The Terwilliger Algebra of a Directed Hamming Graph

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Association Schemes

Definition 1

Let X be a nonempty finite set, and $X \times X = X_0 \cup X_1 \cup \cdots \cup X_d$ be a partition of $X \times X$. Let A_i be matrices rows and columns indexed by X such that $A_i[x, y] = 1$ if $(x, y) \in X_i$ and 0 otherwise. If $A_0, A_1, \ldots, A_d$ satisfy the following conditions, then X with its partition becomes an *association schemes*.

1. $A_0 = I$, the identity matrix.
2. $A_i^\top = A_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$.
3. For each $i, j \in \{0, 1, \ldots, d\}$, there exist constants $p_{i,j}^0, \ldots, p_{i,j}^d$ such that $A_iA_j = \sum_{k=0}^d p_{i,j}^k A_k$.

The algebra $\mathcal{B} = \langle A_0, A_1, \ldots, A_d \rangle \subseteq \mathrm{Mat}_X(\mathbb{C})$ is called the *Bose-Mesner algebra* of the association schme. If $\mathcal{B}$ is commutative, it is called a *commutative* association scheme.

Examples of Association Schemes

$\mathcal{H}(H\backslash G/H)$

Let H be a subgroup of a finite group G. Let $X = G/H$ be the set of left cosets. The orbits of G on $X \times X$ define an association scheme with respect to the following action.

$$g : X \times X \to X \times X : \ (g_1H, g_2H) \mapsto (gg_1H, gg_2H) \quad \text{for } g, g_1, g_2 \in G.$$

$\mathcal{H}(G_x\backslash G/G_x)$

Let G be a transitive permutation group on a finite set X. Then, the orbits of G on $X \times X$ (called orbitals) define an association scheme.

Commutativity

The association scheme defined above is commutative if the permutation character of the permutation representation is multiplicity-free.

Primitive Idempotents

Let $\mathcal{B} = \langle A_0, A_1, \ldots, A_d \rangle \subseteq \mathrm{Mat}_X(\mathbb{C})$ be the Bose-Mesner algebra of a commutative association scheme with $n = |X|$. Then there are primitive idempotents $E_0, E_1, \ldots, E_d$ satisfying the following.

(*i*) $E_0 = \frac{1}{n}J$, where J is the all one's matrix of size n, and $E_iE_j = \delta_{i,j}E_i$ for all $i, j \in \{0, 1, \ldots, d\}$.

(*ii*) $E_j^\top = E_{\hat{j}}$ for some $\hat{j} \in \{0, 1, \ldots, d\}$.

(*iii*) For $i, j \in \{0, 1, \ldots, d\}$, there exists $q^k_{i,j} \in \mathbb{R}$ such that $E_i \circ E_j = \frac{1}{n}\sum_{k=0}^{n} q^k_{i,j}E_k$, where $\circ$ is the entry-wise product, called the Hadamard product.

(*iv*) For $i, j \in \{0, 1, \ldots, d\}$, there exists $p_i(j), q_j(i) \in \mathbb{R}$ such that

$$A_i = \sum_{j=0}^{d} p_i(j)E_j, \quad E_j = \frac{1}{n}\sum_{i=0}^{d} q_j(i)A_i.$$

Definition 2 (DRGs and WDRGs)

- Let $\Gamma = (X, \tilde{E})$, where $\tilde{E}$ is a set of pairs of X. Γ is a *distance-regular graph* if Γ is connected of diamter d, and the following partition defines an association scheme. Here, $\partial(x, y)$ is the path distance.

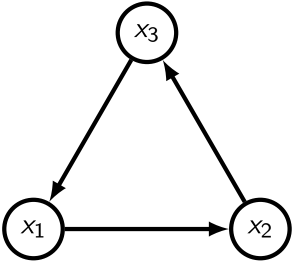
$$X \times X = \bigcup_{i=0}^{d} \Delta_i,\ \Delta_i = \{(x, y) \mid \partial(x, y) = i\}.$$

- Let $\Gamma = (X, E)$, where E is a set of ordered pairs of X. Γ is a *weakly distance-regular digraph* if Γ is strongly connected, and the following partition defines a <u>nonsymmetric</u> association scheme.

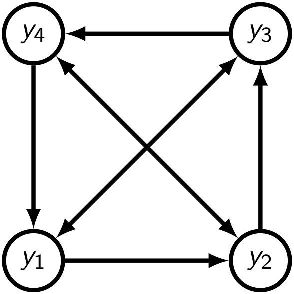
$$X \times X = \bigcup_{(i,j)\in\Delta} \Delta_{i,j},\ \Delta_{i,j} = \{(x, y) \mid \widetilde{\partial}(x, y) = (i, j)\},$$

where $\widetilde{\partial}(x, y)$ denotes the two-way distance of x and y, and $\Delta(\Gamma) = \{\widetilde{\partial}(x, y) \mid x, y \in X\}$.

WDRDGs over K_3 and K_4



$$\Delta = \{(0,0),(1,2),(2,1)\}$$



$$\Delta = \{(0,0),(1,1),(1,2),(2,1)\}$$

WDRDGs over a Clique (Semicomplete WDRDGs)

Proposition 1 (Y. Yang, Q. Zeng and K. Wang [4, Proposition 3.5])

Let $\Gamma = (X, E)$ be a weakly distance-regular digraph (wdrdg) of diameter d and girth g whose underlying graph is a complete graph. Then $d = 2$, $g \leq 3$ and

$$\Delta(\Gamma) = \{(0,0),(1,2),(2,1)\}, \text{ or } \Delta(\Gamma) = \{(0,0),(1,1),(1,2),(2,1)\}.$$

Proof.

Suppose $x, y \in X$ with $\tilde{\partial}(x, y) = (1, p)$ and $p \geq 3$. Note that $X = \{x\} \cup \Gamma_{1,*}(x) \cup \Gamma_{*,1}(x)$. If $z \in \Gamma_{1,*}(y) \setminus (\{x\} \cup \Gamma_{1,*}(x))$, then

$$3 \leq p = \partial(y, x) \leq \partial(y, z) + \partial(z, x) = 2,$$

which is absurd. Hence, $\{y\} \cup \Gamma_{1,*}(y) \subseteq \Gamma_{1,*}(x)$. A contradiction. $\square$

Two Types of Semicomplete WDRDGs

Type I: $d = 2$, $g = 3$, and $\Delta(\Gamma) = \{(0,0), (1,2), (2,1)\}$

- The existence is equivalent to that of a nonsymmetirc association scheme with two classes with $4m - 1$ vertices.
- It exists if and only if a skew Hadamard matrix of size $4m$ exists.

Type II: $d = 2$, $g = 2$, and $\Delta(\Gamma) = \{(0,0), (1,1), (1,2), (2,1)\}$

Let $\mathcal{B} = \langle I, A, A^\top, B = B^\top \rangle$ be the Bose-Mesner Algebra of a nonsymmetric association scheme of class three.

- If the digraph defined by A is connected, it is semicomplete of type II.
- If the digraph defined by A is not connected, each connected component is a semicomplete wdrdg of type I.

Every association scheme of class at most four is commutative.

Weakly Distance-Regular Digraphs of Hamming Type

Theorem 2 (Y. Yang, Q. Zeng and K. Wang [4])

Let Γ be a commutative weakly distance-regular digraph. Then Γ has a Hamming graph as its underlying graph if and only if Γ is isomorphic to one of the following diagrams:

(i) $\mathrm{Cay}(\mathbb{Z}_4, \{1\})$ — $[H(2,2)]$;

(ii) $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1,0),(0,1)\})$ — $[H(3,2)]$;

(iii) Δ^1 *or* $\Delta^1 \square \Delta^2$ — $[H(1,q)$ *or* $H(2,q)]$;

(iv) $\Gamma^1 \square \Gamma^2 \square \cdots \square \Gamma^d$ — $[H(d,q)$ *with* $q \equiv 3 \pmod 4)]$.

Here, Δ^i (resp Γ^i) is a semicomplete weakly distance-regular digraph of girth 2 (resp 3) with the same intersection numbers for each i.

The cases when the underlying graph is a folded cube and a Doob graph are determined. $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_4, \{(1,0),(0,1),(-1,-1)\})$ is the only nontrivial case comming from the Shrikhande graph.

Definition 3 ($H(d, q)$)

Let $N = \{a_1, a_2, \ldots, a_n\}$. The Hamming graph $H(d, n)$ or more specifically, the Hamming graph of diameter d on the set N, $H(d, N)$ is defined by X, the set of vertices, and $\tilde{E}$, the set of edges.

$$\begin{aligned} X &= \{(x_1, \ldots, x_d) \mid \text{ for all } i,\ x_i \in N\}, \\ \tilde{E} &= \{xy \mid \text{exactly 1 coordinate } i,\ x_i \neq y_i\} \subseteq X \times X. \end{aligned}$$

Definition 4 ($H^*(d, 3)$)

The directed graph $H^*(d, 3)$ is defined by X, the set of vertices, and E, the set of arcs.

$$\begin{aligned} X &= \{(x_1, \ldots, x_d) \mid \text{ for all } i,\ x_i \in \mathbb{F}_3\}, \\ E &= \{xy \mid \text{exactly 1 coordinate } i,\ x_i + 1 = y_i\} \subseteq X \times X. \end{aligned}$$

(We assign the trivial direction on each coordinate, $0 \to 1 \to 2 \to 0$.)

Adjacecy Matrix

Definition 5

The adjacency matrix of $H^*(d,3)$, $A = A^{(d)}$ is defined by the following.

$$(A)_{ij} = \begin{cases} 1 & \text{if there exits an arc from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

For x, y in X, $\partial(x,y)$ denotes the distance between x and y, i.e., the smallest number of arcs connecting from x to y. We also define the two-way distance $\tilde{\partial}(x,y) = (\partial(x,y), \partial(y,x))$, and Δ, the set of all two-way distances;

$$\Delta = \{\tilde{\partial}(x,y) \mid x, y \in X\}.$$

Vector Space V

Let $\Gamma = (X, E)$ be $H^*(d, 3)$. Fix a base vertex $x = (0, 0, \ldots, 0) \in X$.

- $V = \mathbb{C}^{|X|}$: the vector space over the complex number field $\mathbb{C}$ whose coordinates are indexed by the elements of X.
- $X_{i,j} = \{y \mid \partial(x, y) = i, \partial(y, x) = j\}$.
- If $y \in X$ has s ones, t twos, and $r = d - s - t$ zeros, then $\partial(x, y) = s + 2t, \partial(y, x) = 2s + t$.
- If $\tilde{\partial}(x, y) = (i, j)$, then $s = (2j - i)/3$, and $t = (2i - j)/3$.
- We also write

$$X_{[s,t]} = \{y \mid \text{there are } s \text{ ones and } t \text{ twos}\} = X_{s+2t,2s+t}.$$

$E^*_{i,j}$

For $(i,j) \in \Delta$, $E^*_{i,j} = E^{(d)*}_{i,j}$ denotes a diagonal matrix such that

$$E^*_{i,j}(z,z) = \begin{cases} 1, & \text{if } \partial(x,z) = (i,j), \\ 0 & \text{othersise}, \end{cases}$$

and the zero matrix of the same size if $(i,j) \notin \Delta$. Then,

$$E^*_{i,j}\mathbf{1} = \{\sum \hat{y} \mid y \in X, \partial(x,y) = i, \partial(y,x) = j\}.$$

We set

$$E^*_{i,j} = E^*_{[(2j-i)/3,(2i-j)/3]}, \text{ and } E^*_{[s,t]} = E^*_{s+2j,2s+j}.$$

If a vector $\boldsymbol{v} \in V$ satisfies $E^*_{[s,t]}\boldsymbol{v} = \boldsymbol{v}$, then we write

$$\text{type}(\boldsymbol{v}) = [r(\boldsymbol{v}), s(\boldsymbol{v}), t(\boldsymbol{v})]: \text{ the type of } \boldsymbol{v}.$$

Note that $\boldsymbol{v}$ can be written as a linear combination of $\hat{y}$ with $y \in X_{[s,t]}$.

The Terwilliger Algebra $\mathcal{T}(x)$

Definition 6

The Terwilliger algebra $\mathcal{T}(x)$ of $H^*(d,3)$ with respect to a base vertex x is a $\mathbb{C}$-algebra generated by A, $A^\top$ and $E^*_{i,j}$ for $(i,j) \in \Delta$. We also write $\mathcal{T}(H^*(d,3))$ to specify the digraph.

Note

1. $\Gamma = H^*(d,3)$ is a Cayley digraph defined on an abelian group $\mathbb{Z}_3^d$. Hence, it is commutative.
2. It is easy to calculate eigenvalues of $A = A^{(d)}$.
3. $H^*(1,3)$ is a directed triangle, and the orientation of arcs of $H^*(d,3)$ is uniquely determined.
4. $A^{(d)} = A^{(1)} \otimes I \otimes \cdots \otimes I + I \otimes A^{(1)} \otimes \cdots \otimes I + I \otimes I \otimes \cdots \otimes A^{(1)}$.

$\mathcal{T}(H^*(1,3))$

Lemma 7

Let $\mathcal{T}_1$ be the Terwilliger algebra of $H^*(1,3)$. Then

$$\mathcal{T}_1 = \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle \simeq \mathrm{Mat}_3(\mathbb{C}).$$

Proof.

This is trivial. If $A^{(1)}$ is the adjacenct matrix of $H^*(1,3)$, then ${A^{(1)}}^2 = {A^{(1)}}^\top$, and ${E^{(1)}}^*_{0,0}$, ${E^{(1)}}^*_{1,2}$ and ${E^{(1)}}^*_{2,1}$ are diagonal matrix units $e_{1,1}$, $e_{2,2}$ and $e_{3,3}$. Hence, every matrix unit is in $\mathcal{T}_1$. $\square$

Note that if the principal module of dimension n is the only irreducible module of the algebra, it is isomorphic to $\mathrm{Mat}_n(\mathbb{C})$.

The Algebraic Structure of $\mathcal{T}(x)$

Theorem 3 (T.Miezaki, H.Suzuki, K.Uchida [5])

Let $\mathcal{T}(x)$ be the Terwilliger algebra of $H^*(d,3)$ with respect to a base vertex x. Then,

$$\mathcal{T}(x) \simeq \mathrm{Sym}^d(\mathrm{Mat}_3(\mathbb{C})).$$

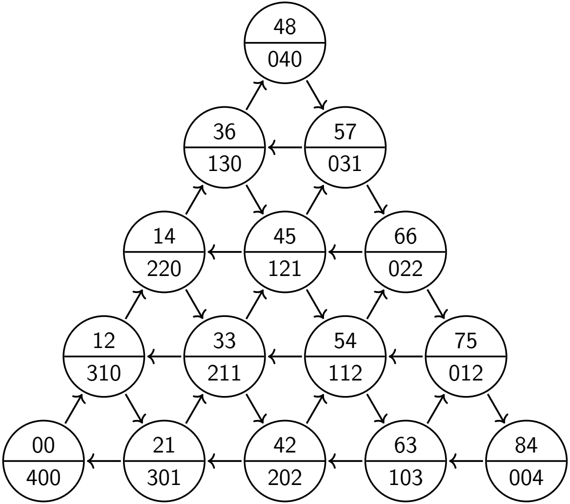
Moreover,

$$\mathcal{T}(x) \simeq \mathrm{Sym}^d(\mathrm{Mat}_3(\mathbb{C})) \simeq \bigoplus_{n\in\Lambda} \mathrm{Mat}_n(\mathbb{C}), \text{ where}$$

$$\Lambda = \left\{ \frac{1}{2}(d-3\ell-2m+1)(m+1)(d-3\ell-m+2) \mid \right.$$
$$\left. 0 \le \ell \le \left[\frac{d}{3}\right], 0 \le m \le \left[\frac{d-3\ell}{2}\right] \right\}.$$

Example: $H^*(4,3)$

Two-way distance: $\tilde{\partial}(x,y) = (i,j)$, and $\mathrm{type}(\boldsymbol{v}) = [r(\boldsymbol{v}), s(\boldsymbol{v}), t(\boldsymbol{v})]$



Elements in $\mathcal{T}(x)$

Let $r = d - s - t$.

$$H_1 = \sum_{s,t}(r-s)E^*_{[s,t]},\ H_2 = \sum_{s,t}(s-t)E^*_{[s,t]},$$

$$H_3 = H_1 + H_2,$$

$$R_1 = \sum_{s,t} E^*_{[s+1,t]}AE^*_{[s,t]},\ R_2 = \sum_{s,t} E^*_{[s-1,t+1]}AE^*_{[s,t]},$$

$$R_3 = \sum_{s,t} E^*_{[s,t+1]}A^\top E^*_{[s,t]},$$

$$L_1 = \sum_{s,t} E^*_{[s-1,t]}A^\top E^*_{[s,t]},\ L_2 = \sum_{s,t} E^*_{[s+1,t-1]}A^\top E^*_{[s,t]},$$

$$L_3 = \sum_{s,t} E^*_{[s,t-1]}AE^*_{[s,t]}.$$

Note that $H_1, H_2, H_3, R_1, R_2, R_3, L_1, L_2, L_3$ are all in $\mathcal{T}(x)$.

Lemma 8

We have the following.

(i) $A = R_1 + R_2 + L_3$ *and* $A^\top = L_1 + L_2 + R_3$.

(ii) *Let* $\tilde{A} = A + A^\top$. *Then* $\tilde{A}$ *is the adjacency matrix of* $H(d,3)$.

(iii) $\tilde{R} = R_1 + R_3$ *is the raising operator,* $\tilde{F} = R_2 + L_2$ *the flat operator, and* $\tilde{L} = L_1 + L_3$ *the lowering operator of* $H(d,3)$.

For matrices M_1, M_2, $[M_1, M_2] = M_1 M_2 - M_2 M_1$. Then $sl_n(\mathbb{C}) = \{M \in \mathrm{Mat}_n(\mathbb{C}), \mathrm{tr}(M) = 0\}$ becomes a simple Lie algebra.

Lemma 9

The following hold.

(i) $[L_1, R_1] = H_1$, $[H_1, L_1] = 2L_1$, *and* $[H_1, R_1] = -2R_1$.

(ii) $[L_2, R_2] = H_2$, $[H_2, L_2] = 2L_2$, *and* $[H_2, R_2] = -2R_2$.

(iii) $[L_3, R_3] = H_3$, $[H_3, L_3] = 2L_3$, *and* $[H_3, R_3] = -2R_3$.

The Relation Between $\mathcal{T}(x)$ and $sl_3(\mathbb{C})$

Proposition 4

The following hold.

(i) *The Lie algebra generated by $H_1, H_2, L_1, L_2, R_1, R_2$ is isomorphic to $sl_3(\mathbb{C})$ with a Cartan subalgebra generated by H_1 and H_2, and a Borel subalgebra generated by H_1, H_2, L_1, L_2, L_3.*

(ii) *The Lie algebra generated by the triple H_1, L_1, R_1 (resp. the triple H_2, L_2, R_2 and H_3, L_3, R_3) is isomorphic to $sl_2(\mathbb{C})$ with a Cartan subalgebra generated by H_1 (resp. H_2, H_3) and a Borel subalgebra generated by H_1, L_1 (resp. H_2, L_2, and H_3, L_3).*

Associative algebra $\mathcal{T}(x)$ and the associated Lie algebra

$$\mathcal{T}(x) = \langle A, A^\top, E^*_{i,j} \mid (i,j) \in \Delta(\Gamma) \rangle = \langle L_1, L_2, R_1, R_2 \rangle \supseteq \langle L_1, L_2, R_1, R_2 \rangle_{Lie}.$$

$\mathcal{T}(x)$-modules and $sl_3(\mathbb{C})$-modules

Proposition 5

The following hold. $\mathbb{C}[A] = \mathbb{C}[A^\top] = \mathbb{C}[A, A^\top]$ *is the Bose-Mesner algebra of* $H^*(d, 3)$.

Proposition 6

The following hold.

(i) $\mathcal{T} = \mathcal{T}(x) = \langle A, E^*_{i,j} \mid (i,j) \in \Delta \rangle$.

(ii) $\mathcal{T}(x) = \langle L_1, L_2, R_1, R_2 \rangle \supseteq \langle L_1, L_2, R_1, R_2 \rangle_{Lie}$.

(iii) *Let* $\mathcal{L}$ *be the Lie algebra generated by* L_1, L_2, R_1, R_2. *Then* $\mathcal{L} \simeq sl_3(\mathbb{C})$, *and* $\mathcal{L}$ *submodules of* V *are* $\mathcal{T}$ *submodules and vice versa. Moreover, an* $\mathcal{L}$*-submodule* W *of* V *is irreducible if and only if a* $\mathcal{T}$*-module* W *of* V *is irreducible.*

Poincaré–Birkhoff–Witt Theorem and $\mathcal{U}(\mathcal{L})$

Proposition 7 ([1, Theorem on page 108])

Let $\mathcal{L} = \langle L_1, L_2, L_2, R_1, R_2, R_3 \rangle_{Lie}$, *and* W *an irreducible submodule of* V. *Then there is a highest weight vector* $\boldsymbol{v} \in W$ *satisfying* $L_1\boldsymbol{v} = L_2\boldsymbol{v} = L_3\boldsymbol{v} = \boldsymbol{0} \neq \boldsymbol{v}$, *determined up to nonzero scalar multiple, and* W *is spanned by the set of vectors*

$$\{R_3^k R_1^j R_2^i \boldsymbol{v} \mid i, j, k \geq 0\}.$$

Higest weight vectors

$$H_1 = \sum_{s,t} (r - s) E^*_{[s,t]}, \quad H_2 = \sum_{s,t} (s - t) E^*_{[s,t]}.$$

Hightst weight vector and the weight of $\boldsymbol{v} = E^*_{[s,t]}\boldsymbol{v}$

Definition 10

Let $\boldsymbol{v} \in V$ be a nonzero vector of V satisfying $H_1\boldsymbol{v} = m_1\boldsymbol{v}$ and $H_2\boldsymbol{v} = m_2\boldsymbol{v}$. Then we call $\lambda = (m_1, m_2)$ the weight of $\boldsymbol{v}$.

Lemma 11

Let W be an irreducible $\mathcal{L}$ submodule of V with the heighest weight vector $\boldsymbol{v} = E^*_{[s,t]}\boldsymbol{v}$ with weight $\lambda = (m_1, m_2)$. Set $r = d - s - t$, and $\boldsymbol{v}_{i_1,i_2,i_3} = R_3^{i_3}R_1^{i_1}R_2^{i_2}\boldsymbol{v}$. Then, the following hold.

(i) *The weight of $\boldsymbol{v}_{i_1,i_2,i_3}$ is $(r - s - 2i_1 + i_2 - i_3, s - t + i_1 - 2i_2 - i_3)$. In particuar, $m_1 = r - s$ and $m_2 = s - t$.*

(ii) $\boldsymbol{v}_{i_1,i_2,i_3} = R_3^{i_3}R_1^{i_1}R_2^{i_2}\boldsymbol{v} \neq \boldsymbol{0}$ *if and only if* $0 \leq i_2 \leq s - t$, $0 \leq i_1 \leq r - s + i_2$ *and* $0 \leq i_3 \leq r - t - i_1 - i_2$.

Highest weights

Lemma 12

Suppose $\lambda = m_1\lambda_1 + m_2\lambda_2$ be the heighest weight of an irreducible $\mathcal{L}$ submodule W of V, and $\boldsymbol{v} = E^*_{[s,t]}\boldsymbol{v}$ a highest weight vector. Set $r = d - s - t$. Then, the following hold.

(i) $(r, s, t) = (m_1 + m_2 + t, m_2 + t, t)$, *and* $r \geq s \geq t$.

(ii) $t \in \{0, 1, \ldots, [\frac{d}{3}]\}$.

(iii) $m_1 = r - s = d - 3t - 2m_2$.

(iv) $m_2 = s - t \in \{0, 1, \ldots, [\frac{d-3t}{2}]\}$.

The structure of an irreducible module

Lemma 13

Let $\boldsymbol{v}$ be a heigest weight vector with weight $\lambda = (m_1, m_2)$ of an irreducible $\mathcal{L}$ submodule of V. Let $\boldsymbol{v}_{i,j,k} = R_3^k R_1^j R_2^i \boldsymbol{v}$ for $0 \leq i \leq m_2,\ 0 \leq j \leq m_1 + i,\ 0 \leq k \leq m_1 + m_2 - i - j$. Let

$$W_{i,j} = \mathrm{Span}(\boldsymbol{v}_{i,j,k} \mid 0 \leq k \leq m_1 + m_2 - i - j),$$

and

$$U_i = \mathrm{Span}(\boldsymbol{v}_{i,j,k} \mid 0 \leq j \leq m_1, 0 \leq k \leq m_1 + m_2 - i - j).$$

Then $W_{i,j}$ is an irreducible $\mathcal{L}_3 = \langle L_3, R_3 \rangle \simeq sl_2(\mathbb{C})$ module of heigest weight $m_1 + m_2 - i - j$ with a heighest weight vector $\boldsymbol{v}_{i,j,0}$ of dimension $m_1 + m_2 - i - j + 1$. Moreover, $W_{i,j}$ and $W_{i',j'}$ are orthogonal if $i + j \neq i' + j'$.
We also have that $R_1 \boldsymbol{v}_{i,m_1,0} \in U_{i-1} + \cdots + U_0$.

The highest weight and the dimension of an irreducible module

Lemma 14

For each (r, s, t) with $r \geq s \geq t$, there is a highest weight vector $\boldsymbol{v}$ with $\lambda = (m_1, m_2) = (r - s, s - t)$ such that $(r(\boldsymbol{v}), s(\boldsymbol{v}), t(\boldsymbol{v})) = (r, s, t)$. Every highest-weight vector is of this type.

Proposition 8

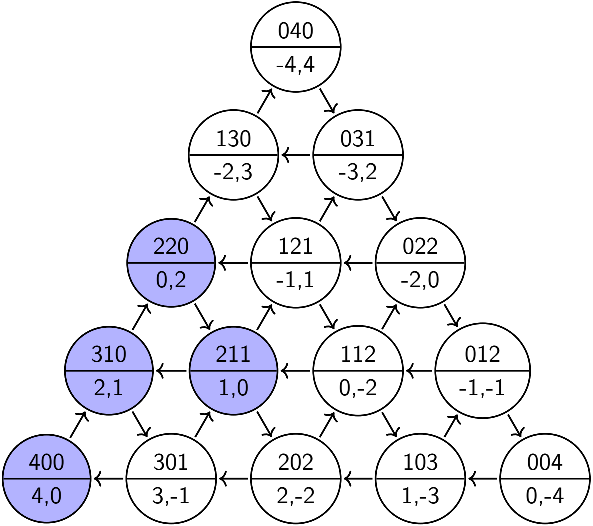
Let $\boldsymbol{v}$ be a heigest weight vector with weight $\lambda = (m_1, m_2)$ of an irreducible $\mathcal{L}$ submodule W of V. Let $\boldsymbol{v}_{i,j,k} = R_3^k R_1^j R_2^i \boldsymbol{v}$. Then the set

$$\{\boldsymbol{v}_{i,j,k} \mid 0 \leq i \leq m_2, 0 \leq j \leq m_1, 0 \leq k \leq m_1 + m_2 - i - j\}$$

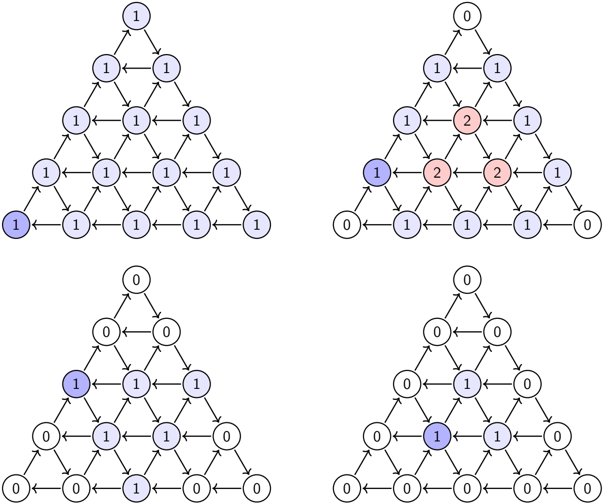
forms a basis of W. Moreover, $\dim W = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)$.

Example: $H^*(4,3)$

type($\boldsymbol{v}$) $= [r(\boldsymbol{v}), s(\boldsymbol{v}), t(\boldsymbol{v})$, and weight($\boldsymbol{v}$) $= (\ell, m) = (r(\boldsymbol{v}) - s(\boldsymbol{v}), s(\boldsymbol{v}) - t(\boldsymbol{v}))$)



Example: $H^*(4, 3)$ - Irreducible Modules



Examples for small d

Example 15

For a small d, we list (r, s, t) of the highest weight vector and the highest weight $\lambda = (m_1, m_2)$ as $[r, s, t], (m_1, m_2) : \dim, m$, where dim denotes the dimension of the corresponding irreducible module, and m the multiplicity in the standard module in the following.

- $d = 1$: $[0, 0, 0], (1, 0) : 3, 1$.
- $d = 2$: $[2, 0, 0], (2, 0) : 6, 1$, $[1, 1, 0], (0, 1) : 3, 1$.
- $d = 3$: $[3, 0, 0], (3, 0) : 10, 1$, $[2, 1, 0], (1, 1) : 8, 2$, $[1, 1, 1], (0, 0) : 1, 1$
- $d = 4$: $[4, 0, 0], (4, 0) : 15, 1$, $[3, 1, 0], (2, 1) : 15, 3$, $[2, 2, 0], (0, 2) : 6, 2$, $[2, 1, 1], (1, 0) : 3, 3$.
- $d = 5$: $[5, 0, 0], (5, 0) : 21, 1$, $[4, 1, 0], (3, 1) : 24, 4$, $[3, 2, 0], (1, 2) : 15, 5$, $[3, 1, 1], (2, 0) : 6, 6$, $[2, 2, 1], (0, 1) : 3, 5$.

Algebra generated by two matrices

Theorem 9

A and A^* Let A be the adjacency matrix of $H^*(d,3)$. Let $\omega \in \mathbb{C}$ such that $\omega^2+\omega+1=0$ and

$$A^* = \sum_{s,t}((d-s-t)+s\omega+t\omega^2)E^*_{[s,t]}.$$

Then, the following hold.

$$[A,[A,[A,A^*]]] = (1-\omega)^3 A^*$$
$$[A^*,[A^*,[A^*,A]]] = -(1-\omega)^3 A.$$

The Algebra defined by generators A, B, and relations

$$\mathcal{L} = \langle A, B \mid [A,[A,[A,B]]] = B,\ [B,[B,[B,A]]] = A\rangle$$

Special type of distance-regular graphs

Q-polynomial distance-regular graphs

Most of the distance regular graphs of large diameter satisfy the following. There is a primitive idempotent E with the following property by setting

$$A^* = \frac{1}{|X|}\text{diagonal}(E_{0,0}, E_{0,1}, \ldots, E_{0,d}).$$

$$[A, [A, [A, A^*]]] = b^2[A, A^*]$$
$$[A^*, [A^*, [A^*, A]]] = b^{*2}[A^*, A].$$

Onsagar Algebra

Let $\mathcal{O}$ be a Lie algebra generated by A and A^* with the relation above. If A and A^* act on a finite dimensional algebra as linear transformations. Then A, A^* acts on V as a tridiagonal pair.

Problems

1. Find the multiplicity formula of the standard module of $H^*(d, 3)$.
2. Study the structure of the irreducible modules of the Terwilliger algebra of $H^*(d, q)$, ($q \equiv 3 \pmod 4$).
3. Study finite dimensional irreducible module of the following algebra generated by two elements and two relations.

$$\mathcal{L} = \langle A, B \mid [A, [A, [A, B]]] = B,\ [B, [B, [B, A]]] = A \rangle$$

4. Develop an algebraic theory of weakly distance-regular digraphs, and find a good class. Or, find a good class of commutative but not symmetric assocation scheme.
5. Are there noncommutative weakly distance-regular digraphs?

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THANK YOU!!

Matrices, Part I

Let $r = d - s - t$.

$$H_1 = \sum_{s,t}(r-s)E^*_{[s,t]} \sim h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$H_2 = \sum_{s,t}(s-t)E^*_{[s,t]} \sim h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$H_3 = H_1 + H_2, \text{ where } r = d - s - t,$$

$$R_1 = \sum_{s,t} E^*_{[s+1,t]}AE^*_{[s,t]} \sim e_{-s} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$R_2 = \sum_{s,t} E^*_{[s-1,t+1]}AE^*_{[s,t]} \sim e_{-t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

Matrices - Part II

$$R_3 = \sum_{s,t} E^*_{[s,t+1]} A^\top E^*_{[s,t]} \sim e_{-s-t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$L_1 = \sum_{s,t} E^*_{[s-1,t]} A^\top E^*_{[s,t]} \sim e_s = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L_2 = \sum_{s,t} E^*_{[s+1,t-1]} A^\top E^*_{[s,t]} \sim e_t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L_3 = \sum_{s,t} E^*_{[s,t-1]} A E^*_{[s,t]} \sim e_{-s-t} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $H_1, H_2, H_3, R_1, R_2, R_3, L_1, L_2, L_3$ are all in $\mathcal{T}(x)$.

$[A, [A, [A, B]]] = B, \ [B, [B, [B, A]]] = A$

Recall $[M, N] = MN - NM$. Hence,

$$[A, [A, [A, B]]] = A^3B - 3A^2BA + 3ABA^2 - BA^3.$$

Suppose A, B are diagonalizable linear transformation on a finite dimentional vector space V, and E_i (resp. F_j) are projections onto the eigenspaces written as a polynomial in A (resp. B). satisfying $AE_i = E_iA = \phi_i E_i$ and $BF_j = F_jB = \psi_j F_j$.
Then, $[A, [A, [A, B]]] = B$ implies

$$\begin{aligned} E_iBE_j &= E_i[A, [A, [A, B]]]E_j = (\phi_i^3 - 3\phi_i^2\phi_j + 3\phi_i\phi_j^2 - \phi_j^3)E_iA^*E_j \\ &= (\phi_i - \phi_j)^3 E_iA^*E_j. \end{aligned}$$

Therefore, if $[A, [A, [A, B]]] = B$, and $[B, [B, [B, A]]] = A$,

$$E_iBE_j \neq O \Rightarrow \phi_i - \phi_j \in \{1, \omega, \omega^2\}, \text{ and}$$
$$F_iAF_j \neq O \Rightarrow \psi_i - \psi_j \in \{1, \omega, \omega^2\},$$

where $1 + \omega + \omega^2 = 0$.

Hexagonal Pairs

Let A and B be linear transformations on a finite dimensional vector space $V \neq 0$ satisfying the following four conditions.

(i) A and B are diagonazable on V.

(ii) There is an indexing $V_{i,j}, (i,j) \in \Phi$ of the eigenspaces of A such that

$$BU_{i,j} \subseteq U_{i+1,j} + U_{i-1,j+1} + U_{i,j-1}, \quad (i,j) \in \Phi.$$

(iii) There is an indexing $V_{i,j}, (i,j) \in \Psi$ of the eigenspaces of A such that

$$BV_{i,j} \subseteq V_{i+1,j} + V_{i-1,j+1} + V_{i,j-1}, \quad (i,j) \in \Psi.$$

(iv) There is no subspace W of V such that $AW \subseteq W$, $BW \subseteq W$, other than $W = 0$ and $W = V$.

Examples with small number of vertices

Hanaki-Miyamoto, Classification of association schemes with a small number of vertices at `http://math.shinshu-u.ac.jp/~hanaki/as/`.

`http://math.shinshu-u.ac.jp/~hanaki/as/data/as07` No. 2

An example of a wdrdg with seven vertices:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 0 & 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 0 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 & 0 \end{bmatrix}$$