An Introduction to Q-polynomial Association Schemes

(Version 2.1)

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November 6, 1997

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[†]This research was partially supported by JSPS–DOST exchange program and the Grant-in-Aid for Scientific Research (No.06640075), the Ministry of Education, Science and Culture, Japan.

Contents

Chapter 1

Introduction

1.1 First Examples

We introduce typical examples for this series of lectures. One of the keys to understand the beautiful theory is to have a good grasp of these examples and check the theory by applying it to them.

Example 1.1.1 *Platonic Solids.* There are five regular polyhedra, which has regular n-gons as faces. They are the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. See Figure 1.1.1. These appear in various fields of mathematics, especially in combinatorics, these are quoted as the best and the most beautiful examples. It is not the exception in this series of lectures on algebraic combinatorics.

In the following we list f, the number of faces, e, the number of edges, v, the number of vertices, n, the number of edges (and of vertices) in each face and k, the degree of each vertex.

Name	f	e	v	n	k
Tetrahedron	4	6	4	3	3
Cube	6	12	8	4	3
Octahedron	8	12	6	3	4
Dodecahedron	12	30	20	5	3
Icosahedron	20	30	12	3	5

Example 1.1.2 *Hyper-cubes.* Let H_n denote the hyper-cube:

$$H_n = \{(x_1, x_2, \ldots, x_n) \mid x_i = \pm 1,\ i = 1, 2, \ldots, n\}.$$

Clearly, the cube can be regarded as H_3. We define the adjacency in H_n as follows. For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in H_n$,

$$\boldsymbol{x} \sim \boldsymbol{y} \iff |\{i \mid x_i \neq y_i,\ i = 1, 2, \ldots, n\}| = 1.$$

Example 1.1.3 *Half Cubes.* Let $\frac{1}{2}H_n$ denote the half cube defined as follows:

$$\frac{1}{2}H_n = \{(x_1, x_2, \ldots, x_n) \mid x_i = \pm 1,\ x_1 x_2 \cdots x_n = 1,\ i = 1, 2, \ldots, n\}.$$

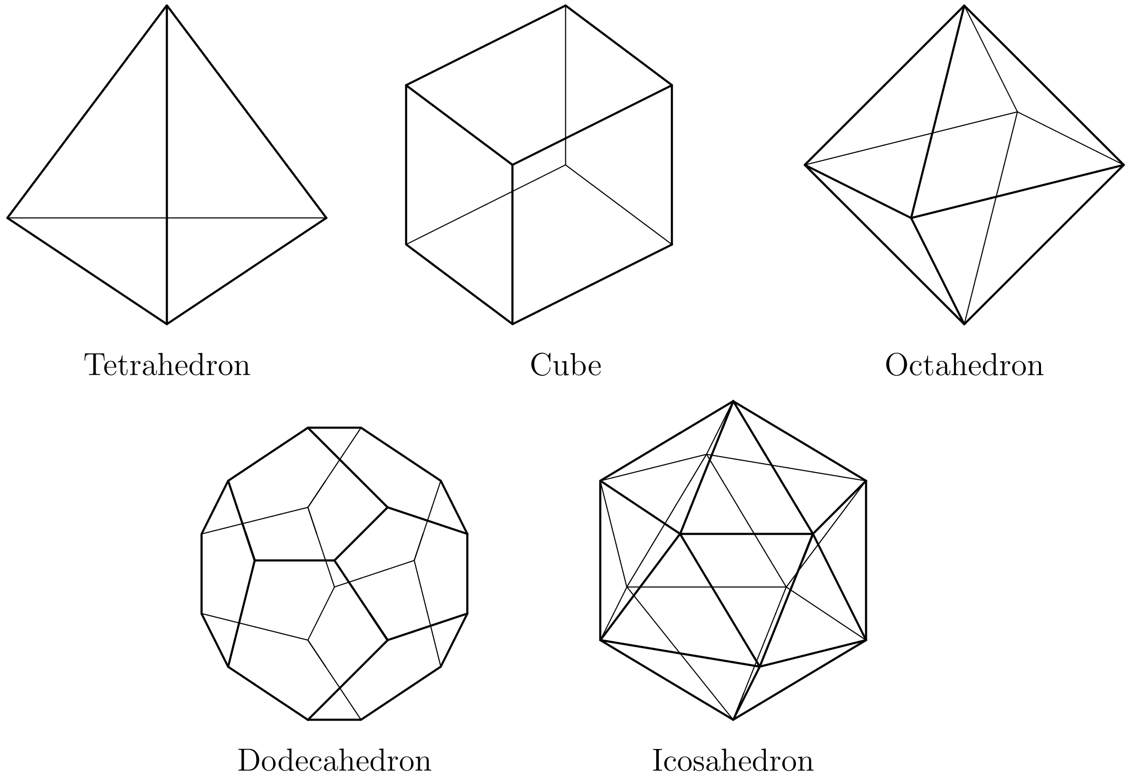


Figure 1.1: Platonic Solids

By definition, it is easy to see that the elements of the half cube is the elements of the hyper-cube with even number of -1's. The adjacency is defined as follows. For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \frac{1}{2}H_n$,

$$\boldsymbol{x} \sim \boldsymbol{y} \Longleftrightarrow |\{i \mid x_i \neq y_i,\ i = 1, 2, \ldots, n\}| = 2.$$

1.2 Graphs

We first give definitions and notational convention of graphs.

Definition 1.2.1 A graph $\Gamma = (V\Gamma, E\Gamma)$ is a pair of sets, $V\Gamma \neq \emptyset$ and $E\Gamma$. $V\Gamma$ is called the *vertex set*, $E\Gamma$ the *edge set*, which is a set of unordered pairs of $V\Gamma$. In general, a graph may include a *loop* $(x, x) \in E\Gamma$. A graph without loops is called a *simple graph*. When we consider a set of *ordered* pairs for $E\Gamma$, Γ is called a *directed graph* or *digraph*, $(x, y) \in E\Gamma$ with $x \neq y$ is called an *arc*.

In the following, assume that all graphs Γ are finite, (i.e., $|V\Gamma| < \infty$), simple graphs otherwise specified.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a graph. When $(x, y) \in E\Gamma$, x and y are said to be *adjacent* and denoted by $x \sim y$. For $x \in V\Gamma$,

$$\Gamma(x) = \{y \in V\Gamma \mid x \sim y\}.$$

A sequence of vertices $x_0, x_1, \ldots, x_l$ is said to be a *walk* (of length l connecting x_0 and x_l), if

$$x_0 \sim x_1 \sim \cdots \sim x_l.$$

A walk above is said to be a *path*, if $x_i \neq x_{i+2}$ for $i = 0, 1, \ldots, l-2$. When $x_0 = x_l$, the walk is said to be *closed*. A closed path is said to be a *circuit*.

Γ is *connected* if, for every pair $x, y \in V\Gamma$ of vertices, there is a walk connecting x and y.

Definition 1.2.2 Let $\Gamma = (V\Gamma, E\Gamma)$ be a graph.

(1) Γ is said to be *k-regular* or *regular* of valency k, if $|\Gamma(x)| = k$ for every $x \in V\Gamma$.

(2) Γ is said to be a *tree*, if Γ is connected and there is no circuit of positive length.

(3) Γ is said to be *planar*, if Γ can be embedded in the sphere in $\boldsymbol{R}^3$ (or plane) without any crossing of edges. Here we say Γ is embedded, when the set of vertices are points in the space and the edges are curves connecting corresponding adjacent points.

(4) Γ is said to be a regular polyhedron, if Γ is a planar graph such that each face (i.e., region surrounded by edges) is an n-gon for a fixed n, each edge is in two faces, and each vertex is in k faces for a fixed k.

The definition of regular polyhedra above is not standard. See an excellent monograph by H.S.M. Coxeter [29].

In the following we leave a few properties of graphs as exercises. The following are standard and easy to prove. For those who are not familiar with graphs consult any standard books, for example see [18].

Exercise 1.2.1 Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph.

1. Show that the following are equivalent.
   (a) Γ is a tree.
   (b) $|V\Gamma| = |E\Gamma| + 1$.
2. There exists a spanning tree, i.e., a tree $\Delta = (V\Delta, E\Delta)$ with $V\Delta = V\Gamma$, and $E\Delta \subset E\Gamma$.

Exercise 1.2.2 [Euler's Equality] Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected planar graph in the sphere with f faces, e edges and v vertices. Then we have

$$f - e + v = 2.$$

Proposition 1.2.1 *Let $\Gamma = (V\Gamma, E\Gamma)$ be a regular polyhedron with v vertices, e edges and f faces of n-gons. Then the parameters coincide with those of a Platonic solid or an n-gon.*

Proof. Let $F\Gamma$ denote the set of faces. Suppose each vertex is in k faces. We may assume that $n \geq 3$.

Since every edge is in two faces and each face has n edges, we have

$$|\{(E, F) \mid E \in E\Gamma,\ F \in F\Gamma,\ E \in F\}| = 2e = nf.$$

Similarly, we have that

$$|\{(P, F) \mid P \in V\Gamma,\ F \in F\Gamma,\ P \in F\}| = kv = nf.$$

Since Γ is a connected planar graph, it follows from Exercise 1.2.2 that $f - e + v = 2$. After eliminating e and v using the equalities, we have the following by dividing the both hand sides by nf:

$$\frac{1}{n} + \frac{1}{k} = \frac{1}{2} + \frac{1}{nf} > \frac{1}{2}.$$

Now it is not difficult to determine the parameters satisfying the conditions above. ■

Exercise 1.2.3 Complete the proof of Proposition 1.2.1.

There is a more geometrical and traditional definition of regular polyhedra:

> A regular polyhedron is a finite connected set of regular n-gons (i.e., the one with equal sides and equal angles), such that every side of each n-gon belongs to just one other n-gon and that there are k n-gons surrounding each vertex form a single circuit.

Since the interior angle of a regular n-gon is $(1 - 2/n)\pi$, we have

$$k(1 - \frac{2}{n})\pi < 2\pi,$$

as the total of k of these angles around the vertex must total less than 2π. By simplifying the above equality, we again have the following:

$$\frac{1}{n} + \frac{1}{k} > \frac{1}{2}.$$

1.3 Distance-Regularity

Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph. For $x, y \in V\Gamma$, let $\partial_\Gamma(x, y) = \partial(x, y)$ denote the distance between x and y, which is defined to be the length of a shortest walk connecting x and y. It is clear that the function ∂ satisfies the usual requirement of the distance functions.

1. $\partial(x, y) \geq 0$, and $\partial(x, y) = 0$ if and only if $x = y$.
2. $\partial(x, y) = \partial(y, x)$.
3. $\partial(x, y) \leq \partial(x, z) + \partial(z, y)$.

For each $x \in X$ and an integer i, let

$$\begin{array}{rcl} \Gamma_i(x) & = & \{y \in V\Gamma \mid \partial(x,y) = i\} \\ d_\Gamma(x) & = & \max\{\partial(x,y) \mid y \in V\Gamma\} \\ d_\Gamma & = & \max\{\partial(x,y) \mid x, y \in V\Gamma\}. \end{array}$$

By definition, $\Gamma(x) = \Gamma_1(x)$. $d_\Gamma(x)$ is called the *local diameter* with respect to x, and d_Γ the *diameter* of the graph Γ.

Definition 1.3.1 A connected graph $\Gamma = (V\Gamma, E\Gamma)$ is said to be *distance regular*, if for $x, y \in V\Gamma$ with $\partial(x,y) = i$,

$$c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|,\ b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$$

depend only on i and do not depend on the choices of x, y as far as $\partial(x,y) = i$.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a distance-regular graph. Since the number

$$k = |\Gamma(x)| = |\Gamma_1(x)| = |\Gamma_1(x) \cap \Gamma_1(x)| = b_0$$

does not depend on the vertex x, Γ is k-regular. Let

$$a_i = |\Gamma_i(x) \cap \Gamma_1(y)| = k - b_i - c_i.$$

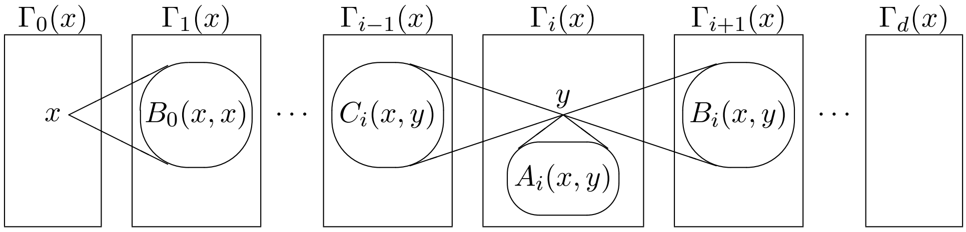
Note that if $\partial(x,y) = i$ and $\partial(z,y) = 1$,

$$i - 1 = \partial(x,y) - \partial(z,y) \leq \partial(x,z) \leq \partial(x,y) + \partial(y,z) = i + 1,$$

or $\Gamma_1(y) \subset \Gamma_{i-1}(x) \cup \Gamma_i(x) \cup \Gamma_{i+1}(x)$.

The following notation is also useful. For vertices x, y of Γ with $\partial(x,y) = i$, let

$$\begin{array}{rcl} C(x,y) & = & C_i(x,y) = \Gamma_{i-1}(x) \cap \Gamma(y), \\ A(x,y) & = & A_i(x,y) = \Gamma_i(x) \cap \Gamma(y), \\ B(x,y) & = & B_i(x,y) = \Gamma_{i+1}(x) \cap \Gamma(y). \end{array}$$



The numbers c_i, a_i and b_i play important roles in the study of distance-regular graphs. The following is called the intersection array of Γ and d denotes the diameter of Γ.

$$\iota(\Gamma) = \left\{ \begin{array}{ccccccc} * & c_1 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_i & \cdots & b_{d-1} & * \end{array} \right\}.$$

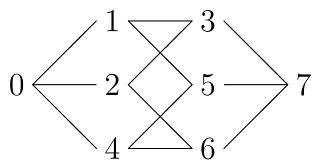
It is clear that $c_0 = a_0 = b_d = 0$ and $c_1 = 1$.

All first examples are distance regular.

Example 1.3.1 Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and

$$E = \{01, 02, 04, 13, 15, 23, 26, 45, 46, 37, 57, 67\}.$$

This is a cube and is distance regular of diameter 3 and valency 3. The intersection array is given below.



$$\left\{\begin{matrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{matrix}\right\}.$$

$\Gamma_0(0) = \{0\}$, $\Gamma_1(0) = \{1, 2, 4\}$, $\Gamma_2(0) = \{3, 5, 6\}$ and $\Gamma_3(0) = \{7\}$.

Exercise 1.3.1 Determine the intersection arrays of the graphs in the first examples.

1.4 Balanced Conditions

In this section we introduce a notion of 'balanced conditions', which is one of the main topics of this lecture note. The conditions will be defined in more general setting in the later chapters.

Let $\boldsymbol{R}^m$ denote the m-dimensional real Euclidean space, i.e., m-dimensional real vector space endowed with the inner product:

$$\boldsymbol{x} \cdot \boldsymbol{y} = (x_1, x_2, \ldots, x_m) \cdot (y_1, y_2, \ldots, y_m) = \sum_{i=1}^{m} x_i y_i.$$

Let $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$. The unit sphere in $\boldsymbol{R}^m$ is the set of unit vectors in $\boldsymbol{R}^m$, i.e.,

$$S^{m-1} = \{\boldsymbol{x} \in \boldsymbol{R}^m \mid \|\boldsymbol{x}\| = 1\}.$$

Let X be a nonempty finite subset of $S^{m-1} \subset \boldsymbol{R}^m$ with

$$\Delta = \{\boldsymbol{x} \cdot \boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{y} \in X\} = \{\gamma_0 = 1, \gamma_1, \ldots, \gamma_s\}.$$

Since $X \subset S^{m-1}$, $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq 1$, and $\boldsymbol{x} \cdot \boldsymbol{y} = 1$ if and only if $\boldsymbol{x} = \boldsymbol{y}$. Hence $-1 \leq \gamma_i < 1$ for $i = 1, 2, \ldots, s$. Since $\|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{2(1-\gamma)}$ when $\boldsymbol{x} \cdot \boldsymbol{y} = \gamma$, the distance is uniquely determined by their inner product. Thus X is sometimes called a Δ-distance set of degree $s = |\Delta| - 1$. For $0 \leq i, j \leq s$ and $\boldsymbol{x}, \boldsymbol{y} \in X$, let

$$P_{i,j}(\boldsymbol{x}, \boldsymbol{y}) = \{\boldsymbol{z} \in X \mid \boldsymbol{x} \cdot \boldsymbol{z} = \gamma_i,\ \boldsymbol{z} \cdot \boldsymbol{y} = \gamma_j\}.$$

For a subset $S \subset \boldsymbol{R}^m$, we write

$$\widehat{S} = \sum_{\boldsymbol{x} \in S} \boldsymbol{x}.$$

For example,

$$\widehat{P_{i,j}(\boldsymbol{x}, \boldsymbol{y})} = \sum_{\boldsymbol{z} \in X,\, \boldsymbol{x} \cdot \boldsymbol{z} = \gamma_i,\, \boldsymbol{z} \cdot \boldsymbol{y} = \gamma_j} \boldsymbol{z}.$$

Definition 1.4.1 Let $X \subset S^{m-1} \subset \boldsymbol{R}^m$ be a Δ-distance set with $\Delta = \{\gamma_0, \gamma_1, \ldots, \gamma_s\}$.

(1) X is said to be *balanced* if

$$P_{i,j}\widehat{(\boldsymbol{x}, \boldsymbol{y})} - P_{j,i}\widehat{(\boldsymbol{x}, \boldsymbol{y})} \in \mathrm{Span}(\boldsymbol{x} - \boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in X$ and for all $0 \leq i, j \leq s$.

(2) X is said to be *strongly balanced* if

$$P_{i,j}\widehat{(\boldsymbol{x}, \boldsymbol{y})} \in \mathrm{Span}(\boldsymbol{x}, \boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in X$ and for all $0 \leq i, j \leq s$.

All first examples are similar to (by rescaling) strongly balanced sets.

Example 1.4.1 The tetrahedron can be embedded in $S^2 \subset \boldsymbol{R}^3$ as follows.

$$\left\{(1,0,0),\ (-\frac{1}{3}, 0, \frac{2\sqrt{2}}{3}),\ (-\frac{1}{3}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}),\ (-\frac{1}{3}, -\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3})\right\}.$$

$\Delta = \{1, -1/3\}$. In this case, both of the conditions are easily checked and the tetrahedron is balanced and strongly balanced. More generally, every finite subset of S^{m-1} is balanced if it is of degree 1.

Exercise 1.4.1 Determine the coordinates of the embedding of the first examples on a unit sphere S^{m-1} for some appropriate m.

Exercise 1.4.2 For each of the first examples, check that they are actually balanced and strongly balanced.

Chapter 2

Association Schemes

2.1 Adjacency Matrices of Distance-Regular Graphs

Let $\Gamma = (V\Gamma, E\Gamma)$ be a distance-regular graph of diameter $d = d_\Gamma$. Let A_i be a matrix of size $|V\Gamma|$, whose rows and columns are indexed by vertices in $V\Gamma$ such that (x, y)-entry is defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{otherwise.} \end{cases}$$

A_i is called the *i-th adjacency matrix*, and $A = A_1$ the *adjacency matrix*. Clearly, $A_j = O$ if $j < 0$ or $j > d$, $A_0 = I$, where O denotes the zero matrix while I is the identity matrix of size $|V\Gamma|$.

Lemma 2.1.1 *Let A_i be the i-th adjacency matrix of a distance-regular graph $\Gamma = (V\Gamma, E\Gamma)$ of diameter d. Let $A = A_1$. Then the following hold.*

(1) *$A_iA = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}$ for all i, where b_j and c_j with $j < 0$ or $j > d$ are indeterminates.*

(2) *Let $v_0(t) = 1$, $v_1(t) = t$, and $v_{i+1}(t)$ is defined by*

$$v_i(t)t = b_{i-1}v_{i-1}(t) + a_iv_i(t) + c_{i+1}v_{i+1}(t)$$

for $i = 1, \ldots, d$ with $c_{d+1} = 1$. Then the polynomial $v_i(t)$ is uniquely determined for $i = 0, 1, \ldots, d+1$ and that $v_i(A) = A_i$. In particular, $v_{d+1}(A) = O$.

Proof.

(1) Let $x, y \in V\Gamma$. Then we have

$$\begin{aligned} (A_iA)_{x,y} &= \sum_{z \in V\Gamma} (A_i)_{x,z}(A)_{z,y} \\ &= |\Gamma_i(x) \cap \Gamma_1(y)| \\ &= \begin{cases} c_{i+1} & \text{if } \partial(x,y) = i+1 \\ a_i & \text{if } \partial(x,y) = i \\ b_{i-1} & \text{if } \partial(x,y) = i-1 \\ 0 & \text{otherwise} \end{cases} \\ &= (b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1})_{x,y}. \end{aligned}$$

Since x and y are arbitrary, we have

$$A_i A = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1}$$

for all i. Recall that $A_j = O$ if $j < 0$ or $j > d$.

(2) By definition, $c_j \neq 0$ for $j = 1, \ldots, d+1$. Hence $v_j(t)$ is uniquely determined by the three term recurrence for $j = 0, 1, \ldots, d+1$. Clearly, $v_0(A) = I = A_0$, and $v_1(A) = A = A_1$. Assume $i \geq 1$. Then by induction hypothesis, we have

$$\begin{aligned} c_{i+1}v_{i+1}(A) &= v_i(A)A - b_{i-1}v_{i-1}(A) - a_i v_i(A) \\ &= A_i A - b_{i-1}A_{i-1} - a_i A_i \\ &= c_{i+1}A_{i+1} \end{aligned}$$

by (1). ■

Exercise 2.1.1 Prove the following.

1. $A_0, A_1, \ldots, A_d$ are linearly independent over the complex number field $\boldsymbol{C}$.
2. $\mathrm{Span}(A_0, A_1, \ldots, A_d) = \mathrm{Span}(I, A, A^2, \ldots, A^d) = \boldsymbol{C}[A]$, where $\boldsymbol{C}[A] = \{p(A) \mid p(t) \in \boldsymbol{C}[t]\}$.
3. $v_{d+1}(t)$ is a minimal polynomial of A, i.e., a polynomial in t, which is of minimal degree subject to the condition that it vanishes at A.
4. $\boldsymbol{C}[A] = \boldsymbol{C}[t]/(v_{d+1}(t))$ as an algebra over $\boldsymbol{C}$.

Exercise 2.1.2 Show that $A_0, A_1, \ldots, A_d$ are $(0,1)$ matrices satisfying the following.

1. $A_0 = I$.
2. $A_0 + A_1 + \cdots + A_d = J$, where J is the all 1's matrix.
3. ${}^tA_i = A_i$ for $i = 0, 1, \ldots, d$.
4. $A_i A_j = \sum_{h=0}^{d} p_{i,j}^h A_h$, where $p_{i,j}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$ with $h = \partial(x, y)$.

2.2 Association Schemes

We now define association schemes which play an essential role in everywhere in algebraic combinatorics.

Definition 2.2.1 Let X be a finite set. Let $\emptyset \neq R_i \subset X \times X$, $i = 0, 1, \ldots, d$ satisfying the following.

1. $R_0 = \{(x, x) \mid x \in X\}$.

2. $X \times X = R_0 \cup \cdots \cup R_d$, $R_i \cap R_j = \emptyset$, if $i \neq j$.

3. ${}^tR_i = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where ${}^tR_i = \{(x, y) \mid (y, x) \in R_i\}$.

4. For $h, i, j \in \{0, 1, \ldots, d\}$ and $(x, y) \in R_h$,

$$p^h_{i,j} = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}$$

depends only on h, i, j and does not depend on the choice of $(x, y) \in R_h$.

5. $p^h_{i,j} = p^h_{j,i}$.

Such a configuration $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is called a *commutative association scheme* of class d on X. If ${}^tR_i = R_i$ for all i, $\mathcal{X}$ is called a *symmetric association scheme.*

Exercise 2.2.1 Let X be a finite set. Let $\emptyset \neq R_i \subset X \times X$, $i = 0, 1, \ldots, d$. Let A_i be a square matrix of size $|X|$, whose rows and columns are indexed by X such that

$$(A_i)_{x,y} = \begin{cases} 1 & (x, y) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is a commutative association scheme if and only if A_i's satisfy the following.

1. $A_0 = I$.

2. $A_0 + A_1 + \cdots + A_d = J$, where J is the all 1's matrix.

3. ${}^tA_i = A_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$ for all $i = 0, 1, \ldots, d$.

4. $A_iA_j = \sum_{h=0}^{d} p^h_{i,j} A_h$ for some $p^h_{i,j} \in \boldsymbol{C}$.

5. $A_iA_j = A_jA_i$ for all $i, j = 0, 1, \ldots, d$.

Example 2.2.1 Let $\Gamma = (V\Gamma, E\Gamma)$ be a distance-regular graph of diameter d. Let

$$R_i = \{(x, y) \in V\Gamma \times V\Gamma \mid \partial(x, y) = i\}.$$

Then $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ becomes a symmetric association scheme. Compare Exercise 2.1.2 and Exercise 2.2.1.

Association schemes are closely related to permutation groups. In fact, if a group G acts on X transitively, then the set of orbits $R_0, R_1, \ldots, R_d$ on $X \times X$ of G satisfies 1, 2, 3, 4 in Definition 2.2.1. See [6].

2.3 Semisimple Matrix Algebras and Modules

In this section we introduce semisimple algebras and their modules. We restrict out attention to the subalgebras of a matrix algebra $\mathrm{Mat}_n(\boldsymbol{C})$ consisting of $n \times n$ matrices over the complex number field. Let $V = \boldsymbol{C}^n$ be the n-dimensional row vector space over $\boldsymbol{C}$. For a matrix $M \in \mathrm{Mat}_n(\boldsymbol{C})$, let $M^* = {}^t\bar{M}$ denote the transpose of the complex conjugate of the matrix M.

Definition 2.3.1

1. An *algebra* $\mathcal{A}$ in $\mathrm{Mat}_n(\boldsymbol{C})$ is a vector subspace of a matrix algebra $\mathrm{Mat}_n(\boldsymbol{C})$ considered as a vector space over the complex number field $\boldsymbol{C}$, which is closed under multiplication.
2. An algebra $\mathcal{A}$ is said to be *semisimple* if $A^* \in \mathcal{A}$ for all $A \in \mathcal{A}$.
3. A (finite dimensional) vector space W is said to be an $\mathcal{A}$-module if every element A of $\mathcal{A}$ is a linear transformation on W, $\boldsymbol{w} \mapsto A \cdot \boldsymbol{w}$, such that
$$(AB) \cdot \boldsymbol{w} = A \cdot (B \cdot \boldsymbol{w}), \text{ for all } A, B \in \mathcal{A}, \ \boldsymbol{w} \in W.$$
4. Let U and W be $\mathcal{A}$-modules and let f be a mapping from U to W. f is said to be a *homomorphism* if f is a linear mapping such that $f(A \cdot \boldsymbol{u}) = A \cdot f(\boldsymbol{u})$ for all $A \in \mathcal{A}$ and $\boldsymbol{u} \in U$. A bijective homomorphism is called an *isomorphism*. If there is an isomorphism between U and W, we say that U and W are *isomorphic*.
5. For an algebra $\mathcal{A}$ in $\mathrm{Mat}_n(\boldsymbol{C})$, $V = \boldsymbol{C}^n$ is an $\mathcal{A}$-module with the natural action of the elements of $\mathcal{A}$, i.e., $A \cdot \boldsymbol{w} = A\boldsymbol{w}$. V is called the *standard* module.
6. For an algebra $\mathcal{A}$, $\mathcal{A}$ itself can be regarded as an $\mathcal{A}$-module by left multiplication. This module is called the *regular* $\mathcal{A}$-module.
7. A vector subspace W of an $\mathcal{A}$-module U is said to be a *submodule* of U, if for all $A \in \mathcal{A}$ and $\boldsymbol{w} \in W$, $A\boldsymbol{w} \in W$. In this case we also say that W is $\mathcal{A}$-invariant.
8. If a nonzero $\mathcal{A}$-submodule W does not have a proper (i.e., different from zero or itself) $\mathcal{A}$-submodule, W is said to be *irreducible*.
9. An $\mathcal{A}$-module U is said to be *completely reducible* if U is a direct sum of irreducible $\mathcal{A}$-modules.

Lemma 2.3.1 *Let $\mathcal{A}$ be an algebra in $\mathrm{Mat}_n(\boldsymbol{C})$ and U be an $\mathcal{A}$-module.*

(1) *The following are equivalent.*

- (*i*) *U is a completely reducible $\mathcal{A}$-module.*
- (*ii*) *U is a sum of irreducible $\mathcal{A}$-submodules.*
- (*iii*) *For every $\mathcal{A}$-submodule W of U, there is a $\mathcal{A}$-submodule W' of U such that $U = W \oplus W'$, i.e., U can be written as a direct sum of W and W', or $U = W + W'$ and $W \cap W' = \{\boldsymbol{0}\}$.*

(2) *If U is a completely reducible $\mathcal{A}$-submodule, then so are the submodules of U.*

(3) *Let $f : U \to W$ be a surjective homomorphism of $\mathcal{A}$-modules. If U is completely reducible, there is a submodule W' of U isomorphic to W, and W is completely reducible as well.*

Proof.

(1) The implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are obvious. Suppose U is a sum of irreducible $\mathcal{A}$-modules W_i's, $U = \sum_i W_i$. Let W be an $\mathcal{A}$-submodule of U. Choose an $\mathcal{A}$-submodule W' of U maximal so that $W \cap W' = \{\mathbf{0}\}$. We claim that $W + W' = U$. Otherwise, there is an irreducible $\mathcal{A}$-submodule W_i in U such that $W_i \not\subset W + W'$. Since W_i is irreducible, $(W + W') \cap W_i = \{\mathbf{0}\}$ and $W \cap (W' + W_i) = \{\mathbf{0}\}$. This contradicts the maximality of W'. Hence $W + W' = U$ and $U = W \oplus W'$.

(2) This is clear from the condition $(1)(iii)$.

(3) Let $U_0 = \{\boldsymbol{u} \in U \mid f(\boldsymbol{u}) = \mathbf{0}\}$, the kernel of the homomorphism f. Now it is easy to check that U_0 is an $\mathcal{A}$-submodule. Hence by completely reducibility and the condition $(1)(iii)$, there is an $\mathcal{A}$-submodule W' of U such that $U = W' + U_0$. Since $W = f(U) = f(W')$, the restriction of f on W' is surjective. Since $U_0 \cap W' = \{\mathbf{0}\}$, f is injective as well. Thus f induces an isomorphism between W' and W. By (2), W' is completely reducible, so W is completely reducible as well. ■

In the following we show the so-called complete reducibility of a semisimple algebra, and the Schur's lemma.

For $\boldsymbol{u}, \boldsymbol{v} \in V$, let $(u, v) = {}^t\bar{\boldsymbol{u}}\boldsymbol{v}$, the Hermitean inner product on V. A set of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_t$ is said to be *orthogonal* if $(\boldsymbol{u}_i, \boldsymbol{u}_j) = 0$ for $i \neq j$, and *orthonormal* if $(\boldsymbol{u}_i, \boldsymbol{u}_j) = \delta_{i,j}$. Here δ denotes the Kronecker's delta. Two vector subspaces U and W are said to be *orthogonal* if each vector in U is orthogonal to each vector in W. The following are easy exercises of linear algebra.

Exercise 2.3.1 Let $V = \boldsymbol{C}^n$ and let $(u, v) = {}^t\bar{\boldsymbol{u}}\boldsymbol{v}$ for $\boldsymbol{u}, \boldsymbol{v} \in V$. Show the following.

1. $(\boldsymbol{u}, \boldsymbol{u})$ is a real nonnegative number, and $(\boldsymbol{u}, \boldsymbol{u}) = 0$ if and only if $\boldsymbol{u} = \mathbf{0}$.

2. Let W be a vector subspace of V and let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_t$ be orthonormal vectors of W. Then there is an orthonormal basis of W containing $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_t$.

3. Let U, W be vector subspaces of V with $W \subset U$. Let
$$W^{\perp} = \{\boldsymbol{v} \in V \mid (\boldsymbol{v}, \boldsymbol{w}) = 0, \ \text{for every } \boldsymbol{w} \in W\}.$$
Then $U = W \oplus (W^{\perp} \cap U)$ (orthogonal direct sum), i.e., $U = W + (W^{\perp} \cap U)$, and $W \cap (W^{\perp} \cap U) = \{\mathbf{0}\}$.

4. Let W be a vector subspace of V and M an $n \times n$ matrix satisfying $M\boldsymbol{w} \in W$ for every $\boldsymbol{w} \in W$. Then there exists a nonzero vector $\boldsymbol{v} \in W$ and a scalar $\lambda \in \boldsymbol{C}$ such that $M\boldsymbol{v} = \lambda\boldsymbol{v}$.

Proposition 2.3.2 *Let $\mathcal{A}$ be a semisimple algebra in* $\mathrm{Mat}_n(\boldsymbol{C})$. *Let U and W be $\mathcal{A}$-submodules of the standard module V with $W \subset U$. Then the following hold.*

(1) *$W^\perp$ and hence $W^\perp \cap U$ are $\mathcal{A}$-submodules such that $U = W \oplus (W^\perp \cap U)$ (orthogonal direct sum).*

(2) *W can be written as an orthogonal direct sum of irreducible $\mathcal{A}$-submodules. In particular, the standard module V is completely reducible.*

Proof. Since $\mathcal{A}$ is semisimple, $A^* \in \mathcal{A}$ for every $A \in \mathcal{A}$.

(1) First we show that $W^\perp$ is an $\mathcal{A}$-submodule. For $A \in \mathcal{A}$, $\boldsymbol{u}, \boldsymbol{v} \in V$, we have

$$(A\boldsymbol{u}, \boldsymbol{v}) = {}^t\bar{\boldsymbol{u}}A^*\boldsymbol{v} = (\boldsymbol{u}, A^*\boldsymbol{v}).$$

Let $\boldsymbol{u} \in W^\perp$. Then for every $A \in \mathcal{A}$, we have

$$(A\boldsymbol{u}, \boldsymbol{w}) = (\boldsymbol{u}, A^*\boldsymbol{w}) = 0,$$

for all $\boldsymbol{w} \in W$ as $A^* \in \mathcal{A}$ implies that $A^*\boldsymbol{w} \in W$. This means that $A\boldsymbol{u} \in W^\perp$ or $W^\perp$ is an $\mathcal{A}$-submodule. Now the rest follows directly from Exercise 2.3.1.

(2) This is a direct consequence of (1) as we proceed by induction. ■

Exercise 2.3.2 For $M = (M_{i,j}), N = (N_{i,j}) \in \mathrm{Mat}_n(\boldsymbol{C})$, let

$$\ll M, N \gg = \mathrm{tr}({}^t\bar{M}N) = \sum_{i,j} \overline{M_{i,j}} N_{i,j}.$$

This is the natural Hermitean inner product used above regarding $\mathrm{Mat}_n(\boldsymbol{C})$ as an n^2-dimensional vector space. For $A \in \mathrm{Mat}_n(\boldsymbol{C})$, A^* coincides with the transposed of the complex conjugate of the matrix of A as a linear transformation

$$A : \mathrm{Mat}_n(\boldsymbol{C}) \to \mathrm{Mat}_n(\boldsymbol{C}),\ (M \mapsto AM)$$

with respect to the matrix units as a basis for $\mathrm{Mat}_n(\boldsymbol{C})$. In particular, $\ll AM, N \gg = \ll M, A^*N \gg$ for all $M, N \in \mathrm{Mat}_n(\boldsymbol{C})$.

Corollary 2.3.3 *Let $\mathcal{A}$ be a semisimple algebra in* $\mathrm{Mat}_n(\boldsymbol{C})$. *Then the regular $\mathcal{A}$-module $\mathcal{A}$ is completely reducible.*

Proof. By Exercise 2.3.2, $\mathcal{A}$ is semisimple as an algebra in $\mathrm{Mat}_{n^2}(\boldsymbol{C})$ with standard module $\mathrm{Mat}_n(\boldsymbol{C})$. Since the regular $\mathcal{A}$-module is a submodule of this standard module, it is completely reducible by Proposition 2.3.2. ■

Proposition 2.3.4 *Let $\mathcal{A}$ be a semisimple algebra in* $\mathrm{Mat}_n(\boldsymbol{C})$. *Then every $\mathcal{A}$-module is completely reducible.*

Proof. Let U be an $\mathcal{A}$-module. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_s$ be a linear basis of U. Let π be a mapping from U_0, s direct sum of the regular $\mathcal{A}$-modules, to U.

$$\pi : U_0 = \underbrace{\mathcal{A} \oplus \cdots \oplus \mathcal{A}}_{s \text{ times}} \to U, \ (A_1, \ldots, A_s) \mapsto A_1 \cdot \boldsymbol{u}_1 + \cdots + A_s \cdot \boldsymbol{u}_s.$$

π is surjective by definition. By Corollary 2.3.3, the regular $\mathcal{A}$-module is completely reducible. Hence U_0 is completely reducible as well. Now it follows from Lemma 2.3.1 (3) that U is completely reducible. ■

Proposition 2.3.5 [Schur's Lemma] *Let $\mathcal{A}$ be an algebra in $\mathrm{Mat}_n(\boldsymbol{C})$. Let W be an irreducible $\mathcal{A}$-module. If a member $A \in \mathcal{A}$ commutes with all elements of $\mathcal{A}$, then A acts on W as a scalar multiple.*

Proof. Let W be an irreducible $\mathcal{A}$-submodule. Let $A \in \mathcal{A}$ be a nonzero element, which commutes with all elements of $\mathcal{A}$. By Exercise 2.3.1, there exists a nonzero vector $\boldsymbol{w} \in W$ and a scalar $\lambda \in \boldsymbol{C}$ such that $A\boldsymbol{w} = \lambda \boldsymbol{w}$. Since W is irreducible and $\boldsymbol{w}$ is nonzero, $\mathcal{A}\boldsymbol{w} = W$. Hence

$$(A - \lambda I)W = (A - \lambda I)\mathcal{A}\boldsymbol{w} = \mathcal{A}(A - \lambda I)\boldsymbol{w} = \boldsymbol{0}.$$

Therefore $A\boldsymbol{v} = \lambda \boldsymbol{v}$ for every $\boldsymbol{v} \in W$, i.e., A acts on W as a scalar multiple. ■

Corollary 2.3.6 *Let $\mathcal{A}$ be a semisimple commutative algebra in $\mathrm{Mat}_n(\boldsymbol{C})$. Then every irreducible $\mathcal{A}$-module is of dimension 1. Moreover, there is an orthonormal basis of V consisting of common eigen vectors of $\mathcal{A}$. In particular, there is a unitary matrix $U \in \mathrm{Mat}_n(\boldsymbol{C})$, i.e., a matrix satisfying $UU^* = U^*U = I$, such that $U^*\mathcal{A}U$ consists of diagonal matrices.*

Proof. Let W be an irreducible $\mathcal{A}$-module. Since $\mathcal{A}$ itself is commutative, by Proposition 2.3.5 every element of $\mathcal{A}$ acts as a scalar on W. Let $\boldsymbol{w} \in W$ be a nonzero vector. Since every element of $\mathcal{A}$ acts as a scalar, $\mathrm{Span}(\boldsymbol{w})$ is an $\mathcal{A}$-submodule. Since W is irreducible, $\mathrm{Span}(\boldsymbol{w}) = W$ and W is of dimension 1.

By Proposition 2.3.2, V can be written as an orthogonal direct sum of irreducible $\mathcal{A}$-modules of dimension 1,

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n.$$

Choose $\boldsymbol{w}_i \in W_i$ so that $(\boldsymbol{w}_i, \boldsymbol{w}_i) = 1$. Then $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ becomes an orthonormal basis of V, and according to this basis, each element A of $\mathcal{A}$ is represented as a diagonal matrix. Note that if we let $U = (\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n)$, U is a unitary matrix satisfying $AU = UD$, where D is a diagonal matrix whose (i,i)-entry is determined by $A\boldsymbol{w}_i = D_{i,i}\boldsymbol{w}_i$. ■

Exercise 2.3.3 Let $\mathcal{A}$ be an algebra in $\mathrm{Mat}_n(\boldsymbol{C})$. Let U and W be irreducible $\mathcal{A}$-modules, and let $f : U \to W$ be a homomorphism. Then f is either the zero mapping, sending every element of U to the zero vector in W, or an isomorphism. (Hint:Recall that the image and the kernel of a homomorphism are submodules.)

2.4 Primitive Idempotents and Eigen Matrices

Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a commutative association scheme and let $A_0, A_1, \ldots, A_d$ be the adjacency matrices. Set $\mathcal{M} = \mathrm{Span}(A_0, A_1, \ldots, A_d)$. Then $\mathcal{M}$ is an algebra of dimension $d+1$ over the complex number field $\boldsymbol{C}$ with basis $A_0, A_1, \ldots, A_d$. $\mathcal{M}$ is called the *Bose-Mesner algebra* of $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$. As an application of the results in the previous section, we are going to construct another basis of this algebra consisting of primitive orthogonal idempotents.

Let $|X| = n$ and $V = \boldsymbol{C}^n$, the n-dimensional row vector space over $\boldsymbol{C}$. Since $\mathcal{M} \subset \mathrm{Mat}_X(\boldsymbol{C}) \simeq \mathrm{Mat}_n(\boldsymbol{C})$, $\mathcal{M}$ is an algebra in $\mathrm{Mat}_X(\boldsymbol{C})$. For $M \in \mathcal{M}$, let $M^* = {}^t\bar{M}$. Since $A_i^* = {}^tA_i = A_{i'} \in \mathcal{M}$ and every element of $\mathcal{M}$ is a $\boldsymbol{C}$-linear combination of A_i's, $M^* \in \mathcal{M}$ for every $M \in \mathcal{M}$. Hence $\mathcal{M}$ is a commutative semisimple algebra in the sense of Definition 2.3.1.

By Corollary 2.3.6, V can be decomposed into $n = |X|$ orthogonal direct sum of irreducible $\mathcal{M}$-submodules W_i.

$$V = W_1 \oplus \cdots \oplus W_n.$$

Let $W_i = \mathrm{Span}(\boldsymbol{w}_i)$ for $i = 1, \ldots, n$. Since W_i is an $\mathcal{M}$-submodule of dimension 1, there is a $\boldsymbol{C}$-valued function λ_i defined on $\mathcal{M}$ such that $M\boldsymbol{w}_i = \lambda_i(M)\boldsymbol{w}_i$. Let $\mu_0, \ldots, \mu_t$ be distinct functions in the set $\{\lambda_1, \ldots, \lambda_n\}$. Now we define a subspace V_i of V as follows.

$$V_i = \{\boldsymbol{v} \in V \mid M\boldsymbol{v} = \mu_i(M)\boldsymbol{v}, \text{ for all } M \in \mathcal{M}\}.$$

In other words, V_i is a sum of W_j's such that $\lambda_j = \mu_i$. By construction, we have

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_t, \text{ (orthogonal direct sum)}.$$

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be the unit vectors corresponding to the elements of $X = \{x_1, \ldots, x_n\}$. Recall that

$$J = A_0 + A_1 + \cdots + A_d \in \mathcal{M}.$$

Let $\boldsymbol{j} = \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_n$, the all 1's vector. Since $J\boldsymbol{x}_i = \boldsymbol{j}$, we have

$$J\boldsymbol{j} = n\boldsymbol{j}, \ J(\boldsymbol{x}_1 - \boldsymbol{x}_i) = \boldsymbol{0}, \text{ for all } i = 2, \ldots, n.$$

Assume that $\mu_0(J) = n$. Since

$$\dim\{\boldsymbol{v} \in V \mid J\boldsymbol{v} = n\boldsymbol{v}\} = \dim \mathrm{Span}(\boldsymbol{j}) = 1,$$

$\dim V_0 = 1$ and $V_0 = \mathrm{Span}(\boldsymbol{j})$. Since

$$V_0^\perp = V_1 + \cdots + V_t = \mathrm{Span}(\boldsymbol{x}_1 - \boldsymbol{x}_2, \ldots, \boldsymbol{x}_1 - \boldsymbol{x}_n),$$

$\mu_i(J) = 0$ for $i = 1, 2, \ldots, t$. Let $m_i = \dim V_i$, and let $\boldsymbol{u}_1^{(i)}, \ldots, \boldsymbol{u}_{m_i}^{(i)}$ be an orthonormal basis of V_i. (See Exercise 2.3.1.) Let U_i be $n \times m_i$ matrix $U_i = (\boldsymbol{u}_1^{(i)}, \ldots, \boldsymbol{u}_{m_i}^{(i)})$, and let $E_i = U_i{}^t\bar{U}_i$ for $i = 0, 1, \ldots, t$. Since $\boldsymbol{u}_j^{(i)}$'s are orthonormal basis of V_i and V_i's are mutually orthogonal, we have

$$({}^t\bar{U}_iU_j)_{h,k} = {}^t\overline{\boldsymbol{u}_h^{(i)}}\boldsymbol{u}_k^{(j)} = (\boldsymbol{u}_h^{(i)}, \boldsymbol{u}_k^{(j)}) = \delta_{i,j}\delta_{h,k}.$$

Hence ${}^t\bar{U}_iU_j = \delta_{i,j}I_{m_i}$, where I_{m_i} is the identity matrix of size m_i.

Lemma 2.4.1 (1) $E_0 = \dfrac{1}{|X|}J$.

(2) $E_iE_j = \delta_{i,j}E_i$ *for all* $i, j = 0, 1, \ldots, t$.

(3) ${}^tE_i = \bar{E}_i$.

(4) *Let* $\boldsymbol{v}_j \in V_j$. *Then* $E_i\boldsymbol{v}_j = \delta_{i,j}\boldsymbol{v}_j$.

(5) $I = E_0 + E_1 + \cdots + E_t$.

(6) $ME_i = \mu_i(M)E_i$ *for* $M \in \mathcal{M}$ *and* $i = 0, 1, \ldots, t$.

Proof.

(1) Since $(1/\sqrt{|X|})\boldsymbol{j}$ is the orthonormal basis of V_0, we have

$$E_0 = {}^t(\frac{1}{\sqrt{|X|}}\boldsymbol{j})(\frac{1}{\sqrt{|X|}}\boldsymbol{j}) = \frac{1}{|X|}{}^t\boldsymbol{j}\boldsymbol{j} = \frac{1}{|X|}J.$$

(2) $E_iE_j = U_i{}^t\bar{U}_iU_j{}^t\bar{U}_j = \delta_{i,j}U_i{}^t\bar{U}_j = \delta_{i,j}E_i$.

(3) ${}^tE_i = {}^t(U_i{}^t\bar{U}_i) = \bar{U}_i{}^tU_i = \overline{U_i{}^t\bar{U}_i} = \bar{E}_i$.

(4) $E_i\boldsymbol{u}_k^{(j)} = U_i{}^t\bar{U}_i\boldsymbol{u}_k^{(j)} = \delta_{i,j}U_i\boldsymbol{e}_k = \delta_{i,j}\boldsymbol{u}_k^{(i)}$, where $\boldsymbol{e}_k$ is the k-th unit vector of length m_i. Since $\boldsymbol{u}_1^{(j)}, \ldots, \boldsymbol{u}_{m_j}^{(j)}$ is a basis of V_j, we have the desired conclusion.

(5) Let $\boldsymbol{v} \in V$ and $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_t$ with $\boldsymbol{v}_i \in V_i$. Then by (4),

$$(E_0 + E_1 + \cdots + E_t)\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_t = \boldsymbol{v} = I\boldsymbol{v}.$$

(6) Since $MU_i = \mu_i(M)U_i$, we have $ME_i = \mu_i(M)E_i$. ■

Proposition 2.4.2 *Under the notation above,* $t = d$ *and*

$$\mathcal{M} = \mathrm{Span}(A_0, A_1, \ldots, A_d) = \mathrm{Span}(E_0, E_1, \ldots, E_d).$$

Proof. Let $\mu_0, \mu_1, \ldots, \mu_t$ be the functions defined above. Then by Lemma 2.4.1(5), (6),

$$A_i = A_i(E_0 + \cdots + E_t) = \mu_0(A_i)E_0 + \cdots + \mu_t(A_i)E_t.$$

Hence $\mathrm{Span}(A_0, A_1, \ldots, A_d) \subset \mathrm{Span}(E_0, E_1, \ldots, E_t)$ and $d \le t$.

Now fix i. For each $j \neq i$, there exists A_{l_j} such that $\mu_i(A_{l_j}) \neq \mu_j(A_{l_j})$, as μ_i and μ_j are distinct linear functions on the space spanned by $A_0, A_1, \ldots, A_d$. Let

$$F_i = \prod_{j \neq i}(A_{l_j} - \mu_j(A_{l_j})I).$$

By definition, $F_i \in \mathcal{M} \subset \mathrm{Span}(E_0, E_1, \ldots, E_t)$, $F_i E_j = O$ if $i \neq j$ and

$$F_i E_i = \prod_{j \neq i} (\mu_i(A_{l_j}) - \mu_j(A_{l_j})) E_i \neq O.$$

Hence F_i is a nonzero scalar multiple of E_i. Thus

$$\mathrm{Span}(E_0, E_1, \ldots, E_t) \subset \mathrm{Span}(A_0, A_1, \ldots, A_d).$$

Note that $E_0, E_1, \ldots, E_t$ are linearly independent, as we have $\alpha_0 E_0 + \alpha_1 E_1 + \cdots + \alpha_t E_t = O$ implies $\alpha_i E_i = O$, by multiplying E_i on both hand sides. Thus $t \leq d$ as desired. ■

Lemma 2.4.3 ${}^t E_i = \bar{E}_i = E_{\hat{\imath}}$ *for some* $0 \leq \hat{\imath} \leq d$.

Proof. The first equality is already obtained in Lemma 2.4.1. Since $\mathcal{M}$ is closed under complex conjugate and $E_0, E_1, \ldots, E_d$ is a basis of $\mathcal{M}$, we can write as follows:

$${}^t E_i = \sum_{h=0}^{d} \alpha_{i,h} E_h.$$

Now we have

$$\begin{aligned} \delta_{i,j} \sum_{h=0}^{d} \alpha_{i,h} E_h &= \delta_{i,j} {}^t E_i \\ &= {}^t(E_j E_i) = {}^t E_i {}^t E_j \\ &= (\sum_{h=0}^{d} \alpha_{i,h} E_h)(\sum_{l=0}^{d} \alpha_{j,l} E_l) \\ &= \sum_{h=0}^{d} \alpha_{i,h} \alpha_{j,h} E_h. \end{aligned}$$

Hence $\alpha_{i,h}\alpha_{j,h} = \delta_{i,j}\alpha_{i,h}$ and $\alpha_{i,h} = 0$ or 1. Let $I_i = \{h \mid \alpha_{i,h} \neq 0\}$. Then $I_i \neq \emptyset$ and $I_i \cap I_j = \emptyset$ if $i \neq j$. Since

$$I_0 \cup I_1 \cup \cdots \cup I_d \subset \{0, 1, \ldots, d\},$$

${}^t E_i = E_{\hat{\imath}}$ for some $\hat{\imath}$. ■

We have seen that the Bose-Mesner algebra $\mathcal{M}$ has two bases, namely, the set of adjacency matrices, $A_0, A_1, \ldots, A_d$, and the set of primitive idempotents, $E_0, E_1, \ldots, E_d$ constructed above. Hence we can find the following expressions.

$$A_i = \sum_{j=0}^{d} p_i(j) E_j, \;\; E_i = \frac{1}{|X|} \sum_{j=0}^{d} q_i(j) A_j.$$

Let P be a $(d+1) \times (d+1)$ matrix such that $P_{j,i} = p_i(j)$. Let Q be a $(d+1) \times (d+1)$ matrix such that $Q_{j,i} = q_i(j)$. Then it is easy to see that

$$PQ = QP = |X| I.$$

The matrix P is called the *P-matrix* or the *first eigen matrix*, and the matrix Q is called the *Q-matrix* or the *second eigen matrix* of the commutative association scheme $\mathcal{X}$.

We define another operation on the Bose-Mesner algebra $\mathcal{M}$. Since one of the bases $A_0, A_1, \ldots, A_d$ is a set of $(0,1)$ matrices, $\mathcal{M}$ is closed under the entry-wise product, which is often called a Hadamard product or $\circ$-product and denoted by $\circ$. We have $A_i \circ A_j = \delta_{i,j} A_i$. So A_i's are the orthogonal idempotents with respect to this product and the all 1's matrix $J = |X| E_0$ is the identity element. Let

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{d} q_{i,j}^h E_h.$$

The parameters $q_{i,j}^h$'s are called Krein parameters. Compared with the parameters $p_{i,j}^h$ which has combinatorial interpretation, Krein parameters are not well understood. The study of these parameters is one of the main topics of this lecture note.

Chapter 3

P-polynomial and Q-polynomial Association Schemes

3.1 Terwilliger Algebra and Inner Product

Let $\mathrm{Mat}_X(\boldsymbol{C})$ denote the set of square matrices over $\boldsymbol{C}$, whose rows and columns are indexed by the finite set X. For $M \in \mathrm{Mat}_X(\boldsymbol{C})$, let $\mathrm{tr}(M)$ denote the trace of M, i.e., the sum of the diagonal entries of M. Let $\tau(M)$ denote the sum of the all entries of M.

Exercise 3.1.1 Show the following on the trace function.

1. For an $n \times m$ matrix M and an $m \times n$ matrix N, $\mathrm{tr}(MN) = \mathrm{tr}(NM)$.
2. Using the definition of E_i and 1 above, show the following.

$$\mathrm{tr}(E_i) = \mathrm{rank}(E_i) = \dim(V_i) = m_i.$$

We now define a Hermitean form on $\mathrm{Mat}_X(\boldsymbol{C})$. For $M, N \in \mathrm{Mat}_X(\boldsymbol{C})$, let

$$\ll M, N \gg = \mathrm{tr}({}^t\bar{M}N).$$

.

Exercise 3.1.2 For $L, M, N \in \mathrm{Mat}_X(\boldsymbol{C})$, show the following.

1. $\ll M, N \gg = \tau(\bar{M} \circ N) = \overline{\tau(\bar{N} \circ M)} = \overline{\ll N, M \gg}$.
2. $\ll M, M \gg$ is nonnegative real and is zero if and only if $M = O$.
3. $\ll L \circ M, N \gg = \ll L, \bar{M} \circ N \gg$.
4. $\ll LM, N \gg = \ll L, N{}^t\bar{M} \gg = \ll M, {}^t\bar{L}N \gg$.

We prove some properties of the parameters using this Hermitean form restricting on the Bose-Mesner algebra $\mathcal{M}$ of a commutative association scheme. Let $k_i = \mu_0(A_i)$. k_i is known as *i-th valency*. Since $J = |X|E_0$, $A_iJ = \mu_0(A_i)J = k_iJ$. Hence k_i is the constant row and column sum of A_i, which is equal to $p^0_{i,i'}$. In particular, we have $k_i = k_{i'}$.

Lemma 3.1.1 *The following hold.*

(1) $\ll E_i, E_j \gg = m_i\delta_{i,j} = q^0_{\hat{i},j}$.

(2) $\ll A_i, A_j \gg = |X|k_i\delta_{i,j} = |X|p^0_{i',j}$.

(3) $\ll E_i, A_j \gg = m_i p_j(i) = k_j\overline{q_i(j)}$.

Proof.

(1) $\ll E_i, E_j \gg = \mathrm{tr}({}^t\bar{E}_i E_j) = \mathrm{tr}(E_i E_j) = \delta_{i,j}\mathrm{tr}(E_i) = m_i\delta_{i,j}$. On the other hand,

$$\ll E_i, E_j \gg = \tau(E_{\hat{i}} \circ E_j) = \tau((E_{\hat{i}} \circ E_j) \circ (|X|E_0)) = q^0_{\hat{i},j}\mathrm{tr}(E_0) = q^0_{\hat{i},j}.$$

(2) $\ll A_i, A_j \gg = \tau(A_i \circ A_j) = \delta_{i,j}\tau(A_i) = |X|k_i\delta_{i,j}$. On the other hand,

$$\ll A_i, A_j \gg = \mathrm{tr}(A_{i'}A_j) = \tau((A_{i'}A_j) \circ A_0) = p^0_{i',j}\tau(A_0) = |X|p^0_{i',j}.$$

(3) $\ll E_i, A_j \gg = \mathrm{tr}(E_i A_j) = \mathrm{tr}(p_j(i)E_i) = m_i p_j(i)$. On the other hand, since

$$E_i = \frac{1}{|X|}\sum_{h=0}^{d} q_i(h)A_h,$$

$$\ll E_i, A_j \gg = \frac{\overline{q_i(j)}}{|X|}\tau(A_j) = k_j\overline{q_i(j)}.$$

■

Exercise 3.1.3 1. Compute $\ll E_j, E_h \circ E_i \gg$ to show the following.

$$m_h q^h_{\hat{i},j} = m_j q^j_{h,i} = m_i q^i_{\hat{h},j},\ m_h = m_{\hat{h}},\ \text{ and } q^h_{i,j} = q^{\hat{h}}_{\hat{i},\hat{j}}.$$

2. Compute $\ll A_j, A_h A_i \gg$ to show the following.

$$k_h p^h_{i',j} = k_j p^j_{h,i} = k_i p^i_{h',j},\ k_h = k_{h'},\ \text{ and } p^h_{i,j} = p^{h'}_{i',j'}.$$

Let $x \in X$, and $M \in \mathrm{Mat}_X(\boldsymbol{C})$. Then x_M denotes a diagonal matrix such that $(x_M)_{y,y} = M_{x,y}$.

Definition 3.1.1 The *dual Bose-Mesner algebra* with respect to x is

$$x_{\mathcal{M}} = \{x_M \mid M \in \mathcal{M}\}.$$

The *Terwilliger algebra* or *subconstituent algebra* $T(x)$ with respect to x is a subalgebra of $\mathrm{Mat}_X(\boldsymbol{C})$ generated by $\mathcal{M}$ and $x_{\mathcal{M}}$, i.e.,

$$T(x) = < \mathcal{M}, x_{\mathcal{M}} >_{alg} = < M, x_M \mid M \in \mathcal{M} >_{alg} \subset \mathrm{Mat}_X(\boldsymbol{C}).$$

We now consider the inner product of special elements of the Terwilliger algebra.

Lemma 3.1.2 *Let $M, N \in \mathrm{Mat}_X(\boldsymbol{C})$, and $x, y \in X$. Then the following hold.*

(1) $\ll E_h x_M E_i, E_j y_N E_k \gg = \delta_{h,j}\delta_{i,k}(\bar{M}(E_h \circ E_{\hat{i}})^t N)_{x,y}$.

(2) $\ll E_i x_{E_h} E_j, E_i x_{E_h} E_j \gg = \frac{1}{|X|^2} q^{\hat{h}}_{i,\hat{j}} m_{\hat{h}} = \frac{1}{|X|^2} q^j_{h,i} m_j \geq 0$. *In particular, the Krein parameters $q^j_{h,i}$'s are all nonnegative real numbers.*

(3) $\ll E_i x_{E_h} E_j, E_i x_{E_h} E_j \gg = 0$, *if and only if* $E_i x_{E_h} E_j = 0$, *if and only if* $q^j_{h,i} = 0$.

Proof.

(1) Since $E_i E_j = \delta_{i,j} E_i$, applying the properties of the inner product in Exercise 3.1.2 we have

$$\begin{aligned}
\ll E_h x_M E_i, E_j y_N E_k \gg &= \ll E_h x_M, E_j y_N E_k E_i \gg \\
&= \delta_{i,k} \ll x_M, E_h E_j y_N E_i \gg \\
&= \delta_{i,k}\delta_{h,j} \ll x_M, E_h y_N E_i \gg \\
&= \delta_{i,k}\delta_{h,j} \mathrm{tr}(\overline{x_M} E_h y_N E_i) \\
&= \delta_{i,k}\delta_{h,j} \sum_{z \in X} (\overline{x_M} E_h y_N E_i)_{z,z} \\
&= \delta_{i,k}\delta_{h,j} \sum_{z,w \in X} (\overline{x_M})_{z,z} (E_h)_{z,w} (y_N)_{w,w} (E_i)_{w,z} \\
&= \delta_{i,k}\delta_{h,j} \sum_{z,w \in X} \bar{M}_{x,z} (E_h \circ E_{\hat{i}})_{z,w} {}^t N_{w,y} \\
&= \delta_{h,j}\delta_{i,k}(\bar{M}(E_h \circ E_{\hat{i}})^t N)_{x,y}.
\end{aligned}$$

(2) Applying (1), we have

$$\begin{aligned}
\ll E_i x_{E_h} E_j, E_i x_{E_h} E_j \gg &= (\bar{E}_h (E_i \circ E_{\hat{j}})^t E_h)_{x,x} \\
&= ((E_i \circ E_{\hat{j}}) E_{\hat{h}})_{x,x} \\
&= \frac{q^{\hat{h}}_{i,\hat{j}}}{|X|} (E_{\hat{h}})_{x,x} \\
&= \frac{q^{\hat{h}}_{i,\hat{j}} m_{\hat{h}}}{|X|^2} = \frac{q^j_{h,i} m_j}{|X|^2},
\end{aligned}$$

by Exercise 3.1.3. Note that

$$(E_h)_{x,x} = \left(\frac{1}{|X|} \sum_{i=0}^{d} q_h(i) A_i \right)_{x,x} = \frac{q_h(0)}{|X|} = \frac{1}{|X|} \mathrm{tr}(E_h) = \frac{m_h}{|X|}.$$

(3) This is clear from Exercise 3.1.2. ■

Exercise 3.1.4 Compute $\ll x_{A_i} A_h x_{A_j}, x_{A_l} A_k x_{A_m} \gg$ to show an analogous result.

Terwilliger algebras are studied in depth and applied to investigate some problems in algebraic combinatorics already. But most of the work is done by P. Terwilliger and his students to study P- and Q-polynomial association schemes which we define at the end of this section. See references at the end of this lecture note. Here we only give the semisimplicity of this algebra as an exercise.

Exercise 3.1.5 Let $T(x)$ be a Terwilliger algebra with respect to x. Let $V = \boldsymbol{C}^n$ be the Hermitean space of dimension $n = |X|$. Define $T(x)$-submodules and irreducible $T(x)$-submodules by mimicking the definitions of $\mathcal{M}$-submodules and irreducible ones. Show that every $T(x)$-submodule of V can be decomposed into an orthogonal direct sum of irreducible $T(x)$-submodules.
(Hint: For $M \in T(x)$, it is easy to show that $M^* = {}^t\bar{M} \in T(x)$. Use this fact and apply Proposition 2.3.2.)

3.2 P and Q Matrices

In this section, we collect several properties of the parameters $p_i(j)$, $q_i(j)$, $p_{i,j}^h$ and $q_{i,j}^h$ for the convenience of the readers. In the following for $x \in X$, by abuse of notation, x also denotes the unit column vector of size $|X|$ whose x position is 1 and 0 elsewhere.

Lemma 3.2.1 *Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a commutative association scheme. For $i = 0, 1 \ldots, d$ the following hold.*

(1) $p_0(i) = q_0(i) = 1$.

(2) $p_i(0) = k_i = \tau(A_i)/|X|$, *and* $|X| = k_0 + k_1 + \cdots + k_d$.

(3) $q_i(0) = m_i = \mathrm{tr}(E_i) = |X|(E_i)_{x,x} = |X|(E_i x, E_i x)$, *for* $x \in X$, *and* $|X| = m_0 + m_1 + \cdots + m_d$.

Proof. These are all straightforward from the definitions. ■

Proposition 3.2.2 *Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a commutative association scheme. Then for all $0 \leq i, j \leq d$, the following hold.*

(1) $\gamma_j(i) = \dfrac{q_j(i)}{m_j} = \dfrac{\overline{p_i(j)}}{k_i} = \dfrac{(E_j x, E_j y)}{(E_j x, E_j x)}$, *for* $(x, y) \in R_i$.

(2) $\dfrac{m_i}{|X|} \displaystyle\sum_{h=0}^{d} \dfrac{1}{k_h} p_h(i)\overline{p_h(j)} = \dfrac{m_i}{|X|} \sum_{h=0}^{d} k_h \gamma_i(h)\overline{\gamma_j(h)} = \delta_{i,j}$.

(3) $\dfrac{1}{k_i|X|} \displaystyle\sum_{h=0}^{d} m_h p_i(h)\overline{p_j(h)} = \dfrac{k_i}{|X|} \sum_{h=0}^{d} m_h \gamma_h(i)\overline{\gamma_h(j)} = \delta_{i,j}$.

Proof.

(1) This follows from Lemma 3.1.1.

(2) Since $(P)_{i,h} = p_h(i)$ and $(Q)_{h,i} = q_i(h) = m_i\overline{p_h(i)}/k_h$, the orthogonality comes from the equation $PQ = |X|I$.

(3) Since $PQ = |X|I$, P is nonsingular and we have $QP = |X|I$. We have this second orthogonality from this equation. ■

Lemma 3.2.3 (1) $q_{i,j}^h = \dfrac{m_i m_j}{|X|}\displaystyle\sum_{l=0}^{d}\frac{1}{k_l^2}p_l(i)p_l(j)\overline{p_l(h)}$.

(2) $p_{i,j}^h = \dfrac{k_i k_j}{|X|}\displaystyle\sum_{l=0}^{d}\frac{1}{m_l^2}q_l(i)q_l(j)\overline{q_l(h)}$.

Proof.

(1) By Lemma 3.1.1,

$$\frac{1}{|X|}q_{i,j}^h m_h = \ll E_h, E_i \circ E_j \gg = \tau(\bar{E}_h \circ E_i \circ E_j).$$

Now the formula can be derived using the linear expression of E_l's in terms of A_l's.

(2) This is similar to (1). ■

Exercise 3.2.1 Complete the proof of Lemma 3.2.3.

3.3 C-algebras

We have already seen some duality and similarities of the two bases of the Bose-Mesner algebra, namely the adjacency matrices and the primitive idempotents. In this section we introduce a notion of C-algebras or character algebra, which was first introduced by Y. Kawada in 1942 [89]. This was introduced by P. Delsarte to explain the duality in commutative association schemes in 1973 [34], possibly without knowing the work of Y. Kawada.

Definition 3.3.1 Let $\mathcal{A}$ be an algebra over $\boldsymbol{C}$ with a basis $x_0, x_1, \ldots, x_d$ in the linear sense. $\mathcal{A}$ together with $x_0, x_1, \ldots, x_d$ is called a *C-algebra* if the following conditions hold.

1. $x_i x_j = \displaystyle\sum_{h=0}^{d}\alpha_{i,j}^h x_h = x_j x_i$, (commutative algebra).

2. x_0 is the identity element.

3. $\alpha_{i,j}^h \in \boldsymbol{R}$.

4. There is a permutation $i \to i'$ such that $(i')' = i$ and $\alpha_{i,j}^h = \alpha_{i',j'}^{h'}$, i.e., the mapping $x_i \to x_{i'}$ induces an automorphism,

$$x_{i'}x_{j'} = \sum_{h=0}^{d}\alpha_{i,j}^h x_{h'}.$$

5. $\alpha_{i,j}^0 = \delta_{i,j'}\kappa_i$ with $\kappa_i > 0$ for all i, j.

6. The mapping $x_h \to \kappa_h$ induces a homomorphism.

A C-algebra is said to be *symmetric* if $i' = i$ for all i. A C-algebra is said to be of *positive type* if all structure constants $\alpha_{i,j}^h$ are nonnegative.

Example 3.3.1 Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme.

1. The Bose-Mesner algebra $\mathcal{M}$ together with a basis $A_0, A_1, \ldots, A_d$ is a C-algebra of positive type with $p_{i,j}^h \in \boldsymbol{Z}$, i.e., the structure constants are all nonnegative integers.

2. The Bose-Mesner algebra with $\circ$-product, which is sometimes denoted by $\mathcal{M}^*$, together with a basis $|X|E_0, |X|E_1, \ldots, |X|E_d$ is a C-algebra of positive type. In this case the structure constants $q_{i,j}^h$ are all nonnegative real numbers but they are not integers in general.

Exercise 3.3.1 Check each of the conditions of C-algebras for the examples above.

Exercise 3.3.2 Let $\mathcal{A}$ with the basis $x_0, x_1, \ldots, x_d$ be a C-algebra. Compute the coefficient of x_0 in the expression of $(x_ix_j)x_h = (x_jx_h)x_i$ and show the following.

$$\kappa_h\alpha_{\hat{i},j}^h = \kappa_j\alpha_{h,i}^j = \kappa_i\alpha_{\hat{h},j}^i.$$

Definition 3.3.2 Let $\mathcal{A}$ with the basis $x_0, x_1, \ldots, x_d$ be a C-algebra.

1. For $h = 0, 1, \ldots, d$, let B_h be a $(d+1)\times(d+1)$ matrix such that $(B_h)_{i,j} = \alpha_{h,i}^j$.

2. Suppose $h' = h$. The the *distribution diagram* $\mathcal{D}(\mathcal{A}, x_h)$ is a graph with $d+1$ vertices $\{0, 1, \ldots, d\}$ such that $i \sim j$ if and only if $\alpha_{h,i}^j \neq 0$.

Since $h' = h$, $\alpha_{h,i}^j \neq 0$ if and only if $\alpha_{h,j}^i \neq 0$ by Exercise 3.3.2. Hence $i \sim j$ if and only if $j \sim i$.

The following proposition is a basic result of C-algebra giving the duality. We state it without proof. See [6, Section II-5]. It is possible to prove using the argument in Proposition 2.3.2.

Proposition 3.3.1 *Let $\mathcal{A}$ with the basis $x_0, x_1, \ldots, x_d$ be a C-algebra. Let $x_i \circ x_j = \delta_{i,j}x_i$ for all i, j. Then there exists a basis $e_0, e_1, \ldots, e_d$ of $\mathcal{A}$ such that $e_ie_j = \delta_{i,j}e_i$. Moreover if $n = \sum_{i=0}^d \kappa_i$, then $ne_0, ne_1, \ldots, ne_d$ is a C-algebra with respect to $\circ$-product.*

3.4 C-algebras of Polynomial Type

Definition 3.4.1 (1) A C-algebra $\mathcal{A}$ with the basis $x_0, x_1, \ldots, x_d$ is of *polynomial type* if it is symmetric and there are polynomials $v_i(t) \in \boldsymbol{R}[t]$, $i = 0, 1, \ldots, d$ of degree i such that $x_i = v_i(x_1)$.

(2) A symmetric association scheme is said to be *P-polynomial* if and only if the Bose-Mesner algebra $\mathcal{M}$ together with a basis $A_0, A_1, \ldots, A_d$ is a C-algebra of polynomial type.

(3) A symmetric association scheme is said to be *Q-polynomial* if and only if the Bose-Mesner algebra with $\circ$-product (i.e., $\mathcal{M}^*$) and with a basis $|X|E_0, |X|E_1, \ldots, |X|E_d$ is a C-algebra of polynomial type.

For a C-algebra of polynomial type it is defined with a fixed ordering of the basis. We note that there are symmetric association schemes which are P-polynomial (or Q-polynomial) with respect to two different orderings of basis elements.

Proposition 3.4.1 *Let $\mathcal{A}$ with the basis $x_0, x_1, \ldots, x_d$ be a C-algebra. Then the following are equivalent.*

(i) *$\mathcal{A}$ is of polynomial type.*

(ii) *The matrix B_1 is tridiagonal with nonzero offdiagonal, i.e, $\alpha_{1,i}^{j} = 0$ if $|j-i| > 1$ and $\alpha_{1,i}^{j} \neq 0$ if $|j-i| = 1$.*

(iii) *$x_1 = x_{1'}$ and the distribution diagram $\mathcal{D}(\mathcal{A}, x_1)$ is a path,*

$$x_0 \sim x_1 \sim \cdots \sim x_d.$$

(iv) *$\alpha_{h,i}^{j} = 0$ if one of h, i, j is greater than the sum of the other two, and $\alpha_{h,i}^{j} \neq 0$ if one of h, i, j is equal to the sum of the other two, for all $h, i, j \in \{0, 1, \ldots, d\}$.*

Proof. The equivalence (ii) $\Leftrightarrow$ (iii) is obvious. (iv) $\Rightarrow$ (ii) is trivial by setting $h = 1$.

(ii) $\Rightarrow$ (i) By definition of B_1, we have

$$x_1 x_i = \alpha_{1,i}^{i-1} x_{i-1} + \alpha_{1,i}^{i} x_i + \alpha_{1,i}^{i+1} x_{i+1}.$$

Since x_0 is an identity element, $\alpha_{1,i}^{0} \neq 0$ only when $i = 1$. Hence $1' = 1$. Since $\alpha_{1,i}^{i+1} x_{i+1} \neq 0$ for $1 \leq i+1 \leq d$ by assumption, we can show easily by induction that x_{i+1} can be written as a polynomial of x_1 of degree $i+1$. Thus $\mathcal{A}$ is symmetric and is of polynomial type.

(i) $\Rightarrow$ (iv) Since $\mathcal{A}$ is symmetric, by Exercise 3.3.2

$$\alpha_{i,j}^{h} = 0 \Leftrightarrow \alpha_{h,j}^{i} = 0 \Leftrightarrow \alpha_{h,i}^{j} = 0.$$

Hence we may assume that $h \geq i + j$. Since $x_0, x_1, \ldots, x_d$ are linearly independent, $x_1^0, x_1^1, \ldots, x_1^d$ are linearly independent. Hence the polynomial in x_1 of degree $i+j$, $x_i x_j = v_i(x_1) v_j(x_1)$, has x_h as a linear summand if $h = i + j$ and does not have x_h as a linear summand if $h > i + j$. ∎

Exercise 3.4.1 The association scheme attached to a distance-regular graph is a P-polynomial association scheme. Conversely, if $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ is a P-polynomial association scheme, then the graph $\Gamma = (X, R_1)$, the vertex set X with edge set R_1 defines a distance-regular graph.

Exercise 3.4.2 Let $\mathcal{A}$ be a C-algebra of polynomial type. Let

$$\beta_i = \alpha^i_{i+1,1},\ \alpha_i = \alpha^i_{i,1},\ \gamma_i = \alpha^i_{i-1,1}.$$

Show the following.

1. $\alpha^h_{i+1,j}\gamma_{i+1} = \alpha^h_{i,j-1}\beta_{j-1} + \alpha^h_{i,j}(\alpha_j - \alpha_i) + \alpha^h_{i,j+1}\gamma_{j+1} - \alpha^h_{i-1,j}\beta_{i-1}$.
2. $\alpha^0_{i,j} = \delta_{i,j}\kappa_i$, $\kappa_i\beta_i = \kappa_{i+1}\gamma_{i+1}$.
3. $\gamma_h\alpha^{i+h}_{i,h} = \gamma_{i+1}\alpha^{i+h}_{i+1,h-1}$.
4. $\alpha^{i+h}_{i,h+1}\gamma_{h+1} = \alpha^{i+h}_{i,h}(\alpha_i + \cdots + \alpha_{i+h} - \alpha_1 - \cdots - \alpha_h)$.

(Hint: For (1) compute the coefficient of x_h in the expression of $(x_1x_i)x_j = (x_1x_j)x_i$ in two ways.)

Exercise 3.4.3 Let d be a positive integer and let $\gamma_{i+1}, \alpha_i, \beta_{i-1}$ be real numbers satisfying the following.

1. $\alpha_0 = \beta_{-1} = \gamma_{d+1} = 0$, $\gamma_1 = 1$.
2. $\gamma_i + \alpha_i + \beta_i = \beta_0 = \gamma_d + \alpha_d$ for $i = 1, \ldots, d-1$.
3. $\beta_i\gamma_{i+1} > 0$.

Define multiplication by $x_0x_0 = x_0$, and

$$x_ix_1 = \beta_{i-1}x_{i-1} + \alpha_ix_i + \gamma_{i+1}x_{i+1}.$$

Then this algebra becomes a C-algebra of polynomial type with

$$\beta_i = \alpha^i_{i+1,1},\ \alpha_i = \alpha^i_{i,1},\ \gamma_i = \alpha^i_{i-1,1}.$$

Chapter 4

Balanced Conditions

4.1 Subspaces of Terwilliger Algebra

Let $\mathcal{X}$ be a commutative association scheme. Recall some notation.

- $A_0, A_1, \ldots, A_d$, i-th adjacency matrices.
- $E_0, E_1, \ldots, E_d$, primitive orthogonal idempotents.
- $\mathcal{M} = \mathrm{Span}(A_0, A_1, \ldots, A_d) = \mathrm{Span}(E_0, E_1, \ldots, E_d)$, the Bose-Menser algebra.
- For $x \in X$ and $M \in \mathcal{M}$, x_M denotes a diagonal matrix such that $(x_M)_{y,y} = M_{x,y}$.
- $T(x) =< M, x_M \mid M \in \mathcal{M} >_{alg} \subset Mat_X(\boldsymbol{C})$, the Terwilliger algebra with respect to x.

Let E be a positive semidefinite symmetric matrix in the Bose-Mesner algebra, i.e.,

$$E = \alpha_0 E_0 + \alpha_1 E_1 + \cdots + \alpha_d E_d,$$

where α_i's are nonnegative real numbers and that $O \neq E = {}^tE$. Note that in this case, E_i and $E_{\hat{i}} = {}^tE_i = \bar{E}_i$ appear in pair with the same coefficients. In particular, all entries of E are real. When $E_h = {}^tE_h$, E_h is a typical positive semidefinite symmetric element. Though most of the parts below can be extended to a general positive semidefinite symmetric matrix, we restrict our attention to a symmetric primitive idempotent to avoid lengthy calculation.

For simplicity, let $E = E_1 = {}^tE_1$.

Exercise 4.1.1 Let $E = {}^tE$ be a symmetric matrix belonging to the Bose-Mesner algebra.

1. Then the eigenvalues of E are all nonnegative and real, if and only if E is positive semidefinite in the sense above.
2. If E is positive semidefinite then there is a positive semidefinite symmetric matrix $\tilde{E}$ in the Bose-Mesner algebra such that $E = {}^t\tilde{E}\tilde{E}$. Hence E can be regarded as a Gram matrix. Conversely, any Gram matrix of real vectors of the same length becomes a positive semidefinite symmetric matrix.

In the following we consider four subspaces of $T(x)$.

$$\begin{aligned}
\mathcal{L} &= \mathcal{M}x_E\mathcal{M} = \mathrm{Span}(E_ix_EE_j \mid 0 \le i,j \le d),\\
\mathcal{N} &= x_E\mathcal{M} + \mathcal{M}x_E = \mathrm{Span}(x_EA_i, A_ix_E \mid 0 \le i \le d),\\
\mathcal{L}_0 &= \mathrm{Span}(E_ix_EE_j - E_jx_EE_i \mid 0 \le i,j \le d),\\
\mathcal{N}_0 &= \mathrm{Span}(x_EA_i - A_ix_E \mid 0 \le i \le d).
\end{aligned}$$

It is clear that $\mathcal{L}$ is the largest and $\mathcal{N}_0$ is the smallest among them.
Recall that by Lemma 3.1.2, and Exercise 3.1.3, we have the following.

1. $\ll E_ix_EE_j, E_kx_EE_l \gg = \delta_{i,k}\delta_{j,l}\frac{1}{|X|^2}q^j_{1,i}m_j \ge 0$.

2. $\ll E_ix_EE_j, E_ix_EE_j \gg = 0$, if and only if $E_ix_EE_j = 0$, if and only if $q^j_{1,i} = 0$.

Hence the following lemma is a direct consequence of the above. Recall that the set of nonzero orthogonal elements in a vector space are linearly independent.

Lemma 4.1.1 *The following hold.*

(1) $\dim \mathcal{L} = |\{(i,j) \mid 0 \le i,j \le d,\ \text{such that } q^j_{1,i} \ne 0\}|$.

(2) $\dim \mathcal{L}_0 = |\{(i,j) \mid 0 \le i < j \le d,\ \text{such that } q^j_{1,i} \ne 0\}|$.

Lemma 4.1.2 *The following hold for A_ix_E and x_EA_i.*

(1) $(x_EA_i)_{yz} = \begin{cases} E_{x,y} & (y,z) \in R_i \\ 0 & (y,z) \notin R_i, \end{cases} \qquad (A_ix_E)_{yz} = \begin{cases} E_{x,z} & (y,z) \in R_i \\ 0 & (y,z) \notin R_i. \end{cases}$

In particular, for $(x,y),(z,x) \in R_i$,

(a) $(x_EA_i)_{x,y} = E_{x,x} = q_1(0)/|X| = m_1/|X|$,
(b) $(A_ix_E)_{x,y} = E_{x,y} = q_1(i)/|X|$,
(c) $(x_EA_i)_{z,x} = E_{x,z} = q_1(i)/|X|$, *and*
(d) $(A_ix_E)_{z,x} = E_{x,x} = q_1(0)/|X| = m_1/|X|$.

(2) $\mathcal{N} = \bigoplus_{i=0}^{d} \mathrm{Span}(x_EA_i, A_ix_E)$.

Proof. (1) The assertions are all clear from the definition.
(2) Suppose there is a linear combination expressing O:

$$\sum_{i=0}^{d}(\alpha_ix_EA_i + \beta_iA_ix_E) = O.$$

For $(y,z) \in R_i$,

$$0 = \left(\sum_{i=0}^{d}(\alpha_ix_EA_i + \beta_iA_ix_E)\right)_{y,z} = (\alpha_ix_EA_i + \beta_iA_ix_E)_{y,z}.$$

Hence $\alpha_ix_EA_i + \beta_iA_ix_E = O$, because if $(y,z) \notin R_i$, the (y,z)-entry of this matrix is automatically 0. This proves the assertion. ■

Lemma 4.1.3 *The following hold.*

(1) $1 \leq \dim \mathrm{Span}(x_E A_i, A_i x_E) \leq 2$.

(2) $\dim \mathrm{Span}(x_E A_i, A_i x_E) = 1$ *if and only if* $q_1(0) = \pm q_1(i)$.

(3) *Suppose* $q_1(0) \neq q_1(i)$ *for all* $i = 1, 2, \ldots, d$. *Let* $z \in X$ *denote the unit column vector whose* z*-position is 1 and 0 elsewhere. Then the following hold.*

  (a) *If* $q_1(i) = -q_1(0)$ *for some* i, *then* $q_1(j) \neq -q_1(0)$ *for* $j \neq i$, $\dim \mathcal{N} = 2d$, *and* $\dim \mathcal{N}_0 = d$. *Moreover* $\tilde{X} = \{Ez \mid z \in X\}$ *satisfies,* $\tilde{X} = -\tilde{X}$. *In this case we call* antipodal.

  (b) *If* $q_1(i) \neq -q_1(0)$ *for all* i, *then* $\mathcal{N} = 2d + 1$ *and* $\dim \mathcal{N}_0 = d$.

Proof. (1) By Lemma 4.1.2, $x_E A_i \neq O$. Hence we have (1).

(2) Suppose $\alpha x_E A_i + \beta A_i x_E = O$ for some $(\alpha, \beta) \neq (0, 0)$. For $(x, y), (z, x) \in R_i$, considering (x, y) and (z, x) entries of the matrix equation, we have

$$\begin{cases} \alpha q_1(0) + \beta q_1(i) = 0 \\ \alpha q_1(i) + \beta q_1(0) = 0. \end{cases}$$

Thus we have $q_1(0)^2 - q_1(i)^2 = 0$ or $q_1(0) = \pm q_1(i)$.

Conversely, suppose $q_1(0) = \pm q_1(i)$ for some i. Then by Lemma 4.1.2, for $(y, z) \in R_i$,

$${}^t y {}^t E E z = {}^t y E z = E_{y,z} = \pm E_{y,y} = \pm {}^t y E y = \pm {}^t y {}^t E E y.$$

Hence $Ey = \pm Ez$ for all $(y, z) \in R_i$. Therefore,

$$(x_E A_i)_{y,z} = E_{x,y} = {}^t x E y = \pm {}^t x E z = \pm E_{x,z} = (A_i x_E)_{y,z}$$

for all $(y, z) \in R_i$. We thus have $x_E A_i = \pm A_i x_E$ in this case.

(3) (a) By the observation above, $\tilde{X} = -\tilde{X}$. By our assumption $Ey = Ez$ if and only if $(y, z) \in R_0$, i.e., $y = z$. Thus i with $q_1(0) = -q_1(i)$ is uniquely determined. Therefore, $\dim \mathrm{Span}(x_E A_j, A_j x_E) = 1$ if and only if $j \in \{0, i\}$. This means, $\dim \mathcal{N} = 2d$. $\dim \mathcal{N}_0 = d$ is obvious as $x_E A_j - A_j x_E = O$ if and only if $j = 0$ by our assumption. See Lemma 4.1.2 (2).

(b) This is similar to (a). ■

4.2 Representation Diagrams

In this section, we define representation diagrams and prove some basic properties of them.

Definition 4.2.1 Suppose $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is a commutative association scheme. Suppose $E = E_1 = {}^t E_1$ is symmetric. *The representation diagram* of $\mathcal{X}$ with respect to E, denoted by $\mathcal{D}^* = \mathcal{D}^*(\mathcal{X}, E)$, is a diagram with $\{0, 1, \ldots, d\}$ as the vertex set such that the adjacency is defined as follows.

$$i \sim j \Leftrightarrow E(E_i \circ {}^t E_j) \neq 0 \ \Leftrightarrow q_{1,i}^j \neq 0.$$

The adjacent pair i, j is said to be an edge if $i \neq j$ and a loop if $i = j$. Let $\partial(i,j)$ denote the distance in $\mathcal{D}^*$. Let $d_{\mathcal{D}^*}(i) = \max\{\partial(i,j) \mid j = 0, 1, \ldots, d\}$ and $d_{\mathcal{D}^*} = \max\{\partial(i,j) \mid i, j = 0, 1, \ldots, d\}$.

Lemma 4.2.1 *The representation diagram $\mathcal{D}^* = \mathcal{D}^*(\mathcal{X}, E)$ is connected if and only if $q_1(0) \neq q_1(i)$ for all $i \geq 1$. In this case the vectors $\{Ex \mid x \in X\} \subset V_i$ are all distinct.*

Proof. Since $Ex = Ey$ for $(x,y) \in R_i$ if and only if $(E)_{x,y} = (E)_{x,x}$, the last assertion follows from Proposition 3.2.2.

Next we claim that

$$C_{\mathcal{M}}(x_E) = \{M \in \mathcal{M} \mid Mx_E = x_E M\} = \mathrm{Span}(A_i \mid i = 0, 1, \ldots, d,\ q_1(0) = q_1(i)).$$

Let $M = \sum_{h=0}^{d} \alpha_h A_h \in \mathcal{M}$. Then

$$x_E M - M x_E = \sum_{h=0}^{d} \alpha_h (x_E A_h - A_h x_E) = O$$

if and only if $\alpha_h(q_1(0) - q_1(i)) = 0$ for every h, by Lemma 4.1.3 and its proof. Thus we have the claim.

Let Δ be a connected component of $\mathcal{D}^*$. Let $F = \sum_{i \in \Delta} E_i$. Recall that

$$E_j x_E E_i = O \Leftrightarrow q^i_{1,j} = 0 \Leftrightarrow j \not\sim i.$$

Thus $E_j x_E E_i = O$ if $i \in \Delta$ and $j \notin \Delta$. Therefore we have

$$\begin{aligned}
x_E F &= I x_E F = \left(\sum_{j=0}^{d} E_j\right) x_E \left(\sum_{i \in \Delta} E_i\right) \\
&= \left(\sum_{j \in \Delta}^{d} E_j\right) x_E \left(\sum_{i \in \Delta} E_i\right) \\
&= \left(\sum_{j \in \Delta}^{d} E_j\right) x_E \left(\sum_{i=0}^{d} E_i\right) \\
&= F x_E
\end{aligned}$$

Suppose $q_1(0) \neq q_1(i)$ for all $i \geq 1$. Then by our claim above, $F \in \mathrm{Span}(I)$ or $\Delta = \{0, 1, \ldots, d\}$.

We leave the converse as an exercise for the readers. ∎

Exercise 4.2.1 Suppose $S = \{i \mid q_1(0) = q_1(i)\} \neq \{0\}$. Show the following.

1. For $i, j \in S$, $A_i A_j$ can be written as a linear combination of A_h's with $h \in S$.

2. Let $F = \sum_{i \in S} A_i$, and $f = \sqrt{\sum_{i \in S} k_i}$. Then $F^2 = f^2 F$.

3. There is a subset $T \subset \{0, 1, \ldots, d\}$ containing 0 and 1 such that $(1/f)F = \sum_{j \in T} E_j$.

4. For $i, j \in T$, $E_i \circ E_j$ can be written as a linear combination of E_h's with $h \in T$.

5. The connected component of $\mathcal{D}^*$ containing 0 is contained in T and $T \neq \{0, 1, \ldots, d\}$.

If $S \neq \{0\}$, the commutative association scheme $\mathcal{X}$ is said to be *imprimitive.*

Lemma 4.2.2 *Suppose $q_1(i) \neq q_1(0)$ for all $i \geq 1$ and $q_1(h) = -q_1(0)$ for some $h \geq 0$. Then $E_i x_E E_i = O$ for every $i = 0, 1, \ldots, d$, i.e., the representation graph $\mathcal{D}^*(\mathcal{X}, E)$ does not have a loop.*

Proof. By Lemma 4.1.3, we have $x_E A_h = -A_h x_E$. Hence $A_h^2 \in C_{\mathcal{M}}(x_E)$. So by the claim in Lemma 4.2.1, A_h^2 can be written as a nonzero scalar multiple of I. In particular, A_h is nonsingular and the eigenvalue $p_h(i) \neq 0$ for all i. Note that $A_h E_i = p_h(i) E_i$. Now,

$$p_h(i) E_i x_E E_i = A_h E_i x_E E_i = E_i A_h x_E E_i = -E_i x_E A_h E_i = -p_h(i) E_i x_E E_i.$$

Hence we must have $E_i x_E E_i = O$, or $q_{1,i}^i = 0$. ∎

Lemma 4.2.3 *Let $\Delta = \{q_1(i) \mid i = 0, 1, \ldots, d\}$ and let $\delta = |D| - 1$, the number one less than the different values $q_1(i)$ takes. Then*

$$d_{\mathcal{D}^*} = d_{\mathcal{D}^*}(0) = \delta.$$

In particular, $\mathcal{D}^*(\mathcal{X}, E)$ is a path (i.e., Q-polynomial), if and only if $\delta = d$.

Proof. Let $\Delta = \{\alpha_0 = q_1(0), \alpha_1, \ldots, \alpha_\delta\}$. Set

$$F = (E - \frac{\alpha_0}{|X|} J) \circ (E - \frac{\alpha_1}{|X|} J) \circ \cdots \circ (E - \frac{\alpha_\delta}{|X|} J).$$

Since $A_i \circ F = O$ and $F \in \mathcal{M} = \mathrm{Span}(A_0, A_1, \ldots, A_d)$, we have $F = O$. It is clear that this gives a minimal polynomial of E with respect to $\circ$-product. Let $\mathcal{E}$ be the subalgebra of $\mathcal{M}^*$ generated by E with respect to $\circ$-product. Then $\dim \mathcal{E} = \delta + 1$, and $\mathcal{E}$ is spanned by 0-th to δ-th power of E with respect to $\circ$-product. Hence if E_j appears as a linear summand of some power of E, it must appear as a linear summand of the power of E_1 at most δ. This is our assertion. ∎

4.3 Tree Condition

In this section, we give a proof of Terwilliger's results on balanced conditions. It connects three conditions. The first is the condition that the representation diagram $\mathcal{D}^*(\mathcal{X}, E)$ is a tree. The second is the conditions on the subspaces of the Terwilliger algebra. The third is the balanced condition of the finite set on the sphere introduced in the first chapter. The second condition can be regarded as an algebraic condition and the third a geometric condition. Since a Q-polynomial association scheme is the one with a representation diagram which is a path, this theorem gives an algebraic and geometric characterization of Q-polynomial association schemes.

We first interpret Lemma 4.1.1 using the terminologies of representation diagram.

Lemma 4.3.1 *Let $\mathcal{D}^* = \mathcal{D}^*(\mathcal{X}, E)$ be a representation graph. Let* $\mathrm{edge}(\mathcal{D}^*)$ *denote the number of edges in $\mathcal{D}^*$ and let* $\mathrm{loop}(\mathcal{D}^*)$ *denote the number of loops in $\mathcal{D}^*$. Then the following hold.*

(1) $\dim \mathcal{L} = 2 \cdot \mathrm{edge}(\mathcal{D}^*) + \mathrm{loop}(\mathcal{D}^*)$.

(2) $\dim \mathcal{L}_0 = \mathrm{edge}(\mathcal{D}^*)$.

Before stating the results, we give a definition of balanced sets and strongly balanced set in a little more general setting than the one we introduced before.

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a finite set X in a real Euclidean space $\boldsymbol{R}^m$ together with some relations on it:

$$\emptyset \neq R_i \subset X \times X,\ i = 0, 1, \ldots, d.$$

We assume the following conditions.

1. $\mathrm{Span}(X) = \boldsymbol{R}^m$.
2. $R_0 = \{(x, x) \mid x \in X\}$.
3. $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$, (disjoint union).
4. ${}^tR_i = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where
$$ {}^tR_i = \{(x, y) \mid (y, x) \in R_i\}. $$
5. For $x, y \in X$, the inner product $x \cdot y = \gamma(i)$ depends only on i such that $(x, y) \in R_i$.

Let $A_i \in Mat_X(\boldsymbol{C})$ defined by

$$(A_i)_{xy} = \begin{cases} 1 & (x, y) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

For $x, y \in X$, $0 \leq i, j \leq d$, let

$$\begin{aligned} P_{i,j}(x, y) &= \{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\} \subset \boldsymbol{R}^m, \text{ and} \\ \widehat{P_{i,j}(x, y)} &= \sum_{z \in X, (x,z) \in R_i, (z,y) \in R_j} z \in \boldsymbol{R}^m. \end{aligned}$$

Definition 4.3.1 Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be as above.

(1) $\mathcal{X}$ defines a commutative association scheme if the following hold. For all i, j,

$$A_i A_j = A_j A_i \in \mathrm{Span}(A_0, A_1, \ldots, A_d).$$

(2) $\mathcal{X}$ is *eutactic* if

$$\widehat{P_{i,i}(x, x)} \in \mathrm{Span}(x),\ \text{for all } x \in X,\ 0 \leq i \leq d.$$

(3) $\mathcal{X}$ is *balanced* if for all $x, y \in X$, and $0 \le i, j \le d$,

$$P_{i,j}\widehat{(x,y)} - P_{j,i}\widehat{(x,y)} \in \mathrm{Span}(x-y).$$

(4) $\mathcal{X}$ is *strongly balanced* if for all $x, y \in X$, and $0 \le i, j \le d$,

$$P_{i,j}\widehat{(x,y)} \in \mathrm{Span}(x,y).$$

Exercise 4.3.1 Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme and let E be a positive semidefinite symmetric matrix. Let $\hat{X} = \{\hat{x} \mid x \in X\}$ with $\hat{x} = Ex$. Suppose $|\hat{X}| = |X|$, i.e., the embedding is faithful.

1. Show that $\mathcal{X} = (\hat{X}, \{R_i\}_{0\le i\le d})$ satisfies the conditions above and $\mathcal{X}$ defines a commutative association scheme in the sense defined above.

2. Show that $\mathcal{X}$ is eutactic if and only if E is a (positive real) scalar multiple of a primitive idempotent.

The balanced conditions defined in the introduction is a little too strong, as the partition is given by the values of the inner product. In that case by Lemma 4.2.3, if $\mathcal{X}$ defines a commutative association scheme and if it is eutactic, it automatically becomes a Q-polynomial association scheme.

Theorem 4.3.2 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme with a symmetric primitive idempotent $E = E_1 = {}^tE_1$. Let $\mathcal{M}$ be the Bose-Mesner algebra. Suppose $q_1(0) \ne q_1(i)$ for all $i \ge 1$. Then the following are equivalent.*

(i) The representation graph $\mathcal{D}^* = \mathcal{D}^*(\mathcal{X}, E)$ is a tree and $\mathrm{loop}(\mathcal{D}^*) \le 1$.

(ii) $\mathcal{M}x_E\mathcal{M} = x_E\mathcal{M} + \mathcal{M}x_E$, i.e., $\mathcal{L} = \mathcal{N}$ for every $x \in X$.

(iii) $\hat{X} = \{\hat{x} \mid x \in X\}$ with $\hat{x} = Ex$ is strongly balanced, i.e., for all $u, v \in X$, and $0 \le i, j \le d$,

$$P_{i,j}\widehat{(u,v)} \in \mathrm{Span}(\hat{u}, \hat{v}).$$

Proof. By the condition $q_1(0) \ne q_1(i)$ for all $i \ge 1$, $\mathcal{D}^*$ is connected by Lemma 4.2.1.

By Lemma 4.3.1, (i) is equivalent to the condition $\dim \mathcal{L} = 2d$ or $2d+1$. Now by Lemma 4.1.3 and Lemma 4.2.2, this is equivalent to (ii).

Assume (ii). Then $A_i x_E A_j \in x_E\mathcal{M} + \mathcal{M}x_E$. Hence there are constants $\beta^h_{i,j}, \gamma^h_{i,j}$ such that

$$A_i x_E A_j = x_E\left(\sum_{h=0}^{d} \beta^h_{i,j} A_h\right) + \left(\sum_{h=0}^{d} \gamma^h_{i,j} A_h\right) x_E.$$

Let $(u, v) \in R_h$. Then

$$(A_i x_E A_j)_{u,v} = \sum_{z\in X,\,(u,z)\in R_i,\,(z,v)\in R_j} (E)_{x,z} = (\hat{x}, P_{i,j}\widehat{(u,v)}).$$

On the other hand,

$$\left(x_E\left(\sum_{h=0}^{d}\beta_{i,j}^h A_h\right)+\left(\sum_{h=0}^{d}\gamma_{i,j}^h A_h\right)x_E\right)_{u,v}=(\hat{x},\beta_{i,j}^h\hat{u}+\gamma_{i,j}^h\hat{v}).$$

By setting $u=x$ and $v=x$, we have the following.

$$\begin{aligned} p_{i,j}^h q_1(i) &= \beta_{i,j}^h q_1(0)+\gamma_{i,j}^h q_1(h) \\ p_{i,j}^h q_1(j) &= \beta_{i,j}^h q_1(h)+\gamma_{i,j}^h q_1(0) \end{aligned}$$

Therefore we can choose these constants $\beta_{i,j}^h, \gamma_{i,j}^h$ so that they do not depend on the choices of $x \in X$. Hence we have the desired conclusion as $\hat{X}$ spans V_1.

Assume (iii). Let

$$P_{i,j}\widehat{(u,v)}=\beta_{i,j}^h\hat{u}+\gamma_{i,j}^h\hat{v}.$$

This means that for every $x \in X$

$$(\hat{x}, P_{i,j}\widehat{(u,v)})=(\hat{x},\beta_{i,j}^h\hat{u}+\gamma_{i,j}^h\hat{v}).$$

By setting $x=u$ and $x=v$, we have that these constants $\beta_{i,j}^h, \gamma_{i,j}^h$ can be chosen so that they do not depend on the choices of $(u,v) \in R_h$. Therefore we have the following by comparing the entries of both sides.

$$A_i x_E A_j = x_E\left(\sum_{h=0}^{d}\beta_{i,j}^h A_h\right)+\left(\sum_{h=0}^{d}\gamma_{i,j}^h A_h\right)x_E.$$

for every $x \in X$. ■

Exercise 4.3.2 Determine the constants $\beta_{i,j}^h, \gamma_{i,j}^h$ in terms of $p_{i,j}^h$'s and $q_1(h)$'s.

Theorem 4.3.3 *Let $\mathcal{X}=(X,\{R_i\}_{0\le i\le d})$ be a commutative association scheme with a symmetric primitive idempotent $E=E_1={}^tE_1$. Let $\mathcal{M}$ be the Bose-Mesner algebra. Suppose $q_1(0)\neq q_1(i)$ for all $i\geq 1$. Then the following are equivalent.*

(i) *The representation graph $\mathcal{D}^*=\mathcal{D}^*(\mathcal{X},E)$ is a tree.*

(ii) *$\mathcal{L}_0=\mathcal{N}_0$ for every $x\in X$.*

(iii) *$\hat{X}=\{\hat{x}\mid x\in X\}$ with $\hat{x}=Ex$ is balanced, i.e., for all $u,v\in X$, and $0\le i,j\le d$,*

$$P_{i,j}\widehat{(u,v)}-P_{j,i}\widehat{(u,v)}\in \mathrm{Span}(\hat{u}-\hat{v}).$$

Exercise 4.3.3 Prove Theorem 4.3.3.

4.4 Distribution Diagrams

In Definition 3.3.2, the distribution diagrams can be defined for C-algebras. Note that the representation graph defined in Definition 4.2.1 can be regarded as a distribution diagram for a C-algebra. Now we define the distribution diagram for a commutative association scheme using another C-algebra structure.

Definition 4.4.1 Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme. Let $A = {}^tA$ be a nonzero element of the Bose-Mesner algebra with nonnegative real entries. *The distribution diagram* of $\mathcal{X}$ with respect to A, denoted by $\mathcal{D} = \mathcal{D}(\mathcal{X}, A)$, is a diagram with $\{0, 1, \ldots, d\}$ as the vertex set such that the adjacency is defined as follows.

$$i \sim j \Leftrightarrow A \circ (A_i{}^tA_j) \neq 0 \; (\Leftrightarrow A \circ (A_j{}^tA_i) \neq 0)$$

The adjacent pair i, j is said to be an edge if $i \neq j$ and a loop if $i = j$.

Recall that $\mathcal{X}$ is *a P-polynomial association scheme* if $\mathcal{D}(\mathcal{X}, A)$ is a path (possibly with some loops) for some A.

Exercise 4.4.1 Prove a corresponding result in the previous section for the distribution graphs.

In [135], P. Terwilliger showed the following by a complicated calculation.

Theorem 4.4.1 *Suppose one of the diagrams $\mathcal{D}(\mathcal{X}, A_1)$ and $\mathcal{D}^*(\mathcal{X}, E_1)$ is a path, then the other has at most one leaf besides the node* 0.

Corollary 4.4.2 *Suppose both diagrams $\mathcal{D}^*(\mathcal{X}, E_1)$ and $\mathcal{D}(\mathcal{X}, A_1)$ are trees. Then the one is a path if and only if the other is a path.*

These results state that if one of the diagrams $\mathcal{D}^*(\mathcal{X}, E_1)$ and $\mathcal{D}(\mathcal{X}, A_1)$ is very nice, then the structure of the other diagram is also restricted. There is also a result of P. Terwilliger to give a condition for $\mathcal{D}(\mathcal{X}, A_1)$ to be a tree. See [135].

Chapter 5

Examples of Balanced Sets and Related Structures

In this chapter we give examples of Q-polynomial association schemes and balanced sets. We also introduce several related structures.

5.1 s-distance Sets and Absolute Bound Conditions

Let $V = \boldsymbol{C}^n$ be the n-dimensional vector space of column vectors. For a matrix $A \in \mathrm{Mat}_n(\boldsymbol{C})$, and a nonnegative integer h, let $A^{\circ h}$ denote the h-th power of A with respect to the $\circ$-product. Let $\mathcal{C}(A)$ denote the column space of A, i.e., the subspace of V spanned by the columns of A. We also use $\circ$-product for entry-wise product of column vectors.

Lemma 5.1.1 *Let A and B be matrices in $\mathrm{Mat}_n(\boldsymbol{C})$ of rank m and l respectively. Let $\mathcal{C}(B) = \mathrm{Span}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)$ and $\mathcal{C}(A) = \mathrm{Span}(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l)$. Then the following hold.*

(1) $\mathcal{C}(A^{\circ h}) \subset \mathrm{Span}(\boldsymbol{a}_{i_1} \circ \cdots \circ \boldsymbol{a}_{i_h} \mid 1 \leq i_1, \ldots, i_h \leq m)$. *In particular,*

$$\dim \mathcal{C}(A^{\circ h}) = \mathrm{rank}(A^{\circ h}) \leq \binom{m+h-1}{h}.$$

(2) $\mathcal{C}(A \circ B) \subset \mathrm{Span}(\boldsymbol{a}_i \circ \boldsymbol{b}_j \mid 1 \leq i \leq m,\ 1 \leq j \leq l)$. *In particular,*

$$\dim \mathcal{C}(A \circ B) = \mathrm{rank}(A \circ B) \leq m \cdot l.$$

Proof. All assertions are straightforward. Note that

$$|\{(i_1, i_2, \ldots, i_h) \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_h \leq m\}| = \binom{m+h-1}{h}.$$

■

Proposition 5.1.2 *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a commutative association scheme. Then for fixed i and j, the following hold.*

$$\sum_{h : q_{i,j}^h > 0} m_h \leq \begin{cases} \frac{1}{2} m_i(m_i + 1) & \text{if } i = j, \\ m_i m_j & \text{if } i \neq j. \end{cases}$$

Proof. Since $\mathcal{C}(E_i) = V_i$, and V is an orthogonal direct sum of V_i's. we have

$$\mathrm{rank}(E_i \circ E_j) = \mathrm{rank}(\frac{1}{|X|}\sum_{h=0}^{d} q_{i,j}^h E_h) = \sum_{h:q_{i,j}^h>0} \mathrm{rank}E_h = \sum_{h:q_{i,j}^h>0} m_h.$$

On the other hand by Lemma 5.1.1,

$$\mathrm{rank}(E_i \circ E_j) \leq \begin{cases} \frac{1}{2}m_i(m_i+1) & \text{if } i = j \\ m_i m_j & \text{if } i \neq j. \end{cases}$$

Thus we have the bound. ∎

Exercise 5.1.1 Apply Proposition 5.1.2 to show that Q-polynomial association scheme with $m_1 = k_1^* = 2$ is a polygon.

Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset S^{m-1} \subset \boldsymbol{R}^m$ be a nonempty subset of cardinality n. Let

$$\Delta = \{\boldsymbol{x}\cdot\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{y} \in X\} = \{\gamma_0 = 1, \gamma_1, \ldots, \gamma_s\}.$$

So X is an s-distance set. For each i, let A_i be the matrix in $\mathrm{Mat}_X(\boldsymbol{C})$, whose $(\boldsymbol{x}, \boldsymbol{y})$-entry is defined by the following.

$$(A_i)_{\boldsymbol{x},\boldsymbol{y}} = \begin{cases} 1 & \boldsymbol{x}\cdot\boldsymbol{y} = \gamma_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $U = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ be the $m \times n$ matrix, and let $E = {}^tUU$ be the Gram matrix of X, i.e., $E \in \mathrm{Mat}_X(\boldsymbol{C})$, whose $(\boldsymbol{x}, \boldsymbol{y})$-entry is $\boldsymbol{x}\cdot\boldsymbol{y}$. Then

$$E = \sum_{i=0}^{s} \gamma_i A_i$$

and E is a real symmetric matrix.

Lemma 5.1.3 *Let $\mathcal{C}(E) = \mathrm{Span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)$, and let $\boldsymbol{j}$ be the all 1's vector in $\boldsymbol{R}^n$. Then*

$$\boldsymbol{j} \in \mathcal{C}^{(2)} = \mathrm{Span}(\boldsymbol{u}_i \circ \boldsymbol{u}_j \mid 1 \leq i, j \leq m).$$

Proof. Since U is a real matrix and ${}^tUU = E$, $\mathcal{C}({}^tU) = \mathcal{C}(E)$. In fact, since $E = {}^tUU$, $\mathcal{C}(E) \subset \mathcal{C}({}^tU)$. Moreover, since

$$U\boldsymbol{v} = \boldsymbol{0} \Rightarrow E\boldsymbol{v} = {}^tUU\boldsymbol{v} = \boldsymbol{0} \Rightarrow \|U\boldsymbol{v}\|^2 = {}^t\boldsymbol{v}{}^tUU\boldsymbol{v} = 0 \Rightarrow U\boldsymbol{v} = \boldsymbol{0},$$

we have

$$\mathrm{rank}{}^tU = \mathrm{rank}U = n - \dim\{\boldsymbol{v} \mid U\boldsymbol{v} = \boldsymbol{0}\} = n - \dim\{\boldsymbol{v} \mid E\boldsymbol{v} = \boldsymbol{0}\} = \mathrm{rank}E.$$

Thus $\mathcal{C}({}^tU) = \mathcal{C}(E) = \mathrm{Span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)$.

Let ${}^t\boldsymbol{x}_i = (x_{i1}, \ldots, x_{im})$ for $i = 1, \ldots, n$. Since $\boldsymbol{x}_i \in S^{m-1}$, $x_{i1}^2 + \cdots + x_{im}^2 = 1$. Hence the sum of the columns of ${}^tU \circ {}^tU$ is $\boldsymbol{j}$. In other words,

$$\boldsymbol{j} \in \mathcal{C}({}^tU \circ {}^tU) \subset \mathcal{C}^{(2)} = \mathrm{Span}(\boldsymbol{u}_i \circ \boldsymbol{u}_j \mid 1 \leq i, j \leq m).$$

Thus we have the lemma. ∎

Lemma 5.1.4 *Let* $\mathcal{C}(E) = \mathrm{Span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)$*. Let*

$$\mathcal{C}^{(h)} = \mathrm{Span}(\boldsymbol{u}_{i_1} \circ \cdots \circ \boldsymbol{u}_{i_h} \mid 1 \le i_1, \ldots, i_h \le m).$$

Then $\mathcal{C}(E^{\circ h-2i}) \subset \mathcal{C}^{(h-2i)} \subset \mathcal{C}^{(h)}$ *for all nonnegative integer* $i \le h/2$*.*

Proof. It suffices to show the inclusion for $i = 1$ and $h \ge 2$. By Lemma 5.1.3,

$$\boldsymbol{j} \in \mathcal{C}^{(2)} = \mathrm{Span}(\boldsymbol{u}_{i_{h-1}} \circ \boldsymbol{u}_{i_h} \mid 1 \le i_{h-1}, i_h \le m).$$

Hence

$$\begin{aligned}
\mathcal{C}^{(h)} &= \mathrm{Span}(\boldsymbol{u}_{i_1} \circ \cdots \circ \boldsymbol{u}_{i_h} \mid 1 \le i_1, \ldots, i_h \le m) \\
&= \mathrm{Span}(\boldsymbol{u}_{i_1} \circ \cdots \circ \boldsymbol{u}_{i_{h-2}} \circ \boldsymbol{v} \mid 1 \le i_1, \ldots, i_{h-2} \le m,\ \boldsymbol{v} \in \mathcal{C}^{(2)}) \\
&\supset \mathrm{Span}(\boldsymbol{u}_{i_1} \circ \cdots \circ \boldsymbol{u}_{i_{h-2}} \circ \boldsymbol{j} \mid 1 \le i_1, \ldots, i_{h-2} \le m) \\
&\supset \mathrm{Span}(\boldsymbol{u}_{i_1} \circ \cdots \circ \boldsymbol{u}_{i_{h-2}} \mid 1 \le i_1, \ldots, i_{h-2} \le m) \\
&= \mathcal{C}^{(h-2)} \\
&\supset \mathcal{C}(E^{\circ h-2}),
\end{aligned}$$

as desired. ■

Corollary 5.1.5 *Let* $f(t)$ *be a polynomial of degree* h *and let* $f^\circ(E)$ *be the matrix obtained by substituting* E *in* t*, where the product of matrices is* $\circ$*-product.*

(1) $\mathrm{rank}(f^\circ(E)) \le \binom{m+h-1}{h} + \binom{m+h-2}{h-1}$.

(2) *Suppose the coefficients of* t^{h-2i-1} *in* $f(t)$ *are zero for* $i = 0, 1, \ldots, [(h-1)/2]$*. Then*

$$\mathrm{rank}(f^\circ(E)) \le \binom{m+h-1}{h}.$$

Proof. By Lemma 5.1.4,

$$\mathcal{C}(E^{\circ h}) + \mathcal{C}(E^{\circ h-2}) + \cdots + \mathcal{C}(E^{\circ h-2[h/2]}) \subset \mathcal{C}^{(h)}.$$

Hence $\mathcal{C}(f^\circ(E)) \subset \mathcal{C}^{(h)} + \mathcal{C}^{(h-1)}$ in the first case and $\mathcal{C}(f^\circ(E)) \subset \mathcal{C}^{(h)}$ in the second case. Since $\dim \mathcal{C}^{(h)} \le \binom{m+h-1}{h}$, the both assertions hold. ■

The following theorem is proved in [35].

Theorem 5.1.6 *Let* $X \subset S^{m-1} \subset \boldsymbol{R}^m$ *be a s-distance set. Then the following hold.*

(1) $|X| \le \binom{m+s-1}{s} + \binom{m+s-2}{s-1}$.

(2) *Suppose* X *is antipodal, i.e.,* $X = -X$*. Then we have*

$$|X| \le 2\binom{m+s-2}{s-1}.$$

Proof. We use the notation above.

(1) Let $f(t) = (t-\gamma_1)\cdots(t-\gamma_s)$. Since

$$E = \sum_{i=0}^{s} \gamma_i A_i,\ A_i \circ A_j = \delta_{i,j},$$

$$\begin{aligned} f^\circ(E) &= (E-\gamma_1 J)\circ\cdots\circ(E-\gamma_s J) \\ &= f^\circ(\sum_{i=0}^{s}\gamma_i A_i) \\ &= \sum_{i=0}^{s} f(\gamma_i)A_i \\ &= f(\gamma_0)A_0. \end{aligned}$$

Since $f(\gamma_0) \neq 0$, $\mathrm{rank}(f^\circ(E)) = |X| = n$. On the other hand, since f is a polynomial of degree s, by Corollary 5.1.5,

$$\mathrm{rank}(f^\circ(E)) \leq \binom{m+s-1}{s} + \binom{m+s-2}{s-1}.$$

Therefore we have the bound of $|X|$.

(2) Suppose X is antipodal. We may assume that $\gamma_s = -1$. Let $f(t) = (t-\gamma_1)\cdots(t-\gamma_{s-1})$. Then in this case we have

$$f^\circ(E) = \sum_{i=0}^{s} f(\gamma_i)A_i = f(\gamma_0)A_0 + f(\gamma_s)A_s.$$

Since $\boldsymbol{x}\cdot\boldsymbol{y} = -\boldsymbol{x}\cdot(-\boldsymbol{y})$, $-\gamma_i \in \Delta$ for every i. Hence the coefficients of t^{h-2i-1} in $f(t)$ above are zero for $i = 0, 1, \ldots, [(h-1)/2]$. Hence by Corollary 5.1.5,

$$\mathrm{rank}(f^\circ(E)) \leq \binom{m+s-1}{s}.$$

Moreover, $\gamma_0 = 1$ and $\gamma_s = -1$. Hence either $f(\gamma_0) = -f(\gamma_s)$ or $f(\gamma_s)$ depending on whether $0 \in \Delta$ or not. By the rearrangement of the ordering of the elements of X, we have

$$A_s = \begin{pmatrix} O & I \\ I & O \end{pmatrix},\ f^\circ(E) = f(\gamma_0)\begin{pmatrix} I & \pm I \\ \pm I & I \end{pmatrix}.$$

Since $f(\gamma_0) \neq 0$, $\mathrm{rank}(f^\circ(E)) = |X|/2$. Therefore we have the bound of $|X|$ for this case. ■

It is known that if one of the absolute bounds is attained, then the set X becomes a *tight spherical* $(2s-1)$*-design* and $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ becomes a Q-polynomial association scheme. The octahedron gives rise to a tight spherical 3-design and the icosahedron a tight spherical 5-design.

5.2 Representation of Graphs

We start with a definition of graphs related to association schemes.

Definition 5.2.1 1. A k-regular graph $\Gamma = (V\Gamma, E\Gamma)$ with $v = |V\Gamma|$ is said to be *edge regular* with parameter (v, k, λ), if any two adjacent vertices have precisely λ common neighbors, i.e.,

$$|\Gamma(x) \cap \Gamma(y)| = \lambda, \text{ for all } x, y \in V\Gamma \text{ with } x \sim y.$$

2. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a commutative association scheme. Suppose ${}^tR_i = R_i$. Then the graph $\Gamma^{(i)} = (X, R_i)$ with vertex set X and edge set defined by R_i is called the *i-th relation graph.*

Remarks.

1. Since
$$\Gamma^{(i)}(x) = R_i(x) = \{y \in X \mid (x, y) \in R_i\},$$
$\Gamma^{(i)}$ is an edge-regular graph with parameter $(v, k_i, p^i_{i,i})$.

2. It is easy to see from the definition that a regular graph $\Gamma = (V\Gamma, E\Gamma)$ is edge regular if and only if for a fixed integer λ, the induced subgraph on $\Gamma(x)$ is λ-regular for all $x \in V\Gamma$.

3. A regular triangle-free graph is always edge regular.

Definition 5.2.2 A *quasi-regular polyhedron* of type $(p, q; r)$, $(p < q)$ is a regular plane graph in S^2 such that each face is either a regular p-gon or a regular q-gon such that each vertex is surrounded by r p-gons and r q-gons cyclically and alternately.

Lemma 5.2.1 *The parameters $(p, q; r)$ of a quasi-regular polyhedron is either $(3, 4; 2)$ or $(3, 5; 2)$. A quasi-regular polyhedron with either one of these parameters exists and unique up to isomorphism. Both of them are edge-regular graphs and the parameters (v, k, λ) are $(12, 4, 1)$ and $(30, 4, 1)$.*

Proof. Since the inner angle of a regular p-gon in S^1 is $(1 - 2/p)\pi$,

$$r(1 - 2/p)\pi + r(1 - 2/q)\pi < 2\pi.$$

Therefore, we have
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

Hence one of p, q, r is 2. Since p and q cannot be less than 3, $r = 2$ and (p, q) is either $(3, 4)$ or $(3, 5)$.

Suppose they exist. Then by definition, it is $2r$-regular edge-regular graph with $\lambda = 1$. Let v be the number of vertices, e the number of edges, f_p the number of p-gonal faces and f_q the number of q-gonal faces. Then we have

$$rv = pf_p = qf_q,\ 2rv = 2e,\ v - e + f_p + f_q = 2.$$

The last equality comes from Euler's equality (Exercise 1.2.2). Now we have either $v = 12$ or 30 according as $q = 4$ or 5. We left the rest as an exercise. ■

Exercise 5.2.1 Show the existence and the uniqueness of two quasi-regular polyhedra in Lemma 5.2.1.

The quasi-regular polyhedron with parameter $(3, 4; 2)$ is called the *cuboctahedron* and the one with parameter $(3, 5; 2)$ the *icosidodecahedron*.

5.3 Association Schemes with $m = 3$

In this section, we study a faithful representation of a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$, and an edge-regular graph on the sphere defined by a relation corresponding to the nearest points in it. As an application we prove a result of E. Bannai [3, 4] classifying all symmetric association schemes with 3-dimensional faithful representation.

Throughout this section, assume the following.

- $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ is a symmetric association scheme with $|X| > 1$.
- $E = E_1 = \dfrac{1}{|X|}\displaystyle\sum_{i=0}^{d} q_1(i)A_i$: a primitive idempotent of $\mathcal{X}$.
- $m = m_1 = q_1(0) > q_1(1) \ge q_1(2) \ge \cdots \ge q_1(d)$.
- $\Gamma = \Gamma^{(1)} = (X, R_1)$. Let $R = R_1$ and $k = k_1$.
- $\pi : V \to W = V_1$ is the orthogonal projection onto the eigen space corresponding to the primitive idempotent E.
- $\hat{X} = \{\hat{x} \mid x \in X\}$, where $\hat{x} = (\sqrt{|X|/m})\pi(x) \in W$.
- $S = S^{m-1} \subset W$: the unit sphere in W.
- $\gamma_i = q_1(i)/m$, and $\gamma = \gamma_1$.
- $\rho_i = \sqrt{2(1-\gamma_i)}$ and $\rho = \rho_1$.

When $x, y \in S$, $\angle xy$ denotes the angle between the vectors x and y.

Lemma 5.3.1 *The following hold.*

(1) *Let $(x, y) \in R_i$. Then $\hat{x} \cdot \hat{y} = \gamma_i$. In particular, $\hat{X}$ is a subset of the unit sphere S with $|\hat{X}| = |X|$, i.e., X is faithfully embedded in the unit sphere S.*

(2) *Let* $(x, y) \in R_i$. *Then*

$$\rho_i = \|\hat{x} - \hat{y}\| = \sqrt{(\hat{x} - \hat{y}) \cdot (\hat{x} - \hat{y})},$$

and $0 = \rho_0 < \rho_1 \leq \cdots \leq \rho_d$.

(3) $\gamma = -1$ *if and only if* $|X| = 2$.

(4) $\sum_{y \in R_i(x)} \hat{y} = k_i \gamma_i \hat{x}$, *for every* $x \in X$.

(5) *For* $x \in X$ *let* $H_i = \mathrm{Span}(\hat{x})^\perp + o_i$ *with* $o_i = \gamma_i \hat{x}$. *Then*

$$\{\hat{y} \mid y \in R_i(x)\} \subset S_i = H_i \cap S.$$

Moreover, $\sum_{y \in R_i(x)} (\hat{y} - o_i) = 0$.

Proof. (1) and (2) are straightforward.

Suppose $\gamma = -1$. Since $\hat{x} \cdot \hat{y} = -1$ implies $\hat{y} = -\hat{x}$ and $\gamma_i \leq \gamma$ for $i \geq 1$, $\hat{X} = \{\hat{x}, -\hat{x}\}$. Conversely, if $|X| = 2$, then $d = m = k_1 = 1$ and $\gamma = -1$. This is (3).

Since $W = V_1$ is an eigen space of A_i corresponding to an eigenvalue $p_i(1) = k_i q_1(i)/m_1 = k_i \gamma_i$, we have

$$k_i \gamma_i \pi(x) = k_i \gamma_i \pi(Ex) = \pi(EA_i x) = \pi(E \sum_{y \in R(x)} y) = \sum_{y \in R(x)} \pi(y).$$

Thus we have (4).

(5) follows from (4), as $\hat{y} \cdot \hat{x} = \gamma_i = o_i \cdot \hat{x}$. ■

The following is an elementary result on the vectors on the sphere S^2, but it is a key to what follows.

Lemma 5.3.2 *Let* a, x, y, z, w *be vectors on the sphere* S^2 *satisfying the following.*

1. $a \in \mathrm{Span}(x, y) \cap \mathrm{Span}(z, w)$.
2. $0 < \angle xy = \angle zw = \theta < \pi$.
3. $\angle xy = \angle xa + \angle ay$, *and* $\angle zw = \angle za + \angle aw$.

Then one of $\|x - z\|$, $\|z - y\|$, $\|y - w\|$ *or* $\|w - x\|$ *is less than or equal to* $\|x - y\|/\sqrt{2}$. *Equality holds if and only if*

$$\angle xa = \angle ay = \angle za = \angle aw, \ \mathrm{Span}(x, y) \perp \mathrm{Span}(z, w),$$

and $\|x - z\| = \|z - y\| = \|y - w\| = \|w - x\|$.

Proof. Let $\theta = \angle xy = \angle zw$. Let $\alpha = \angle xa$ and $\beta = \angle za$. Let $0 \leq \delta \leq \pi/2$ be the angle between the planes $\mathrm{Span}(x, y)$ and $\mathrm{Span}(z, w)$. By symmetry we may assume that $\alpha \leq \theta/2$ and $\beta \leq \theta/2$. We now take the coordinates so that

$$a = (1, 0, 0),\ x = (\cos\alpha, \sin\alpha, 0),\ z = (\cos\beta, \sin\beta\cos\delta, \sin\beta\sin\delta).$$

Then we have

$$y = (\cos(\alpha-\theta), \sin(\alpha-\theta), 0),\ w = (\cos(\beta-\theta), \sin(\beta-\theta)\cos\delta, \sin(\beta-\theta)\sin\delta).$$

Now we have

$$\begin{aligned} \|x - z\|^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta\cos\delta)^2 + \sin^2\beta\sin^2\delta \\ &= 2(1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\delta). \end{aligned}$$

On the other hand

$$\|x - y\|^2 = (2\sin\frac{\theta}{2})^2 = 2(1 - \cos\theta).$$

By our choice, we have $\cos\alpha\cos\beta \geq \cos^2(\theta/2)$ and equality holds if and only if $\alpha = \beta = \theta/2$. Moreover, $0 \leq \alpha, \beta \leq \theta/2 < \pi/2$. Hence

$$\begin{aligned} \|x - z\|^2 &= 2(1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\delta) \\ &\leq 2(1 - \cos^2\frac{\theta}{2} - \sin\alpha\sin\beta\cos\delta) \\ &\leq 2(\sin^2\frac{\theta}{2}) \\ &\leq \frac{1}{2}\|x - y\|^2. \end{aligned}$$

Clearly the equality holds above if and only if

$$\alpha = \beta = \theta/2,\ \delta = \pi/2.$$

Thus we have the assertion. ∎

Lemma 5.3.3 *Under the notation above, the following hold.*

(1) *If $k_i = 1$ for $i > 0$, then $\gamma_i = -1$. Moreover if $i = 1$, then $m = 1$ and $|X| = 2$.*

(2) *If $k_i = 2$, then every connected component C of $\Gamma^{(i)}$ is an n_i-gon for a fixed n_i such that $\dim\mathrm{Span}(\hat{x} \mid x \in C) = 2$. Moreover if the subspaces spanned by the connected components of Γ intersect in a nonzero subspace, then there is an index $0 < j < i$ such that $\rho_j \leq \rho_i/\sqrt{2}$.*

(3) *If $m = 1$, then $\gamma = -1$, $k = 1$ and $|X| = 2$.*

(4) *If $m = 2$, then Γ is an n-gon with $n = |X|$ and $k = 2$.*

Proof.

(1) Let $R_i(x) = \{y\}$. Since $\hat{y} = \gamma_i \hat{x}$ by Lemma 5.3.1 (4), we have $\gamma_i = -1$ as $1 = \gamma_0 > \gamma_i$ for $i > 0$. If $i = 1$, then $X = \{x, y\}$ by Lemma 5.3.1 (3).

(2) Let $R_i(x) = \{y, z\}$. Since the embedding is faithful and $\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \gamma_i$, $y, z \neq \pm x$. Since $\hat{y} + \hat{z} = 2\gamma_i \hat{x}$ by Lemma 5.3.1, $\mathrm{Span}(\hat{x}, \hat{y}) = \mathrm{Span}(\hat{x}, \hat{z})$. By induction, we see that every vertex in a connected component of $\Gamma^{(i)}$ containing $\hat{x}$ is in the 2-dimensional space $\mathrm{Span}(\hat{x}, \hat{y})$. Since the inner product γ_i is constant, the size n_i of the connected component does not depend on each component. Suppose $i = 1$ and the space spanned by the connected components C and C' intersect nontrivially. Then they span 3-dimensional space. Since C and C' are in 2-dimensional space, they intersect in one dimensional space. Let a be a vector on S in the intersection. Then we can find $\hat{x}, \hat{y} \in C$ and $\hat{z}, \hat{w} \in C'$ such that

$$\angle xy = \angle xa + \angle ay, \text{ and } \angle zw = \angle za + \angle aw$$

with $(x, y), (z, w) \in R$. This contradicts Lemma 5.3.2.

(3) If $m = 1$, $|X| = |\hat{X}| \leq |S| = 2$. Hence we have the assertion.

(4) If $m = 2$, $k \leq 2$ and $k > 1$ by (1). Hence we have (4). ∎

We now state the result of E. Bannai.

Theorem 5.3.4 *Let $E = E_1$ be a primitive idempotent of a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$. Let*

$$E = \frac{1}{|X|} \sum_{i=0}^{d} q_1(i) A_i.$$

Suppose that $3 = m_1 = q_1(0) > q_1(1) \geq q_1(2) \geq \cdots \geq q_1(d)$. Let $\Gamma = \Gamma^{(1)} = (X, R_1)$. Then Γ is isomorphic to one of the following edge-regular graphs with parameters (v, k, λ).

(*i*) *Tetrahedron* $(4, 3, 2)$,

(*ii*) *Octahedron* $(6, 4, 2)$,

(*iii*) *Cube* $(8, 3, 0)$,

(*iv*) *Dodecahedron* $(20, 3, 0)$,

(*v*) *Icosahedron* $(12, 5, 2)$,

(*vi*) *Cuboctahedron* $(12, 4, 1)$*; or*

(*vii*) *Icosidodecahedron* $(30, 4, 1)$.

For the proof, we identify $x \in X$ with $\hat{x}$, and X with $\hat{X}$.

We define edges on the sphere. First we note that $1 > \gamma > -1$ by Lemma 5.3.1.

Let $(x, y) \in R$. Since $\gamma \neq -1$, the geodesic, i.e., the shortest arc between x and y is uniquely determined. We call this arc *edge* on the sphere S. We assume Γ is embedded on S with these edges. In other words, Γ is a graph with vertex set $V\Gamma = X$ and edge set $E\Gamma$ on the sphere S.

The following result is in [49].

Lemma 5.3.5 *The edges of* Γ *on the sphere* S *do not cross each other. In particular,* Γ *is embedded in* S *as a plane graph.*

Proof. Let $(x, y), (z, w) \in R$. Suppose that the edges corresponding to (x, y) and (z, w) intersect at $a \in S$. Then by Lemma 5.3.2, one of $\|x - z\|$, $\|z - y\|$, $\|y - w\|$ or $\|w - x\|$ is less than ρ. This contradicts the choice of R. ■

Lemma 5.3.6 Γ *is an edge-regular graph of valency at most* 5.

Proof. Let $\Gamma(x) = \{x_1, \ldots, x_k\}$. Let $o = \gamma x$, $H_0 = \mathrm{Span}(x)^{\perp}$ and $H = H_0 + o$. Let $S_1 = H \cap S$. Since for $i = 1, \ldots, k$,

$$(x_i - o) \cdot x = \gamma - \gamma = 0,$$

$x_i \in S_1$. Since S_1 is a circle of radius $\rho' = \sqrt{\rho^2 - (1-\gamma)^2} < \rho$, $k < 6$. Note that the mutual distances of the points in $\Gamma(x)$ are at least ρ, which means that the angle between $x_i - o$ and $x_j - o$ $(1 \le i \ne j \le k)$ is strictly larger than the angle of the equilateral triangle. ■

Lemma 5.3.7 *If* $k \le 3$*, then* $k = 3$ *and* Γ *is a regular plyhedron and is isomorphic to either the tetrahedron, the cube or the dodecahedron.*

Proof. By Lemma 5.3.3, we may assume that $k = 3$. Let $x \in X$ and $\Gamma(x) = \{y_1, y_2, y_3\}$. Suppose $(y_1, y_2) \in R_i$. Since the graph on $\Gamma(x)$ with vertex set defined by R_i is regular of valency at least 1, it is 2-regular and $y_j \cdot y_l = \gamma_i$ for $1 \le j \ne l \le 3$. By Lemma 5.3.5, Γ is planar. By our observation above, every face is a regular polygon of same size. Hence Γ is a regular polyhedra. Therefore by Proposition 1.2.1, we have either the tetrahedron, the cube or the dodecahedron. ■

Lemma 5.3.8 *If* $k = 4$*, then* Γ *is either a regular polyhedron or a quasi-regular polyhedron and is isomorphic to either the octahedron, the cuboctahedron or the icosidodecahedron.*

Proof. Let $x \in X$ and $\Gamma(x) = \{y_1, y_2, y_3, y_4\}$. $\Gamma(x)$ is in a circle S_1 in the proof of Lemma 5.3.6. If $\|y_i - y_{i-1}\|$ with $2 \le i \le 4$ are all equal, we have a regular polyhedron. We have the octahedron in this case.

Since the graph on $\Gamma(x) \subset S_1$ with edge set defined by R_i is $p^1_{1,i}$-regular, we may assume that $(y_1, y_2), (y_3, y_4) \in R_i$, $(y_2, y_3), (y_4, y_1) \in R_j$, $(y_1, y_3), (y_2, y_4) \in R_l$ with $\rho_i < \rho_j \le \rho_l$ with $i < j < l$. Note that $y_1 + y_4 = y_2 + y_3 = 2\gamma x$. In particular, by Lemma 5.3.2, $\rho_i < \rho/\sqrt{2}$.

If $i = 1$, then we have a quasi-regular polyhedron by Lemma 5.3.5. Hence by Lemma 5.2.1, Γ is isomorphic to either the cuboctahedron or the icosidodecahedron.

Now assume that $1 < i$. Then $p^1_{1,i} > 0$. We note that Since $\gamma_i > \gamma_j \ge -1$, we can define the edges of $\Gamma^{(i)}$ on the sphere. Let $z \in X$ such that $(y_1, z) \in R$, $(z, x) \in R_i$. Suppose z is in between y_1 and y_2. Then the edges (y_1, y_2) and (x, z) on the sphere intersect. But then by Lemma 5.3.2, one of the sides of the square xy_1zy_2 is less than or equal to $\rho_i/\sqrt{2} < \rho$. This is absurd. Hence z and y_4 are on the same side of the plane $\mathrm{Span}(x, y_1) = \mathrm{Span}(x, y_3)$. Since the triangles y_1xz and xy_3y_4 are isometric, $\|z - y_4\| < \rho$. This is a contradiction. ■

Lemma 5.3.9 *If $k = 5$, then Γ is a regular polyhedron and is isomorphic to the icosahedron.*

Proof. Let $x \in X$ and $\Gamma(x) = \{y_1, y_2, y_3, y_4, y_5\}$. $\Gamma(x)$ is in a circle S_1 in the proof of Lemma 5.3.6. Suppose $\|y_1 - y_2\|$ is the smallest among $\|y_j - y_l\|$ with $j \neq l$. Say $(y_1, y_2) \in R_i$. Then the graph on $\Gamma(x)$ with vertex set defined by R_i is regular of valency 2. Hence again Γ becomes a regular polyhedra. We have the icosahedron in this case. ■

By Lemmas 5.3.6, 5.3.7, 5.3.8, 5.3.9, we have the assertion of the theorem. ■

Exercise 5.3.1

1. Determine symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ such that one of the graphs $\Gamma^{(i)} = (X, R_i)$ is isomorphic to one of the edge-regular graphs in Theorem 5.3.4.
2. Determine commutative association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ whose symmetrization has the property in 1.

5.4 Balanced Sets in S^2 and Finite Reflection Groups

5.5 Examples

Chapter 6

Parametrical Restrictions

6.1 Parametrical Conditions on P-polynomial Association Schemes

In this chapter we treat the parametrical restrictions of commutative association schemes and Q-polynomial association schemes. We first view how some of the parametrical restrictions on P-polynomial association schemes based on the graph structures are obtained.

Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a P-polynomial association scheme and let $\Gamma = (V\Gamma, E\Gamma)$ with $V\Gamma = X$ and $E\Gamma = \{\{x,y\} \mid (x,y) \in R_1\}$ be the distance-regular graph associated with $\mathcal{X}$. Recall that for vertices x, y of Γ with $\partial(x,y) = h$,

$$\begin{aligned} C(x,y) &= C_h(x,y) = \Gamma_{h-1}(x) \cap \Gamma(y), \\ A(x,y) &= A_h(x,y) = \Gamma_h(x) \cap \Gamma(y), \\ B(x,y) &= B_h(x,y) = \Gamma_{h+1}(x) \cap \Gamma(y), \\ P_{i,j}(x,y) &= P^h_{i,j}(x,y) = \Gamma_i(x) \cap \Gamma_j(y). \end{aligned}$$

We have $|C(x,y)| = c_h$, $|A(x,y)| = a_h$, $|B(x,y)| = b_h$ and $|P_{i,j}(x,y)| = p^h_{i,j}$ do not depend on the choices of x and y as for as $\partial(x,y) = h$. When x and y are fixed, we also write D^i_j for $P_{i,j}(x,y)$. For $S, T \subset X$, let

$$\mathrm{edge}(S,T) = |\{(s,t) \in S \times T \mid s \sim t\}|.$$

Recall that by Proposition 3.4.1 $p^h_{i,j}$'s satisfy the following conditions.

$(P1)$ $p^h_{i,j} = 0$ if one of h, i, j is greater than the sum of the other two.

$(P2)$ $p^h_{i,j} \neq 0$ if one of h, i, j is equal to the sum of the other two for $0 \le h, i, j \le d$.

Lemma 6.1.1 *Suppose $\partial(x,y) = h$. Let $z \in C(y,x)$. Then the following hold.*

(1) *$B(x,y) \subset B(z,y)$. In particular, $b_i \le b_{i-1}$.*

(2) *$C(x,y) \supset C(z,y)$. In particular, $c_i \ge c_{i-1}$.*

Proof. These are obvious geometrically. We can prove also by triangular inequalities. For example, if $w \in B(x, y) = \Gamma_{h+1}(x) \cap \Gamma_1(y)$,

$$h = \partial(x, w) - \partial(x, z) \leq \partial(z, w) \leq \partial(z, y) + \partial(y, w) = h.$$

Hence $w \in \Gamma_h(z) \cap \Gamma_1(y) = B(z, y)$. We have (1), and (2) is similar. ■

Lemma 6.1.2 *Suppose $\partial(x, y) = h = i + j$, and $z \in P_{i,j}(x, y)$, i.e., z is on a geodesic (i.e., shortest path) connecting x and y. Then the following hold.*

(1) *$C(x, z) \subset B(y, z)$. In particular, $c_i \leq b_j$.*

(2) *$c_i = b_j$ if and only if* $\mathrm{edge}(z, P_{i,j+1}(x, y) \cup P_{i+1,j+1}(x, y)) = 0$. *In particular, if $p^h_{i,j+1} = p^h_{i+1,j+1} = 0$, then $c_i = b_j$.*

Proof. Let $w \in C(x, z)$. Since $\partial(x, z) + \partial(z, y) = \partial(x, y)$, $\partial(w, y) = \partial(z, y) + 1$ and $w \in B(y, z)$. In particular, $C(x, z) = P_{i-1,j+1}(x, y) \cap \Gamma(z)$. This is (1).

On the other hand, let $w \in B(y, z)$. So $\partial(y, w) = j + 1$. Since $\partial(x, z) = i$ and $\partial(z, w) = 1$, we have $i - 1 \leq \partial(x, w) \leq i + 1$. This means

$$\begin{aligned} B(y, z) &= (P_{i-1,j+1}(x, y) \cup P_{i,j+1}(x, y) \cup P_{i+1,j+1}(x, y)) \cap \Gamma(z) \\ &= C(x, z) \cup (((P_{i,j+1}(x, y) \cup P_{i+1,j+1}(x, y)) \cap \Gamma(z)). \end{aligned}$$

Therefore, $b_j = c_i + \mathrm{edge}(z, P_{i,j+1}(x, y) \cup P_{i+1,j+1}(x, y))$. Thus we have the assertion in (2). ■

Note that the proofs of the lemmas above make full use of the geometrical interpretation of the parameters. We give another type of argument which we will apply to treat Krein parameters $q^h_{i,j}$.

Proposition 6.1.3 *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a P-polynomial association scheme and $\Gamma = (V\Gamma, E\Gamma)$ be the associated distance-regular graph. Suppose that*

$$\{l \mid p^l_{j,h+i} p^l_{i-j,h+j} \neq 0\} \subset \{h + i - j\}.$$

Then for $h \geq 0$, $i \geq j \geq 1$, and $h + i + j \leq d$, $p^{h+i}_{i,h+j} = 0$ implies that $p^{h+j}_{j,h+j} = 0$.

Proof. Suppose $p^{h+j}_{j,h+j} \neq 0$. Then there are vertices $x, y, z \in X$ such that

$$(x, y),\ (x, z) \in R_{h+j} \ \text{ and } \ (y, z) \in R_j.$$

Since $p^{h+j}_{i-j,h+i} \neq 0$ by $(P2)$, there exists a vertex $u \in X$ such that $(x, u) \in R_{i-j}$ and $(u, y) \in R_{h+i}$. Consider two triples (u, y, z) and (u, x, z). Since $\{l \mid p^l_{h+i,j} p^l_{i-j,h+j} \neq 0\} \subset \{h + i - j\}$, $(u, z) \in R_{h+i-j}$. Since $(y, u) \in R_{h+i}$ and $p^{h+i}_{h+i+j,j} \neq 0$ by $(P2)$, there exists a vertex $v \in X$ such that $(y, v) \in R_{h+i+j}$ and $(v, u) \in R_j$. Consider two triples (v, y, x) and (v, u, x). Since $\{l \mid p^l_{h+i+j,h+j} p^l_{j,i-j} \neq 0\} \subset \{i\}$ by $(P2)$, we have $(v, x) \in R_i$.

Next consider two triples (v, y, z) and (v, u, z). Since $\{l \mid p^l_{h+i+j,j} p^l_{j,h+i-j} \neq 0\} \subset \{h+i\}$ by $(P2)$, we have $(v, z) \in R_{h+i}$. Finally consider a triple (v, x, z). Since

$$(v, z) \in R_{h+i},\ (v, x) \in R_i, \ \text{ and } \ (x, z) \in R_{h+j},$$

$p^{h+i}_{i,h+j} \neq 0$, which is a contradiction. ■

6.2 Vanishing Conditions of Krein Parameters

Some of the formulas below can be generalized to commutative association schemes, we restrict our attention to symmetric ones.

The following lemma gives the basis of what follows.

Lemma 6.2.1 *Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a symmetric association scheme. For $x \in X$, the following are equivalent.*

(i) $q^j_{h,i} = 0$.

(ii) $E_i x_{E_h} E_j = O$.

(iii) $\sum_{u\in X} (E_h)_{u,x}(E_i)_{u,y}(E_j)_{u,z} = 0$ *for all* $y, z \in X$.

Proof. (i) $\Leftrightarrow$ (ii) This is already appeared many times. See Lemma 3.1.2.

(ii) $\Leftrightarrow$ (iii) This can be seen by taking the (y, z) entry of the matrix equation in (ii).

■

Lemma 6.2.2 *Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a symmetric association scheme. Suppose $\{i \mid q^i_{j,k}q^i_{l,m} \neq 0\} \subset \{h\}$. Then for all integers $0 \leq h, i, j, k, l, m \leq d$ and the vertices a, a', b, b', the following hold.*

(1) $\displaystyle\sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb} = \frac{q^h_{l,m}}{|X|}\sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_h)_{eb}$.

(2) $\displaystyle\sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb'} = \sum_{e,e'\in X} (E_j)_{ea}(E_k)_{ea'}(E_h)_{ee'}(E_l)_{e'b}(E_m)_{e'b'}$.

Proof. Suppose $\{i \mid q^i_{j,k}q^i_{l,m} \neq 0\} \subset \{h\}$.

(1) By Lemma 6.2.1, we have

$$\begin{aligned}
&\sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb} \\
&\quad = \sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_l \circ E_m)_{eb} \\
&\quad = \frac{1}{|X|}\sum_{i=0}^{d} q^i_{l,m}\sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_i)_{eb} \\
&\quad = \frac{q^h_{l,m}}{|X|}\sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_h)_{eb}.
\end{aligned}$$

Note that if $q^i_{l,m} \neq 0$, then by our assumption,

$$\sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_i)_{eb} \neq 0 \ \text{ only if } i = h.$$

(2) Since $I = E_0 + E_1 + \cdots + E_d$, similarly we have

$$\begin{aligned}
&\sum_{e\in X}(E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb'} \\
&= \sum_{e,e'\in X}(E_j)_{ea}(E_k)_{ea'}(I)_{ee'}(E_l)_{e'b}(E_m)_{e'b'} \\
&= \sum_{i=0}^{d}\sum_{e,e'\in X}(E_j)_{ea}(E_k)_{ea'}(E_i)_{ee'}(E_l)_{e'b}(E_m)_{e'b'} \\
&= \sum_{i=0}^{d}\sum_{e'\in X}\left(\sum_{e\in X}(E_j)_{ea}(E_k)_{ea'}(E_i)_{ee'}\right)(E_l)_{e'b}(E_m)_{e'b'} \\
&= \sum_{i=0}^{d}\sum_{e\in X}(E_j)_{ea}(E_k)_{ea'}\left(\sum_{e'\in X}(E_i)_{ee'}(E_l)_{e'b}(E_m)_{e'b'}\right).
\end{aligned}$$

Comparing the last two expressions using Lemma 6.2.1, we have the right hand side of (2) by our assumption. ■

Proposition 6.2.3 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme. Suppose the following.*

$$(1)\ \{t \mid q^t_{i,k}q^t_{h,m} \neq 0\} \subset \{l\}, \quad (2)\ \{t \mid q^t_{j,k}q^t_{h,l} \neq 0\} \subset \{m\}.$$

Then $q^h_{i,j} \neq 0$ implies that $q^i_{k,l} = q^j_{k,m}$.

Proof. Let $X(h,i,j,k,l,m)$ be the following sum.

$$\sum_{w,x,y,z\in X}(E_h)_{w,x}(E_i)_{w,y}(E_j)_{w,z}(E_k)_{y,z}(E_l)_{x,y}(E_m)_{x,z}.$$

We evaluate $X(h,i,j,k,l,m)$ using Lemma 6.2.2 under our assumption.

Rearranging first the order of the product and apply Lemma 6.2.2 (2) by our assumption (1), we have the following.

$$\begin{aligned}
&X(h,i,j,k,l,m) \\
&= \sum_{w,z\in X}(E_j)_{w,z}\sum_{x,y\in X}(E_i)_{y,w}(E_k)_{y,z}(E_l)_{y,x}(E_h)_{x,w}(E_m)_{x,z} \\
&= \sum_{w,z\in X}(E_j)_{w,z}\sum_{x\in X}(E_i)_{x,w}(E_k)_{x,z}(E_h)_{x,w}(E_m)_{x,z} \\
&= \sum_{w,x\in X}(E_i)_{x,w}(E_h)_{x,w}\sum_{z\in X}(E_k)_{x,z}(E_m)_{x,z}(E_j)_{w,z} \\
&= \sum_{w,x\in X}(E_i)_{x,w}(E_h)_{x,w}\sum_{z\in X}(E_k\circ E_m)_{x,z}(E_j)_{z,w} \\
&= \sum_{w,x\in X}(E_i)_{x,w}(E_h)_{x,w}((E_k\circ E_m)E_j)_{x,w} \\
&= \frac{q^j_{k,m}}{|X|}\sum_{w,x\in X}(E_i)_{x,w}(E_h)_{x,w}(E_j)_{x,w}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^j_{k,m}}{|X|}\sum_{w\in X}\sum_{x\in X}(E_i\circ E_j)_{w,x}(E_h)_{x,w}\\
&= \frac{q^j_{k,m}}{|X|}\sum_{w\in X}((E_i\circ E_j)E_h)_{w,w}\\
&= \frac{q^j_{k,m}}{|X|}\frac{q^h_{i,j}}{|X|}\sum_{w\in X}(E_h)_{w,w}\\
&= \frac{q^j_{k,m}q^h_{i,j}}{|X|^2}m_h.
\end{aligned}
$$

Now by symmetry we replace i with j, and l with m to obtain the following.

$$X(h,i,j,k,l,m) = \frac{q^i_{k,l}q^h_{i,j}}{|X|^2}m_h.$$

Therefore if $q^h_{i,j}\neq 0$, we have $q^i_{k,l} = q^j_{k,m}$ as desired. ■

Let $\mathcal{X}$ be a Q-polynomial association scheme with respect to a fixed ordering of primitive idempotents $E_0, E_1, \ldots, E_d$. By Proposition 3.4.1, we have the following.

$(Q1)$ $q^h_{i,j} = 0$ if one of h,i,j is greater than the sum of the other two.

$(Q2)$ $q^h_{i,j}\neq 0$ if one of h,i,j is equal to the sum of the other two for $0\leq h,i,j\leq d$.

Imitating the notation for distance-regular graphs, let $c^*_i = q^i_{i-1,1}$, $a^*_i = q^i_{i,1}$, $b^*_i = q^i_{i+1,1}$ and $k^*_i = m_i = q^0_{i,i}$ for $i = 0,1,\ldots,d$.

Corollary 6.2.4 *Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. If $q^h_{i,j}\neq 0$, then the following hold.*

(1) *If $q^h_{i-1,j} = q^h_{i-1,j-1} = q^h_{i,j+1} = q^h_{i+1,j+1} = 0$, then $c^*_i = b^*_j$.*

(2) *If $q^h_{i-1,j} = q^h_{i-1,j+1} = q^h_{i,j-1} = q^h_{i+1,j-1} = 0$, then $c^*_i = c^*_j$.*

(3) *If $q^h_{i-1,j+1} = q^h_{i,j+1} = q^h_{i+1,j-1} = q^h_{i+1,j} = 0$, then $b^*_i = b^*_j$.*

Proof. It is easy to check each of the following.

(1) $\{t \mid q^t_{i,1}q^t_{h,j+1}\neq 0\}\subset\{i-1\}$, and $\{t\mid q^t_{j,1}q^t_{h,i-1}\neq 0\}\subset\{j+1\}$.

(2) $\{t \mid q^t_{i,1}q^t_{h,j-1}\neq 0\}\subset\{i-1\}$, and $\{t\mid q^t_{j,1}q^t_{h,i-1}\neq 0\}\subset\{j-1\}$.

(3) $\{t \mid q^t_{i,1}q^t_{h,j+1}\neq 0\}\subset\{i+1\}$, and $\{t\mid q^t_{j,1}q^t_{h,i+1}\neq 0\}\subset\{j+1\}$.

Hence we have the assertions as direct consequences of the previous proposition. ■

Let $h = i+j$ in (1) above. Then $q^h_{i-1,j} = q^h_{i-1,j-1} = 0$. Hence this states a result similar to Lemma 6.1.2. We give another corollary below.

Corollary 6.2.5 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. If $k_1^* > 2$, then the following hold.*

(1) *If $q^h_{2,h} = q^h_{1,h} = q^h_{2,h-1} = 0$ with $2 \le h \le d$, then $h = d$.*

(2) *If $q^h_{2,h} = q^h_{1,h} = q^h_{2,h+1} = 0$ with $2 \le h \le d$, then $h = d$.*

Proof. (1) By Corollary 6.2.4, $b_1^* = c^*_{h-1}$, $c_1^* = c^*_{h+1}$. Suppose $h < d$. Then $q^2_{h+1,h-1} \ne 0$ and $q^2_{h,h-1} = q^2_{h,h} = 0$ by our assumption. Hence by Corollary 6.2.4, $c^*_{h+1} = c^*_{h-1}$. Thus $b_1^* = c_1^* = 1$. This implies $k_1^* = 2$, because $a_h^* = q^h_{1,h} = 0$ implies $a_1^* = 0$ by Corollary 6.2.7. This is not the case. Therefore $h = d$.

(2) In this case, we have $b_1^* = b^*_{h+1}$ and $c_1^* = b^*_{h-1}$. Similarly, if $h < d$, then we have $b^*_{h+1} = b^*_{h-1}$ and $k_1^* = 2$. This is a contradiction. ■

Exercise 6.2.1 Prove the counter part of Proposition 6.2.3 for $p^h_{i,j}$'s and the corresponding corollaries for P-polynomial association schemes.

Now we give a result similar to Lemma 6.1.3 for Q-polynomial association schemes.

Proposition 6.2.6 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. Suppose that*

$$\{l \mid q^l_{j,h+i} q^l_{i-j,h+j} \ne 0\} \subset \{h+i-j\}.$$

Then for $h \ge 0$, $i \ge j \ge 1$ with $h+i+j \le d$, $q^{h+i}_{i,h+j} = 0$ implies that $q^{h+j}_{j,h+j} = 0$.

Proof. Since $q^{h+i}_{i,h+j} = 0$, by Lemma 6.2.1,

$$\begin{aligned}
0 &= \frac{q^i_{j,i-j}}{|X|} \sum_{u\in X} (E_{h+i})_{ux}(E_i)_{uy}(E_{h+j})_{uz} \\
&= \sum_{u\in X} \left(\frac{q^i_{j,i-j}}{|X|} (E_i)_{uy} \right) (E_{h+i})_{ux}(E_{h+j})_{uz} \\
&= \sum_{u\in X} ((E_j \circ E_{i-j})E_i)_{uy}(E_{h+i})_{ux}(E_{h+j})_{uz} \\
&= \sum_{u\in X} \sum_{v\in X} (E_j)_{uv}(E_{i-j})_{uv}(E_i)_{vy}(E_{h+i})_{ux}(E_{h+j})_{uz} \\
&= \sum_{v\in X} (E_i)_{vy} \left(\sum_{u\in X} (E_j)_{uv}(E_{h+i})_{ux}(E_{i-j})_{uv}(E_{h+j})_{uz} \right).
\end{aligned}$$

Since $\{l \mid q^l_{j,h+i} q^l_{i-j,h+j} \ne 0\} \subset \{h+i-j\}$, by Lemma 6.2.2 (2),

$$= \sum_{u\in X} \sum_{v\in X} \sum_{w\in X} (E_i)_{vy}(E_j)_{uv}(E_{h+i})_{ux}(E_{h+i-j})_{uw}(E_{i-j})_{wv}(E_{h+j})_{wz}.$$

Since this holds for arbitrary x, y, z, we have

$$\begin{aligned}
0 &= \sum_{x,y,z\in X}(E_{h+i+j})_{xy}(E_{h+j})_{yz}(E_j)_{xz} \\
&\quad \sum_{u,v,w\in X}(E_i)_{vy}(E_j)_{uv}(E_{h+i})_{ux}(E_{h+i-j})_{uw}(E_{i-j})_{wv}(E_{h+j})_{wz} \\
&= \sum_{y,z,v,w\in X}(E_{h+j})_{yz}(E_i)_{vy}(E_{i-j})_{wv}(E_{h+j})_{wz} \\
&\quad \sum_{x,u\in X}(E_{h+i+j})_{xy}(E_j)_{xz}(E_{h+i})_{ux}(E_j)_{uv}(E_{h+i-j})_{uw}.
\end{aligned}$$

Since $\{l \mid q^l_{h+i+j,j}q^l_{j,h+i-j} \neq 0\} \subset \{h+i\},$ by Lemma 6.2.2 (2) we have

$$\begin{aligned}
&= \sum_{y,z,v,w\in X}(E_{h+j})_{yz}(E_i)_{vy}(E_{i-j})_{wv}(E_{h+j})_{wz}\sum_{x\in X}(E_{h+i+j})_{xy}(E_j)_{xz}(E_j)_{xv}(E_{h+i-j})_{xw} \\
&= \sum_{x,z,w\in X}(E_{h+j})_{wz}(E_j)_{xz}(E_{h+i-j})_{xw}\sum_{y,v\in X}(E_{h+i+j})_{yx}(E_{h+j})_{yz}(E_i)_{yv}(E_j)_{xv}(E_{i-j})_{wv}.
\end{aligned}$$

Since $\{l \mid q^l_{h+i+j,h+j}q^l_{j,i-j} \neq 0\} \subset \{i\},$ by Lemma 6.2.2 (2) we have

$$\begin{aligned}
&= \sum_{x,z,w\in X}(E_{h+j})_{wz}(E_j)_{xz}(E_{h+i-j})_{xw}\sum_{y\in X}(E_{h+i+j})_{yx}(E_{h+j})_{yz}(E_j)_{xy}(E_{i-j})_{wy} \\
&= \sum_{x,z,w\in X}(E_{h+j})_{wz}(E_j)_{xz}(E_{h+i-j})_{xw}\sum_{y\in X}(E_{h+j})_{yz}(E_{i-j})_{yw}(E_j)_{yx}(E_{h+i+j})_{yx}.
\end{aligned}$$

Since $\{l \mid q^l_{h+j,i-j}q^l_{j,h+i+j} \neq 0\} \subset \{h+i\},$ by Lemma 6.2.2 (1) we have

$$\begin{aligned}
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{x,z,w\in X}(E_{h+j})_{wz}(E_j)_{xz}(E_{h+i-j})_{xw}\sum_{y\in X}(E_{h+j})_{yz}(E_{i-j})_{yw}(E_{h+i})_{yx} \\
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{y,z\in X}(E_{h+j})_{yz}\sum_{x,w\in X}(E_{h+i})_{xy}(E_j)_{xz}(E_{h+i-j})_{xw}(E_{i-j})_{wy}(E_{h+j})_{wz}.
\end{aligned}$$

Since $\{l \mid q^l_{h+i,j}q^l_{i-j,h+j} \neq 0\} \subset \{h+i-j\},$ by Lemma 6.2.2 (2), we have

$$\begin{aligned}
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{y,z\in X}(E_{h+j})_{yz}\sum_{x\in X}(E_{h+i})_{xy}(E_j)_{xz}(E_{i-j})_{xy}(E_{h+j})_{xz} \\
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{x,z\in X}(E_j)_{xz}(E_{h+j})_{xz}\sum_{y\in X}(E_{i-j})_{xy}(E_{h+i})_{xy}(E_{h+j})_{yz} \\
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{x,z\in X}(E_j)_{xz}(E_{h+j})_{xz}\sum_{y\in X}((E_{i-j})\circ(E_{h+i}))_{xy}(E_{h+j})_{yz} \\
&= \frac{q^{h+i}_{j,h+i+j}q^{h+j}_{i-j,h+i}}{|X|^2}\sum_{x,z\in X}(E_j)_{xz}(E_{h+j})_{xz}(E_{h+j})_{xz} \\
&= \frac{q^{h+i}_{j,h+i+j}q^{h+j}_{i-j,h+i}}{|X|^2}\sum_{x\in X}\left(\sum_{z\in X}((E_j)\circ(E_{h+j}))_{xz}(E_{h+j})_{zx}\right) \\
&= \frac{q^{h+i}_{j,h+i+j}q^{h+j}_{i-j,h+i}q^{h+j}_{j,h+j}}{|X|^3}\sum_{x\in X}(E_{h+j})_{xx}
\end{aligned}$$

$$= \frac{q^{h+i}_{j,h+i+j} q^{h+j}_{i-j,h+i} q^{h+j}_{j,h+j}}{|X|^3} \cdot m_{h+j}.$$

Moreover, $q^{h+i}_{j,h+i+j} \neq 0$ and $q^{h+j}_{i-j,h+i} \neq 0$ by $(Q2)$. Hence $q^{h+j}_{j,h+j} = 0$. ■

Corollary 6.2.7 *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents.*

(1) *For $h \geq 0$, $i \geq 1$ with $h + i + 1 \leq d$,*

$$q^{h+i}_{i,h+1} = q^{h+i}_{1,h+i} = 0 \ \textit{ implies that } \ q^{h+1}_{1,h+1} = 0.$$

In particular, if $a^*_i = 0$ for $i \geq 1$, then $a^*_1 = 0$.

(2) *For $h \geq 0$, $i \geq 2$ with $h + i + 2 \leq d$,*

$$q^{h+i}_{i,h+2} = q^{h+i}_{2,h+i} = q^{h+i}_{2,h+i-1} = 0 \ \textit{ implies that } \ q^{h+2}_{2,h+2} = 0.$$

Proof. (1) Since $q^{h+i}_{1,h+i} = 0$, by $(Q2)$ we have the following.

$$\{l \mid q^l_{1,h+i} q^l_{i-1,h+1} \neq 0\} \subset \{h + i - 1\}.$$

Hence we have the assertion from Proposition 6.2.6 by setting $j = 1$. The last line follows by setting $h = 0$.

(2) Since $q^{h+i}_{2,h+i} = q^{h+i}_{2,h+i-1}$, by $(Q2)$ we have the following.

$$\{l \mid q^l_{2,h+i} q^l_{i-2,h+2} \neq 0\} \subset \{h + i - 2\}.$$

Hence we have the assertion from Proposition 6.2.6 by setting $j = 2$. ■

6.3 Neumaier Diagram

In the previous section, we proved several vanishing conditions of Krein parameters using the equations of matrix entries. Sometimes it is more convenient to symbolize the conjugation of the equations in the following way. I was introduced this diagram, called Neumaier diagram by P. Terwilliger through G. Dickie.

It is possible to give rigorous definition, instead we just give the idea below.

We represent the identities of $(E_h)_{x,y}$'s in the following way. The vertices $x \in X$ are represented by two kinds of circles. Solid circles represent constants in the identity and hollow circles represent variables. The edges or loops labeled h are attached between x and y, if $(E_h)_{x,y}$ appears in the monomial.

It is much easier to use examples for explanation. For example,

$$\sum_{u \in X} (E_h)_{u,x} (E_i)_{u,y} (E_j)_{u,z}$$

is represented by a star graph. (Figure 6.1)

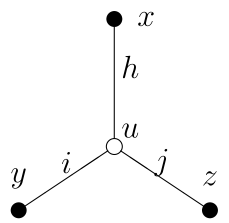


Figure 6.1: $\sum_{u \in X}(E_h)_{u,x}(E_i)_{u,y}(E_j)_{u,z}$

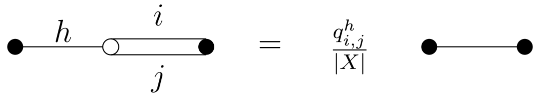


Figure 6.2: $\sum_{u \in X}(E_h)_{u,x}((E_i) \circ (E_j))_{u,y} = \frac{q^h_{i,j}}{|X|}(E_h)_{x,y}$

Hence the Lemma 6.2.1 states that this figure represents 0 if and only if $q^j_{h,i} = 0$.

The condition $E_h(E_i \circ E_j) = \frac{q^h_{i,j}}{|X|}E_h$ can be represented as in Figure 6.2 by considering the (x, y)-entry of this equation.

Figure 6.3 is an interpretation of Lemma 6.2.2 (1), and Figure 6.4 is an interpretation of Lemma 6.2.2 (2). Compare with the formulas below.

(1) $\displaystyle\sum_{e \in X}(E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb} = \frac{q^h_{l,m}}{|X|}\sum_{e \in X}(E_j)_{ea}(E_k)_{ea'}(E_h)_{eb}.$

(2) $\displaystyle\sum_{e \in X}(E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb'} = \sum_{e,e' \in X}(E_j)_{ea}(E_k)_{ea'}(E_h)_{ee'}(E_l)_{e'b}(E_m)_{e'b'}.$

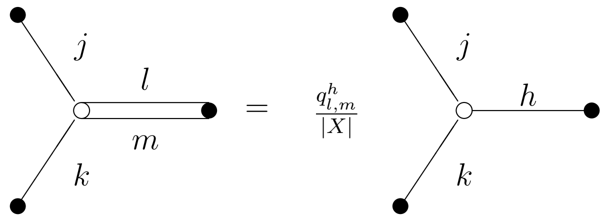


Figure 6.3: Lemma 6.2.2 (1)

The proof of Proposition 6.2.3 starts with Figure 6.5 to conjugate the diagram using the formulas represented by the diagram. Note that a hollow circle can be regarded as a solid circle by fixing the variable, but not vice versa.

As for Proposition 6.2.6, use Figure 6.6.

Exercise 6.3.1 Prove Proposition 6.2.3 and Proposition 6.2.6 using these Neumaier diagram.

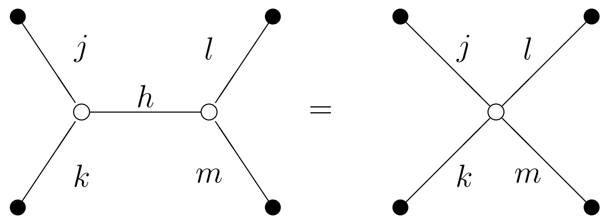


Figure 6.4: Lemma 6.2.2 (2)

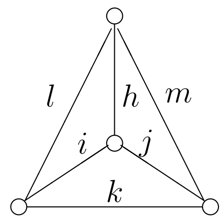


Figure 6.5: Proposition 6.2.3

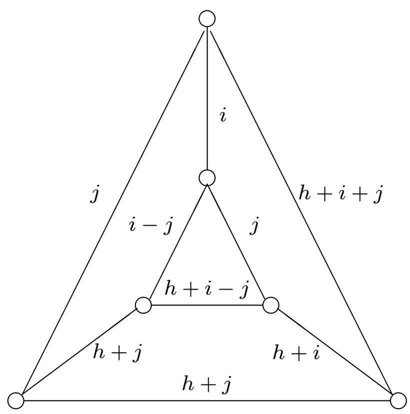


Figure 6.6: Proposition 6.2.6

Chapter 7

Group Schemes

7.1 The Center of a Semisimple Algebra

In Section 2.3, we proved that every $\mathcal{A}$-module of a semisimple algebra $\mathcal{A}$ in $\mathrm{Mat}_n(\boldsymbol{C})$ is completely reducible. In this section, we study the center of $\mathcal{A}$.

Let $\mathcal{A}$ be an algebra in $\mathrm{Mat}_n(\boldsymbol{C})$. Let

$$\mathcal{Z} = \{A \in \mathcal{A} \mid AX = XA, \quad \text{for all } X \in \mathcal{A}\}$$

denote the center of an algebra $\mathcal{A}$ in $\mathrm{Mat}_n(\boldsymbol{C})$.

We need one more definition. Let W be an irreducible $\mathcal{A}$-module and let U be an arbitrary $\mathcal{A}$-module. Then $U(W)$ denote the sum of all irreducible $\mathcal{A}$-submodules of U isomorphic to W.

Lemma 7.1.1 *Let $\mathcal{A}$ be a semisimple algebra in* $\mathrm{Mat}_n(\boldsymbol{C})$*, and let $\mathcal{Z}$ be its center. Then the following hold.*

(1) *$\mathcal{Z}$ is semisimple.*

(2) *Let W be an irreducible $\mathcal{A}$-module. Then every element of $\mathcal{Z}$ acts as a scalar multiple on W.*

Proof. (1) follows from the definition, and (2) is a direct consequence of Proposition 2.3.5. ∎

Lemma 7.1.2 *Let $\mathcal{A}$ be a semisimple algebra in* $\mathrm{Mat}_n(\boldsymbol{C})$*. Let W be an irreducible $\mathcal{A}$-submodule. Then the following hold.*

(1) *$\mathcal{A}(W)$, the sum of all irreducible submodules of the regular $\mathcal{A}$-module isomorphic to W, is a nonzero two-sided ideal of $\mathcal{A}$.*

(2) *For an $\mathcal{A}$-module U, $U(W)$ is a sum of all irreducible $\mathcal{A}$-submodules isomorphic to W in a direct sum decomposition of U into irreducible $\mathcal{A}$-submodules.*

(3) *There are finitely many nonisomorphic irreducible $\mathcal{A}$-modules. Let $W_1, \dots, W_s$ be the representatives of nonisomorphic irreducible $\mathcal{A}$-modules. Then*

$$\mathcal{A} = \mathcal{A}(W_1) \oplus \cdots \oplus \mathcal{A}(W_s), \quad \text{with } \mathcal{A}(W_i)\mathcal{A}(W_j) = \{O\} \ \text{ for } i \neq j.$$

(4) *Let $1 = e_1 + \cdots + e_s$ with $e_i \in \mathcal{A}(W_i)$. Then $e_i e_j = \delta_{i,j} e_i$ with $e_i \neq O$ and $\mathcal{Z} = \mathrm{Span}(e_1, \dots, e_s)$. Moreover $\mathcal{A}(W_i) = e_i \mathcal{A} e_i$ is a minimal two-sided ideal.*

(5) *Let U be an $\mathcal{A}$-module. Then $U(W_i) = e_i U$.*

Proof.

(1) First we show that $\mathcal{A}(W)$ is nonzero. Let $\boldsymbol{w}$ be a nonzero vector in W. Then a mapping $\mathcal{A} \to W$, $(A \mapsto A\boldsymbol{w})$ is a surjective homomorphism. Hence by Lemma 2.3.1, there is an $\mathcal{A}$-submodule W_1 of the regular $\mathcal{A}$-module isomorphic to W. Hence $\mathcal{A}(W)$ contains W_1 and is nonzero.

Next we show that $\mathcal{A}(W)$ is a two-sided ideal. Let $A \in \mathcal{A}$. Let U be an irreducible $\mathcal{A}$-submodule of the regular $\mathcal{A}$-module isomorphic to W. Then the mapping $U \to UA$ sending $X \mapsto XU$, (for $X \in U$) is a homomorphism. Hence it is either the zero mapping or $\mathcal{A}$-isomorphism by Exercise 2.3.3. Hence in any case, $UA \subset \mathcal{A}(W)$ and $\mathcal{A}(W)$ is a two-sided ideal of $\mathcal{A}$.

(2) Let $U = W_1 \oplus \cdots \oplus W_t$ be a direct sum decomposition. Let N be the partial sum collecting W_i's isomorphic to W. It is clear that $N \subset U(W)$. Let $\pi_i : U \to W_i$ be a projection. Let W' be an irreducible $\mathcal{A}$-submodule of $U(W)$ such that $\pi_i(W') \neq 0$. Since W' and W_i are irreducible, we have $W' \simeq W_i$ by Exercise 2.3.3. Hence if $W' \simeq W$, then

$$W' = (\pi_1 + \cdots + \pi_t)W' \subset N.$$

We have (2).

(3) By Lemma 7.1.2 (2), $\mathcal{A}$ is a direct sum of $\mathcal{A}(W_i)$'s. Since

$$\mathcal{A}(W_i)\mathcal{A}(W_j) \subset \mathcal{A}(W_i) \cap \mathcal{A}(W_j) = \{O\} \ \text{ for } i \neq j,$$

and the dimension of $\mathcal{A}$ is finite, there are finitely many nonzero $\mathcal{A}(W_i)$'s. By (1), we conclude that there are finitely many irreducible $\mathcal{A}$-modules.

(4) Let $1 = e_1 + \cdots + e_s$ with $e_i \in \mathcal{A}(W_i)$. Then $e_i e_j = \delta_{i,j} e_i$. Since $\mathcal{A}(W_i)$'s are nonzero, $e_i \neq O$, as $\mathcal{A}(W_i) = \mathcal{A}(W_i)e_i = e_i \mathcal{A}(W_i) e_i$. Since every element of $\mathcal{A}(W_i)$ can be written as $e_i A e_i$ with $A \in \mathcal{A}$, e_i behaves as the identity element in $\mathcal{A}(W_i)$ and zero element in $\mathcal{A}(W_j)$ for $j \neq i$. Therefore $e_i \in \mathcal{Z}$. Let $A \in \mathcal{Z}$. By Lemma 2.3.5, A acts as a scalar on W_i and $\mathcal{A}(W_i)$ is a direct sum of irreducible $\mathcal{A}$-modules isomorphic to W_i, A acts on $\mathcal{A}(W_i)$ as a scalar. Hence A can be written as a linear combination of e_i's. Let I be a two sided ideal properly contained in $\mathcal{A}(W_i)$. Then there is a submodule W of $\mathcal{A}(W_i)$ isomorphic to W_i such that $W \not\subset I$. Since W is irreducible, $IW \subset I \cap W = \{O\}$. Hence $IW_i = \{O\}$ and

$$I = e_i I e_i \subset (e_i I e_i)(e_i \mathcal{A} e_i) = I\mathcal{A}(W_i) = \{O\}.$$

This is a contradiction.

(5) Let $j \neq i$. Since $\mathcal{A}(W_j)\mathcal{A}(W_i) = \{O\}$, every element of $\mathcal{A}(W_j)$ acts trivially on an irreducible module isomorphic to W_i. Hence $\mathcal{A}(W_j)W_i = \{\mathbf{0}\}$. Thus we have the assertion. ∎

e_i's in the previous lemma are called *central primitive idempotents.*

Lemma 7.1.3 *Let $\mathcal{A}$ be a semisimple algebra in $\mathrm{Mat}_n(\boldsymbol{C})$. Let W be an irreducible $\mathcal{A}$-module. Then $\mathcal{A}(W) \simeq \mathrm{Mat}_f(\boldsymbol{C})$, where $f = \dim W$. In particular, $\dim \mathcal{A}(W) = f^2$.*

Proof. We may assume that $W \subset \mathcal{A}(W)$. Let $u \in W$ be a nonzero element. Since $\mathcal{A}(W)$ is a minimal two-sided ideal, $\mathcal{A}u\mathcal{A} = \mathcal{A}(W)$. Choose $a_i, b_i \in \mathcal{A}$ so that $e = \sum_{i=1}^n a_i u b_i$, where e is the central primitive idempotent in $\mathcal{A}(W)$. Let $\mathrm{End}(W)$ denote the set of linear transformations on W. So, $\mathrm{End}(W) \simeq \mathrm{Mat}_f(\boldsymbol{C})$. Since each element of $\mathcal{A}(W)$ acts on W, there is a mapping from $\mathcal{A}(W)$ to $\mathrm{End}(W)$. We show this mapping is surjective. Let ϕ be a linear transformation on W. We show in the following that for $w \in W$,

$$\phi(w) = \left(\sum_{i=1}^n \phi(a_i u) b_i\right) \cdot w.$$

First for $w \in W$, the mapping $\alpha_w : W \to W$, $(x \mapsto xw)$ is a homomorphism which commutes with the action of $\mathcal{A}$. Hence by Proposition 2.3.5, α_w is a scalar. Since $a_i u \in W$ and $b_i w \in W$, we have

$$\begin{aligned}
\phi(w) &= \phi(ew) \\
&= \phi(\sum_{i=1}^n a_i u b_i w) \\
&= \phi(\sum_{i=1}^n \alpha_{b_i w}(a_i u)) \\
&= \sum_{i=1}^n \alpha_{b_i w} \phi(a_i u) \\
&= \sum_{i=1}^n \phi(a_i u) b_i w \\
&= \left(\sum_{i=1}^n \phi(a_i u) b_i\right) \cdot w,
\end{aligned}$$

as desired. Since $\mathcal{A}(W) \subset \mathrm{Mat}_n(\boldsymbol{C})$ and every element of $\mathcal{A}(W)$ acts as zero element on the irreducible modules which are not isomorphic to W, nonzero element of $\mathcal{A}(W)$ acts nontrivially on W, and we have the assertion. ∎

7.2 Group Algebras and Representation

In this section we introduce the representation theory of finite groups and the character theory. The Bose-Mesner algebra of a commutative association scheme is, in one sense, a generalization of the center of the group algebra.

Definition 7.2.1 Let G be a finite group. The *group algebra* of G, denoted by $\boldsymbol{C}[G]$ is a vector space over the complex number field $\boldsymbol{C}$ of dimension $|G|$, whose basis elements are labeled by the elements of G endowed with the product defined as follows.

$$(\sum_{x\in G}\alpha_x x)(\sum_{y\in G}\beta_y y) = \sum_{x,y\in G}\alpha_x\beta_y xy.$$

Hence the coefficient of z in the product is equal to $\sum_{w\in G}\alpha_w\beta_{w^{-1}z}$.

Let $V = \boldsymbol{C}[G]$. We regard V as a Hermitean space with a set of orthonormal basis G, i.e., the inner product is defined as follows.

$$\left(\sum_{x\in G}\alpha_x x, \sum_{y\in G}\beta_y y\right) = \sum_{x\in G}\overline{\alpha_x}\beta_x.$$

By abuse of notation, we view V as $|G|$-dimensional row vector space with unit vectors x, $(x\in G)$, whose x-coordinate is 1 and 0 elsewhere. Then $\boldsymbol{C}[G]$ acts on V by left multiplication and $\boldsymbol{C}[G] \subset \mathrm{Mat}_G(\boldsymbol{C})$. Thus $\boldsymbol{C}[G]$ can be regarded as an algebra in the sense of Definition 2.3.1. Hence we can consider $\boldsymbol{C}[G]$-modules.

Exercise 7.2.1 Suppose U is a vector space and the action of G is defined, i.e., every element x of G is a linear transformation U sending $\boldsymbol{u}$ to $x\cdot\boldsymbol{u}$ satisfying

$$(xy)\cdot\boldsymbol{u} = x\cdot(y\cdot\boldsymbol{u}), \quad \text{for all } x, y\in G.$$

Then U becomes a $\boldsymbol{C}[G]$-module by the following.

$$\left(\sum_{x\in G}\alpha_x x\right)\cdot\boldsymbol{u} = \sum_{x\in G}\alpha_x(x\cdot\boldsymbol{u}).$$

In other words, if there is a group homomorphism from G to $GL(U)$, the general linear group on U consisting of nonsingular linear transformations, then U becomes a $\boldsymbol{C}[G]$-module. Conversely, if U is a $\boldsymbol{C}[G]$-module, every element of G acts as a nonsingular linear transformation on U and induces a group homomorphism from G to $GL(U)$.

Let $x\in G$. Then the matrix in $\mathrm{Mat}_G(\boldsymbol{C})$ corresponding to x has (y,z)-entry $\delta_{xy,z} = \delta_{y,x^{-1}z}$, or equivalently $(xy,z) = \delta_{xy,z} = \delta y, x^{-1}z$. Hence ${}^t x = x^{-1}$ or ${}^t\overline{x} = x^{-1}$. In other words, the elements of G are represented as unitary matrices.

We define * mapping on $\boldsymbol{C}[G]$ as follows.

$$\left(\sum_{x\in G}\alpha_x x\right)^* = \sum_{x\in G}\overline{\alpha_x}{}^t x = \sum_{x\in G}\overline{\alpha_x}x^{-1}.$$

Then for $u, v\in V$ and $A\in\boldsymbol{C}[G]$, $A^*\in\boldsymbol{C}[G]$ and $(Au,v) = (u, A^*v)$. Thus $\boldsymbol{C}[G]$ is semisimple.

By Proposition 2.3.4, we have the following.

Proposition 7.2.1 *Every $\boldsymbol{C}[G]$-module is completely reducible.*

Example 7.2.1 Let $\boldsymbol{C}[G]$ be a group algebra.

1. Let $\boldsymbol{C} = \mathrm{Span}(\boldsymbol{u})$ be a one dimensional vector space. Then it becomes a $\boldsymbol{C}[G]$-module by the following action.

$$\left(\sum_{x\in G}\alpha_x x\right)\cdot(\lambda\boldsymbol{u}) = \left(\lambda\sum_{x\in G}\alpha_x\right)\boldsymbol{u}.$$

   This $\boldsymbol{C}[G]$-module is called a *trivial* module. This module is defined by the trivial group homomorphism from G to $\boldsymbol{C}^*$ sending every element of G to 1.

2. Suppose U and W be $\boldsymbol{C}[G]$-modules.

   (a) The direct sum $U \oplus W$ becomes a $\boldsymbol{C}[G]$-module.

   (b) The tensor product $U \otimes W$ becomes a $\boldsymbol{C}[G]$-module.

Let $C_0, C_1, \ldots, C_d$ be the conjugacy classes of the group G. Let $c_0 = 1, c_1, \ldots, c_d$ be the representatives. We write the elements in the group algebra corresponding to these classes as follows.

$$\widehat{C_i} = \sum_{x\in C_i} x.$$

Exercise 7.2.2 Let $\boldsymbol{C}[G]$ be a group algebra. Show the following.

1. Let $\mathscr{Z} = Z(\boldsymbol{C}[G]) = \{\alpha \in \boldsymbol{C}[G] \mid \alpha\beta = \beta\alpha \text{ for all } \beta \in \boldsymbol{C}[G]\}$ be the center of $\boldsymbol{C}[G]$. Then the dimension is $d+1$ and $\{\widehat{C_0}, \widehat{C_1}, \ldots, \widehat{C_d}\}$ is a basis, where $\widehat{C_i} = \sum_{x\in C_i} x$.

2. $Z(\boldsymbol{C}[G])$ is a subalgebra, i.e., it is a vector subspace which is closed under the multiplication.

Since the center of the algebra is commutative, $Z(\boldsymbol{C}[G])$ becomes a commutative algebra. Let

$$\widehat{C_i}\widehat{C_j} = \sum_{h=0}^{d} c_{i,j}^h \widehat{C_j}.$$

Proposition 7.2.2 *Let $\boldsymbol{C}[G]$ be a group algebra of a finite group G with $d+1$ conjugacy classes $C_0, C_1, \ldots, C_d$.*

(1) *There are $d+1$ nonisomorphic irreducible $\boldsymbol{C}[G]$-modules. Let $W_0, W_1, \ldots, W_d$ be the representatives of nonisomorphic irreducible $\boldsymbol{C}[G]$-modules. Then*

$$\boldsymbol{C}[G] = \boldsymbol{C}[G](W_0) \oplus \boldsymbol{C}[G](W_1) \oplus \cdots \oplus \boldsymbol{C}[G](W_d),$$

with $\boldsymbol{C}[G](W_i)\boldsymbol{C}[G](W_j) = \{O\}$ for $i \neq j$.

(4) *Let $1 = \mathcal{E}_0 + \cdots + \mathcal{E}_d$ with $\mathcal{E}_i \in \boldsymbol{C}[G](W_i)$. Then $\boldsymbol{C}[G](W_i) = \mathcal{E}_i\boldsymbol{C}[G]\mathcal{E}_i$, $\mathcal{E}_i\mathcal{E}_j = \delta_{i,j}\mathcal{E}_i$ with $\mathcal{E}_i \neq 0$ and*

$$\mathscr{Z} = Z(\boldsymbol{C}[G]) = \mathrm{Span}(\mathcal{E}_0, \mathcal{E}_1 \ldots, \mathcal{E}_d) = \mathrm{Span}(\widehat{C_0}, \widehat{C_1}, \ldots, \widehat{C_d}).$$

Proof. This is a direct consequence of Exercise 7.2.2 and Lemma 7.1.2. ■

Definition 7.2.2 Let G be a finite group and let U be a $\boldsymbol{C}[G]$-module. The *character* $\chi = \chi_U$ of U is a function $\boldsymbol{C}[G] \to \boldsymbol{C}$ defined by $\chi(A) = \mathrm{tr}(A)$. $\mathrm{tr}(A)$ is the trace of A regarding A as a linear transformation on U. By the same symbol χ, we sometimes denote the restriction of χ to G.

Exercise 7.2.3 Show that the character is well defined i.e., it does not depend on the choice of the basis. Show also that isomorphic $\boldsymbol{C}[G]$-modules have same characters.

Lemma 7.2.3 *The following hold.*

(1) *Let χ be a character of U. Then $\chi(x^{-1}yx) = \chi(y)$ for every $x, y \in G$. In particular, χ is constant on each conjugacy class of G.*

(2) *Let 1 denote the character of the trivial module. Then $1(x) = 1$ for every $x \in G$.*

(3) *Let ϕ and ψ be characters of $\boldsymbol{C}[G]$-modules U and W respectively.*

 (*a*) *The character of $U \oplus W$ denoted by $\phi + \psi$ satisfies $(\phi + \psi)(x) = \phi(x) + \psi(x)$.*

 (*b*) *The character of $U \otimes W$ denoted by $\phi\psi$ satisfies $(\phi\psi)(x) = \phi(x)\psi(x)$.*

Exercise 7.2.4 Prove Lemma 7.2.3.

7.3 Definition of Group Schemes

In this section, we give a definition of group schemes and show some basic properties.

Definition 7.3.1 Let G be a finite group. Let $C_0 = \{1\}, C_1, \ldots, C_d$ be the conjugacy classes of G. Since the set of the inverse of the elements of a conjugacy class C_i becomes a conjugacy class, let $C_{i'} = C_i^{-1}$. Let

$$R_i = \{(x, y) \in G \times G \mid yx^{-1} \in C_i\}.$$

Then $\mathcal{G} = (G, \{R_i\}_{0 \le i \le d})$ becomes a commutative association scheme, which is called a *group scheme*, or a *group association scheme*.

Since the center of the algebra is commutative, $Z(\boldsymbol{C}[G])$ becomes a commutative algebra. Let

$$\widehat{C_i}\widehat{C_j} = \sum_{h=0}^{d} c_{i,j}^h \widehat{C_j}.$$

Lemma 7.3.1 *Let $(x, y) \in R_i$, i.e., $yx^{-1} \in C_i$. Then*

$$c_{i,j}^h = |\{z \in G \mid (x, z) \in R_i,\ (z, y) \in R_j\}|,$$

and $\mathcal{G} = (G, \{R_i\}_{0 \le i \le d})$ becomes a commutative association scheme, whose Bose-Mesner algebra is isomorphic to the center of the group algebra $Z(\boldsymbol{C}[G])$.

Proof. Suppose $yx^{-1} \in C_h$. We define a bijection between the following sets.

$$\begin{aligned} S_1 &= \{(u,v) \in C_i \times C_j \mid uv = yx^{-1}\} \\ S_2 &= \{z \in G \mid (x,z) \in R_i,\ (z,y) \in R_j\} \end{aligned}$$

Let $(u,v) \in S_1$. Then $\phi(u,v) = ux = yx^{-1}v^{-1}x \in S_2$. Let $z \in S_2$. Then $\psi(z) = (zx^{-1}, xz^{-1}yx^{-1}) \in S_1$ as $yz^{-1} \in C_j$ implies $xz^{-1}yx^{-1} = (zx^{-1})^{-1}(yz^{-1})(zx^{-1}) \in C_j$. Hence these correspondences ϕ and ψ are inverses each other. Since $c_{i,j}^h = |S_2|$, $|S_1|$ depends only on h, i, j and does not depend on the choice of $(x,y) \in R_h$. Hence $\mathcal{G} = (G, \{R_i\}_{0 \le i \le d})$ becomes an commutative association scheme whose Bose-Mesner algebra is isomorphic to $Z(\boldsymbol{C}[G])$, and $p_{i,j}^h = c_{i,j}^h$ for all $0 \le h, i, j \le d$. ∎

Let $A_0, A_1, \ldots, A_d$ be the i-th adjacency matrices, and let $E_0, E_1, \ldots, E_d$ be the primitive orthogonal idempotents of $\mathcal{G}$. Let $\mathcal{Z} = Z(\boldsymbol{C}[G])$. Then by Lemma 7.3.1, $\mathcal{Z}$ is isomorphic to the Bose-Mesner algebra of $\mathcal{G}$ by the correspondence $\widehat{C_i} \leftrightarrow A_i$. Let $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_d$ be the orthogonal primitive idempotents of $\mathcal{Z}$ corresponding to $E_0, E_1, \ldots, E_d$.

Let W be an irreducible $\boldsymbol{C}[G]$-module. Since every element of $\mathcal{Z}$ acts as a scalar on W, we assume that this scalar for $M \in \mathcal{Z}$ is $\omega_W(M)$. Then for $\boldsymbol{0} \neq \boldsymbol{w} \in W$, $M\boldsymbol{w} = \omega_W(M)\boldsymbol{w}$. Clearly, this mapping ω from $\mathcal{Z}$ to $\boldsymbol{C}$ is an algebra homomorphism.

Lemma 7.3.2 *Let W be an irreducible $\boldsymbol{C}[G]$-module. Let $c_i \in C_i$ and let χ be the character of W. Hence $\chi_W(c_i)$ is the trace of the action of c_i on W. Then the following hold.*

(1) $\omega_W(\widehat{C_i}) = k_i\chi_W(c_i)/\dim(W)$, *where* $k_i = |C_i|$.

(2) $\omega_W(\widehat{C_i})\omega_W(\widehat{C_j}) = \displaystyle\sum_{h=0}^{d} c_{i,j}^h \omega_W(\widehat{C_h})$.

(3) *Let $\mathcal{E}_j$ be the primitive idempotent of $\mathcal{A}$ such that $\mathcal{E}_j W \neq O$. Then W determines j uniquely and $\mathcal{E}_j$ determines the isomorphism class of W.*

Proof.

(1) By Lemma 7.2.3, χ is constant on each conjugacy class. Hence

$$\dim(W)\omega_W(\widehat{C_i}) = \mathrm{tr}(\widehat{C_i}) = \chi(\widehat{C_i}) = k_i\chi(c_i).$$

We have the formula.

(2) Since ω_W is an algebra homomorphism, this is clear.

(3) There are $d+1$ isomorphism classes of irreducible modules. Thus we have the assertion. ∎

By the lemma above, there are $d+1$ isomorphism classes of irreducible $\boldsymbol{C}[G]$-modules. Let $W_0, W_1, \ldots, W_d$ be nonisomorphic irreducible modules. We may assume that W_0 is the trivial module. Let χ_i be the character of W_i and $\omega_i = \omega_{W_i}$. Set $\mathrm{Irr}(G) = \{\chi_0, \chi_1, \ldots, \chi_d\}$.

Lemma 7.3.3 *Let $\mathcal{G} = (G, \{R_i\}_{0\le i\le d})$ be a group scheme of a finite group G. Let $f_i = \chi_i(1)$. Then the following hold.*

(1) $m_i = f_i^2$.

(2) $p_i(j) = \omega_j(\widehat{C_i}) = k_i\chi_j(c_i)/f_j$.

(3) $q_i(j) = f_i\overline{\chi_i(c_j)}$.

Proof. (1) follows from Lemma 7.1.3. Since $\widehat{C_i}\mathcal{E}_j = \omega_j(\widehat{C_i})\mathcal{E}_j$, we have (2). (3) is straightforward. ∎

Definition 7.3.2 1. A function $\phi : G \to \boldsymbol{C}$ on a finite group G, which is constant on each conjugacy class, is called a *class function.*

2. Let ϕ, ψ be class functions of G. We define a Hermitean inner product as follows.

$$< \phi, \psi >= \frac{1}{|G|}\sum_{i=0}^{d} k_i\overline{\phi(c_i)}\psi(c_i) = \frac{1}{|G|}\sum_{x\in G}\overline{\phi(x)}\psi(x).$$

Lemma 7.3.4 *The following hold.*

(1) $< \chi_i, \chi_j >= \dfrac{1}{|G|}\displaystyle\sum_{h=0}^{d} k_h\overline{\chi_i(c_h)}\chi_j(c_h) = \delta_{i,j}$.

(2) $\dfrac{1}{|G|}\displaystyle\sum_{h=0}^{d}\overline{\chi_h(c_i)}\chi_h(c_j) = \delta_{i,j}\dfrac{1}{k_i}$.

Proof. This is a direct consequence of Proposition 3.2.2 using the interpretation of parameters in Lemma 7.3.3. ∎

Lemma 7.3.5 (1) $q_{i,j}^h = \dfrac{f_if_j}{f_h|G|}\displaystyle\sum_{l=0}^{d} k_l\overline{\chi_h(c_l)}\chi_i(c_l)\chi_j(c_l) = \dfrac{f_if_j}{f_h} < \chi_h, \chi_i\chi_j >$.

(2) $p_{i,j}^h = \dfrac{k_ik_j}{|G|}\displaystyle\sum_{l=0}^{d}\frac{1}{f_l}\overline{\chi_h(c_l)}\chi_i(c_l)\chi_j(c_l)$.

(3) *$p_{i,j}^h$'s are nonnegative integers, and $q_{i,j}^h$'s are nonnegative reals. Moreover, if $q_{i,j}^h \neq 0$, then $q_{i,j}^h \ge 1$.*

Proof. These are direct consequences of Lemma 3.2.3, except the last assertion. If $q_{i,j}^h \neq 0$, then the irreducible module affording χ_h is a direct summand of the module affording $\chi_i\chi_j$ with multiplicity $< \chi_h, \chi_i\chi_j >$. Hence

$$f_if_j = \dim W_i \otimes W_j \ge < \chi_h, \chi_i\chi_j > \dim W_h =< \chi_h, \chi_i\chi_j > f_h.$$

Hence $q_{i,j}^h \ge (< \chi_h, \chi_i\chi_j >)^2 \ge 1$. ∎

7.4 Representation Diagrams of Group Schemes

We consider the following.

- G : a finite group
- $\mathrm{Irr}(G) := \{\chi_0, \chi_1, \ldots, \chi_d\}$: the set of (absolutely) irreducible (ordinary) characters
- χ : a character (not necessarily irreducible) such that $\chi = \bar{\chi}$

Since $\chi\chi_i$ corresponds to a character of the tensor product, $\chi\chi_i$ can be written as a linear combination of irreducible characters with nonnegative integers as coefficients.

$$\chi\chi_i = \sum_{j=0}^{d} \alpha_{ji}\chi_j$$

$$\alpha_{ji} =< \chi\chi_i, \chi_j >= \frac{1}{|G|}\sum_{g\in G}\chi(g)\chi_i(g)\overline{\chi_j(g)}$$

Definition 7.4.1 *The representation diagram* of a group G with respect to a real character χ, denoted by $\mathcal{D}^* = \mathcal{D}^*(G, \chi)$, is a diagram with $\mathrm{Irr}(G)$ as the vertex set such that the adjacency is defined as follows.

$$\chi_i \sim \chi_j \Leftrightarrow \alpha_{ji} \neq 0 \ (\Leftrightarrow \alpha_{ij} \neq 0)$$

The adjacent pair χ_i, χ_j is said to be an edge if $i \neq j$ and a loop if $i = j$.

By Lemma 7.3.5, if χ corresponds to the primitive idempotent $E = E_i$ of the group scheme, $\mathcal{D}^*(G, \chi)$ is isomorphic to $\mathcal{D}^*(\mathcal{G}, E)$.

The following is the interpretation of Lemma 4.2.3.

Lemma 7.4.1 *Let $\mathcal{D}^* = \mathcal{D}^*(G, \chi)$ be a representation diagram.*

(1) *$\mathcal{D}^*$ is connected if and only if χ is faithful, i.e.,*

$$Ker\chi = \{g \in G \mid \chi(g) = \chi(1)\} = 1.$$

(2) *Suppose $\mathcal{D}^*$ is connected. Then the diameter of $\mathcal{D}^*$ is equal to the number of values of χ on $G - \{1\}$.*

Example 7.4.1 $G = SL_2(3)$, with $\chi = \chi_3$ such that $\chi(1) = 2$.

	$1a$	$2a$	$4a$	$3a$	$3b$	$6a$	$6b$
χ_0	1	1	1	1	1	1	1
χ_1	1	1	1	ω^2	ω	ω	ω^2
χ_2	1	1	1	ω	ω^2	ω^2	ω
χ_3	2	-2	0	-1	-1	1	1
χ_4	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2
χ_5	2	-2	0	$-\omega$	$-\omega^2$	ω^2	ω
χ_6	3	3	-1	0	0	0	0

$\chi\chi_0 = \chi_3$
$\chi\chi_1 = \chi_4$
$\chi\chi_2 = \chi_5$
$\chi\chi_3 = \chi_0 + \chi_6$
$\chi\chi_4 = \chi_1 + \chi_6$
$\chi\chi_5 = \chi_2 + \chi_6$
$\chi\chi_6 = \chi_3 + \chi_4 + \chi_5$

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & & \chi_5 \arrow[r, dash] & \chi_2 \\
\chi_0 \arrow[r, dash] & \chi_3 \arrow[r, dash] & \chi_6 \arrow[ur, dash] \arrow[dr, dash] & & \\
 & & & \chi_4 \arrow[r, dash] & \chi_1
\end{tikzcd}
$$

Figure 7.1: $\mathcal{D}^*(SL_2(3), \chi_3)$

Hence in this case $D^*(SL_2(3), \chi)$ becomes as in Figure 7.1.

$$
\begin{pmatrix}
 & & & 1 & & & \\
 & & & & 1 & & \\
 & & & & & 1 & \\
1 & & & & & & 1 \\
 & 1 & & & & & 1 \\
 & & 1 & & & & 1 \\
 & & & 1 & 1 & 1 &
\end{pmatrix}
\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \\ \chi_6 \end{pmatrix}
= \chi
\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \\ \chi_6 \end{pmatrix}
$$

By setting $g = 1$, we have that

$$
\begin{pmatrix}
 & & & 1 & & & \\
 & & & & 1 & & \\
 & & & & & 1 & \\
1 & & & & & & 1 \\
 & 1 & & & & & 1 \\
 & & 1 & & & & 1 \\
 & & & 1 & 1 & 1 &
\end{pmatrix}
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 3 \end{pmatrix}
= 2 \cdot
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 3 \end{pmatrix}
$$

Exercise 7.4.1 Let $\mathcal{G} = (G, \{R_i\}_{0 \le i \le d})$ be the group scheme of $SL(2,3)$, where R_i's correspond to the conjugacy classes in the character table above. Show that the symmetrization of $\mathcal{G}$, i.e., $\mathcal{G}' = (G, \{R_0, R_1, R_2, R_3 \cup R_4, R_5 \cup R_6)$, is a symmetric association scheme and it is Q-polynomial with respect to a suitable ordering to the primitive idempotents.

The following result is one of the McKay observations. See [96, 56, 86, 16].

Theorem 7.4.2 *Let G be a finite group and χ a real faithful character with $\chi(1) = 2$. Then G and $\mathcal{D}^*(G, \chi)$ is one of the following.*

(1) $\boldsymbol{Z}_{n+1}$*, a cyclic group of order* $n+1$*, and* $\mathcal{D}^* \simeq \tilde{A}_n$*. (Figure 7.2)*

(2) $D_{4(n-2)}$*, a dihedral group, or* $Q_{4(n-2)} = (2, 2, n-2)$*, a generalized quaternion group, and* $\mathcal{D}^* \simeq \tilde{D}_n$*. (Figure 7.3)*

(3) $SL_2(3) = (2, 3, 3)$*, and* $\mathcal{D}^* \simeq \tilde{E}_6$*. (Figure 7.4)*

(4) $2 \cdot S_4 = (2, 3, 4)$*, and* $\mathcal{D}^* \simeq \tilde{E}_7$*. (Figure 7.5)*

(5) $SL_2(5) = (2, 3, 5)$*, and* $\mathcal{D}^* \simeq \tilde{E}_8$*. (Figure 7.6)*

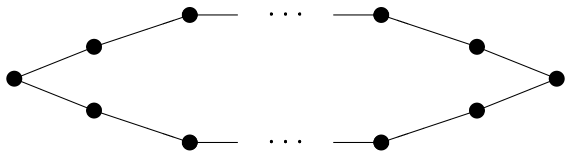


Figure 7.2: $\tilde{A}_n$

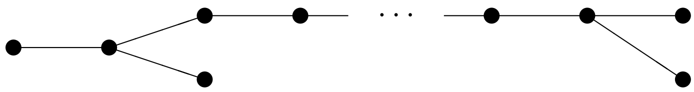


Figure 7.3: $\tilde{D}_n$

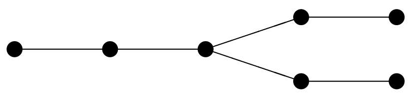


Figure 7.4: $\tilde{E}_6$

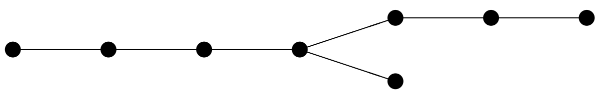


Figure 7.5: $\tilde{E}_7$

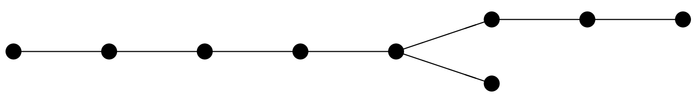


Figure 7.6: $\tilde{E}_8$

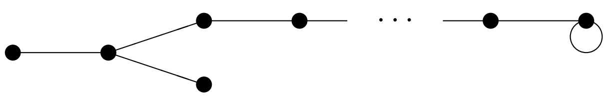


Figure 7.7: $\tilde{DL}_n$

(6) $D_{2(2l+1)}$, *a dihedral group, and* $\mathcal{D}^* \simeq \tilde{DL}_n$. *(Figure 7.7)*

Therefore we know that except the first reducible one, the group scheme of each of the groups above has a representation diagram which is a tree. Hence there is a strongly balanced set for those group schemes.

The examples above appear in many places in Mathematics.

Theorem 7.4.3 *A connected graph with largest eigenvalue 2 is in the graphs above.*

Theorem 7.4.4 *A binary polyhedral group*

$$(l, m, n) =< x, y, z \mid x^l = y^m = z^n = xyz >$$

is finite if and only if $\dfrac{1}{l} + \dfrac{1}{m} + \dfrac{1}{n} > 1$, *i.e.,*

$$(l, m, n) = (2, 2, n), (2, 3, 3), (2, 3, 4) \text{ or } (2, 3, 5)$$

7.5 Representation with Real Character

In this section we consider the representation of a finite group G with real a character and the embedding of the group elements on the sphere.

The following is well-known. See for example [71].

Lemma 7.5.1 *Let G be a finite group and let χ be an irreducible character. Then $\chi^2 = \mathrm{Sym}^2\chi + \mathrm{Alt}^2\chi$ and the following hold.*

(1) $< \chi^2\chi, 1 >= 0$ *if and only if χ is not real valued.*

(2) $< \mathrm{Sym}^2\chi, 1 >= 1$ *if and only if χ is real representable.*

(3) $< \mathrm{Alt}^2\chi, 1 >= 1$ *if and only if χ is real valued but not real representable.*

Lemma 7.5.2 *Let χ be a real valued irreducible characters of a finite group G. Suppose for some positive integer m and $\psi \in \mathrm{Irr}(G)$, $\chi^2 = 1+m\psi$. Then χ is not real representable and $\psi(1) = \chi(1) + 1$.*

Proof. Let $n = \chi(1)$. By Lemma 7.5.1, $< \chi^2, 1 >= 1$. Let $\epsilon = 1$ if $< \mathrm{Sym}^2\chi, 1 >= 1$, and $\epsilon = -1$ if $< \mathrm{Alt}^2\chi, 1 >= 1$. Since $\chi^2 = 1 + m\psi$, either one of the following holds.

(i) Both $\mathrm{Sym}^2\chi(1) - 1$ and $\mathrm{Alt}^2\chi(1)$ are multiples of $\psi(1)$, when $< \mathrm{Sym}^2\chi, 1 >= 1$.

(ii) Both $\mathrm{Sym}^2\chi(1)$ and $\mathrm{Alt}^2\chi(1) - 1$ are multiples of $\psi(1)$, when $< \mathrm{Alt}^2\chi, 1 >= 1$.

Hence the diference of the quantities above are also divisible by $\psi(1)$. Since $\mathrm{Sym}^2\chi(1) = n(n+1)/2$ and $\mathrm{Alt}^2\chi(1) = n(n-1)/2$, for some integer t we have

$$\mathrm{Sym}^2\chi(1) - \epsilon - \mathrm{Alt}^2\chi(1) = \frac{n(n+1)}{2} - \epsilon - \frac{n(n-1)}{2} = n - \epsilon = t\psi(1).$$

We have

$$m\psi(1) = \chi^2(1) - 1 = n^2 - 1 = (n-1)(n+1) = (n+\epsilon)t\psi(1)$$

or $m = (n+\epsilon)t$. Since $< \chi, \chi\psi >=< \chi\overline{\chi}, \psi >=< \chi^2, \psi >= m$, we have the following.

$$n(n-\epsilon) = t\chi(1)\psi(1) \geq tm\chi(1) = n(n+\epsilon)t^2.$$

Thus $\epsilon = -1$ and $< \mathrm{Alt}^2\chi, 1 >= 1$. Hence χ is not real representable by Lemma 7.5.1. More over we have either $n = \chi(1) = 1$, or $t = 1$ and $\psi(1) = n+1$. Since $\psi(1) \neq 0$, $n > 1$ and we have the desired conclusion. ∎

Lemma 7.5.3 *Let G be a nontrivial finite group and let χ be a faithful irreducible character. Then the following are equivalent.*

(i) *χ has exactly two nonzero values.*

(ii) *χ is real valued and for every irreducible character $\psi \neq \chi$ of G, $\chi\psi =< \chi\psi, \chi > \chi$.*

Proof. First assume (i). Let $\chi(G) \subset \{\chi(1), -l, 0\}$. Since

$$\chi^2 + l\chi = \chi(\chi + l1)$$

vanishes at each nonidentity element of G, it is a multiple of the regular character ρ_G, i.e. $\chi^2 + l1 = a\rho_G$ for some integer a. Since $< \chi, 1 >= 0$, l is a rational number, which is also an algebraic integer. Hence l is a positive integer such that $l \leq \chi(1)$. We have

$$a\chi(1) =< a\rho_G, \chi >=< \chi^2 + l\chi, \chi >\leq \chi(1) + l \leq 2\chi(1).$$

If $a = 2$, then $l = \chi(1)$ and $< \chi^2, \chi >< \chi(1)^3/|G| < \chi(1)$, which is impossible. Hence $a = 1$ and $\rho_G = \chi^2 + l\chi$. Let $\psi \neq \chi$ be an irreducible character. Then

$$< \chi\psi, \chi >=< \chi^2, \psi >=< \rho_G - l\chi, \psi >=< \rho_G, \psi >= \psi(1).$$

Comparing the degrees of the characters, we have $\chi\psi = \psi(1)\chi$. This is (i).

Assume (ii). Let $< \chi^2, \chi_i >= n_i$ for $i = 1, \ldots, d$, where $\mathrm{Irr}(G) = \{\chi, \chi_1, \ldots, \chi_d\}$. Since

$$\chi\chi_i =< \chi\chi_i, \chi > \chi =< \chi^2, \chi_i > \chi = n_i\chi,$$

we have $n_i = \chi_i(1)$. Thus there is an integer l such that $\rho_G = \chi^2 + l\chi$. Since ρ_G takes value zero at each nonidentity element of G, χ takes at most two nonzero values, namely $\chi(1)$ and $-l$. $l = 0$ implies $|G| = \chi(1)^2$ and $|G| = 1$, which is not the case. If $l = -\chi(1)$,

$$\chi(1) \geq< \chi^2, \chi >=< \rho_G + \chi(1)\chi, \chi >= 2\chi(1).$$

This is impossible. Thus $\chi(1), -l, 0$ are all different and we have (ii) in this case. ∎

Remarks. If the assumption of the previous lemma is satisfied, then the order of the center of G is at most 2 as the value at the center has same absolute value as the one at the identity element, and we must have $l = \chi(1)$.

The author is imfromed that this class of groups is fully studied by Gagola.

Chapter 8

Solutions to Exercises

8.1 Chapter 1. Introduction

1.2.1

1.2.2

1.2.3 We have $(n-2)(k-2) < 3$. Hence

$$(n,k) = (3,3), (4,3), (3,4), (5,3), \text{ or } (3,5).$$

1.3.1 The intersection arrays become as follows.

Tetrahedron : $\left\{\begin{matrix} * & 1 \\ 0 & 2 \\ 3 & * \end{matrix}\right\}$.

Cube : $\left\{\begin{matrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{matrix}\right\}$.

Octahedron : $\left\{\begin{matrix} * & 1 & 4 \\ 0 & 2 & 0 \\ 4 & 1 & * \end{matrix}\right\}$.

Dodecahedron : $\left\{\begin{matrix} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{matrix}\right\}$.

Icosahedron : $\left\{\begin{matrix} * & 1 & 2 & 5 \\ 0 & 2 & 2 & 0 \\ 5 & 2 & 1 & * \end{matrix}\right\}$.

Hyper-Cube : $b_i = n - i$, $a_i = 0$, $c_i = i$ for $i = 0, 1, \ldots, n$.

Half Cube : $b_i = \binom{n-2i}{2}$, $a_i = 2i(n-2)$, $c_i = \binom{2i}{2}$ for $i = 0, 1, \ldots, [n/2]$.

1.4.1 Platonic solids can be naturally embedded in $S^2 \subset \boldsymbol{R}^3$.

Octahedron : $X = \{\pm \boldsymbol{e}_i \mid i = 1, 2, 3\}$, where $\boldsymbol{e}_i$'s are unit vectors.

Dodecahedron : $X = \{x_1, x_2, \ldots, x_{20}\}$, where $x_{21-i} = -x_i$ for $i = 1, 2, \ldots, 10$.

$$\begin{array}{ll}
x_1 = (1,0,0) & x_2 = (\frac{\sqrt{5}}{3}, \frac{2}{3}, 0), \\
x_3 = (\frac{\sqrt{5}}{3}, -\frac{1}{3}, \frac{\sqrt{3}}{3}), & x_4 = (\frac{\sqrt{5}}{3}, -\frac{1}{3}, -\frac{\sqrt{3}}{3}), \\
x_5 = (\frac{1}{3}, \frac{\sqrt{5}}{3}, \frac{\sqrt{3}}{3}), & x_6 = (\frac{1}{3}, \frac{\sqrt{5}}{3}, -\frac{\sqrt{3}}{3}), \\
x_7 = (\frac{1}{3}, \frac{3-\sqrt{5}}{6}, \frac{\sqrt{3}(\sqrt{5}+1)}{6}), & x_8 = (\frac{1}{3}, \frac{3-\sqrt{5}}{6}, -\frac{\sqrt{3}(\sqrt{5}+1)}{6}), \\
x_9 = (\frac{1}{3}, -\frac{3+\sqrt{5}}{6}, \frac{\sqrt{3}(\sqrt{5}-1)}{6}), & x_{10} = (\frac{1}{3}, -\frac{3+\sqrt{5}}{6}, -\frac{\sqrt{3}(\sqrt{5}-1)}{6}).
\end{array}$$

In this case, $\Delta(X) = \{1, \frac{\sqrt{5}}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{\sqrt{5}}{3}, -1\}$.

Icosahedron : $X = \{x_1, x_2, \ldots, x_{12}\}$, where $x_{13-i} = -x_i$ for $i = 1, 2, \ldots, 6$.

$$\begin{array}{ll}
x_1 = (1,0,0) & x_2 = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0), \\
x_3 = (\frac{1}{\sqrt{5}}, \frac{5-\sqrt{5}}{10}, \sqrt{\frac{5+\sqrt{5}}{10}}), & x_4 = (\frac{1}{\sqrt{5}}, \frac{5-\sqrt{5}}{10}, -\sqrt{\frac{5+\sqrt{5}}{10}}), \\
x_5 = (\frac{1}{\sqrt{5}}, -\frac{5+\sqrt{5}}{10}, \sqrt{\frac{5-\sqrt{5}}{10}}), & x_6 = (\frac{1}{\sqrt{5}}, -\frac{5+\sqrt{5}}{10}, -\sqrt{\frac{5-\sqrt{5}}{10}}).
\end{array}$$

In this case, $\Delta(X) = \{1, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -1\}$.

Hyper-Cube : Since the length of the vectors of the hyper-cube H_n is $\sqrt{n}$. Each vector devided by $\sqrt{n}$ is in $S^{n-1} \subset \boldsymbol{R}^n$. The cube is a special case of this hyper-cube with $n = 3$.

Half Cube : Since a half cube is a subset of a hyper-cube, the embedding is given above.

1.4.2 Cube :

Octahedron :

Dodecahedron :

Icosahedron :

Hyper-Cube : We may assume that $\boldsymbol{x} = (1, \ldots, 1)$ and $\boldsymbol{y} = (1, \ldots, 1, -1, \ldots, -1)$ with h -1's. Then we have

$$P_{i,j}\widehat{(\boldsymbol{x}, \boldsymbol{y})} = (\alpha, \ldots, \alpha, \beta, \ldots, \beta)$$

with h β's , where

$$\begin{aligned}
\alpha &= \left(\binom{n-h-1}{l-1} - \binom{n-h-1}{l}\right)\binom{h}{i-l}, \\
\beta &= \left(\binom{h-1}{i-l} - \binom{h-1}{i-l-1}\right)\binom{n-h}{l},
\end{aligned}$$

with $2l = i + j - h$. Note that $P_{i,j}\widehat{(\boldsymbol{x}, \boldsymbol{y})} = \boldsymbol{0}$ if $i + j - h$ is odd.

Half Cube :

8.2 Chapter 2. Association Schemes

2.1.1

2.1.2

2.2.1

2.3.1

2.3.3

8.3 Chapter 3. P-polynomial and Q-polynomial Association Schemes

3.1.1

3.1.2

3.1.3

3.1.4

3.1.5

3.2.1

3.3.1

3.3.2

3.4.1

3.4.2

1. Compute the coefficient of x_h in the expression of $(x_1x_i)x_j = (x_1x_j)x_i$ in two ways, and we obtain the formula.
2. These follow directly from the definition of C-algebra and Exercise 3.3.2.
3. This is a direct consequence of 1 using the condition in Proposition 3.4.1 (iv).
4. We prove by induction on h. If $h = 0$, both hand sides become α_i. Now using 1 and induction hypothesis, we have

$$
\begin{aligned}
&\gamma_{h+1}\alpha^{i+h}_{i,h+1} \\
&\quad= \gamma_{i+1}\alpha^{i+h}_{i+1,h} + \alpha^{i+h}_{i,h}(\alpha_i - \alpha_h) \\
&\quad= \frac{\gamma_{i+1}}{\gamma_h}\alpha^{i+h}_{i+1,h-1}(\alpha_{i+1} + \cdots + \alpha_{i+h} - \alpha_1 - \cdots - \alpha_{h-1}) + \alpha^{i+h}_{i,h}(\alpha_i - \alpha_h) \\
&\quad= \frac{\gamma_{i+1}}{\gamma_h}\frac{\gamma_h\alpha^{i+h}_{i,h}}{\gamma_{i+1}}(\alpha_{i+1} + \cdots + \alpha_{i+h} - \alpha_1 - \cdots - \alpha_{h-1}) + \alpha^{i+h}_{i,h}(\alpha_i - \alpha_h) \\
&\quad= \alpha^{i+h}_{i,h}(\alpha_i + \cdots + \alpha_{i+h} - \alpha_1 - \cdots - \alpha_h).
\end{aligned}
$$

3.4.3

8.4 Chapter 4. Balanced Conditions

4.1.1

4.2.1

4.3.1

4.3.2

4.3.3

4.4.1

8.5 Chapter 5. Examples of Balanced Sets and Related Structures

5.2.1

5.3.1

8.6 Chapter 6. Parametrical Restrictions

6.2.1

6.3.1

5.1.1

8.7 Chapter 7. Group Schemes

7.2.1

7.2.2

7.2.3

7.2.4

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