Solutions to Take-Home Quiz 4 (*October 9, 2010*)

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$ and B be the matrices given below.

$$A = \begin{bmatrix} -3 & 5 & 0 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -2 & 1 \\ -3 & 5 & 1 & 0 \\ 2 & 1 & 2 & -1 \\ -1 & 2 & 3 & 3 \end{bmatrix}.$$

1. Determine $\det(A)$ by cofactor expansion along the <u>third column</u>.

   Sol. $\det(A) = A_{1,3}C_{1,3} + A_{2,3}C_{2,3} + A_{3,3}C_{3,3}$

   $$= 0(-1)^{1+3}\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)(-1)^{2+3}\begin{vmatrix} -3 & 5 \\ -1 & 2 \end{vmatrix} + 3(-1)^{3+3}\begin{vmatrix} -3 & 5 \\ 2 & 1 \end{vmatrix} = -1 - 39 = -40.$$

2. Find $\operatorname{adj}(A)$. (Solution only!)

   Sol. $C_{i,j} = (-1)^{i+j}M_{i,j}$ and $M_{i,j}$ is the determinant of the submatrix that remains after the ith row and jth column are deleted from A.

   $$\operatorname{adj}(A) = \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ C_{1,3} & C_{2,3} & C_{3,3} \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 0 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 0 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} -3 & 0 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & 5 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} -3 & 5 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & -15 & -5 \\ -5 & -9 & -3 \\ 5 & 1 & -13 \end{bmatrix}$$

3. Use Cramer's Rule to express x_3 as a quotient of two determinants for the equation $A\boldsymbol{x} = \boldsymbol{b}$, and evaluate x_3. (Solution only!)

   Sol.

   $$x_3 = \frac{\begin{vmatrix} -3 & 5 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} -3 & 5 & 0 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{vmatrix}} = \frac{-32}{-40} = \frac{4}{5}$$

4. Express $\det(B)$ by the cofactor expansion along the <u>first row</u> writing each of $C_{i,j}$ as a determinant. (Don't evaluate the determinant involved in $C_{i,j}$.)

   Sol. $\det(B) = 1 \cdot C_{1,1} + 0 \cdot C_{1,2} + (-2) \cdot C_{1,3} + 1 \cdot C_{1,4}$

   $$= \begin{vmatrix} 5 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 3 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} -3 & 5 & 0 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} -3 & 5 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix} = 40 - 2 \cdot (-40) - (-32) = 152.$$

5. Find $\det(B)$.

   Sol. See above.