Linear Algebra I *November 15, 2018*

Solutions to Final Exam 2018

(Total: 100 pts, 50% of the grade)

1. Let $\boldsymbol{u} = [1, -1, -2]^\top$, $\boldsymbol{v} = [-1, -2, 1]^\top$, $\boldsymbol{w} = [2, 1, -2]^\top$, $\boldsymbol{e}_1 = [1, 0, 0]^\top$, $\boldsymbol{e}_2 = [0, 1, 0]^\top$ and $\boldsymbol{e}_3 = [0, 0, 1]^\top$. (15 pts)

   (a) Find $\boldsymbol{u} \times \boldsymbol{v}$ and the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$. Show work.
   Solution.
$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} 1 & -1 & \boldsymbol{e}_1 \\ -1 & -2 & \boldsymbol{e}_2 \\ -2 & 1 & \boldsymbol{e}_3 \end{vmatrix} = \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}.$$
$$\text{Volume} = |(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}| = |(-5) \cdot 2 + 1 \cdot 1 + (-3) \cdot (-2)| = |-3| = 3.$$

   (b) Find (i) $\boldsymbol{u} \times \boldsymbol{v}$ and (ii) the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Show work.
   Solution. Let $B = [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$. Then
$$AB = [A\boldsymbol{u}, A\boldsymbol{v}, A\boldsymbol{w}] = [T(\boldsymbol{u}), T(\boldsymbol{v}), T(\boldsymbol{w})] = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] = I.$$
   By the Invertible Matrix Theorem, $A = B^{-1}$.
$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 & 1 & 0 \\ -2 & 1 & -2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{[1,3;1],[1;-1]} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ -1 & -2 & 1 & 0 & 1 & 0 \\ -2 & 1 & -2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{[2,1;1],[3,1;2],[2,3]}$$
$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & -2 & -2 & 0 & -1 \\ 0 & -2 & 1 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{[3,2;2],[3;-1/3]} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & -2 & -2 & 0 & -1 \\ 0 & 0 & 1 & 5/3 & -1/3 & 1 \end{bmatrix} \xrightarrow{[2,3;-2]}$$
$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 4/3 & -2/3 & 1 \\ 0 & 0 & 1 & 5/3 & -1/3 & 1 \end{bmatrix}. \quad A = \begin{bmatrix} -1 & 0 & -1 \\ 4/3 & -2/3 & 1 \\ 5/3 & -1/3 & 1 \end{bmatrix}.$$
   Since $\det(B) = -3$ by (a), it is not difficult to find $A = B^{-1}$ using $B^{-1} = -\frac{1}{3}\mathrm{adj}(B)$.

2. Consider the system of linear equations with augmented matrix $C = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7]$, where $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7$ are the columns of C. Let $A = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_6]$ be its coefficient matrix. We obtained a row echelon form G after applying a sequence of elementary row operations to the matrix C. (35 pts)
$$C = \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 1 & -4 \\ 2 & -3 & 12 & -4 & 3 & 3 & 0 \\ 0 & 1 & -2 & 0 & -1 & -1 & -6 \\ -3 & 0 & -9 & 3 & 1 & 5 & 9 \end{bmatrix}, \; G = \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 1 & -4 \\ 0 & 1 & -2 & 0 & -1 & -1 & -6 \\ 0 & 0 & 0 & -2 & 0 & -2 & -10 \\ 0 & 0 & 0 & 0 & 1 & 8 & -3 \end{bmatrix}.$$

   (a) Describe each step of a sequence of elementary row operations to obtain G from C by $[i, j], [i, j; c], [i; c]$ notation. Show work.
   Solution.
$$C \xrightarrow{[2,1;-2]} \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 1 & -4 \\ 0 & -3 & 6 & -2 & 3 & 1 & 8 \\ 0 & 1 & -2 & 0 & -1 & -1 & -6 \\ -3 & 0 & -9 & 3 & 1 & 5 & 9 \end{bmatrix} \xrightarrow{[4,1;3]} \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 1 & -4 \\ 0 & -3 & 6 & -2 & 3 & 1 & 8 \\ 0 & 1 & -2 & 0 & -1 & -1 & -6 \\ 0 & 0 & 0 & 0 & 1 & 8 & -3 \end{bmatrix}$$
$$\xrightarrow{[2,3]} \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 1 & -4 \\ 0 & 1 & -2 & 0 & -1 & -1 & -6 \\ 0 & -3 & 6 & -2 & 3 & 1 & 8 \\ 0 & 0 & 0 & 0 & 1 & 8 & -3 \end{bmatrix} \xrightarrow{[3,2;3]} \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 1 & -4 \\ 0 & 1 & -2 & 0 & -1 & -1 & -6 \\ 0 & 0 & 0 & -2 & 0 & -2 & -10 \\ 0 & 0 & 0 & 0 & 1 & 8 & -3 \end{bmatrix} = G.$$
   Hence the sequence of operations above is $[2, 1; -2]$, $[4, 1; 3]$, $[2, 3]$, $[3, 2; 3]$. There are many other solutions. One of them is $[2, 3]$, $[3, 1; -2]$, $[4, 1; 3]$, $[3, 2; 3]$.

(b) Find an invertible matrix P of size 4 such that $G = PC$ and express P as a product of <u>four</u> elementary matrices. Do not forget writing P. Show work.

Solution. P is the matrix obtained by applying the sequence of row operations $[2, 1; -2]$, $[4, 1; 3]$, $[2, 3]$, $[3, 2; 3]$ to the identity matrix of size 4 in this order. Hence,

$$\begin{aligned} P &= E(3,2;3)E(2,3)E(4,1;3)E(2,1;-2) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 3 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

(c) Explain that the invertible matrix P of size 4 such that $G = PC$ is unique.

Solution. Let $G = [\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4, \boldsymbol{g}_5, \boldsymbol{g}_6, \boldsymbol{g}_7]$, $G' = [\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_4, \boldsymbol{g}_5]$ and $C' = [\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_4, \boldsymbol{c}_5]$. Then

$$G' = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_4, \boldsymbol{g}_5] = P[\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_4, \boldsymbol{c}_5] = PC'$$

Since $|G'| = -2$, G' is invertible. Hence $G'^{-1}PC' = I$ and by the invertible matrix theorem C' is invertible. Thus $P = G'C'^{-1}$ and P is unique. (To show that C' is invertible, you can also quote a theorem stating that if a product of two square matrices is invertible, both of them are invertible.)

(d) Find the reduced row echelon form of the matrix C. Show work.

Solution.

$$C \to\to G \xrightarrow{[3;-1/2]} \begin{bmatrix} 1 & 0 & 3 & -1 & 0 & 1 & -4 \\ 0 & 1 & -2 & 0 & -1 & -1 & -6 \\ 0 & 0 & 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 8 & -3 \end{bmatrix} \xrightarrow{[1,3;1],[2,4;1]} \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 2 & 1 \\ 0 & 1 & -2 & 0 & 0 & 7 & -9 \\ 0 & 0 & 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 8 & -3 \end{bmatrix}.$$

(e) Find all solutions of the system of linear equations.

Solution. Let $x_3 = s$ and $x_6 = t$ be free parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3s - 2t + 1 \\ 2s - 7t - 9 \\ s \\ -t + 5 \\ -8t - 3 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 0 \\ 5 \\ -3 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ -7 \\ 0 \\ -1 \\ -8 \\ 1 \end{bmatrix}.$$

(f) Explain that the linear transformation defined by $T : \mathbb{R}^6 \to \mathbb{R}^4$ $(\boldsymbol{x} \mapsto A\boldsymbol{x})$, i.e., $T(\boldsymbol{x}) = A\boldsymbol{x}$ is NOT one-to-one.

Solution. Using the notation in (c), $[\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4, \boldsymbol{g}_5, \boldsymbol{g}_6]$ is an echelon form of A. Since $\boldsymbol{g}_3, \boldsymbol{g}_6$ are not pivot columns, the columns of A are not linearly independent and T is not one-to-one.

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. (20 pts)

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ -2 & -1 & -5 & 5 \\ 4 & 1 & 3 & -3 \\ -2 & 3 & 1 & 4 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 8 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Show work.

Solution.

$$\begin{aligned}\det(A) &= \begin{vmatrix} 2 & 0 & 1 & 4 \\ -2 & -1 & -5 & 5 \\ 4 & 1 & 3 & -3 \\ -2 & 3 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 1 & 4 \\ -1 & -1 & -5 & 5 \\ 2 & 1 & 3 & -3 \\ -1 & 3 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 1 & 4 \\ 1 & 0 & -2 & 2 \\ 2 & 1 & 3 & -3 \\ -7 & 0 & -8 & 13 \end{vmatrix} \\ &= -2\begin{vmatrix} 1 & 1 & 4 \\ 1 & -2 & 2 \\ -7 & -8 & 13 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 & 4 \\ 0 & -3 & -2 \\ 0 & -1 & 41 \end{vmatrix} = (-2)\cdot(-123-2) = 250.\end{aligned}$$

(b) Express x_4 as a quotient (*bun-su*) of determinants when $A\boldsymbol{x} = \boldsymbol{b}$, and write $\mathrm{adj}(A)$, the adjugate of A. Don't evaluate the determinants.

$$x_4 = \frac{\begin{vmatrix} 2 & 0 & 1 & 2 \\ -2 & -1 & -5 & 0 \\ 4 & 1 & 3 & 1 \\ -2 & 3 & 1 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 1 & 4 \\ -2 & -1 & -5 & 5 \\ 4 & 1 & 3 & -3 \\ -2 & 3 & 1 & 4 \end{vmatrix}} \left(= \frac{116}{250} = \frac{58}{125}\right), \quad \left(\mathrm{adj}(A) = \begin{bmatrix} 10 & 45 & 75 & -10 \\ -70 & 60 & 100 & 70 \\ 54 & -82 & -70 & -4 \\ 44 & -2 & -20 & 6 \end{bmatrix}\right)$$

$$\mathrm{adj}(A) = \begin{bmatrix} \begin{vmatrix} -1 & -5 & -5 \\ 1 & 3 & -3 \\ 3 & 1 & 4 \end{vmatrix}, & -\begin{vmatrix} 0 & 1 & 4 \\ 1 & 3 & -3 \\ 3 & 1 & 4 \end{vmatrix}, & \begin{vmatrix} 0 & 1 & 4 \\ -1 & -5 & -5 \\ 3 & 1 & 4 \end{vmatrix}, & -\begin{vmatrix} 0 & 1 & 4 \\ -1 & -5 & -5 \\ 1 & 3 & -3 \end{vmatrix} \\ -\begin{vmatrix} -2 & -5 & 5 \\ 4 & 3 & -3 \\ -2 & 1 & 4 \end{vmatrix}, & \begin{vmatrix} 2 & 1 & 4 \\ 4 & 3 & -3 \\ -2 & 1 & 4 \end{vmatrix}, & -\begin{vmatrix} 2 & 1 & 4 \\ -2 & -5 & 5 \\ -2 & 1 & 4 \end{vmatrix}, & \begin{vmatrix} 2 & 1 & 4 \\ -2 & -5 & 5 \\ 4 & 3 & -3 \end{vmatrix} \\ \begin{vmatrix} -2 & -1 & 5 \\ 4 & 1 & -3 \\ -2 & 3 & 4 \end{vmatrix}, & -\begin{vmatrix} 2 & 0 & 4 \\ 4 & 1 & -3 \\ -2 & 3 & 4 \end{vmatrix}, & \begin{vmatrix} 2 & 0 & 4 \\ -2 & -1 & 5 \\ -2 & 3 & 4 \end{vmatrix}, & -\begin{vmatrix} 2 & 0 & 4 \\ -2 & -1 & 5 \\ 4 & 1 & -3 \end{vmatrix} \\ -\begin{vmatrix} -2 & -1 & -5 \\ 4 & 1 & 3 \\ -2 & 3 & 1 \end{vmatrix}, & \begin{vmatrix} 2 & 0 & 1 \\ 4 & 1 & 3 \\ -2 & 3 & 1 \end{vmatrix}, & -\begin{vmatrix} 2 & 0 & 1 \\ -2 & -1 & -5 \\ -2 & 3 & 1 \end{vmatrix}, & \begin{vmatrix} 2 & 0 & 1 \\ -2 & -1 & -5 \\ 4 & 1 & 3 \end{vmatrix} \end{bmatrix}.$$

4. Let A, $\boldsymbol{v}$ and P be given below. (30 pts)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix}, \ \boldsymbol{v} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{bmatrix}, \ P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 & {x_5}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & {x_4}^3 & {x_5}^3 \\ {x_1}^4 & {x_2}^4 & {x_3}^4 & {x_4}^4 & {x_5}^4 \end{bmatrix}.$$

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + x^5$.

(a) Find the determinant of P. Show work. You may use the following:
$\det(Q) = (x_3 - x_2)(x_4 - x_2)(x_5 - x_2)(x_4 - x_3)(x_5 - x_3)(x_5 - x_4)$, where Q is the matrix obtained from P by deleting the 5th row and the 1st column.
Solution.

$$\begin{aligned}|P| &= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 & {x_5}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & {x_4}^3 & {x_5}^3 \\ {x_1}^4 & {x_2}^4 & {x_3}^4 & {x_4}^4 & {x_5}^4 \end{vmatrix} \cdots \text{Column operations } [2,1;-1]_c, [3,1;-1]_c, [4,1;-1]_c, [5,1;-1]_c \\ &= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ x_1 & x_2 - x_1 & x_3 - x_1 & x_4 - x_1 & x_5 - x_1 \\ {x_1}^2 & {x_2}^2 - {x_1}^2 & {x_3}^2 - {x_1}^2 & {x_4}^2 - {x_1}^2 & {x_5}^2 - {x_1}^2 \\ {x_1}^3 & {x_2}^3 - {x_1}^3 & {x_3}^3 - {x_1}^3 & {x_4}^3 - {x_1}^3 & {x_5}^3 - {x_1}^3 \\ {x_1}^4 & {x_2}^4 - {x_1}^4 & {x_3}^4 - {x_1}^4 & {x_4}^4 - {x_1}^4 & {x_5}^4 - {x_1}^4 \end{vmatrix} \ldots \text{First row cofactor expansion}\end{aligned}$$

$$
\begin{aligned}
&= \begin{vmatrix} x_2-x_1 & x_3-x_1 & x_4-x_1 & x_5-x_1 \\ {x_2}^2-{x_1}^2 & {x_3}^2-{x_1}^2 & {x_4}^2-{x_1}^2 & {x_5}^2-{x_1}^2 \\ {x_2}^3-{x_1}^3 & {x_3}^3-{x_1}^3 & {x_4}^3-{x_1}^3 & {x_5}^3-{x_1}^3 \\ {x_2}^4-{x_1}^4 & {x_3}^4-{x_1}^4 & {x_4}^4-{x_1}^4 & {x_5}^4-{x_1}^4 \end{vmatrix} \cdots [4,3;-x_1],[3,2;-x_1],[2,1;-x_1] \\
&= \begin{vmatrix} x_2-x_1 & x_3-x_1 & x_4-x_1 & x_5-x_1 \\ {x_2}^2-x_1x_2 & {x_3}^2-x_1x_3 & {x_4}^2-x_1x_4 & {x_5}^2-x_1x_5 \\ {x_2}^3-x_1{x_2}^2 & {x_3}^3-x_1{x_3}^2 & {x_4}^3-x_1{x_4}^2 & {x_5}^3-x_1{x_5}^2 \\ {x_2}^4-x_1{x_2}^3 & {x_3}^4-x_1{x_3}^3 & {x_4}^4-x_1{x_4}^3 & {x_5}^4-x_1{x_5}^3 \end{vmatrix} \cdots \text{Factoring each column} \\
&= (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_2 & x_3 & x_4 & x_5 \\ {x_2}^2 & {x_3}^2 & {x_4}^2 & {x_5}^2 \\ {x_2}^3 & {x_3}^3 & {x_4}^3 & {x_5}^3 \end{vmatrix} \\
&= (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)\det(Q) \cdots\cdots \text{Apply the formula for } \det(Q) \\
&= (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_3-x_2)(x_4-x_2)(x_5-x_2)(x_4-x_3)(x_5-x_3)(x_5-x_4) \\
&= \prod_{i=1}^{4}\prod_{j=i+1}^{5}(x_j-x_i).
\end{aligned}
$$

(b) Show the following: If $\boldsymbol{v}$ is an eigenvector of A corresponding to an eigenvalue α, then $\alpha=\lambda$ and $f(\alpha)=0$.
Solution.

$$
\alpha\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{bmatrix} = A\boldsymbol{v} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix}\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \\ -a_0-a_1\lambda-a_2\lambda^2-a_3\lambda^3-a_4\lambda^4 \end{bmatrix}.
$$

Hence, if $A\boldsymbol{v}=\alpha\boldsymbol{v}$, $\alpha=\lambda$ and $-a_0-a_1\lambda-a_2\lambda^2-a_3\lambda^3-a_4\lambda^4=\alpha\lambda^4=\lambda^5$. Therefore, $0=a_0+a_1\lambda+a_2\lambda^2+a_3\lambda^3+a_4\lambda^4+\lambda^5=f(\lambda)=f(\alpha)$. ■

(c) Suppose x_1,x_2,x_3,x_4,x_5 are distinct and $f(x_1)=f(x_2)=f(x_3)=f(x_4)=f(x_5)=0$. Show that A is diagonalizable.
Solution. If $f(x_1)=f(x_2)=f(x_3)=f(x_4)=f(x_5)=0$, then for $i=1,2,3,4,5$, $\boldsymbol{v}_i=[1,x_i,{x_i}^2,{x_i}^3,{x_i}^4]^\top$ is an eigenvector of A corresponding to the eigenvalue x_i by the equation in the solution for (b). Since x_1,x_2,x_3,x_4,x_5 are distinct, the set of eigenvectors $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4,\boldsymbol{v}_5\}$ is linearly independent and A is diagonalizable. (Since $P=[\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4,\boldsymbol{v}_5]$ and $|P|\neq 0$ when x_1,x_2,x_3,x_4,x_5 are distinct, P is invertible and

$$
AP=[A\boldsymbol{v}_1,A\boldsymbol{v}_2,A\boldsymbol{v}_3,A\boldsymbol{v}_4,A\boldsymbol{v}_5]=[x_1\boldsymbol{v}_1,x_2\boldsymbol{v}_2,x_3\boldsymbol{v}_3,x_4\boldsymbol{v}_4,x_5\boldsymbol{v}_5]=PD,
$$

where D is a diagonal matrix with x_1,x_2,x_3,x_4,x_4 in the main diagonal. Hence $P^{-1}AP=D$ and A is diagonalizable.) ■

(d) Show that $\det(xI-A)=f(x)$.
Solution.

$$
\begin{aligned}
&|xI-A| \\
&= \begin{vmatrix} x & -1 & 0 & 0 & 0 \\ 0 & x & -1 & 0 & 0 \\ 0 & 0 & x & -1 & 0 \\ 0 & 0 & 0 & x & -1 \\ a_0 & a_1 & a_2 & a_3 & x+a_4 \end{vmatrix} = x\begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a_1 & a_2 & a_3 & x+a_4 \end{vmatrix} + a_0\begin{vmatrix} -1 & 0 & 0 & 0 \\ x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \end{vmatrix} \\
&= x^2\begin{vmatrix} x & -1 & 0 \\ 0 & x & -1 \\ a_2 & a_3 & x+a_4 \end{vmatrix} - a_1x\begin{vmatrix} -1 & 0 & 0 \\ x & -1 & 0 \\ 0 & x & -1 \end{vmatrix} + a_0 = x^3\begin{vmatrix} x & -1 \\ a_3 & x+a_4 \end{vmatrix} + a_2x^2+a_1x+a_0 \\
&= x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0 \\
&= f(x).
\end{aligned}
$$

■