Lecture Note on Terwilliger Algebra

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Setting

sudo This note is created by `bookdown` package on RStudio.

For `bookdown` See (Xie, 2015), (Xie, 2017), (Yihui Xie, 2018).

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3. Use This Template
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Chapter 1

Subconstituent Algebra of a Graph

Wednesday, January 20, 1993

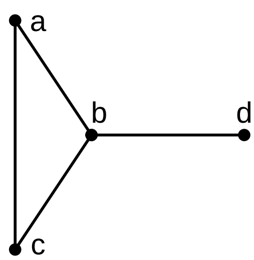
A graph (undirected, without loops or multiple edges) is a pair $\Gamma = (X, E)$, where

$$X = \text{finite set (of vertices)} \tag{1.1}$$

$$E = \text{set of (distinct) 2-element subsets of } X \text{ (= edges of) } \Gamma. \tag{1.2}$$

vertices x and $y \in X$ are adjacent if and only if $xy \in E$.

Example 1.1. Let Γ be a graph. $X = \{a, b, c, d\}$, $E = \{ab, ac, bc, bd\}$.



Set $n = |X|$, the order of Γ.

Pick a field K (= $\mathbb{R}$ or $\mathbb{C}$). Then $\mathrm{Mat}_X(K)$ denotes the K algebra of all $n \times n$ matrices with entries in K. (rows and columns are indexed by X)

Adjacency matrix $A \in \mathrm{Mat}_X(K)$ is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E \\ 0 & \text{else .} \end{cases} \tag{1.3}$$

Example 1.2. Let a, b, c, d be labels of rows and columns. Then

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The subalgebra M of $\mathrm{Mat}_X(K)$ generated by A is called the *Bose-Mesner algebra* of Γ.

Set $V = K^n$, the set of n-dimensional column vectors, the coorinates are indexed by X.

Let $\langle\ ,\ \rangle$ denote the Hermitean inner product:

$$\langle u, v\rangle = u^{\top} \cdot v \quad (u, v \in V)$$

V with $\langle\ ,\ \rangle$ is the *standard module* of Γ.

M acts on V: For every $x \in X$, write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where 1 is at the x position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since A is a real symmetrix matrix,

$$V = V_0 + V_1 + \dots + V_r \quad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal A-eigenspaces.

Let $E_i \in \mathrm{Mat}_X(K)$ denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then $E_0, \dots, E_r$ are the primitive idempotents of M.

$$M = \mathrm{Span}_K(E_0, \dots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all} \;\; i, j, \quad E_0 + \dots + E_r = I.$$

Let θ_i denote the eigenvalue of A for V_i in $\mathbb{R}$. Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \cdots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \ \theta_i = \dim V_i = \mathrm{rank} E_i.$$

Set

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \cdots, & \theta_r \\ m_0, & m_1, & \cdots, & m_r \end{pmatrix}.$$

Problem. What can we say about Γ when $\mathrm{Spec}(\Gamma)$ is given?

The following Lemma1.1, is an example of Problem.

For every $x \in X$,

$$k(x) \equiv \text{ valency of } x \equiv \text{ degree of } x \equiv |\{y \mid y \in X,\ xy \in E\}|.$$

Definition 1.1. The graph Γ is regular of valency k if $k = k(x)$ for every $x \in X$.

Lemma 1.1. *With the above notation,*

(i) $\theta_0 \leq \max\{k(x) \mid x \in X\} = k^{\max}$.
(ii) If Γ *is regular of valency* k, *then* $\theta_0 = k$.

Proof.

(i) Without loss of generality we may assume that $\theta_0 > 0$, else done. Let $v := \sum_{x \in X} \alpha_x \hat{x}$ denote the eivenvector for θ_0.

Pick $x \in X$ with $|\alpha_x|$ maximal. Then $|\alpha_x| \neq 0$.

Since $Av = \theta_0 v$,

$$\theta_0 \alpha_x = \sum_{y \in X, xy \in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \leq \sum_{y \in X, xy \in E} |\alpha_y| \leq k(x)|\alpha_x| \leq k^{\max}|\alpha_x|.$$

(ii) All 1's vector $v = \sum_{x \in X} \hat{x}$ satisfies $Av = kv$.

$\square$

Subconstituent Algebra

Let $x, y \in X$ and $\ell \in \mathbb{Z}^{\geq 0}$.

Definition 1.2. A path of length ℓ connecting x, y is a sequence

$$x = x_0, x_1, \ldots, x_\ell = y, \quad x_i \in X,\ 0 \leq i \leq \ell$$

such that $x_i x_{i+1} \in E$ for $0 \leq i \leq \ell - 1$.

Definition 1.3. The distance $\partial(x, y)$ is the length of a shortest path connecting x and y.

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

Definition 1.4. The graph Γ is connected if and only if $\partial(x, y) < \infty$ for all $x, y \in X$.

From now on, assume that Γ is connected with $|X| \geq 2$.

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \text{the diameter of } \Gamma.$$

Fix a 'base' vertex $x \in X$.

Definition 1.5.

$$d(x) = \text{the diameter with respect to } x = \max\{\partial(x, y) \mid y \in X\} \leq d.$$

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad \text{(orthogonal direct sum)},$$

where

$$V_i^* = \mathrm{Span}_K(\hat{y} \mid \partial(x, y) = i) \equiv V_i * (x)$$

and $V_i^* = V_i^*(x)$ is called the i-the subconstituent with respect to x.

Let $E_i^* = E_i^*(x)$ denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View $E_i^*(x) \in \mathrm{Mat}_X(K)$. So, $E_i^*(x)$ is diagonal with yy entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{else,} \end{cases} \qquad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \mathrm{Span}_K(E_0^*(x), \ldots, E_{d(x)}^*(x)).$$

Then $M^*(x)$ is a commutative subalgebra of $\mathrm{Mat}_X(K)$ and is calle the *dual Bose-Mesner algbara with respect to* x.

Definition 1.6 (Subconstituent Algebra)**.** Let $\Gamma = (X, E)$, x, M, $M^*(x)$ be as above. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(K)$ generated by M and $M^*(x)$. T is the *subconstituent algebra* of Γ with respect to x.

Definition 1.7. A T-module is any subspace $W \subset V$ such that $aw \in W$ for all $a \in T$ and $w \in W$.

T-module W is *irreducible* if and only if $W \neq 0$ and W does not properly contain a nonzero T-module.

For any $a \in \mathrm{Mat}_X(K)$, let a^* denbote the conjugate transpose of a.

Observe that

$$\langle au, v\rangle = \langle u, a^* v\rangle \quad \text{for all } \ a \in \mathrm{Mat}_X(K), \ \text{and for all } \ u, v \in V.$$

Lemma 1.2. *Let $\Gamma = (X, E)$, $x \in X$ and $T \equiv T(x)$ be as above.*

(i) If $a \in T$, then $a^* \in T$.

(ii) For any T-module $W \subset V$,

$$W^{\perp} := \{v \in V \mid \langle w, v\rangle = 0, \ \textit{for all } w \in W\}$$

is a T-module.

(iii) V decomposes as an orthogonal direct sum of irreducible T-modules.

Proof.

(i) It is becase T is generated by symmetric real matrices

$$A, E_0^*(x), E_1^*(x), \ldots, E_{d(x)(x)}^*.$$

(ii) Pick $v \in W^{\perp}$ and $a \in T$, it suffices to show that $av \in W^{\perp}$. For all $w \in W$,

$$\langle w, av\rangle = \langle a^* w, v\rangle = 0$$

as $a^* \in T$.

(iii) This is proved by the induction on the dimension of T-modules. If W is an irreducible T-module of V, then

$$V = W + W^{\perp} \quad \text{(orthogonal direct sum)}.$$

□

Problem. What does the structure of the $T(x)$-module tell us about Γ?

Study those Γ whose modules take ‘simple’ form. The Γ’s involved are highly regular.

Remark.

1. The subconstituent algebra T is semisimple as the left regular representation of T is completely reducible. See Curtis-Reiner 25.2 (Charles W. Curtis, 2006).
2. The inner product $\langle a, b\rangle_T = \mathrm{tr}(a^{\top}\bar{b})$ is nondegenerate on T.

3. In general,

$$
\begin{aligned}
T\colon \text{Semisimple and Artinian} &\Leftrightarrow T\colon \text{Artinian with } J(T)=0\\
&\Leftarrow T\colon \text{Artinian with nonzero nilpotent element}\\
&\Leftarrow T \subset \mathrm{Mat}_X(K) \text{ such that for all } a \in T \text{ is normal.}
\end{aligned}
$$

Chapter 2

Perron-Frobenius Theorem

Friday, January 22, 1993

In this lecture we use the Perron Frobenius theory of nonnegative matrices to obtain informaiton on eigenvalues of a graph.

Let $K = \mathbb{R}$. For $n \in \mathbb{Z}^{>0}$, pick a symmetrix matrix $C \in \mathrm{Mat}_n(\mathbb{R})$.

Definition 2.1. The matrix C is *reducible* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i \in X^+$, and for all $j \in X^-$, and for all $i \in X^-$, and for all $j \in X^+$,i.e.,

$$C \sim \begin{pmatrix} * & O \\ O & * \end{pmatrix}.$$

Definition 2.2. The matrix C is *bipartite* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i, j \in X^+$, and for all $i, j \in X^-$, i.e.,

$$C \sim \begin{pmatrix} O & * \\ * & O \end{pmatrix}.$$

Note.

1. If C is bipatite, for every eigenvalue θ of C, $-\theta$ is an eigenvalue of C such that $\mathrm{mult}(\theta) = \mathrm{mult}(-\theta)$.

Indeed, let $C = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$,

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix},$$

where $Ay = \theta x$ and $Bx = \theta y$.

2. If C is bipartite, C^2 is reducible.
3. The matrix C is irreducible and C^2 is reducible, if $C_{ij} \geq 0$ for all i, j and C is reducible. (Exercise)

Remark. Note 1. Even if C is not symmetric

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

holds. So the geometrix multiplicities coincide. How about the algebraic multiplicities?

Note 3. Set $x \sim y$ if and only if $C_{xy} > 0$. So the graph may have loops. Then

$$(C^2)_{xy} > 0 \Leftrightarrow \text{ if there exists } z \in X \text{ such that } x \sim z \sim y.$$

Note that C is irreducible if and only if $\Gamma(C)$ is connected. Let

$$X^+ = \{y \mid \text{there is a path of even length from } x \text{ to } y\} \tag{2.1}$$
$$X^- = \{y \mid \text{there is no path of even length from } x \text{ to } y\} \neq \emptyset. \tag{2.2}$$

If there is an edge $y \sim z$ in X^+ and $w \in X^-$. Then there would be a path from x to y of even length. So $\mathrm{e}(X^+, X^+) = \mathrm{e}(X^-, X^-) = 0$..

Theorem 2.1 (Perron-Frobenius)**.** *Given a matrix C in $\mathrm{Mat}_n(\mathbb{R})$ such that*

(a) C is symmetric.

(b) C is irreducible.

(c) $C_{ij} \geq 0$ for all i, j.

Let θ_0 be the maximal eigenvalue of C with eigenspace $V_0 \subseteq \mathbb{R}^n$, and let θ_r be the maximal eigenvalue of C with eigenspace $V_r \subseteq \mathbb{R}^n$. Then the following hold.

(i) Suppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$. *Then $\alpha_0 > 0$ for all i, or $\alpha_i < 0$ for all i.*

(ii) $\dim V_0 = 1$.

(iii) $\theta_r \geq -\theta_0$.

(iv) $\theta_r = \theta_0$ if and only if C is bipartite.

First, we prove the following lemma.

Lemma 2.1. *Let $\langle\ ,\ \rangle$ be the dot product in $V = \mathbb{R}^n$. Pick a symmetric matrix $B \in \mathrm{Mat}_n(\mathbb{R})$. Suppose all eigenvalues of B are nonnegative. (i.e., B is positive semidefinite.) Then there exist vectors $v_1, v_2, \ldots, v_n \in V$ such that $B_{ij} = \langle v_i, v_j \rangle$ for $(1 \leq i, j \leq n)$.*

Proof. By elementary linear algebra, there exists an orthonormal basis $w_1, w_2, \dots, w_n$ of V consisting of eigenvectors of B. Set the i-th column of P is w_i and $D = \mathrm{diag}(\theta_1, \dots, \theta_n)$. Then $P^\top P = I$ and $BP = PD$.

Hence,
$$B = PDP^{-1} = PDP^\top = QQ^\top,$$
where
$$Q = P \cdot \mathrm{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \dots, \sqrt{\theta_n}) \in \mathrm{Mat}_n(\mathbb{R}).$$
Now, let v_i be the i-th column of $Q^\top$. Then
$$B_{ij} = v_i^\top \cdot v_j^- = \langle v_i, v_j \rangle.$$
$\square$

Now we start the proof of Theorem 2.1.

Proof of Theorem 2.1(*i*)

Let $\langle , \rangle$ denote the dot product on $V = \mathbb{R}^n$. Set

$$\begin{aligned} B &= \theta I - C && (2.3)\\ &= \text{symmetric matrix with eigenvalues } \theta_0 - \theta_i \geq 0 && (2.4)\\ &= (\langle v_i, v_j \rangle)_{1\leq i,j\leq n} && (2.5)\end{aligned}$$

with the same $v_1, \dots, v_n \in V$ by Lemma 2.1.

Observe: $\sum_{i=1}^n \alpha_i v_i = 0$.

Pf.

$$\begin{aligned} \|\sum_{i=1}^n \alpha_i v_i\|^2 &= \langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \rangle && (2.6)\\ &= \begin{pmatrix} \alpha_1 & \dots & \alpha_n \end{pmatrix} B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} && (2.7)\\ &= v^\top B v && (2.8)\\ &= 0, && (2.9)\end{aligned}$$

since $Bv = (\theta_0 I - C)v = 0$.

Now set
$$s = \text{the number of indices} i, \text{ where } \alpha_i > 0.$$

Replacing v by $-v$ if necessary, without loss of generality we may assume that $s \geq 1$. We want to show $s = n$.

Assume $s < n$. Without loss of generality, we may assume that $\alpha_i > 0$ for $1 \leq s \leq s$ and $\alpha_i = 0$ for $s + 1 \leq i \leq n$. Set

$$\rho = \alpha_1 v_1 + \dots + \alpha_s v_s = -\alpha_{s+1} v_{s+1} - \dots - \alpha_n v_n.$$

Then, for $i = 1, \dots, s$,

$$\langle v_i, \rho \rangle = \sum_{j=s+1}^{n} -\alpha_j \langle v_i, v_j \rangle \quad (\langle v_i, v_j \rangle = B_{ij}, B = \theta_0 I - C) \tag{2.10}$$

$$= \sum_{j=s+1}^{n} (-\alpha_{ij})(-C_{ij}) \tag{2.11}$$

$$\leq 0. \tag{2.12}$$

Hence

$$0 \leq \langle \rho, \rho \rangle = \sum_{i=1}^{s} \alpha_i \langle v_i, \rho \rangle \leq 0,$$

as $\alpha > 0$ and $\langle v_i, \rho \rangle \leq 0$. Thus, we have $\langle, \rho, \rho \rangle = 0$ and $\rho = 0$. For $j = s+1, \dots, n$,

$$0 = \langle \rho, v_j \rangle = \sum_{i=1}^{s} \alpha_i \langle v_i, v_j \rangle \leq 0,$$

as $\langle v_i, v_j \rangle = -C_{ij}$.

Therefore,

$$0 = \langle v_i, v_j \rangle = -C_{ij} \text{ for } 1 \leq i, \leq s,\ s+1 \leq j \leq n.$$

Since C is symmetric,

$$C = \begin{pmatrix} * & O \\ O & * \end{pmatrix}$$

Thus C is reducible, which is not the case. Hence $s = n$.

Proof of Theorem 2.1 (ii).

Suppose $\dim V_0 \geq 2$. Then,

$$\dim \left(V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{\perp} \right) \geq 1.$$

So, there is a vector

$$0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with $\alpha_1 = 0$. This contradicts 1.

Now pick

$$0 \neq w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r.$$

Proof of Theorem 2.1 (iii).

Suppose $\theta_r < -\theta_0$. Since the eigenvalues of C^2 are the squares of those of C, θ_r^2 is the maximal eigenvalue of C^2.

Also we have $C^2 w = \theta_r^2 w$.

Observe that C^2 is irreducible. (As otherwise, C is bipartite by Note 3, and we must have $\theta_r = -\theta_0$.) Therefore, $\beta_i > 0$ for all i or $\beta_i < 0$ for all i. We have

$$\langle v, w \rangle = \sum_{i=1}^{n} \alpha_i \beta_j \neq 0.$$

This is a contradiction, as $V_0 \perp V_r$.

Proof of Theorem 2.1 (iv)

$\Rightarrow$: Let $\theta_r = -\theta_0$. Then $\theta = \theta_1^2 = \theta_0^2$ is the maximal eigenvalue of C^2, and v and w are linearly independent eigenvalues for θ for C^2. Hence, for C^2, $\text{mult}(\theta) \geq 2$.

Thus by 2, C^2 must be reducible. Therefore, C is bipartite by Note 3.

$\Leftarrow$: This is Note 1. $\square$

Let $\Gamma = (X, E)$ be any graph.

Definition 2.3. Γ is said to be *bipartite* if the adjacency matrix A is bipartite. That is, X can be written as a disjoint union of X^+ and X^- such that X^+, X^- contain no edges of Γ.

Corollary 2.1. *For any (connected) graph* Γ *with*

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_1 & m_1 & \cdots & m_r \end{pmatrix} \quad \textit{with } \theta_0 > \theta_1 > \cdots > \theta_r.$$

Let V_i *be the eigenspace of* θ_i*. Then the following holds.*

1. *Supppose* $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \in \mathbb{R}^n$*. Then* $\alpha_i > 0$ *for all* i *or* $\alpha_i < 0$ *for all* i*.*
2. $m_0 = 1$*.*
3. $\theta_r \geq -\theta_0$ *if and only if* Γ *is bipartite. In this case,*

$$-\theta_i = \theta_{r-i} \textit{ and } m_i = m_{r-i} \quad (0 \leq i \leq r)$$

Proof. This is a direct consequences of Theorem 2.1 and Note 3. $\square$

Chapter 3

Cayley Graphs

Monday, January 25, 1993

Given graphs $\Gamma = (X, E)$ and $\Gamma' = (X', E')$.

Definition 3.1. A map $\sigma : X \to X'$ is an *isomorphism*\index{isomorophism of graphs whenever;

i. σ is one-to-one and onto,
ii. $xy \in E$ if and only if $\sigma x \sigma y \in E'$ for all $x, y \in X$.

We do not distinguish between isomorphic graphs.

Definition 3.2. Suppose $\Gamma = \Gamma'$. Above isomorphism σ is called an *automorphism* of Γ. Then set $\mathrm{Aut}(\Gamma)$ of all automorphisms of Γ becomes a finite group under composition.

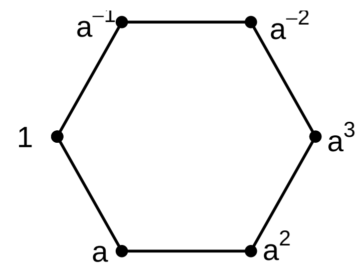
Definition 3.3. If $\mathrm{Aut}(\Gamma)$ acts transitive on X, Γ is called *vertex transitive*.

Example 3.1. A Cayley graphs:

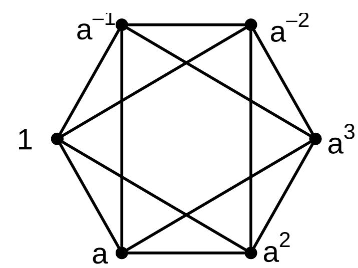
Definition 3.4 (Cayley Graphs)**.** Let G be any finite group, and Δ any generating set for G such that $1_G \notin \Delta$ and $g \in \Delta \to g^{-1} \in \Delta$. Then Cayley graph $\Gamma = \Gamma(G, \Delta)$ is defined on the vetex set $X = G$ with the edge set E define by the following.

$$E = \{(h_1, h_2) \mid h_1, h_2 \in G, h_1^{-1} h_2 \in \Delta\} = \{(h, hg) \mid h \in G, g \in \Delta\}$$

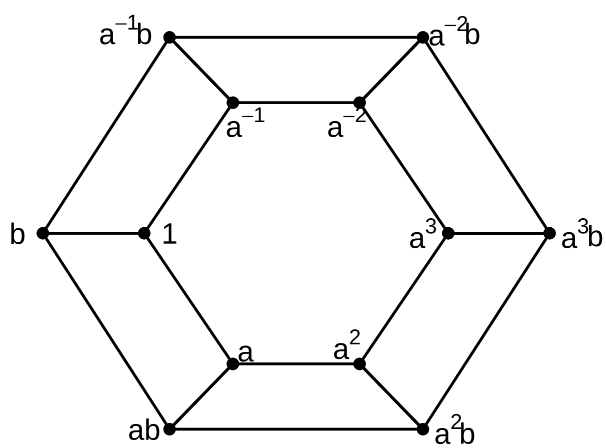
Example 3.2. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}\}$.



Example 3.3. $G = \langle a \mid a^6 = 1\rangle$, $\Delta = \{a, a^{-1}, a^2, a^{-2}\}$.



Example 3.4. $G = \langle a, b \mid a^6 = 1, b^2, ab = ba\rangle$, $\Delta = \{a, a^{-1}, b\}$.



Remark. $\mathrm{Aut}(\Gamma) \simeq D_6 \times \mathbb{Z}_2$ contains two regular subgroups isomorphic to D_6 and $\mathbb{Z}_5 \times \mathbb{Z}_2$ and Γ is obtained as Cayley graphs in two ways.

Cayley graphs are vertex transitive, indeed.

Theorem 3.1. *The following hold.*

(i) For any Cayley graph $\Gamma = \Gamma(G, \Delta)$*, the map*

$$G \to \mathrm{Aut}(\Gamma)\ (g \mapsto \hat{g})$$

is an injective homomorphism of groups, where

$$\hat{g}(x) = gx \quad \textit{for all } g \in G \textit{ and for all } x \in X(= G).$$

Also, the image $\hat{G}$ is regular on X. i.e., the image $\hat{G}$ acts transitively on X with trivial vertex stabilizers.

(ii) For any graph $\Gamma = (X, E)$, suppose there exists a subgroup $G \subseteq \mathrm{Aut}(\Gamma)$ that is regular on X. Pick $x \in X$, and let

$$\Delta = \{g \in G \mid \langle x, g(x) \in E\}.$$

Then $1 \notin \Delta$, $g \in \Delta \to g^{-1} \in \Delta$, and Δ generates G. Moreover, $\Gamma \simeq \Gamma(G, \Delta)$.

Proof. (i) Let $g \in G$. We want to show that $\hat{g} \in \mathrm{Aut}(\Gamma)$. Let $h_1, h_2 \in X = G$. Then,

$$\begin{aligned} (h_1, h_2) \in E &\to h_1^{-1} h_2 \in \Delta && (3.1)\\ &\to (gh_1)^{-1}(gh_2) \in \Delta && (3.2)\\ &\to (gh_1, gh_2) \in E && (3.3)\\ &\to (\hat{g}(h_1), \hat{g}(h_2)) \in E. && (3.4) \end{aligned}$$

Hence, $\hat{g} \in \mathrm{Aut}(\Gamma)$.

Observe: $g \mapsto \hat{g}$ is a homomorphism of groups:

$$\hat{1}_G = 1,\ \widehat{g_1 g_2} = \widehat{g_1}\widehat{g_2}.$$

Observe: $g \mapsto \hat{g}$ is one-to-one:

$$\widehat{g_1} = \widehat{g_2} \to g_1 = \widehat{g_1}(1_G) = \widehat{g_2}(1_G) = g_2.$$

Observe: $\hat{G}$ is regular on X: Clear by construction.

(ii) $1_G \notin \Delta$: Since Γ has not loops, $(x, 1_G x) \notin E$.

$g \in \Delta \to g^{-1} \in \Delta$:

$$g \in \Delta \to (x, g(x)) \in E \to E \ni (g^{-1}(x), g^{-1}(g(x))) = (g^{-1}(x), x).$$

Δ generates G: Suppose $\langle \Delta \subsetneq G$. Let $\hat{X} = \{g(x) \mid g \in \langle \Delta \rangle\} \subsetneq X$. ($\hat{X} \subsetneq X$ as G acts regularly on X.)

Since Γ is connected, there exists $y \in \hat{X}$ and $z \in X \quad \hat{X}$ with $yz \in E$.

Let $y = g(x)$, $g \in \langle \Delta \rangle$, $z \in h(x)$, $h \in G \quad \langle \Delta \rangle$. Then

$$(y, z) = (g(x), h(x)) \in E \to (x, g^{-1}h(x)) \in E \to g^{-1}h \in \langle \Delta \rangle \to h \in \langle \Delta \rangle.$$

This is a contradition. Therefore, Δ generates G.

Let $\Gamma' = (X', E')$ denote $\Gamma(G, \Delta)$. We shall show that

$$\theta : X' \to X \ (g \mapsto g(x))$$

is an isomorphism of graphs.

θ is one-to-one: For $h_1, h_2 \in X' = G$,

$\theta(h_1) = \theta(h_2) \to h_1(x) = h_2(x) \to h_2^{-1}h_1(x) = x \to h_2^{-1}h_1 \in \mathrm{Stab}_G(x) = \{1_G\} \to h_1 = h_2.$

$(\mathrm{Stab}_G = \{g \in G \mid g(x) = x\}.)$

θ is onto: Since G is transitive,

$$X = \{g(x) \mid g \in G\} = \theta(X') = \theta(G).$$

θ respects adjacency: For $h_1, h_2 \in X' = G$,

$(h_1, h_2) \in E' \leftrightarrow h_1^{-1}h_2 \in \Delta \leftrightarrow (x, h_1^{-1}h_2(x)) \in E \leftrightarrow (h_1(x), h_2(x)) \in E \leftrightarrow (\theta(h_1), \theta(h_2)) \in E.$

Therefore θ is an isomorphism between graphs $\Gamma(G, \Delta)$ and $\Gamma(X, E)$. □

How to compute the eigenvalues of the Cayley graph of and abelian group.

Let G be any finite abelian group. Let $\mathbb{C}^*$ be the multiplicative group on $\mathbb{C}$ $\{0\}$.

Definition 3.5. A (linear) G-character is any group homomorphism $\theta : G \to \mathbb{C}^*$.

Example 3.5. $G = \langle a \mid a^3 = 1 \rangle$ has three characters, $\theta_0, \theta_1, \theta_2$.

$\theta_i(a^j)$	1	a	a^2
θ_0	1	1	1
θ_1	1	ω	ω^2
θ_2	1	ω^2	ω

, with $\omega = \dfrac{-1 + \sqrt{-3}}{2}$.

Here ω is a primitive cube root pf q in $\mathbb{C}^*$, i.e., $1 + \omega + \omega^2 = 0$.

For arbitraty group G, let $X(G)$ be the set of all characters of G.

Observe: For $\theta_1, \theta_2 \in X(G)$, one can define product \$ _1 _2:

$$\theta_1\theta_2(g) = \theta_1(g)\theta_2(g) \quad \text{for all} \ \ g \in G.$$

Then $\theta_1\theta_2 \in X(G)$.

Observe: $X(G)$ with this product is an (abelian) group.

Lemma 3.1. *The groups G and $X(G)$ are isomorphic for all finite abelian groups G.*

Proof. G is a direct sum of cyclic groups;

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m, \quad \text{where } G_i = \langle a_i \mid a_i^{d_i} = 1 \rangle \quad (1 \leq i \leq m).$$

Pick any alement ω_i of order d_i in $\mathbb{C}^*$, i.e., a primitive d_i-the root of 1. Define

$$\theta_i : G \to \mathbb{C}^* \quad (a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \mapsto \omega_i^{\varepsilon_i} \quad \text{where } 0 \leq \varepsilon_i < d_i, 1 \leq i \leq m).$$

Then $\theta_i \in X(G)$. (Exercise)

Claim: There exists an isomorphism of groups $G \to X(G)$ that sends a_i to θ_i.

Observe: $\theta_i^{d_i} = 1$. For every $g = a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \in G$,

$$\theta_i^{d_i}(g) = (\theta_i(g))^{d_i} = (\omega_i^{\varepsilon_i})^{d_i} = (\omega_i^{d_i})^{\varepsilon_i} = 1.$$

Observe: If $\theta_1^{\varepsilon_1}\theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m} = 1$ for some $0 \leq \varepsilon_i < d_i, 1 \leq i \leq m$. Then $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = 0$.

Pf. $1 = \theta_1^{\varepsilon_1}\theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m}(a_i) = \omega_i^{\varepsilon_i}$, Since ω_i is a primitive d_i-th root of 1, $\varepsilon_i = 0$ for $1 \leq i \leq m$.

Observe: $\theta_1, \dots, \theta_m$ generate $X(G)$. Pick $\theta \in X(G)$. Since $a_i^{d_i} = 1$, $1 = \theta(a_i^{d_i}) = \theta(a_i)^{d_i}$.

Hence $\theta(a_i) = \omega^{\varepsilon_i}$ for some ε_i with $0 \leq \varepsilon_i < d_i$.

Now $\theta = \theta_1^{\varepsilon_1} \cdots \theta_m^{\varepsilon_m}$, since these are both equal to $\omega_i^{\varepsilon_i}$ at a_i for $1 \leq i \leq m$.

Therefore,

$$G \to X(G) \quad (a_i \mapsto \theta_i)$$

is an isomorphism of groups. □

Note. The correspondence above is clearly a group homomorphism.

Chapter 4

Examples

Wednesday, January 27, 1993

Theorem 4.1. *Given a Cayley graph $\Gamma = \Gamma(G, \Delta)$. View the standard module $V \equiv \mathbb{C}G$ (the group algebra), so*

$$\left\langle \sum_{g \in G} \alpha_g g, \sum_{g \in G} \beta_g g \right\rangle = \sum_{g \in G} \alpha_g \overline{\beta_g}, \quad \text{with } \alpha_g, \beta_g \in \mathbb{C}.$$

For any $\theta \in X(G)$, write

$$\hat{\theta} = \sum_{g \in G} \theta(g^{-1}) g.$$

Then the following hold.

(i) $\langle \hat{\theta}_1, \hat{\theta}_2 \rangle = |G|$ if $\theta_1 = \theta_2$ and 0 othewise for $\theta_1, \theta_2 \in X(G)$. In particular, $\{\hat{\theta} \mid \theta \in X(G)\}$ forms a basis for V.

(ii) $A\hat{\theta} = \Delta_\theta \hat{\theta}$ for $\theta \in X(G)$, where A is the adjacency matrix and

$$\Delta_\theta = \sum_{g \in \Delta} \theta(g).$$

In particular, the eigenvalues of Γ are precisely

$$\Delta_\theta \mid \theta \in X(G)\}.$$

Proof.

(i) Claim: For every $\theta \in X(G)$, let

$$s := \sum_{g \in G} \theta(g^{-1}) = \begin{cases} |G| & \text{if } \theta = 1 \\ 0 & \text{if } \theta \neq 1. \end{cases}$$

Pf. Clear if $\theta = 1$.

Let $\theta \neq 1$. Then $\theta(h) \neq 1$ for some $h \in G$.

$$s \cdot \theta(h) = \left(\sum_{g \in G} \theta(g^{-1}) \right) \theta(h) = \sum_{g \in G} \theta(g^{-1}h) = \sum_{g' \in G} \theta(g'^{-1}) = s.$$

Since $\theta(h) \neq 1$, $s = 0$.

Claim. $\theta(g^{-1}) = \overline{\theta(g)}$ for every $\theta \in X(G)$ and every $g \in G$.

Since $\theta(g) \in \mathbb{C}$ is a root of 1,

$$|\theta(g)|^2 = \theta(g)\overline{\theta(g)} = 1.$$

On the other hand, since θ is a homomorphism,

$$\theta(g)\theta(g^{-1}) = \theta(1) = 1.$$

Hence $\theta(g^1) = \overline{\theta(g)}$.

Now

$$\begin{aligned}
\langle \widehat{\theta_1}, \widehat{\theta_2} \rangle &= \sum_{g \in G} \theta_1(g^{-1}) \overline{\theta_2(g^{-1})} && (4.1)\\
&= \sum_{g \in G} \theta_1(g^{-1}) \theta_2(g) && (4.2)\\
&= \sum_{g \in G} \theta_1 \theta_2^{-1}(g^{-1}) && (4.3)\\
&= \begin{cases} |G| & \text{if} \quad \theta_1\theta_2^{-1} = 1 \\ 0 & \text{if} \quad \theta_1\theta_2^{-1} \neq 1. \end{cases} && (4.4)
\end{aligned}$$

Since $|G| = |X(G)|$ by Lemma 3.1, and $\widehat{\theta_i}$'s are orthogonal nonzero elements in V, thet form a basis of V.

(ii) Let $\Delta = \{g_1, \dots, g_r\}$. Then

$$
\begin{aligned}
A\hat{\theta} &= A\left(\sum_{g\in G}\theta(g^{-1}g)\right) && (4.5)\\
&= \sum_{g\in G}\theta(g^{-1})(gg_1+\cdots+gg_r) \quad (\Gamma(g)=\{gg_1,\ldots,gg_r\}) && (4.6)\\
&= \sum_{i=1}^{r}\left(\sum_{g\in G}\theta(g^{-1})(gg_i)\right) && (4.7)\\
&= \sum_{i=1}^{r}\left(\sum_{g\in G}\theta(g_ig_i^{-1}g^{-1})(gg_i)\right) && (4.8)\\
&= \sum_{i=1}^{r}\left(\sum_{g\in G}\theta(g_i)\theta((gg_i)^{-1})gg_i\right) && (4.9)\\
&= \sum_{i=1}^{r}\theta(g_i)\sum_{h\in G}\theta(h^{-1})h && (4.10)\\
&= \Delta_\theta\cdot\hat{\theta}. && (4.11)
\end{aligned}
$$

Since $\{\hat{\theta}\mid\theta\in X(G)\}$ forms a basis, the eigenvalues of Γ are precisely,

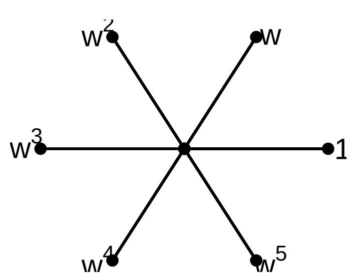
$$\{\Delta_\theta \mid \theta\in X(G)\}.$$

This completes the proof.

□

Example 4.1. Let $G=\langle a \mid a^6=1\rangle$, and $\Delta=\{a,a^{-1}\}$. Pick a primitive 6-th root of 1, ω. Then

$$X(G)=\{\theta^i \mid 0\le i\le 5\} \quad \text{such that} \quad \theta(a)=\omega,\ \omega+\omega^{-1}=1.$$



$\varphi \in X(G)$	$\varphi(a)$	$\Delta_\varphi = \theta(a) + \theta(a)^{-1}$
1	1	2
θ	ω	$\omega + \omega^{-1} = 1$
θ^2	ω^2	-1
θ^3	$\omega^3 = -1$	-2
θ^4	ω^4	-1
θ^5	ω^5	1

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

Example 4.2. D-cube, $H(D,2)$. Let

$$X = \{(a_1, \dots, a_D) \mid a_i \in \{1, -1\},\ 1 \leq i \leq D\},$$

$$E = \{xy \mid x, y \in X,\ x, y\text{: different in exactly one coordinate}\}.$$

Also $H(D,2)$ is a Cayley graph $\Gamma(G, \Delta)$, where

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_D,$$

$$G_i = \langle a_i \mid a_i^2 = 1 \rangle, \quad \Delta = \{a_1, \dots, a_D\}.$$

Homework: The spectrum of $H(D,2)$ is

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_D \\ m_0 & m_1 & \cdots & m_D \end{pmatrix},$$

where

$$\theta_i = D - 2i \quad (0 \leq i \leq D), \quad m_i = \binom{D}{i}.$$

Remark. Let $\theta \in X(G)$. Then $\theta : X \to \{\pm 1\}$. If

$$\nu(\theta) = |\{i \mid \theta(a_i) = -1\}|,$$

then $\Delta_\theta = D - 2i$. Since there are $\binom{D}{i}$ such θ, we have te assertion.

We want to compute the subconstituent algebra for $H(D,2)$. First, we make a few observations about arbitrary graphs.

Let $\Gamma = (X, E)$ be any graph, A, the adjacemcy matrix of Γ, and V, the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$. Write $E_i^* = E_i^*(x)$, and

$$T \equiv T(x) = \text{the algebra generated by } A, E_0^*, E_1^*, \dots.$$

Definition 4.1. Let W be any orreducible T-module ($\subseteq V$). Then the endpoint $r \equiv r(W)$ satisfied

$$r = \min\{i \mid E_i^* W \neq 0\}.$$

The diameter $d = d(W)$ satisfied

$$d = |\{i \mid E_i^* W \neq 0\}| - 1.$$

Lemma 4.1. *With the above notation, let W be an irreducible T-module. Then*

(i) $E_i^* A E_j^* = 0$ *if* $|i-j| = 1$, $E_i^* A E_j^* \neq 0$ *if* $|i-j| = 1$, $\quad 0 \leq i, j \leq d(x)$.
(ii) $AE_j^* W \subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$, $0 \leq j \leq d(x)$. *(*$E_i^* W = 0$ *if* $i < j$ *or* $i > d(x)$.*)*
(iii) $E_j^* W \neq 0$ *if* $r \leq j \leq r + d$, $= 0$ *if* $0 \leq j \leq r$ *or* $r + d < j \leq d(x)$.
(iv) $E_i^* A E_j^* W \neq 0$, *if* $|i - j| = 1$ $(r \leq i, j \leq r + d)$.

Proof.

(i) Pick $y \in X$ with $\partial(x, y) = j$. We want to find $E_i^* A E_j^* \hat{y}$. Note,

$$E_j^* \hat{y} = \begin{cases} 0 & \text{if } \partial(x.y) \neq j \\ \hat{y} & \text{if } \partial(x,y) = j. \end{cases}.$$

$$\begin{aligned} E_i^* A E_j^* \hat{y} &= E_i^* A \hat{y} && (4.12) \\ &= E_i^* \sum_{z \in X, yz \in E} \hat{z} && (4.13) \\ &= \sum_{z \in X, yz \in E, \partial(x,z) = i} \hat{z} && (4.14) \\ &= 0 \ \text{ if } \ |i-j| > 1 \qquad \text{by triangle inequality.} && (4.15) \end{aligned}$$

If $|i - j| = 1$, there exist $y, y' \in X$ such that $\partial(x, y) = j$, $\partial(x, y') = i$, $yy' \in E$ by connectivity of Γ. Hence (4.14) contains $\widehat{y'}$ and (4.14) is not equal to zero.

(ii) We have

$$\begin{aligned} AE_j^* W &= \left(\sum_{i=0}^{d(x)} E_i^* \right) AE_j^* W && (4.16) \\ &= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W && (4.17) \\ &\subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W. && (4.18) \end{aligned}$$

(iii) Suppose $E_j^* W = 0$ for some j $(r \leq j \leq r + d)$. Then $r < j$ by the definition of r. Set

$$\widetilde{W} = E_r^*W + E_{r+1}^*W + \cdots + E_{j-1}^*W.$$

Observe $0 \subsetneq \widetilde{W} \subsetneq W$. Also $A\widetilde{W} \subseteq \widetilde{W}$ by (ii), and $E_i^*\widetilde{W} \subseteq \widetilde{W}$ for every i by construction.

Thus, $T\widetilde{W} \subseteq \widetilde{W}$, contradicting W beging irreducible.

□

Chapter 5

T-Modules of $H(D,2)$, I

Friday, January 29, 1993

Let $\Gamma = (X, E)$ be a graph, A the adjacency matrix, and V the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$ and write $E_i^* \equiv E_i^*(x)$, and $T \equiv T(x)$.

Let W be an irreducible T-module with endpoint $r := \min\{i \mid E_i^* W \neq 0\}$ and diameter $d := |\{i \mid E_i^* W \neq 0\}| - 1$.

We have

$$\begin{aligned} E_i^* W &\neq 0 & r \leq i \leq r+d \qquad (5.1)\\ &= 0 & 0 \leq i < r \;\text{ or }\; r+d < i \leq d(x). \qquad (5.2) \end{aligned}$$

Claim: $E_i^* A E_j^* W \neq 0$ if $|i-j| = 1$ for $r \leq i, j \leq r+d$. (See Lemma 4.1.)

Suppose $E_{j+1}^* A E_j^* W = 0$ for some j with $r \leq j < r+d$. Observe that

$$\tilde{W} = E_r^* W + \cdots + E_j^* W$$

is T-invariant with

$$0 \subsetneq \tilde{W} \subsetneq W.$$

Becase $A\tilde{W} \subseteq \tilde{W}$ since $AE_j^* W \subseteq E_{j-1}^* W + E_j^* W$,

$$E_k^* \tilde{W} \subseteq \tilde{W} \quad \text{for all } \; k,$$

we have $T\tilde{W} \subseteq W$.

Suppose $E_{i-1}^* A E_i^* W = 0$ for some i with $r \leq i < r+d$.

Similarly,

$$\tilde{W} = E_i^* W + \cdots + E_{r+d}^* . W$$

is a T-module with $0 \subsetneq \tilde{W} \subsetneq W$.

Definition 5.1. Let Γ, E_i^*, and T be as above. Irreducible T-modules W and W' are isomorphic whenever there is an isomorphism $\sigma : W \to W'$ of vector spaces such that $a\sigma = \sigma a$ for all $a \in T$.

Recall that the standard module V is an orthogonal direct sum of irreducible T-modules $W_1 \oplus W_2 \oplus \cdots$. Given W in this list, the multiplicity of W in V is

$$|\{j \mid W_j \simeq W\}|.$$

Remark. It is known that the multiplicity does not depend on the decomposition.

Now assume that Γ is the D-cube, $H(D,2)$ with $D \geq 1$. Vew

$$X = \{a_1 \cdots a_D \mid a_i \in \{1,-1\}, 1 \leq i \leq D\}, \tag{5.3}$$
$$E = \{xy \mid x, y \in X,\ x, y \text{ differ in exactly 1 coordinate.}\}. \tag{5.4}$$

Find T-modules.

Claim: $H(D,2)$ is bipartite with a partition $X = X^+ \cup X^-$, where

$$X^+ = \{a_1 \cdots a_D \in X \mid \prod a_i > 0\} \tag{5.5}$$
$$X^- = \{a_1 \cdots a_D \in X \mid \prod a_i < 0\} \tag{5.6}$$

Observe: for all $y, z \in X$,

$$\partial(y,z) = i \Leftrightarrow y, z \text{ differ in exactly in } i \text{ coorinates with } 0 \leq i \leq D.$$

Here, the diameter of $H(D,2) = D = d$ for all $x \in X$.

Theorem 5.1. *Let $\Gamma = H(D,2)$ be as above. Fix $x \in X$, and write $E_i^* = E_i^*(x)$, and $T = T(x)$.*

Let W be an irreducible T-module with endpoint r, and diameter d with $0 \leq r \leq r + d \leq D$.

(i) W has a basis $w_0, w_1, \ldots, w_d$ with $w_i \in E_{i+r}^* W$ for $0 \leq i \leq d$. With respect to which the matrix representing A is

$$\begin{pmatrix}
0 & d & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & d-1 & \cdots & 0 & 0 & 0 \\
0 & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 3 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots \\
0 & 0 & 0 & \ddots & 0 & 2 & 0 \\
0 & 0 & 0 & \cdots & d-1 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & d & 0
\end{pmatrix}$$

(ii) $d = D - 2r$. *In particular,* $0 \leq r \leq D/2$.

(iii) Let W' denote an irreducible T-module with endpoint r'. Then W and W' are isormorphic as T-modules if and only if $r = r'$.

(iv) The multiplicity of the irreducible T-module with endpoint r is

$$\binom{D}{r} - \binom{D}{r-1} \quad \textit{if } 1 \leq r \leq R/2,$$

and 1 *if* $r = 0$.

Proof. Recall that Γ is vertex transitive. It is a Cayley graph.

Hence without loss of generality, we may assume that $x = \overbrace{11 \cdots 1}^{D}$.

Notation: Set $\Omega = \{1, 2, \dots, D\}$. For every subset $S \subseteq \Omega$, let

$$\hat{S} = a_1 \cdots a_d \in X \quad a_i = \begin{cases} -1 & \text{if } \ i \in S \\ 1 & \text{if } i \notin S. \end{cases}$$

In particular, $\hat{\emptyset} = x$ and

$$|S| = i \Leftrightarrow \partial(x, \hat{S}) = i \Leftrightarrow \hat{S} \in E_i^* V.$$

For all $S, T \subseteq \Omega$, we say S covers T if and only if $S \supseteq T$ and $|S| = |T| + 1$.

Observe that $\hat{S}, \hat{T}$ are adjacent in Γ if and only if either T coverse S or S coverr T.

Define the 'raising matrix'

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*.$$

Observe that

$$RE_i^* V \subseteq E_{i+1}^* V \ \text{ for } \ 0 \leq i \leq D, \ \text{ and } E_{D+1}^* V = 0.$$

Indeed for any $S \subseteq \Omega$ with $|S| = i$,

$$\begin{aligned} R\hat{S} &= RE_i^* \hat{S} && (5.7)\\ &= E_{i+1}^* A \hat{S} && (5.8)\\ &= \sum_{T_1 \subseteq \Omega, S \text{ covers } T_1} E_{i+1}^* \widehat{T_1} + \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T} && (5.9)\\ &= \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T}. && (5.10) \end{aligned}$$

Define the 'lowering matrix'

$$L = \sum_{i=0}^{D} E^*_{i-1} A E^*_i.$$

Observe that

$$LE^*_i V \subseteq E^*_{i-1} V \quad \text{for} \quad 0 \le i \le D, \quad \text{and } E^*_{-1} V = 0.$$

Indeed for any $S \subseteq \Omega$,

$$L\hat{S} = \sum_{T \subseteq \Omega, S \text{ covers } T} \hat{T}.$$

Observe that $A = L + R$.

For convenience, set

$$A^* = \sum_{i=0}^{D} (D - 2i) E^*_i.$$

Claim: The following hold.

(a) $LR - RL = A^*$.
(b) $A^*L - LA^* = 2L$.
(c) $A^*R - RA^* = -2R$.

In particular $\mathrm{Span}(R, L, A^*)$ is a 'representation of Lie algebra $\mathrm{sl}_2(\mathbb{C})$.

Remark (Lie Algebra sl2(C)).

$$\mathrm{sl}_2(\mathbb{C}) = \{X \mid \mathrm{Mat}(\mathbb{C} \mid \mathrm{tr}(X) = 0\}.$$

For $X, Y \in \mathrm{sl}_2(\mathbb{C})$, define a binary operation $[X, Y] = XY - YX$.

$$A^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then these satisfy the relations (a) - (c) above.

Proof of Claim. Apply both sides to $\hat{S} \quad (S \subseteq \Omega)$. Say $|S| = i$.

Proof of (a):

$$(LR - RL)\hat{S} = L \left(\sum_{\substack{T \subseteq \Omega, T \text{ covers } S \\ (D-i \text{ of them})}} \hat{T} \right) - R \left(\sum_{\substack{U \subseteq \Omega, S \text{ covers } U \\ (i \text{ of them})}} \hat{T} \right) \tag{5.11}$$

$$= (D - i)\hat{S} + \sum_{V \subseteq \Omega, |V| = i, |S \cap V| = i-1} \hat{V} - \left(i\hat{S} + \sum_{V \subseteq \Omega, |V| = i, |S \cap V| = i-1} \hat{V} \right) \tag{5.12}$$

$$= (D - 2i)\hat{S} \tag{5.13}$$

$$= A^*\hat{S}. \tag{5.14}$$

Proof of (b)*:*

$$(A^*L - LA^*)\hat{S} = (D-2(i-1))L\hat{S} - (D-2i)L\hat{S} \quad (\text{since } L\hat{S} \in E^*_{i-1}V) \tag{5.15}$$
$$= 2L\hat{S}. \tag{5.16}$$

Proof of (c)*:*

$$(A^*R - RA^*)\hat{S} = (D-2(i+1))R\hat{S} - (D-2i)R\hat{S} \quad (\text{since } R\hat{S} \in E^*_{i+1}V) \tag{5.17}$$
$$= 2R\hat{S}. \tag{5.18}$$

Let W be an irreducible T-module with endpoint r and diameter d $(0 \le r \le r+d \le D)$.

Proof of (i) *and* (ii)*:*

Pick $0 \neq w \in E^*_r W$.

Claim: $LRw = (D-2r)w$.

Pf.

$$LRw = (A^* + RL)w \quad (\text{by Claim } (a)) \tag{5.19}$$
$$= A^*w \quad (Lw \in E^*_{r-1}W = 0) \tag{5.20}$$
$$(D-2r)w. \tag{5.21}$$

Define

$$w_i = \frac{1}{i!}R^i w \in E^*_{r+i}W \quad (0 \le i \le d).$$

Then,

$$Rw_i = (i+1)w_{i+1} \quad (0 \le i \le d) \tag{5.22}$$
$$Rw_d = 0 \quad (\text{by definition of } d) \tag{5.23}$$

Claim: $Lw_0 = 0$ and

$$Lw_i = (D-2r-i+1)w_{i-1} \quad (1 \le i \le d).$$

Pf. We prove by induction on i. The case $i=0$ is trivial, and the case $i=1$ follows from above claim. Let $i \ge 2$,

$$Lw_i = \frac{1}{i}LRw_{i-1} = \frac{1}{i}(A^* + RL)w_{i-1} \quad (\text{by Claim (a)}) \tag{5.24}$$
$$(\text{by induction hypothesis}) \tag{5.25}$$
$$= \frac{1}{i}((D-2(r+i-1))w_{i-1} + (D-2r-(i-1)+1)Rw_{i-2} \quad (Rw_{i-2} = (i-1)w_{i-1}) \tag{5.26}$$
$$= \frac{1}{i}i(D-2r-i+1)w_{i-1} \tag{5.27}$$
$$= (D-2r-i+1)w_{i-1}. \tag{5.28}$$

Claim: $w_0, \dots, w_d$ is a basis for W.

Pf. Let $W' = \mathrm{Span}\{w_0, \dots, w_d\}$. Then W' is R and L invariant. So it is $A = R + L$ invariant.

Also it is E_i^*-invariant for every i.

Hence W' is a T-module.

Since W is irreducible, $W' = W$.

As w_i's are orthogonal, theyy are linearly independent. Note that $w_i \neq 0$ by the definition of d and Lemma 4.1 (iv).

Claim: $d = D - 2r$.

Pf. By (a),

$$\begin{aligned}
0 &= (LR - RL - A^*)w_d && (5.29)\\
&= 0 - (D - 2r - d + 1)Rw_{d-1} - (D - 2(r+d))w_d && (5.30)\\
&= -d(D - 2r - d + 1)w_d - (D - 2(r+d))w_d && (5.31)\\
&= (-dD + 2rd + d^2 - d - D + 2r + 2d)w_d && (5.32)\\
&= (d^2 + (2r - D + 1)d + 2r - D)w_d && (5.33)\\
&= (d + 2r - D)(d+1)w_d. && (5.34)
\end{aligned}$$

Hence $d = D - 2r$.

Therefore, with respect to a bais $w_0, w_1, \dots, w_d$, $A = L + R$, $w_{-1} = w_{d+1} = 0$,

$$Lw_i = (d - i + 1)w_{i-1}, \quad Rw_i = (i+1)w_{i+1}.$$

$$L = \begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0 \\ 0 & 0 & d-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & d & 0 \end{pmatrix}.$$

This completes the proof of (i) and (ii). □

Chapter 6

T-Modules of $H(D, 2)$, II

Monday, February 1, 1993

Proof of Theorem 5.1 Continued.

(*iii*) Let $r = r'$,

$w_0, \ldots, w_d$: a basis for W with $w_i \in E_i^* W$, and

$w_0', \ldots, w_d'$: a basis for W' with $w_i' \in E_i^* W'$.

Then $d = D - 2r = D - 2r' = d'$, and

$$\sigma : W \to W' \quad (w_i \mapsto w_i')$$

is an isomorsphism of T-modules by (*i*).

If $r \neq r'$, then

$$d = D - 2r \neq D - 2r' = d',$$

hence, $\dim W \neq \dim W'$.

(*iv*) Let W_i be the irreducible T-module with endpoint i. Then

$$\dim E_r^* V = \binom{D}{r} = \sum_{i=0}^{r} \mathrm{mult}(W_i).$$

Hence, we have that

$$\mathrm{mult}(W_r) = \binom{D}{r} - \binom{D}{r-1}$$

by induction on r.

□

Theorem 6.1. *Let $\Gamma = H(D,2)$ with $D \geq 1$. Fix a vertex $x \in X$ and write*

$$E_i^* \equiv E_i^*(x), \quad T = T(x), and A^* \equiv \sum_{i=0}^{D}(D-2i)E_i^*.$$

Let W be an irreducible T-module with endpoint r with $0 \leq r \leq D/2$. Then,
(i) W has a basis

$$w_0^*, w_1^*, \dots, w_d^* \quad (d = D-2r), \quad \textit{such that} \;\; w_i^* \in E_{i+r}W \; (0 \leq i \leq d)$$

with respect to which the matrix corresponding to A^* is

$$\begin{pmatrix} 0 & d & 0 & & & & \\ 1 & 0 & d-1 & & & & \\ 0 & 2 & 0 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 0 & 2 & 0 \\ & & & & d-1 & 0 & 1 \\ & & & & 0 & d & 0 \end{pmatrix}.$$

In particular, / (ii) $E_iA^*E_j = 0$ if $|i-j| \neq 1$ for $0 \leq i,j \leq D$.

Proof. We use the notation,

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \; (= -[\beta, \alpha]).$$

Recall that

(a) $[L, R] = A^*$,
(b) $[A^*, L] = wL$,
(c) $[A^*, R] = -2R$,

and $A = L + R$.

Write $(a) - (c)$ in terms of A and A^*, we have,

$$[A, A^*] = [L, A^*] + [R, A^*] = 2(R - L).$$

$$\begin{cases} R + L & = A \\ R - L & = [A, A^*]/2. \end{cases}.$$

Hence,

$$R = \frac{1}{4}(2A + [A, A^*]) \quad \text{and} \tag{6.1}$$

$$L = \frac{1}{4}(2A - [A, A^*]). \tag{6.2}$$

Now (a), (b) become

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0 \tag{6.3}$$
$${A^*}^2A - 2A^*AA^* + A{A^*}^2 - 4A = 0 \tag{6.4}$$

Pf. By (b),

$$\begin{align} 2A - AA^* + A^*A &= 4L \tag{6.5}\\ &= 2[A^*, L] \tag{6.6}\\ &= A^*\frac{2A - [A, A^*]}{2} - \frac{2A - [A, A^*]}{2}A^* \tag{6.7}\end{align}$$

So we have ((6.4))

$${A^*}^2A - 2A^*AA^* + A{A^*}^2 - 4A = 0.$$

By (a),

$$\begin{align} -16A^* &= [2A + [A, A^*], 2A - [A, A^*]] \tag{6.8}\\ &(2A + [A, A^*])(2A - [A, A^*]) - (2A - [A, A^*])(2A + [A, A^*]) \tag{6.9}\\ &= [4A^2 - 2A[A, A^*] + [A, A^*](2A) - [A, A^*]^2 \tag{6.10}\\ &\quad - 4A^2 - 2A[A, A^*] + [A, A^*](2A) + [A, A^*]^2 \tag{6.11}\\ &= -4A^2A^* + 4AA^*A + 4AA^*A - 4A^*A^2. \tag{6.12}\end{align}$$

So,

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0.$$

Claim: $E_i^*A^*E_j = 0$ if $|i - j| \neq 1$ for $0 \leq i, j \leq D$.

Pf. We have,

$$\begin{align} 0 &= E_i(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_j \tag{6.13}\\ &= E_iA^*E_j(\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4) \tag{6.14}\\ &\quad (AE_j = \theta_jE_j,\ E_iA = (AE_j)^\top = (\theta_iE_i)^\top = \theta_iE_i) \tag{6.15}\\ &= E_iA^*E_j(\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) \tag{6.16}\\ &= E_iA^*E_j(D - 2i - (D - 2j) - 2)(D - 2i - (D - 2j) + 2) \tag{6.17}\\ &\quad (\theta_k = D - 2k) \tag{6.18}\\ &= E_iA^*E_j \cdot 4(i - j + 1)(i - j - 1) \tag{6.19}\end{align}$$

and $i - j + 1 \neq 0$, $i - j - 1 \neq 0$. Hence, $E_i^*A^*E_j = 0$.

Now define "dual raising matrix",

$$R^* = \sum_{i=0}^{D} E_{i+1}A^*E_i.$$

So,

$$R^* E_i V \subseteq E_{i+1} V, \quad (0 \leq i \leq D,\ E_{D+1} V = 0).$$

Define "dual lowering matrix"

$$L^* = \sum_{i=0}^{D} E_{i-1} A^* E_i.$$

Then

$$L^* E_i V \subseteq E_{i-1} V \quad (0 \leq i \leq D,\ E_{-1} V = 0).$$

Observe that

$$A^* = \left(\sum_{i=0}^{D} E_i \right) A^* \left(\sum_{j=0}^{D} E_j \right) = L^* + R^*$$

by Claim 1.

Claim 2. We have | (a) $[L^*, R^*] = A$, | (b) $[A, L^*] = 2L^*$, | (c) $[A, R^*] = -2R^*$.

Pf. (b)

$$\begin{aligned}
AL^* - L^* A &= \sum_{i=0}^{D} (AE_{i-1} A^* E_i - E_{i-1} A^* E_i A) && (6.20)\\
&= \sum_{i=0}^{D} E_{i-1} A^* E_i (\theta_{i-1} - \theta_i) && (6.21)\\
&\quad (\theta_k = D - 2k, \quad \theta_{i-1} - \theta_i = 2I - 2(i-1) = 2 && (6.22)\\
&= 2L^*. && (6.23)
\end{aligned}$$

(c) Similar.

Remark.

$$\begin{aligned}
AR^* - R^* A &= \sum_{i=0}^{D} (AE_{i+1} A^* E_i - E_{i+1} A^* E_i A) && (6.24)\\
&= \sum_{i=0}^{D} E_{i+1} A^* E_i (\theta_{i+1} - \theta_i) && (6.25)\\
&= 2R^*. && (6.26)
\end{aligned}$$

(a) We have, by (b), (c)

$$[A, A^*] = [A, L^*] + [A, R^*] = 2(L^* - R^*). \qquad (6.27)$$

Since $A^* = L^* + R^*$,

$$R^* = \frac{2A^* + [A^*, A]}{4}, \quad L^* = \frac{2A^* - [A^* - A]}{4}.$$

Now (a) is seen to be equivalent to ((6.4)) upon evaluation. This proves Claim 2.

Remark.

$$[L^*, R^*] = \frac{1}{16}((2A^* - [A^*, A])(2A^* + [A^*, A]) - (2A^* + [A^*, A])(2A^* - [A, A^*])) \tag{6.28}$$

$$= \frac{1}{16}(4{A^*}^2 + 2A^*[A^*, A] - [A^*, A]2A^* - [A^*, A]^2 - 4{A^*}^2 + 2A^*[A^*, A] - [A^*, A]2A^* + [A^*, A]^2) \tag{6.29}$$

$$= \frac{1}{4}({A^*}^2 A - 2A^*AA^* + A{A^*}^2) \tag{6.30}$$

$$= A, \tag{6.31}$$

by ((6.4)).

Now apply same argument as for ((6.3)), ((6.4)) of Theorem 5.1 and observe A^* has $D+1$ distinct eigenvalues. So,

$$A^* = \sum_{i=0}^{D} (D - 2i)E_i^*$$

generates

$$M^* = \mathrm{Span}(E_0^*, \dots, E_D^*).$$

Hence, $E_0, \dots, E_D,\ A^*$ generates T.

Take an irreducible T-module W with endpoint r with $0 \leq r \leq D/2$. Set $t = \min\{i \mid E_i W\}$.

Pick $0 \neq w_0^* \in E_t W$. Set

$$w_i^* = \frac{1}{i!}{R^*}^i w_0^* \in E_{t+i}W \quad \text{for all } i.$$

Then,

$$R^* w_i^* = (i+1)w_{i+1}^* \quad \text{for all } i.$$

By (a), we get by induction, $L^* w_i^* = (D - 2t - i + 1)w_{i-1}^*$,

$$L^* w_i^* = \frac{1}{i} L^* R^* w_{i-1}^* \tag{6.32}$$

$$= \frac{1}{i}(A + R^* L^*)w_{i-1}^* \tag{6.33}$$

$$= \frac{1}{i}((D - 2(t + i - 1))w_{i-1}^* + (i-1)(D - 2t - i + 2)w_{i-1}^*) \tag{6.34}$$

$$= (D - 2t - i + 1)w_{i-1}^*. \tag{6.35}$$

So $\mathrm{Span}(w_0^*, w_1^*, \dots)$ is L^*, R^*, A^*-invariant. Hence, $W = (Span)(w_0^*, w_1^*, \dots, w_d^*)$, $w_0^*, w_1^*, \dots, w_d^* \neq 0$, $w_i^* = 0$ for every $i > d$ by dimension.

Thus $d = D - 2t$.

Pf.

$$
\begin{aligned}
(D-2(t+d))w_d^* &= Aw_d^* && (6.36)\\
&= (L^*R^* - R^*L^*)w_d^* && (6.37)\\
&= -(D-2t-d+1)R^*w_{d-1}^* && (6.38)\\
&= -(D-2t-d+1)dw_d^*. && (6.39)
\end{aligned}
$$

Hence,

$$0 = d^2 + (2t - D - 1 + 2)d - (D - 2t) = (d - D + 2t)(d + 1)$$

So $d = D - 2t$. □

Definition 6.1. For any graph $\Gamma = (X, E)$, pick a vertex $x \in X$ and set $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

(i) an irreducible T-module W is thin if $\dim E_i^* W \leq 1$ for every i,

(ii) Γ is thin with respet to x, if every irreducible $T(x)$-module is thin,

(iii) an irreducible T-module W is dual thin if $\dim E_i W \leq 1$ for every i,

(iv) Γ is dual thin with respect to x, if every irreducible $T(x)$-module is dual thin.

Observe: $H(D, 2)$ is thin, dual thin with respect to each $x \in X$.

Definition 6.2. With above notation, write $D \equiv D(x)$.

(i) an ordering $E_0, E_1, \ldots, E_R$ of primitive idempotents of Γ is restricted if E_0 corresponds to the maximal eigenvalue.

Fix a restricted ordering,

(ii) Γ is Q-polynomial with respect to x, above ordering if there exists $A^* \equiv A^*(x)$ such that

- (a) $E_0^* V, \ldots, E_D^* V$ are the maximal eigenspaces for A^*.
- (b) $E_i A^* E_j = 0$ if $|i - j| > 1$ for $0 \leq i, j \leq R$.

Observe $H(D, 2)$ is Q-polynomial with respect to the natural ordering of the idempotents and every vetex.

Program. Study graphs that are thin and Q-polynomial with respect to each vertex.

(In fact, thin with respect to x implies dual thin with respect to x.)

Get a situation like $H(D, 2)$, where T is generated by A, A^*. Except $\mathrm{sl}_s(\mathbb{C})$ is repalaced by a quantum Lie algebra.

Chapter 7

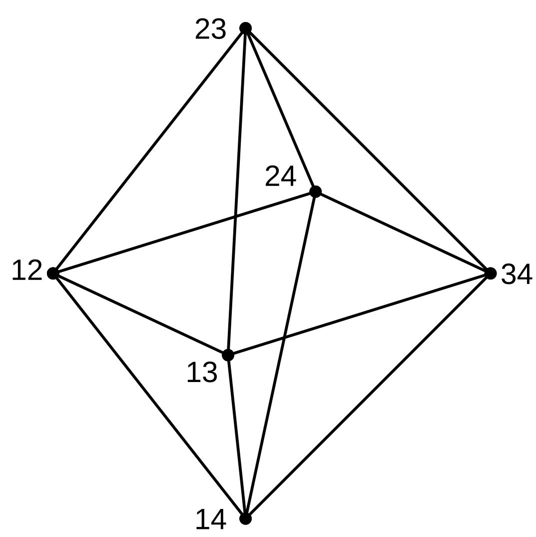
The Johnson Graph $J(D, N)$

Wednesday, February 3, 1993

Definition 7.1. The Johnson graph, $\Gamma = J(D, N)$ $(1 \leq D \leq N - 1)$ satisfies

$$X = \{S \mid S \subset \Omega,\ |S| = D\} \quad \text{where} \ \ \Omega = \{1, 2, \dots, N\} \tag{7.1}$$
$$E = \{ST \mid S, T \in X, \quad |S \cap T| = D - 1\}. \tag{7.2}$$

Example 7.1. $J(2, 4)$



Note 1. The symmetric group S_N acts on Ω. $S_N \subseteq \mathrm{Aut}(\Gamma)$ acts vertex transitively on Γ.

Note 2. $\Gamma = J(D,N)$ is isomorphic to $\Gamma' = J(N-D,N)$.

$$\Gamma = (X,E) \qquad\qquad \Gamma' = (X',E') \tag{7.3}$$
$$X \ni S \quad \longrightarrow \quad \bar{S} = \Omega \quad S \in X' \tag{7.4}$$

This correspondence induces an isomorphism of graphs.

Pf.

$$ST \in E \Leftrightarrow |S \cap T| = D-1 \tag{7.5}$$
$$\Leftrightarrow |\Omega - (S \cup T)| = N-D-1 \tag{7.6}$$
$$\Leftrightarrow |\bar{S} \cap \bar{T}| = N-D-1 \tag{7.7}$$
$$\Leftrightarrow \bar{S}\bar{T} \in E' \tag{7.8}$$

Hence, without loss of generality, assume

$$D \leq N/2 \quad \text{for } J(D,N).$$

We sill need the eigenvalues of $J(D,N)$ for certain problem later in the course. We can get these eigenvalues from our study of $H(D,2)$.

Lemma 7.1. *The eigenvalues for $J(D,N)$ with $1 \leq D \leq N/2$ are give by*

$$\theta_i = (N-D-i)(D-i) - i \quad (0 \leq i \leq D) \tag{7.9}$$
$$m_i = \binom{D}{i} - \binom{N}{i-1}. \tag{7.10}$$

Proof. Let

$$\Gamma_J \equiv J(D,N) = (X_J, E_J) \tag{7.11}$$
$$\Gamma_H \equiv H(N,2) = (X_H, E_H). \tag{7.12}$$

Set $x \equiv 11\cdots 1 \in X_H$.

Define $\tilde{\Gamma} \equiv (\tilde{X}, \tilde{E})$, where

$$\tilde{X} = \{y \in X_H \mid \partial_H(x,y) = D\} \quad \partial_H : \text{distance in } \Gamma_H \tag{7.13}$$
$$\tilde{E} = \{yz \in X_H \mid \partial_H(y,z) = 2\}. \tag{7.14}$$

Observe

$$X_J \rightarrow \qquad \tilde{X} \tag{7.15}$$
$$S \mapsto \qquad \hat{S}, \tag{7.16}$$

where

$$\hat{S} = a_1 \cdots a_N, \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$$

induces an isomorphism of graphs $\Gamma_J \to \tilde{\Gamma}$.

Pf.

$$\begin{aligned}
ST \in E_J &\Leftrightarrow |S \cap T| = D - 1 && (7.17)\\
&\Leftrightarrow \partial_H(\hat{S}, \hat{T}) = 2 && (7.18)\\
&\Leftrightarrow (\hat{S}, \hat{T}) \in \tilde{E}. && (7.19)
\end{aligned}$$

Identify, Γ_J with $\tilde{\Gamma}$. Then the standard module V_J of Γ_J becomes $\tilde{V} = E_D^* V_H$, where V_H is the standard module of Γ_H, and $E_D^* \equiv E_D^*(x)$.

Let R be the raising matrix with respect to x in Γ_H, and

let L be the lowering matrix with respect to x in Γ_H.

Recall

$$(RL - DE_D^*)|_{\tilde{V}}$$

is the adjacency map in $\tilde{\Gamma}$.

To find eigenvalues of $\tilde{A}$, pick any irreducible $T(x)$-module W with the endpoint $r \leq D$. Then by Theorem 5.1

$$\mathrm{diam}(W) = N - 2r + 1.$$

Let $w_0, w_1, \dots, w_{N-2r}$ denote a basis for W as in Theorem 5.1. Then,

$$w_{D-r} \in E_D^* W \subseteq \tilde{V}.$$

Observe:

$$\begin{aligned}
\tilde{A} w_{D-r} &= RLw_{D-r} - DE_D^* w_{D-r} && (7.20)\\
&= R(N - 2r - D + r + 1) w_{D-r-1} - D w_{D-r} && (7.21)\\
&= ((N - D - r + 1)(D - r) - D) w_{D-r}. && (7.22)
\end{aligned}$$

Note that this is valid for $D = r$ as well.

Hence,

$$\tilde{A} w_{D-r} = ((N - D - r)(D - r) - r) w_{D-r}.$$

Let

$$V_H = \sum W \quad \text{(direct sum of irreducible } T(x) - \text{modules.)}$$

Then,

$$\begin{aligned}
V_J &= E_D^* V_H && (7.23)\\
&= \sum_{W : r(W) \leq D} E_D^* W && (7.24)\\
&= \text{a direct sum of 1 dimensional eigenspaces for } \tilde{A}. && (7.25)
\end{aligned}$$

The eigenspace for eigenvalue

$$(N - D - r)(D - r) - r \quad \text{(monotonously decreasing with respec to } r\text{)}$$

appears with multiplicity

$$\binom{N}{r} - \binom{N}{r-1}$$

in this sum by Theorem 5.1 (iv). ☐

Theorem 7.1. *Let $\Gamma = (X, E)$ be any graph. For a fixed vertex $x \in X$, let*

$$E_i^* \equiv E_i^*(x), \quad T \equiv T(x), \quad D \equiv D(x), \text{ and } K = \mathbb{C}.$$

Then we have the following implications of conditions:

$$TH \Leftrightarrow C \Leftarrow S \Leftarrow G.$$

where

(TH) Γ is thinn with respect to x.

(C) $E_i^* T E_i^*$ is commutative for every i, $(0 \leq i \leq D)$.

(S) $E_i^* T E_i^*$ is symmetric for every i, $(0 \leq i \leq D)$.

(G) For every $y, z \in X$ with $\partial(x, y) = \partial(x, z)$, there exists $g \in \mathrm{Aut}(\Gamma)$ such that

$$gx = x, \ gy = z, \ gz = y.$$

Proof.

(TH) $\Rightarrow$ (C)

Fix i with $0 \leq i \leq D$. Let

$V = \sum W$. The standard module written as a direct sum of irreducible T-modules.

The,

$E_i^* V = \sum E_i^* W$. The direct sum of 1-dimensional $E_i^* T E_i^*$-modules.

Since $\dim E_i^* W = 1$, for $a, b \in E_i^* T E_i^*$, $ab - ba_{|E_i^* W} = 0$. Hence $ab - ba = 0$.

(C) $\Rightarrow$ (TH)

Suppose $\dim E_i^* W \geq 2$ for some irreducible T-module W with some i with $1 \leq i \leq D$.

Claim: $E_i^* W$ is an irreducible $E_i^* T E_i^*$-module.

Pf. Suppose

$$0 \subsetneq U \subsetneq E_i^* W,$$

where U is a $E_i^*TE_i^*$-module. Then by the irreducibility,

$$TU = W.$$

So

$$U \supseteq E_i^*TE_i^*U = E_i^*TU = E_i^*W.$$

This is a contradiction.

Claim 2: Each irreducible $S = E_i^*TE_i^*$-module U has dimension 1. In particular, Γ is thin with respect to x.

Pf. Pick

$$0 \neq a \in E_i^*TE_i^*.$$

Since $\mathbb{C}$ is algebraicallt closed, a has an eigenvector $w \in U$ with eigenvalue θ. Then,

$$\begin{aligned}
(a - \theta I)U &= (a - \theta I)Sw && (7.26)\\
&= S(a - \theta I)w && (7.27)\\
&= 0. && (7.28)
\end{aligned}$$

Hence,

$$a_{|U} = \theta I_{|U} \quad \text{for all} \quad a \in S.$$

Thus each 1 dimensional subspace of U is an S-module. We have

$$\dim U = 1.$$

By Claim 1 and Claim 2, we hat (TH).

$\square$

Chapter 8

Thin Graphs

Friday, February 5, 1993

Proof of Theorem 7.1 continued.

(S) $\Rightarrow$ (C)

Fix i and pick $a, b \in E_i^* T E_i^*$.

Since a, b and ab are symmetric,

$$ab = (ab)^\top = b^\top a^\top = ba.$$

Hence $E_i^* T E_i^*$ is commutative.

(G) $\Rightarrow$ (S)

Fix i and pick $a \in E_i^* T E_i^*$. Pick vertices $y, z \in X$.

We want to show that

$$a_{yz} = a_{zy}.$$

We may assume that

$$\partial(x, y) = \partial(x, z) = i,$$

othewise

$$a_{yz} = a_{zy} = 0.$$

By our assumption, there exists $g \in G$ such that

$$g(y) = z, \quad g(z) = y, \quad g(x) = x.$$

Let $\hat{g}$ denote the permutation matrix representing g, i.e.,

$$\hat{g}\hat{y} = \widehat{g(y)} \quad \text{for all} \quad y \in X, \quad \hat{y} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow y.$$

If $g \in \mathrm{Aut}(\Gamma)$, then

$$\hat{g}A = A\hat{g} \quad \text{Exercise.}$$

Also we have

$$\hat{g}E_j^* = E_j^*\hat{g} \quad (0 \leq j \leq D),$$

since

$$\partial(x,y) = \partial(g(x), g(y)) = \partial(x, g(y)).$$

Hence $\hat{g}$ commutes with each element of T. We have

$$a_{yz} = (\hat{g}^{-1}a\hat{g})_{yz}, \quad (\hat{g})_{yz} = \begin{cases} 1 & g(z) = y \\ 0 & \text{else.} \end{cases} \tag{8.1}$$

$$= \sum_{y',z'} (\hat{g}^{-1})_{yy'} a_{y'z'} \hat{g}_{z'z} \tag{8.2}$$

$$\text{(zero except for } g^{-1}(y') = y,\ g(z) = z'.\text{)} \tag{8.3}$$

$$= a_{g(y)g(z)} \tag{8.4}$$

$$a_{zy}. \tag{8.5}$$

This proves Theorem 7.1.

□

Open Problem: Find all the graphs that satisfy the condition (G) for every vertex x.

$H(N,2)$ is one example, because

$$\mathrm{Aut}\Gamma_{1\cdots 1} \simeq S_\Omega, \quad x = (1\cdots 1), \Gamma_i(x) = \{\hat{S} \mid |S| = i\}.$$

Property (G) is clearly related to the distance-transitive property.

Definition 8.1. Let $\Gamma = (X,E)$ be any graph. Γ with $G \subseteq \mathrm{Aut}(\Gamma)$ is said to be distance-transitive (or two-point homogeneous), whenever

$$\text{for all } x, x', y, y' \in X \ \text{ with } \partial(x,y) = \partial(x',y'),$$

there exists $g \in G$ such that

$$g(x) = y, \quad g(x') = y'.$$

(This means G is as close to being doubly transitive as possible.)

Lemma 8.1. *Suppose a graph $\Gamma = (X,E)$ satisfies the property (G) = (G(x)) for every $x \in X$. Then,*

(i) either
(ia) Γ is vertex transitive; or
(iia) Γ is bipartite ($X = X^+ \cup X^-$) with X^+, X^- each an orbit of $\mathrm{Aut}(\Gamma)$.

(ii) if (ia) holds, then Γ is distance-transitive.

Proof. (i) Claim. Suppose $y, z \in X$ are conneced by a path of even length. Then y, z are in the same orbit of $\mathrm{Aut}(\Gamma)$.

Pf. It suffices to assume that the path has lenght 2, $y \sim w \sim z$.

Now $\partial(y, w) = \partial(w, z) = 1$. So there exits $g \in \mathrm{Aut}(\Gamma)$ such that \$$gw = w, \quad gy = z, \quad gz = y$. This proves Claim.

Fix $x \in X$. Now suppose that Γ is not vertex transitive, and we shall show (ib).

Observe that $X = X^+ \cup X^-$, where

$$X^+ = \{y \in X \mid \text{there exists a path of even length connecting } x \text{ and } y\} \quad (8.6)$$
$$X^- = \{y \in X \mid \text{there exists a path of odd length connecting } x \text{ and } y\} \quad (8.7)$$

Asi X^+ is contained in an orbit O^+ of $\mathrm{Aut}(\Gamma)$, and X^- is contained in an orbit O^- of $\mathrm{Aut}(\Gamma)$.

Now $O^+ \cap O^- = \emptyset$ (else $O^+ = O^- = X$ and vertex transitive). So, $X = O^+$, and $X^- = O^-$.

Also $X^+ \cup X^- = X$ is a bipartition by construction.

(ii) Fix x, y, x', y' with $\partial(x, y) = \partial(x', y')$.

By vertex transitivity, there exists an element

$$g_1 \in G \text{ such that } g_1 x = x'.$$

Observe that

$$\partial(x', y') = \partial(x, y) = \partial(g_1 x, g_1 y) = \partial(x', g_1 y).$$

Hence, there exisits an element

$$g_2 \in G \text{ such that } g_1 x' = x', g_2 y' = g_1 y', g_2 g_1 y = y'$$

by $(\mathrm{G}(x'))$ property.

Set $g = g_2 g_1$. Then

$$gx = x', gy = y'$$

by construction. □

The following graphs $\Gamma = (X, E)$ are vertex transitive, and satisfy the property $(\mathrm{G}(x))$ for all $x \in X$.

$$J(D, N), \quad H(D, r), \quad J_q(D, N),$$

where

$H(D, r)$:

$$X = \{a_1 \cdots a_D \mid a_i \in F, 1 \le i \le D\} \tag{8.8}$$
$$F: \text{ any set of cardinality } r \tag{8.9}$$
$$E = \{xy \mid y, x \in X,\ x \text{ and } y \text{ differ in exactly one coordiate}\}. \tag{8.10}$$

$J_q(D, N)$:

$$X = \text{the set of all } D\text{-dimensional subspaces of } N\text{-dimensional vector space over } GF(q). \tag{8.11}$$
$$F: \text{ any set of cardinality } r \tag{8.12}$$
$$E = \{xy \mid y, x \in X,\ \dim(x \cap y) = D - 1\}. \tag{8.13}$$

The following graph is distance-transitive but does not satisify (G(x)) for any $x \in G$.

$H_q(D, N)$:

$$X = \text{the set of all } D \times N \text{ matrices with entries in } GF(q). \tag{8.14}$$
$$E = \{xy \mid y, x \in X,\ \operatorname{rank}(x - y) = 1.\}. \tag{8.15}$$

Remark.

$H(D, r)$: $G = S_r \mathrm{wr} S_D$, $G_x = S_{r-1} \mathrm{wr} S_D$,

For $x, y \in X$ with $\partial(x, y) = \partial(x, z) = i$,

$$Y = \{j \in \Omega \mid x_j \neq y_j\} \leftrightarrow Z = \{j \in \Omega \mid x_j \neq z_j\} \tag{8.16}$$
$$(y_{j_1}, \dots, y_{j_i}) \leftrightarrow (z_{\ell_1}, \dots, z_{\ell_i}) \tag{8.17}$$

$J(D, N)$: $G = S_N$, $G_x = S_D \times S_{N-D}$.

$$X \cap Y \leftrightarrow X \cap Z \tag{8.18}$$
$$(\Omega \quad X) \cap Y \leftrightarrow (\Omega \quad X) \cap Z. \tag{8.19}$$

The following graph is distance-transitive but does not satisify (G(x)) for any $x \in G$.

$J_q(D, N)$:

$$X \cap Y \leftrightarrow X \cap Z.$$

The theory of single thin irreducible T-module.

Let $\Gamma = (X, E)$ be any graph.

$$M = \text{Bose-Mesner algebra over } K/\mathbb{C} \text{ generated by the adjacency matrix } A. \tag{8.20}$$

$$= \text{Span}(E_0, \dots, E_R). \tag{8.21}$$

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix $x \in X$, let $D \equiv D(x)$ be the x-diameter, and $k = k(x)$ be the valency of x.

Chapter 9

Thin T-Module, I

Monday, February 8, 1993

Let $\Gamma = (X, E)$ be any graph.

M: Bose-Mesner algebra over $K/\mathbb{C}$ generated by the adjacency matrix A.

$M = \mathrm{Span}(E_0, \dots, E_R)$.

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix $x \in X$, let $D \equiv D(x)$ be the x-diameter, and $k = k(x)$ be the valency of x.

Definition 9.1. Pick $x \in X$ and write $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

Let W be an irreducible thin T-module with endpoint r, diameter d.

Let $a_i = a_i(W) \in \mathbb{C}$ satisfying

$$E^*_{r+i} A E^*_{r+i}|_{E^*_{r+i}W} = a_i 1|_{E^*_{r+i}} \quad (0 \le i \le d).$$

Let $x_i = x_i(W) \in \mathbb{C}$ satisfying

$$E^*_{r+i-1} A E^*_{r+i} A E^*_{r+i-1}|_{E^*_{r+i}W} = x_i 1|_{E^*_{r+i}} \quad (0 \le i \le d).$$

Lemma 9.1. *With above notation, the following hold.*

(i) $a_i \in \mathbb{R} \quad (0 \le i \le d)$.

(ii) $x_i \in \mathbb{R}^{>0} \quad (0 \le i \le d)$.

(iii) Pick $0 \ne w_0 \in E_r^* W$. *Set* $w_i = E^*_{r+i} A^i w_0$ *for all* i. *Then*

(iiia) $w_0, w_1, \dots, w_d$ *is a basis for* W, $w_{-1} = w_{d+1} = 0$.

(iiib) $A w_i = w_{i+1} + a_i w_i + x_i w_{i-1} \quad (0 \le i \le d)$.

(iv) Define $p_0, p_1, \dots, p_{d+1} \in \mathbb{R}[\lambda]$ *by*

$$p_0 = 1, \quad \lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1} \quad (0 \le i \le d), \quad p_{-1} = 0.$$

(iva) $p_i(A) w_0 = w_i, \quad (0 \le i \le d+1)$.
(ivb) p_{d+1} *is the minimal polynomial of* $A|_W$.

Proof. (*i*) a_i is an eigenvalue of a real symmetric matrix $E^*_{r+i} A E^*_{r+i}$.

(*ii*) x_i is an eigenvalue of a real symmetrix matrix $B^\top B$, where

$$B = E^*_{r+i} A E^*_{r+i-1}.$$

Hence, $x_i \in \mathbb{R}$.

Since $B^\top B$ is positive semidefinite,

$$x_i \ge 0.$$

Pf. If $B^\top B v = \sigma v$ for some $\sigma \in \mathbb{R}$, $v \in \mathbb{R}^m \quad \{0\}$, then

$$0 \le \|Bv\|^2 = v^\top B^\top B v = \sigma v^\top v = \sigma \|v\|^2, \quad \|v\|^2 > 0.$$

Hence, $\sigma \ge 0$.

Moreover, $x_i \neq 0$ by Lemma 4.1 (*iv*).

(*iiia*) Observe

$$w_i = E^*_{r+i} A E^*_{r+i-1} w_{i-1} \quad (1 \le i \le d).$$

So $w_i \neq 0 \quad (1 \le i \le d)$ by Lemma 4.1 (*iv*).

Hence,

$$W = \mathrm{Span}(w_0, \dots, w_d)$$

by Lemma 4.1. (*iii*).

(*iiib*) We have that

$$\begin{aligned} Aw_i &= E^*_{r+i+1} A w_i + E^*_{r+i} A w_i + E^*_{r+i-1} A w_i && (9.1)\\ &= w_{i+1} + E^*_{r+i} A E^*_{r+i} w_i + E^*_{r+i-1} A E^*_{r+i} A E^*_{r+i-1} w_{i-1} && (9.2)\\ &= w_{i+1} + a_i w_i + x_i w_{i-1} && (9.3) \end{aligned}$$

(*iva*) Clear for $i = 0$. Assume it is valid for $0, \dots, i$.

$$p_{i+1}(A) w_0 = (A - a_i I) w_i - x_i w_{i-1} = w_{i+1}.$$

(*ivb*) By definition,

$$p_{d+1}(A) w_0 = 0.$$

Moreover, $p_{d+1}(A)W = 0$. For every $w \in W$, write

$$w = \sum_{i=0}^{d} \alpha_i w_i \tag{9.4}$$

$$= \sum_{i=0}^{d} \alpha_i p_i(A) w_0 \qquad \text{for some } \alpha_i \in \mathbb{C} \tag{9.5}$$

$$= p(A) w_0 \qquad \text{for some } p \in \mathbb{C}[\lambda] \tag{9.6}$$

Hence,

$$\begin{aligned} p_{d+1}(A)w &= p_{d+1}(A)p(A)w_0 && (9.7)\\ &= p(A)p_{d+1}(A)w_0 && (9.8)\\ &= 0. && (9.9) \end{aligned}$$

Note that p_{d+1} is the minimal polynomial.

Pf. Suppose $q(A)W = 0$ for some $0 \neq q \in \mathbb{C}[\lambda]$ with $\deg q < \deg p_{d+1} = d + 1$. Then,

$$q = \sum_{i=0}^{d} \beta_i p_i \quad \text{for some } \beta_i \in \mathbb{C}.$$

We have,

$$0 = q(A)w_0 = \sum_{i=0}^{d} \beta_i w_i.$$

Hence $\beta_0 = \cdots = \beta_d = 0$ by $(iiia)$. Thus $q = 0$ and a contradiction. $\square$

Corollary 9.1. *Let* Γ, W, r, d *be as above. Then*

(i) W *is dual thin, that is,*

$$\dim E_i W \leq 1 \quad (1 \leq i \leq d).$$

(ii) $d = |\{i \mid E_i W \neq 0\}| - 1.$

Proof. (i) Set as in Lemma 9.1,

$$w_i = p_i(A) w_0 \in E^*_{r+i} W.$$

Then $w_0, w_1, \ldots, w_d$ is a basis for W. We have

$$W = M w_0.$$

So,

$$E_i W = E_i M w_0 = \mathrm{Span}(E_i w_0).$$

Thus,

$$\dim E_i W = \begin{cases} 1 & \text{if } E_i w_0 \neq 0 \\ 0 & \text{if } E_i w_0 = 0. \end{cases}$$

In particular,

$$\dim E_i^* W \leq 1.$$

(ii) Immediate as

$$\dim W = d + 1.$$

This proves the lemma. ☐

Lemma 9.2. *Given an irreducible $T(x)$-module W with endpoint $r = r(W)$, diameter $d = d(W)$. Write*

$$x_i = x_i(W)\ (0 \leq i \leq d), \quad w_i = p_i(A) w_0 \in E^*_{r+i} W\ (0 \leq i \leq d), \quad 0 \neq w_0 \in E^*_r W.$$

Then,

$$\frac{\|w_i\|^2}{\|w_0\|^2} = x_1 x_2 \cdots x_i \quad (1 \leq i \leq d).$$

Proof. It suffices to show that

$$\|w_i\|^2 = x_i \|w_i\|^2 \quad (1 \leq i \leq d).$$

Recall by Lemma 9.1 $(iiib)$ that

$$Aw_j = w_{j+1} + a_j w_j + x_j w_{j-1} \quad (0 \leq j \leq d), \quad w_{-1} = w_{d+1} = 0.$$

Now observe,

$$\begin{aligned} \langle w_{i-1}, Aw_i \rangle &= \langle w_{i-1}, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle && (9.10) \\ &= \overline{x_i} \|w_{i-1}\|^2 && (9.11) \\ &= x_i \|w_{i-1}\|^2. && (9.12) \end{aligned}$$

by Lemma 9.1 (ii). Also,

$$\begin{aligned} \langle w_{i-1}, Aw_i \rangle &= \langle Aw_{i-1}, w_i \rangle \quad (textsince\ \bar{A}^\top = A) && (9.13) \\ &= \langle x_i + a_{i-1} w_{i-1} + x_{i-1} x_{i-2}, w_i \rangle && (9.14) \\ &= \|w_i\|^w. && (9.15) \end{aligned}$$

This proves the lemma. ☐

Definition 9.2. Let W be an irreducible thin $T(x)$ module with endpoint r, $E_i^* \equiv E_i^*(x)$.

The measure $m = m_W$ is the function

$$m : \mathbb{R} \to \mathbb{R}$$

such that

$$m(\theta) = \begin{cases} \frac{\|E_i w\|^2}{\|w\|^2} & \text{where } 0 \neq w \in E_r^* W \\ & \text{ if } \theta = \theta_i \text{ is an eigenvalue for } \Gamma \\ 0 & \text{if } \theta \text{ is not an eigenvalue for } \Gamma. \end{cases}$$

Chapter 10

Thin T-Module, II

Wednesday, February 10, 1993

Let $\Gamma = (X, E)$ be any graph.

Fix a vertex $x \in X$. Let $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$, the subconstituent algebra over $\mathbb{C}$, and $V = \mathbb{C}^{|X|}$ the standard module.

Lemma 10.1. *With above notation, let W denote a thin irreducible $T(x)$-module with endpoint r and diameter d. Let*

$$a_i = a_i(W) \quad (0 \le i \le d) \tag{10.1}$$

$$x_i = x_i(W) \quad (1 \le i \le d) \tag{10.2}$$

$$p_i = p_i(W) \quad (0 \le i \le d+1) \tag{10.3}$$

be from Lemma 9.1, and measure $m = m_W$. Then,

(i) $p_0, \dots, p_{d+1}$ are orthogonal with respect to m, i.e.,

$$\sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \le i, j \le d+1) \text{ with } x_{d+1} = 0.$$

(ia) $\displaystyle\sum_{\theta \in \mathbb{R}} p_i(\theta)^2 m(\theta) = x_1 \cdots x_i \quad (0 \le i \le d).$

(iia) $\displaystyle\sum_{\theta \in \mathbb{R}} m(\theta) = 1.$

(iiia) $\displaystyle\sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) = x_1 \cdots x_i a_i \quad (0 \le i \le d).$

Proof. Pick $0 \ne w_0 \in E_r^* W$. Set

$$w_i = p_i(A) w_0 \in E_{r+i}^* W.$$

Since E_i^*W and E_j^*W are orthogonal if $i \neq j$,

$$
\begin{aligned}
\delta_{ij}\|w_i\|^2 &= \langle w_i, w_j \rangle && (10.4)\\
&= \langle p_i(A)w_0, p_j(A)w_0 \rangle && (10.5)\\
&= \left\langle p_i(A)\left(\sum_{\ell=0}^{R} E_\ell\right) w_0, p_j(A)\left(\sum_{\ell=0}^{R} E_\ell\right) w_0 \right\rangle && (10.6)\\
&= \left\langle \sum_{\ell=0}^{R} p_i(\theta_\ell)E_\ell w_0, \sum_{\ell=0}^{R} p_j(\theta_\ell)E_\ell w_0 \right\rangle \qquad (\text{as } AE_j = \theta_j E_j) && (10.7)\\
&= \sum_{\ell=0}^{R} p_i(\theta_\ell)\overline{p_j(\theta_\ell)}\|E_\ell w_0\|^2 && (10.8)\\
&\qquad (\text{as } \ p_j \in \mathbb{R}[\lambda], \quad \theta_\ell \in \mathbb{R}, \quad m(\theta_i)\|w_0\|^2 = \|E_i w_0\|^2) && (10.9)\\
&= \sum_{\theta\in\mathbb{R}} p_i(\theta)p_j(\theta)m(\theta)\|w_0\|^2. && (10.10)
\end{aligned}
$$

Now we are done by Lemma 9.2 as

$$\|w_i\|^2 = \|w_0\|^2 x_1 x_2 \dots x_i.$$

For (ia), set $i = j$, and for (ib), set $i = j = 0$.

(ii) We have

$$
\begin{aligned}
\langle w_i, Aw_i \rangle &= \langle w_i, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle && (10.11)\\
&= \overline{a_i}\|w_i\|^2 && (10.12)\\
&= a_i x_1 \cdots x_i \|w_0\|^2, && (10.13)
\end{aligned}
$$

as $a_i \in \mathbb{R}$ by Lemma 9.1.

Also,

$$
\begin{aligned}
\langle w_i, Aw_i \rangle &= \langle p_i(A)w_0, Ap_i(A)w_0 \rangle && (10.14)\\
&= \left\langle p_i(A)\left(\sum_{\ell=0}^{R} E_\ell\right) w_0, Ap_i(A)\left(\sum_{\ell=0}^{R} E_\ell\right) w_0 \right\rangle \qquad (\text{ as in } (i)) && (10.15)\\
&= \sum_{\ell=0}^{D} p_i(\theta_\ell)^2\theta_\ell\|E_\ell w_0\|^2 && (10.16)\\
&= \sum_{\theta\in\mathbb{R}} p_i(\theta)^2\theta m(\theta)\|w_0\|^2. && (10.17)
\end{aligned}
$$

Thus, we have (ii). $\square$

Lemma 10.2. *With above notation, let W be a thin irreducible $T(x)$-module with measure m. Then m determines diameter $d(W)$,*

$$a_i = a_i(W) \quad (0 \leq i \leq d) \tag{10.18}$$
$$x_i = x_i(W) \quad (1 \leq i \leq d) \tag{10.19}$$
$$p_i = p_i(W) \quad (0 \leq i \leq d+1). \tag{10.20}$$

Proof. Note that $d+1$ is the number of $\theta \in \mathbb{R}$ such that $m(\theta) \neq 0$. Hence m determines d.

Apply (ia), (ii) of Lemma 10.1.

$$\sum_{\theta\in\mathbb{R}} m(\theta) = 1 \qquad\qquad p_0 = 1. \tag{10.21}$$
$$\sum_{\theta\in\mathbb{R}} \theta m(\theta) = a_0 \qquad\qquad p_1 = \lambda - a_0 \tag{10.22}$$
$$\sum_{\theta\in\mathbb{R}} p_1(\theta)^2 m(\theta) = x_1 \tag{10.23}$$
$$\sum_{\theta\in\mathbb{R}} p_1(\theta)^2 \theta m(\theta) = x_1 a \qquad\qquad \to a_1 \tag{10.24}$$
$$p_2 = (\lambda - a_1)p_1 - x_1 p_0 \tag{10.25}$$
$$\sum_{\theta\in\mathbb{R}} p_2(\theta)^2 m(\theta) = x_1 x_2 \qquad\qquad \to x_2 \tag{10.26}$$
$$\sum_{\theta\in\mathbb{R}} p_2(\theta)^2 \theta m(\theta) = x_1 x_2 a_2 \qquad\qquad \to a_2 \tag{10.27}$$
$$p_3 = (\lambda - a_2)p_2 - x_2 p_1 \tag{10.28}$$
$$\vdots \tag{10.29}$$
$$\sum_{\theta\in\mathbb{R}} p_d(\theta)^2 m(\theta) = x_1 x_2 \cdots x_d \qquad\qquad \to x_d \tag{10.30}$$
$$\sum_{\theta\in\mathbb{R}} p_d(\theta)^2 \theta m(\theta) = x_1 x_2 \cdots x_d a_d \qquad\qquad \to a_d \tag{10.31}$$
$$p_{d+1} = (\lambda - a_d)p_d - x_d p_{d-1}. \tag{10.32}$$
$$\tag{10.33}$$

This proves the assertions. □

Corollary 10.1. *With above notation, let W, W' denote thin irreducible $T(x)$-modules. The following are equivalent.*

(i) W, W are isomorpphic as T-modoles.

(ii) $r(W) = r(W')$ and $m_W = m_{W'}$.

(iii) $r(W) = r(W')$, $d(W) = d(W')$, $a_i(W) = a_i(W')$ amd $x_i(W) = x_i(W')$ $(0 \leq i \leq d)$.

Proof. (*i*) $\Rightarrow$ (*iii*) Write $r \equiv r(W)$, $r' \equiv r(W')$, $d = d(W)$, $d' = d(W')$, $a_i = a_i(W)$, $a_i' = a_i(W')$, $x_i = x_i(W)$ and $x_i' = x_i(W')$.

Let $\sigma : W \to W'$ denote an isomorphism of T-modules. (See Definition 5.1.)

For every i,

$$\sigma E_i^* W = E_i^* \sigma W = E_i^* W'.$$

So, $r = r'$ and $d = d'$.

To show $a_i = a_i'$, pick $w \in E_{r+i}^* W \quad \{0\}$. Then,

$$E_{r+i}^* A E_{r+i}^* \sigma(W) = \sigma(E_{r+i}^* A E_{r+i}^* w) = \sigma(a_i w) = a_i \sigma(w),$$

and $\sigma w \neq 0$. So,

$$a_i = \text{eigenvalue of } E_{r+i}^* A E_{r+i}^* \text{ on } E_{r+i}^* W \tag{10.34}$$

$$= a_i' \tag{10.35}$$

It is similar to show $x = x'$.

Remark. Pick $w \in E_{r+i-1}^* W \quad \{0\}$

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* \sigma(W) = \sigma(E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* w) = x_i \sigma(w).$$

Hence, x_i is the eigenvalue of $E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^*$ on $E_{r+i-1}^* W = x_i'$.

(*iii*) $\Rightarrow$ (*i*)

Pick $0 \neq w_0 \in E_r^* W$, $0 \neq w_0' \in E_r^* W'$. Let p_i be in Lemma 9.1, and set

$$w_i = p_i(A) w_0 \in E_{r+i}^* W \quad (0 \leq i \leq d) \tag{10.36}$$

$$w_i' = p_i'(A) w_0' \in E_{r+i}^* W \quad (0 \leq i \leq d) \tag{10.37}$$

Define a linear transformation,

$$\sigma : W \to W' \quad (w_i \mapsto w_i').$$

Since $\{w_i\}$ and $\{w_i'\}$ are bases with $d = d'$, σ is an isomorphism of vector spaces.

We need to show

$$a\sigma = \sigma a \quad (\text{for all } \ a \in T).$$

Take $a = E_j^*$ for some j $(0 \leq j \leq d(x))$. Then for all i, we have

$$E_j^* \sigma w_i = E_j^* w_i' = \delta_{ij} w_i',$$

$$\sigma E_j^* w_i = \delta_{ij} \sigma(w_i) = \delta_{ij} w_i'.$$

$$E_j^* \sigma w_i = \sigma E_j^* w_i ?$$

Take an adjacency matrix A of a. Then,

$$A \sigma w_i = A w_i' = w_{i+1}' + a_i' w_i' + x_i' w_{i-1}' = \sigma(w_{i+1} + a_i w_i + x_i w_{i-1}) = \sigma A w_i.$$

$(ii) \Rightarrow (iii)$ Lemma 10.2.

$(iii) \Rightarrow (ii)$ Given d, a_i, x_i, we can compute the polynomial sequence

$$p_0, p_1, \dots, p_{d+1}$$

for W.

Show $p_0, p_1, \dots, p_{d+1}$ determines $m = m_W$. Set

$$\Delta = \{\theta \in \mathbb{R} \mid p_{d+1}(\theta) = 0\}.$$

Observe: $|\Delta| = d + 1$. See 'An Introcuction to Interlacing'.

$m(\theta) = 0$ if $\theta \notin \Delta \quad (\theta \in \mathbb{R})$. So it suffices to find $m(\theta)$, $\theta \in \Delta$.

By Lemma 10.1 (i),

$$\begin{cases} \sum_{\theta\in\Delta} m(\theta)p_0(\theta) & = 1 \\ \sum_{\theta\in\Delta} m(\theta)p_1(\theta) & = 0 \\ \quad\vdots & \\ \sum_{\theta\in\Delta} m(\theta)p_d(\theta) & = 0 \end{cases}$$

$d + 1$ linear equation with $d + 1$ unknowns $m(\theta)$ $(\theta \in \Delta)$.

But the coefficient matrix is essentially Vander Monde (since $\deg p_i = i$). Hence the system is nonsingular and there are unique values for $m(\theta)$ $(\theta \in \Delta)$. $\square$

Remark.

$$\begin{pmatrix} \theta - a_0 & -1 & \cdots & 0 & 0 \\ -x_1 & \theta - a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta - a_{d-1} & -1 \\ 0 & 0 & \cdots & -x_d & \theta - a_d \end{pmatrix} \begin{pmatrix} p_0(\theta) \\ \vdots \\ \vdots \\ \vdots \\ p_d(\theta) \end{pmatrix} = 0,$$

where θ is an eigenvalue of a diagonalizable matrix

$$L = \begin{pmatrix} a_0 & 1 & \cdots & 0 & 0 \\ x_1 & a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{d-1} & 1 \\ 0 & 0 & \cdots & x_d & \theta a_d \end{pmatrix}$$

with multiplicity $\dim(\mathrm{Ker}(\theta I - L) = 1)$.

Chapter 11

Examples of T-Module

Friday, February 12, 1993

Let $\Gamma = (X, E)$ be a connected graph.

Let θ_0 be the maximal eigenvalue of Γ, and δ its corresponding eigenvector.

$$\delta = \sum_{y \in X} \delta_y \hat{y}.$$

Without loss of generality, we may assume that $\delta_y \in \mathbb{R}^*$ for all $y \in X$.

Lemma 11.1. *Fix a vertex $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$.*

(i) $T\delta = T\hat{x}$ is an irreducible T-module.

(ii) Given any irreducible T-module W, the following are equivalent:
(iia) $W = T\delta$.
(iib) The diameter $d(W) = d(x)$.
(iic) The endpoint $r(W) = 0$.

Proof. (*i*) Observe: there exists an irreducible T-module W that contains δ.

Let $V = \sum_i W_i$ be a direct sum decomposition of the standard module. Then

$$\mathrm{Span}(\delta) = E_0 V = \sum_i E_0 W_i.$$

So, $E_0 W_i \neq 0$ for some i. Then,

$$\delta \in E_0 W_i \subseteq W_i.$$

Observe: $T\delta$ is an irreducible T-module.

Since $\delta \in W$, where W is a T-module. As $T\delta \subseteq W$ and W is irreducible, $T\delta = W$.

Observe: $T\delta = T\hat{x}$.

Since $\hat{x} = \delta_x^{-1} E_0^* \delta \in T\delta$, $T\hat{x} \subseteq T\delta$. Since $T\delta$ is irreducible, $T\hat{x} = T\delta$.

(ii) $(a) \to (b)$:

$$E_i^*\delta = \sum_{y \in X, \partial(x,y)=i} \delta_y \hat{y} \neq 0, \quad (0 \leq i \leq d(x)),$$

because $\delta_y > 0$ for every $y \in X$.

Hence,

$$E_i^* T\delta \neq 0, \quad (0 \leq i \leq d(x)).$$

Thus, $d(x) = d(W)$.

$(b) \to (c)$: Immediate.

$(c) \to (a)$: Since $r(W) = 0$, $E_0^* W \neq 0$. Hence, $\hat{x} \in W$ and $T\hat{x} \subseteq W$.

By the irreduciblity, we have $T\hat{x} = W$. $\square$

Lemma 11.2. *Assume Γ is bipartite ($X = X^+ \cup X^-$) (X^+ and X^- are nonempty). Then the following are equivalent.*

(i) *There exist α^+ and $\alpha^- \in \mathbb{R}$ such that*

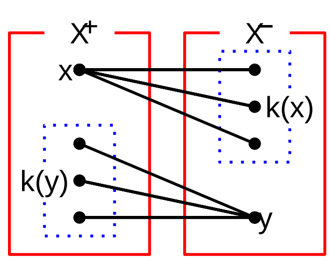
$$\delta_x = \begin{cases} \alpha^+ & \text{if } x \in X^+ \\ \alpha^- & \text{if } x \in X^-. \end{cases}$$

/ (ii) *There exist $k+$ and $k^- \in \mathbb{Z}^{>0}$ such that*

$$k(x) = \begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^-. \end{cases}$$

In this xase, $k^+k^- = \theta_0^2$, and Γ is called bi-regular.

Proof. $(i) \to (ii)$



$$
\begin{aligned}
A\delta &= A\left(\alpha^+ \sum_{x\in X^+} \hat{x} + \alpha^- \sum_{y\in X^-} \hat{y}\right) && (11.1)\\
&= \alpha^+ \sum_{y\in X^-} k(y)\hat{y} + \alpha^- \sum_{x\in X^+} k(x)\hat{x} && (11.2)\\
&= \theta_0\delta. && (11.3)
\end{aligned}
$$

So,

$$k(x)\alpha^- = \theta_0\alpha^+, \quad k(y)\alpha^+ = \theta_0\alpha^-.$$

As $\alpha^+ \neq=$ and $\alpha^- \neq 0$,

$$
\begin{aligned}
k^+ &:= k(x) \text{ is independent of the choice of } x \in X^+, \text{ and} && (11.4)\\
k^- &:= k(y) \text{ is independent of the choice of } y \in X^-. && (11.5)
\end{aligned}
$$

Moreover, $k^+k^- = \theta_0^2$.

$(ii) \to (i)$ Set

$$\delta' = \sum_{y\in X} \alpha_y \hat{y} \quad \text{where } \alpha = \begin{cases} 1/\sqrt{k^-} & \text{if } y \in X^+ \\ 1/\sqrt{k^+} & \text{if } y \in X^-. \end{cases}$$

Then one checks

$$
\begin{aligned}
A\delta' &= A\left(\frac{1}{\sqrt{k^-}} \sum_{y\in X^+} \hat{y} + \frac{1}{\sqrt{k^+}} \sum_{y\in X^-} \hat{y}\right) && (11.6)\\
&= \frac{k^-}{\sqrt{k^-}} \sum_{y\in X^-} \hat{y} + \frac{k^+}{\sqrt{k^+}} \sum_{y\in X^+} \hat{y} && (11.7)\\
&= \sqrt{k^+k^-}\delta' && (11.8)
\end{aligned}
$$

Since $\delta' > 0$, $\delta' \in \mathrm{Span}(\delta)$, and $\theta_0 = \sqrt{k^+k^-}$. $\square$

Definition 11.1. For any graph $\Gamma = (X, E)$, fix a vertex $x \in X$. Set $d = d(x)$.

Γ is distance-regular with respect to x, if for all i : (0 i d), and all $y \in X$ such that $\partial(x, y) = i$:

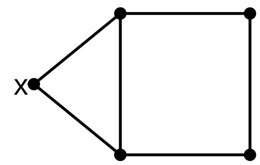
$$
\begin{aligned}
c_i(x) &:= |\{z \in X \mid \partial(x, z) = i - 1,\ \partial(y, z) = 1\}| && (11.9)\\
a_i(x) &:= |\{z \in X \mid \partial(x, z) = i,\ \partial(y, z) = 1\}| && (11.10)\\
b_i(x) &:= |\{z \in X \mid \partial(x, z) = i + 1,\ \partial(y, z) = 1\}| && (11.11)
\end{aligned}
$$

depends only on i, x, and not on y.

(In this case, $c_0(x) = a_0(x) = b_d(x) = 0$, $c_1(x) = 1$, $b_0(x) = k(x)$ is the valency of x.)

We call $c_i(x)$, $a_i(x)$ and $b_i(x)$ the intersection numbers with respect to x.

Example 11.1.



$$c_0 = 1 \qquad c_1 = 1 \qquad c_2 = 1 \tag{11.12}$$
$$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = 1 \tag{11.13}$$
$$b_0 = 2 \qquad b_1 = 1 \qquad b_2 = 0 \tag{11.14}$$

Chapter 12

Distance-Regular

Monday, February 15, 1993

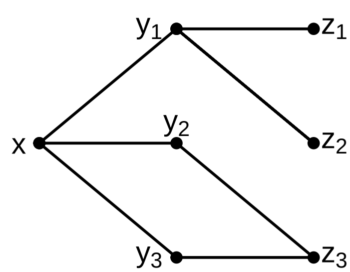
Lemma 12.1. *For any connected graph $\Gamma = (X, E)$, the following are equivalent.*

(i) The trivial $T(x)$-module is thin for all $x \in X$.

(ii) $\left\{ \sum_{y \in X, \partial(x,y)=i} \hat{y} \,|0 \leq i \leq d(x) \right\}$ *is a basis for the trivial $T(x)$-module for every $x \in X$.*

(iii) Γ is distance-regular with respect to x for all $x \in X$.

Note. Let $\Gamma = (X, E)$ be a graph, with $X = \{x, y_1, y_2, y_3, z_1, z_2, z_3\}$, $E = \{xy_1, xy_2, xy_3, y_1z_1, y_1z_2, y_2z_3, y_3z_3\}$.



Then (i), (ii) are not equivalent for a single vertex x.

$$E_0^* T\hat{x} = \langle \hat{x} \rangle, \tag{12.1}$$
$$E_1^* T\hat{x} = \langle y_1 + y_2 + y_3 \rangle, \tag{12.2}$$
$$E_2^* T\hat{x} = \langle z_1 + z_2 + 2z_3 \rangle. \tag{12.3}$$

Proof of Lemma 12.1. $(i) \to (ii)$ Let $\delta = \sum_{y \in X} \delta_y \hat{y}$ be an eigenvector for the maximal eigenvalue θ_0. Then,

$$\sum_{y \in X, \partial(x,y)=1} \hat{y} = A\hat{x} \in T(x)\hat{x} = T(x)\delta \ni E_1^* \delta \tag{12.4}$$

$$= \sum_{y \in X, \partial(x,y)=1} \delta_h \hat{y} \tag{12.5}$$

If the trivial $T(x)$-module is thin,

$$\delta_y = \delta_z \;\text{ for }\; y, z \in X,\ \partial(x,y) = \partial(x,z) = 1.$$

Hence, $\delta_y = \delta_z$ if y and z in X are connected by a path of even length.

So, Γ is regular or bipartite biregular by Lemma 11.2.

In particular, $\delta_y = \delta_z$ if $\partial(x,y) = \partial(x,z)$, as there is a path of length $2 \cdot \partial(x,y)$;

$$y \sim \cdots \sim x \sim \cdots \sim z.$$

Hence,

$$E_i^* \delta \in \operatorname{Span} \left(\sum_{y \in X, \partial(x,y)=i} \hat{y} \right).$$

Since $E_0^*\delta, E_1^*\delta, \ldots, E_d^*\delta$ forms a basis for $T(x)\delta$, we have (ii).

$(ii) \to (iii)$ Fix $x \in X$, and let $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, and $d \equiv d(x)$.

$$A \sum_{y \in X, \partial(x.y)=i} \hat{y} = \sum_{z \in X} |\{y \in X \mid \partial(y,z) = 1,\ \partial(x,y) = i\}| \hat{z} \tag{12.6}$$

$$= \sum_{z \in X, \partial(x,y)=i-1} b_{i-1}(x,z)\hat{z} \tag{12.7}$$

$$+ \sum_{z \in X, \partial(x,y)=i} a_i(x,z)\hat{z} \tag{12.8}$$

$$+ \sum_{z \in X, \partial(x,y)=i+1} c_{i+1}(x,z)\hat{z} \tag{12.9}$$

$$\in \operatorname{Span} \left\{ \sum_{z \in X, \partial(x,z)=j} \hat{z} \;\middle|\; j = 0, 1, \ldots, d \right\}. \tag{12.10}$$

Hence, $b_{i-1}(x,z)$, $a_i(x,z)$ and $c_{i+1}(x,z)$ depend only on i and x, and not on z. Therefore, Γ is distance-regular with respect to x.

$(iii) \to (i)$ Fix $x \in X$, and let $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, and $d \equiv d(x)$. By defintion of distance-regularity, for every i $(0 \le i \le d)$,

$$A\left(\sum_{y\in X,\partial(x,y)=i} \hat{y}\right) = b_{i-1}(x) \sum_{y\in X,\partial(x,y)=i-1} \hat{y} \tag{12.11}$$

$$+ a_i(x) \sum_{y\in X,\partial(x,y)=i} \hat{y} \tag{12.12}$$

$$+ c_{i+1}(x) \sum_{y\in X,\partial(x,y)=i+1} \hat{y}. \tag{12.13}$$

Hence,

$$W = \left\{ \sum_{y\in X,\partial(x,y)=i} \hat{y} \;\middle|\; 0 \le i \le d \right\}$$

is A-invariant and so T-invariant. Since $\hat{x} \in W$, $T\hat{x} = W$ is the trivial module and $T\hat{x}$ is thin. □

Next, we show more is true if $(i) - (iii)$ hold in Lemma 12.1.

In fact, $d(x)$, $a_i(x)$, $c_i(x)$, and $b_i(x)$ are

$$\begin{cases} \text{independent of } X & \text{if } \Gamma \text{ is regular; or} \\ \text{constant over } X^+ \text{ and } X^- & \text{if } \Gamma \text{ is biregular.} \end{cases}$$

Let $\Gamma = (X, E)$ be any (connected) graph. Pick vertices $x, y \in X$.

Let W be a thin, irreducible $T(x)$-module, and measure $m : \mathbb{R} \to \mathbb{R}$ determined by W.

Let W' be a thin, irreducible $T(y)$-module, and measure $m : \mathbb{R} \to \mathbb{R}$ determined by W'.

Recall W, W' are orthogonal if

$$\langle w, w' \rangle = 0 \quad \text{for all } w \in W, w' \in W'.$$

We shall show if W and W' are note orthogonal, then m and m' are related:

$$m \cdot \mathrm{poly}_1 = m' \cdot \mathrm{poly}_2$$

for some polynomials with

$$\deg \mathrm{poly}_1 + \deg \mathrm{poly}_2 \le 2 \cdot \partial(x, y).$$

Notation. V: standard module of Γ.

H: any subspace of V.

$$V = H + H^{\perp} \quad \text{orthogonal direct sum},$$

and for $v = v_1 + v_2$ $\text{proj}_H : V \to H$ $(v \mapsto v_1)$: linear transformation.

Observe: For every $v \in V$,

$$v - \text{proj}_H v \in H^{\perp}.$$

So,

$$\langle v - \text{proj}_H v, h\rangle = 0 \quad \text{for all } \ h \in H \ \text{or},$$

$$\langle v, h\rangle = \langle \text{proj}_H v, h\rangle \quad \text{for all } \ v \in V, \ \text{ and for all } \ h \in H.$$

Theorem 12.1. *Let $\Gamma = (X, E)$ be any graph. Pick vertices $x, y \in X$ and set $\Delta = \partial(x, y)$. Assume*

W: thin irreducible $T(x)$-module with endpoint r, diameter d, and measure m.

W': thin irreducible $T(y)$-module with endpoint r', diameter d', and measure m'.

W and W' are not orghotonal.

Now pick

$$0 \neq w \in E^*_r(x)W, \quad 0 \neq w \in E^*_{r'}(x)W'.$$

Then,

(i) $\text{proj}_{W'} w = p(A)\dfrac{\|w\|}{\|w'\|}w'$

for some $0 \neq p \in \mathbb{C}[\lambda]$ with $\deg p \leq \Delta - r' + r, d'$,

$\text{proj}_W w' = p'(A)\dfrac{\|w'\|}{\|w\|}w$

for some $0 \neq p' \in \mathbb{C}[\lambda]$ with $\deg p \leq \Delta - r + r', d$.

(ii) For all eigenvalues θ_i of Γ,

$$\frac{\langle E_i w, E_i w'\rangle}{\|w\|\|w'\|} = m(\theta_i)\overline{p'(\theta_i)}.$$

(iii) For all eigenvalues θ_i of Γ,

$$p(\theta_i)p'(\theta_i)$$

is in a real number in interval $[0, 1]$.

Proof. (i) Since W, W' are not orthogonal, there exist

$$v \in W,\ v' \in W' \ \text{ sich that } \ \langle v, v' \rangle \neq 0.$$

Then there exists $a \in M$ such that

$$v' = aw'.$$

(This is becase $w_i' = p_i'(A)w_0'$ and hence for every $v' \in W'$, there is a polynomial $q \in \mathbb{C}[\lambda]$, $q(A)w_0' = v$.)

We have

$$0 \neq \langle v', v \rangle = \langle aw', v \rangle = \langle w', a^* v \rangle$$

and $a^* v \in W$.

Hence, $\mathrm{proj}_W w' \neq 0$.

Let $p_0, \dots, p_d \in \mathbb{C}[\lambda]$ be from Lemma 9.1.

Then, $w_i = p_i(A)w$ is a basis for $E^*_{r+i}(x)W \quad (0 \leq i \leq d)$.

Hence,

$$\mathrm{proj}_W w' = \alpha_0 w_0 + \cdots + \alpha_d w_d \quad \text{for some } \ \alpha_j \in \mathbb{C}.$$

Set

$$p' := \frac{\|w\|}{\|w'\|} \sum_{i=0}^{d} \alpha_i p_i.$$

Then $0 \neq p' \in \mathbb{C}[\lambda]$ and $\deg p' \leq d$.

Claim: $\alpha_i = 0$ $(\Delta - r + r' < i \leq d)$.

In particular, $\deg p' \leq \Delta - r + r'$.

Pf. Obseve:

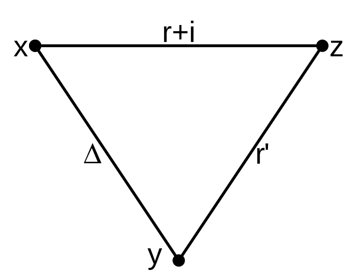
$$w' \in E^*_{r'}(y)V, \quad w \in E^*_r(x)V,$$

for $\partial(x, y) = \Delta$.

$$E^*_{r'}(y)V \cap E^*_{r+i}(x)V = 0$$

by triangle inequality.

$(\Delta = \partial(x, y) < r + i - r'$ or $\Delta + r' < r + i$ by our choice of i.)



Hence,

$$E^*_{r'}(y)V \perp E^*_{r+i}(x)V,$$

or

$$\begin{aligned} 0 &= \langle w', w_i \rangle && (12.14)\\ &= \langle \mathrm{proj}_W w', w_i \rangle && (12.15)\\ &= \sum_{j=0}^{d} \alpha_j \langle w_j, w_i \rangle && (12.16)\\ &= \alpha_i \|w_i\|^2. && (12.17) \end{aligned}$$

Hence, $\alpha_i = 0$. Thus,

$$\begin{aligned} \mathrm{proj}_W w' &= \sum_{i=0}^{\Delta+r'-r} \alpha_i w_i && (12.18)\\ &= \sum_{i=0}^{\Delta+r'-r} \alpha_i p_i(A) w_0 && (12.19)\\ &= p'(A)\frac{\|w'\|}{\|w\|} w. && (12.20) \end{aligned}$$

(ii) We have

$$\begin{aligned} \frac{\langle E_i w, E_i w' \rangle}{\|w\|\|w'\|} &= \frac{\langle E_i w, w' \rangle}{\|w\|\|w'\|} && (12.21)\\ &= \frac{\langle E_i w, \mathrm{proj}_W w' \rangle}{\|w\|\|w'\|} \qquad \text{as } \mathrm{proj}_W w' = p'(A)\frac{\|w\|}{\|w'\|} w && (12.22)\\ &= \frac{\langle E_i w, p'(A) w \rangle}{\|w\|^2} && (12.23)\\ &= \frac{\langle E_i w, E_i p'(A) w \rangle}{\|w\|^2} && (12.24)\\ &= \overline{p'(\theta_i)} \frac{\|E_i W\|^2}{\|w\|^2} && (12.25)\\ &= \overline{p'(\theta_i)} m(\theta_i). && (12.26) \end{aligned}$$

Moreover, as $m(\theta_i)$, $m'(\theta_i) \in \mathbb{R}$,

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\|\|w'\|} = \frac{\overline{\langle E_i w, E_i w' \rangle}}{\|w'\|\|w\|} = \overline{\overline{p(\theta_i)} m'(\theta_i)} = p(\theta_i) m'(\theta_i).$$

(iii) Sicne,

$$\frac{|\langle E_i w, E_i w' \rangle\|^2}{\|w\|^2\|w'\|^2} = p(\theta_i) p'(\theta_i) m(\theta_i) m'(\theta_i),$$

$$p(\theta_i)p'(\theta_i) = \frac{|\langle E_i w, E_i w' \rangle\|^2}{m(\theta_i)m'(\theta_i)\|w\|^2\|w'\|^2} \in \mathbb{R} \tag{12.27}$$

$$= \frac{|\langle E_i w, E_i w' \rangle\|^2}{\frac{\|E_i w\|^2}{\|w\|^2}\frac{\|E_i w'\|^2}{\|w'\|^2}\|w\|^2\|w'\|^2}. \tag{12.28}$$

By Cauchy-Schwartz inequality,

$$(|\langle a, b \rangle| \leq \|a\|\|b\|,)$$

$$\frac{|\langle E_i w, E_i w' \rangle\|^2}{\|E_i w\|^2\|E_i w'\|^2} \leq 1.$$

Hence, we have the assertion. $\square$

Chapter 13

Modules of a DRG

Wednesday, February 17, 1993

Lemma 13.1. *Let $\Gamma = (X, E)$ be any graph. Pick an edge $xy \in E$.*
Assume the trivial $T(x)$-module $T(x)\delta$ is thin with measure m_x,
and the trivial $T(y)$-module $T(y)\delta$ is thin with measure m_y.
Then,

(ia) $\dfrac{m_x(\theta)}{k_x} = \dfrac{m_y(\theta)}{k_y}$ *for all* $\theta \in \mathbb{R}\ \ \{0\}$.

(ib) $\dfrac{m_x(0) - 1}{k_x} = \dfrac{m_y(0) - 1}{k_y}$ *for all* $\theta \in \mathbb{R}\ \ \{0\}$.

$$(\delta = \sum_{y \in X} \delta_y \hat{y} \quad \textit{eigenvector corresponding to the maximal eigenvalue})$$

Proof. Apply Theorem 12.1,

$$W = T(x)\delta \quad r = 0, \quad d = d(x) \tag{13.1}$$
$$W' = T(y)\delta \quad r' = 0, \quad d' = d(y). \tag{13.2}$$

Take $w = \hat{x}$, $w' = \hat{y}$.

Claim. $\text{proj}_{T(y)\delta}\hat{x} = k_y^{-1} A\hat{y}$.

Pf. Since

$$\hat{y} \in T(y)\delta, \quad A\hat{y} \in T(y)\delta.$$

Show

$$(\hat{x} - {k_y}^{-1} A\hat{y}) \perp (T(y)\delta).$$

Recall

$$A\hat{y} = \sum_{z \in X, yz \in E} \hat{z}.$$

$$\hat{x} - {k_y}^{-1} Ay \in E_1^*(y)V.$$

So,

$$\hat{x} - \frac{1}{k_y} A\hat{y} \perp E_j^*(y)T(y)\delta \quad \text{if } j \neq 1 \ (0 \leq j \leq k(y)).$$

And we have,

$$\left\langle \hat{x} - \frac{1}{k_y} A\hat{y}, A\hat{y} \right\rangle = \left\langle \hat{x}, \sum_{z \in X, yz \in E} \hat{z} \right\rangle - \frac{1}{k_y} \left\| \sum_{z \in X, yz \in E} \hat{z} \right\|^2 \tag{13.3}$$

$$= 1 - 1 \tag{13.4}$$

$$= 0 \tag{13.5}$$

This proves Claim.

Similarly,

$$\text{prof}_{T(x)\delta} \hat{y} = {k_x}^{-1} A\hat{x}.$$

Hence, the polynomials $p, p' \in \mathbb{C}[\lambda]$ from Theorem 12.1 equal

$$\frac{\lambda}{k_y} \quad \text{and} \quad \frac{\lambda}{k_x}$$

respectively.

By Theorem 12.1,

$$\frac{m_x(\theta)\theta}{k_x} = m_x(\theta)\overline{p'(\theta)} = m_y(\theta)\overline{p(\theta)} = \frac{m_y(\theta)\theta}{k_y}.$$

If $\theta \neq 0$, we have (ia).

Also,

$$\frac{1 - m_x(0)}{k_x} = \left(\sum_{\theta \in \mathbb{R} \ \{0\}} m_x(0) \right) \frac{1}{k_x} \qquad \text{by } (ia) \tag{13.6}$$

$$= \left(\sum_{\theta \in \mathbb{R} \ \{0\}} m_y(0) \right) \frac{1}{k_y} \tag{13.7}$$

$$= \frac{1 - m_y(0)}{k_y} \tag{13.8}$$

Hence, we have (ib). □

Theorem 13.1. *Suppose any graph $\Gamma = (X, E)$ is distance-regular with respect to every vertex $x \in X$. (So Γ is regular or biregular by Lemma 12.1.)*

Then,

Case Γ is regular: the diameter $d(x)$ and the intersection numbers $a_i(x)$, $b_i(x)$, $c_i(x)$ $(0 \leq i \leq d(x))$ are independent of $x \in X$.

(And Γ is called distance-regular.)

Case Γ is biregular: $(X = X^+ \cup X^-)$

$d(x)$ and $a_i(x)$, $b_i(x)$, $c_i(x)$ $(0 \leq i \leq d(x))$ are constant over X^+ and X^-. (And Γ is called distance-biregular.)

Proof. We apply Lemma 13.1.

Case Γ: regular.

Then $m_x = m_y$ for all $xy \in E$. Hence, the measure of the trivial $T(x)$-module is independent of $x \in X$.

Case Γ is biregular.

Then $m_x = m_{x'}$ for all $x, x' \in X$ with $\partial(x, x') = 2$.

Hence, the measure of the trivial $T(x)$-module is constant over $x \in X^+$, X^-.

Fix $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, $W = T\delta$ with measure m, diameter $d = d(x)$.

We know by Corollary 10.1 that m determines

$$d, \quad a_i(W)\ (0 \leq i \leq d), \quad x_i(W)\ (1 \leq i \leq d)$$

(as $d = D(x) = d(W)$ by Lemma 11.1.)

We shall show that m determines

$$a_i(x),\ c_i(x),\ b_i(x) \quad (0 \leq i \leq d).$$

Observe:

$$a_i(W) = a_i(x) \quad (0 \leq i \leq d) \tag{13.9}$$

$$x_i(W) = b_{i-1}c_i(x) \quad (1 \leq i \leq d) \tag{13.10}$$

Remark. $a_i = a_i(W)$ is an eigenvalue of

$$E_i^* A E_i^* \text{ on } E_i^* W = \langle \sum_{y \in \Gamma_i(x)} \hat{y} \rangle.$$

(See Lemma 12.1.)

$x_i = x_i(W)$ is an eigenvalue of

$$E^*_{i-1}AE^*_iAE^*_{i-1} \text{ on } E^*_{i-1}W,$$

and

$$A \sum_{y\in X,\partial(x,y)} \hat{y} = b_{i-1}(x) \sum_{y\in X,\partial(x,y)=i-1} \hat{y} \tag{13.11}$$

$$+ a_i(x) \sum_{y\in X,\partial(x,y)=i} \hat{y} \tag{13.12}$$

$$+ c_{i+1} \sum_{y\in X,\partial(x,y)=i+1} \hat{y} \tag{13.13}$$

So $x_i = b_{i-1}(x)c_i(x)$.

Set $k^+ = k_x$. Define

$$k^- = \frac{{\theta_0}^2}{k^+},$$

where θ_0 is the maximal eigenvalue. (See Lemma 11.1.)

(So, $k^+ = k^-$ is the valency, if Γ is regular.)

For every i $(0 \le i \le d)$ and for every $z \in X$ with $\partial(x,z) = i$,

$$k_z = c_i(x) + a_i(x) + b_i(x) \tag{13.14}$$

$$= \begin{cases} k^+ & \text{if } i \text{ is even,} \\ k^- & \text{if } i \text{ is odd.} \end{cases} \tag{13.15}$$

Now m determines

$$c_0(x) = a_0(x) = 0, \quad c_1(x) = 1,$$

$$b_0(x) = b_0(x)c_1(x) = x_1(W).$$

$$k^+ = b_0(x) \tag{13.16}$$

$$k^- = {\theta_0}^2/k^+ \tag{13.17}$$

$$c_i(x) = x_i(W)/b_{i-1}(x) \quad (1 \le i \le d) \tag{13.18}$$

$$b_i(x) = \begin{cases} k^+ - a_i(x) - c_i(c) & i\text{; even,} \\ k^- - a_i(x) - c_i(x) & i\text{: odd.} \end{cases} \tag{13.19}$$

This proves the assertions. □

Proposition 13.1. *Under the assumption of Theorem 13.1, the following hold.*

Case Γ*: regular.*

(i) $\dim E_i V = |X| m(\theta_i)$.
(ii) Γ *has exactly* $d+1$ *distinct eigenvalues*

($d = \text{diam}\Gamma = d(x)$, *for all* $x \in X$*).*

Case Γ*: biregular.*

(i) $\dim E_V = |X^+| m^+(\theta_i) + |X^-| m^-(\theta_i)$.
(ii) Γ *has exactly* $d^+ + 1$ *distinct eigenvalues (*$d^+ \geq d^-$*).*
(iii) If d^+ *is odd, the* Γ *is regular.*
(iv) $d^+ = d^-$*, or* $d^+ = d^- + 1$ *is even.*
(v) $a_i(x) = 0$ *for all* i *and for all* x*.*

Proof. *(i)* Suppose Γ is regular.

Let m_x be the measure of the trivial $T(x)$-module,

$$m_x(\theta_i) = \|E_i \hat{x}\|^2, \quad \text{as } \|\hat{x}\| = 1.$$

Now,

$$\begin{aligned} |X| m_x(\theta_i) &= \sum_{x \in X} m_x(\theta_i) && (13.20) \\ &= \sum_{x \in X} \|E_i \hat{x}\|^2 && (13.21) \\ &= \sum_{y,z \in X} |(E_i)_{yz}|^2 && (13.22) \\ &= \text{trace} E_i \overline{E_i}^\top. && (13.23) \end{aligned}$$

Since A is real symmetric and

$$E_i \overline{E_i}^\top = E_i^2 = E_i$$

with E_i symmetric

$$E_i \sim \begin{pmatrix} I & O \\ O & I \end{pmatrix}.$$

$$\text{trace} E_i = \text{rank} E_i = \dim E_i V.$$

Thus, we have the assertion in this case.

Suppose Γ is biregular.

Then, same except,

$$\sum_{x \in X} m_x(\theta_i) = |X^+| m^+(\theta_i) + |X^-||m^-(\theta_i).$$

(ii) Γ: regular. Immediately, if θ is an eigenvalue of Γ, then $m(\theta) \neq 0$.

Γ: biregular. For each $\theta = \theta_i \in \mathbb{R}\ \ \{0\}$,

$$m^-(\theta) \neq 0 \Leftrightarrow m^+(\theta) \neq 0 \tag{13.24}$$

$$\Leftrightarrow \theta \ \text{ is an eigenvalue of } \Gamma \tag{13.25}$$

$$\left(\frac{m^+(\theta)}{k^+} = \frac{m^-(\theta)}{k^-} \right) \tag{13.26}$$

(iv) and (v) are clear.

Remark. (iii) If d^+ is odd, $d^+ = d^-$ and Γ has even number of eigenvalues, i.e., 0 is not an eigenvalue. So A is nonsingular, and Γ is regular.

$\square$

Chapter 14

Parameters of Thin Modules, I

Friday, February 19, 1993

Summary.

Definition 14.1. Assume $\Gamma = (X, E)$ is distance-regular with respect to every vertex $x \in X$.

Notation: Let $x \in X$. The data of the trivial $T(x)$-module.

	Case DR	Case DBR
valencyk_x	k	$\begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^- \end{cases}$
x-diameter D_x	D	$\begin{cases} D^+ & \text{if } x \in X^+ \\ D^- & \text{if } x \in X^- \end{cases}$
measure m_x	m	$\begin{cases} m^+ & \text{if } x \in X^+ \\ m^- & \text{if } x \in X^- \end{cases}$
int. number $c_i(x)$	c_i	$\begin{cases} c_i^+ & \text{if } x \in X^+ \\ c_i^- & \text{if } x \in X^- \end{cases}$
int. number $b_i(x)$	b_i	$\begin{cases} b_i^+ & \text{if } x \in X^+ \\ b_i^- & \text{if } x \in X^- \end{cases}$
int. number $a_i(x)$	a_i	$\begin{cases} a_i^+ & \text{if } x \in X^+ \\ a_i^- & \text{if } x \in X^- \end{cases}$

Call m, $m^{\pm 1}$ the measure of Γ.

Assume $\Gamma = (X, E)$ is distance-regular.

To what extent do a_i's, b_i's and c_i's determine the structure of irreducible $T(x)$-modules? In general the following hold.

Lemma 14.1. *Assume $\Gamma = (X, E)$ is distance-regular. Pick $x \in X$. Let X be a thin irreducible $T(x)$-module with endpoint r, diameter d and measure m_W.*

(i) There is a unique polynomial $f_W \in \mathbb{C}[\lambda]$ with the following properties.

(ia) $\deg f_W \le D$ (diameter of Γ).

(ib) $m_W(\theta) = m(\theta) f_W(\theta)$ for every $\theta \in \mathbb{R}$, where m is the measure of Γ.

Moreover, $f_W \in \mathbb{R}[\lambda]$, and

(ii) $\deg f_W \le 2r$.
(iii) For all eigenvalues θ_i of Γ, $\lambda - \theta_i$ is a factor of f_W whenever, $E_i W = 0$.

In particular, $2r - D + d \ge 0$.

Proof. Let $\theta_0, \dots, \theta_D$ denote distinct eigenvalues of Γ. Then $m(\theta_i) \ne 0$ $(0 \le i \le D)$ by Proposition 13.1.

There exists a unique $f_W \in \mathbb{C}[\lambda]$ with $\deg f_W \le D$ such that

$$f_W(\theta_i) = \frac{m_W(\theta_i)}{m(\theta_i)} \quad (0 \le i \le D)$$

by polynomial interpolation.

$f_W \in \mathbb{R}[\lambda]$ since

$$\theta_0, \dots, \theta_D \in \mathbb{R} \quad \text{and} \quad f_W(\theta_0), \dots, f_W(\theta_D) \in \mathbb{R}.$$

(ii) Without loss of generality, we may assume $r < D/2$, else trivial.

Pick $0 \ne w \in E^*_r(x)W$.

$$w = \sum_{y \in W, \partial(x,y) = r} \alpha_y \hat{y} \quad \text{some } \alpha_y \in \mathbb{C}.$$

Pick $y \in X$ such that $\alpha_y \ne 0$.

Set W' be the trivial $T(y)$-module. ($\langle w, \hat{y} \rangle \ne 0$, as $W \not\perp W'$.)

$$r' = 0, \quad m' = m, \quad \Delta = r.$$

Apply Theorem 12.1, we have

$$\deg p \le \Delta - r' + r = 2r, \quad p \ne 0 \tag{14.1}$$
$$\deg p' \le \Delta - r + r' = 0, \quad p' \ne 0. \tag{14.2}$$

$$m_W(\theta)\overline{p'(\theta)} = m(\theta)p(\theta) \quad (\text{ for all } \theta \in \mathbb{R}).$$

So,

$$\deg p/\overline{p}' \leq 2r,$$

and $p/\overline{p}'$ satisfies the conditions of f_W.

$$\left(\frac{p(\theta)}{\overline{p}'(\theta)} = \frac{m_W(\theta)}{m(\theta)}\right)$$

(iii)

$$E_iW = 0 \to m_W(\theta_i) = 0 \to f_W(\theta_i) = 0.$$

that is, $E_iW = 0$. Hence θ_i is a root of $f_W(\lambda) = 0$. So,

$$2r \geq \deg f_W \geq |\{\theta_i \mid E_iW = 0\}| = D - d.$$

Hence,

$$2r - D + d \geq 0.$$

This proves the assertions. □

Lemma 14.2. *Let $\Gamma = (X, E)$ be any distance-regular graph with valency k, diameter D ($d \geq 2$), measure m, and eigenvalues*

$$k = \theta_0 > \theta_1 > \cdots > \theta_D.$$

Pick $x \in X$. Let W be a thin irreducible $T(x)$-module with endpoint $r = 1$, diameter D and measure $m_W = mf_W$. Then one fo the following cases $(i) - (iv)$ occurs.

Case	d	$f_W(\lambda)$	$a_0(W)$
(i)	$D-2$	$\frac{(\lambda-k)(\lambda-\theta_1)}{k(\theta_1+1)}$	$-\frac{b_1}{\theta_1+1} - 1$
(ii)	$D-2$	$\frac{(\lambda-k)(\lambda-\theta_D)}{k(\theta_D+1)}$	$-\frac{b_1}{\theta_1+1} - 1$
(iii)	$D-1$	$\frac{k-\lambda}{k}$	-1
(iv)	$D-1$	$\frac{(\lambda-k)(\lambda-\beta)}{k(\beta+1)}$	$-\frac{b_1}{\beta+1} - 1$

for some $\beta \in \mathbb{R}$ with $\beta \in (-\infty, \theta_D) \cup (\theta_1, \infty)$. Moreover, the isomorphism class of W is determined by $a_0(W)$.

Note. By (iii), the possible "shapes" of a thin irreducible $T(x)$-modules are:

$$r = 0 \quad d = D \tag{14.3}$$
$$r = 1 \quad d = D-1 \tag{14.4}$$
$$r = 1 \quad d = D-2 \tag{14.5}$$

Chapter 15

Parameters of Thin Modules, II

Monday, February 22, 1993

Proof of Lemma 14.2 Continued.

We have $\deg f_W \leq 2$ by Lemma 14.1 (*ii*).

Also bt Lemma 11.1, $E_0 W = 0$.

(As otherwise $\langle \delta \rangle = E_0 V \subseteq W$ and $r(W) = 0$.)

Hence, $\lambda - \theta_0 = \lambda - k$ is a factor of f_W by Lemma 14.1 (*iii*).

Let $p_0, p_1, \ldots, p_D$ denote the polynomials for the trivial $T(x)$-module from Lemma 9.1.

Recall,

$$\sum_{\theta \in \mathbb{R}} m(\theta) p_i(\theta) p_j(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \leq i, j \leq D) \tag{15.1}$$

$$= \delta_{ij} b_0 b_1 \cdots b_{i-1} c_1 c_2 \cdots c_i. \tag{15.2}$$

Note that $x_i = b_{i-1} c_i$ is in the proof of Theorem 7.1.

By construction,

$$p_0(\lambda) = 1.p_1(\lambda) \qquad = \lambda.p_2(\lambda)\lambda^2 - a_1 \lambda - k. \tag{15.3}$$

Apparently,

$$f_W = \sigma_0 p_0 + \sigma p_1 + \sigma_2 p_2$$

for some $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{C}$.

Claim:

$$\sigma_0 = 1, \tag{15.4}$$

$$\sigma_1 = \frac{a_0(W)}{k}, \tag{15.5}$$

$$\sigma_2 - \frac{1 + a_0(W)}{kb_1}. \tag{15.6}$$

Pf of Claim.

$$\begin{aligned}
1 &= \sum_{\theta \in \mathbb{R}} m_W(\theta) && (15.7)\\
&= \sum_{\theta \in \mathbb{R}} m(\theta) f_W(\theta) && (15.8)\\
&= \sum_{j=0}^{2} \sigma_j \left(\sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) \right) && (15.9)\\
&= \sigma_0. && (15.10)
\end{aligned}$$

We applied Lemma 10.1 (ib), Lemma 14.1 (ib), and Lemma 10.1 (i) in this order.

Next by Lemma 10.1 (ii), and $p_1(\theta) = \theta$,

$$\begin{aligned}
a_0(W) &= \sum_{\theta \in \mathbb{R}} m_W(\theta)\theta && (15.11)\\
&= \sum_{\theta \in \mathbb{R}} f_W(\theta)\theta && (15.12)\\
&= \sum_{j=0}^{2} \sigma_j \sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) p_1(\theta) && (15.13)\\
&= \sigma_1 x_1(T\delta) && (15.14)\\
&= \sigma_1 b_0 c_1 && (15.15)\\
&= \sigma_1 k. && (15.16)
\end{aligned}$$

So for,

$$f_W(\lambda) = 1 + \frac{a_0(W)}{k}\lambda + \sigma_2(\lambda^2 - a_1\lambda - k).$$

But,

$$\begin{aligned}
0 &= f_W(k) && (15.17)\\
&= 1 + a_0(W) + \sigma_2 k(k - a_1 - 1) && (15.18)\\
& 1 + a_0(W) + \sigma_2 k b_1. && (15.19)
\end{aligned}$$

Thus,

$$\sigma_2 = -\frac{1 + a_0(W)}{kb_1}.$$

This proves Claim.

Case: $a_0(W) = -1$.

Here, $\sigma_2 = 0$ and

$$f_W(\lambda) = 1 + \frac{a_0(W)\lambda}{k} = 1 - \frac{\lambda}{k}.$$

Also,

$$d + 1 = |\{\theta \mid \theta \text{ is an eigenvalue of } \Gamma,\ f_W(\theta) \neq 0\} = D.$$

Case: $a_0(W) \neq -1$.

Here, $\sigma_2 \neq 0$, and $\deg f_W = 2$. So,

$$f_W(\lambda) = (\lambda - k)(\lambda - \beta)\alpha$$

for some $\alpha, \beta \in \mathbb{C},\ \alpha \neq 0$.

Comparing the coefficients in

$$(\lambda - k)(\lambda - \beta)\alpha = 1 + \frac{a_0(W)}{k}\lambda - \frac{a_0(W)+1}{kb_1}(\lambda^2 - a_1\lambda - k),$$

we find

$$\alpha = -\frac{a_0(W)+1}{kb_1}, \tag{15.20}$$

$$-(k+\beta)\alpha = \frac{a_0(W)}{k} + \frac{a_0(W)+1}{kb_1}a_1, \tag{15.21}$$

$$k\beta\alpha = 1 + \frac{a_0(W)+1}{b_1}. \tag{15.22}$$

Hence,

$$-\beta(a_0(W)+1) = b_1 + (a_0(W)+1).$$

Thus, we have

$$(1 + a_0(W))(1+\beta) = -b_1. \tag{15.23}$$

In particular, $\beta \neq -1$, and

$$\alpha = -\frac{1 + a_0(W)}{kb_1} = \frac{1}{k(\beta+1)}.$$

Also, by Definition 9.2,

$$0 \leq m_W(\theta) \tag{15.24}$$

$$= m(\theta) f_W(\theta) \quad \text{(for all } \theta \in \mathbb{R}). \tag{15.25}$$

But if θ is an eigenvalue of Γ,

$$0 < m(\theta).$$

So,

$$
\begin{align}
0 &\le f_W(\theta) \tag{15.26}\\
&= \frac{(\theta-k)(\theta-\beta)}{k(\beta+1)}. \tag{15.27}
\end{align}
$$

Either

$$\beta+1>0 \to \theta-\beta\le 0 \ \text{ or } \ \beta\ge\theta_1,$$

or

$$\beta+1<0 \to \theta-\beta\ge 0 \ \text{ or } \ \beta\le\theta_D.$$

If $\beta=\theta_1$,

$$
\begin{align}
a_0(W) &= -\frac{b_1}{\beta+1}-1 = -\frac{b_1}{\theta_1+1}-1 \tag{15.28}\\
f_W(\lambda) &= \frac{(\lambda-k)(\lambda-\theta_1)}{k(\theta_1+1)}, \tag{15.29}
\end{align}
$$

and we have (i).

If $\beta=\theta_D$,

$$
\begin{align}
a_0(W) &= -\frac{b_1}{\theta_D+1}-1 \tag{15.30}\\
f_W(\lambda) &= \frac{(\lambda-k)(\lambda-\theta_D)}{k(\theta_D+1)}, \tag{15.31}
\end{align}
$$

and we have (ii).

If $\beta\notin\{\theta_1,\theta_2\}$,

$$\theta\in(-\infty,\theta_D)\cup(\theta_1,\infty),$$

we have (iv).

Note using (15.23), we have (iv).

$\square$

Note. Using (15.23),

$$a_0(W)\to\beta\to f_W\to m_W\to \text{isomorphism class of } W.$$

Note on Lemma 14.2. In fact, $\theta_1>-1$, $\theta_D<-1$ if $D\ge 2$.

Definition 15.1. The complete graph K_n has n vertices and diameter $D=1$, i.e., $xy\in E$ for all vertices x,t.

K_n is distance-regular with valency $k = n-1$ and $a_1 = n-2$, $D = 1$. Moreover, it has two distince eigenvalues θ_0, θ_1.

Recall, $\theta_0, \ldots, \theta_D$ are roots of p_{D+1}, i.e., $D+1$ st polynomial for the trivial module/

$$\begin{aligned} p_0 &= 1 && (15.32)\\ p_1 &= \lambda && (15.33)\\ p_2 &= \lambda^2 - a_1\lambda - k && (15.34)\\ &= \lambda^2 - (n-2)\lambda - (n-1) && (15.35)\\ &= (\lambda - (n-1))(\lambda+1). && (15.36) \end{aligned}$$

The roots are $\theta_0 = n-1 = k$ and $\theta_1 = -1$.

Lemma 15.1. *Let* $\Gamma = (X, E)$ *be distance-regular of diameter* $D \geq 1$ *with distinct eigenvalues*

$$k = \theta_0 > \theta_1 > \cdots > \theta_D.$$

(i) $\theta_D \leq -1$ *with equality if and only if* $D = 1$.

(ii) $\theta_1 \geq -1$ *with equality if and only if* $D = 1$.

Proof. *(i)* Suppose $\theta_D \geq -1$.

Then $I + A$ is positive semi-definite.

By Lemma 2.1, there exists vectors $\{v_x \mid x \in X\}$ in a Euclidean space such that

$$\begin{aligned} \langle v_x, v_y\rangle &= (I+A)_{xy} && (15.37)\\ &= \begin{cases} 1 & \text{if } x = y \text{ or } xy \in E,\\ 0 & \text{othewise.} \end{cases} && (15.38) \end{aligned}$$

For every $xy \in E$,

$$\langle v_x, v_y\rangle = \|v_x\|\|v_y\| = 1.$$

Hence, $v_x = v_y$, and v_x is independent of $x \in X$.

Shus $\langle v_x, v_y\rangle = 1$ for all $x, y \in X$.

We have $I + A = J$, (all 1's matrix), and $D = 1$.

(ii) Let m be the trivial measure. Then,

$$\begin{aligned} 1 &= \sum_{\theta\in\mathbb{R}} m(\theta) + \sum_{\theta\in\mathbb{R}} m(\theta)\theta && (15.39)\\ &= \sum_{\theta\in\mathbb{R}} m(\theta)(\theta+1) && (15.40)\\ &= m(k)(k+1) + \sum_{\theta\neq k} m(\theta)(\theta+1) && (15.41)\\ &\leq (k+1)|X|^{-1}. && (15.42) \end{aligned}$$

Note that $m(k) = |X|^{-1} \dim d_0 V = |X|^{-1}$.

So $k+1 \geq |X|$ or $k = |X|-1$. Thus, $xy \in E$ for every $x, y \in X$, and $D = 1$. $\square$

Note. Lemma 15.1 does not require distance-regular assumption.

Chapter 16

Thin Modoles of a DRG

Wednesday, February 24, 1993

Let $\Gamma = (X, E)$ denote any graph of diameter D.

Definition 16.1. For all integer i, the i-th incidence matrix $A_i \in \mathrm{Mat}_X(\mathbb{C})$ satisfies

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i, \end{cases} \quad (x, y \in X).$$

Observe,

$$\begin{aligned} A_0 &= I & &\text{(identity)} & (16.1)\\ A_1 &= A & &\text{(adjacency matrix)} & (16.2)\\ A_0 + A_1 + \cdots + A_D &= J & &\text{(all 1's matrix).} & (16.3) \end{aligned}$$

In general, A_i may not belong to Bose-Mesner algebra.

Lemma 16.1. *Assume $\Gamma = (X, E)$ is distance-regular with diameter $D \geq 1$ and intersection numbers c_i, a_i, b_i.*

(i)

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1}, \quad (0 \leq i \leq D,\ A_{-1} = A_{D+1} = O).$$

(ii) $A_i = \dfrac{p_i(A)}{c_1c_2\cdots c_i}$, $\quad (0 \leq i \leq D)$, *where $p_0, p_1, \ldots, p_D$ are polynomials for the trivial module from Lemma 9.1.*

(iii) $A_0, A_1, \ldots, A_D$ form a bais for Bose-Mesner algebra M.

(iv) For all distances h, i, j $\quad (0 \leq, i, j, h \leq D)$, and for all vertices $x, y \in X$ with $\partial(x,y) = h$, the constant

$$p^h_{i,j} = |\{z \in X \mid \partial(x,z) = i,\ \partial(y,z) = j\}|$$

depends only on h, i, j *and not on* x, y.

(v) $E_0 = \dfrac{1}{|X|} J$.

Proof.

(i) Pick $x \in X$. Apply each side to $\hat{x}$, we want to show that

$$AA_i\hat{x} = c_{i+1}A_{i+1}\hat{x} + a_iA_i\hat{x} + b_{i-1}A_{i-1}\hat{x}.$$

$$\begin{aligned}
\text{LHS} &= A\left(\sum_{y\in X, \partial(x,y)=i} \hat{y}\right) && (16.4)\\
&= c_{i+1}\left(\sum_{z\in X, \partial(x,z)=i+1} \hat{z}\right) + a_i\left(\sum_{z\in X, \partial(x,z)=i} \hat{z}\right) + b_{i-1}\left(\sum_{z\in X, \partial(x,z)=i-1} \hat{z}\right) && (16.5)\\
&= \text{RHS}. && (16.6)
\end{aligned}$$

(ii) Recall (Lemma 9.1)

$$Ap_i(A) = p_{i+1}(A) + a_ip_i(A) + b_{i-1}c_ip_{i-1}(A) \quad (0 \le i \le D).$$

Dividing by $c_1c_2\cdots c_i$, we have

$$A\frac{p_i(A)}{c_1c_2\cdots c_i} = c_{i+1}\frac{p_{i+1}(A)}{c_1c_2\cdots c_{i+1}} + a_i\frac{p_i(A)}{c_1c_2\cdots c_i} + b_{i-1}\frac{p_{i-1}(A)}{c_1c_2\cdots c_i}.$$

So, A_i, $p_i(A)/(c_1c_2\cdots c_i)$ satisfy the same recurrence.

Also boundary condition,

$$A_0 = p_0(A) = I.$$

Hence,

$$A_i = \frac{p_i(A)}{c_1c_2\cdots c_i} \quad (0 \le i \le D).$$

(iii) Since $E_0, E_1, \dots, E_D$ form a basis for M, $\dim M = D + 1$.

Observe $A_0, A_1, \dots, A_D \in M$ by (ii), $A_0, A_1, \dots, A_D$ are linearly independent, since $p_0, p_1, \dots, p_D$ are linearly independent.

Thus, $A_0, A_1, \dots, A_D$ form a basis for M.

(iv) $A_0, A_1, \dots, A_D$ form a basis for an algebra M,

$$A_iA_j = \sum_{\ell=0}^{D} p^{\ell_{ij}} A_\ell \quad \text{for some } \; p_{ij}^{\ell} \in \mathbb{C}. \tag{16.7}$$

Fix $h \quad (0 \le h \le D)$. Pick $x, y \in X$ with $\partial(x, y) = h$.

Compute x, y entry in (16.7),

$$(A_iA_j)_{xy} = \sum_{z\in X} (A_i)_{xz}(A_j)_{zy} \tag{16.8}$$

$$= \sum_{z\in X, \partial(x,z)=i, \partial(y,z)=j} 1 \cdot 1 \tag{16.9}$$

$$= |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|. \tag{16.10}$$

On the other hand,

$$\left(\sum_{\ell=0}^{D} p_{ij}^{\ell} A_\ell\right)_{xy} = p_{ij}^{h}(A_h)_{xy} = p_{ij}^{h}.$$

(v) $\frac{1}{|X|}J$ is the orthogonal projection onto $\mathrm{Span}(\delta) = E_0V$. Hence,

$$\frac{1}{|X|} = E_0.$$

This proves the assertions.

□

Theorem 16.1. *Let $\Gamma = (X, E)$ be distance-regular with diameter $D \ge 2$ and intersection numbers c_i, a_i, b_i. Pick a vertex $x \in X$. Let W be a thin irreducible $T(x)$-module with endpoint $r = 1$ and diameter d ($d = D-2$ or $D-1$). Set $\gamma_0 = a_0(W) + 1$.*

(i) The scalars

$$\gamma_i := \frac{c_2c_3 \cdots c_{i+1}b_2b_3 \cdots b_{i+1}\gamma_0}{x_1(W)x_2(W) \cdots x_i(W)} \quad (0 \le i \le d) \tag{16.11}$$

$a_i(W), x_i(W)$ are algebraic integers in $\mathbb{Q}[\gamma_0]$. In particular, if $\gamma_0 \in \mathbb{Q}$, then γ_i, $a_i(W)$ and $x_i(W)$ are integers for all i.

(ii) The numbers, $\gamma_i, a_i(W), x_i(W)$ can all be determined from γ_0 and the intersection numbers of Γ in order

$$x_1(W), \gamma_1, a_1(W), x_2(W), \gamma_2, a_2(W), ...$$

using (i),

$$x_i(W) = c_ib_i + \gamma_{i-1}(a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad (1 \le i \le D-1), \tag{16.12}$$

and

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad (1 \le i \le D). \tag{16.13}$$

Note.

$$p_i = p_1^W + \gamma_{i-1} p_{i-1}^W - c_i(p_{i-1}^W + \gamma_{i-2}^W),\ (\gamma_{-1} = -\gamma_{-2} = 0,\ 0 \le i \le d+1).$$

Proof. Set

$$\tilde{A}_i = A_0 + A_1 + \cdots + A_i \quad (0 \le i \le D).$$

Claim 1. $A\tilde{A}_i = c_{i+1}\tilde{A}_{i+1} + (a_i - c_{i+1} + c_i)\tilde{A}_i + b_i\tilde{A}_{i-1} \quad (0 \le i \le D-1).$

Proof of Claim 1.

$$\begin{aligned}
\text{LHS} &= \sum_{j=0}^{i} AA_j && (16.14)\\
&= \sum_{j=0}^{i} (c_{j+1}A_{j+1} + a_jA_j + b_{j-1}A_{j-1}) && (16.15)\\
&= \sum_{j=0}^{i-1} A_j(c_j + a_j + b_j) + A_i(c_i + a_i) + A_{i+1}c_{i+1} && (16.16)\\
&= k(A_0 + \cdots + A_{i-1}) + (a_i + c_i)A_i + c_{i+1}A_{i+1}. && (16.17)
\end{aligned}$$

$$\begin{aligned}
\text{RHS} = {} & c_{i+1}(A_0 + A_1 + \cdots + A_{i-1} + A_i + A_{i+1}) && (16.18)\\
& + (a_i - c_{i+1} + c_i)(A_0 + A_1 + \cdots + A_{i-1} + A_i) && (16.19)\\
& + b_i(A_0 + A_1 + \cdots + A_{i-1}) && (16.20)\\
= {} & k(A_0 + \cdots + A_{i-1}) + A_i(a_i + c_i) + A_{i+1}c_{i+1}. && (16.21)
\end{aligned}$$

This proves Claim 1.

Now pick $0 \ne w \in E_1^*(x)W$ and let

$$w = \sum_{z \in X, \partial(x,z)=1} \alpha_z \hat{z}.$$

Pick y, where $\alpha_y \ne 0$.

For i $(0 \le i \le D)$, define

$$\begin{aligned}
B_i &= \tilde{A}_i(\hat{x} - \hat{y}) && (16.22)\\
&= \sum_{z \in X, \partial(x,z) \le i} \hat{z} - \sum_{z \in X, \partial(y,z) \le i} \hat{z} && (16.23)\\
&= \sum_{z \in X, \partial(x,z)=i, \partial(y,z)=i+1} \hat{z} - \sum_{z \in X, \partial(y,z)=i+1, \partial(y,z)=i} \hat{z}. && (16.24)
\end{aligned}$$

Note that $B_D = O$, $B_0 = \hat{x} - \hat{y}$, and

$$\langle B_0, w_0\rangle = -\alpha_y \neq 0.$$

From Claim 1,

$$AB_i = c_{i+1}B_{i+1} + (a_i - c_{i+1} + c_i)B_i + b_i B_{i-1}\ (0\ eqi \leq D),\ B_{-1} = O.$$

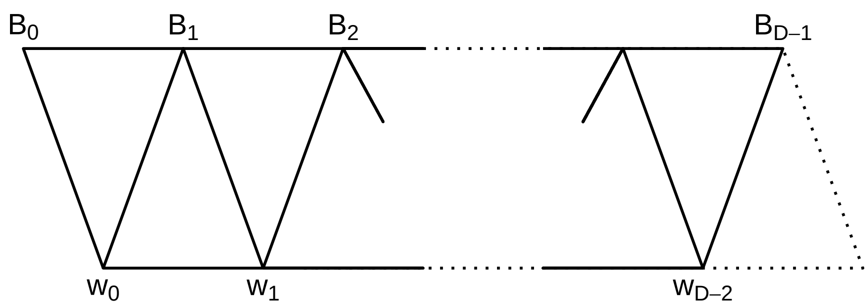
Let $p_0^W, \dots, p_d^W$ denote polynomials for W from Lemma 9.1. So,

$$w_i = p_i^W(A)w \in E^*_{1+i}(x)W, \quad (0 \leq i \leq d).$$

Claim 2. $\langle w_i, B_j\rangle = 0$ if $j \notin \{i, i+1\}$, $(0 \leq i \leq d, 0 \leq j \leq D)$.

Proof of Claim 2.

$$w_i \in E^*_{1+i}W, \quad B_j \in E^*_j(x)W + E^*_{j+1}(x)W.$$



Vertical lines indicate possible non-orthogonality.

Compute

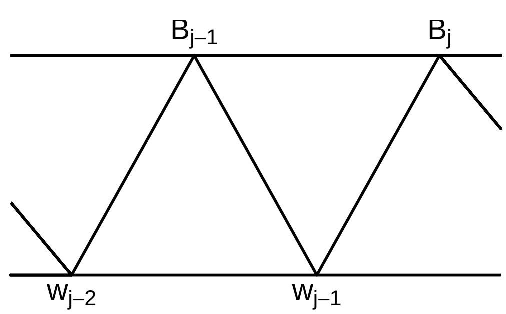
$$\langle Aw_i, B_j\rangle = \langle w_i, AB_j\rangle, quad(0 \leq i \leq D,\ 0 \leq j \leq D-1). \tag{16.25}$$

$$\text{LHS} = \langle w_{i+1}, B_j\rangle + a_i(W)\langle w_i, B_j\rangle + x_i(W)\langle w_{i-1}, B_j\rangle \tag{16.26}$$

$$\text{RHD} = b_j\langle w_i, B_{j-1}\rangle + (a_j - c_{j+1} + c_j)\langle w_i, B_j\rangle + c_{j+1}\langle w_i, B_{j+1}\rangle. \tag{16.27}$$

Evaluate for $i = j-2, j-1, j, j+1$.

Set $i = j-2$.



Then (16.25) becomes

$$\langle w_{j-1}, B_j\rangle = b_j\langle w_{j-2}, B_{j-1}\rangle \quad (2 \leq j \leq D-1).$$

By induction,

$$\langle w_{j-1}, B_j\rangle = b_2 b_3 \cdots b_j \langle w_0, B_1\rangle \quad (1 \leq j \leq D-1).$$

Define

$$\gamma_0 = \frac{\langle w_0, B_1\rangle}{\langle w_0, B_0\rangle}.$$

(We will show $\gamma_0 = 1 + a_0(W)$.)

Then,

$$\langle w_{j-1}, B_j\rangle = b_2 b_3 \cdots b_j \gamma_0 \langle w_0, B_0\rangle. \tag{16.28}$$

Set $i = j+1$. Then (16.25) becomes

$$x_{j+1}(W)\langle w_j, B_j\rangle = c_{j+1}\langle w_0, B_{j+1}\rangle \quad (0 \leq j \leq d).$$

Hence,

$$\langle w_j, B_j\rangle = \frac{x_1(W)\cdots w_j(W)}{c_1 c_2 \cdots c_j}\langle w_0, B_0\rangle \quad (0 \leq j \leq d). \tag{16.29}$$

Set $i = j-1$. Then (16.25) becomes

$$\langle w_j, B_j\rangle + a_{j-1}(W)\langle w_{j-1}, B_j\rangle = (a_j - c_{j+1} + c_j)\langle w_{j-1}, B_j\rangle + b_j\langle w_{j-1}, B_{j-1}\rangle.$$

Evaluate this using (16.28) and (16.29). ($\langle w_0, B_0\rangle \neq 0$). Then we have

$$\frac{w_1(W)\cdots x_j(W)}{c_1\cdots c_j} + (a_{j-1}(W) - a_j + c_{j+1} - c_j)b_2\cdots b_j\gamma_0 = b_j\frac{x_1(W)\cdots x_{j-1}(W)}{c_1\cdots c_{j-1}},$$

$$\left(\gamma_i := \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1}\gamma_0}{x_0(W)x_2(W)\cdots x_i(W)}\right).$$

$$\frac{x_j(W)}{c_j} = b_j + \frac{c_1 c_3 \cdots c_{j-1} b_2 b_3 \cdots b_j \gamma_0}{x_0(W)x_2(W)\cdots x_{j-1}(W)}(a_j + c_j - c_{j+1} - a_{j-1}).$$

So,

$$x_j(W) = c_j b_j + \gamma_{j-1}(a_j + c_j - c_{j+1} - a_{j-1}(W)).$$

This proves (16.12).

Set $i = j$. Then (16.25) becomes

$$a_j(W)\langle w_j, B_j\rangle + x_j(W)\langle w_{j-1}, B_j\rangle = (a_j - c_{j+1} + c_j)\langle w_j, B_j\rangle + c_{j+1}\langle w_j, B_{j+1}\rangle.$$

$$(a_j(W) - (a_j - c_{j+1} + c_j))\frac{x_1(W)\cdots x_j(W)}{c_1\cdots c_j}x_j(W)b_2\cdots b_j\gamma_0 - c_{j+1}b_2\cdots b_{j+1}\gamma_0 = 0.$$

Thus,

$$a_j(W) - (a_j - c_{j+1} + c_j) + \frac{c_1 \cdots c_j b_2 \cdots b_j \gamma_0}{x_1(W) \cdots x_{j-1}(W)} - \frac{c_1 \cdots c_j c_{j+1} b_2 \cdots b_{j+1} \gamma_0}{x_1(W) \cdots x_j(W)} = 0,$$

or

$$a_j(W) = a_j + c_j - c_{j+1} - \gamma_{j-1} + \gamma_j.$$

This proves (16.13).

Also by setting $i = j = 0$, we have

$$\begin{aligned} a_0(W)\langle w_0, B_0\rangle &= (a_0 - c_1 + c_0)\langle w_0, B_0\rangle + c_1\langle w_0, B_1\rangle && (16.30)\\ &= -\langle w_0, B_0\rangle + \gamma_0\langle w_0, B_0\rangle. && (16.31) \end{aligned}$$

Hence,

$$\gamma_0 = 1 + a_0(W).$$

Both $a_i(W)$ and $x_i(W)$ are algebraic integers, since they are eigenvalues of matrices with integer entries, namely,

$$E^*_{i+1}(x)AE^*_{i+1}(x) \text{ and } E^*_i(x)AE^*_{i+1}(x)AE^*_i(x).$$

Also $\gamma_0 = 1 + a_0(W)$ is an algebraic integer, and $\gamma_i - \gamma_{i-1}$ is an algebraic integer by (16.12).

Hence, γ_i is an algebraic integer by induction.

This completes the proof of Theorem 16.1. □

Example 16.1 (D=2)**.**

$$D = 2 \Leftrightarrow \text{strongly regular.}$$

Free parameters are k, a_1, c_2. Let W be an irreducible module of endpoint 1. The matrix representation of $A|_W$ is

$$\begin{pmatrix} a_0(W) & x_1(W) \\ 1 & a_1(W) \end{pmatrix}.$$

$a_0(W)$: free.

$$\begin{aligned} x_1(W) &= c_1 b_1 + (a_0(W) + 1)(a_1 + c_1 - c_2 - a_0(W)) && (16.32)\\ &= k - a_1 - 1 + a_1 a_0(W) + a_0(W) - c_2 a_0(W) - a_0(W)^2 + a_1 + a - c_2 - a_0(W) && (16.33)\\ &= a_1 a_0(W) - c_2 a_0(W) + k - c_2 - a_0(W)^2, && (16.34)\\ \gamma_1 &= 0, && (16.35)\\ a_1(W) &= -(a_0(W) + 1) + a_1 + c_1 - c_2 && (16.36)\\ &= -a_0(W) + a_1 - c_2. && (16.37) \end{aligned}$$

Then the matrix has eigenvalues θ, θ_1. There is one feasible condition: $a_0(W)$ is an algebraic integer.

Example 16.2 (D=3)**.** Free parameters c_2, c_3, k, a_1, a_2. The matrix representation becomes

$$A|_W = \begin{pmatrix} a_0(W) & x_1(W) & 0 \\ 1 & a_1(W) & x_2(W) \\ 0 & 1 & a_2(W) \end{pmatrix}.$$

Here, $a_0(W)$ is free $(= \gamma - 1)$

$$\begin{aligned} x_1(W) &= k - 1 - a_1 + \gamma_0(a_1 + 1 - c_2 - a_0(W)) && (16.38) \\ &= \gamma_0(a_1 - c_2 - a_0(W)) + k - a_1 + a_0(W). && (16.39) \end{aligned}$$

Set

$$\gamma_1(W) = \frac{c_2 b_2 \gamma_0}{x_1(W)}.$$

$$\begin{aligned} a_1(W) &= \gamma_1 - \gamma_0 + a_1 + 1 - c_2 && (16.40) \\ x_2(W) &= \gamma_1(a_2 - c_3 - a_1(W)) + c_2(\gamma_0 + b_1 - a_2 + a_1(W)) && (16.41) \\ a_2(W) &= -\gamma_1 + a_2 + c_2 - c_2. && (16.42) \end{aligned}$$

The matrix has eigenvalues, $\theta, \theta_2, \theta_3$.

There are two feasibility conditions; γ_0, γ_1 are algebraic integers.

For arbitrary D, there are $D-1$ feasibility conditions; $\gamma_0, \gamma_1, \dots, \gamma_{D-1}$ are algebraic integers.

Lemma 16.2. *With the notation of Theorem 16.1, suppose*

$$f_W = \frac{k - \lambda}{k} \quad (so,\ \ a_0(W) = -1).$$

Then,

$$\begin{aligned} a_i(W) &= a_i + c_i - c_{i+1} \quad (0 \le i \le D-1) && (16.43) \\ x_i(W) &= b_i c_i \quad (1 \le i \le D-1) && (16.44) \\ \gamma_i(W) &= 0. && (16.45) \end{aligned}$$

Proof. Since $\gamma_0 = a_0(W) = 1$, $\gamma_i = 0$. □

Chapter 17

Association Schemes

Monday, March 1, 1993

Review

Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 2$. Pick a vertex $x \in X$.

Let W be a thin irreducible $T(x)$-module with endpoint $r = 1$, diameter $d = D - 1$ or $D - 2$, and $r_0 = a_(W) + 1$.

Show

$$\gamma_i = \frac{c_2 c_2 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_1(W) \cdots x_i(W)},$$

$a_i(W)$ and $x_i(W)$ are all algebraic integers in $\mathbb{Q}[\gamma_0]$, where

$$x_i(W) = c_i b_i + \gamma_{i-1}(a_i + c_i - c_{i+1} - a_{i-1}(W)) \qquad (1 \leq i \leq d) \qquad (17.1)$$
$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \qquad (1 \leq i \leq d) \qquad (17.2)$$

Certainly, $x_i(W)$, γ_i, and $a_i(W)$ are in $\mathbb{Q}[\gamma_0]$ by the above lines and so on.

$$\gamma_0 \to a_0(W) \to x_1(W) \to \gamma_1 \to a_1(W) \to x_1(W) \to \cdots.$$

Recall some $B \in \mathrm{Mat}_n(\mathbb{C})$ is integral whenever

$$B \in \mathrm{Mat}_n(\mathbb{Z}).$$

In this case, the characteristic polynomial

$$\det(\lambda I - B) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0, \quad \text{some } \alpha_0, \ldots, \alpha_{n-1} \in \mathbb{Z}.$$

Hence, eigenvalues of B are algebraic integers. But $a_i(W)$ is an eigenvalue of an integral matrices,

$$B = E^*_{i+1}(x) A E^*_{i+1}(x).$$

Hence, $a_i(W)$ is an algebraic integer.

Also, $x_i(W)$ is an eigenvalue of an integral matrix

$$B = E_i^*(x)AE_{i+1}^*(x)AE_i^*(x).$$

So $x_i(W)$ is an algebraic integer.

$$\gamma_i - \gamma_{i-1} = a_i(W) - a_i - c_i + c_{i+1}$$

is an algebraic integer.

Since $\gamma_0 = a_0(W) + 1$ is an algebraic integer, we find γ is an algebraic integer for all i.

Definition 17.1. A (commutative) association scheme is a configuration $Y = (X, \{R_i\}_{0\le i\le D})$, where X is a finite nonempty set (of vertices), $R_0, R_1, \dots, R_D$ are nonempty subsets of $X \times X$ such that

(*i*) $R_0 = \{(x,x) \mid x \in X\}$,

(*ii*) $R_0 \cup \cdots \cup R_D = X \times X$ (disjoint union),

(*iii*) for every i, $R_i^\top = \{(y,x) \mid xy \in R\} = R_{i'}$ some $i' \in \{0, 1, \dots, D\}$,

(*iv*) for every h, i, j $(0 \le h, i, j \le D)$, and every $x, y \in X$ such that $(x,y) \in R_h$,

$$p_{ij}^h = |\{z \in X \mid (x,z) \in R_i,\ (z,y) \in R_j\}|$$

depends only on h, i, j and not on x, y; and

(*v*) $p_{ij}^h = p_{ji}^h$ for all h, i, j.

If $i' = i$ for all i, we say Y is symmetric. We call D the class of scheme and R_i, the ith relation of Y. We say vertices $x, y \in X$ are i-related, or 'at distance i', whenever $(x,y) \in R_i$.

We always assume that a 'scheme' is a commutative association scheme.

Let $Y = (X, \{R_i\}_{0\le i\le D})$ be an association scheme.

Definition 17.2. The i-the association matrix $A_i \in \mathbb{Mat}_X(\mathbb{C})$

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{if } (x,y) \notin R_i, \end{cases} \qquad (x, y \in X, 0 \le i \le D) \tag{17.3}$$

Then,

(*i'*) $A_0 = I$.

(*ii'*) $A_0 + A_1 + \cdots + A_D = J$ (= all 1's matrix).

(iii') $A_i^\top = A_{i'}$ $(0 \le i \le D)$.

(iv') $A_iA_j = \displaystyle\sum_{h=0}^{D} p_{ij}^h A_h \quad (0 \le i, j \le D)$.

(v') $A_iA_j = A_jA_i$.

$M := \mathrm{Span}_{\mathbb{C}}(A_0, \dots, A_D)$ (Bose-Mesner algebra of Y) is a commutative $\mathbb{C}$-algebra of dimension $D+1$.

Observe:

$$Y \text{ is symmetric} \leftrightarrow A_i^\top = A_i \text{ for all } i \leftrightarrow M \text{ is symmetric.}$$

Example 17.1. Let $\Gamma = (X, E)$ be distance-regular of diameter D. Set

$$R_i = \{(x, y) \mid \partial(x, y) = i\} \qquad (0 \le i \le D). \tag{17.4}$$

Then,

$$Y = (X, \{R_i\}_{0 \le i \le D})$$

is a symmetric scheme.

$$i\text{-th association matrix} = i\text{-th distance matrix} \quad \text{for all } i.$$

Example 17.2. Suppose a group G acts transitively on a seet X. Assume G is generously transitive, i.e.,

$$\text{for all } x, y \in X, \text{ there exists } g \in G \text{ such that } gx = y, gy = x.$$

Then G acts on $X \times X$ by rule;

$$g(x, y) = (gx, gy), \quad \text{for all } \ g \in G, \text{ and for all } x, y \in X.$$

Let $R_0, \dots, R_D$ denote orbits of G on $X \times X$.

Observe that $R_i^\top = R_i$ for all i by generously transitivity, and

$$Y = (X, \{R_i\}_{0 \le i \le D})$$

is a symmetric scheme.

Exercise 17.1. In Example Example 17.2, Bose-Mesner algebra

$$M = \{B \in \mathrm{Mat}_X(\mathbb{C}) \mid Bg = gB, \text{ for all } g \in G\} \tag{17.5}$$
$$= \text{the commuting algebra of } G \text{ on } X. \tag{17.6}$$

Here, we view each $g \in G$ as a permutation matrix in $\mathrm{Mat}_X(\mathbb{C})$ satisfying

$$g\hat{x} = \widehat{gx}, \quad \text{for all } x \in G.$$

Example 17.3. Let G be any finite group. G acts on $X = G$ by conjugation.

$$G \times X \to X, \quad (g, x) \mapsto gxg^{-1}.$$

Let $C_0, C_1, \dots, C_D$ denote orbits (i.e., conjugacy classes), and let $C_0 = \{1_G\}$. Claim that $Y = (X, \{R_i\}_{0 \le i \le D})$ is a commutative scheme (not symmetric in general).

(i) $R_0 = \{xx \mid x \in X\}$ as $C_0 = \{1_G\}$.

(ii) $R_0, \dots, R_D$ is a partition of $X \times X$ since $C_0, \dots, C_D$ is a partition of $X = G$.

(iii) $R_i^\top = R_{i'}$, where $C_{i'} = \{g^{-1} \mid g \in C_i\}$.

(iv) Set $H = G \oplus G$, the direct sum. Then H acts on $X = G$:

$$\text{for all } h = (g, gz), \text{for all } x \in X, \quad h(x) = gx(gx)^{-1} = gxz^{-1}g^{-1}.$$
$$R_i = \{(x, y) \mid x^{-1}y \in C_i\},\ h_i \in C_i,\ x^{-1}y = gh_ig^{-1}.$$

$$\begin{aligned} (x, y) &= (x, xgh_ig^{-1}) && (17.7)\\ &= (xgg^{-1}, xgh_ig^{-1}) && (17.8)\\ &= (xg, g)(1, h_i). && (17.9) \end{aligned}$$

So, $R_0, \dots, R_D$ are the orbits of H on $X \times X$.

(v) $p^h_{ij} = p^h_{ji}$?

Fix i, j, h and $x, y \in X$ with $(x, y) \in R_h$. Set

$$\begin{aligned} S &= \{z \in X \mid (x, z) \in R_i,\ (z, y) \in R_j\} && (17.10)\\ T &= \{z \in X \mid (x, z) \in R_j,\ (z, y) \in R_i\}. && (17.11) \end{aligned}$$

Show $|S| = |T|$.

$$\text{For all } z \in S,\ \text{set } \hat{z} = xz^{-1}y.$$

Observe, $\hat{z} \in T$.

$$\begin{aligned} & x^{-1}z \in C_i x^{-1}\hat{z} = x^{-1}xz^{-1}y \in C_j && (17.12)\\ & z^{-1}y \in C_j \hat{z}^{-1}y = y^{-1}zx^{-1}x^{-1}y = y^{-1}x(x^{-1}z)x^{-1}y \in C_i. && (17.13) \end{aligned}$$

Observe

$$S \to T \quad (z \mapsto z^{-1}) \quad \text{is one-to-one and onto.}$$

Chapter 18

Polynomial Schemes

Wednesday, March 3, 1993

Lemma 18.1. *Let* $Y = (X, \{R_i\}_{0 \le i \le D})$ *denote the symmetric scheme with associated matrices* $A_0, A_1, \dots, A_D$. *Then the following are equivalent.*

(i) *The graph* $\Gamma = (X, R_1)$ *is distance-regular, and* $R_0, \dots, R_D$ *are labelled so that*

$$R_i = \{xy \mid \partial(x,y) = i\}.$$

(ii) *There exists* $f_i \in \mathbb{C}[\lambda]$, $\deg f_i = i$ *such that* $f_i(A_1) = A_i$ *for all* i *with* $0 \le i \le D$.

(iii) *The parameter* p^h_{ij}

$$\begin{cases} = 0 & \text{if one of } h,i,j \text{ is larger than the sum of the other two} \\ \neq 0 & \text{if one of } h,i,j \text{ is equal to the sum of the other two.} \end{cases}$$

Proof.

$(i) \Rightarrow (ii)$: Lemma 16.1.

$(ii) \Rightarrow (iii)$: Define

$$k_i \equiv p^0_{ii} = |\{z \mid z]inX,\ \partial(x,z) = i\ ((x,z) \in R_i)\}|$$

for any $x \in X$. Then $k_i \neq 0$ $(0 \le i \le D)$, $k_0 = 1$.

(By symmetricity, $(x,y) \in R_i$ if and only if $(y,x) \in R_i$.)

Claim.

$$k_h p^h_{ij} = k_i p^i_{hj} = k_j p^j_{ih} \tag{18.1}$$
$$= |X|^{-1} |\{xyz \in X^3 \mid \partial(x,y) = h, \partial(x,z) = i, \partial(y,z) = j\}|. \tag{18.2}$$

Pf. The number of $xyz \in X^3$, $\partial(x,y) = h, \partial(x,z) = i, \partial(y,z) = j$ is equal to

$$|X| k_h p^h_{ij} = |X| k_i p^i_{hj} = k_j p^j_{ih}.$$

In particular,

$$p^h_{ij} = 0 \leftrightarrow p^i_{hj} = 0 \leftrightarrow p^j_{ih} = 0.$$

Hence, it suffices to show

$$\begin{cases} p^h_{ij} = 0 & \text{if } \ h > i+j \\ p^h_{ij} \neq 0 & \text{if } \ h = i+j. \end{cases}$$

Fix i, j. Without loss of generality, we may assume that $i + j \leq D$ as trivial otherwise.

$$f_i(A) f_j(A) = A_i A_j = \sum_{\ell=0}^{D} p^\ell_{ij} A_\ell = \sum_{\ell=0}^{D} p^\ell_{ij} f_\ell(A).$$

$$i + j = \deg \mathrm{LHS} \tag{18.3}$$
$$= \deg \mathrm{RHS} \tag{18.4}$$
$$= \max\{\ell \mid p^\ell_{ij} \neq 0\}. \tag{18.5}$$

$(iii) \Rightarrow (i)$

Let $A = A_1$, and consider a graph Γ with adjacency matrix A.

$$AA_j = \sum_h p^h_{1j} A_h \tag{18.6}$$
$$= p^{j+1}_{1j} A_{j+1} + p^j_{1j} A_j + p^{j-1}_{1j} A_{j-1}. \tag{18.7}$$

Then, $p^{j+1}_{1j} \neq 0 \neq p^{j-1}_{1j}$.

Fix a vertex $x \in X$, and set $R_i(x) = \{y \mid (x,y) \in R_i\}$.

Then each $y \in R_i(x)$ is adjacent in Γ to exactly

$$p^i_{1,i+1} \neq 0 \quad \text{vertices in } \ R_i(x), \tag{18.8}$$
$$p^i_{1i} \qquad \text{vertices in } \ R_{i+1}(x), \tag{18.9}$$
$$p^i_{1,i-1} \neq 0 \quad \text{vertices in } \ R_{i-1}(x). \tag{18.10}$$

Hence, by induction,

$$R_i(x) = \{y \mid \partial(x,y) = i \text{ in } \Gamma\} \qquad (0 \leq i \leq D), \tag{18.11}$$

and Γ is distance regular.

□

Chapter 19

Commutative Association Schemes

Friday, March 5, 1993

Lemma 19.1. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme with Bose-Mesner algebra M.*

Then there exists a basis $E_0, E_1, \ldots, E_D$ for M such that

(i) $E_0 = |X|^{-1}J$.

(ii) $E_iE_j = E_jE_i = \delta_{ij}E_i \quad (0 \le i, j \le D)$.

(iii) $E_0 + E_1 + \cdots + E_D = I$.

(iv) $E_i^\top = \overline{E_i} = E_{\hat{i}}$ *for some* $\hat{i} \in \{0, 1, \ldots, D\}$.

Proof. M acts on Hermitean space $V = \mathbb{C}^n$ $(n = |X|)$.

If W is an M-module, so is $W^\perp$.

Each irreducible M-module is 1 dimensional by commutativity of M. So V is orthognal direct sum of 1-dimensional M-modules.

Let $v_1, \ldots, v_n$ be an orthonormal basis for V consisiting of eigenvectors for all $m \in M$.

Set $P \in \mathrm{Mat}_X(\mathbb{C})$ so that the i-th column of P is equal to v_i. So,

$$\bar{P}^\top P = I = P\bar{P}^\top = \bar{P}P^\top,$$

and P is unitary.

Also, for all $m \in M$,

$$\begin{aligned} P^{-1}mP &= \text{diagonal} \qquad (19.1) \\ &= \text{diag}(\theta_1(m), \dots, \theta_n(m)). \qquad (19.2) \end{aligned}$$

for some functions

$$\theta_i : M \longrightarrow \mathbb{C}.$$

Observe: each $\theta = \theta_i$ is a character of M, i.e.,

$$\theta : M \longrightarrow \mathbb{C}$$

is a $\mathbb{C}$-algebra homomorphism.

Observe: the $\theta_1, \dots, \theta_n$ are not all distinct.

Let $\sigma_0, \dots, \sigma_r$ denote distinct elements of

$$\theta_1, \dots, \theta_n.$$

Say σ_i appears m_i times. Without loss of generality, we may assume that

$$P^{-1}mP = \begin{pmatrix} \sigma_0(m)I_{m_0} & O & O & O \\ O & \sigma_1(m)I_{m_1} & O & O \\ O & O & \ddots & O \\ O & O & O & \sigma_r(m)I_{m_r} \end{pmatrix}.$$

Set

$$E_i = P \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix} P^{-1},$$

where I_{m_i} is in the i-th block.

Then,

$$E_iE_j = \delta_{ij}E_i \quad (0 \le i, j \le r),$$
$$E_0 + E_1 + \cdots + E_r = I.$$

Hence for all $m \in M$,

$$m = \sum_{i=0}^{r} \sigma_i(m)E_i \in \text{Span}(E_0, \dots, E_r).$$

So,

$$M \subseteq \text{Span}(E_0, \dots, E_r).$$

Since $E_0, \dots, E_r$ are linearly independent, $r \ge D$.

Show $E_i \in M$.

Claim 1. For all distinct $i, j \quad (0 \le i, j \le D)$, there exists $m \in M$ such that $\sigma_i(m) \ne 0$, $\sigma_j(m) = 0$.

Pf of Claim 1. $\sigma_i \neq \sigma_j$ implies that there exists $m' \in M$ such that $\sigma_i(m') \neq \sigma_j(m')$.

Set $m = m' - \sigma_j(m')I$. Then,

$$\sigma_j(m)\sigma_j(m') - \sigma_j(m') \qquad = 0, \tag{19.3}$$

$$\sigma_i(m)\sigma_i(m') - \sigma_j(m') \qquad \neq 0. \tag{19.4}$$

Claim 2. $E_i \in M \quad (0 \leq i \leq D)$.

Pf of Claim 2. Fix a vertex $x \in X$. For all $j \neq i$, there exists $m_j \in M$ such that $\sigma_i(m_j) \neq 0, \quad \sigma_j(m_j) = 0, \quad i \neq j$.\$ Observe

$$s = \sigma_i \left(\prod_{\ell \neq i} m_\ell \right) \neq 0.$$

Set

$$m^* = \sigma_i \left(\prod_{\ell \neq i} m_\ell \right) s^{-1}.$$

Observe

$$\sigma_i(m^*) = 1, \quad \sigma_j(m^*) = 0, \quad \text{for all } j \neq i \quad (0 \leq j \leq D).$$

So

$$P^{-1}m^*P = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}.$$

We have

$$E_i = m^* \in M.$$

Now $r = D$, $M = \mathrm{Span}(E_0, \dots, E_D)$ and $E_0, \dots, E_D$ is a basis for M.

Observe

$$P^{-1}E_iP = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}$$

implies

$$P^{-1}\overline{E_i}^{\top}P = \bar{P}^{\top}\overline{E_i}^{\top}\overline{P^{-1}}^{\top} = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}^{\top} = P^{-1}E_iP.$$

Hence,

$$\overline{E_i}^{\top} = E_i.$$

$E_0^{\top}, \dots, E_D^{\top}$ are nonzero matrices satisfying

$$E_i^{\top}E_j^{\top} = \delta_{ij}E_i^{\top},$$

$$E_0^\top + E_1^\top + \cdots + E_D^\top = I.$$

Each $E_i^\top$ is a linear combination of $E_0, \dots, E_D$ with coefficientss that are 0 or 1, and for no two E_i's are coefficients of any E_j both 1's.

So, $E_0^\top, \dots, E_D^\top$ is a permutation of $E_0, \dots, E_D$.

Observe $J = A_0 + \cdots + A_D \in M$.

The matrix $|X|^{-1}J$ is an idempotent of rank 1.

So, without loss of generality we may assume that

$$E_0 = \frac{1}{|X|}J.$$

We have the assertions. □

Define entry-wise product $\circ$ on $\mathrm{Mat}_X(\mathbb{C})$.

$$A_i \circ A_j = \delta_{ij} A_i.$$

So, M is closed under $\circ$.

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^h E_h.$$

The numbers q_{ij}^h is called Krein parameters of Y.

Claim. $q_{ij}^h \in \mathbb{R}$.

Pf.

$$\begin{aligned}
\frac{1}{|X|} \sum_{h=0}^{D} \overline{q_{ij}^h} E_h &= \frac{1}{|X|} \sum_{h=0}^{D} \overline{q_{ij}^h} \overline{E_h}^\top && (19.5)\\
&= (\overline{E_i \circ E_j})^\top && (19.6)\\
&= E_i \circ E_j && (19.7)\\
&= \frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^h E_h. && (19.8)
\end{aligned}$$

Hence, $q_{ij}^h = \overline{q_{ij}^h}$.

Observe $A_0, \dots, A_D$, $E_0, \dots, E_D$ are bases for M. Hence, there exist $p_i(j)$, $q_i(j) \in \mathbb{C}$ such that

$$A_i = \sum_{j=0}^{D} p_i(j) E_j \tag{19.9}$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j. \tag{19.10}$$

Taking transpose and conjugate we find,

$$\overline{p_i(j)} = p_i(j) = p_{i'}(\hat{j}) \qquad (0 \le i, j \le D) \tag{19.11}$$

$$\overline{q_i(j)} = q_i(j) = q_{\hat{i}}(j') \qquad (0 \le i, j \le D). \tag{19.12}$$

Fix a vertex $x \in X$. Define

$$E_i^* \equiv E_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$$

to be a diagonal matrix such that

$$(E_i^*)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases} \quad (0 \le i \le D, y \in X.)$$

Then,

$$E_i^* E_j^* = \delta_{ij} E_i^*,$$

$$E_0^* + \cdots + E_D^* = I,$$

$$(E_i^*)^\top = \overline{E_i^*} = E_i^*.$$

Definition 19.1. Dual Bose-Mesner algebra $M^* \equiv M^*(x)$ with respect to x is

$$\mathrm{Span}(E_0^*, \dots, E_D^*).$$

Define dual associate matrices $A_0^*, \dots, A_D^*$. Indeed $A_i^* \equiv A_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$ is a diagonal matrix with

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad (y \in X).$$

A_i^* is a diagonal matrix having the row x of E_i^* on the diagonal.

Observe

$$A_i^* = \sum_{j=0}^{D} q_i(j) E_j^* \quad \left(E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j \right) \tag{19.13}$$

$$E_i^* = \frac{1}{|X|} \sum_{j=0}^{D} p_i(j) A_j^* \quad \left(A_i = \sum_{j=0}^{D} p_i(j) E_j \right). \tag{19.14}$$

So, $A_0^*, \dots, A_D^*$ form a basis for M^*.

Also,

$$A_i^* E_j^* = q_i(j) E_j^*.$$

$$\left(A_i^* E_j^* = \sum_{h=0}^{D} q_i(h) E_h^* E_j^* = q_i(j) E_j^*. \right)$$

So, $q_i(j)$ are dual eigenvalues of A_i^*.

Observe,

$$A_0^* = I, \quad A_0^* + \cdots + A_D^* = |X|E_0^*, \quad \overline{A_i^*} = A_{\hat{i}}^*,$$

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \quad (0 \le i, j \le D).$$

Remark. Proof.

$$(A_0^*)_{yy} = |X|(E_0)_{xy} = (J)_{xy} = 1.$$

$$A_0^* + \cdots + A_D^* = \sum_{i=0}^{D} \sum_{j=0}^{D} q_i(j) E_j^* = |X|E_0^*.$$

Note that

$$I = E_0 + \cdots + E_D = \frac{1}{|X|} \sum_{i=0}^{D} \sum_{j=0}^{D} q_i(j) A_j.$$

$$\sum_{i=0}^{D} q_i(j) = \delta_{j0}|X|.$$

$$\overline{A_i^*} = \sum_{j=0}^{D} \overline{q_i(j)} E_j^* = \sum_{j=0}^{D} q_{\hat{i}}(j) E_j^* = A_{\hat{i}}^*.$$

$$\begin{aligned}
(A_i^* A_j^*)_{yy} &= |X|^2 (E_i)_{xy} (E_j)_{xy} && (19.15)\\
&= |X|^2 (E_i \circ E_j)_{xy} && (19.16)\\
&= |X| \sum_{h=0}^{D} q_{ij}^h (E_h)_{xy} && (19.17)\\
&= \sum_{h=0}^{D} q_{ij}^h (A_h^*)_{yy}. && (19.18)
\end{aligned}$$

The following statements will be proved after a couple of lemmas in the next lecture.

Lemma. Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme. Fix a vertex $x \in X$, and set $E^* \equiv E_i^*(x)$ and $A_i^* \equiv A^*(x)$. Then the following hold.

(i) $E_i^* A_j E_k^* = O$ if and only if $p_{ij}^k = 0$ for $0 \le i, j, k \le D$.

(ii) $E_i A_j^* E_k = O$ if and only if $q_{ij}^k = 0$ for $0 \le i, j, k \le D$.

Chapter 20

Vanishing Conditions

Monday, March 15, 1993 (Monday after Spring break)

Lemma 20.1. *Let* $Y = (X, \{R_i\}_{0 \le i \le D})$ *be a commutative scheme.*

(i) $p_0(i) = 1$.

(ii) $p_i(0) = k_i$, *where*

$$k_i = p^0_{ii'} = |\{y \in X \mid (x, y) \in R_i\}|.$$

(iii) $q_0(i) = 1$.

(iv) $q_i(0) = m_i$, *where*

$$m_i = \mathrm{rank} E_i.$$

Proof.

(i) Since $A_0 = I$ and

$$A_0 = p_0(0)E_0 + p_0(1)E_1 + \cdots + p_0(D)E_D \tag{20.1}$$
$$I = E_0 + E_1 + \cdots + E_D, \tag{20.2}$$

$p_0(i) = 1$ for all i.

(ii) Since

$$A_i = p_i(0)E_0 + p_i(1)E_1 + \cdots + p_i(D)E_D,$$

$A_iE_0 = p_i(0)E_0$, and

$$k_iJ = A_iJ = p_i(0)J$$

as there are k_i 1's in each row of A_i, we have $k_i = p_i(0)$.

(iii) Since $E_0 = |X|^{-1}J$ and

$$E_0 = |X|^{-1}(q_0(0)A_0 + q_0(1)A_1 + \cdots + q_0(D)A_D) \tag{20.3}$$
$$|X|^{-1}J = |X|^{-1}(A_0 + A_1 + \cdots + A_D), \tag{20.4}$$

$q_0(i) = 1$ for all i.

(iv) $E_i = |X|^{-1}(q_i(0)A_0 + q_i(1)A_1 + \cdots + q_i(D)A_D)$, $E_i^2 = E_i$, and E_i is similar to a matrix

$$\begin{pmatrix} I_{m_i} & O \\ O & O \end{pmatrix}.$$

So,

$$m_i = \mathrm{rank}E_i = \mathrm{trace}E_i = \sum_{x \in X}(E_i)_{xx} = |X||X|^{-1}q_i(0) = q_i(0).$$

Note that as

$$E_i = \frac{1}{|X|}\sum_{j=0}^{D} q_i(j)A_j \to (E_i)_{xx} = \frac{1}{|X|}q_i(0)(A_0)_{xx}.$$

Hence, we have all formulas.

□

Lemma 20.2. *With the above notation*

(i) $p_{ij}^h = p_{j'i'}^{h'}$.

(ii) $k_hp_{ij}^h = k_jp_{i'h}^j = k_{hj'}^i$.

(iii) $q_{ij}^h = q_{\hat{j}\hat{i}}^{\hat{h}}$.

(iv) $m_hq_{ij}^h = m_jq_{\hat{i}h}^j = m_iq_{h\hat{j}}^i$.

Proof.

(i) We have

$$
\begin{aligned}
\sum_{h=0}^{D} p_{ij}^{h} A_{h'} \left(\sum_{h=0}^{D} p_{ij}^{h} A_{h} \right)^{\top} & \qquad (20.5)\\
= (A_i A_j)^{\top} & \qquad (20.6)\\
= A_j^{\top} A_i^{\top} & \qquad (20.7)\\
= A_{j'} A_{i'} & \qquad (20.8)\\
= \sum_{h=0}^{D} p_{j'i'}^{h'} A_h'. & \qquad (20.9)
\end{aligned}
$$

(ii) Count the following number,

$$
\begin{aligned}
&|\{xyz \in X^3 \mid (x,y) \in R_h, (x,z) \in R_i, (z,y) \in R_j\}| \qquad (20.10)\\
&= |X| k_h p_{ij}^{h} = |X| k_j p_{i'h}^{j} = |X| k_{hj'}^{i}. \qquad (20.11)
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^{h} E_{\hat{h}} = \left(\frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^{h} E_h \right)^{\top} & \qquad (20.12)\\
= (E_i \circ E_j)^{\top} & \qquad (20.13)\\
= E_j^{\top} \circ E_i^{\top} & \qquad (20.14)\\
= E_{\hat{j}} E_{\hat{i}} & \qquad (20.15)\\
= \frac{1}{|X|} \sum_{h=0}^{D} q_{\hat{j}\hat{i}}^{\hat{h}} E_{\hat{h}}. & \qquad (20.16)
\end{aligned}
$$

(iv) Let $\tau(B)$ denote the sum of the entries in the matrix B.

Observe: $\tau(B \circ C) = \mathrm{trace}(BC^{\top})$.

Observe

$$
\tau(E_i \circ E_j \circ E_{\hat{k}}) = \tau((E_i \circ E_j \circ E_{\hat{k}})^{\top}) = \tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}).
$$

Compute each one.

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \mathrm{trace}((E_i \circ E_j)E_k) = \mathrm{trace}\left(\left(\frac{1}{|X|}\sum_h q_{ij}^h E_h\right) E_k\right) \tag{20.17}$$

$$= \mathrm{trace}\left(\frac{1}{|X|} q_{ij}^k E_k\right) = \frac{1}{|X|} m_k q_{ij}^k, \tag{20.18}$$

$$\tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \mathrm{trace}((E_{\hat{i}} \circ E_k)E_{\hat{j}}) = \mathrm{trace}\left(\left(\frac{1}{|X|}\sum_h q_{\hat{i}k}^h E_h\right) E_{\hat{j}}\right) \tag{20.19}$$

$$= \mathrm{trace}\left(\frac{1}{|X|} q_{\hat{i}k}^j E_k\right) = \frac{1}{|X|} m_j q_{\hat{i}k}^j, \tag{20.20}$$

$$\tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}) = \mathrm{trace}((E_k \circ E_{\hat{j}})E_i) = \mathrm{trace}\left(\left(\frac{1}{|X|}\sum_h q_{k\hat{j}}^h E_h\right) E_i\right) \tag{20.21}$$

$$= \mathrm{trace}\left(\frac{1}{|X|} q_{k\hat{j}}^i E_i\right) = \frac{1}{|X|} m_i q_{k\hat{j}}^i. \tag{20.22}$$

Hence, we have (iv).

□

Lemma 20.3. *Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a commutative scheme. Fix a vertex $x \in X$, and set $E^* \equiv E_i^*(x)$ and $A_i^* \equiv A^*(x)$. Then the following hold.*

(i) $E_i^* A_j E_k^* = O$ if and only if $p_{ij}^k = 0$ for $0 \le i, j, k \le D$.

(ii) $E_i A_j^* E_k = O$ if and only if $q_{ij}^k = 0$ for $0 \le i, j, k \le D$.

Proof.

(i) Partition rows and columns by $R_0(x), R_1(x), \ldots, R_D(x)$. Then,

$$E_i^*(x) A_j E_h^*(x)$$

is the (i, h) block of A_j.

Hence this submatrix is zero if and only if there exists no $y, z \in X$ such that $(x, y) \in R_i$, $(x, z) \in R_h$ and $(y, z) \in R_j$. This is exactly when $p_{ij}^h = 0$.

(ii) The sum of the squares of norms of entries in $E_i A_j^* E_k$

$$
\begin{aligned}
&= \tau((E_iA_j^*E_k) \circ (\overline{E_jA_j^*E_k})) && (20.23)\\
&= \mathrm{trace}(E_iA_j^*E_k(\overline{E_jA_j^*E_k})^\top) && (20.24)\\
&= \mathrm{trace}(E_iA_j^*E_kA_{\hat{j}}^*E_i) && (20.25)\\
&= \mathrm{trace}(E_iA_j^*E_kA_{\hat{j}}^*) \qquad \text{as } \mathrm{trace}(XY) = \mathrm{trace}(YX) && (20.26)\\
&= \sum_{y\in X}(E_iA_j^*E_kA_{\hat{j}}^*)_{yy} && (20.27)\\
&= \sum_{y\in X}\left(\sum_{z\in X}(E_i)_{yz}(A_j^*)_{zz}(E_k)_{zy}(A_{\hat{j}}^*)_{yy}\right) && (20.28)\\
&= \sum_{y\in X}\left(\sum_{z\in X}(E_{\hat{i}})_{zy}(|X|(E_j)_{xz})(E_k)_{zy}(|X|(E_j)_{yx})\right) && (20.29)\\
&= |X|^2(E_j(E_{\hat{i}} \circ E_k))E_j)_{xx} && (20.30)\\
&= |X|q_{\hat{i}k}^j(E_j)_{xx} && (20.31)\\
&= q_{\hat{i}k}^j m_j && (20.32)\\
&= m_kq_{ij}^k. && (20.33)
\end{aligned}
$$

Note that since $|X|E_j = q_j(0)A_0 + q_j(1)A_1 + \cdots q_j(D)A_D$,

$$(E_j)_{xx} = \frac{1}{|X|}q_j(0) = \frac{m_j}{|X|}.$$

Thus, we have (ii).

□

Corollary 20.1 (Krein Condition)**.** *For any commutative scheme $Y = (X, \{R_i\}_{0\le i\le D})$, q_{ij}^h is a non-negative real number for $0 \le h, i, j \le D$.*

Proof. Since $q_{ij}^h m_h$ is a non-negative real by the proof of Lemma 20.3 (ii).

Note that m_h is a positive integer. □

An interpretation of the Krein parameters.

Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a commutative scheme with standard module V.

Pick a vector $v \in V$ with

$$v = \sum_{x\in X}\alpha_x\hat{x}.$$

View v as a function

$$v : X \longrightarrow \mathbb{C} \quad (x \mapsto \alpha_x).$$

View V as the set of all functions $V \longrightarrow \mathbb{C}$. Then the vector space V together with product of functions is a $\mathbb{C}$-algebra.

For
$$v = \sum_{x \in X} \alpha_x \hat{x}, \quad w = \sum_{x \in X} \beta_x \hat{x} \in V,$$
write
$$v \circ w = \sum_{x \in X} \alpha_x \beta_x \hat{x}$$
to represent the product of v and w viewed as functions.

Lemma 20.4. *With the above notation,*

(i) $A_j^*(x)v = |X|(E_j\hat{x} \circ v)$ *for all* $v \in V$ *and for all* $x \in X$.

(ii) $E_iV \circ E_jV \subseteq \sum_{h: q_{ij}^h \neq 0} E_hV$ *for all* $0 \leq i, j \leq D$.

(iii) $E_h(E_i \circ E_jV) = E_hV$ *if* $q_{ij}^h \neq 0$ *for all* $0 \leq h, i, j \leq D$.

Chapter 21

Norton Algebras

Wednesday, March 17, 1993

Proof of Lemma 20.4.

(i) Suppose

$$v = \sum_{x \in X} \alpha_x \hat{x}.$$

Pick a vertex $z \in X$ and compare z-coordinate of each side in (i).

$$(A_j^*(x)v)_z = (A_j^*(x))_{zz} v_z = |X|(E_j)_{xz}\alpha_z. \tag{21.1}$$

$$|X|(E_{\hat{j}}\hat{x} \circ v)_z = |X|(E_{\hat{j}}\hat{x})_z \cdot \alpha_z = |X|(E_j)_{xz}\alpha_z. \tag{21.2}$$

Note that $E_{\hat{j}}\hat{x}$ is the column x of $E_{\hat{j}}$ is the row x of E_j.

(ii) Fix i, j, h such that $q_{ij}^h = 0$.

Claim. $E_h(E_iV \circ E_jV) = 0$.

$$\begin{aligned}
E_h(E_iV \circ E_jV) &= E_h(\mathrm{Span}(v \circ w \mid v \in E_iV, w \in E_jV)) && (21.3)\\
&= E_h(\mathrm{Span}(E_i\hat{y} \circ E_j\hat{z} \mid y, z \in X)) && (21.4)\\
&= \mathrm{Span}(E_h(E_j\hat{z} \circ E_i\hat{y} \mid y, z \in X) && (21.5)\\
&= \mathrm{Span}((E_hA_{\hat{j}}^*(z)E_i)\hat{y} \mid y, z \in X) \qquad \text{by } (i) && (21.6)
\end{aligned}$$

But $q_{ij}^h = 0$ implies $q_{\hat{j}\hat{i}}^{\hat{h}} = 0$.

So, by Lemma 20.3 (ii),

$$0 = (E_{\hat{i}}A_{\hat{j}}^*E_{\hat{h}})^\top = E_hA_{\hat{j}}^*E_i.$$

Hence, $E_h(E_iV \circ E_jV) = 0$.

(iii) Fix i, j, h such that $q^h_{ij} \neq 0$. Then,

$$E_h(E_iV \circ E_jV) \subseteq E_hV$$

is clear. We show the other inclusion. Since

$$\begin{aligned}
E_i\hat{y} \circ E_j\hat{y} &= (\text{column } y \text{ of } E_i \circ \text{column } y \text{ of } E_j) && (21.7)\\
&= \text{column } y \text{ of } E_i \circ E_j && (21.8)\\
&= (E_i \circ E_j)\hat{y} && (21.9)\\
&= \left(\frac{1}{|X|}\sum_{h=0}^{D} q^h_{ij}E_h\right)\hat{y}, && (21.10)
\end{aligned}$$

we have,

$$\begin{aligned}
E_h(E_iV \circ E_jV) &= E_h\text{Span}(E_i\hat{y} \circ E_j\hat{z} \mid y, z \in X) && (21.11)\\
&\supseteq E_h\text{Span}(E_i\hat{y} \circ E_j\hat{y} \mid y \in X) && (21.12)\\
&= \text{Span}(q^h_{ij}E_h\hat{y} \mid y \in X) && (21.13)\\
&= \text{Span}(E_h\hat{y} \mid y \in X) \qquad \text{since } q^h_{ij} \neq 0 && (21.14)\\
&= E_hV. && (21.15)
\end{aligned}$$

This proves the assertion.

□

Lemma 21.1. *Given a commutative scheme $Y = (X, \{R_i\}_{0 \le i \le D})$, fix j $(0 \le j \le D)$. Define binary multiplication:*

$$E_jV \times E_jV \longrightarrow E_jV \quad ((v, w) \mapsto v * w = E_j(v \circ w)).$$

Then,

(i) $v * w = w * v$, for all $v, w \in E_jV$,

(ii) $v * (w + w') = v * w + v * w'$ for all $v, w, w' \in E_jV$, and

(iii) $(\alpha v) * w = \alpha(v * w)$ for all $\alpha \in \mathbb{C}$.

In particular, the vector space E_jV together with $*$ is a commutative $\mathbb{C}$-algebra, (not associative in general).

($N_j : (E_jV, *)$ is called the Norton algebra on E_jV.)

(iv) $v * w = 0$ for all $v, w \in E_jV$ if and only if $q^j_{jj} = 0$.

Proof.

$(i) - (iii)$ Immediate.

(iv) Immediate from Lemma 20.4 (ii), (iii).

□

Let Y, j, N_j be as in Lemma 21.1, and M Bose-Mesner algebra of Y. Let

$$\mathrm{Aut}Y = \{\sigma \in \mathrm{Mat}_X(\mathbb{C}) \mid \sigma : \text{ permutation matrix }, \sigma \cdot m = m \cdot \sigma \text{ for all } m \in M\} \tag{21.16}$$
$$= \{\sigma \in \mathrm{Mat}_X(\mathbb{C}) \mid \sigma : \text{ permutation matrix }, \tag{21.17}$$
$$(x, y) \in R_i \to (\sigma x, \sigma y) \in R_i, \text{ for all } i, \text{ and for all } x, y \in X\} \tag{21.18}$$
$$\mathrm{Aut}(N_j) = \{\sigma : E_jV \to E_jV \mid \sigma \text{ is } \mathbb{C}\text{-algebra isomorphims}, i.e., \tag{21.19}$$
$$\sigma(v * w) = \sigma(v) * \sigma(w) \text{ for all } v, w \in E_jV\}. \tag{21.20}$$

Lemma 21.2. *Let $Y, j, *$ be as in Lemma 21.1.*

(i) E_jV *is a module for* $\mathrm{Aut}(Y)$.

(ii) $\sigma|_{E_jV} \in \mathrm{Aut}(N_j)$ *for all* $\sigma \in \mathrm{Aut}(Y)$.

(iii) $\mathrm{Aut}Y \longrightarrow \mathrm{Aut}(N_j)$, $(\sigma \mapsto \sigma|_{E_j})$ *is a homomorphism of groups,*

(i.e., a representation of $\mathrm{Aut}(Y)$*).*

(iv) *Suppose $R_0, \ldots, R_D$ are orbits of* $\mathrm{Aut}(Y)$ *acting on $X \times X$, (so, we are in Example 17.2) then above representation is irreducible.*

Proof.

(i) Pick $\sigma \in \mathrm{Aut}Y$ and $v \in V$. Then,

$$\sigma E_j v = E_j \sigma v,$$

since σ commutes with each element of M.

(ii) $\sigma|_{E_jV} : E_jV \to E_jV$ is an isomorphism of a vector space. Since σ is invertible,for all $v, w \in E_jV$,

$$\sigma(v*w) = \sigma(E_j(E_jv \circ E_jw)) = E_j\sigma(E_jv \circ E_jw) = E_j(E_j\sigma v \circ E_j\sigma w) = \sigma(v)*\sigma(w).$$

(iii) Immediate from (i) and (ii).

(iv) Here Bose-Mesner algebra M is the full commuting algebra, i.e.,

$$M = \{m \in \mathrm{Mat}_X(\mathbb{C}) \mid \sigma \cdot m = m \cdot \sigma, \text{ for all } \sigma \in \mathrm{Aut}(Y)\}.$$

Suppose there sia a nonzero proper subspace $0 \neq W \subsetneq E_jV$ that is $\mathrm{Aut}(Y)$-invariant.

Set
$$W^\perp = \{v \in E_jV \mid \langle w, v\rangle = 0, \text{ for all } w \in W\}.$$

Then, $W^\perp$ is a module for $\mathrm{Aut}(Y)$, since $\mathrm{Aut}(Y)$ is closed under transpose conjugate.

Let $e : V \to W$ and $f : V \to W^\perp$ be orthogonal projection such that $e + f = E_j$,

$$e^e = e, f^e = f, ef = fe = 0, eE_h = 0, \text{ if } h \neq j.$$

Since e commutes with all $\sigma \in \mathrm{Aut}(Y)$, $e \in M$ and

$$e = \sum_{i=0}^{D} \alpha_i E_i.$$

If $h \neq j$, then $0 = eE_h$ and $\alpha_h = 0$. Thus, $e = \alpha_j E_j$, i.e., $e = 0$ or $f = 0$.

A contradiction.

□

Norton algebras were used in original construction of Monster, a finite simple group G.

Compute character table of G,

$\to$ p^h_{ij}, q^h_{ij} of group scheme on G,

$\to$ find j where $m_j = \dim E_jV$ is small and $q^j_{jj} \neq 0$,

$\to$ guess abstract structure of N_j using the knowlege of p^h_{ij}'s and q^h_{ij}'s,

$\to$ compute $\mathrm{Aut}(N_j)$,

$\to$ G.

Chapter 22

Q-Polynomial Schemes

Friday, March 19, 1993

Lemma 22.1. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme.*

(i) $p^h_{0j} = p^h_{j0} = \delta_{jh}..$

(ii) $p^0_{ij} = \delta_{ij'} k_i.$

(iii) $q^h_{0j} = q^h_{j0} = \delta_{jh}.$

(iv) $q^0_{ij} = \delta_{i\hat{j}} m_i.$

(v) $\displaystyle\sum_{j=0}^{D} p^h_{ij} = k_i.$

(vi) $\displaystyle\sum_{j=0}^{D} q^h_{ij} = m_i.$

Proof.

(i), (ii) These are trivial.

(iii) We have

$$|X|^{-1} \sum_{\ell=0}^{D} q^\ell_{0j} E_\ell = E_0 \circ E_j = |X|^{-1} J \circ E_j = |X|^{-1} E_j.$$

(iv) Recall from Lemma 20.2

$$|X|^{-1} m_h q^h_{ij} = \tau(E_i \circ E_j \circ E_{\hat{h}}),$$

(where $\tau(B)$ is the sum of entries in matrix B.)

$$
\begin{aligned}
|X|^{-1}m_0 q_{ij}^0 &= \tau(E_i \circ E_j \circ E_0) && (22.1)\\
&= |X|^{-1}\tau(E_i \circ E_j) \qquad\qquad (E_0 = |X|^{-1}J) && (22.2)\\
&= |X|^{-1}\mathrm{trace}(E_i E_{\hat{j}}) && (22.3)\\
&= |X|^{-1}\delta_{i\hat{j}}\mathrm{trace}E_i && (22.4)\\
&= |X|^{-1}\delta_{i\hat{j}}m_i. && (22.5)
\end{aligned}
$$

(v) Pick $x, y \in X$ with $(x, y) \in R_h$. Then,

{j=0}^D p^h{ij} & = |{z X (x,z) R_i, ; (z,y) R_j ; for some j}\ & = |{z X (x,z) R_i}|\ & k_i. \end{align}

(vi)

$$E_i \circ E_j = |X|^{-1}\sum_{h=0}^{D} q_{ij}^h E_h.$$

So,

$$
\begin{aligned}
\sum_{j=0}^{D} E_i \circ E_j &= |X|^{-1}\sum_{h=0}^{D}\left(\sum_{j=0}^{D} q_{ij}^h\right) E_h && (22.6)\\
&= E_i \circ \sum_{j=0}^{D} E_j && (22.7)\\
&= E_i \circ I && (22.8)\\
&= |X|^{-1}(q_i(0)A_0 + q_i(1)A_1 + \cdots + q_i(0)A_D) \circ I && (22.9)\\
&= |X|^{-1}q_i(0)I && (22.10)\\
&= |X|^{-1}m_i(E_0 + E_1 + \cdots + E_D). && (22.11)
\end{aligned}
$$

This proves the assertions.

□

Definition 22.1. Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a commutative scheme.

Y is Q-polynomial with respect to ordering $E_0, E_1, \ldots, E_D$ of primitive idempotents, if

$$
q_{ij}^h \begin{cases} = 0 & \text{if one of } h, i, j \text{ is greater than the sum of the other two} \\ \neq 0 & \text{if one of } h, i, j \text{ is equal to the sum of the other two.} \end{cases}
$$

In this case, set

$$c_i^* = q_{1,i-1}^i,\ a_i^* = q_{1,i}^i,\ b_i^* = q_{1,i+1}^i \quad (0 \le i \le D),\ (c_0^* = b_D^* = 0).$$

Observe: Q-polynomial $\to Y$ is symmetric.

Suppose $i \neq \hat{i}$ for some i. Then, by the condition in Definition 22.1,

$$0 = q^0_{i\hat{i}} = m_i \ (\neq 0)$$

by Lemma 22.1 (iv). This is a contradiction.

Hence, ${E_i}^\top = E_{\hat{i}} = E_i$ for all i.

Therefore M is symmetric and Y is symmetric.

Observe: If Y is Q-polynomial,

$$c^*_i + a^*_i + b^*_i = m_1 \quad (0 \leq i \leq D)$$

(just as $c_i + a_i + b_i = k$ for P-polynomial.)

By Lemma 22.1 (iv),

$$m_1 = q^i_{10} + q^i_{11} + \cdots + q^i_{1,i-1} + q^i_{1i} + q^i_{1,i+1} + \cdots$$

and $q^i_{10} = q^i_{11} = 0$, $q^i_{1,i-1} = c^*_i$, $q^i_{1i} = a^*_i$, and $q^i_{1,i+1} = b^*_i$.

Lemma 22.2. *Assume $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is a symmetric scheme. Pick $x \in X$, and set $E^*_i \equiv E^*_i(x)$, $A^* \equiv A^*(x)$. Then the following are equivalent.*

(i) Γ is Q-polynomial with respect to $E_0, \ldots, E_D$.

*(ii) $$qˆh_{1j} \begin{cases} = 0 & if |h − j| > 0 *

0 & if |h − j| = 1. \end{cases} (0 h,j D).$$

(iii) There exists $f^*_i \in \mathbb{C}[\lambda]$, $\deg f^*_i = i$, and

$$A^*_i = f^*_i(A^*_1) \quad (0 \leq i \leq D).$$

(iv) $E^*_0V, \ldots, E^*_DV$ are maximal eigenspaces of A^*_1, and

$$E_i A^*_1 E_j = O \quad \textit{if} \ \ |i-j| > 0, \quad (0 \leq i, j \leq D).$$

(Compare (iv) with the definition of Q-polynomial in Definition 6.2.)

Proof.

$(i) \to (ii)$ Clear.

$(ii) \to (iii)$ $A^*_0 = I$,

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \tag{22.12}$$

$$A_1^* A_j^* = q_{1j}^{j-1} A_{j-1}^* + q_{1j}^j A_j^* + q_{1j}^{j+1} A_{j+1}^* \qquad (q_{1j}^{j+1} \neq 0, 1 \leq j \leq D-1). \tag{22.13}$$

Hence A_j^* is a polynomial of degree exactly j in A_1^* by induction on j.

$$\lambda f_j^*(\lambda) = b_{j-1}^* f_{j-1}^*(\lambda) + a_j^* f_j^*(\lambda) + c_{j+1}^* f_{j+1}^*(\lambda) \quad \text{with } c_{j+1}^* \neq 0,$$

and $f_{-1}^* = 0$, $f_0^*(\lambda) = 1$.

$(iii) \to (i)$ Pick i, j, h with $0 \leq i, j, h \leq D$ and $h \geq i + j$. Since

$$m_h q_{ij}^h = m_j q_{ih}^j = m_i q_{hj}^i$$

by Lemma 20.2, it suffices to show that

$$q_{ij}^h \begin{cases} = 0 & \text{if } \ h > i + j \\ \neq 0 & \text{if } \ h = i + j. \end{cases}$$

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \tag{22.14}$$

$$f_i^*(A_1) f_j^*(A_1) = \sum_{h=0}^{D} q_{ij}^h f_h^*(A_1^*). \tag{22.15}$$

Hence,

$$f_i^*(\lambda) f_j^*(\lambda) = \sum_{h=0}^{D} q_{ij}^h f_h^*(\lambda).$$

Note that since $A_0^*, A_1^*, \ldots, A_D^*$ are linearly independent, $f(A_1^*) = 0$ implies $\deg f > D$.

$$\deg \mathrm{LHS} = i + j \to q_{ij}^{i+j} \neq 0, \ q_{ij}^h = 0, \ \text{if } \ h > i + j.$$

$(iii) \to (iv)$ Recall

$$A_1^* = q_1(0) E_0^* + q_1(1) E_1^* + \cdots.$$

Each A_i^* is a polynomial in A_1^*. Then A_1^* generates the dual Bose-Mesner algebra. So, $q_1(0), q_1(1), \ldots, q_1(D)$ are distinct.

So, $E_0^* V, \ldots, E_D^* V$ are maximal eigenspaces.

Also, $|i - j| > 1$ implies $q_{11}^j = 0$.

Thus, $E_i A^*_1 E_j = 0$ by Lemma 20.3 (ii).

$(iv) \to (ii)$ $q^i_{1j} = 0$ if $|i-j| > 1$. since in this case,

$E_i A^*_1 E_j = O$ implies $q^i_{1j} = 0$ by Lemma 20.3 (ii).

Suppose $q^{j+1}_{1j} = 0$ for some j $(0 \leq j \leq D-1)$.

Without loss of generalith, choose j minimum. Then A^*_h is a polynomial of degree h in A^*_1 $(0 \leq h \leq j)$, and

$$A^*_1 A^*_j - q^{j-1}_{1j} A^*_{j-1} - q^j_{1j} A^*_j = O.$$

the left hand side is a polynomial in A^*_1 of degree $j+1$.

Hence, the minimal polynomial of A^*_1 has degree less than or equal to $j+1 \leq D$. But %A^*_1$ has $D+1$ distince eigenvalues.

This is a contradiction.

$\square$

Chapter 23

Representation of a Scheme

Monday, March 22, 1993

Theorem 23.1. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a symmetric scheme. (View the standard module V as an algebra of functions from X to $\mathbb{C}$.) Then the following are equivalent.*

(i) Y is Q-polynomial with respect to ordering $E_0, E_1, \dots, E_D$ of primitive idempotents.

(ii) For all i $(0 \le i \le D)$,

$$E_0V + E_1V + (E_1V)^2 + \cdots + (E_1V)^i = E_0V + E_1V + \cdots + E_iV.$$

Proof.

By Lemma 20.4 (ii), (iii).

$$E_h(E_iV \circ E_jV) = 0 \text{ if and only if } q_{ij}^h = 0 \quad (0 \le i, j, h \le D).$$

$(i) \to (ii)$ By our assumption,

$$q_{1j}^h = 0 \text{ if } |h - j| > 1, \text{ and } q_{1j}^{j+1} \ne 0.$$

So,

$$E_1V \circ E_jV \subseteq E_{j-1}V + E_jV + E_{j+1}V \quad (0 \le j \le D), \tag{23.1}$$

$$E_{j+1}(E_1V \circ E_jV) = E_{j+1}V \quad (0 \le j \le D-1), \tag{23.2}$$

by Lemma 20.4.

Also $E_0 V \subseteq \text{Span}(\delta)$, where δ is all 1's vector, i.e., 1 as a function $X \to \mathbb{C}$. So,

$$E_0 \circ E_j V = E_j V \quad (0 \leq j \leq D). \tag{23.3}$$

Show (ii) by induction on i.

The cases $i = 0, 1$ are trivial.

$i > 1$: $\subseteq$.

$$\begin{aligned} & E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^i && (23.4)\\ &= E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + (E_1 V)^{i-1}) && (23.5)\\ &= E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + E_{i-1} V) && (23.6)\\ &\subseteq E_0 V + E_1 V + \cdots + E_i V && (23.7) \end{aligned}$$

by (23.1).

$\supseteq$.

Claim. $E_i \subseteq E_1 V \circ E_{i-1} V + E_{i-1} V + E_{i-2} V \quad (2 \leq i \leq D)$.

Proof of Claim. By (23.2),

$$E_i (E_1 V \circ E_{i-1} V) = E_i V.$$

For all $v \in E_i V$, there exists $u \in E_1 V \circ E_{i-1} V$ such that $E_i u = v$.

On the other hand, by (23.1),

$$E_1 V \circ E_{i-1} V \subseteq E_{i-2} V + E_{i-1} V + E_{i-2} V.$$

So, $u = w + v$, where $w \in E_{i-2} V + E_{i-1} V$. We have,

$$w = u - v \in E_1 V \circ E_{i-1} V + E_{i-1} V + E_{i-2} V$$

as desired.

Remark.

$$E_i V \circ E_j V = \text{Span}(u \circ v \mid u \in E_i V, v \in E_j V).$$

By claim,

$$\begin{aligned} & E_0 V + E_1 V + \cdots + E_i V && (23.8)\\ &\subseteq E_0 V + E_1 V + \cdots + E_i V + E_1 V \circ E_{i-1} V && (23.9)\\ &\subseteq E_0 V + E_1 V + \cdots + (E_1 V)^{i-1} + E_1 V (E_0 V + E_1 V + \cdots + (E_1 V)^{i-1}) && (23.10)\\ &\subseteq E_0 V + E_1 V + \cdots + (E_1 V)^{i-1} + (E_1 V)^i. && (23.11) \end{aligned}$$

$(ii) \to (i)$

Claim 1. Pick i, j ($0 \leq i, j \leq D$) with $j > i + 1$. Then $q^j_{1i} = 0$.

Proof of Claim 1.

$$\begin{aligned} E_j(E_1 \circ E_j V) &\subseteq E_j(E_1 V \circ (E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^i)) && (23.12)\\ &\subseteq E_j(E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^{i+1}) && (23.13)\\ &= E_j(E_0 V + E_1 V + \cdots + E_{i+1} V) && (23.14)\\ &= 0. && (23.15) \end{aligned}$$

So $q^j_{1i} = 0$ by Lemma 20.4.

Claim 2. $q^{i+1}_{1i} \neq 0$ ($0 \leq i < D$).

Proof of Claim 2.

$$\begin{aligned} &E_0 V + E_1 V + \cdots + E_{i+1} V && (23.16)\\ &\quad = E_0 V + E_1 V + \cdots + (E_1 V)^{i+1} && (23.17)\\ &\quad = E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + (E_1 V)^i) && (23.18)\\ &\quad = E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + E_i V) && (23.19)\\ &\quad = E_0 V + E_1 V \circ (E_0 V + \cdots + E_i V). && (23.20) \end{aligned}$$

So,

$$\begin{aligned} E_{i+1} V &= E_{i+1}(E_1 V \circ (E_0 V + \cdots + E_i V)) && (23.21)\\ &= E_{i+1}(E_1 V \circ E_i V) && (23.22) \end{aligned}$$

by Claim 1 and Lemma 20.4.

Hence, $q^{i+1}_{1i} \neq 0$ by Lemma 20.4.

□

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme with standard module V.

Definition 23.1. A representation of Y is a pair (ρ, H), where H is a non-zero Hermitean space (with inner product $\langle\ ,\ \rangle$) and $\rho : X \to H$ is a map satisfying the following.

R1. $H = \mathrm{Span}(\rho(x) \mid x \in X)$.

R2. $\langle \rho(x), \rho(y) \rangle$ depends only on i for which $(x, y) \in R_i$ ($x, y \in X$).

R3. For every $x \in X$ and for all i ($0 \leq i \leq D$),

$$\sum_{y \in X, (y,x) \in R_i} \rho(y) \in \mathrm{Span}(\rho(x)).$$

Above representation is nondegenerate if $\{\rho(x) \mid x \in X\}$ are distinct.

Example 23.1. $Y = H(D,2)$, $X = \{a_1 \cdots a_D \mid a_i \in \{1,-1\}, 1 \le i \le D\}$. Let $H = \mathbb{C}^D$ and $\langle\ ,\ \rangle$ usual Hermitean dot product.

For a vertex $x = a_1 \cdots a_D \in X$, define

$$\rho(x) = a_1 \cdots a_D \in H.$$

Then, $\mathrm{R1} - \mathrm{R3}$ hold.

Remark. R1, R2 are obvious. For R3, we may assume that $x = 1 \cdots 1$. Restrict

$$\sum_{y \in X, (y,x) \in R_i} \rho(y)$$

on the first coordinate. Then,

$$-1 \quad \text{appers} \quad \binom{D-1}{i-1} \quad \text{times} \tag{23.23}$$

$$1 \quad \text{appers} \quad \binom{D-1}{i} \quad \text{times.} \tag{23.24}$$

Hence,

$$\sum_{y \in X, (y,x) \in R_i} \rho(y) = \left(\binom{D-1}{i} - \binom{D-1}{i-1} \right) \rho(x).$$

Let (ρ, H) be a representation of arbitrary commutative scheme Y. Set

$$E = (\langle \rho(x), \rho(y) \rangle)_{x,y \in X}$$

Gram matrix of the representation.

Definition 23.2. Representations (ρ, H), (ρ', H') of Y are equivalent, whenever, Gram matrices are related by

$$E' \in \mathrm{Span} E.$$

We do not distinguish between equivalent representations.

Note. Suppose (ρ, H) is a representation of a symmetric scheme Y. Pick $x, y \in X$ with $(x,y) \in R_j$.

Then $(y,x) \in R_j$. So, by R2,

$$\langle \rho(x), \rho(y) \rangle = \langle \rho(y), \rho(x) \rangle = \overline{\langle \rho(x), \rho(y) \rangle},$$

since $\langle\ ,\ \rangle$ is Hermitean.

Hence, the Gram matrix E of ρ is real symmetirc. Without loss of generality, we can view H as a real Euclidean space in this case.

Lemma 23.1. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme and V a standard module.*

Let E_j be any primitive idempotent of Y.

(i) (ρ, H) is a representation of Y, where $H = E_j V$ (with inner product inherited from Y).

$$\rho : X \to H \quad (x \mapsto E_j \hat{x})$$

(i.e., $\rho(x)$ is the x-th column of E_j.)

(ii) $\langle \rho(x), \rho(y) \rangle = |X|^{-1} q_j(i)$, if $(x, y) \in R_i$, $(x, y \in X)$.

(iii) For $0 \le i \le D$ and $x, y \in X$,

$$\sum_{y \in X, (y,x) \in R_i} \rho(y) = p_i(j) \rho(x).$$

(iv) (ρ, H) is nondegenerate if and only if $q_j(i) \ne q_j(0)$ for all i, $(0 \le i \le D)$.

(v) Every representation of Y is equivalent to a representation of the above type for some j $(0 \le j \le D)$, and j is unique.

Proof.

$(i) - (iii)$.

R1: Span(ρX) is the column space of E_j which is equal to H.

R2:

$$\begin{aligned}
\langle \rho(x), \rho(y) \rangle &= \langle E_j \hat{x}, E_j \hat{y} \rangle && (23.25)\\
&= (\overline{E_j \hat{x}})^\top E_j \hat{y} && (23.26)\\
&= \hat{x}^\top \overline{E_j}^\top E_j \hat{y} && (23.27)\\
&= \hat{x}^\top E_j \hat{y} && (23.28)\\
&(E_j)_{xy}. && (23.29)
\end{aligned}$$

Note that $\overline{E_j}^\top = E_j$ by Lemma 19.1.

Recall

$$E_j = |X|^{-1}(q_j(0) A_0 + \cdots + q_j(D) A_D).$$

So,

$$(E_j)_{xy} = |X|^{-1} q_j(i), \quad \text{where } (x, y) \in R_i.$$

R2: Recall

$$A_i = p_i(0) E_0 + \cdots + p_i(D) E_D.$$

So, $E_jA_i = p_i(j)E_j$, and

$$p_i(j)\rho(x) = p_i(j)E_j\hat{x} = E_jA_i\hat{x} = E_j \sum_{y\in X,(y,x)\in R_i} \hat{y} = \sum_{y\in X,(y,x)\in R_i} \rho(y).$$

Note.

$$A_i\hat{x} = \sum_{y\in X,(x,y)\in R_{i'}} \hat{y}.$$

Pf.

$$\begin{aligned} z \text{ entry of LHS} &= (A_i\hat{x})_z && (23.30)\\ &= \sum_{w\in X}(A_i)_{zw}\hat{x}_w && (23.31)\\ &= (A_i)_{zx} && (23.32)\\ &= \begin{cases} 1 & \text{if } (x,z)\in R_{i'} \\ 0 & \text{else.}\end{cases} && (23.33)\end{aligned}$$

$$\begin{aligned} z \text{ entry of RHS} &= \sum_{y\in X,(x,y)\in R_{i'},z=y} 1 && (23.34)\\ &= \begin{cases} 1 & \text{if } (x,z)\in R_{i'} \\ 0 & \text{else.}\end{cases} && (23.35)\end{aligned}$$

(iv) By (ii),

$$\begin{aligned} \|\rho(x)\|^2 &= \langle \rho(x), \rho(y)\rangle && (23.36)\\ &|X|^{-1}q_j(0) && (23.37)\\ &|X|^{-1}m_j, && (23.38)\end{aligned}$$

as $m_j = \dim E_jV$, and is independent of $x \in X$.

Pick distinct $x, y \in X$ such that $(x, y) \in R_i$ with $i \neq 0$.

Then,

$$\begin{aligned} \rho(x) = \rho(y) &\Leftrightarrow \langle \rho(x), \rho(y)\rangle = \|\rho(x)^2\| = |X|^{-1}q_j(0) && (23.39)\\ &\Leftrightarrow |X|^{-1}q_j(i) = |X|^{-1}q_j(0) && (23.40)\\ &\Leftrightarrow q_j(i) = q_j(0). && (23.41)\end{aligned}$$

Hence, we have (iv). To be continued.

□

Chapter 24

Balanced Conditions, I

Wednesday, March 23, 1993

No Class on Friday (another conference).

Proof of Lemma 23.1 continued. Let E_j be a primitive idempotent, $H = E_jV$ and

$$\rho : X \to H \quad (x \mapsto E_j\hat{x}).$$

(v) Every representation (ρ, H) of Y is equivalent to a representation of above type, for some j $(0 \le j \le D)$ and j is unique.

Let $E := (\langle \rho(x), \rho(y)\rangle)_{x,y\in X}$.

By R2,

$$E = \sum_{i=0}^{D} \sigma_i A_i, \quad \text{some } \sigma_0, \sigma_1, \ldots, \sigma_D \in \mathbb{C}.$$

Hence, E belongs to the Bose-Mesner algebra \$M of Y.

We want to show that E is a scalar multiple of a primitive idempotent.

Fix $x \in X$ and fix i $(0 \le i \le D)$.

By R3,

$$\sum_{y\in X,(y,x)\in R_i} \rho(y) = \alpha\rho(x), \quad \text{some } \ \alpha \in \mathbb{C}. \tag{24.1}$$

So,

$$k_i\overline{\sigma_i} = \left\langle \sum_{y\in X,(y,x)\in R_i} \rho(y), \rho(x) \right\rangle = \bar{\alpha}\langle \rho(x), \rho(x)\rangle = \bar{\alpha}\sigma_0.$$

Hence, α is independent of x. In maatrix form (24.1) becomes

$$EA_i\hat{x} = \alpha E\hat{x}.$$

Remark.

$$Eu = Ev \Leftrightarrow \langle z, Eu\rangle = \langle z, Ev\rangle \text{ for all } z \in X \Leftrightarrow (Eu)_z = (Ev)_z \text{ for all } z \in X.$$

$$\begin{aligned}
(EA_i\hat{x})_z &= \left\langle \rho(z), \sum_{y\in X,(y,x)\in R_i} \rho(y) \right\rangle && (24.2)\\
&= \alpha\langle \rho(z), \rho(x)\rangle && (24.3)\\
&= (\alpha E\hat{x})_z. && (24.4)
\end{aligned}$$

Hence,

$$EA_i\hat{x} = \alpha E\hat{x}.$$

Since x is arbitrary,

$$EA_i = \alpha E.$$

So,

$$EA_i \in \mathrm{Span}E \quad\text{and}\quad EM = \mathrm{Span}E.$$

We have $E \in \mathrm{E_j}$ for unique j $(0 \le j \le D)$. □

Remark.

$$E = \tau_0 E_0 + \cdots + \tau)DE_D,\ \tau_j \in \mathbb{C} \quad (0 \le j \le D).$$

And, at least one of τ_j is nonzero, and

$$\tau_j E_j = EE_j \in \mathrm{Span}E.$$

So,

$$\tau_j E_j = E$$

as $E_0, \dots, E_D$ are linearly independent.

Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a symmetric scheme, and let E be a primitive idempotent.

Definition 24.1. Y is Q-polynomial with respect to E, if and only if Y is Q-polynomial with respect to some ordering $E_0, E_1, \dots, E_D$ of primitive idempotents, where $E_0 = |X|^{-1}J$, and $E_1 = E$.

Theorem 24.1. *Assume $Y = (X, \{R_i\}_{0\le i\le D})$ is P-polynomial (i.e., (X, R_1) is distance-regular). Let E be any primitive idempotent of Y. Let (ρ, H) be the corresponding representation.*

(i) *The following are equivalent.*

(ia) *Y is Q-polyonimial with respect to E.*

(ib) *(ρ, H) is nondegenerate and for all $x, y \in X$, and for all i, j $(0 \le i, j \le D)$,*

$$\sum_{z\in X,(x,z)\in R_i,(y,z)\in R_j}\rho(z)-\sum_{z'\in X,(x,z')\in R_j,(y,z')\in R_i}\rho(z')\in \mathrm{Span}(\rho(x)-\rho(y)).$$

(ic) (ρ,H) is nondegenerate and for all $x,y\in X$,

$$\sum_{z\in X,(x,z)\in R_1,(y,z)\in R_2}\rho(z)-\sum_{z'\in X,(x,z')\in R_2,(y,z')\in R_1}\rho(z')\in \mathrm{Span}(\rho(x)-\rho(y)).$$

(ii) Wirte

$$E=|X|^{-1}\sum_{j=0}^{D}\theta_j^*A_j,$$

and suppose $(ia)-(ic)$ hold. Then the coefficient in (ib) is

$$p_{ij}^h\frac{\theta_i^*-\theta_j^*}{\theta_0^*-\theta_h^*}\quad (1\le h\le D, 0\le i,j\le D).$$

Proof.

$(ia)\to(ib)$ Without loss of generality, assume $E\equiv E_1$, and Y is Q-polynomial with respect to E.

Then by Lemma 22.2, $\theta_0^*,\dots,\theta_D^*$ are distinct. So $\theta_h^*\ne\theta_0^*$ for all $h\in\{1,2\dots,D\}$, and (ρ,H) is nondegenerate.

Fix $x\in X$, write $E_i^*\equiv E_i^*(x)$, $A_i^*\equiv A_i^*(x)$, $A^*\equiv A_1^*$.

Let M be the Bose-Mesner algebra. Set

$$L=\{mA^*n-nA^*m\mid m,n\in M\}.$$

Claim 1. $\dim L\le D$.

Proof of Claim 1.

$$\begin{aligned} L&=\mathrm{Span}(E_iA^*E_j-E_jA^*E_i\mid 0\le i<j\le D) &(24.5)\\ &=\mathrm{Span}(E_iA^*E_{i+1}-E_{i+1}A^*E_i\mid 0\le i\le D-1). &(24.6)\end{aligned}$$

Since $E_iA^*E_j=O$ if $q_{ij}^1=0$ by Lemma 20.2 and Lemma 20.3, and this occurs if $|i-j|>1$ by Q-polynomial property.

Hence, $\dim L\le D$.

Claim 2. (i) $\{A^*A_h-A_hA^*\mid 1\le h\le D\}$ is a basis for L. In particular,

(ii) there exist $r_{ij}^h \in \mathbb{C}$ $(1 \le h \le D, 0 \le i, j \le D)$ such that

$$A_i A^* A_j - A_j A^* A_i = \sum_{h=1}^{D} r_{ij}^h (A^* A_h - A_h A^*).$$

Proof of Claim 2.

(i) The column x of $A^* A_h - A_h A^*$ is a nonzero scalar $\theta_h^* - \theta_0^*$ times the column x of A_h.

Remark.

$$((A^* A_h - A_h A^*)\hat{x})_y = E_{xy}(A_h)_{yx} - (A_h)_{yx} E_{xx} = (\theta_h^* - \theta_0^*)(A_h)_{yz}.$$

Also the column x of $A_0, A_1, \dots, A_D$ are linearly independent.

Hence, the matrices given are linearly independent.

They are in L by construction, so they form a basis for L by Claim 1.

(ii) This is immediate since

$$A_i A^* A_j - A_j A^* A_i \in L, \quad \text{for all } i, j.$$

Cloim 3.

$$r_{ij}^\ell = p_{ij}^\ell \left(\frac{\theta^* - \theta_j^*}{\theta_0^* - \theta_\ell^*} \right) \quad (1 \le \ell \le D, 0 \le i, j \le D).$$

Proof of Claim 3. Fix i, j,

$$A_i A^* A_j - A_j A^* A_i - \sum_{h=1}^{D} r_{ij}^h (A^* A_h - A_h A^*) = 0.$$

Pick ℓ $(1 \le \ell \le D)$. Pick $y \in X$ such that $(x, y) \in R_\ell$.

$$\begin{aligned} (A_i A^* A_j)_{xy} &= \sum_{z \in X} (A_i)_{xz} (A^*)_{zz} (A_j)_{zy} && (24.7) \\ &= \sum_{z \in X, (x,z) \in R_i, (y,z) \in R_j} (A^*)_{zz} && (24.8) \\ &= |X|^{-1} p_{ij}^\ell \theta_i^*. && (24.9) \end{aligned}$$

Similarly,

$$(A_j A^* A_i)_{xy} = |X|^{-1} p_{ij}^\ell \theta_j^*.$$

$$\begin{aligned}(A^*A_h - A_hA^*)_{xy} &= (A_0A^*A_h - A_hA^*A_0)_{xy} &(24.10)\\ &= |X|^{-1}p^{\ell}_{0h}(\theta^*_0 - \theta^*_h) &(24.11)\\ &= \begin{cases} 0 & \text{if } \ell \neq h \\ |X|^{-1}(\theta^*_0 - \theta^*_h) & \text{if } \ell = h. \end{cases} &(24.12)\end{aligned}$$

Hence,

$$\sum_{h=1}^{D} r^h_{ij}(A^*A_h - A_hA^*)_{xy} = |X|^{-1}r^{\ell}_{ij}(\theta^*_0 - \theta^*_{\ell}).$$

Comparing terms, we have

$$p^{\ell}_{ij}(\theta^*_i - \theta^*_j) - r^{\ell}_{ij}(\theta^*_0 - \theta^*_{\ell}) = 0.$$

Claim 4. For all h $(1 \leq h \leq D)$, for all i, j $(0 \leq i, j \leq D)$, for all $w, y \in X$, $(w, y) \in R_h$,

$$\sum_{z \in X, (w,z) \in R_i, (y,z) \in R_j} \rho(z) - \sum_{z' \in X, (w,z') \in R_j, (y,z) \in R_i} \rho(z') - r^h_{ij}(\rho(w) - \rho(y)) = 0. \tag{24.13}$$

Proof of Claim 4. Set $L = \langle \text{LHS of (24.13)}, \rho(x) \rangle$ It suffices to show that $L = 0$. Note that since x is arbitrary, if LHS of (24.13) is zero.

$$\begin{aligned}L = &\sum_{z \in X, (w,z) \in R_i, (y,z) \in R_j} \langle \rho(z), \rho(x) \rangle - \sum_{z' \in X, (w,z') \in R_j, (y,z) \in R_i} \langle \rho(z'), \rho(x) \rangle &(24.14)\\ &- r^h_{ij}\langle \rho(w) - \rho(y), \rho(x) \rangle &(24.15)\\ = &\,|X|^{-1}(A_iA^*A_j)_{wy} - |X|^{-1}(A_jA^*A_i)_{wy} - |X|^{-1}\sum_{\ell=1}^{D} r^{\ell}_{ij}(A^*A_{\ell} - A_{\ell}A^*)_{wy} &(24.16)\\ = &\,|X|^{-1}\text{times } wy \text{ entry of a matrix known to be zero by Claim 2} &(24.17)\\ = &\,0. &(24.18)\end{aligned}$$

□

Remark.

$$\begin{aligned}|X|^{-1}\sum_{\ell=1}^{D} r^{\ell}_{ij}(A^*A_{\ell} - A_{\ell}A^*)_{wy} &= |X|^{-1}r^h_{ij}(A^*A_h - A_hA^*)_{wy} &(24.19)\\ &= r^h_{ij}(\langle \rho(x), \rho(w) \rangle - \langle \rho(x), \rho(y) \rangle) &(24.20)\end{aligned}$$

Chapter 25

Balanced Conditions, II

Monday, March 29, 1993

Proof of Theorem 24.1 continued.

$(ib) \to (ic)$ Obvious.

$(ic) \to (ia)$ Without loss of generality, we may assume $D \geq 3$, else trivial.

Remark. The case $D = 2$ should be treated somewhere, but the assumption $D \geq 3$ is not used.

Fix $w \in X$, and write $E_i^* \equiv E_i^*(w)$, $A_i^* \equiv A_i^*(w)$, $A^* \equiv A_1^*$, and A_i, i-th distance matrix. Set

$$E \equiv E_1 = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i.$$

Since (ρ, H) is nondegenerate,

$$\theta_0^* \neq \theta_h^*, \text{ for all } h \in \{1, 2, \dots, D\}$$

See Lemma 23.1 (iv).

Claim 1. Pick h $(1 \leq h \leq D)$, and x, y with $(x, y) \in R_h$. Then

$$\sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z') = r_{12}^h(\rho(x) - \rho(y)),$$

where

$$r_{12}^h = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}.$$

Proof of Claim 1. By our assumption,

$$\sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z') = \alpha(\rho(x) - \rho(y)).$$

Hence,

$$|X|^{-1}p^h_{12}(\theta^*_1-\theta^*_2) = \left\langle \sum_{z\in X,(x,z)\in R_1,(y,z)\in R_2} \rho(z) - \sum_{z'\in X,(x,z')\in R_2,(y,z')\in R_1} \rho(z'), \rho(x) \right\rangle \tag{25.1}$$

$$= \alpha\langle \rho(x)-\rho(y), \rho(x) \tag{25.2}$$

$$= \alpha|X|^{-1}(\theta^*_0-\theta^*_h). \tag{25.3}$$

We have

$$\alpha = p^h_{12}\frac{\theta^*_1-\theta^*_2}{\theta^*_0-\theta^*_h}.$$

Claim 2. $A_1A^*A_2 - A_2A^*A_1 = \sum_{h=1}^{D} r^h_{12}(A^*A_h - A_hA^*)$.

Proof of Claim 2. The xy entry of the LHS − RHS is

$$|X| \left\langle \sum_{z\in X,(x,z)\in R_1,(y,z)\in R_2} \rho(z) - \sum_{z'\in X,(x,z')\in R_2,(y,z')\in R_1} \rho(z') - r^h_{12}(\rho(x)-\rho(y)), \rho(w) \right\rangle,$$

where $(x,y)\in R_h$, $h=1,2,\dots,D$, and the xy entry of the LHS − RHS is 0 if $x=y$.

But the vector on the left in the above inner product is 0 by Claim 1, so the inner product is 0.

Thus, the xy entry of the LHS − RHS is always 0, and we have Claim 2.

Claim 3. $A^*A_3 - A_3A^* \in \mathrm{Span}(AA^*A_2 - A_2A^*A, A^*A_2 - A_2A^*, A^*A - AA^*)$.

Proof of Claim 3. Since $p^h_{12}=0$, if $h>3$, and $p^h_{12}\neq 0$, if $h=3$, we have $r^h_{12}=0$ if $h>0$, and $r^h_{12}\neq 0$, if $h=3$. Note that $\theta^*_1\neq\theta^*_2$.

Now we are done by Claim 2.

Claim 4. There exist $\beta,\gamma,\delta\in\mathbb{R}$ such that

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^*+A^*A) - \delta A^*] \tag{25.4}$$

$$= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A). \tag{25.5}$$

Proof of Claim 4. There exists $f_i\in\mathbb{R}[\lambda]$, $\deg f_i = i$ such that $A_i = f_i(A_1)$.

Writing A_2, A_3 as polynomials in A in Claim 3 and simplifying, we find

$$A^3A^* - A^*A^3 \in \mathrm{Span}(A^2A^*A - AA^*A^2, A^2A^* - A^*A^2, AA^* - A^*A).$$

Remark. Let $A_3 = \beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I$ with $\beta_3 \neq 0$, and $A_2 = \gamma_2 A^2 + \gamma_1 A + \gamma_0 I$, with $\gamma_2 \neq 0$. Then

$$A^*A_3 - A_3A^* = A^*(\beta_3A^3 + \beta_2A^2 + \beta_1A + \beta_0I) - (\beta_3A^3 + \beta_2A^2 + \beta_1A + \beta_0I)A^*. \tag{25.6}$$

$$A^3A^* - A^*A^3 \in \mathrm{Span}(A^*A_3 - A_3A^*, A^2A^* - A^*A^2, AA^* - A^*A) \tag{25.7}$$

$$\subseteq \mathrm{Span}(AA^*A_2 - A_2A^*A, A^*A_2 - A_2A^*, A^2A^* - A^*A^2, AA^* - A^*A) \tag{25.8}$$

$$A^*A_2 - A_2A^* = A^*(\gamma_2A^2 + \gamma_1A + \gamma_0I) - (\gamma_2A^2 + \gamma_1A + \gamma_0I)A^* \tag{25.9}$$

$$AA^*A_2 - A_2A^*A = AA^*(\gamma_2A^2 + \gamma_1A + \gamma_0I) - (\gamma_2A^2 + \gamma_1A + \gamma_0I)A^*A \tag{25.10}$$

$$A^*A_2 - A_2A^* \in \mathrm{Span}(A^2A^* - A^*A^2, AA^* - AA^*) \tag{25.11}$$

$$AA^*A_2 - A_2A^*A \in \mathrm{Span}(A^2A^*A - AA^*A^2, AA^* - AA^*) \tag{25.12}$$

$$A^3A^* - A^*A^3 \in \mathrm{Span}(A^2A^*A - AA^*A^2, A^2A^* - A^*A^2, AA^* - A^*A). \tag{25.13}$$

Hence, we can find δ, γ, δ satisfying

$$0 = A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A).$$

On the other hand,

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*] \tag{25.14}$$

$$= A^3A^* - A^2A^*A - \beta A^2A^*A + \beta AA^*A^2 + AA^*A^2 - A^*A^3 \tag{25.15}$$

$$- \gamma A^2A^* - \gamma AA^*A + \gamma AA^*A + \gamma A^*A^2 - \delta AA^* + \delta A^*A \tag{25.16}$$

$$= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A). \tag{25.17}$$

Thus we have (i) and (ii).

Define a diagram D_E on nodes $0, 1, \dots, D$.

Connect distinct nodes , by undirected arc if $q^1_{ij} \neq 0$. (Note $q^1_{ij} = q^1_{ji}$).

Since $q^1_{0j} = \delta_{1j}$, the 0-node is adjacent to the 1-node and no other node.

Y is Q-polynomial with respect to E if and only if E_E is a path.

Claim 5. D_E is connected.

Proof of Claim 5. Suppose there exists $\Delta \subseteq \{0, 1, \dots, D\}$ such that i, j not connected for every $i \in \Delta$ and $j \in \{0, 1, \dots, D\} \quad D$.

Set

$$f = \sum_{i \in \Delta} E_i.$$

Observe

$$fA^* = \sum_{i\in\Delta} E_i A^* \left(\sum_{j=0}^{D} E_j\right) \tag{25.18}$$

$$= \sum_{i\in\Delta, j\in\Delta} E_i A^* E_j \quad (\text{since } E_i A^* E_j = O \text{ if } q^1_{ij} = 0) \tag{25.19}$$

$$= fA^*f. \tag{25.20}$$

Also, $A^*f = fA^*f$.

Hence, f commutes with A^*.

But f is an element of the Bose-Mesner algebra

$$f = \sum_{i=0}^{D} \alpha_i A_i \quad \text{for some } \alpha_0, \dots, \alpha_D \in \mathbb{C}.$$

We have

$$0 = fA^* - A^*f = \sum_{i=1}^{D} \alpha_i (A_i A^* - A^* A_i).$$

But $\{A_h A^* - A^* A_h \mid 1 \le h \le D\}$ are linearly independent. (The column w of $A_h A^* - A^* A_h$ is $\theta^*_h - \theta^*_0$ times the column w of A_h.)

Hence, $\alpha_1 = \dots = \alpha_D = 0$, and $f = \alpha_0 I$. Since $f^2 = f$, α_0 or 1.

If $\alpha_0 = 0$, $f = O$ and $\Delta = \emptyset$.

If $\alpha_0 = 1$, $f = I$ and $\Delta = \{0, 1, \dots, D\}$.

This proves Claim 5.

□

Remark. Claim 5 proves the following in general.

Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a symmetric association scheme. Fix a vertex $x \in X$, and let

$$E = \frac{1}{|X|}\sum_{j=0}^{D} \theta^*_j A_j \quad (\theta^*_j = q_1(j) \;\text{ if } E = E_1)$$

be a primitive idempotent and $E^*_j \equiv E^*_j(x)$.

$$A^* = \sum_{j=0}^{D} \theta^*_j E^*_j.$$

If $\theta_0 = \theta^*_h$, $h = 1, \dots, D$, then the following hold.

(i) $\{A_h A^* - A^* A_h \mid 1 \le h \le D\}$ are linearly independent.

(ii) The diagram D_E on nodes $0, 1, \dots, D$ defined by

$$i \sim j \Leftrightarrow E(E_i \circ E_j) \neq O$$

is connected.

(iii) $C_M(A^*) = \{L \in M \mid LA^* = A^*L\} = \mathrm{Span}(I)$.

Proof. | *(i)* The column x of $A_h A^* - A^*(A_h)$ is $\theta_0^* - \theta_h^*$ times the column x of A_h.

(iii) $0 = [\sum_{h=0}^{D} \alpha_h A_h, A^*] = \sum_{h=1}^{D} \alpha_h (A_h A^* - A^* A_h)$. Hence, $\alpha_0 = \cdots = \alpha_D = 0$.

(ii) Δ is a connected component. Let $f = \sum_{i \in \Delta} E_i$, then $f \in C_M(A^*)$.

Let $Y = (X, \{R_i\}_{0 \leq i \leq 2})$ be a symmetric association scheme with $D = 2$. Let

$$E = \frac{1}{|X|} \sum_{j=0}^{2} \theta_j^* A_j$$

be a primitive idempotent. If $\theta_0^*, \theta_1^*, \theta_2^*$.

Then Y us Q-polynomial with respect to E.

Proof. By the previous lemma, D_E is connected.

Note. It seems $\theta_1^* \neq \theta_2^*$ is necessary. Clarify the condition $\theta_1^* = \theta_2^*$.

Terwilliger claims that $\theta_1^* = \theta_2^*$ does not occur under the assumption *(ic)*. (March 7, 1995)

Chapter 26

Title of the Chapter

Wednesday, February 17, 1993 # Edit Date

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