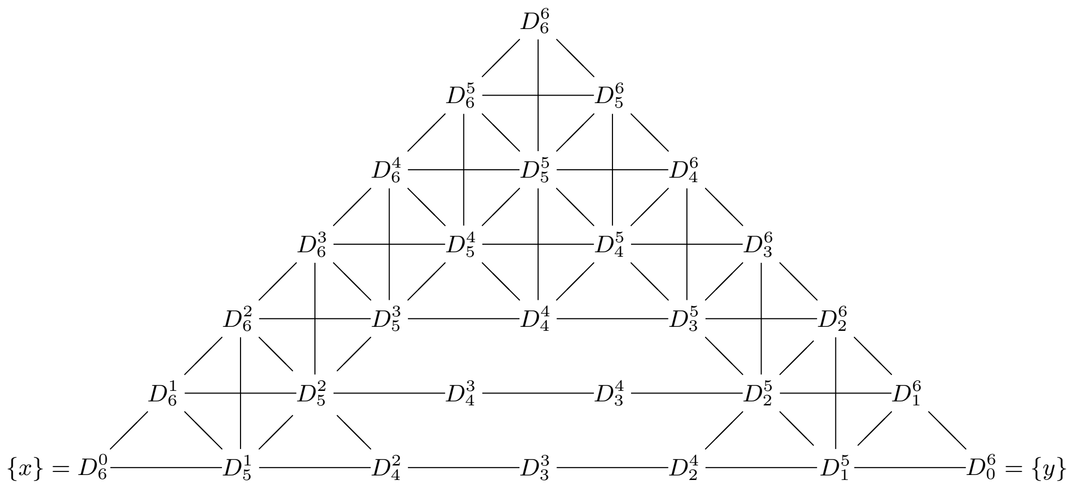


Figure 3.2:

a contradiction. Hence, $D_5^4 = D_4^5 = \emptyset$.

If $D_5^5 \neq \emptyset$. Pick a vertex $x_6 \in D_5^5$. Then

$$c_5 = e(x_6, D_4^6) \leq e(x_6, D_4^6 \cup D_5^6 \cup D_6^6) = b_5 \leq b_4,$$

a contradiction.

Step 5 $D_j^i = \emptyset$ whenever $i + j \geq 8$.

Since $e(D_6^4, \Gamma_5(y)) = 0$, we have $D_6^4 = \emptyset$. Note that $e(D_6^3, \Gamma_4(x)) = 0$, which implies that $D_6^3 = \emptyset$. In a similar way, $D_6^2 = \emptyset$. By the symmetry, we have $D_j^i = \emptyset$ if $8 \leq i + j \leq 10$. Since Γ is connected, $D_6^6 = D_5^6 = D_6^5 = \emptyset$.

Hence the intersection diagram of rank 6 has the shape in Figure 3.3.

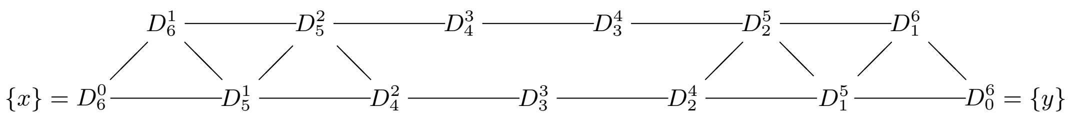


Figure 3.3:

If $D_6^1 = \emptyset$, $|\Gamma_6(x)| = 1$, as desired. If $D_6^1 \neq \emptyset$, then $D_5^2 \neq \emptyset$. Again $D_4^3 \neq \emptyset$. By the proof of Step 2, $b_3 = c_4$ and $k_2 = k_3 = k_4$. Moreover, $|\Gamma_6(x)| = a_6 + 1$, consequently $\Gamma_6(x)$ is a clique. ■

The case (ii) in Propostion 3.2.1 may be eliminated by the following result.

Proposition 3.2.2 ([128]) *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d with $k_i = k_j, i < j$ and $i + j \leq d$. Then one of the following hold.*

(i) $k_d = 1$*; or*

(ii) $d \leq 3i - 2$*, in particular,* $i \geq 3$.

3.3 Graphs with $b_1 = c_{d-1}$

Problem 1 *Let Γ be a distance-regular graph of diameter d. If $b_1 = c_{d-1}$ with $d \geq 3$, then Γ is antipodal, i.e., $b_i = c_{d-i}$ for all $i \neq [d/2]$.*

Proposition 3.3.1 ([127]) *Let Γ be a distance-regular graph with diameter d and valency k. Suppose*

$$b_1 = c_{d-1},\ b_2 = c_{d-2}, \ldots, b_i = c_{d-i},\ \textit{for some } i \geq 1.$$

Then the following hold.

(i) $b_{d-1} = c_1,\ b_{d-2} = c_2, \ldots, b_{d-i} = c_i$.

(ii) *If $a = a_d \neq 0$, then $k = a(a+1)$, $b_1 = \cdots = b_i = a^2$ and $1 = c_1 = c_2 = \cdots = c_i = c_{i+1}$.*

Lemma 3.3.2 ([21, Propositions 5.4.3-5.4.4]) *Let Γ be a distance-regular graph with diameter d. If $b_1 = b_i$, then $c_i = 1$.*

Proof. Suppose $i = 2$. Suppose to the contrary that $c_2 \geq 2$. For vertices x and y at distance 2, pick any two distinct vertices $u, v \in \Gamma(x) \cap \Gamma(y)$. Since $b_1 = b_2$, we have $B(v, y) = B(x, y)$; and so $\partial(u, v) = 1$. It follows that the induced subgraph on $\Gamma(x) \cap \Gamma(y)$ is a clique. For any vertex $z \in A(x, y)$, $B(v, y) = B(x, y)$ implies that $\partial(v, z) = 1$. In a similar way, we have $\partial(u, z) = 1$. It follows that

$$a_1 = |\Gamma(u) \cap \Gamma(v)| \geq c_2 + a_2,$$

i.e., $1 + b_1 \leq b_2$, a contradiction. Hence the result holds for $i = 2$.

Now suppose $i \geq 3$. Suppose to the contrary that $c_i \geq 2$. For two vertices x and y, pick a path from x to y

$$(x = x_0, x_1, \ldots, x_i = y).$$

Since $c_i \geq 2$, there exists a vertex $x_{i+1} \in \Gamma(y) \cap \Gamma_{i-1}(x) \setminus \{x_{i-1}\}$. Since $b_1 = b_2 = \cdots = b_i$, $B(x, y) = B(x_1, y) = \cdots = B(x_{i-1}, y)$. It follows that $\partial(x_{i-1}, x_{i+1}) = 1$. In a similar way, we obtain $\partial(x_{i-2}, x_{i+1}) = 1$; and so $x_{i+1}, x_{i-1} \in \Gamma(x_{i-2}) \cap \Gamma(y)$. Hence $c_2 \geq 2$, a contradiction. ∎

Corollary 3.3.3 *Let Γ be a distance-regular graph with diameter d. If $b_1 = c_j$ $(j < d)$, then $b_j = 1$.*

Proof. By Lemma 3.1.2, for each $t = 1, 2, \ldots, d - j$, we have

$$c_j \leq c_{d-t} \leq b_t \leq b_1.$$

Since $b_1 = c_j$, we obtain $b_t = c_{d-t}$, i.e.,

$$b_1 = c_{d-1}, b_2 = c_{d-2}, \ldots, b_{d-j} = c_j.$$

By Proposition 3.3.1, $b_j = c_{d-j}$. Since $b_{d-j} = c_j = b_1$, $c_{d-j} = 1$ by Lemma 3.3.2; consequently $b_j = 1$. ∎

Proposition 3.3.4 ([2]) *Let Γ be a distance-regular graph with diameter d. If $b_i = c_{d-i}$ for all $i \in \{1, 2, \ldots, e\}$ with $d \leq e$, then $k_d = 1$.*

Proposition 3.3.5 ([177]) *Let Γ be a distance-regular graph with diameter d. If $b_i = c_{d-i}$ for all $i \in \{1, 2, \ldots, e\}$ with $d \leq 3e + 5$ and $a_d \neq 0$, then $k_d = a_d + 1$.*

In the case (ii) of Proposition 3.3.1, $\Gamma_d(u)$ is a disjoint union of cliques of size $a_d + 1$. If $\Gamma_d(u)$ is a clique, then Γ has another P-polynomial ordering, i.e., $\Gamma^{(d)} = (X, R_d)$ is also distance-regular with $a_1 = \cdots = a_{d-1} = 0 \neq a_d$. (See [21, Proposition 4.2.10]). But the case when $\Gamma_d(u)$ is a disjoint union of several cliques of size at least two is very rare and in all known cases $a_d = 1$ and $b_{d-1} > 1$. If Γ is a strongly regular graph, i.e., a distance-regualr graph of diameter two, then $\Gamma_2(u)$ is a disjoint union of m cliques of size $a_2 + 1$ if and only if either $m = 1$ or $a_2 = 0$. This is becase if we take the complement of the graph, we obtain a strongly regular graph whose neighborhood of a vertex is complete multipartite. See [21, Proposition 1.1.5, Corollary 1.1.6, Section 5.7].

Problem 3.3.1 *Show that the case* (ii) *of Proposition 3.3.1 does not occur.*

Problem 3.3.2 *Is there a distance-regular graph such that $\Gamma_d(u)$ is a disjoint union of several cliques of size at least three?*

The following special case arises in [21, Theorem 1.11.1 (vii)] as a special case of a bipartite double of a distance-regular graph with special parameters.

Problem 3.3.3 *Let Γ be a bipartite distance-regular graph with diameter $d = 2e + 1$ and valency k. Suppose*

$$b_1 = c_{d-1},\ b_2 = c_{d-2}, \ldots, b_{e-1} = c_{d-e+1}.$$

Show that Γ is antipodal, i.e., $b_e = c_{e+1}$.

The following is also a problem related to antipodal graphs.

Problem 3.3.4 *Suppose Γ is a bipartite distance-regular graph of valency k with $c_e = 1$ and $c_{e+1} = k - 1$ for some $e \leq d - 2$. Then is Γ antipodal?*

3.4 Graphs of Order (s, t)

In this section, we study distance-regular graphs having no induced subgraphs isomorphic to $K_{2,1,1}$, a graph with 4 vertices with five edges.

Lemma 3.4.1 *Let Γ be a distance-regular graph of diameter $d \geq 2$ and valency k. Then the following are equivalent.*

(i) *There are no induced subgraphs isomorphic to $K_{2,1,1}$.*

(ii) *Each connected component of $\Gamma(x)$ is a clique of size $a_1 + 1$ for every vertex x.*

In this case, set $s = a_1 + 1$ and write $k = s(t + 1)$. Then each maximal clique is of size $s + 1$, and each vertex is contained in $t + 1$ maximal cliques.

Definition 3.4.1 A distance-regular graph Γ is said to be of *order* (s, t) if one of (i) and (ii) in Lemma 3.4.1 holds.

If $c_2 = 1$, $a_1 = 0$ or $a_1 = 1$, then Γ is of order (s, t).

Problem 3.4.1 Is there a constant R such that $r(\Gamma) \leq R$ for all distance-regular graphs?

In order to study Problem 3.4.1, we may assume that $r(\Gamma) \geq 2$, then $c_2 = 1$ and there is no induced subgraph ismorphic to $K_{2,1,1}$. Hence we only need to study distance-regular graphs of order (s, t).

Problem 3.4.2 Let $r = l(c_1, a_1, b_1) \geq 2$. Prove that $l(c, a, b) \leq r$ for any distance-regular graph of order (s, t).

The *geometric girth* $gg(\Gamma)$ of a distance-regular graph is the length of a shortest reduced circuit. The notion of geometric girth is very useful for distance-regular graphs of order (s, t). For such graphs with $r = r(\Gamma)$, $gg(\Gamma) = 2r + 2$ if $c_{r+1} > 1$, and $gg(\Gamma) = 2r + 3$ if $c_{r+1} = 1$.

3.5 (s, c, a, k)-bound

This result is proved when the girth is at least 4 in [255] and it is generalized to the form below replacing paths by reduced paths in [157]. Recall that a reduced path is a path without an induced triangle. It is obtained by counting the number of reduced closed paths $v_0 \sim v_1 \sim \cdots \sim v_{2s} = v_0$ such that $v_0 \in M_0$ and $v_s \in M_s$ in two ways, and apply Cauchy-Schwartz inequality.

Theorem 3.5.1 *Let $\Gamma = (X, R)$ be a connected graph, and let c, p, and s be integers at least 2. Suppose X can be partitioned into $s + 1$ disjoint sets $X = \bigcup_{i=0}^{s} M_i$, where for any u, $v \in X$, $u \in M_i$, $v \in M_j$, and $u \sim v$ implies $|i - j| \leq 1$. For $i = 1$ or s let l_i and L_i denote the minimum and maximum number of vertices in M_{i-1} any vertex in M_i is adjacent to, and for $i = 0$ or $s - 1$, let r_i and R_i denote the minimum and maximum number of vertices in M_{i+1} any vertex in M_i is adjacent to. Also assume*

(i) *$\partial(u, v) = s$ for some $u \in M_0$ and $v \in M_s$,*

(ii) *for integers $i, j \in [0, s]$, and for any $u \in M_i$ and $v \in M_j$, there are either c or 0 reduced paths of length s connecting them if $|j - i| = s$ and either 0 or 1 reduced paths of length $|j - i|$ connecting them if $1 \leq |j - i| \leq s - 1$, and*

(iii) *for any $u, v \in X$ with $u \in M_1$, $v \in M_{s-1}$, and $\partial(u, v) > s - 2$, there are at most p reduced paths $(u = v_0, v_1, \cdots, v_{s-1}, v_s = v)$, where $v_1 \in M_0$ or $v_{s-1} \in M_s$.*

Then
$$\frac{p}{c-1} \geq \frac{r_{s-1}}{R_0 - 1} + \frac{l_1}{L_s - 1}.$$

Proof. Let $Z = \{(u, v) \in M_1 \times M_{s-1} \mid \partial(u, v) > s - 2\}$. For any two real functions F and G on Z, define
$$F \cdot G = \sum_{x \in Z} F(x)G(x).$$

For $x = (u, v) \in Z$, define

$$\begin{aligned} U(x) &= \text{the number of reduced paths } (u = v_0, v_1, \ldots, v_s = v) \text{ with } v_{s-1} \in M_s, \\ D(x) &= \text{the number of reduced paths } (u = v_0, v_1, \ldots, v_s = v) \text{ with } v_1 \in M_0, \\ I(x) &= 1. \end{aligned}$$

By assumption (iii) we have
$$U(x) + D(x) \leq pI(x), x \in Z.$$

Let
$$E = \{(u_0, u_1, \ldots, u_{s-2}) \mid u_i \in M_{i+1}\},$$

and
$$Y = \{\mathrm{reduced\,circuit\,}(\mathrm{v_0}, \mathrm{v_1}, \ldots, \mathrm{v_{2s}} = \mathrm{v_0}) \mid \mathrm{v_0} \in \mathrm{M_0}, \mathrm{v_s} \in \mathrm{M_s}\}.$$

For $e = (u_0, u_1, \ldots, u_{s-2}) \in E$, define

$$l(e) = |\Gamma(u_0) \cap M_0|, \ r(e) = \Gamma(u_{s-2}) \cap M_s|.$$

For each reduced circuit $(v_0, v_1, \ldots, v_{2s} = v_0)$ in Y, we must have $\partial(v_1, v_{s+1}) > s - 2$. This means

$$U \cdot D = |Y| = (c-1)(\sum_{e \in E} l(e)r(e)).$$

By definition $l_1 \leq l(e)$ and $r_{s-1} \leq r(e)$, so

$$|Y| \geq (c-1)l_1 \sum_{e \in E} r(e), \ |Y| \geq (c-1)r_{s-1} \sum_{e \in E} l(e). \tag{3.1}$$

Note that

$$U \cdot I \leq (L_s - 1) \sum_{e \in E} r(e), \ D \cdot I \leq (R_0 - 1) \sum_{e \in E} l(e). \tag{3.2}$$

By Chauchy-Schwarz inequality,

$$(U \cdot D)^2 \leq (U \cdot U)(D \cdot D) \leq (U \cdot (pI - D))(D \cdot (pI - U)) = (p(U \cdot I) - U \cdot D)(p(D \cdot I) - D \cdot U).$$

Solving for $\frac{p}{U \cdot D}$ we obtain

$$\frac{p}{U \cdot D} \geq \frac{1}{D \cdot I} + \frac{1}{U \cdot I}.$$

Since $|Y| = U \cdot D$, we can write

$$\frac{p}{c-1} \geq \frac{|Y|}{(c-1)D \cdot I} + \frac{|Y|}{(c-1)U \cdot I}.$$

Using (3.1) and (3.2) we obtain

$$\frac{p}{c-1} \geq \frac{r_{s-1}}{R_0 - 1} + \frac{l_1}{L_s - 1},$$

as desired. ■

Theorem 3.5.2 *Let Γ be a distance-regular graph of order (s,t) with diameter d, valency k and geometric girth $2s$. Let $c = c_s$. Then we have the following.*

$$\begin{aligned} \frac{c}{c-1} &\geq \frac{b_y}{b_{y-s+1} - 1} + \frac{c_{x-s+1}}{c_x - 1}, \ (s \leq x \leq y+1 \leq d) \\ \frac{c}{c-1} &\geq \frac{c_{y-s+1}}{c_y - 1} + \frac{c_{x-s+1}}{c_x - 1}, \ (s \leq x \leq d - y + s \leq d) \end{aligned}$$

Proof. For the first inequality, set $\partial(u,v) = y + 1 - x$ and $M_i = \Gamma_{y+1-s+i}(u) \cap \Gamma_{x-s+i}(v)$ for $i = 0, 1, \ldots, s$.

Let $\alpha \in M_1$ and $\beta \in M_{s-1}$ be two vertices with distance at least $s-1$. If $\partial(\alpha, \beta) = s$, then there are at most c reduced paths

$$(\alpha = v_0, v_1, \cdots, v_{s-1}, v_s = \beta) \tag{3.3}$$

where $v_1 \in M_0$ or $v_{s-1} \in M_s$. If $\partial(\alpha, \beta) = s - 1$, then there are no reduced paths satisfying (3.3) by Exercise 20. Hence then there are at most c reduced paths satisfying (3.3).

Since the induced subgraph on $M_0 \cup M_1 \cup \ldots \cup M_s$ satisfies the conditions in Theorem 3.5.1, we have the assertion. Here the constants in the previous theorem are

$$p = c, r_{s-1} = b_y,\ R_0 = b_{y-s+1},\ l_1 = c_{x-s+1},\ \text{ and } L_s = c_x.$$

For the second inequality, set $\partial(u,v) = x+y-s$ and $M_i = \Gamma_{x-s+i}(u) \cap \Gamma_{y-i}(v)$ for $i = 0, 1, \ldots, s$. By Theorem 3.5.1, we have the assertion. Here the constants in the previous theorem are

$$p = c, r_{s-1} = c_{y-s+1},\ R_0 = c_y,\ l_1 = c_{x-s+1},\ \text{ and } L_s = c_x.$$

■

Corollary 3.5.3 *Let Γ be a distance-regular graph of order (s,t) with diameter d, and geometric girth $2r+2 \geq 4$. Then $b_i > b_{i+r}$, and $c_i < c_{i+r}$ for any $i = 0, 1, \ldots, d-r$.*

Proof. In this case we have $s = r+1$ and $c = c_{r+1} > 1$ in Theorem 3.5.2. By setting $x = r+1$, we have $b_i > b_{i+r}$, and $c_i < c_{i+r}$ as desired. ■

3.6 Exercises

1. Show the following.
   (a) The Johnson graph $J(n,d)$ is antipodal for $n = 2d$.
   (b) The Johnson graph $J(n,d)$ is primitive for $n > 2d$.
2. Show the following.
   (a) The Hamming graph $H(d,q)$ is bipartite and antipodal for $q = 2$.
   (b) The Hamming graph $H(d,q)$ is primitive for $q \geq 3$.
3. Show that the Odd graph O_k is primitive.
4. Let Γ be the Johnson graph $J(2d+1,d)$. Show that $\Gamma^{(d)}$ is the Odd graph O_{d+1}.
5. Determine all primitive cubic distance-regular graphs.
6. Draw rank d diagrams of Five Platonic solids, where d is the diameter of each graph.
7. Draw rank d diagrams of the following Johnson graphs, where d is the diameter of each graph.
   (a) $J(n,2)$, $J(n,3)$, $J(n,4)$ and $J(n,d)$ with $n \geq 2d$.

   Is anything different when $n = 2d$.
8. Draw rank d diagrams of the following Hamming graphs, where d is the diameter of each graph.
   (a) $H(2,2)$, $H(3,2)$, $H(4,2)$ and $H(d,2)$.
   (b) $H(2,q)$, $H(3,q)$, $H(4,q)$ and $H(d,q)$ with $q \geq 3$.
9. Draw rank d diagrams of the following Odd graphs, where d is the diameter of each graph.
   (a) O_2, O_3, O_4 and O_d.

10. Draw rank d diagrams of all cubic distance-regular graphs, i.e., those of valency three, where d is the diameter of each graph. (See [21, Theorem 7.5.1].)

11. Show that $\Gamma_d(u) \simeq J(n-d, d)$ if $\Gamma \simeq J(n, d)$.

12. Show that $\Gamma_d(u) \simeq H(d, q-1)$ if $\Gamma \simeq H(d, q)$.

13. Let Γ be a distance-regular graph of diameter $d \geq 2$ and valency k. Show that the following are equivalent.

    (i) There are no induced subgraphs isomorphic to $K_{2,1,1}$.

    (ii) Each connected component of $\Gamma(x)$ is a clique of size $a_1 + 1$ for every vertex x.

14. Show that distance-regular graphs Γ of valency k and diameter $d \geq 2$ with one of the following properties are of order (s, t). Find s and t in each case.

    (a) Γ is of triangular free, i.e., $a_1 = 0$.

    (b) $c_2 = 1$.

    (c) $a_1 = 1$.

15. The Hamming graph $H(d, q)$ is a distance-regular graph of order $(s, t) = (q-1, d-1)$.

16. Show that the Hamming graphs $H(d, q)$ are distance-regular graphs of order $(q-1, d-1)$.

17. Let $\Gamma = (X, R)$ be a distance-regular graph of diameter 2 with the following intersection array:

$$\iota(\Gamma) = \left\{ \begin{array}{ccc} * & 1 & 2 \\ 0 & 2 & 4 \\ 6 & 3 & * \end{array} \right\}.$$

    Show the following.

    (a) If Γ is of order (s, t), then it is isomorphic to the Hamming graph $H(2, 4)$.

    (b) If Γ is not of order (s, t), then it is uniquely determined (up to isomorphism) and the induced subgraph on $\Gamma(u)$ is the hexagon, i.e., 6-gon, for every vertex $u \in X$. This graph is called the Shrikhande graph.

18. Show that a distance-regular graph of valency 6, $a_1 = 2$ and $c_2 = 2$ is isomorphic to $H(2, 4)$ or the Shrikhande graph.

19. Let $\Gamma = (X, R)$ be a distance-regular graph of order (s, t) of diameter d. Let $r = \ell(c_1, a_1, b_1)$. Suppose $c_{r+1} = 1$. Show $\mathrm{gg}(\Gamma) = 2r + 3$, where $\mathrm{gg}(\Gamma)$ denotes the geometric girth of Γ. i.e., the length of a shortest reduced (without triangle) circuit.

20. Let $\Gamma = (X, R)$ be a distance-regular graph of order (s, t) of diameter d. Let $r = \ell(c_1, a_1, b_1)$. Suppose $c_{r+1} \geq 2$. Show the following.

    (a) $\mathrm{gg}(\Gamma) = 2r + 2$.

    (b) For every $u, v \in X$ with $\partial(u, v) = 1$, there is no edge between $\Gamma_{r+1}(u) \cap \Gamma_r(v)$ and $\Gamma_r(u) \cap \Gamma_r(v)$.

21. Let $\Gamma = (X, R)$ be a distance-regular graph of order (s, t) of diameter d. Let $r = \ell(c_1, a_1, b_1)$. Suppose $c_{r+1} \geq 2$. By applying the '(s, c, a, k)-bound' of Terwilliger, show that

$$d \leq (k-1)r + 1.$$

Chapter 4

The Q-Polynomial Condition

4.1 Primitive Idempotents and Eigen Matrices

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d and let A_i be adjacency matrices of Γ. The algebra $\mathcal{M}$ =Span $(A_0, A_1, \ldots, A_d)$ is said to be the *Bose-Mesner algebra* of Γ.

Let $V = \boldsymbol{R}^n$ be the vector space of dimension n consisting of the column vectors of size n. By Exercise 21 in Chapter 2, real symmetric matrix A exactly has $d+1$ distinct eigenvalues. Let $\theta_0 > \theta_1 > \cdots > \theta_d$ be the distinct eigenvalues of A and let V_i be the subspace of V spanned by the eigenvectors corresponding to the eigenvalue θ_i. Then $V_i = \{\boldsymbol{v} \mid A\boldsymbol{v} = \theta_i \boldsymbol{v}\}$ and

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d \quad \text{(orthogonal direct sum)}.$$

It is 'orthogonal' with respect to the natural inner product $<\boldsymbol{u}, \boldsymbol{v}> = {}^t\boldsymbol{u}\boldsymbol{v}$. Note that V has an orthonormal basis consisting of eigenvectors of A, as A is a real symmetric matrix. Let $\boldsymbol{u}_{i_1}, \ldots, \boldsymbol{u}_{i_{m(i)}}$ be an orthonormal basis of V_i, and let

$$U_i = (\boldsymbol{u}_{i_1}, \ldots, \boldsymbol{u}_{i_{m(i)}}),$$

i.e., a matrix of size $n \times m(i)$ whose columns consist of an orthonormal basis of V_i. Let $E_i = U_i {}^tU_i$. Since ${}^tU_i U_j = \delta_{i,j} I_{m(i)}$, we have

$$E_i E_j = \delta_{i,j} E_i, \ E_0 + E_1 + \cdots + E_d = I, \text{ and } AE_i = \theta_i E_i.$$

E_i's are called the *primitive idempotents* or *orthogonal projections* with respect to the subspaces V_i. In particular, we have that $V_i = E_i V$, and that if $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_d$, where $\boldsymbol{v}_i \in V_i$ for $i = 0, 1, \ldots, d$, then $E_i \boldsymbol{v} = \boldsymbol{v}_i$. In particular, we have

$$\mathrm{tr}(E_i) = \mathrm{rank}(E_i) = \dim(V_i) = m(i).$$

Since $A = AI = A(E_0 + E_1 + \cdots + E_d)$,

$$A = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_d E_d.$$

Moreover we have

$$A^j = \theta_0^j E_0 + \theta_1^j E_1 + \cdots + \theta_d^j E_d.$$

Applying the polynomials $f_i(t)$ in the previous lemma, we have

$$f_i(A) = E_i.$$

In particular, each E_i can be written as a polynomial of A.

Suppose Γ is a distance-regular graph. By Exercise 21 in Chapter 2, the i-th adjacency matrix A_i can be written as a polynomial of A of degree exactly equal to i. Hence

$$\mathcal{M} = \mathrm{Span}(A_0, A_1, \ldots, A_d) = \mathrm{Span}(I, A, A^2, \ldots, A^d) = \mathrm{Span}(E_0, E_1, \ldots, E_d).$$

Since Γ is regular, $AJ = JA = kJ$. Since rank$J = 1$ and $J^2 = nJ$, $\frac{1}{n}J$ is a primitive idempotent of $\mathcal{M}$. Hence we may assume that $E_0 = \frac{1}{n}J$.

We have seen that the Bose-Mesner algebra $\mathcal{M}$ of a distance-regular graph has two bases, namely, the set of adjacency matrices, $A_0, A_1, \ldots, A_d$, and the set of primitive idempotents, $E_0, E_1, \ldots, E_d$ constructed above. Hence we can find the following expressions.

$$A_i = \sum_{j=0}^{d} p_i(j)E_j, \; E_i = \frac{1}{|X|}\sum_{j=0}^{d} q_i(j)A_j.$$

Let P be a $(d+1)\times(d+1)$ matrix such that $P_{j,i} = p_i(j)$. Let Q be a $(d+1)\times(d+1)$ matrix such that $Q_{j,i} = q_i(j)$. Then it is easy to see that

$$PQ = QP = |X|I.$$

The matrix P is called the *P-matrix* or the *first eigen matrix*, and the matrix Q is called the *Q-matrix* or the *second eigen matrix*. Note that $p_i(j)$ is the eigenvalue of A_i on V_j, i.e., $A_iE_j = p_i(j)E_j$. By Exercise 21 in Chapter 2, for each $i = 0, 1, \ldots, d$, there exists a polynomial $v_i(t)$ of degree i such that $A_i = v_i(A)$. Then

$$\sum_{j=0}^{d} p_i(j)E_j = A_i = v_i(A) = v_i(\sum_{j=0}^{d} p_1(j)E_j) = \sum_{j=0}^{d} v_i(p_1(j))E_j;$$

and so $p_i(j) = v_i(p_1(j)) = v_i(\theta_j)$.

We define another operation on the Bose-Mesner algebra $\mathcal{M}$. Since one of the bases $A_0, A_1, \ldots, A_d$ is a set of $(0,1)$ matrices, $\mathcal{M}$ is closed under the entry-wise product, which is often called a Hadamard product or $\circ$-product and denoted by $\circ$. We have $A_i \circ A_j = \delta_{i,j}A_i$. So A_i's are the orthogonal idempotents with respect to this product and the all 1's matrix $J = |X|E_0$ is the identity element. Let

$$E_i \circ E_j = \frac{1}{|X|}\sum_{h=0}^{d} q_{i,j}^h E_h.$$

The parameters $q_{i,j}^h$'s are called Krein parameters. Compared with the parameters $p_{i,j}^h$ which has combinatorial interpretation, Krein parameters are not well understood.

4.2 Q-Polynomial Distance-Regular Graphs and Balanced Conditions

Proposition 4.2.1 *Let Γ be a distance-regular graph of diameter d with primitive idempotents $E_0, E_1, \ldots, E_d$. Then the following are equivalent:*

(i) *There is a polynomial $v_j^\circ(t)$ of degree j such that*

$$q_j(i) = v_j^\circ(\theta_j^*), \theta_j^* = q_1(j).$$

(ii) $q^h_{i,j} = 0$, *if* $i + j < h, h + i < j$ *or* $j + h < i$;
$q^h_{i,j} \neq 0$, *if* $i + j = h, h + i = j$ *or* $j + h = i$.

(iii) *There is a polynomial* $v^\circ_j(t)$ *of degree* j *such that* $E_j = v^\circ_j(E_1)$. *($\circ$-product).*

A distance-regular graph Γ is said to be *Q-polynomial* if it satisfies one of the conditions in above proposition.

For a distance-regular graph Γ, the following equality holds.

$$\frac{q_j(i)}{q_j(0)} = \frac{p_i(j)}{p_i(0)}.$$

Moreover, if Γ is Q-polynomial, then

$$\frac{v^\circ_j(\theta^*_i)}{v^\circ_j(\theta^*_0)} = \frac{v_i(\theta_j)}{v_i(\theta_0)}.$$

Theorem 4.2.2 *(Leonard 1980, Bannai-Ito 1984) Let* Γ *be a Q-polynomial distance-regular graph of diameter* d *and valency* k*. Then all parameters can be written by the following* 5 *parameters*

$$d, k, c_d, b = \frac{b_1}{\theta + 1}, b' = \frac{b_2}{\theta - 1 - a_1 + c_2 - b}, (AE_1 = \theta E_1).$$

Proof. We follow Terwilliger's proof. See ([264],[266]).

Let Γ be a distance-regular graph. Then the following are equivalent:

(i) Γ is Q-polynomial.

(ii) $\sum_{z \in \Gamma_1(x) \cap \Gamma_2(y)} E\hat{z} - \sum_{z' \in \Gamma_2(x) \cap \Gamma_1(y)} E\hat{z'} \in \mathrm{Span}(E\hat{x} - E\hat{y})$, for any $x, y \in X$.

(iii) $\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} - \sum_{z' \in \Gamma_j(x) \cap \Gamma_i(y)} E\hat{z'} \in \mathrm{Span}(E\hat{x} - E\hat{y})$, for any $i, j = 0, 1, \ldots, d$ and $x, y \in X$.

Note that $\langle E\hat{u}, E\hat{v} \rangle = \frac{\theta^*_h}{|X|}$ if $\partial(u, v) = h$. Let $\partial(x, y) = 3$ and let $w \in \Gamma_{i+3}(x) \cap \Gamma_i(y)$. Then

$$\langle \sum_{z \in \Gamma_1(x) \cap \Gamma_2(y)} E\hat{z} - \sum_{z' \in \Gamma_2(x) \cap \Gamma_1(y)} E\hat{z'}, E\hat{w} \rangle = \langle \gamma(E\hat{x} - E\hat{y}), E\hat{w}) \rangle.$$

By computation, we have

$$\frac{1}{|X|} c_3(\theta^*_{i+2} - \theta^*_{i+1}) = \frac{1}{|X|} \gamma(\theta^*_{i+3} - \theta^*_i).$$

Then $\gamma \neq 0$ and

$$\theta^*_{i+3} - \theta^*_{i+1} = \frac{c_3}{\gamma}(\theta^*_{i+2} - \theta^*_i).$$

Let $p = -1 + \frac{c_3}{\gamma}$. Then

$$\theta^*_{i+3} - p\theta^*_{i+2} + \theta^*_{i+1} = \theta^*_{i+2} - p\theta^*_{i+1} + \theta^*_i$$

is constant independent of i, denoted by r. Note that

$$\theta^*_{i+2} = r + p\theta^*_{i+1} + \theta^*_i,$$

.... ■

Primitive distance-regular graph with $d \geq 4,\ k \geq 3$

(Classical, Exceptional)

$J(v,d)$	Johnson graph $(v \geq 2d+1)$
O_{d+1}	Odd graph $(J(2d+1,d)^{(d)})$
$J(2d,d)'$	folded graph of $J(2d,d)$
$J_q(v,q)$	generalized Johnson graph $(v \geq 2d)$
$B_d(q)$	dual polar graph (DPG) of type B
$D_d(q)^{(2)}$	bipartite half of DPG of type D
$C_d(q)$	DPG of type C
$C_n(q)^{(1,2)}$	Ustimenko graph
${}^2D_{d+1}(q)$	DPG of type ${}^2D$
${}^2A_{2d}(q)$	DPG of type ${}^2A$ with even dimension
${}^2A_{2d-1}(q)$	DPG of type ${}^2A$ with odd dimension
$H(d,q)$	Hamming graph $(q \neq 2)$
$H(2d+1,2)^{(2)}$	bipartite half of Hamming cube
$H(2d+1,2)'$	folded Hamming cube
$H(4d,2)'^{(2)}$	bipartite half of folded Hamming cube
$H(4d+2,2)'^{(2)}$	bipartite half of folded Hamming cube
$Doob(d)$	Doob graph $(\iota(\Gamma) = \iota(H(d,4)))$
$Bilin_q(d,n)$	bilinear forms graph $(n \geq d)$
$Alt_q(d)$	alternating bilinear forms graph
$Her_q(d)$	hermitian forms graph
$Quad_q(d)$	quadratic forms graph (not DTG)
$GO(s,t)$	point graph of gen. octagon $(d=4)$
$GD(s,1)$	point graph of gen. dodecagon $(d=6)$

Primitive Sporadic distance-regular graphs with $d \geq 4$

Biggs-Smith graph $L_2(17),\ d=7,\ v=102$	$\left\{\begin{matrix} * & 1 & 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 3 & 2 & 2 & 2 & 1 & 1 & 1 & * \end{matrix}\right\}$
Patterson graph $Suz,\ d=4,\ v=22880$	$\left\{\begin{matrix} * & 1 & 8 & 90 & 280 \\ 0 & 36 & 28 & 180 & 0 \\ 280 & 243 & 144 & 10 & * \end{matrix}\right\}$
Livingston graph $J_1,\ d=4,\ v=266$	$\left\{\begin{matrix} * & 1 & 1 & 5 & 11 \\ 0 & 0 & 4 & 5 & 0 \\ 11 & 10 & 6 & 1 & * \end{matrix}\right\}$
Leonard graph $\mathrm{Aut}(L_3(4)),\ d=4,\ v=280$	$\left\{\begin{matrix} * & 1 & 1 & 3 & 8 \\ 0 & 0 & 2 & 3 & 1 \\ 9 & 8 & 6 & 3 & * \end{matrix}\right\}$
M_{22} graph $\mathrm{Aut}(M_{22}),\ d=4,\ v=330$	$\left\{\begin{matrix} * & 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 2 & 1 \\ 7 & 6 & 4 & 4 & * \end{matrix}\right\}$
Coxeter graph $\mathrm{Aut}(L_2(7)),\ d=4,\ v=28$	$\left\{\begin{matrix} * & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 3 & 2 & 2 & 1 & * \end{matrix}\right\}$
$NO(2,4)$ $\mathrm{Aut}(J_2),\ d=4,\ v=315$	$\left\{\begin{matrix} * & 1 & 1 & 4 & 5 \\ 0 & 1 & 1 & 4 & 5 \\ 10 & 8 & 8 & 2 & * \end{matrix}\right\}$

4.3 Homogeneity Properties

Definition 4.3.1 Let $\Gamma = (X, E)$ be a distance-regular graph of diameter d.

(i) $X = \bigcup_{\lambda\in\Lambda} C_\lambda$ (disjoint union) is called an *equitable partition* of X if $e(x_\lambda, C_\mu)$ depends only on $\lambda, \mu \in \Lambda$, and does not depend on $x_\lambda \in C_\lambda$.

(ii) Γ is called *h-homogeneous* if $\{D_i^j \mid 0 \leq i, j \leq d\}$ gives an equitable partition of X, where $D_i^j = D_i^j(x,y)$, $\partial(x,y) = h$.

Example 4.3.1 (i) *$H(d,q)$ is 1-homogeneous. Is it 2-homogeneous?*

(ii) *$J(7,3)$ is not 1-homogeneous.*

(ii) *$H(d,q)$ and $J(n,d)$ are d-homogeneous.*

Definition 4.3.2 Let Γ be a distance-regular graph of order (s,t) with diamter d, and let $\mathcal{L}$ be the set of all maximal cliques of size $s+1$.

(i) Γ is said to be a *regular near polygon* if $\partial(x,l) = i < d$ implies $|\Gamma_i(x) \cap l| = 1$, for any $l \in \mathcal{L}$.

(ii) Γ is said to be a *regular near 2d-gon* if $\partial(x,l) = i \leq d$ implies $|\Gamma_i(x) \cap l| = 1$, for any $l \in \mathcal{L}$.

Example 4.3.2 (i) *Bipartite distance-regular graphs are RN 2d-gons of order $(1,t)$.*

(ii) *Almost bipartite distance-regular graphs are RNP.*

(iii) $H(d,q)$ *is RN* $2d$*-gon.*

In the following we list some developments in this direction:

- In [205], Miklavič proved that Q-polynomial distance-regular graphs with $a_1 = 0$ are 1-homogeneous.
- In [206], Miklavič proved that Q-polynomial bipartite distance-regular graphs with $c_2 = 1$ have an equitable partition with $4d-4$ cells. Moreover, he also proved that Q-polynomial distance-regular graphs of negative type have an equitable partition with $4d-1$ cells.
- In [217], Nomura proved that an Γ is RN $2d$-gon of order (s,t) if and only if Γ is 1-homogeneous.
- In [172] Jurišić, Koolen and Miklavic proved that triangle- and pentagon-free distance-regular graphs with an eigenvalue multiplicity equal to the valency are 2-homogeneous.
- In [218], Nomura proved that an almost bipartite distance-regular graph with an eigenvalue multiplicity equal to the valency is 2-homogeneous, and classified all these graphs.

Problem 4.3.1 Classify all primitive distance-regular graphs of large diameter with $a_1 \neq 0$.

Problem 4.3.2 Derive some homogeneity when Γ is of order (s,t), $s = m$. (Hermitian form graphs satisfy this condition).

Problem 4.3.3 Classify all distance-regular graphs of order (s,t) such that γ_i exists for any $i < d$.

Problem 4.3.4 Classify all distance-regular graphs such that γ_i exists for any $i < d$.

Proposition 4.3.1 *(Suzuki) Let* Γ *be a* Q*-polynomial RN* $2d$*-gon of order* (s,t)*, and let* $E = \frac{1}{|X|}\sum_{i=0}^{d} \theta_i^* A_i$ *be an idempotent. If*

$$s \neq -\frac{\theta_j^* - \theta_{j-1}^*}{\theta_{j+1}^* - \theta_j^*}$$

Then Γ *is* d*-homogeneous.*

Problem 4.3.5 Classify all d-homogeneous RN $2d$-gons.

Problem 4.3.6 Classify all d-homogeneous Q-polynomial distance-regular graphs.

Problem 4.3.7 Classify all distance-regular graphs of diameter d such that $\Gamma_d(x) \simeq H(m,q)$.

Problem 4.3.8 Classify all distance-regular graphs of diameter d such that $\Gamma_d(x) \simeq J(m,s)$.

Problem 4.3.9 Classify all d-homogeneous distance-regular graphs of diameter d such that $\Gamma_d(x) \simeq H(m,q)$.

Problem 4.3.10 Classify all d-homogeneous distance-regular graphs of diameter d such that $\Gamma_d(x) \simeq J(m,s)$.

Chapter 5

Terwilliger Algebras and their Modules

5.1 Terwilliger Algebras

We now introduce the Terwilliger algebra of a distance-regular graph with respect to a nonempty subset of vertices.

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d, and let Y be a nonempty subset of X. The number

$$w(Y) = \max\{\partial(x, y) \mid x, y \in Y\}$$

is called the *width* of Y in Γ. In particular, $w(X) = d$.

For $x \in X$, let

$$\partial(x, Y) = \partial(Y, x) = \min\{\partial(x, y) \mid y \in Y\}.$$

Set

$$\tau = \tau(Y) = \max\{\partial(x, Y) \mid x \in X\}.$$

The number τ is often called the *covering radius* of Y in X. For $i \in \{0, 1, \ldots, \tau\}$, let

$$\Gamma_i(Y) = \{x \in X \mid \partial(x, Y) = i\},$$

and let $E_i^* = E_i^*(Y)$ denote the diagonal matrix in $\mathrm{Mat}_X(\boldsymbol{C})$ with (x, x)-entry

$$(E_i^*)_{x,x} = \begin{cases} 1 & \text{if } x \in \Gamma_i(Y), \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Throughout this section, we adopt the convention that $E_i = 0$ and $E_j^* = 0$ for any integers i and j such that $i < 0$, $j < 0$, $i > d$ or $j > \tau$.

For a vector $\boldsymbol{v} = \sum_{x \in X} \alpha(x)\hat{x} \in V$ expressed as a linear combination of $\hat{x}$'s, $\mathrm{supp}(\boldsymbol{v})$ denotes the *support* of $\boldsymbol{v}$, i.e., $\mathrm{supp}(\boldsymbol{v}) = \{x \in X \mid \alpha(x) \neq 0\}$.

By definition, $E_i^*\boldsymbol{v} = \boldsymbol{v}$ if and only if $\mathrm{supp}(\boldsymbol{v}) \subset \Gamma_i(Y)$. Moreover, for a subset Z of X, $\mathbf{1}_Z = \sum_{z \in Z} \hat{z}$ is called the *characteristic vector* of Z.

Let $\Gamma_i(Y) = Y_i$ $(0 \leq i \leq \tau)$. Note that

$$X = Y \cup Y_1 \cup \cdots \cup Y_\tau.$$

Lemma 5.1.1 *For $h, i, j \in \{0, 1, \ldots, \tau\}$, the following hold.*

(i) $E_i^* E_j^* = \delta_{i,j} E_i^*$ $(0 \le i, j \le \tau)$.

(ii) $E_0^* + E_1^* + \cdots + E_\tau^* = I$.

(iii) ${}^t(E_i^*) = \overline{E_i^*} = E_i^*$.

(iv) $E_h^* A_i E_j^* \neq 0$ *if and only if there exist vertices* $x \in \Gamma_h(Y)$ *and* $y \in \Gamma_j(Y)$ *such that* $\partial(x, y) = i$.

(v) *If* $E_h^* A_i E_0^* \neq 0$, *then* $h \le i \le h + w(Y)$.

Proof. Parts (i) – (iv) are immediate from (5.1). Since $E_h^* A_i E_0^*$, there exist $x \in \Gamma_h(Y)$ and $y \in Y$ such that $\partial(x, y) = i$. Pick $z \in Y$ such that $\partial(x, z) = h$. Then $\partial(z, y) \le w(Y)$. By (1.1), $h \le \partial(x, y) \le h + w(Y)$. Thus (v) follows. ■

Let $V = C^X$. We have the decompositions

$$\begin{aligned} V &= E_0V + E_1V + \cdots + E_dV \text{ (orthogonal direct sum)} \\ &= E_0^*V + E_1^*V + \cdots + E_\tau^*V \text{ (orthogonal direct sum).} \end{aligned}$$

Definition 5.1.1 Let $\Gamma = (X, R)$ be a distance-regular graph, and let Y be a nonempty subset of X such that $\tau = \tau(Y)$. Let $\mathcal{T} = \mathcal{T}(Y)$ denote the subalgebra of $\mathrm{Mat}_X(\boldsymbol{C})$ generated by the Bose-Mesner algebra $\mathcal{M}$ and $E_0^*, E_1^*, \ldots, E_\tau^*$. We call $\mathcal{T}$ the *Terwilliger algebra* (or *subconstituent algebra*) of Γ with respect to Y.

Definition 5.1.2 A submodule of V is a subspace of V which is invariant under the action of $\mathcal{T}$ by the usual matrix multiplication. A submodule W of V is *irreducible* if there is no proper submodule in W.

The vector space V is called the *standard module* of $\mathcal{T}$. It is a fact that every $\mathcal{T}$-module is isomorphic to a submodule of V. Thus, the term $\mathcal{T}$-module shall refer only to submodule of V.

Since $\mathcal{T} = \mathcal{T}(Y)$ is generated by symmetric real matrices, it is semisimple, i.e., V may be written as a directed sum of irreducible $\mathcal{T}$-modules.

Definition 5.1.3 Let W be an irreducible $\mathcal{T}(Y)$-module. W is said to be *thin* whenever

$$\dim E_i^* W \le 1 \qquad \text{for all } i \in \{0, 1, \ldots, d\}.$$

W is said to be *dual-thin* whenever

$$\dim E_i W \le 1 \qquad \text{for all } i \in \{0, 1, \ldots, d\}.$$

Definition 5.1.4 Let W be an irreducible $\mathcal{T}(Y)$-module. The *endpoint* ν and *diameter* δ of W are the nonnegative integers defined by the following.

$$\nu = \min\{i \mid E_i^* W \neq 0\}, \ \nu + \delta = \max\{i \mid E_i^* W \neq 0\}.$$

Lemma 5.1.2 ([269, Lemma 3.9]) *Let* $\mathcal{T} = \mathcal{T}(Y)$ *and let* W *denote an irreducible* $\mathcal{T}$*-module of endpoint* ν *and diameter* δ. *Then*

$$E_j^* W \neq 0 \text{ if and only if } \nu \le j \le \nu + \delta.$$

Moreover, the following hold.

(i) $AE_j^* W \subset E_{j-1}^* W + E_j^* W + E_{j+1}^* W$ *for every* $j \in \{0, 1, \ldots, d\}$.

(ii) $E_i^* A E_j^* W \neq 0$ *if* $|i-j| = 1 \quad (\nu \leq i, j \leq \nu + \delta)$.

(iii) *Suppose* W *is thin. Then for every* $i \in \{0, 1, \ldots, \delta\}$

$$\begin{aligned} & E_\nu^* W + E_{\nu+1}^* W + \cdots + E_{\nu+i}^* W \\ = \quad & E_\nu^* W + A E_\nu^* W + \cdots + A^i E_\nu^* W. \end{aligned}$$

(iv) *Suppose* W *is thin. Then*

$$E_j W = E_j E_\nu^* W, \text{ for } j \in \{0, 1, \ldots, d\}.$$

(v) *If* W *is thin, then* W *is dual thin.*

Proof. Let $j \in \{0, 1, \ldots, d\}$. By Lemma 5.1.1 (ii) and (iv), we have

$$\begin{aligned} A E_j^* W &= \sum_{i=0}^{d} E_i^* A E_j^* W \\ &= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W \\ &\subset E_{j-1}^* W + E_j^* W + E_{j+1}^* W. \end{aligned}$$

Hence we have (i).

Let $\nu' = \max\{i \mid E_i^* W \neq 0, \text{ and } E_{i-1}^* A E_i^* W = 0\}$. Let δ' be the least nonnegative integer satisfying the following.

$$E_{\nu'+\delta'}^* W \neq 0 \quad \text{ and that } \quad E_{\nu'+\delta'+1}^* A E_{\nu'+\delta'}^* W = 0.$$

Let $W' = E_{\nu'}^* W + E_{\nu'+1}^* W + \cdots + E_{\nu'+\delta'}^* W$. Then $AW' \subset W'$ by (i) in this lemma. Since $E_j^* W' \subset W'$ for every $j \in \{0, 1, \ldots, d\}$, W' is a non-zero $\mathcal{T}$-invariant subspace of W. Since W is irreducible, we have $W = W'$. This shows $\nu = \nu'$ and $\delta = \delta'$.

Since

$$A E_\nu^* W = E_\nu A E_\nu^* W + E_{\nu+1}^* A E_\nu^* W,$$

$E_{\nu+1}^* A E_\nu^* W \neq 0$. Then $E_{\nu+1}^* W \neq 0$. By induction, $E_j^* W \neq 0$ for $j = \nu, \nu+1, \ldots, \nu+\delta$. Therefore,

$$E_{j+1}^* A E_j^* W \neq 0 \quad (\nu \leq j < \nu + \delta), \tag{5.2}$$

and

$$E_{j-1}^* A E_j^* W \neq 0 \quad (\nu < j \leq \nu + \delta).$$

Thus we have (ii). In particular, we have $E_j^* W \neq 0$ if and only if $\nu \leq j \leq \nu + \delta$.

If W is thin, the property (5.2) implies (iii). Thus we have $W = \mathcal{M} E_\nu^* W$. Multiplying both sides of this equation on the left by E_j gives (iv). Since $\dim E_j W = \dim E_j E_\nu^* W$ by (iv), (v) holds. ■

Lemma 5.1.3 *Let* $\mathcal{T} = \mathcal{T}(Y)$ *and let* W *be an irreducible* $\mathcal{T}$*-module. Then the following are equivalent.*

(i) $W = \mathcal{M}\boldsymbol{v}$ *for some vector* $\boldsymbol{v}$.

(ii) W *is dual thin.*

Moreover, if W *is thin, then* W *is dual-thin and the vector* $\boldsymbol{v}$ *above can be taken from* $E_\nu^* W$, *where* ν *is the endpoint of* W.

Proof. Suppose $W = \mathcal{M}\boldsymbol{v}$ for some vector $\boldsymbol{v} \in V$. Then for every $i \in \{0, 1, \ldots, d\}$,

$$E_i W = E_i \mathcal{M}\boldsymbol{v} \subset \mathrm{Span}(E_i \boldsymbol{v}).$$

Hence W is dual thin.

Conversely, suppose that W is dual thin. Then $\dim E_i W \leq 1$. Hence we can choose $\boldsymbol{v}_i$ so that $E_i W = \mathrm{Span}(\boldsymbol{v}_i)$ for every $i \in \{0, 1, \ldots, d\}$. Let $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_d$. Since $E_i \boldsymbol{v} = \boldsymbol{v}_i$, $E_i W \subset \mathcal{M}\boldsymbol{v}$ and $W \subset \mathcal{M}\boldsymbol{v}$. Therefore, $W = \mathcal{M}\boldsymbol{v}$.

If W is thin, then $\dim E_\nu^* W = 1$. Hence by Lemma 5.1.2 (v) W is dual thin and the last assertion is obvious. ■

Theorem 5.1.4 ([269, Lemma 3.9]) *Let $\Gamma = (X, R)$ be a distance-regular graph, Y be a nonempty subset of X and $\mathcal{T} = \mathcal{T}(Y)$. For a vector $\boldsymbol{v} \in E_0^* V$, set*

$$\rho_{\boldsymbol{v}(t)} = \frac{1}{|X|} \sum_{j=0}^{w(Y)} \eta^{(j)}(\boldsymbol{v}) \frac{v_j(t)}{k_j},$$

where $\eta^{(j)}(\boldsymbol{v}) = \frac{{}^t\boldsymbol{v} A_i \bar{\boldsymbol{v}}}{{}^t\boldsymbol{v}\bar{\boldsymbol{v}}}$. Then $w(Y) = |\{i \mid \rho_{\boldsymbol{v}}(\theta_i) = 0\}|$ if and only if $\mathcal{T}\boldsymbol{v}$ is thin of dimension $d - w(Y) + 1$. In particular, $\boldsymbol{v}$ is an eigenvector of $E_0^* A_i E_0^*$.

Definition 5.1.5 (i) $\boldsymbol{v} \in E_0^*$ is a tight vector if $w(Y) = |\{i \mid \rho_{\boldsymbol{v}}(\theta_i) = 0\}|$.

(ii) $\boldsymbol{v} \in E_0^*$ is a $\mathcal{T}$-vector if $\mathcal{T}\boldsymbol{v}$ is an irreducible thin module.

(iii) A nonempty set Y of X is a tight subset if $E_0^*(Y)V$ is spanned by tight vectors.

(iv) A nonempty set Y of X is a $\mathcal{T}$-subset if $E_0^*(Y)V$ is spanned by $\mathcal{T}$-vectors.

5.2 Principal Module and Completely Regular Codes

Let

$$\mathbf{1}_i = \sum_{x \in \Gamma_i(Y)} \hat{x} \text{ and } \mathbf{1} = \mathbf{1}_0 + \mathbf{1}_1 + \cdots + \mathbf{1}_\tau.$$

Lemma 5.2.1 *Let W be an irreducible $\mathcal{T}$-module, then the followings are equivalent.*

(i) $E_0 W \neq \mathbf{0}$.

(ii) $\mathbf{1}_0 \in W$.

(iii) $\mathbf{1}_i \in W_i$, *for each i $(0 \leq i \leq \tau)$.*

For $z \in \Gamma_h(Y)$, define

$$\pi_{ij}^h(z) = |\Gamma_i(Y) \cap \Gamma_j(z)| = (E_h^* A_j \mathbf{1}_i)_z.$$

Definition 5.2.1 Y is a completely regular code if $\pi_{0j}^h(z)$ depends only on j, h with $z \in \Gamma_h(Y)$.

Proposition 5.2.2 *Let $\mathcal{T} = \mathcal{T}(Y)$. Then the following are equivalent.*

(i) *Y is a completely regular code.*

(ii) *$\mathcal{T}\mathbf{1}_0$ is a thin irreducible $\mathcal{T}$-module.*

(iii) *$\mathcal{T}\mathbf{1}_0 = $ Span $(\mathbf{1}_0, \mathbf{1}_1, \ldots, \mathbf{1}_\tau)$.*

(iv) $E^*_h A_j \mathbf{1}_i = \pi^h_{ij}\mathbf{1}_h$.

(v) $A\mathbf{1}_j = \pi^{j-1}_{j1}\mathbf{1}_{j-1} + \pi^{j}_{j1}\mathbf{1}_{j} + \pi^{j+1}_{j1}\mathbf{1}_{j+1}$.

Problem 5.2.1 Determine all tight subgraphs of known distance-regular graphs, i.e., subsets Y such that $E^*_0 V$ is spanned by tight vectors with respect to Y. (See [23]).

$H(d, q)$ in $H(D, q)$ with $d < D$ or dual polar subspaces in a dual polar space are some examples.

Problem 5.2.2 Let Y be a subset with $w(Y) + w^*(Y) = d$. Then Γ is Q-polynomial distance-regular. (See [27]). Suppose Y is connected, the Y is geodetically closed. (See Tanaka EKR).

Problem 5.2.3 Develop a theory of a distance-regular graph such that $\Gamma_d(x)$ is a $\mathcal{T}$-subgraph. If we further assume that it is 1-thin, then it should give very strong restriction.

Problem 5.2.4 Classify distance-regular graphs Γ such that $\Gamma_d(x)$ is a $\mathcal{T}$-subgraph isomorphic to a known distance-regular graph, such as $J(n, d)$ or $H(d, q)$. (See [126, 158, 159, 238, 248, 281, 282])

Problem 5.2.5 Study 1-thin distance-regular graphs such that $\Gamma_D(x)$ is a clique. Decide the condition for such graphs to be Q-polynomial. (See [269]).

Problem 5.2.6 Find correspondence between eigenmatrices of Γ and those on $E^*_0(Y)V$ when Y is a tight subgraph (or $\mathcal{T}$-subgraph). (See [23],[146]).

5.3 Geometric Girths and Thin Properties

A distance-regular graph Γ of order (s, t) with diameter $d \geq 2$ is called a *regular near* $2d$*-gon* (or the collinearity graph of a regular near $2d$-gon) if for every vertex x and a maximal clique L with $\partial(x, L) = i$, $|\Gamma_i(x) \cap L| = 1$. A regular near $2d$-gon of order $(1, t)$ is nothing but a bipartite distance-regular graph. A regular near $2d$-gon with geometric girth $2d$ is called a *generalized* $2d$*-gon.*

In [40], B. Collins proved that if a distance-regular graph Γ of valency at least three with $c_3 = 1$ is thin, then Γ is a generalized 8-gon of order $(1, t)$ (i.e., the incidence graph of a generalized quadrangle). In particular, if a distance-regular graph Γ of order (s, t) with $k = s(t + 1) > 2$ is thin, then $\mathrm{gg}(\Gamma) \leq 8$. The following results prove two kinds of refinements of his result. (See also the remark at the end.)

Theorem 5.3.1 *Let Γ be a distance-regular graph of order (s, t) with diameter $d \geq 2$, valency $k = s(t+1) > 2$ and geometric girth $g = \mathrm{gg}(\Gamma)$. Let x be a fixed vertex, $\mathcal{T} = \mathcal{T}(x)$ and e a positive integer. Then the following hold.*

(i) *Let $\boldsymbol{v}$ be a nonzero vector in $E^*_e V$. If $W = \mathcal{T}\boldsymbol{v}$ is a thin irreducible module of endpoint e with $2e + 2 \leq g$, then $E^*_{e-1}A\boldsymbol{v} = \mathbf{0}$ and $E^*_e A\boldsymbol{v} = \mu\boldsymbol{v}$ with $\mu \in \{s - 1, -1\}$.*

(ii) *Let $\boldsymbol{v}$ be a nonzero vector of $E^*_e V$ such that $E^*_e A\boldsymbol{v} = (s-1)\boldsymbol{v}$ and $E^*_{e-1}A\boldsymbol{v} = \mathbf{0}$. If $4 \leq 4e \leq g$, then $\dim \mathcal{M}\boldsymbol{v} \geq d - 1$. In particular, if $\mathcal{T}\boldsymbol{v}$ is a thin irreducible $\mathcal{T}$-module, then $e \leq 2$.*

(iii) *Let $\boldsymbol{v}$ be a nonzero vector of $E^*_e V$ such that $E^*_e A\boldsymbol{v} = -\boldsymbol{v}$ and $E^*_{e-1}A\boldsymbol{v} = \mathbf{0}$. If $4 \leq 4e \leq g$, then $\dim \mathcal{M}\boldsymbol{v} = d$. In particular, if $\mathcal{T}\boldsymbol{v}$ is a thin irreducible $\mathcal{T}$-module, then $e \leq 1$.*

In particular, if there is a thin irreducible $\mathcal{T}$-module of endpoint 3, then $g \leq 11$.

Chapter 6

Solutions to Exercises

6.1 Chapter 1.

1. Let $\partial(u,w) = s$ and $\partial(w,v) = t$. Then there exists a path $(u = w_0, w_1, \ldots, w = w_s)$ of length s from u to w, and there exists a path $(w = w_s, w_{s+1}, \ldots, w = w_{s+t})$ of length t from w to v. Hence $(u = w_0, w_1, \ldots, w = w_{s+t})$ is a path of length $s+t$ from u to v; consequently $\partial(u,v) \leq s+t$ by Definition 1.1.1.

2. Suppose $\partial(u,v) = s$ and $\partial(\sigma(u), \sigma(v)) = t$. It suffices to prove $s = t$. Pick a path $(u = u_0, u_1, \ldots, v = u_s)$ of length s from u to v. Then $(\sigma(u) = \sigma(u_0), \sigma(u_1), \ldots, \sigma(v) = \sigma(u_s))$ is a path of length s from $\sigma(u)$ to $\sigma(v)$; consequently, $t \leq s$.

   On the other hand, since $\sigma^{-1} \in \mathrm{Aut}(\Gamma)$, by the above argument we obtain $s \leq t$. Hence $s = t$.

3. Suppose $|\Gamma_i(u) \cap \Gamma_j(v)| = s$ and $|\Gamma_i(x) \cap \Gamma_j(y)| = t$. It suffices to prove $s = t$. For each vertex $w \in \Gamma_i(u) \cap \Gamma_j(v)$, $z := \sigma(w) \in \Gamma_i(x) \cap \Gamma_j(y)$; and so $s \leq t$.

   On the other hand, since $\sigma^{-1} \in \mathrm{Aut}(\Gamma)$, by the above argument we obtain $s \leq t$. Hence $s = t$.

4. Let Γ be a bipartite graph, i.e., there is a nontrivial bipartition $X = X^+ \cup X^-$ of vertices such that both induced subgraphs on X^+ and X^- are empty. Suppose Γ has a circuit $(u_0, u_1, \ldots, u_{2s}, u_0)$ of length $2s+1$. Then $u_0 \in X^+$ or X^-. Without loss of generality, we assume $u_0 \in X^+$. By assumption, we have

   $$u_1 \in X^-, u_2 \in X^+, \ldots, u_{2s} \in X^+,$$

   a contradiction to the fact that the induced subgraph on X^+ is empty.

   Conversely, suppose that Γ has no circuits of odd length. For a fixed vertex $u \in X$, the vertex set X has the following partition:

   $$X = \Gamma_0(u) \cup \Gamma_1(u) \cup \ldots \cup \Gamma_d(u),$$

   where $\Gamma_i(u) = \{v \in X \mid \partial(u,v) = i\}$. Since Γ has no circuits of odd length, the induced subgraph on $\Gamma_i(u)$ is empty for $0 \leq i \leq d$. Let

   $$X^+ = \{\Gamma_i(u) \mid i \,\text{is even}\} \text{ and } X^+ = \{\Gamma_i(u) \mid i \,\text{is odd}\}.$$

   By 1.1, the both induced subgraphs on X^+ and X^- are empty; and so Γ is bipartite.

6.2 Chapter 2. Basic Theory of Distance-Regular Graphs

1. n is even: $\iota(C_n) = \left\{ \begin{array}{ccccc} * & 1 & \cdots & 1 & 2 \\ 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & \cdots & 1 & * \end{array} \right\}.$

   n is odd: $\iota(C_n) = \left\{ \begin{array}{ccccc} * & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ 2 & 1 & \cdots & 1 & * \end{array} \right\}.$

2. The intersection arrays become as follows.

   Tetrahedron : $\left\{ \begin{array}{cc} * & 1 \\ 0 & 2 \\ 3 & * \end{array} \right\}.$

   Cube : $\left\{ \begin{array}{cccc} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{array} \right\}.$

   Octahedron : $\left\{ \begin{array}{ccc} * & 1 & 4 \\ 0 & 2 & 0 \\ 4 & 1 & * \end{array} \right\}.$

   Dodecahedron : $\left\{ \begin{array}{cccccc} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{array} \right\}.$

   Icosahedron : $\left\{ \begin{array}{cccc} * & 1 & 2 & 5 \\ 0 & 2 & 2 & 0 \\ 5 & 2 & 1 & * \end{array} \right\}.$

3 (a) We prove the result by induction on r. By the definition of Johnson graphs, the result is valid for $r = 1$. Now suppose the result holds for any integer $m \leq r$, i.e.,

$$\partial(\alpha, \beta) = m \text{ if and only if } |\alpha \cap \beta| = d - m.$$

For any two vertices α and β at distance $r + 1$, pick $\gamma \in \Gamma_r(\alpha) \cap \Gamma(\beta)$. By induction hyperthesis, we have $|\alpha \cap \gamma| = d - r$; and so

$$\alpha = \{s_1, \ldots, s_{d-r}, i_1, \ldots, i_r\} \text{ and } \gamma = \{s_1, \ldots, s_{d-r}, j_1, \ldots, j_r\},$$

where i's and j's are all distinct. Since $|\beta \cap \gamma| = d - 1$, β has one of the following possibilities:

(i) $\beta = \{s_1, \ldots, s_{d-r}, j_1, \ldots, j_{r-1}, i_a\}$,
(ii) $\beta = \{s_1, \ldots, s_{d-r}, j_1, \ldots, j_{r-1}, a\}$,
(iii) $\beta = \{s_1, \ldots, s_{d-r-1}, i_a, j_1, \ldots, j_{r-1}, i_a\}$,
(iv) $\beta = \{s_1, \ldots, s_{d-r-1}, a, j_1, \ldots, j_{r-1}, i_a\}$,

where $i_a \in \{i_1, i_2, \ldots, i_r\}$ and $a \notin \alpha \cup \beta$. In Case (i), we have $|\alpha \cap \beta| = d - (r - 1)$, so by induction hypothesis $\partial(\alpha, \beta) = r - 1$, a contradiction. In Cases (ii) and (iii) we have $|\alpha \cap \beta| = d - r$, so again by induction hypothesis $\partial(\alpha, \beta) = r$, a contradiction. Therefore Case (iv) holds, i.e., $|\alpha \cap \beta| = d - (r + 1)$.

Conversely, suppose $|\alpha \cap \beta| = d-(r+1)$. By induction hypothesis $\partial(\alpha,\beta) \geq r+1$, also we may assume that

$$\alpha = \{s_1,\ldots,s_{d-(r+1)}, i_1,\ldots,i_{r+1}\} \text{ and } \beta = \{s_1,\ldots,s_{d-(r+1)}, j_1,\ldots,j_{r+1}\},$$

where i's and j's are all distinct. Let

$$\gamma = \{s_1,\ldots,s_{d-(r+1)}, j_1,\ldots,j_r, i_{r+1}\}.$$

Then $|\alpha \cap \gamma| = d-r$. By induction hypothesis $\partial(\alpha,\gamma) = r$ and $\partial(\beta,\gamma) = 1$. Therefore,

$$\partial(\alpha,\beta) \leq \partial(\alpha,\gamma) + (\beta,\gamma) = r+1.$$

(b) Any permutation σ on V induces a permutation on X as following:

$$\{\sigma_1,\sigma_2,\ldots,\sigma_d\}^\sigma = \{\sigma_1^\sigma,\sigma_2^\sigma,\ldots,\sigma_d^\sigma\}.$$

By Exercise 2 σ is an automorphism of Γ; consequently $S_n \leq \mathrm{Aut}(\Gamma)$. Let $\partial(\alpha,\beta) = \partial(\alpha',\beta') = m$. By (a) we may assume that

$$\alpha = \{s_1,\ldots,s_{k-m}, i_1,\ldots,i_m\} \text{ and } \beta = \{s_1,\ldots,s_{k-m}, j_1,\ldots,j_m\},$$

where where i's and j's are all distinct;

$$\alpha^* = \{s_1^*,\ldots,s_{k-m}^*, i_1^*,\ldots,i_m^*\} \text{ and } \beta^* = \{s_1^*,\ldots,s_{k-m}^*, j_1^*,\ldots,j_m^*\},$$

where where i^*'s and j^*'s are all distinct. Let

$$\sigma = \begin{pmatrix} s_1 & \cdots & s_{k-m} & i_1 & \cdots & i_m & j_1 & \cdots & j_m & t_{k+m+1} & \cdots & t_n \\ s_1^* & \cdots & s_{k-m}^* & i_1^* & \cdots & i_m^* & j_1^* & \cdots & j_m^* & t_{k+m+1} & \cdots & t_n \end{pmatrix}.$$

Then $\alpha^\sigma = \alpha^*$ and $\beta^\sigma = \beta^*$; consequently Γ is distance-transitive.

4 The intersection array of $J(n,d)$ is given by $\iota(J(n,d)) =$

$$\left\{ \begin{array}{ccccccc} * & 1 & \cdots & i^2 & \cdots & (d-1)^2 & d^2 \\ 0 & n-2 & \cdots & (n-2i)i & \cdots & (n-2(d-1))(d-1) & (n-2d)d \\ d(n-d) & (d-1)(n-d-1) & \cdots & (d-i)(n-d-i) & \cdots & n+1 & * \end{array} \right\}$$

5 (a) It is similar to that of Exercise 3 (a) and will be omitted.

(b) Note that $S_q \wr S_d$ acts on the vertex set X as following:

$$(x_1,x_2,\ldots,x_d)^{(\varphi,\sigma)} = (x_{1^{\sigma^{-1}}}^{\varphi_1}, x_{2^{\sigma^{-1}}}^{\varphi_2},\ldots,x_{d^{\sigma^{-1}}}^{\varphi_d}),$$

where $\varphi = (\varphi_1,\varphi_2,\ldots,\varphi_d)$. Then $S_q \wr S_d \leq \mathrm{Aut}(H(d,q))$. It is routine to check that $H(d,q)$ is distance-transitive.

6 The intersection array of $H(d,q)$ is given by $\iota(H(d,q)) =$

$$\left\{ \begin{array}{ccccccc} * & 1 & \cdots & i & \cdots & d-1 & d \\ 0 & q-2 & \cdots & i(q-2) & \cdots & (d-1)(q-2) & d(q-2) \\ d(q-1) & (d-1)(q-1) & \cdots & (d-i)(q-i) & \cdots & q-1 & \end{array} \right\}$$

7 It is similar to that of Exercise 3 and will be omitted.

8 The intersection array of O_k is given by

$$\iota(O_k) = \left\{ \begin{array}{cccccccccc} * & 1 & 1 & \cdots & i & i & \cdots & \frac{1}{2}(k-1) & \frac{1}{2}(k-1) \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{2}(k+1) \\ k & k-1 & k-1 & \cdots & k-i & k-i & \cdots & \frac{1}{2}(k+1) & * \end{array} \right\}$$

for k odd, and

$$\iota(O_k) = \left\{ \begin{array}{cccccccccc} * & 1 & 1 & \cdots & i & i & \cdots & \frac{1}{2}k-1 & \frac{1}{2}k-1 & \frac{1}{2}k \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2}k \\ k & k-1 & k-1 & \cdots & k-i & k-i & \cdots & \frac{1}{2}k & \frac{1}{2}k & * \end{array} \right\}$$

for k is even.

9. Let N be the number of triples (x, y, z) satisfying

$$\partial(x, y) = h, \partial(x, z) = i \text{ and } \partial(z, y) = j.$$

For a fixed vertex x, y has k_h choices. For given x and y at distance h, z has $p^h_{i,j}$ choices. Hence $N = nk_h p^h_{i,j}$, where n is the number of vertices. By similar arguments, we also obtain $N = nk_i p^i_{j,h}$ and $N = nk_j p^j_{h,i}$. If we equate these three expressions for N, we obtain the equalities.

10. Suppose $j = i+1, h = 1$ in Exercise 9. Then $b_i k_i = c_{i+1}k_{i+1}$.

Let N be the number of edges between $\Gamma_i(u)$ and $\Gamma_{i+1}(u)$, i.e., $N = e(\Gamma_i(u), \Gamma_{i+1}(u))$. For any vertex $x \in \Gamma_i(u)$, $e(x, \Gamma_{i+1}(u)) = b_i$; and so $N = k_i b_i$. On the other hand, for any vertex $y \in \Gamma_{i+1}(u)$, $e(y, \Gamma_i(u)) = c_{i+1}$; and so $N = k_{i+1}c_{i+1}$. If we equate these two expressions for N, we obtain $b_i k_i = c_{i+1}k_{i+1}$.

11. (a) By the definition of D^i_j, $|D^i_j| = p^h_{i,j}$.

(b) Let $0 \le i \le d-h$. If $h = d$, then $i = 0$. In this case, $|D^i_{h+i}| = |D^0_d| = p^d_{0,d} = 1$. Hence, $D^0_d \ne \emptyset$. Now suppose $0 \le h \le d-1$. Since $|B(y, x)| = b_h$, $B(y, x) \ne \emptyset$. Pick $x_1 \in B(y, x)$. By induction, there exists vertices $x = x_0, x_1, \ldots, x_i$ such that $x_{j+1} \in B(y, x_j)$ for $0 \le j \le i-1$. Since $x_i \in D^i_{h+i}$, we have $D^i_{h+i} \ne \emptyset$.
Let $(x = x_0, x_1, \ldots, x_h = y)$ be a path of length h from x to y. Then $x_i \in D^i_{h-i}$ for $0 \le i \le h$. Hence, $D^i_{h-i} \ne \emptyset$.

(c) Since $e(D^i_j, D^s_t) = \emptyset$ if $|i-s| > 1$ or $|j-t| > 1$, we have

$$c_i = e(u, \Gamma_{i-1}(x)) = e(u, D^{i-1}_{j+1} \cup D^{i-1}_j \cup D^{i-1}_{j-1}) = e(u, D^{i-1}_{j+1}) + e(u, D^{i-1}_j) + e(u, D^{i-1}_{j-1}).$$

In a similar way, $\partial(y, u) = j$ implies that

$$c_j = e(u, D^{i+1}_{j-1}) + e(u, D^i_{j-1}) + e(u, D^{i-1}_{j-1}).$$

(d) Since $e(D^i_j, D^s_t) = \emptyset$ if $|i-s| > 1$ or $|j-t| > 1$, we have

$$a_i = e(u, \Gamma_i(x) = e(u, D^i_{j+1} \cup D^i_j \cup D^i_{j-1}) = e(u, D^i_{j+1}) + e(u, D^i_j) + e(u, D^i_{j-1}).$$

In a similar way, $\partial(y, u) = j$ implies that $a_j = e(u, D^{i-1}_j) + e(u, D^i_j) + e(u, D^{i+1}_j)$.

(e) Since $e(D^i_j, D^s_t) = \emptyset$ if $|i-s| > 1$ or $|j-t| > 1$, we have

$$b_i = e(u, \Gamma_{i+1}(x)) = e(u, D^{i+1}_{j+1} \cup D^{i+1}_j \cup D^{i+1}_{j-1}) = e(u, D^{i+1}_{j+1}) + e(u, D^{i+1}_j) + e(u, D^{i+1}_{j-1}).$$

In a similar way, $\partial(y, u) = j$ implies that

$$b_j = e(u, D^{i+1}_{j+1}) + e(u, D^i_{j+1}) + e(u, D^{i-1}_{j+1}).$$

12. Let $\partial(x, y) = h$. By Exercise 11 (b), $D^i_j(x, y) \neq \emptyset$ for $i + j = h$. Pick $u \in D^i_j(x, y)$. Since $D^{i-1}_j = D^{i-1}_{j-1} = \emptyset$, by Exercise 11 (c) and (d),

$$c_i = e(u, D^{i-1}_{j+1})+e(u, D^{i-1}_j)+e(u, D^{i-1}_{j-1}) = e(u, D^{i-1}_{j+1}) \leq e(u, D^{i-1}_{j+1})+e(u, D^i_{j+1})+e(u, D^{i+1}_{j+1}) = b_j.$$

Let $\partial(x, y) = h$. Pick a path $(x = x_0, x_1, \ldots, y = x_h)$ of length h from x to y. For any $u \in C(x, x_i)$, we have $\partial(u, y) = j + 1$, i.e., $u \in B(y, x_i)$. Then $C(x, x_i) \subseteq B(y, x_i)$; and so $c_i \leq b_j$.

13. (a) Since $D^s_t = \emptyset$ whenever $|s-t| > 1$, we have

$$a_i = e(v, \Gamma_i(x)) = e(v, D^i_{i-1} \cup D^i_i \cup D^i_{i+1})) = e(v, D^i_{i-1}) + e(v, D^i_i) + e(v, D^i_{i+1}).$$

In a similar way, we have

$$a_i = e(v, \Gamma_i(y)) = e(v, D^{i-1}_i) + e(v, D^i_i) + e(v, D^{i+1}_i).$$

Since $e(D^{i-1}_{i-2}, D^i_i) = 0$, we obtain

$$c_i = e(v, \Gamma_{i-1}(x)) = e(v, D^{i-1}_i \cup D^{i-1}_{i-1} \cup D^{i-1}_{i-2}) = e(v, D^{i-1}_i) + e(v, D^{i-1}_{i-1}).$$

Since $e(D^{i+1}_{i+2}, D^i_i) = 0$, we obtain

$$b_i = e(v, \Gamma_{i+1}(x)) = e(v, D^{i+1}_i \cup D^{i+1}_{i+1} \cup D^{i+1}_{i+2}) = e(v, D^{i+1}_i) + e(v, D^{i+1}_{i+1}).$$

(b) By Exercise 11 (a), we have $|D^i_i| = p^1_{i,i}$. By Exercise 10, $kp^1_{i,i} = k_i p^i_{i,1} = k_i a_i$. Hence $a_i = 0$ if and only if $D^i_i = \emptyset$.
If $a_i = 0 \neq a_{i+1}$, then $D^i_i = \emptyset$ and $D^{i+1}_{i+1} \neq \emptyset$. Pick $w \in D^{i+1}_{i+1}$. Then by (a)

$$c_i \leq c_{i+1} = e(w, D^i_{i+1})+e(w, D^i_i) = e(w, D^i_{i+1}) \leq e(w, D^i_{i+1})+e(w, D^{i+1}_{i+1})+e(w, D^{i+2}_{i+1}) = a_{i+1}.$$

If $a_{i+1} = 0 \neq a_i$, then $D^i_i \neq \emptyset$ and $D^{i+1}_{i+1} = \emptyset$. Pick $z \in D^i_i$. Then by (a)

$$b_i = e(z, D^{i+1}_i) + e(z, D^{i+1}_{i+1}) = e(z, D^{i+1}_i) \leq e(z, D^{i+1}_i) + e(z, D^i_i) + e(z, D^{i-1}_i) = a_i.$$

14. The rank 1 diagrams of Five Platonic solids are listed in Figure 6.1.

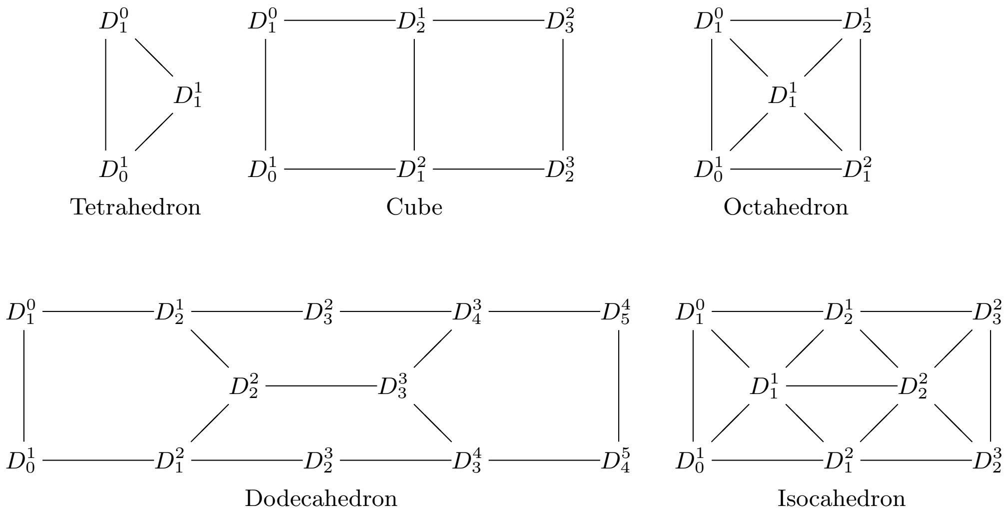


Figure 6.1:

15.

16.

17.

18.

19. Let $J(n,d) = (X,R)$ and $J(n,n-d) = (\bar{X},\bar{R})$. Define

$$\begin{aligned} \sigma:\ X &\longrightarrow \bar{X} \\ \alpha &\longmapsto V \setminus \alpha. \end{aligned}$$

Then σ is an isomorphism from $J(n,d)$ to $J(n,n-d)$; and so $J(n,d) \simeq J(n,n-d)$.

20. (a)-(c) are obvious. Let x and y be vertices of Γ at distance h. Then we have

$$(A_iA_j)_{x,y} = \sum_{z\in X}(A_i)_{x,z}(A_j)_{z,y} = |\{z \in X \mid \partial(x,z) = i,\ \partial(z,y) = j\}| = |\Gamma_i(x)\cap\Gamma_j(y)| = p^h_{i,j}.$$

Then $(A_iA_j)_{x,y}$ coincides with the (x,y)-entry of the right hand side of the equality. Hence, (d) holds.

21. (a) Suppose there exist real numbers $\lambda_0, \lambda_1, \ldots, \lambda_d$ such that

$$\lambda_0A_0 + \lambda_1A_1 + \cdots + \lambda_dA_d = O.$$

For $0 \le i \le d$, let $\partial(x_i, y_i) = i$. Since

$$0 = (\lambda_0A_0 + \lambda_1A_1 + \cdots + \lambda_dA_d)_{x_i,y_i} = \lambda_i$$

for $0 \le i \le d$, $A_0, A_1, \ldots, A_d$ are linearly independent over $\boldsymbol{R}$.

(b) Let $j = 1$ in in Exercise 20 (d). Since $p_{i,1}^h = 0$ for every l such that $|l - i| > 1$ by Lemma 2.1.1, we have at most three nonzero terms in the summation and we have

$$A_i A = p_{i,1}^{i-1} A_{i-1} + p_{i,1}^{i} A_i + p_{i,1}^{i+1} A_{i+1} = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}.$$

(c) By definition, $c_j \neq 0$ for $j = 1, \ldots, d+1$. Hence $v_j(t)$ is a uniquely polynomial of degree j determined by the three term recurrence for $j = 0, 1, \ldots, d+1$. Clearly, $v_0(A) = I = A_0$, and $v_1(A) = A = A_1$. Assume $i \geq 1$. Then by induction hypothesis, we have

$$c_{i+1} v_{i+1}(A) = v_i(A)A - b_{i-1} v_{i-1}(A) - a_i v_i(A) = A_i A - b_{i-1} A_{i-1} - a_i A_i = c_{i+1} A_{i+1}$$

by (b). Hence, $v_i(A) = A_i$ for $i = 0, 1, \ldots, d+1$. It is obvious that $v_{d+1}(t)$ is a minimal polynomial of A.

(d) By (c), it is clear.

22. (a) By Exercise 20 (d) and Exercise 21 (b), we have

$$\begin{aligned}
(AA_i)A_j &= (b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}) A_j \\
&= b_{i-1} (\sum_{h=0}^{d} p_{i-1,j}^h A_h) + a_i (\sum_{h=0}^{d} p_{i,j}^h A_h) + c_{i+1} (\sum_{h=0}^{d} p_{i+1,j}^h A_h) \\
&= \sum_{h=0}^{d} (b_{i-1} p_{i-1,j}^h + a_i p_{i,j}^h + c_{i+1} p_{i+1,j}^h) A_h,
\end{aligned}$$

and

$$\begin{aligned}
A(A_i A_j) &= A(\sum_{h=0}^{d} p_{i,j}^h A_h) \\
&= \sum_{h=0}^{d} p_{i,j}^h A A_h \\
&= \sum_{h=0}^{d} p_{i,j}^h (b_{h-1} A_{h-1} + a_h A_h + c_{h+1} A_{h+1}) \\
&= \sum_{h=0}^{d} (p_{i,j}^{h+1} b_h + p_{i,j}^h a_h + p_{i,j}^{h-1} c_h) A_h.
\end{aligned}$$

Since $(AA_i)A_j = A(A_i A_j)$, we obtain (a) by 21 (a).

(b) By Exercise 20 (d) and Exercise 21 (b), we have

$$\begin{aligned}
(AA_j)A_i &= (b_{j-1} A_{j-1} + a_j A_j + c_{j+1} A_{j+1}) A_i \\
&= b_{j-1} (\sum_{h=0}^{d} p_{j-1,i}^h A_h) + a_j (\sum_{h=0}^{d} p_{j,i}^h A_h) + c_{j+1} (\sum_{h=0}^{d} p_{j+1,i}^h A_h) \\
&= \sum_{h=0}^{d} (b_{j-1} p_{j-1,i}^h A_h + a_j p_{j,i}^h A_h + c_{j+1} p_{j+1,i}^h A_h),
\end{aligned}$$

and

$$(AA_i)A_j = \sum_{h=0}^{d} (b_{i-1} p_{i-1,j}^h + a_i p_{i,j}^h + c_{i+1} p_{i+1,j}^h) A_h.$$

Since $(AA_j)A_i = (AA_i)A_j$, (b) holds by 21 (a).

23. We claim that $p_{i,j}^h$'s are determined by c_i, a_i, b_i $(0 \leq i \leq d)$. We will prove our claim by induction on j. If $j = 0, 1$, it is obvious. Now Suppose $j \geq 2$ and the claim holds for all $l \leq j$. By 22 (b), we have

$$p_{i,j+1}^h = \frac{1}{c_{j+1}} (p_{j,i-1}^h b_{i-1} + p_{j,i}^h (a_i - a_j) + p_{j,i+1}^h c_{i+1} - p_{j-1,i}^h b_{j-1}).$$

By induction hypothesis, $p^h_{i,j+1}$ is determined by c_i, a_i, b_i. Hence our claim is valid. Then $p^h_{i,j}$'s depend only on i, j, k; consequently Γ is distance-regular.

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