

Lecture Note on Terwilliger Algebra

P. Terwilliger, edited by H. Suzuki

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About this lecturenote

Setting

sudo This note is created by `bookdown` package on RStudio.

For `bookdown` See (Xie, 2015), (Xie, 2017), (Yihui Xie, 2018).

1. Log-in to my GitHub Account
2. Go to RStudio/bookdown-demo repository: <https://github.com/rstudio/bookdown-demo>
3. Use This Template
4. Input Repository Name
5. Select Public - default
6. Create repository from template
7. From Code download ZIP
8. Move the extracted folder into a favorite directory
9. Open RStudio Project in the folder
10. Use Terminal in the bottom left pane
 - confirm that the current directory is the home directory of the project by `pwd`
11. (failed to proceed by ssh)
12. Use Console
 1. `library(usethis)`
 2. `use_git()`
 3. `use_github()` — Error
 4. `gh_token_help()`
 5. `create_github_token()`: create a token in the github page. Copy the token
 6. `gitcreds::gitcreds_set()`: paste the token, the token is to be expired in 30 days
13. Use Terminal
 1. `git remote add origin https://github.com/icu-hsuzuki/t-algebra.git`
 2. `git push -u origin main`
 3. type in the password of the computer
14. Use GIT in R Studio

Another Host

1. create a project by version control git
2. git init
3. git remote add origin git@github.com:/.git
4. git branch -r
5. git fetch
6. git pull origin main

Chapter 1

Subconstituent Algebra of a Graph

Wednesday, January 20, 1993

A graph (undirected, without loops or multiple edges) is a pair $\Gamma = (X, E)$, where

$$X = \text{finite set (of vertices)} \quad (1.1)$$

$$E = \text{set of (distinct) 2-element subsets of } X (= \text{edges of } \Gamma). \quad (1.2)$$

vertices x and $y \in X$ are adjacent if and only if $xy \in E$.

Example 1.1. Let Γ be a graph. $X = \{a, b, c, d\}$, $E = \{ab, ac, bc, bd\}$.



Set $n = |X|$, the order of Γ .

Pick a field K ($= \mathbb{R}$ or \mathbb{C}). Then $\text{Mat}_X(K)$ denotes the K algebra of all $n \times n$ matrices with entries in K . (rows and columns are indexed by X)

Adjacency matrix $A \in \text{Mat}_X(K)$ is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E \\ 0 & \text{else .} \end{cases} \quad (1.3)$$

Example 1.2. Let a, b, c, d be labels of rows and columns. Then

$$A = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

The subalgebra M of $\text{Mat}_X(K)$ generated by A is called the *Bose-Mesner algebra* of Γ .

Set $V = K^n$, the set of n -dimensional column vectors, the coordinates are indexed by X .

Let $\langle \cdot, \cdot \rangle$ denote the Hermitean inner product:

$$\langle u, v \rangle = u^\top \cdot v \quad (u, v \in V)$$

V with $\langle \cdot, \cdot \rangle$ is the *standard module* of Γ .

M acts on V : For every $x \in X$, write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where 1 is at the x position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since A is a real symmetrix matrix,

$$V = V_0 + V_1 + \cdots + V_r \quad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal A -eigenspaces.

Let $E_i \in \text{Mat}_X(K)$ denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then E_0, \dots, E_r are the primitive idempotents of M .

$$M = \text{Span}_K(E_0, \dots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all } i, j, \quad E_0 + \cdots + E_r = I.$$

Let θ_i denote the eigenvalue of A for V_i in \mathbb{R} . Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \dots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \theta_i = \dim V_i = \text{rank } E_i.$$

Set

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \dots, & \theta_r \\ m_0, & m_1, & \dots, & m_r \end{pmatrix}.$$

Problem. What can we say about Γ when $\text{Spec}(\Gamma)$ is given?

The following Lemma 1.1, is an example of Problem.

For every $x \in X$,

$$k(x) \equiv \text{valency of } x \equiv \text{degree of } x \equiv |\{y \mid y \in X, xy \in E\}|.$$

Definition 1.1. The graph Γ is regular of valency k if $k = k(x)$ for every $x \in X$.

Lemma 1.1. *With the above notation,*

- (i) $\theta_0 \leq \max\{k(x) \mid x \in X\} = k^{\max}$.
- (ii) *If Γ is regular of valency k , then $\theta_0 = k$.*

Proof.

(i) Without loss of generality we may assume that $\theta_0 > 0$, else done. Let $v := \sum_{x \in X} \alpha_x \hat{x}$ denote the eivenvector for θ_0 .

Pick $x \in X$ with $|\alpha_x|$ maximal. Then $|\alpha_x| \neq 0$.

Since $Av = \theta_0 v$,

$$\theta_0 \alpha_x = \sum_{y \in X, xy \in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \leq \sum_{y \in X, xy \in E} |\alpha_y| \leq k(x) |\alpha_x| \leq k^{\max} |\alpha_x|.$$

(ii) All 1's vector $v = \sum_{x \in X} \hat{x}$ satisfies $Av = kv$.

□

Subconstituent Algebra

Let $x, y \in X$ and $\ell \in \mathbb{Z}^{\geq 0}$.

Definition 1.2. A path of length ℓ connecting x, y is a sequence

$$x = x_0, x_1, \dots, x_\ell = y, \quad x_i \in X, \quad 0 \leq i \leq \ell$$

such that $x_i x_{i+1} \in E$ for $0 \leq i \leq \ell - 1$.

Definition 1.3. The distance $\partial(x, y)$ is the length of a shortest path connecting x and y .

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

Definition 1.4. The graph Γ is connected if and only if $\partial(x, y) < \infty$ for all $x, y \in X$.

From now on, assume that Γ is connected with $|X| \geq 2$.

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \text{the diameter of } \Gamma.$$

Fix a ‘base’ vertex $x \in X$.

Definition 1.5.

$$d(x) = \text{the diameter with respect to } x = \max\{\partial(x, y) \mid y \in X\} \leq d.$$

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad (\text{orthogonal direct sum}),$$

where

$$V_i^* = \text{Span}_K(\hat{y} \mid \partial(x, y) = i) \equiv V_i * (x)$$

and $V_i^* = V_i^*(x)$ is called the i -th subconstituent with respect to x .

Let $E_i^* = E_i^*(x)$ denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View $E_i^*(x) \in \text{Mat}_X(K)$. So, $E_i^*(x)$ is diagonal with yy entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{else,} \end{cases} \quad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \text{Span}_K(E_0^*(x), \dots, E_{d(x)}^*(x)).$$

Then $M^*(x)$ is a commutative subalgebra of $\text{Mat}_X(K)$ and is called the *dual Bose-Mesner algebra with respect to x* .

Definition 1.6 (Subconstituent Algebra). Let $\Gamma = (X, E)$, $x, M, M^*(x)$ be as above. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(K)$ generated by M and $M^*(x)$. T is the *subconstituent algebra* of Γ with respect to x .

Definition 1.7. A T -module is any subspace $W \subset V$ such that $aw \in W$ for all $a \in T$ and $w \in W$.

T -module W is *irreducible* if and only if $W \neq 0$ and W does not properly contain a nonzero T -module.

For any $a \in \text{Mat}_X(K)$, let a^* denote the conjugate transpose of a .

Observe that

$$\langle au, v \rangle = \langle u, a^*v \rangle \quad \text{for all } a \in \text{Mat}_X(K), \text{ and for all } u, v \in V.$$

Lemma 1.2. *Let $\Gamma = (X, E)$, $x \in X$ and $T \equiv T(x)$ be as above.*

(i) *If $a \in T$, then $a^* \in T$.*

(ii) *For any T -module $W \subset V$,*

$$W^\perp := \{v \in V \mid \langle w, v \rangle = 0, \text{ for all } w \in W\}$$

is a T -module.

(iii) *V decomposes as an orthogonal direct sum of irreducible T -modules.*

Proof.

(i) It is because T is generated by symmetric real matrices

$$A, E_0^*(x), E_1^*(x), \dots, E_{d(x)(x)}^*.$$

(ii) Pick $v \in W^\perp$ and $a \in T$, it suffices to show that $av \in W^\perp$. For all $w \in W$,

$$\langle w, av \rangle = \langle a^*w, v \rangle = 0$$

as $a^* \in T$.

(iii) This is proved by the induction on the dimension of T -modules. If W is an irreducible T -module of V , then

$$V = W + W^\perp \quad (\text{orthogonal direct sum}).$$

□

Problem. What does the structure of the $T(x)$ -module tell us about Γ ?

Study those Γ whose modules take ‘simple’ form. The Γ ’s involved are highly regular.

Remark.

1. The subconstituent algebra T is semisimple as the left regular representation of T is completely reducible. See Curtis-Reiner 25.2 (Charles W. Curtis, 2006).
2. The inner product $\langle a, b \rangle_T = \text{tr}(a^\top \bar{b})$ is nondegenerate on T .

3. In general,

T : Semisimple and Artinian $\Leftrightarrow T$: Artinian with $J(T) = 0$

$\Leftrightarrow T$: Artinian with nonzero nilpotent element

$\Leftrightarrow T \subset \text{Mat}_X(K)$ such that for all $a \in T$ is normal.

Chapter 2

Perron-Frobenius Theorem

Friday, January 22, 1993

In this lecture we use the Perron Frobenius theory of nonnegative matrices to obtain information on eigenvalues of a graph.

Let $K = \mathbb{R}$. For $n \in \mathbb{Z}^{>0}$, pick a symmetric matrix $C \in \text{Mat}_n(\mathbb{R})$.

Definition 2.1. The matrix C is *reducible* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i \in X^+$, and for all $j \in X^-$, and for all $i \in X^-$, and for all $j \in X^+$, i.e.,

$$C \sim \begin{pmatrix} * & O \\ O & * \end{pmatrix}.$$

Definition 2.2. The matrix C is *bipartite* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i, j \in X^+$, and for all $i, j \in X^-$, i.e.,

$$C \sim \begin{pmatrix} O & * \\ * & O \end{pmatrix}.$$

Note.

1. If C is bipartite, for every eigenvalue θ of C , $-\theta$ is an eigenvalue of C such that $\text{mult}(\theta) = \text{mult}(-\theta)$.

Indeed, let $C = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$,

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix},$$

where $Ay = \theta x$ and $Bx = \theta y$.

2. If C is bipartite, C^2 is reducible.
3. The matrix C is irreducible and C^2 is reducible, if $C_{ij} \geq 0$ for all i, j and C is reducible. (Exercise)

Remark. Note 1. Even if C is not symmetric

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

holds. So the geometrix multiplicities coincide. How about the algebraic multiplicities?

Note 3. Set $x \sim y$ if and only if $C_{xy} > 0$. So the graph may have loops. Then

$$(C^2)_{xy} > 0 \Leftrightarrow \text{if there exists } z \in X \text{ such that } x \sim z \sim y.$$

Note that C is irreducible if and only if $\Gamma(C)$ is connected. Let

$$X^+ = \{y \mid \text{there is a path of even length from } x \text{ to } y\} \quad (2.1)$$

$$X^- = \{y \mid \text{there is no path of even length from } x \text{ to } y\} \neq \emptyset. \quad (2.2)$$

If there is an edge $y \sim z$ in X^+ and $w \in X^-$. Then there would be a path from x to y of even length. So $e(X^+, X^+) = e(X^-, X^-) = 0$.

Theorem 2.1 (Perron-Frobenius). *Given a matrix C in $\text{Mat}_n(\mathbb{R})$ such that*

- (a) C is symmetric.
- (b) C is irreducible.
- (c) $C_{ij} \geq 0$ for all i, j .

Let θ_0 be the maximal eigenvalue of C with eigenspace $V_0 \subseteq \mathbb{R}^n$, and let θ_r be the maximal eigenvalue of C with eigenspace $V_r \subseteq \mathbb{R}^n$. Then the following hold.

$$(i) \text{ Suppose } 0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0. \text{ Then } \alpha_0 > 0 \text{ for all } i, \text{ or } \alpha_i < 0 \text{ for all } i.$$

$$(ii) \dim V_0 = 1.$$

$$(iii) \theta_r \geq -\theta_0.$$

$$(iv) \theta_r = \theta_0 \text{ if and only if } C \text{ is bipartite.}$$

First, we prove the following lemma.

Lemma 2.1. *Let $\langle \cdot, \cdot \rangle$ be the dot product in $V = \mathbb{R}^n$. Pick a symmetric matrix $B \in \text{Mat}_n(\mathbb{R})$. Suppose all eigenvalues of B are nonnegative. (i.e., B is positive semidefinite.) Then there exist vectors $v_1, v_2, \dots, v_n \in V$ such that $B_{ij} = \langle v_i, v_j \rangle$ for $(1 \leq i, j \leq n)$.*

Proof. By elementary linear algebra, there exists an orthonormal basis w_1, w_2, \dots, w_n of V consisting of eigenvectors of B . Set the i -th column of P is w_i and $D = \text{diag}(\theta_1, \dots, \theta_n)$. Then $P^\top P = I$ and $BP = PD$.

Hence,

$$B = PDP^{-1} = PDP^\top = QQ^\top,$$

where

$$Q = P \cdot \text{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \dots, \sqrt{\theta_n}) \in \text{Mat}_n(\mathbb{R}).$$

Now, let v_i be the i -th column of Q^\top . Then

$$B_{ij} = v_i^\top \cdot v_j = \langle v_i, v_j \rangle.$$

□

Now we start the proof of Theorem 2.1.

Proof of Theorem 2.1(i)

Let \langle, \rangle denote the dot product on $V = \mathbb{R}^n$. Set

$$B = \theta I - C \tag{2.3}$$

$$= \text{symmetric matrix with eigenvalues } \theta_0 - \theta_i \geq 0 \tag{2.4}$$

$$= (\langle v_i, v_j \rangle)_{1 \leq i, j \leq n} \tag{2.5}$$

with the same $v_1, \dots, v_n \in V$ by Lemma 2.1.

Observe: $\sum_{i=1}^n \alpha_i v_i = 0$.

Pf.

$$\left\| \sum_{i=1}^n \alpha_i v_i \right\|^2 = \left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \right\rangle \tag{2.6}$$

$$= (\alpha_1 \quad \dots \quad \alpha_n) B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{2.7}$$

$$= v^\top B v \tag{2.8}$$

$$= 0, \tag{2.9}$$

since $Bv = (\theta_0 I - C)v = 0$.

Now set

$$s = \text{the number of indices } i, \text{ where } \alpha_i > 0.$$

Replacing v by $-v$ if necessary, without loss of generality we may assume that $s \geq 1$. We want to show $s = n$.

Assume $s < n$. Without loss of generality, we may assume that $\alpha_i > 0$ for $1 \leq i \leq s$ and $\alpha_i = 0$ for $s+1 \leq i \leq n$. Set

$$\rho = \alpha_1 v_1 + \dots + \alpha_s v_s = -\alpha_{s+1} v_{s+1} - \dots - \alpha_n v_n.$$

Then, for $i = 1, \dots, s$,

$$\langle v_i, \rho \rangle = \sum_{j=s+1}^n -\alpha_j \langle v_i, v_j \rangle \quad (\langle v_i, v_j \rangle = B_{ij}, B = \theta_0 I - C) \quad (2.10)$$

$$= \sum_{j=s+1}^n (-\alpha_j)(-C_{ij}) \quad (2.11)$$

$$\leq 0. \quad (2.12)$$

Hence

$$0 \leq \langle \rho, \rho \rangle = \sum_{i=1}^s \alpha_i \langle v_i, \rho \rangle \leq 0,$$

as $\alpha > 0$ and $\langle v_i, \rho \rangle \leq 0$. Thus, we have $\langle \rho, \rho \rangle = 0$ and $\rho = 0$. For $j = s+1, \dots, n$,

$$0 = \langle \rho, v_j \rangle = \sum_{i=1}^s \alpha_i \langle v_i, v_j \rangle \leq 0,$$

as $\langle v_i, v_j \rangle = -C_{ij}$.

Therefore,

$$0 = \langle v_i, v_j \rangle = -C_{ij} \text{ for } 1 \leq i \leq s, s+1 \leq j \leq n.$$

Since C is symmetric,

$$C = \begin{pmatrix} * & O \\ O & * \end{pmatrix}$$

Thus C is reducible, which is not the case. Hence $s = n$.

Proof of Theorem 2.1 (ii).

Suppose $\dim V_0 \geq 2$. Then,

$$\dim \left(V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^\perp \right) \geq 1.$$

So, there is a vector

$$0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with $\alpha_1 = 0$. This contradicts 1.

Now pick

$$0 \neq w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r.$$

Proof of Theorem 2.1 (iii).

Suppose $\theta_r < -\theta_0$. Since the eigenvalues of C^2 are the squares of those of C , θ_r^2 is the maximal eigenvalue of C^2 .

Also we have $C^2 w = \theta_r^2 w$.

Observe that C^2 is irreducible. (As otherwise, C is bipartite by Note 3, and we must have $\theta_r = -\theta_0$.) Therefore, $\beta_i > 0$ for all i or $\beta_i < 0$ for all i . We have

$$\langle v, w \rangle = \sum_{i=1}^n \alpha_i \beta_i \neq 0.$$

This is a contradiction, as $V_0 \perp V_r$.

Proof of Theorem 2.1 (iv)

\Rightarrow : Let $\theta_r = -\theta_0$. Then $\theta = \theta_1^2 = \theta_0^2$ is the maximal eigenvalue of C^2 , and v and w are linearly independent eigenvalues for θ for C^2 . Hence, for C^2 , $\text{mult}(\theta) \geq 2$.

Thus by 2, C^2 must be reducible. Therefore, C is bipartite by Note 3.

\Leftarrow : This is Note 1. \square

Let $\Gamma = (X, E)$ be any graph.

Definition 2.3. Γ is said to be *bipartite* if the adjacency matrix A is bipartite. That is, X can be written as a disjoint union of X^+ and X^- such that X^+, X^- contain no edges of Γ .

Corollary 2.1. *For any (connected) graph Γ with*

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_1 & m_1 & \cdots & m_r \end{pmatrix} \quad \text{with } \theta_0 > \theta_1 > \cdots > \theta_r.$$

Let V_i be the eigenspace of θ_i . Then the following holds.

1. Suppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \in \mathbb{R}^n$. Then $\alpha_i > 0$ for all i or $\alpha_i < 0$ for all i .
2. $m_0 = 1$.
3. $\theta_r \geq -\theta_0$ if and only if Γ is bipartite. In this case,

$$-\theta_i = \theta_{r-i} \text{ and } m_i = m_{r-i} \quad (0 \leq i \leq r)$$

Proof. This is a direct consequences of Theorem 2.1 and Note 3. \square

Chapter 3

Cayley Graphs

Monday, January 25, 1993

Given graphs $\Gamma = (X, E)$ and $\Gamma' = (X', E')$.

Definition 3.1. A map $\sigma : X \rightarrow X'$ is an *isomorphism* of graphs whenever;

- i. σ is one-to-one and onto,
- ii. $xy \in E$ if and only if $\sigma x \sigma y \in E'$ for all $x, y \in X$.

We do not distinguish between isomorphic graphs.

Definition 3.2. Suppose $\Gamma = \Gamma'$. Above isomorphism σ is called an *automorphism* of Γ . Then set $\text{Aut}(\Gamma)$ of all automorphisms of Γ becomes a finite group under composition.

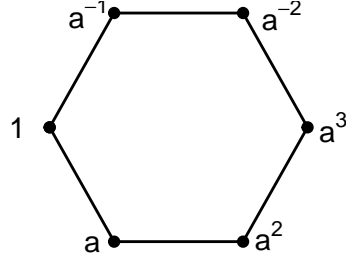
Definition 3.3. If $\text{Aut}(\Gamma)$ acts transitive on X , Γ is called *vertex transitive*.

Example 3.1. A Cayley graphs:

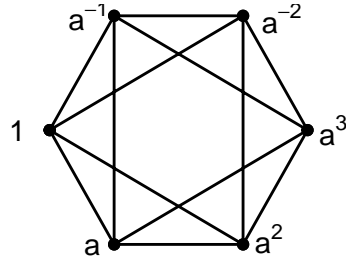
Definition 3.4 (Cayley Graphs). Let G be any finite group, and Δ any generating set for G such that $1_G \notin \Delta$ and $g \in \Delta \rightarrow g^{-1} \in \Delta$. Then Cayley graph $\Gamma = \Gamma(G, \Delta)$ is defined on the vertex set $X = G$ with the edge set E defined by the following.

$$E = \{(h_1, h_2) \mid h_1, h_2 \in G, h_1^{-1}h_2 \in \Delta\} = \{(h, hg) \mid h \in G, g \in \Delta\}$$

Example 3.2. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}\}$.



Example 3.3. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}, a^2, a^{-2}\}$.



Example 3.4. $G = \langle a, b \mid a^6 = 1, b^2 = 1, ab = ba \rangle$, $\Delta = \{a, a^{-1}, b\}$.



Remark. $\text{Aut}(\Gamma) \simeq D_6 \times \mathbb{Z}_2$ contains two regular subgroups isomorphic to D_6 and $\mathbb{Z}_5 \times \mathbb{Z}_2$ and Γ is obtained as Cayley graphs in two ways.

Cayley graphs are vertex transitive, indeed.

Theorem 3.1. *The following hold.*

(i) *For any Cayley graph $\Gamma = \Gamma(G, \Delta)$, the map*

$$G \rightarrow \text{Aut}(\Gamma) \quad (g \mapsto \hat{g})$$

is an injective homomorphism of groups, where

$$\hat{g}(x) = gx \quad \text{for all } g \in G \text{ and for all } x \in X (= G).$$

Also, the image \hat{G} is regular on X . i.e., the image \hat{G} acts transitively on X with trivial vertex stabilizers.

(ii) For any graph $\Gamma = (X, E)$, suppose there exists a subgroup $G \subseteq \text{Aut}(\Gamma)$ that is regular on X . Pick $x \in X$, and let

$$\Delta = \{g \in G \mid \langle x, g(x) \rangle \in E\}.$$

Then $1 \notin \Delta$, $g \in \Delta \rightarrow g^{-1} \in \Delta$, and Δ generates G . Moreover, $\Gamma \simeq \Gamma(G, \Delta)$.

Proof. (i) Let $g \in G$. We want to show that $\hat{g} \in \text{Aut}(\Gamma)$. Let $h_1, h_2 \in X = G$. Then,

$$(h_1, h_2) \in E \rightarrow h_1^{-1}h_2 \in \Delta \quad (3.1)$$

$$\rightarrow (gh_1)^{-1}(gh_2) \in \Delta \quad (3.2)$$

$$\rightarrow (gh_1, gh_2) \in E \quad (3.3)$$

$$\rightarrow (\hat{g}(h_1), \hat{g}(h_2)) \in E. \quad (3.4)$$

Hence, $\hat{g} \in \text{Aut}(\Gamma)$.

Observe: $g \mapsto \hat{g}$ is a homomorphism of groups:

$$\hat{1}_G = 1, \widehat{g_1 g_2} = \widehat{g_1} \widehat{g_2}.$$

Observe: $g \mapsto \hat{g}$ is one-to-one:

$$\widehat{g_1} = \widehat{g_2} \rightarrow g_1 = \widehat{g_1}(1_G) = \widehat{g_2}(1_G) = g_2.$$

Observe: \hat{G} is regular on X : Clear by construction.

(ii) $1_G \notin \Delta$: Since Γ has not loops, $(x, 1_G x) \notin E$.

$g \in \Delta \rightarrow g^{-1} \in \Delta$:

$$g \in \Delta \rightarrow (x, g(x)) \in E \rightarrow E \ni (g^{-1}(x), g^{-1}(g(x))) = (g^{-1}(x), x).$$

Δ generates G : Suppose $\langle \Delta \subsetneq G$. Let $\hat{X} = \{g(x) \mid g \in \langle \Delta \rangle\} \subsetneq X$. ($\hat{X} \subsetneq X$ as G acts regularly on X .)

Since Γ is connected, there exists $y \in \hat{X}$ and $z \in X \setminus \hat{X}$ with $yz \in E$.

Let $y = g(x)$, $g \in \langle \Delta \rangle$, $z \in h(x)$, $h \in G \setminus \langle \Delta \rangle$. Then

$$(y, z) = (g(x), h(x)) \in E \rightarrow (x, g^{-1}h(x)) \in E \rightarrow g^{-1}h \in \langle \Delta \rangle \rightarrow h \in \langle \Delta \rangle.$$

This is a contradiction. Therefore, Δ generates G .

Let $\Gamma' = (X', E')$ denote $\Gamma(G, \Delta)$. We shall show that

$$\theta : X' \rightarrow X \ (g \mapsto g(x))$$

is an isomorphism of graphs.

θ is one-to-one: For $h_1, h_2 \in X' = G$,

$$\theta(h_1) = \theta(h_2) \rightarrow h_1(x) = h_2(x) \rightarrow h_1^{-1}h_2(x) = x \rightarrow h_1^{-1}h_2 \in \text{Stab}_G(x) = \{1_G\} \rightarrow h_1 = h_2.$$

($\text{Stab}_G = \{g \in G \mid g(x) = x\}$.)

θ is onto: Since G is transitive,

$$X = \{g(x) \mid g \in G\} = \theta(X') = \theta(G).$$

θ respects adjacency: For $h_1, h_2 \in X' = G$,

$$(h_1, h_2) \in E' \leftrightarrow h_1^{-1}h_2 \in \Delta \leftrightarrow (x, h_1^{-1}h_2(x)) \in E \leftrightarrow (h_1(x), h_2(x)) \in E \leftrightarrow (\theta(h_1), \theta(h_2)) \in E.$$

Therefore θ is an isomorphism between graphs $\Gamma(G, \Delta)$ and $\Gamma(X, E)$. \square

How to compute the eigenvalues of the Cayley graph of an abelian group.

Let G be any finite abelian group. Let \mathbb{C}^* be the multiplicative group on $\mathbb{C} \setminus \{0\}$.

Definition 3.5. A (linear) G -character is any group homomorphism $\theta : G \rightarrow \mathbb{C}^*$.

Example 3.5. $G = \langle a \mid a^3 = 1 \rangle$ has three characters, $\theta_0, \theta_1, \theta_2$.

$$\begin{array}{c|ccc} \theta_i(a^j) & 1 & a & a^2 \\ \hline \theta_0 & 1 & 1 & 1 \\ \theta_1 & 1 & \omega & \omega^2 \\ \theta_2 & 1 & \omega^2 & \omega \end{array}, \quad \text{with } \omega = \frac{-1 + \sqrt{-3}}{2}.$$

Here ω is a primitive cube root of ω in \mathbb{C}^* , i.e., $1 + \omega + \omega^2 = 0$.

For arbitrary group G , let $X(G)$ be the set of all characters of G .

Observe: For $\theta_1, \theta_2 \in X(G)$, one can define product $\theta_1 \theta_2$:

$$\theta_1 \theta_2(g) = \theta_1(g) \theta_2(g) \quad \text{for all } g \in G.$$

Then $\theta_1 \theta_2 \in X(G)$.

Observe: $X(G)$ with this product is an (abelian) group.

Lemma 3.1. *The groups G and $X(G)$ are isomorphic for all finite abelian groups G .*

Proof. G is a direct sum of cyclic groups;

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m, \quad \text{where } G_i = \langle a_i \mid a_i^{d_i} = 1 \rangle \quad (1 \leq i \leq m).$$

Pick any element ω_i of order d_i in \mathbb{C}^* , i.e., a primitive d_i -th root of 1. Define

$$\theta_i : G \rightarrow \mathbb{C}^* \quad (a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \mapsto \omega_i^{\varepsilon_i} \quad \text{where } 0 \leq \varepsilon_i < d_i, 1 \leq i \leq m).$$

Then $\theta_i \in X(G)$. (Exercise)

Claim: There exists an isomorphism of groups $G \rightarrow X(G)$ that sends a_i to θ_i .

Observe: $\theta_i^{d_i} = 1$. For every $g = a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \in G$,

$$\theta_i^{d_i}(g) = (\theta_i(g))^{d_i} = (\omega_i^{\varepsilon_i})^{d_i} = (\omega_i^{d_i})^{\varepsilon_i} = 1.$$

Observe: If $\theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m} = 1$ for some $0 \leq \varepsilon_i < d_i, 1 \leq i \leq m$. Then $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = 0$.

Pf. $1 = \theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m}(a_i) = \omega_i^{\varepsilon_i}$, Since ω_i is a primitive d_i -th root of 1, $\varepsilon_i = 0$ for $1 \leq i \leq m$.

Observe: $\theta_1, \dots, \theta_m$ generate $X(G)$. Pick $\theta \in X(G)$. Since $a_i^{d_i} = 1$, $1 = \theta(a_i^{d_i}) = \theta(a_i)^{d_i}$.

Hence $\theta(a_i) = \omega_i^{\varepsilon_i}$ for some ε_i with $0 \leq \varepsilon_i < d_i$.

Now $\theta = \theta_1^{\varepsilon_1} \cdots \theta_m^{\varepsilon_m}$, since these are both equal to $\omega_i^{\varepsilon_i}$ at a_i for $1 \leq i \leq m$.

Therefore,

$$G \rightarrow X(G) \quad (a_i \mapsto \theta_i)$$

is an isomorphism of groups. □

Note. The correspondence above is clearly a group homomorphism.

Chapter 4

Examples

Wednesday, January 27, 1993

Theorem 4.1. *Given a Cayley graph $\Gamma = \Gamma(G, \Delta)$. View the standard module $V \equiv \mathbb{C}G$ (the group algebra), so*

$$\left\langle \sum_{g \in G} \alpha_g g, \sum_{g \in G} \beta_g g \right\rangle = \sum_{g \in G} \alpha_g \overline{\beta_g}, \quad \text{with } \alpha_g, \beta_g \in \mathbb{C}.$$

For any $\theta \in X(G)$, write

$$\hat{\theta} = \sum_{g \in G} \theta(g^{-1})g.$$

Then the following hold.

(i) $\langle \hat{\theta}_1, \hat{\theta}_2 \rangle = |G|$ if $\theta_1 = \theta_2$ and 0 otherwise for $\theta_1, \theta_2 \in X(G)$. In particular, $\{\hat{\theta} \mid \theta \in X(G)\}$ forms a basis for V .

(ii) $A\hat{\theta} = \Delta_\theta \hat{\theta}$ for $\theta \in X(G)$, where A is the adjacency matrix and

$$\Delta_\theta = \sum_{g \in \Delta} \theta(g).$$

In particular, the eigenvalues of Γ are precisely

$$\Delta_\theta \mid \theta \in X(G)\}.$$

Proof.

(i) Claim: For every $\theta \in X(G)$, let

$$s := \sum_{g \in G} \theta(g^{-1}) = \begin{cases} |G| & \text{if } \theta = 1 \\ 0 & \text{if } \theta \neq 1. \end{cases}$$

Pf. Clear if $\theta = 1$.

Let $\theta \neq 1$. Then $\theta(h) \neq 1$ for some $h \in G$.

$$s \cdot \theta(h) = \left(\sum_{g \in G} \theta(g^{-1}) \right) \theta(h) = \sum_{g \in G} \theta(g^{-1}h) = \sum_{g' \in G} \theta(g'^{-1}) = s.$$

Since $\theta(h) \neq 1$, $s = 0$.

Claim. $\theta(g^{-1}) = \overline{\theta(g)}$ for every $\theta \in X(G)$ and every $g \in G$.

Since $\theta(g) \in \mathbb{C}$ is a root of 1,

$$|\theta(g)|^2 = \theta(g)\overline{\theta(g)} = 1.$$

On the other hand, since θ is a homomorphism,

$$\theta(g)\theta(g^{-1}) = \theta(1) = 1.$$

Hence $\theta(g^{-1}) = \overline{\theta(g)}$.

Now

$$\langle \widehat{\theta_1}, \widehat{\theta_2} \rangle = \sum_{g \in G} \theta_1(g^{-1}) \overline{\theta_2(g^{-1})} \quad (4.1)$$

$$= \sum_{g \in G} \theta_1(g^{-1}) \theta_2(g) \quad (4.2)$$

$$= \sum_{g \in G} \theta_1 \theta_2^{-1}(g^{-1}) \quad (4.3)$$

$$= \begin{cases} |G| & \text{if } \theta_1 \theta_2^{-1} = 1 \\ 0 & \text{if } \theta_1 \theta_2^{-1} \neq 1. \end{cases} \quad (4.4)$$

Since $|G| = |X(G)|$ by Lemma 3.1, and $\widehat{\theta_i}$'s are orthogonal nonzero elements in V , they form a basis of V .

(ii) Let $\Delta = \{g_1, \dots, g_r\}$. Then

$$A\hat{\theta} = A \left(\sum_{g \in G} \theta(g^{-1}g) \right) \quad (4.5)$$

$$= \sum_{g \in G} \theta(g^{-1})(gg_1 + \cdots + gg_r) \quad (\Gamma(g) = \{gg_1, \dots, gg_r\}) \quad (4.6)$$

$$= \sum_{i=1}^r \left(\sum_{g \in G} \theta(g^{-1})(gg_i) \right) \quad (4.7)$$

$$= \sum_{i=1}^r \left(\sum_{g \in G} \theta(g_i g_i^{-1} g^{-1})(gg_i) \right) \quad (4.8)$$

$$= \sum_{i=1}^r \left(\sum_{g \in G} \theta(g_i) \theta((gg_i)^{-1}) gg_i \right) \quad (4.9)$$

$$= \sum_{i=1}^r \theta(g_i) \sum_{h \in G} \theta(h^{-1}) h \quad (4.10)$$

$$= \Delta_{\theta} \cdot \hat{\theta}. \quad (4.11)$$

Since $\{\hat{\theta} \mid \theta \in X(G)\}$ forms a basis, the eigenvalues of Γ are precisely,

$$\{\Delta_{\theta} \mid \theta \in X(G)\}.$$

This completes the proof.

□

Example 4.1. Let $G = \langle a \mid a^6 = 1 \rangle$, and $\Delta = \{a, a^{-1}\}$. Pick a primitive 6-th root of 1, ω . Then

$$X(G) = \{\theta^i \mid 0 \leq i \leq 5\} \quad \text{such that} \quad \theta(a) = \omega, \quad \omega + \omega^{-1} = 1.$$



$\varphi \in X(G)$	$\varphi(a)$	$\Delta_\varphi = \theta(a) + \theta(a)^{-1}$
1	1	2
θ	ω	$\omega + \omega^{-1} = 1$
θ^2	ω^2	-1
θ^3	$\omega^3 = -1$	-2
θ^4	ω^4	-1
θ^5	ω^5	1

$$\text{Spec}(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

Example 4.2. D -cube, $H(D, 2)$. Let

$$X = \{(a_1, \dots, a_D) \mid a_i \in \{1, -1\}, 1 \leq i \leq D\},$$

$$E = \{xy \mid x, y \in X, x, y: \text{different in exactly one coordinate}\}.$$

Also $H(D, 2)$ is a Cayley graph $\Gamma(G, \Delta)$, where

$$G = G_1 \oplus G_2 \oplus \dots \oplus G_D,$$

$$G_i = \langle a_i \mid a_i^2 = 1 \rangle, \quad \Delta = \{a_1, \dots, a_D\}.$$

Homework: The spectrum of $H(D, 2)$ is

$$\begin{pmatrix} \theta_0 & \theta_1 & \dots & \theta_D \\ m_0 & m_1 & \dots & m_D \end{pmatrix},$$

where

$$\theta_i = D - 2i \quad (0 \leq i \leq D), \quad m_i = \binom{D}{i}.$$

Remark. Let $\theta \in X(G)$. Then $\theta : X \rightarrow \{\pm 1\}$. If

$$\nu(\theta) = |\{i \mid \theta(a_i) = -1\}|,$$

then $\Delta_\theta = D - 2i$. Since there are $\binom{D}{i}$ such θ , we have the assertion.

We want to compute the subconstituent algebra for $H(D, 2)$. First, we make a few observations about arbitrary graphs.

Let $\Gamma = (X, E)$ be any graph, A , the adjacency matrix of Γ , and V , the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$. Write $E_i^* = E_i^*(x)$, and

$$T \equiv T(x) = \text{the algebra generated by } A, E_0^*, E_1^*, \dots$$

Definition 4.1. Let W be any irreducible T -module ($\subseteq V$). Then the endpoint $r \equiv r(W)$ satisfied

$$r = \min\{i \mid E_i^* W \neq 0\}.$$

The diameter $d = d(W)$ satisfied

$$d = |\{i \mid E_i^* W \neq 0\}| - 1.$$

Lemma 4.1. *With the above notation, let W be an irreducible T -module. Then*

- (i) $E_i^* A E_j^* = 0$ if $|i - j| = 1$, $E_i^* A E_j^* \neq 0$ if $|i - j| = 1$, $0 \leq i, j \leq d(x)$.
- (ii) $A E_j^* W \subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$, $0 \leq j \leq d(x)$. ($E_i^* W = 0$ if $i < j$ or $i > d(x)$.)
- (iii) $E_j^* W \neq 0$ if $r \leq j \leq r + d$, $= 0$ if $0 \leq j \leq r$ or $r + d < j \leq d(x)$.
- (iv) $E_i^* A E_j^* W \neq 0$, if $|i - j| = 1$ ($r \leq i, j \leq r + d$).

Proof.

- (i) Pick $y \in X$ with $\partial(x, y) = j$. We want to find $E_i^* A E_j^* \hat{y}$. Note,

$$E_j^* \hat{y} = \begin{cases} 0 & \text{if } \partial(x, y) \neq j \\ \hat{y} & \text{if } \partial(x, y) = j. \end{cases}$$

$$E_i^* A E_j^* \hat{y} = E_i^* A \hat{y} \tag{4.12}$$

$$= E_i^* \sum_{z \in X, yz \in E} \hat{z} \tag{4.13}$$

$$= \sum_{z \in X, yz \in E, \partial(x, z) = i} \hat{z} \tag{4.14}$$

$$= 0 \text{ if } |i - j| > 1 \quad \text{by triangle inequality.} \tag{4.15}$$

If $|i - j| = 1$, there exist $y, y' \in X$ such that $\partial(x, y) = j$, $\partial(x, y') = i$, $yy' \in E$ by connectivity of Γ . Hence (4.14) contains $\widehat{yy'}$ and (4.14) is not equal to zero.

- (ii) We have

$$A E_j^* W = \left(\sum_{i=0}^{d(x)} E_i^* \right) A E_j^* W \tag{4.16}$$

$$= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W \tag{4.17}$$

$$\subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W. \tag{4.18}$$

- (iii) Suppose $E_j^* W = 0$ for some j ($r \leq j \leq r + d$). Then $r < j$ by the definition of r . Set

$$\widetilde{W} = E_r^*W + E_{r+1}^*W + \cdots + E_{j-1}^*W.$$

Observe $0 \subsetneq \widetilde{W} \subsetneq W$. Also $A\widetilde{W} \subseteq \widetilde{W}$ by (ii), and $E_i^*\widetilde{W} \subseteq \widetilde{W}$ for every i by construction.

Thus, $T\widetilde{W} \subseteq \widetilde{W}$, contradicting W being irreducible.

□

Chapter 5

T -Modules of $H(D, 2)$, I

Friday, January 29, 1993

Let $\Gamma = (X, E)$ be a graph, A the adjacency matrix, and V the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$ and write $E_i^* \equiv E_i^*(x)$, and $T \equiv T(x)$.

Let W be an irreducible T -module with endpoint $r := \min\{i \mid E_i^*W \neq 0\}$ and diameter $d := |\{i \mid E_i^*W \neq 0\}| - 1$.

We have

$$E_i^*W \neq 0 \quad r \leq i \leq r + d \quad (5.1)$$

$$= 0 \quad 0 \leq i < r \text{ or } r + d < i \leq d(x). \quad (5.2)$$

Claim: $E_i^*AE_j^*W \neq 0$ if $|i - j| = 1$ for $r \leq i, j \leq r + d$. (See Lemma 4.1.)

Suppose $E_{j+1}^*AE_j^*W = 0$ for some j with $r \leq j < r + d$. Observe that

$$\tilde{W} = E_r^*W + \cdots + E_j^*W$$

is T -invariant with

$$0 \subsetneq \tilde{W} \subsetneq W.$$

Becase $A\tilde{W} \subseteq \tilde{W}$ since $AE_j^*W \subseteq E_{j-1}^*W + E_j^*W$,

$$E_k^*\tilde{W} \subseteq \tilde{W} \quad \text{for all } k,$$

we have $T\tilde{W} \subseteq \tilde{W}$.

Suppose $E_{i-1}^*AE_i^*W = 0$ for some i with $r \leq i < r + d$.

Similarly,

$$\tilde{W} = E_i^*W + \cdots + E_{r+d}^*W$$

is a T -module with $0 \subsetneq \tilde{W} \subsetneq W$.

Definition 5.1. Let Γ , E_i^* , and T be as above. Irreducible T -modules W and W' are isomorphic whenever there is an isomorphism $\sigma : W \rightarrow W'$ of vector spaces such that $a\sigma = \sigma a$ for all $a \in T$.

Recall that the standard module V is an orthogonal direct sum of irreducible T -modules $W_1 \oplus W_2 \oplus \dots$. Given W in this list, the multiplicity of W in V is

$$|\{j \mid W_j \simeq W\}|.$$

Remark. It is known that the multiplicity does not depend on the decomposition.

Now assume that Γ is the D -cube, $H(D, 2)$ with $D \geq 1$. View

$$X = \{a_1 \cdots a_D \mid a_i \in \{1, -1\}, 1 \leq i \leq D\}, \quad (5.3)$$

$$E = \{xy \mid x, y \in X, x, y \text{ differ in exactly 1 coordinate}\}. \quad (5.4)$$

Find T -modules.

Claim: $H(D, 2)$ is bipartite with a partition $X = X^+ \cup X^-$, where

$$X^+ = \{a_1 \cdots a_D \in X \mid \prod a_i > 0\} \quad (5.5)$$

$$X^- = \{a_1 \cdots a_D \in X \mid \prod a_i < 0\} \quad (5.6)$$

Observe: for all $y, z \in X$,

$$\partial(y, z) = i \Leftrightarrow y, z \text{ differ in exactly } i \text{ coordinates with } 0 \leq i \leq D.$$

Here, the diameter of $H(D, 2) = D = d$ for all $x \in X$.

Theorem 5.1. Let $\Gamma = H(D, 2)$ be as above. Fix $x \in X$, and write $E_i^* = E_i^*(x)$, and $T = T(x)$.

Let W be an irreducible T -module with endpoint r , and diameter d with $0 \leq r \leq r + d \leq D$.

(i) W has a basis w_0, w_1, \dots, w_d with $w_i \in E_{i+r}^* W$ for $0 \leq i \leq d$. With respect to which the matrix representing A is

$$\begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & d-1 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \ddots & 0 & 2 & 0 \\ 0 & 0 & 0 & \cdots & d-1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & d & 0 \end{pmatrix}$$

(ii) $d = D - 2r$. In particular, $0 \leq r \leq D/2$.

(iii) Let W' denote an irreducible T -module with endpoint r' . Then W and W' are isomorphic as T -modules if and only if $r = r'$.

(iv) The multiplicity of the irreducible T -module with endpoint r is

$$\binom{D}{r} - \binom{D}{r-1} \quad \text{if } 1 \leq r \leq D/2,$$

and 1 if $r = 0$.

Proof. Recall that Γ is vertex transitive. It is a Cayley graph.

Hence without loss of generality, we may assume that $x = \overbrace{11 \cdots 1}^D$.

Notation: Set $\Omega = \{1, 2, \dots, D\}$. For every subset $S \subseteq \Omega$, let

$$\hat{S} = a_1 \cdots a_d \in X \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S. \end{cases}$$

In particular, $\hat{\emptyset} = x$ and

$$|S| = i \Leftrightarrow \partial(x, \hat{S}) = i \Leftrightarrow \hat{S} \in E_i^* V.$$

For all $S, T \subseteq \Omega$, we say S covers T if and only if $S \supseteq T$ and $|S| = |T| + 1$.

Observe that \hat{S}, \hat{T} are adjacent in Γ if and only if either T covers S or S covers T .

Define the ‘raising matrix’

$$R = \sum_{i=0}^D E_{i+1}^* A E_i^*.$$

Observe that

$$R E_i^* V \subseteq E_{i+1}^* V \quad \text{for } 0 \leq i \leq D, \quad \text{and } E_{D+1}^* V = 0.$$

Indeed for any $S \subseteq \Omega$ with $|S| = i$,

$$R \hat{S} = R E_i^* \hat{S} \tag{5.7}$$

$$= E_{i+1}^* A \hat{S} \tag{5.8}$$

$$= \sum_{T_1 \subseteq \Omega, S \text{ covers } T_1} E_{i+1}^* \hat{T}_1 + \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T} \tag{5.9}$$

$$= \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T}. \tag{5.10}$$

Define the ‘lowering matrix’

$$L = \sum_{i=0}^D E_{i-1}^* A E_i^*.$$

Observe that

$$L E_i^* V \subseteq E_{i-1}^* V \text{ for } 0 \leq i \leq D, \text{ and } E_{-1}^* V = 0.$$

Indeed for any $S \subseteq \Omega$,

$$L \hat{S} = \sum_{T \subseteq \Omega, S \text{ covers } T} \hat{T}.$$

Observe that $A = L + R$.

For convenience, set

$$A^* = \sum_{i=0}^D (D - 2i) E_i^*.$$

Claim: The following hold.

- (a) $LR - RL = A^*$.
- (b) $A^*L - LA^* = 2L$.
- (c) $A^*R - RA^* = -2R$.

In particular $\text{Span}(R, L, A^*)$ is a ‘representation of Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Remark (Lie Algebra $\mathfrak{sl}_2(\mathbb{C})$).

$$\mathfrak{sl}_2(\mathbb{C}) = \{X \mid \text{Mat}(\mathbb{C}) \mid \text{tr}(X) = 0\}.$$

For $X, Y \in \mathfrak{sl}_2(\mathbb{C})$, define a binary operation $[X, Y] = XY - YX$.

$$A^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then these satisfy the relations (a) - (c) above.

Proof of Claim. Apply both sides to \hat{S} ($S \subseteq \Omega$). Say $|S| = i$.

Proof of (a):

$$(LR - RL)\hat{S} = L \left(\sum_{\substack{T \subseteq \Omega, T \text{ covers } S \\ (D-i \text{ of them})}} \hat{T} \right) - R \left(\sum_{\substack{U \subseteq \Omega, S \text{ covers } U \\ (i \text{ of them})}} \hat{T} \right) \quad (5.11)$$

$$= (D - i)\hat{S} + \sum_{V \subseteq \Omega, |V|=i, |S \cap V|=i-1} \hat{V} - \left(i\hat{S} + \sum_{V \subseteq \Omega, |V|=i, |S \cap V|=i-1} \hat{V} \right) \quad (5.12)$$

$$= (D - 2i)\hat{S} \quad (5.13)$$

$$= A^*\hat{S}. \quad (5.14)$$

Proof of (b):

$$(A^*L - LA^*)\hat{S} = (D - 2(i - 1))L\hat{S} - (D - 2i)L\hat{S} \quad (\text{since } L\hat{S} \in E_{i-1}^*V) \quad (5.15)$$

$$= 2L\hat{S}. \quad (5.16)$$

Proof of (c):

$$(A^*R - RA^*)\hat{S} = (D - 2(i + 1))R\hat{S} - (D - 2i)R\hat{S} \quad (\text{since } R\hat{S} \in E_{i+1}^*V) \quad (5.17)$$

$$= 2R\hat{S}. \quad (5.18)$$

Let W be an irreducible T -module with endpoint r and diameter d ($0 \leq r \leq r + d \leq D$).

Proof of (i) and (ii):

Pick $0 \neq w \in E_r^*W$.

Claim: $LRw = (D - 2r)w$.

Pf.

$$LRw = (A^* + RL)w \quad (\text{by Claim (a)}) \quad (5.19)$$

$$= A^*w \quad (Lw \in E_{r-1}^*W = 0) \quad (5.20)$$

$$(D - 2r)w. \quad (5.21)$$

Define

$$w_i = \frac{1}{i!}R^i w \in E_{r+i}^*W \quad (0 \leq i \leq d).$$

Then,

$$Rw_i = (i + 1)w_{i+1} \quad (0 \leq i \leq d) \quad (5.22)$$

$$Rw_d = 0 \quad (\text{by definition of } d) \quad (5.23)$$

Claim: $Lw_0 = 0$ and

$$Lw_i = (D - 2r - i + 1)w_{i-1} \quad (1 \leq i \leq d).$$

Pf. We prove by induction on i . The case $i = 0$ is trivial, and the case $i = 1$ follows from above claim. Let $i \geq 2$,

$$Lw_i = \frac{1}{i}LRw_{i-1} = \frac{1}{i}(A^* + RL)w_{i-1} \quad (\text{by Claim (a)}) \quad (5.24)$$

$$(\text{by induction hypothesis}) \quad (5.25)$$

$$= \frac{1}{i}((D - 2(r + i - 1))w_{i-1} + (D - 2r - (i - 1) + 1)Rw_{i-2}) \quad (Rw_{i-2} = (i - 1)w_{i-1}) \quad (5.26)$$

$$= \frac{1}{i}i(D - 2r - i + 1)w_{i-1} \quad (5.27)$$

$$= (D - 2r - i + 1)w_{i-1}. \quad (5.28)$$

Claim: w_0, \dots, w_d is a basis for W .

Pf. Let $W' = \text{Span}\{w_0, \dots, w_d\}$. Then W' is R and L invariant. So it is $A = R + L$ invariant.

Also it is E_i^* -invariant for every i .

Hence W' is a T -module.

Since W is irreducible, $W' = W$.

As w_i 's are orthogonal, they are linearly independent. Note that $w_i \neq 0$ by the definition of d and Lemma 4.1 (iv).

Claim: $d = D - 2r$.

Pf. By (a),

$$0 = (LR - RL - A^*)w_d \quad (5.29)$$

$$= 0 - (D - 2r - d + 1)Rw_{d-1} - (D - 2(r + d))w_d \quad (5.30)$$

$$= -d(D - 2r - d + 1)w_d - (D - 2(r + d))w_d \quad (5.31)$$

$$= (-dD + 2rd + d^2 - d - D + 2r + 2d)w_d \quad (5.32)$$

$$= (d^2 + (2r - D + 1)d + 2r - D)w_d \quad (5.33)$$

$$= (d + 2r - D)(d + 1)w_d. \quad (5.34)$$

Hence $d = D - 2r$.

Therefore, with respect to a basis w_0, w_1, \dots, w_d , $A = L + R$, $w_{-1} = w_{d+1} = 0$,

$$Lw_i = (d - i + 1)w_{i-1}, \quad Rw_i = (i + 1)w_{i+1}.$$

$$L = \begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0 \\ 0 & 0 & d-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & d & 0 \end{pmatrix}.$$

This completes the proof of (i) and (ii). \square

Chapter 6

T -Modules of $H(D, 2)$, II

Monday, February 1, 1993

Proof of Theorem 5.1 Continued.

(iii) Let $r = r'$,

w_0, \dots, w_d : a basis for W with $w_i \in E_i^* W$, and

w'_0, \dots, w'_d : a basis for W' with $w'_i \in E_i^* W'$.

Then $d = D - 2r = D - 2r' = d'$, and

$$\sigma : W \rightarrow W' \quad (w_i \mapsto w'_i)$$

is an isomorphism of T -modules by (i).

If $r \neq r'$, then

$$d = D - 2r \neq D - 2r' = d',$$

hence, $\dim W \neq \dim W'$.

(iv) Let W_i be an irreducible T -module with endpoint i . Then

$$\dim E_r^* V = \binom{D}{r} = \sum_{i=0}^r \text{mult}(W_i).$$

Hence, we have that

$$\text{mult}(W_r) = \binom{D}{r} - \binom{D}{r-1}$$

by induction on r .

□

Theorem 6.1. *Let $\Gamma = H(D, 2)$ with $D \geq 1$. Fix a vertex $x \in X$ and write*

$$E_i^* \equiv E_i^*(x), \quad T = T(x), \quad \text{and} \quad A^* \equiv \sum_{i=0}^D (D - 2i) E_i^*.$$

Let W be an irreducible T -module with endpoint r with $0 \leq r \leq D/2$. Then,

(i) W has a basis

$$w_0^*, w_1^*, \dots, w_d^* \quad (d = D - 2r), \quad \text{such that} \quad w_i^* \in E_{i+r} W \quad (0 \leq i \leq d)$$

with respect to which the matrix corresponding to A^ is*

$$\begin{pmatrix} 0 & d & 0 & & & & & \\ 1 & 0 & d-1 & & & & & \\ 0 & 2 & 0 & & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & & 0 & 2 & 0 & \\ & & & & d-1 & 0 & 1 & \\ & & & & 0 & d & 0 & \end{pmatrix}.$$

In particular, / (ii) $E_i A^ E_j = 0$ if $|i - j| \neq 1$ for $0 \leq i, j \leq D$.*

Proof. We use the notation,

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \quad (= -[\beta, \alpha]).$$

Recall that

$$(a) \quad [L, R] = A^*,$$

$$(b) \quad [A^*, L] = wL,$$

$$(c) \quad [A^*, R] = -2R,$$

and $A = L + R$.

Write (a) – (c) in terms of A and A^* , we have,

$$[A, A^*] = [L, A^*] + [R, A^*] = 2(R - L).$$

$$\begin{cases} R + L &= A \\ R - L &= [A, A^*]/2. \end{cases}$$

Hence,

$$R = \frac{1}{4}(2A + [A, A^*]) \quad \text{and} \tag{6.1}$$

$$L = \frac{1}{4}(2A - [A, A^*]). \tag{6.2}$$

Now (a), (b) become

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0 \quad (6.3)$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0 \quad (6.4)$$

Pf. By (b),

$$2A - AA^* + A^*A = 4L \quad (6.5)$$

$$= 2[A^*, L] \quad (6.6)$$

$$= A^* \frac{2A - [A, A^*]}{2} - \frac{2A - [A, A^*]}{2} A^* \quad (6.7)$$

So we have ((6.4))

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0.$$

By (a),

$$-16A^* = [2A + [A, A^*], 2A - [A, A^*]] \quad (6.8)$$

$$(2A + [A, A^*])(2A - [A, A^*]) - (2A - [A, A^*])(2A + [A, A^*]) \quad (6.9)$$

$$= [4A^2 - 2A[A, A^*] + [A, A^*](2A) - [A, A^*]^2 \quad (6.10)$$

$$- 4A^2 - 2A[A, A^*] + [A, A^*](2A) + [A, A^*]^2 \quad (6.11)$$

$$= -4A^2A^* + 4AA^*A + 4AA^*A - 4A^*A^2. \quad (6.12)$$

So,

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0.$$

Claim: $E_i^*A^*E_j = 0$ if $|i - j| \neq 1$ for $0 \leq i, j \leq D$.

Pf. We have,

$$0 = E_i(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_j \quad (6.13)$$

$$= E_iA^*E_j(\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4) \quad (6.14)$$

$$(AE_j = \theta_jE_j, E_iA = (AE_j)^\top = (\theta_iE_i)^\top = \theta_iE_i) \quad (6.15)$$

$$= E_iA^*E_j(\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) \quad (6.16)$$

$$= E_iA^*E_j(D - 2i - (D - 2j) - 2)(D - 2i - (D - 2j) + 2) \quad (6.17)$$

$$(\theta_k = D - 2k) \quad (6.18)$$

$$= E_iA^*E_j \cdot 4(i - j + 1)(i - j - 1) \quad (6.19)$$

and $i - j + 1 \neq 0$, $i - j - 1 \neq 0$. Hence, $E_i^*A^*E_j = 0$.

Now define “dual raising matrix”,

$$R^* = \sum_{i=0}^D E_{i+1}A^*E_i.$$

So,

$$R^*E_iV \subseteq E_{i+1}V, \quad (0 \leq i \leq D, E_{D+1}V = 0).$$

Define “dual lowering matrix”

$$L^* = \sum_{i=0}^D E_{i-1}A^*E_i.$$

Then

$$L^*E_iV \subseteq E_{i-1}V \quad (0 \leq i \leq D, E_{-1}V = 0).$$

Observe that

$$A^* = \left(\sum_{i=0}^D E_i \right) A^* \left(\sum_{j=0}^D E_j \right) = L^* + R^*$$

by Claim 1.

Claim 2. We have

$$(a) [L^*, R^*] = A,$$

$$(b) [A, L^*] = 2L^*,$$

$$(c) [A, R^*] = -2R^*.$$

Pf. (b)

$$AL^* - L^*A = \sum_{i=0}^D (AE_{i-1}A^*E_i - E_{i-1}A^*E_iA) \quad (6.20)$$

$$= \sum_{i=0}^D E_{i-1}A^*E_i(\theta_{i-1} - \theta_i) \quad (6.21)$$

$$(\theta_k = D - 2k, \quad \theta_{i-1} - \theta_i = 2I - 2(i-1) = 2) \quad (6.22)$$

$$= 2L^*. \quad (6.23)$$

(c) Similar.

Remark.

$$AR^* - R^*A = \sum_{i=0}^D (AE_{i+1}A^*E_i - E_{i+1}A^*E_iA) \quad (6.24)$$

$$= \sum_{i=0}^D E_{i+1}A^*E_i(\theta_{i+1} - \theta_i) \quad (6.25)$$

$$= 2R^*. \quad (6.26)$$

(a) We have, by (b), (c)

$$[A, A^*] = [A, L^*] + [A, R^*] = 2(L^* - R^*). \quad (6.27)$$

Since $A^* = L^* + R^*$,

$$R^* = \frac{2A^* + [A^*, A]}{4}, \quad L^* = \frac{2A^* - [A^*, A]}{4}.$$

Now (a) is seen to be equivalent to ((6.4)) upon evaluation. This proves Claim 2.

Remark.

$$[L^*, R^*] = \frac{1}{16}((2A^* - [A^*, A])(2A^* + [A^*, A]) - (2A^* + [A^*, A])(2A^* - [A^*, A])) \quad (6.28)$$

$$= \frac{1}{16}(4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* - [A^*, A]^2 - 4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* + [A^*, A]^2) \quad (6.29)$$

$$= \frac{1}{4}(A^{*2}A - 2A^*AA^* + AA^{*2}) \quad (6.30)$$

$$= A, \quad (6.31)$$

by ((6.4)).

Now apply same argument as for ((6.3)), ((6.4)) of Theorem 5.1 and observe A^* has $D + 1$ distinct eigenvalues. So,

$$A^* = \sum_{i=0}^D (D - 2i)E_i^*$$

generates

$$M^* = \text{Span}(E_0^*, \dots, E_D^*).$$

Hence, E_0, \dots, E_D, A^* generates T .

Take an irreducible T -module W with endpoint r with $0 \leq r \leq D/2$. Set $t = \min\{i \mid E_i W\}$.

Pick $0 \neq w_0^* \in E_t W$. Set

$$w_i^* = \frac{1}{i!} R^{*i} w_0^* \in E_{t+i} W \quad \text{for all } i.$$

Then,

$$R^* w_i^* = (i + 1) w_{i+1}^* \quad \text{for all } i.$$

By (a), we get by induction, $L^* w_i^* = (D - 2t - i + 1) w_{i-1}^*$,

$$L^* w_i^* = \frac{1}{i} L^* R^* w_{i-1}^* \quad (6.32)$$

$$= \frac{1}{i} (A + R^* L^*) w_{i-1}^* \quad (6.33)$$

$$= \frac{1}{i} ((D - 2(t + i - 1)) w_{i-1}^* + (i - 1)(D - 2t - i + 2) w_{i-1}^*) \quad (6.34)$$

$$= (D - 2t - i + 1) w_{i-1}^*. \quad (6.35)$$

So $\text{Span}(w_0^*, w_1^*, \dots)$ is L^* , R^* , A^* -invariant. Hence, $W = (\text{Span})(w_0^*, w_1^*, \dots, w_d^*)$, $w_0^*, w_1^*, \dots, w_d^* \neq 0$, $w_i^* = 0$ for every $i > d$ by dimension.

Thus $d = D - 2t$.

Pf.

$$(D - 2(t + d))w_d^* = Aw_d^* \quad (6.36)$$

$$= (L^*R^* - R^*L^*)w_d^* \quad (6.37)$$

$$= -(D - 2t - d + 1)R^*w_{d-1}^* \quad (6.38)$$

$$= -(D - 2t - d + 1)dw_d^*. \quad (6.39)$$

Hence,

$$0 = d^2 + (2t - D - 1 + 2)d - (D - 2t) = (d - D + 2t)(d + 1)$$

So $d = D - 2t$. □

Definition 6.1. For any graph $\Gamma = (X, E)$, pick a vertex $x \in X$, and set $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

- (i) An irreducible T -module W is thin if $\dim E_i^*W \leq 1$ for every i .
- (ii) Γ is thin with respect to x , if every irreducible $T(x)$ -module is thin,
- (iii) An irreducible T -module W is dual thin if $\dim E_iW \leq 1$ for every i .
- (iv) Γ is dual thin with respect to x , if every irreducible $T(x)$ -module is dual thin.

Observe: $H(D, 2)$ is thin, dual thin with respect to each $x \in X$.

Definition 6.2. With above notation, write $D \equiv D(x)$.

- (i) An ordering E_0, E_1, \dots, E_R of primitive idempotents of Γ is restricted if E_0 corresponds to the maximal eigenvalue.

Fix a restricted ordering,

- (ii) Γ is Q -polynomial with respect to x , above ordering if there exists $A^* \equiv A^*(x)$ such that

$$(a) E_0^*V, \dots, E_D^*V \text{ are the maximal eigenspaces for } A^*.$$

$$(b) E_iA^*E_j = 0 \text{ if } |i - j| > 1 \text{ for } 0 \leq i, j \leq R.$$

Observe $H(D, 2)$ is Q -polynomial with respect to the natural ordering of the idempotents and every vertex.

Program. Study graphs that are thin and Q -polynomial with respect to each vertex.

(In fact, thin with respect to x implies dual thin with respect to x .)

Get a situation like $H(D, 2)$, where T is generated by A, A^* . Except $\mathfrak{sl}_2(\mathbb{C})$ is replaced by a quantum Lie algebra.

Chapter 7

The Johnson Graph $J(D, N)$

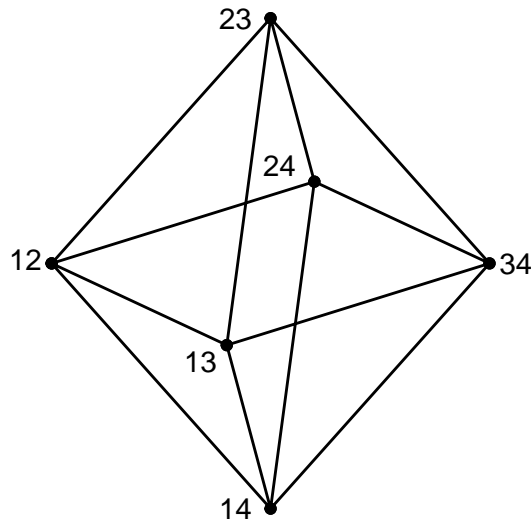
Wednesday, February 3, 1993

Definition 7.1. The Johnson graph, $\Gamma = J(D, N)$ ($1 \leq D \leq N - 1$) satisfies

$$X = \{S \mid S \subset \Omega, |S| = D\} \quad \text{where } \Omega = \{1, 2, \dots, N\} \quad (7.1)$$

$$E = \{ST \mid S, T \in X, |S \cap T| = D - 1\}. \quad (7.2)$$

Example 7.1. $J(2, 4)$



Note 1. The symmetric group S_N acts on Ω . $S_N \subseteq \text{Aut}(\Gamma)$ acts vertex transitively on Γ .

Note 2. $\Gamma = J(D, N)$ is isomorphic to $\Gamma' = J(N - D, N)$.

$$\Gamma = (X, E) \quad \longrightarrow \quad \Gamma' = (X', E') \quad (7.3)$$

$$X \ni S \quad \longmapsto \quad \bar{S} = \Omega \quad S \in X' \quad (7.4)$$

This correspondence induces an isomorphism of graphs.

Pf.

$$ST \in E \Leftrightarrow |S \cap T| = D - 1 \quad (7.5)$$

$$\Leftrightarrow |\Omega - (S \cup T)| = N - D - 1 \quad (7.6)$$

$$\Leftrightarrow |\bar{S} \cap \bar{T}| = N - D - 1 \quad (7.7)$$

$$\Leftrightarrow \bar{S}\bar{T} \in E' \quad (7.8)$$

Hence, without loss of generality, assume

$$D \leq N/2 \quad \text{for} \quad J(D, N).$$

We still need the eigenvalues of $J(D, N)$ for certain problem later in the course. We can get these eigenvalues from our study of $H(D, 2)$.

Lemma 7.1. *The eigenvalues for $J(D, N)$ with $1 \leq D \leq N/2$ are given by*

$$\theta_i = (N - D - i)(D - i) - i \quad (0 \leq i \leq D), \quad (7.9)$$

$$m_i = \binom{N}{i} - \binom{N}{i-1}. \quad (7.10)$$

Proof. Let

$$\Gamma_J \equiv J(D, N) = (X_J, E_J) \quad (7.11)$$

$$\Gamma_H \equiv H(N, 2) = (X_H, E_H). \quad (7.12)$$

Set $x \equiv 11 \cdots 1 \in X_H$.

Define $\tilde{\Gamma} \equiv (\tilde{X}, \tilde{E})$, where

$$\tilde{X} = \{y \in X_H \mid \partial_H(x, y) = D\} \quad \partial_H : \text{distance in } \Gamma_H \quad (7.13)$$

$$\tilde{E} = \{yz \in X_H \mid \partial_H(y, z) = 2\}. \quad (7.14)$$

Observe

$$X_J \quad \rightarrow \quad \tilde{X} \quad (7.15)$$

$$S \quad \mapsto \quad \hat{S}, \quad (7.16)$$

where

$$\hat{S} = a_1 \cdots a_N, \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$$

induces an isomorphism of graphs $\Gamma_J \rightarrow \tilde{\Gamma}$.

Pf.

$$ST \in E_J \Leftrightarrow |S \cap T| = D - 1 \quad (7.17)$$

$$\Leftrightarrow \partial_H(\hat{S}, \hat{T}) = 2 \quad (7.18)$$

$$\Leftrightarrow (\hat{S}, \hat{T}) \in \tilde{E}. \quad (7.19)$$

Identify, Γ_J with $\tilde{\Gamma}$. Then the standard module V_J of Γ_J becomes $\tilde{V} = E_D^* V_H$, where V_H is the standard module of Γ_H , and $E_D^* \equiv E_D^*(x)$.

Let R be the raising matrix with respect to x in Γ_H , and

let L be the lowering matrix with respect to x in Γ_H .

Recall

$$(RL - DE_D^*)|_{\tilde{V}}$$

is the adjacency map in $\tilde{\Gamma}$.

To find eigenvalues of \tilde{A} , pick any irreducible $T(x)$ -module W with the endpoint $r \leq D$. Then by Theorem 5.1

$$\text{diam}(W) = N - 2r.$$

Let $w_0, w_1, \dots, w_{N-2r}$ denote a basis for W as in Theorem 5.1. Then,

$$w_{D-r} \in E_D^* W \subseteq \tilde{V}.$$

Observe:

$$\tilde{A}w_{D-r} = RLw_{D-r} - DE_D^*w_{D-r} \quad (7.20)$$

$$= R(N - 2r - D + r + 1)w_{D-r-1} - Dw_{D-r} \quad (7.21)$$

$$= ((N - D - r + 1)(D - r) - D)w_{D-r}. \quad (7.22)$$

Note that this is valid for $D = r$ as well.

Hence,

$$\tilde{A}w_{D-r} = ((N - D - r)(D - r) - r)w_{D-r}.$$

Let

$$V_H = \sum W \quad (\text{direct sum of irreducible } T(x)\text{-modules.})$$

Then,

$$V_J = E_D^* V_H \quad (7.23)$$

$$= \sum_{W: r(W) \leq D} E_D^* W \quad (7.24)$$

$$= \text{a direct sum of 1 dimensional eigenspaces for } \tilde{A}. \quad (7.25)$$

The eigenspace for eigenvalue

$$(N - D - r)(D - r) - r \quad (\text{monotonously decreasing with respect to } r)$$

appears with multiplicity

$$\binom{N}{r} - \binom{N}{r-1}$$

in this sum by Theorem 5.1 (iv). \square

Theorem 7.1. *Let $\Gamma = (X, E)$ be any graph. For a fixed vertex $x \in X$, let*

$$E_i^* \equiv E_i^*(x), \quad T \equiv T(x), \quad D \equiv D(x), \quad \text{and } K = \mathbb{C}.$$

Then we have the following implications of conditions:

$$TH \Leftrightarrow C \Leftarrow S \Leftarrow G,$$

where

(TH) Γ is thinn with respect to x .

*(C) $E_i^*TE_i^*$ is commutative for every i , $(0 \leq i \leq D)$.*

*(S) $E_i^*TE_i^*$ is symmetric for every i , $(0 \leq i \leq D)$.*

(G) For every $y, z \in X$ with $\partial(x, y) = \partial(x, z)$, there exists $g \in \text{Aut}(\Gamma)$ such that

$$gx = x, \quad gy = z, \quad gz = y.$$

Proof.

$(TH) \Rightarrow (C)$

Fix i with $0 \leq i \leq D$. Let

$V = \sum W$. The standard module written as a direct sum of irreducible T -modules.

Then,

$$E_i^*V = \sum E_i^*W. \text{ The direct sum of 1-dimensional } E_i^*TE_i^*\text{-modules.}$$

Since $\dim E_i^*W = 1$, for $a, b \in E_i^*TE_i^*$, $ab - ba|_{E_i^*W} = 0$. Hence $ab - ba = 0$.

$(C) \Rightarrow (TH)$

Suppose $\dim E_i^*W \geq 2$ for some irreducible T -module W with some i with $1 \leq i \leq D$.

Claim 1. E_i^*W is an irreducible $E_i^*TE_i^*$ -module.

Proof of Claim 1. Suppose

$$0 \subsetneq U \subsetneq E_i^*W,$$

where U is an $E_i^*TE_i^*$ -module. Then by the irreducibility,

$$TU = W.$$

So,

$$U \supseteq E_i^*TE_i^*U = E_i^*TU = E_i^*W.$$

This is a contradiction.

Claim 2. Each irreducible $S = E_i^*TE_i^*$ -module U has dimension 1. In particular, Γ is thin with respect to x .

Proof of Claim 2. Pick

$$0 \neq a \in E_i^*TE_i^*.$$

Since \mathbb{C} is algebraically closed, a has an eigenvector $w \in U$ with eigenvalue θ . Then,

$$(a - \theta I)U = (a - \theta I)Sw \tag{7.26}$$

$$= S(a - \theta I)w \tag{7.27}$$

$$= 0. \tag{7.28}$$

Hence,

$$a|_U = \theta I|_U \quad \text{for all } a \in S.$$

Thus each 1 dimensional subspace of U is an S -module. We have

$$\dim U = 1.$$

By Claim 1 and Claim 2, we have (TH).

□

Remark. Claim 1 shows the following: *If W is an irreducible T -module, then E_i^*W is either 0 or an irreducible $E_i^*TE_i^*$ -module.*

Chapter 8

Thin Graphs

Friday, February 5, 1993

Proof of Theorem 7.1 continued.

(S) \Rightarrow (C)

Fix i and pick $a, b \in E_i^* T E_i^*$.

Since a , b and ab are symmetric,

$$ab = (ab)^\top = b^\top a^\top = ba.$$

Hence $E_i^* T E_i^*$ is commutative.

(G) \Rightarrow (S)

Fix i and pick $a \in E_i^* T E_i^*$. Pick vertices $y, z \in X$.

We want to show that

$$a_{yz} = a_{zy}.$$

We may assume that

$$\partial(x, y) = \partial(x, z) = i,$$

otherwise

$$a_{yz} = a_{zy} = 0.$$

By our assumption, there exists $g \in G$ such that

$$g(y) = z, \quad g(z) = y, \quad g(x) = x.$$

Let \hat{g} denote the permutation matrix representing g , i.e.,

$$\hat{g}\hat{y} = \widehat{g(y)} \quad \text{for all } y \in X, \quad \hat{g} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow y.$$

If $g \in \text{Aut}(\Gamma)$, then

$$\hat{g}A = A\hat{g} \quad (\text{Exercise}).$$

Also, we have

$$\hat{g}E_j^* = E_j^*\hat{g} \quad (0 \leq j \leq D),$$

since

$$\partial(x, y) = \partial(g(x), g(y)) = \partial(x, g(y)).$$

Hence \hat{g} commutes with each element of T . We have

$$a_{yz} = (\hat{g}^{-1}a\hat{g})_{yz}, \quad (\hat{g})_{yz} = \begin{cases} 1 & g(z) = y \\ 0 & \text{else.} \end{cases} \quad (8.1)$$

$$= \sum_{y', z'} (\hat{g}^{-1})_{yy'} a_{y'z'} \hat{g}_{z'z} \quad (8.2)$$

$$(\text{zero except for } g^{-1}(y') = y, g(z) = z'.) \quad (8.3)$$

$$= a_{g(y)g(z)} \quad (8.4)$$

$$= a_{zy}. \quad (8.5)$$

This proves Theorem 7.1. □

Open Problem: Find all the graphs that satisfy the condition (G) for every vertex x .

$H(N, 2)$ is one example, because

$$\text{Aut}\Gamma_{1\dots 1} \simeq S_\Omega, \quad x = (1\dots 1), \quad \Gamma_i(x) = \{\hat{S} \mid |S| = i\}.$$

Property (G) is clearly related to the distance-transitive property.

Definition 8.1. Let $\Gamma = (X, E)$ be any graph. Γ with $G \subseteq \text{Aut}(\Gamma)$ is said to be distance-transitive (or two-point homogeneous), whenever

$$\text{for all } x, x', y, y' \in X \text{ with } \partial(x, y) = \partial(x', y'),$$

there exists $g \in G$ such that

$$g(x) = x', \quad g(y) = y'.$$

(This means G is as close to being doubly transitive as possible.)

Lemma 8.1. Suppose a graph $\Gamma = (X, E)$ satisfies the property $(G) = (G(x))$ for every $x \in X$. Then,

- (i) either
- (ia) Γ is vertex transitive; or
- (iia) Γ is bipartite ($X = X^+ \cup X^-$) with X^+, X^- each an orbit of $\text{Aut}(\Gamma)$.
- (ii) if (ia) holds, then Γ is distance-transitive.

Proof. (i) Claim. Suppose $y, z \in X$ are connected by a path of even length. Then y, z are in the same orbit of $\text{Aut}(\Gamma)$.

Pf. It suffices to assume that the path has length 2, $y \sim w \sim z$.

Now $\partial(y, w) = \partial(w, z) = 1$. So there exists $g \in \text{Aut}(\Gamma)$ such that $gw = w$, $gy = z$, $gz = y$. This proves Claim.

Fix $x \in X$. Now suppose that Γ is not vertex transitive, and we shall show (ib).

Observe that $X = X^+ \cup X^-$, where

$$X^+ = \{y \in X \mid \text{there exists a path of even length connecting } x \text{ and } y\} \quad (8.6)$$

$$X^- = \{y \in X \mid \text{there exists a path of odd length connecting } x \text{ and } y\} \quad (8.7)$$

As X^+ is contained in an orbit O^+ of $\text{Aut}(\Gamma)$, and X^- is contained in an orbit O^- of $\text{Aut}(\Gamma)$.

Now $O^+ \cap O^- = \emptyset$ (else $O^+ = O^- = X$ and vertex transitive). So, $X = O^+$, and $X^- = O^-$.

Also $X^+ \cup X^- = X$ is a bipartition by construction.

(ii) Fix x, y, x', y' with $\partial(x, y) = \partial(x', y')$.

By vertex transitivity, there exists an element

$$g_1 \in G \text{ such that } g_1x = x'.$$

Observe that

$$\partial(x', y') = \partial(x, y) = \partial(g_1x, g_1y) = \partial(x', g_1y).$$

Hence, there exists an element

$$g_2 \in G \text{ such that } g_1x' = x', g_2y' = g_1y', g_2g_1y = y'$$

by $(G(x'))$ property.

Set $g = g_2g_1$. Then

$$gx = x', gy = y'$$

by construction. □

The following graphs $\Gamma = (X, E)$ are vertex transitive, and satisfy the property $(G(x))$ for all $x \in X$.

$$J(D, N), \quad H(D, r), \quad J_q(D, N),$$

where

$$H(D, r):$$

$$X = \{a_1 \cdots a_D \mid a_i \in F, 1 \leq i \leq D\} \quad (8.8)$$

$$F : \text{ any set of cardinality } r \quad (8.9)$$

$$E = \{xy \mid y, x \in X, x \text{ and } y \text{ differ in exactly one coordinate}\}. \quad (8.10)$$

$J_q(D, N)$:

X = the set of all D -dimensional subspaces of N -dimensional vector space over $GF(q)$. (8.11)

$$F : \text{ any set of cardinality } r \quad (8.12)$$

$$E = \{xy \mid y, x \in X, \dim(x \cap y) = D - 1\}. \quad (8.13)$$

The following graph is distance-transitive but does not satisfy $(G(x))$ for any $x \in G$.

$H_q(D, N)$:

$$X = \text{the set of all } D \times N \text{ matrices with entries in } GF(q). \quad (8.14)$$

$$E = \{xy \mid y, x \in X, \text{rank}(x - y) = 1.\}. \quad (8.15)$$

Remark.

$$H(D, r): G = S_r \text{wr} S_D, G_x = S_{r-1} \text{wr} S_D,$$

For $x, y \in X$ with $\partial(x, y) = \partial(x, z) = i$,

$$Y = \{j \in \Omega \mid x_j \neq y_j\} \leftrightarrow Z = \{j \in \Omega \mid x_j \neq z_j\} \quad (8.16)$$

$$(y_{j_1}, \dots, y_{j_i}) \leftrightarrow (z_{\ell_1}, \dots, z_{\ell_i}) \quad (8.17)$$

$$J(D, N): G = S_N, G_x = S_D \times S_{N-D}.$$

$$X \cap Y \leftrightarrow X \cap Z \quad (8.18)$$

$$(\Omega - X) \cap Y \leftrightarrow (\Omega - X) \cap Z. \quad (8.19)$$

The following graph is distance-transitive but does not satisfy $(G(x))$ for any $x \in G$.

$J_q(D, N)$:

$$X \cap Y \leftrightarrow X \cap Z.$$

The theory of single thin irreducible T -module.

Let $\Gamma = (X, E)$ be any graph.

$$M = \text{Bose-Mesner algebra over } K/\mathbb{C} \text{ generated by the adjacency matrix } A. \quad (8.20)$$

$$= \text{Span}(E_0, \dots, E_R). \quad (8.21)$$

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix $x \in X$, let $D \equiv D(x)$ be the x -diameter, and $k = k(x)$ be the valency of x .

Chapter 9

Thin T -Module, I

Monday, February 8, 1993

Let $\Gamma = (X, E)$ be any graph.

M : Bose-Mesner algebra over K/\mathbb{C} generated by the adjacency matrix A .

$$M = \text{Span}(E_0, \dots, E_R).$$

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix $x \in X$, let $D \equiv D(x)$ be the x -diameter, and $k = k(x)$ be the valency of x .

Definition 9.1. Pick $x \in X$ and write $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

Let W be an irreducible thin T -module with endpoint r , diameter d .

Let $a_i = a_i(W) \in \mathbb{C}$ satisfying

$$E_{r+i}^* A E_{r+i}^* |_{E_{r+i}^* W} = a_i 1 |_{E_{r+i}^* W} \quad (0 \leq i \leq d).$$

Let $x_i = x_i(W) \in \mathbb{C}$ satisfying

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* |_{E_{r+i}^* W} = x_i 1 |_{E_{r+i}^* W} \quad (0 \leq i \leq d).$$

Lemma 9.1. *With above notation, the following hold.*

(i) $a_i \in \mathbb{R} \quad (0 \leq i \leq d)$.

(ii) $x_i \in \mathbb{R}^{>0} \quad (0 \leq i \leq d)$.

(iii) Pick $0 \neq w_0 \in E_r^* W$. Set $w_i = E_{r+i}^* A^i w_0$ for all i . Then

(iiia) w_0, w_1, \dots, w_d is a basis for W , $w_{-1} = w_{d+1} = 0$.

(iiib) $A w_i = w_{i+1} + a_i w_i + x_i w_{i-1} \quad (0 \leq i \leq d)$.

(iv) Define $p_0, p_1, \dots, p_{d+1} \in \mathbb{R}[\lambda]$ by

$$p_0 = 1, \quad \lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1} \quad (0 \leq i \leq d), \quad p_{-1} = 0.$$

(iva) $p_i(A)w_0 = w_i$, $(0 \leq i \leq d+1)$.

(ivb) p_{d+1} is the minimal polynomial of $A|_W$.

Proof. (i) a_i is an eigenvalue of a real symmetric matrix $E_{r+i}^* A E_{r+i}^*$.

(ii) x_i is an eigenvalue of a real symmetric matrix $B^\top B$, where

$$B = E_{r+i}^* A E_{r+i-1}^*.$$

Hence, $x_i \in \mathbb{R}$.

Since $B^\top B$ is positive semidefinite,

$$x_i \geq 0.$$

Pf. If $B^\top B v = \sigma v$ for some $\sigma \in \mathbb{R}$, $v \in \mathbb{R}^m \setminus \{0\}$, then

$$0 \leq \|Bv\|^2 = v^\top B^\top B v = \sigma v^\top v = \sigma \|v\|^2, \quad \|v\|^2 > 0.$$

Hence, $\sigma \geq 0$.

Moreover, $x_i \neq 0$ by Lemma 4.1 (iv).

(iiia) Observe

$$w_i = E_{r+i}^* A E_{r+i-1}^* w_{i-1} \quad (1 \leq i \leq d).$$

So $w_i \neq 0$ $(1 \leq i \leq d)$ by Lemma 4.1 (iv).

Hence,

$$W = \text{Span}(w_0, \dots, w_d)$$

by Lemma 4.1. (iii).

(iiib) We have that

$$Aw_i = E_{r+i+1}^* Aw_i + E_{r+i}^* Aw_i + E_{r+i-1}^* Aw_i \quad (9.1)$$

$$= w_{i+1} + E_{r+i}^* A E_{r+i}^* w_i + E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* w_{i-1} \quad (9.2)$$

$$= w_{i+1} + a_i w_i + x_i w_{i-1} \quad (9.3)$$

(iva) Clear for $i = 0$. Assume it is valid for $0, \dots, i$.

$$p_{i+1}(A)w_0 = (A - a_i I)w_i - x_i w_{i-1} = w_{i+1}.$$

(ivb) By definition,

$$p_{d+1}(A)w_0 = 0.$$

Moreover, $p_{d+1}(A)W = 0$. For every $w \in W$, write

$$w = \sum_{i=0}^d \alpha_i w_i \quad (9.4)$$

$$= \sum_{i=0}^d \alpha_i p_i(A)w_0 \quad \text{for some } \alpha_i \in \mathbb{C} \quad (9.5)$$

$$= p(A)w_0 \quad \text{for some } p \in \mathbb{C}[\lambda] \quad (9.6)$$

Hence,

$$p_{d+1}(A)w = p_{d+1}(A)p(A)w_0 \quad (9.7)$$

$$= p(A)p_{d+1}(A)w_0 \quad (9.8)$$

$$= 0. \quad (9.9)$$

Note that p_{d+1} is the minimal polynomial.

Pf. Suppose $q(A)W = 0$ for some $0 \neq q \in \mathbb{C}[\lambda]$ with $\deg q < \deg p_{d+1} = d + 1$. Then,

$$q = \sum_{i=0}^d \beta_i p_i \quad \text{for some } \beta_i \in \mathbb{C}.$$

We have,

$$0 = q(A)w_0 = \sum_{i=0}^d \beta_i w_i.$$

Hence $\beta_0 = \dots = \beta_d = 0$ by (iii). Thus $q = 0$ and a contradiction. \square

Corollary 9.1. *Let Γ , W , r , d be as above. Then*

(i) *W is dual thin, that is,*

$$\dim E_i W \leq 1 \quad (1 \leq i \leq d).$$

(ii) $d = |\{i \mid E_i W \neq 0\}| - 1$.

Proof. (i) Set as in Lemma 9.1,

$$w_i = p_i(A)w_0 \in E_{r+i}^* W.$$

Then w_0, w_1, \dots, w_d is a basis for W . We have

$$W = Mw_0.$$

So,

$$E_i W = E_i M w_0 = \text{Span}(E_i w_0).$$

Thus,

$$\dim E_i W = \begin{cases} 1 & \text{if } E_i w_0 \neq 0 \\ 0 & \text{if } E_i w_0 = 0. \end{cases}$$

In particular,

$$\dim E_i^* W \leq 1.$$

(ii) Immediate as

$$\dim W = d + 1.$$

This proves the lemma. \square

Lemma 9.2. *Given an irreducible $T(x)$ -module W with endpoint $r = r(W)$, diameter $d = d(W)$. Write*

$$x_i = x_i(W) \ (0 \leq i \leq d), \quad w_i = p_i(A)w_0 \in E_{r+i}^* W \ (0 \leq i \leq d), \quad 0 \neq w_0 \in E_r^* W.$$

Then,

$$\frac{\|w_i\|^2}{\|w_0\|^2} = x_1 x_2 \cdots x_i \quad (1 \leq i \leq d).$$

Proof. It suffices to show that

$$\|w_i\|^2 = x_i \|w_{i-1}\|^2 \quad (1 \leq i \leq d).$$

Recall by Lemma 9.1 (iiib) that

$$Aw_j = w_{j+1} + a_j w_j + x_j w_{j-1} \quad (0 \leq j \leq d), \quad w_{-1} = w_{d+1} = 0.$$

Now observe,

$$\langle w_{i-1}, Aw_i \rangle = \langle w_{i-1}, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle \quad (9.10)$$

$$= \overline{x_i} \|w_{i-1}\|^2 \quad (9.11)$$

$$= x_i \|w_{i-1}\|^2. \quad (9.12)$$

by Lemma 9.1 (ii). Also,

$$\langle w_{i-1}, Aw_i \rangle = \langle Aw_{i-1}, w_i \rangle \quad (\text{since } \overline{A}^\top = A) \quad (9.13)$$

$$= \langle x_i + a_{i-1} w_{i-1} + x_{i-1} w_{i-2}, w_i \rangle \quad (9.14)$$

$$= \|w_i\|^2. \quad (9.15)$$

This proves the lemma. \square

Definition 9.2. Let W be an irreducible thin $T(x)$ module with endpoint r , $E_i^* \equiv E_i^*(x)$.

The measure $m = m_W$ is the function

$$m : \mathbb{R} \rightarrow \mathbb{R}$$

such that

$$m(\theta) = \begin{cases} \frac{\|E_i w\|^2}{\|w\|^2} & \text{where } 0 \neq w \in E_r^* W \\ & \text{if } \theta = \theta_i \text{ is an eigenvalue for } \Gamma \\ 0 & \text{if } \theta \text{ is not an eigenvalue for } \Gamma. \end{cases}$$

Chapter 10

Thin T -Module, II

Wednesday, February 10, 1993

Let $\Gamma = (X, E)$ be any graph.

Fix a vertex $x \in X$. Let $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$, the subconstituent algebra over \mathbb{C} , and $V = \mathbb{C}^{|X|}$ the standard module.

Lemma 10.1. *With above notation, let W denote a thin irreducible $T(x)$ -module with endpoint r and diameter d . Let*

$$a_i = a_i(W) \quad (0 \leq i \leq d) \quad (10.1)$$

$$x_i = x_i(W) \quad (1 \leq i \leq d) \quad (10.2)$$

$$p_i = p_i(W) \quad (0 \leq i \leq d+1) \quad (10.3)$$

be from Lemma 9.1, and measure $m = m_W$. Then,

(i) p_0, \dots, p_{d+1} are orthogonal with respect to m , i.e.,

$$\sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \leq i, j \leq d+1) \text{ with } x_{d+1} = 0.$$

$$(ia) \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 m(\theta) = x_1 \cdots x_i \quad (0 \leq i \leq d).$$

$$(iia) \sum_{\theta \in \mathbb{R}} m(\theta) = 1.$$

$$(iiia) \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) = x_1 \cdots x_i a_i \quad (0 \leq i \leq d).$$

Proof. Pick $0 \neq w_0 \in E_r^* W$. Set

$$w_i = p_i(A) w_0 \in E_{r+i}^* W.$$

Since E_i^*W and E_j^*W are orthogonal if $i \neq j$,

$$\delta_{ij}\|w_i\|^2 = \langle w_i, w_j \rangle \quad (10.4)$$

$$= \langle p_i(A)w_0, p_j(A)w_0 \rangle \quad (10.5)$$

$$= \left\langle p_i(A) \left(\sum_{\ell=0}^R E_\ell \right) w_0, p_j(A) \left(\sum_{\ell=0}^R E_\ell \right) w_0 \right\rangle \quad (10.6)$$

$$= \left\langle \sum_{\ell=0}^R p_i(\theta_\ell) E_\ell w_0, \sum_{\ell=0}^R p_j(\theta_\ell) E_\ell w_0 \right\rangle \quad (\text{as } AE_j = \theta_j E_j) \quad (10.7)$$

$$= \sum_{\ell=0}^R p_i(\theta_\ell) \overline{p_j(\theta_\ell)} \|E_\ell w_0\|^2 \quad (10.8)$$

$$(\text{as } p_j \in \mathbb{R}[\lambda], \quad \theta_\ell \in \mathbb{R}, \quad m(\theta_i)\|w_0\|^2 = \|E_i w_0\|^2) \quad (10.9)$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) \|w_0\|^2. \quad (10.10)$$

Now we are done by Lemma 9.2 as

$$\|w_i\|^2 = \|w_0\|^2 x_1 x_2 \dots x_i.$$

For (ia), set $i = j$, and for (ib), set $i = j = 0$.

(ii) We have

$$\langle w_i, Aw_i \rangle = \langle w_i, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle \quad (10.11)$$

$$= \overline{a_i} \|w_i\|^2 \quad (10.12)$$

$$= a_i x_1 \dots x_i \|w_0\|^2, \quad (10.13)$$

as $a_i \in \mathbb{R}$ by Lemma 9.1.

Also,

$$\langle w_i, Aw_i \rangle = \langle p_i(A)w_0, Ap_i(A)w_0 \rangle \quad (10.14)$$

$$= \left\langle p_i(A) \left(\sum_{\ell=0}^R E_\ell \right) w_0, Ap_i(A) \left(\sum_{\ell=0}^R E_\ell \right) w_0 \right\rangle \quad (\text{as in (i)}) \quad (10.15)$$

$$= \sum_{\ell=0}^D p_i(\theta_\ell)^2 \theta_\ell \|E_\ell w_0\|^2 \quad (10.16)$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) \|w_0\|^2. \quad (10.17)$$

Thus, we have (ii). \square

Lemma 10.2. *With above notation, let W be a thin irreducible $T(x)$ -module with measure m . Then m determines diameter $d(W)$,*

$$a_i = a_i(W) \quad (0 \leq i \leq d) \quad (10.18)$$

$$x_i = x_i(W) \quad (1 \leq i \leq d) \quad (10.19)$$

$$p_i = p_i(W) \quad (0 \leq i \leq d+1). \quad (10.20)$$

Proof. Note that $d+1$ is the number of $\theta \in \mathbb{R}$ such that $m(\theta) \neq 0$. Hence m determines d .

Apply (ia), (ii) of Lemma 10.1.

$$\sum_{\theta \in \mathbb{R}} m(\theta) = 1 \quad p_0 = 1. \quad (10.21)$$

$$\sum_{\theta \in \mathbb{R}} \theta m(\theta) = a_0 \quad p_1 = \lambda - a_0 \quad (10.22)$$

$$\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 m(\theta) = x_1 \quad (10.23)$$

$$\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 \theta m(\theta) = x_1 a \quad \rightarrow a_1 \quad (10.24)$$

$$p_2 = (\lambda - a_1)p_1 - x_1 p_0 \quad (10.25)$$

$$\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 m(\theta) = x_1 x_2 \quad \rightarrow x_2 \quad (10.26)$$

$$\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 \theta m(\theta) = x_1 x_2 a_2 \quad \rightarrow a_2 \quad (10.27)$$

$$p_3 = (\lambda - a_2)p_2 - x_2 p_1 \quad (10.28)$$

$$\vdots \quad (10.29)$$

$$\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 m(\theta) = x_1 x_2 \cdots x_d \quad \rightarrow x_d \quad (10.30)$$

$$\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 \theta m(\theta) = x_1 x_2 \cdots x_d a_d \quad \rightarrow a_d \quad (10.31)$$

$$p_{d+1} = (\lambda - a_d)p_d - x_d p_{d-1}. \quad (10.32)$$

$$(10.33)$$

This proves the assertions. \square

Corollary 10.1. *With above notation, let W, W' denote thin irreducible $T(x)$ -modules. The following are equivalent.*

(i) W, W' are isomorphic as T -modules.

(ii) $r(W) = r(W')$ and $m_W = m_{W'}$.

(iii) $r(W) = r(W')$, $d(W) = d(W')$, $a_i(W) = a_i(W')$ and $x_i(W) = x_i(W')$ ($0 \leq i \leq d$).

Proof. (i) \Rightarrow (iii) Write $r \equiv r(W)$, $r' \equiv r(W')$, $d = d(W)$, $d' = d(W')$, $a_i = a_i(W)$, $a'_i = a_i(W')$, $x_i = x_i(W)$ and $x'_i = x_i(W')$.

Let $\sigma : W \rightarrow W'$ denote an isomorphism of T -modules. (See Definition 5.1.)

For every i ,

$$\sigma E_i^* W = E_i^* \sigma W = E_i^* W'.$$

So, $r = r'$ and $d = d'$.

To show $a_i = a'_i$, pick $w \in E_{r+i}^* W \setminus \{0\}$. Then,

$$E_{r+i}^* A E_{r+i}^* \sigma(W) = \sigma(E_{r+i}^* A E_{r+i}^* w) = \sigma(a_i w) = a_i \sigma(w),$$

and $\sigma w \neq 0$. So,

$$a_i = \text{eigenvalue of } E_{r+i}^* A E_{r+i}^* \text{ on } E_{r+i}^* W \quad (10.34)$$

$$= a'_i \quad (10.35)$$

It is similar to show $x = x'$.

Remark. Pick $w \in E_{r+i-1}^* W \setminus \{0\}$

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* \sigma(W) = \sigma(E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* w) = x_i \sigma(w).$$

Hence, x_i is the eigenvalue of $E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^*$ on $E_{r+i-1}^* W = x'_i$.

(iii) \Rightarrow (i)

Pick $0 \neq w_0 \in E_r^* W$, $0 \neq w'_0 \in E_r^* W'$. Let p_i be in Lemma 9.1, and set

$$w_i = p_i(A)w_0 \in E_{r+i}^* W \quad (0 \leq i \leq d) \quad (10.36)$$

$$w'_i = p'_i(A)w'_0 \in E_{r+i}^* W' \quad (0 \leq i \leq d) \quad (10.37)$$

Define a linear transformation,

$$\sigma : W \rightarrow W' \quad (w_i \mapsto w'_i).$$

Since $\{w_i\}$ and $\{w'_i\}$ are bases with $d = d'$, σ is an isomorphism of vector spaces.

We need to show

$$a\sigma = \sigma a \quad (\text{for all } a \in T).$$

Take $a = E_j^*$ for some j ($0 \leq j \leq d(x)$). Then for all i , we have

$$E_j^* \sigma w_i = E_j^* w'_i = \delta_{ij} w'_i,$$

$$\sigma E_j^* w_i = \delta_{ij} \sigma(w_i) = \delta_{ij} w'_i.$$

$$E_j^* \sigma w_i = \sigma E_j^* w_i?$$

Take an adjacency matrix A of a . Then,

$$A\sigma w_i = A w'_i = w'_{i+1} + a'_i w'_i + x'_i w'_{i-1} = \sigma(w_{i+1} + a_i w_i + x_i w_{i-1}) = \sigma A w_i.$$

(ii) \Rightarrow (iii) Lemma 10.2.

(iii) \Rightarrow (ii) Given d, a_i, x_i , we can compute the polynomial sequence

$$p_0, p_1, \dots, p_{d+1}$$

for W .

Show p_0, p_1, \dots, p_{d+1} determines $m = m_W$. Set

$$\Delta = \{\theta \in \mathbb{R} \mid p_{d+1}(\theta) = 0\}.$$

Observe: $|\Delta| = d + 1$. See ‘An Introduction to Interlacing’.

$m(\theta) = 0$ if $\theta \notin \Delta$ ($\theta \in \mathbb{R}$). So it suffices to find $m(\theta)$, $\theta \in \Delta$.

By Lemma 10.1 (i),

$$\begin{cases} \sum_{\theta \in \Delta} m(\theta) p_0(\theta) &= 1 \\ \sum_{\theta \in \Delta} m(\theta) p_1(\theta) &= 0 \\ \vdots & \\ \sum_{\theta \in \Delta} m(\theta) p_d(\theta) &= 0 \end{cases}$$

$d + 1$ linear equation with $d + 1$ unknowns $m(\theta)$ ($\theta \in \Delta$).

But the coefficient matrix is essentially Vander Monde (since $\deg p_i = i$). Hence the system is nonsingular and there are unique values for $m(\theta)$ ($\theta \in \Delta$). \square

Remark.

$$\begin{pmatrix} \theta - a_0 & -1 & \cdots & 0 & 0 \\ -x_1 & \theta - a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta - a_{d-1} & -1 \\ 0 & 0 & \cdots & -x_d & \theta - a_d \end{pmatrix} \begin{pmatrix} p_0(\theta) \\ \vdots \\ \vdots \\ \vdots \\ p_d(\theta) \end{pmatrix} = 0,$$

where θ is an eigenvalue of a diagonalizable matrix

$$L = \begin{pmatrix} a_0 & 1 & \cdots & 0 & 0 \\ x_1 & a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{d-1} & 1 \\ 0 & 0 & \cdots & x_d & \theta a_d \end{pmatrix}$$

with multiplicity $\dim(\text{Ker}(\theta I - L)) = 1$.

Chapter 11

Examples of T -Module

Friday, February 12, 1993

Let $\Gamma = (X, E)$ be a connected graph.

Let θ_0 be the maximal eigenvalue of Γ , and δ its corresponding eigenvector.

$$\delta = \sum_{y \in X} \delta_y \hat{y}.$$

Without loss of generality, we may assume that $\delta_y \in \mathbb{R}^*$ for all $y \in X$.

Lemma 11.1. *Fix a vertex $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$.*

- (i) $T\delta = T\hat{x}$ is an irreducible T -module.
- (ii) *Given any irreducible T -module W , the following are equivalent:*
 - (iia) $W = T\delta$.
 - (iib) *The diameter $d(W) = d(x)$.*
 - (iic) *The endpoint $r(W) = 0$.*

Proof. (i) Observe: there exists an irreducible T -module W that contains δ .

Let $V = \sum_i W_i$ be a direct sum decomposition of the standard module. Then

$$\text{Span}(\delta) = E_0 V = \sum_i E_0 W_i.$$

So, $E_0 W_i \neq 0$ for some i . Then,

$$\delta \in E_0 W_i \subseteq W_i.$$

Observe: $T\delta$ is an irreducible T -module.

Since $\delta \in W$, where W is a T -module. As $T\delta \subseteq W$ and W is irreducible, $T\delta = W$.

Observe: $T\delta = T\hat{x}$.

Since $\hat{x} = \delta_x^{-1} E_0^* \delta \in T\delta$, $T\hat{x} \subseteq T\delta$. Since $T\delta$ is irreducible, $T\hat{x} = T\delta$.

(ii) (a) \rightarrow (b):

$$E_i^* \delta = \sum_{y \in X, \partial(x,y)=i} \delta_y \hat{y} \neq 0, \quad (0 \leq i \leq d(x)),$$

because $\delta_y > 0$ for every $y \in X$.

Hence,

$$E_i^* T\delta \neq 0, \quad (0 \leq i \leq d(x)).$$

Thus, $d(x) = d(W)$.

(b) \rightarrow (c): Immediate.

(c) \rightarrow (a): Since $r(W) = 0$, $E_0^* W \neq 0$. Hence, $\hat{x} \in W$ and $T\hat{x} \subseteq W$.

By the irreducibility, we have $T\hat{x} = W$. □

Lemma 11.2. Assume Γ is bipartite ($X = X^+ \cup X^-$) (X^+ and X^- are nonempty). Then the following are equivalent.

(i) There exist α^+ and $\alpha^- \in \mathbb{R}$ such that

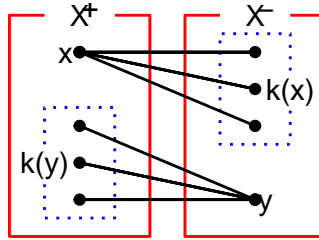
$$\delta_x = \begin{cases} \alpha^+ & \text{if } x \in X^+ \\ \alpha^- & \text{if } x \in X^-. \end{cases}$$

/(ii) There exist k^+ and $k^- \in \mathbb{Z}^{>0}$ such that

$$k(x) = \begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^-. \end{cases}$$

In this case, $k^+ k^- = \theta_0^2$, and Γ is called bi-regular.

Proof. (i) \rightarrow (ii)



$$A\delta = A \left(\alpha^+ \sum_{x \in X^+} \hat{x} + \alpha^- \sum_{y \in X^-} \hat{y} \right) \quad (11.1)$$

$$= \alpha^+ \sum_{y \in X^-} k(y) \hat{y} + \alpha^- \sum_{x \in X^+} k(x) \hat{x} \quad (11.2)$$

$$= \theta_0 \delta. \quad (11.3)$$

So,

$$k(x)\alpha^- = \theta_0\alpha^+, \quad k(y)\alpha^+ = \theta_0\alpha^-.$$

As $\alpha^+ \neq 0$ and $\alpha^- \neq 0$,

$$k^+ := k(x) \text{ is independent of the choice of } x \in X^+, \text{ and} \quad (11.4)$$

$$k^- := k(y) \text{ is independent of the choice of } y \in X^-. \quad (11.5)$$

Moreover, $k^+k^- = \theta_0^2$.

(ii) \rightarrow (i) Set

$$\delta' = \sum_{y \in X} \alpha_y \hat{y} \quad \text{where } \alpha = \begin{cases} 1/\sqrt{k^-} & \text{if } y \in X^+ \\ 1/\sqrt{k^+} & \text{if } y \in X^- \end{cases}.$$

Then one checks

$$A\delta' = A \left(\frac{1}{\sqrt{k^-}} \sum_{y \in X^+} \hat{y} + \frac{1}{\sqrt{k^+}} \sum_{y \in X^-} \hat{y} \right) \quad (11.6)$$

$$= \frac{k^-}{\sqrt{k^-}} \sum_{y \in X^-} \hat{y} + \frac{k^+}{\sqrt{k^+}} \sum_{y \in X^+} \hat{y} \quad (11.7)$$

$$= \sqrt{k^+k^-} \delta' \quad (11.8)$$

Since $\delta' > 0$, $\delta' \in \text{Span}(\delta)$, and $\theta_0 = \sqrt{k^+k^-}$. \square

Definition 11.1. For any graph $\Gamma = (X, E)$, fix a vertex $x \in X$. Set $d = d(x)$.

Γ is distance-regular with respect to x , if for all $i : (0 \leq i \leq d)$, and all $y \in X$ such that $\partial(x, y) = i$:

$$c_i(x) := |\{z \in X \mid \partial(x, z) = i-1, \partial(y, z) = 1\}| \quad (11.9)$$

$$a_i(x) := |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = 1\}| \quad (11.10)$$

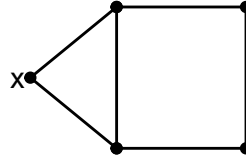
$$b_i(x) := |\{z \in X \mid \partial(x, z) = i+1, \partial(y, z) = 1\}| \quad (11.11)$$

depends only on i , x , and not on y .

(In this case, $c_0(x) = a_0(x) = b_d(x) = 0$, $c_1(x) = 1$, $b_0(x) = k(x)$ is the valency of x .)

We call $c_i(x)$, $a_i(x)$ and $b_i(x)$ the intersection numbers with respect to x .

Example 11.1.



$$c_0 = 1 \qquad c_1 = 1 \qquad c_2 = 1 \qquad (11.12)$$

$$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = 1 \qquad (11.13)$$

$$b_0 = 2 \qquad b_1 = 1 \qquad b_2 = 0 \qquad (11.14)$$

Chapter 12

Distance-Regular

Monday, February 15, 1993

Lemma 12.1. *For any connected graph $\Gamma = (X, E)$, the following are equivalent.*

(i) *The trivial $T(x)$ -module is thin for all $x \in X$.*

(ii) *$\left\{ \sum_{y \in X, d(x,y)=i} \hat{y} \mid 0 \leq i \leq d(x) \right\}$ is a basis for the trivial $T(x)$ -module for every $x \in X$.*

(iii) *Γ is distance-regular with respect to x for all $x \in X$.*

Note. Let $\Gamma = (X, E)$ be a graph, with $X = \{x, y_1, y_2, y_3, z_1, z_2, z_3\}$, $E = \{xy_1, xy_2, xy_3, y_1z_1, y_1z_2, y_2z_3, y_3z_3\}$.



Then (i), (ii) are not equivalent for a single vertex x .

$$E_0^* T \hat{x} = \langle \hat{x} \rangle, \quad (12.1)$$

$$E_1^* T \hat{x} = \langle y_1 + y_2 + y_3 \rangle, \quad (12.2)$$

$$E_2^* T \hat{x} = \langle z_1 + z_2 + 2z_3 \rangle. \quad (12.3)$$

Proof of Lemma 12.1. (i) \rightarrow (ii) Let $\delta = \sum_{y \in X} \delta_y \hat{y}$ be an eigenvector for the maximal eigenvalue θ_0 . Then,

$$\sum_{y \in X, \partial(x,y)=1} \hat{y} = A\hat{x} \in T(x)\hat{x} = T(x)\delta \ni E_1^* \delta \quad (12.4)$$

$$= \sum_{y \in X, \partial(x,y)=1} \delta_y \hat{y} \quad (12.5)$$

If the trivial $T(x)$ -module is thin,

$$\delta_y = \delta_z \text{ for } y, z \in X, \partial(x, y) = \partial(x, z) = 1.$$

Hence, $\delta_y = \delta_z$ if y and z in X are connected by a path of even length.

So, Γ is regular or bipartite biregular by Lemma 11.2.

In particular, $\delta_y = \delta_z$ if $\partial(x, y) = \partial(x, z)$, as there is a path of length $2 \cdot \partial(x, y)$;

$$y \sim \dots \sim x \sim \dots \sim z.$$

Hence,

$$E_i^* \delta \in \text{Span} \left(\sum_{y \in X, \partial(x,y)=i} \hat{y} \right).$$

Since $E_0^* \delta, E_1^* \delta, \dots, E_d^* \delta$ forms a basis for $T(x)\delta$, we have (ii).

(ii) \rightarrow (iii) Fix $x \in X$, and let $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, and $d \equiv d(x)$.

$$A \sum_{y \in X, \partial(x,y)=i} \hat{y} = \sum_{z \in X} |\{y \in X \mid \partial(y, z) = 1, \partial(x, y) = i\}| \hat{z} \quad (12.6)$$

$$= \sum_{z \in X, \partial(x,y)=i-1} b_{i-1}(x, z) \hat{z} \quad (12.7)$$

$$+ \sum_{z \in X, \partial(x,y)=i} a_i(x, z) \hat{z} \quad (12.8)$$

$$+ \sum_{z \in X, \partial(x,y)=i+1} c_{i+1}(x, z) \hat{z} \quad (12.9)$$

$$\in \text{Span} \left\{ \sum_{z \in X, \partial(x,z)=j} \hat{z} \mid j = 0, 1, \dots, d \right\}. \quad (12.10)$$

Hence, $b_{i-1}(x, z)$, $a_i(x, z)$ and $c_{i+1}(x, z)$ depend only on i and x , and not on z . Therefore, Γ is distance-regular with respect to x .

(iii) \rightarrow (i) Fix $x \in X$, and let $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, and $d \equiv d(x)$. By definition of distance-regularity, for every i ($0 \leq i \leq d$),

$$A \left(\sum_{y \in X, \partial(x,y)=i} \hat{y} \right) = b_{i-1}(x) \sum_{y \in X, \partial(x,y)=i-1} \hat{y} \quad (12.11)$$

$$+ a_i(x) \sum_{y \in X, \partial(x,y)=i} \hat{y} \quad (12.12)$$

$$+ c_{i+1}(x) \sum_{y \in X, \partial(x,y)=i+1} \hat{y}. \quad (12.13)$$

Hence,

$$W = \left\{ \sum_{y \in X, \partial(x,y)=i} \hat{y} \mid 0 \leq i \leq d \right\}$$

is A -invariant and so T -invariant. Since $\hat{x} \in W$, $T\hat{x} = W$ is the trivial module and $T\hat{x}$ is thin. \square

Next, we show more is true if (i) – (iii) hold in Lemma 12.1.

In fact, $d(x)$, $a_i(x)$, $c_i(x)$, and $b_i(x)$ are

$$\begin{cases} \text{independent of } X & \text{if } \Gamma \text{ is regular; or} \\ \text{constant over } X^+ \text{ and } X^- & \text{if } \Gamma \text{ is biregular.} \end{cases}$$

Let $\Gamma = (X, E)$ be any (connected) graph. Pick vertices $x, y \in X$.

Let W be a thin, irreducible $T(x)$ -module, and measure $m : \mathbb{R} \rightarrow \mathbb{R}$ determined by W .

Let W' be a thin, irreducible $T(y)$ -module, and measure $m' : \mathbb{R} \rightarrow \mathbb{R}$ determined by W' .

Recall W, W' are orthogonal if

$$\langle w, w' \rangle = 0 \quad \text{for all } w \in W, w' \in W'.$$

We shall show if W and W' are not orthogonal, then m and m' are related:

$$m \cdot \text{poly}_1 = m' \cdot \text{poly}_2$$

for some polynomials with

$$\deg \text{poly}_1 + \deg \text{poly}_2 \leq 2 \cdot \partial(x, y).$$

Notation. V : standard module of Γ .

H : any subspace of V .

$$V = H + H^\perp \quad \text{orthogonal direct sum,}$$

and for $v = v_1 + v_2$ $\text{proj}_H : V \rightarrow H$ ($v \mapsto v_1$): linear transformation.

Observe: For every $v \in V$,

$$v - \text{proj}_H v \in H^\perp.$$

So,

$$\langle v - \text{proj}_H v, h \rangle = 0 \quad \text{for all } h \in H \text{ or,}$$

$$\langle v, h \rangle = \langle \text{proj}_H v, h \rangle \quad \text{for all } v \in V, \text{ and for all } h \in H.$$

Theorem 12.1. *Let $\Gamma = (X, E)$ be any graph. Pick vertices $x, y \in X$ and set $\Delta = \partial(x, y)$. Assume*

W : thin irreducible $T(x)$ -module with endpoint r , diameter d , and measure m .

W' : thin irreducible $T(y)$ -module with endpoint r' , diameter d' , and measure m' .

W and W' are not orthogonal.

Now pick

$$0 \neq w \in E_r^*(x)W, \quad 0 \neq w' \in E_{r'}^*(x)W'.$$

Then,

$$(i) \quad \text{proj}_{W'} w = p(A) \frac{\|w\|}{\|w'\|} w'$$

for some $0 \neq p \in \mathbb{C}[\lambda]$ with $\deg p \leq \Delta - r' + r, d'$,

$$\text{proj}_W w' = p'(A) \frac{\|w'\|}{\|w\|} w$$

for some $0 \neq p' \in \mathbb{C}[\lambda]$ with $\deg p' \leq \Delta - r + r', d$.

(ii) For all eigenvalues θ_i of Γ ,

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = m(\theta_i) \overline{p'(\theta_i)}.$$

(iii) For all eigenvalues θ_i of Γ ,

$$p(\theta_i) p'(\theta_i)$$

is in a real number in interval $[0, 1]$.

Proof. (i) Since W, W' are not orthogonal, there exist

$$v \in W, v' \in W' \text{ such that } \langle v, v' \rangle \neq 0.$$

Then there exists $a \in M$ such that

$$v' = aw'.$$

(This is because $w'_i = p'_i(A)w'_0$ and hence for every $v' \in W'$, there is a polynomial $q \in \mathbb{C}[\lambda]$, $q(A)w'_0 = v$.)

We have

$$0 \neq \langle v', v \rangle = \langle aw', v \rangle = \langle w', a^*v \rangle$$

and $a^*v \in W$.

Hence, $\text{proj}_W w' \neq 0$.

Let $p_0, \dots, p_d \in \mathbb{C}[\lambda]$ be from Lemma 9.1.

Then, $w_i = p_i(A)w$ is a basis for $E_{r+i}^*(x)W$ ($0 \leq i \leq d$).

Hence,

$$\text{proj}_W w' = \alpha_0 w_0 + \dots + \alpha_d w_d \text{ for some } \alpha_j \in \mathbb{C}.$$

Set

$$p' := \frac{\|w\|}{\|w'\|} \sum_{i=0}^d \alpha_i p_i.$$

Then $0 \neq p' \in \mathbb{C}[\lambda]$ and $\deg p' \leq d$.

Claim: $\alpha_i = 0$ ($\Delta - r + r' < i \leq d$).

In particular, $\deg p' \leq \Delta - r + r'$.

Pf. Observe:

$$w' \in E_{r'}^*(y)V, \quad w \in E_r^*(x)V,$$

for $\partial(x, y) = \Delta$.

$$E_{r'}^*(y)V \cap E_{r+i}^*(x)V = 0$$

by triangle inequality.

($\Delta = \partial(x, y) < r + i - r'$ or $\Delta + r' < r + i$ by our choice of i .)



Hence,

$$E_{r'}^*(y)V \perp E_{r+i}^*(x)V,$$

or

$$0 = \langle w', w_i \rangle \quad (12.14)$$

$$= \langle \text{proj}_W w', w_i \rangle \quad (12.15)$$

$$= \sum_{j=0}^d \alpha_j \langle w_j, w_i \rangle \quad (12.16)$$

$$= \alpha_i \|w_i\|^2. \quad (12.17)$$

Hence, $\alpha_i = 0$. Thus,

$$\text{proj}_W w' = \sum_{i=0}^{\Delta+r'-r} \alpha_i w_i \quad (12.18)$$

$$= \sum_{i=0}^{\Delta+r'-r} \alpha_i p_i(A) w_0 \quad (12.19)$$

$$= p'(A) \frac{\|w'\|}{\|w\|} w. \quad (12.20)$$

(ii) We have

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = \frac{\langle E_i w, w' \rangle}{\|w\| \|w'\|} \quad (12.21)$$

$$= \frac{\langle E_i w, \text{proj}_W w' \rangle}{\|w\| \|w'\|} \quad \text{as } \text{proj}_W w' = p'(A) \frac{\|w\|}{\|w'\|} w \quad (12.22)$$

$$= \frac{\langle E_i w, p'(A) w \rangle}{\|w\|^2} \quad (12.23)$$

$$= \frac{\langle E_i w, E_i p'(A) w \rangle}{\|w\|^2} \quad (12.24)$$

$$= \overline{p'(\theta_i)} \frac{\|E_i W\|^2}{\|w\|^2} \quad (12.25)$$

$$= \overline{p'(\theta_i)} m(\theta_i). \quad (12.26)$$

Moreover, as $m(\theta_i), m'(\theta_i) \in \mathbb{R}$,

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = \frac{\overline{\langle E_i w, E_i w' \rangle}}{\|w'\| \|w\|} = \overline{p(\theta_i) m'(\theta_i)} = p(\theta_i) m'(\theta_i).$$

(iii) Since,

$$\frac{|\langle E_i w, E_i w' \rangle|^2}{\|w\|^2 \|w'\|^2} = p(\theta_i) p'(\theta_i) m(\theta_i) m'(\theta_i),$$

$$p(\theta_i)p'(\theta_i) = \frac{|\langle E_i w, E_i w' \rangle|^2}{m(\theta_i)m'(\theta_i)\|w\|^2\|w'\|^2} \in \mathbb{R} \quad (12.27)$$

$$= \frac{|\langle E_i w, E_i w' \rangle|^2}{\frac{\|E_i w\|^2}{\|w\|^2} \frac{\|E_i w'\|^2}{\|w'\|^2} \|w\|^2 \|w'\|^2}. \quad (12.28)$$

By Cauchy-Schwartz inequality,

$$(|\langle a, b \rangle| \leq \|a\| \|b\|,)$$

$$\frac{|\langle E_i w, E_i w' \rangle|^2}{\|E_i w\|^2 \|E_i w'\|^2} \leq 1.$$

Hence, we have the assertion. \square

Chapter 13

Modules of a DRG

Wednesday, February 17, 1993

Lemma 13.1. *Let $\Gamma = (X, E)$ be any graph. Pick an edge $xy \in E$.*

Assume the trivial $T(x)$ -module $T(x)\delta$ is thin with measure m_x ,

and the trivial $T(y)$ -module $T(y)\delta$ is thin with measure m_y .

Then,

$$(ia) \quad \frac{m_x(\theta)}{k_x} = \frac{m_y(\theta)}{k_y} \text{ for all } \theta \in \mathbb{R} \setminus \{0\}.$$

$$(ib) \quad \frac{m_x(0) - 1}{k_x} = \frac{m_y(0) - 1}{k_y} \text{ for all } \theta \in \mathbb{R} \setminus \{0\}.$$

$$(\delta = \sum_{y \in X} \delta_y \hat{y} \quad \text{eigenvector corresponding to the maximal eigenvalue})$$

Proof. Apply Theorem 12.1,

$$W = T(x)\delta \quad r = 0, \quad d = d(x) \tag{13.1}$$

$$W' = T(y)\delta \quad r' = 0, \quad d' = d(y). \tag{13.2}$$

Take $w = \hat{x}$, $w' = \hat{y}$.

Claim. $\text{proj}_{T(y)\delta} \hat{x} = k_y^{-1} A \hat{y}$.

Pf. Since

$$\hat{y} \in T(y)\delta, \quad A\hat{y} \in T(y)\delta.$$

Show

$$(\hat{x} - k_y^{-1} A \hat{y}) \perp (T(y)\delta).$$

Recall

$$A\hat{y} = \sum_{z \in X, yz \in E} \hat{z}.$$

$$\hat{x} - k_y^{-1}Ay \in E_1^*(y)V.$$

So,

$$\hat{x} - \frac{1}{k_y}A\hat{y} \perp E_j^*(y)T(y)\delta \quad \text{if } j \neq 1 \quad (0 \leq j \leq k(y)).$$

And we have,

$$\left\langle \hat{x} - \frac{1}{k_y}A\hat{y}, A\hat{y} \right\rangle = \left\langle \hat{x}, \sum_{z \in X, yz \in E} \hat{z} \right\rangle - \frac{1}{k_y} \left\| \sum_{z \in X, yz \in E} \hat{z} \right\|^2 \quad (13.3)$$

$$= 1 - 1 \quad (13.4)$$

$$= 0 \quad (13.5)$$

This proves Claim.

Similarly,

$$\text{prof}_{T(x)\delta} \hat{y} = k_x^{-1}A\hat{x}.$$

Hence, the polynomials $p, p' \in \mathbb{C}[\lambda]$ from Theorem 12.1 equal

$$\frac{\lambda}{k_y} \quad \text{and} \quad \frac{\lambda}{k_x}$$

respectively.

By Theorem 12.1,

$$\frac{m_x(\theta)\theta}{k_x} = m_x(\theta)\overline{p'(\theta)} = m_y(\theta)\overline{p(\theta)} = \frac{m_y(\theta)\theta}{k_y}.$$

If $\theta \neq 0$, we have (ia).

Also,

$$\frac{1 - m_x(0)}{k_x} = \left(\sum_{\theta \in \mathbb{R} \setminus \{0\}} m_x(0) \right) \frac{1}{k_x} \quad \text{by (ia)} \quad (13.6)$$

$$= \left(\sum_{\theta \in \mathbb{R} \setminus \{0\}} m_y(0) \right) \frac{1}{k_y} \quad (13.7)$$

$$= \frac{1 - m_y(0)}{k_y} \quad (13.8)$$

Hence, we have (ib). □

Theorem 13.1. *Suppose any graph $\Gamma = (X, E)$ is distance-regular with respect to every vertex $x \in X$. (So Γ is regular or biregular by Lemma 12.1.)*

Then,

Case Γ is regular: the diameter $d(x)$ and the intersection numbers $a_i(x)$, $b_i(x)$, $c_i(x)$ ($0 \leq i \leq d(x)$) are independent of $x \in X$.

(And Γ is called distance-regular.)

Case Γ is biregular: ($X = X^+ \cup X^-$)

$d(x)$ and $a_i(x)$, $b_i(x)$, $c_i(x)$ ($0 \leq i \leq d(x)$) are constant over X^+ and X^- . (And Γ is called distance-biregular.)

Proof. We apply Lemma 13.1.

Case Γ : regular.

Then $m_x = m_y$ for all $xy \in E$. Hence, the measure of the trivial $T(x)$ -module is independent of $x \in X$.

Case Γ is biregular.

Then $m_x = m_{x'}$ for all $x, x' \in X$ with $\partial(x, x') = 2$.

Hence, the measure of the trivial $T(x)$ -module is constant over $x \in X^+$, X^- .

Fix $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, $W = T\delta$ with measure m , diameter $d = d(x)$.

We know by Corollary 10.1 that m determines

$$d, \quad a_i(W) \ (0 \leq i \leq d), \quad x_i(W) \ (1 \leq i \leq d)$$

(as $d = D(x) = d(W)$ by Lemma 11.1.)

We shall show that m determines

$$a_i(x), \ c_i(x), \ b_i(x) \quad (0 \leq i \leq d).$$

Observe:

$$a_i(W) = a_i(x) \quad (0 \leq i \leq d) \tag{13.9}$$

$$x_i(W) = b_{i-1}c_i(x) \quad (1 \leq i \leq d) \tag{13.10}$$

Remark. $a_i = a_i(W)$ is an eigenvalue of

$$E_i^* A E_i^* \text{ on } E_i^* W = \langle \sum_{y \in \Gamma_i(x)} \hat{y} \rangle.$$

(See Lemma 12.1.)

$x_i = x_i(W)$ is an eigenvalue of

$$E_{i-1}^* A E_i^* A E_{i-1}^* \text{ on } E_{i-1}^* W,$$

and

$$A \sum_{y \in X, \partial(x,y)} \hat{y} = b_{i-1}(x) \sum_{y \in X, \partial(x,y)=i-1} \hat{y} \quad (13.11)$$

$$+ a_i(x) \sum_{y \in X, \partial(x,y)=i} \hat{y} \quad (13.12)$$

$$+ c_{i+1} \sum_{y \in X, \partial(x,y)=i+1} \hat{y} \quad (13.13)$$

So $x_i = b_{i-1}(x)c_i(x)$.

Set $k^+ = k_x$. Define

$$k^- = \frac{\theta_0^2}{k^+},$$

where θ_0 is the maximal eigenvalue. (See Lemma 11.1.)

(So, $k^+ = k^-$ is the valency, if Γ is regular.)

For every i ($0 \leq i \leq d$) and for every $z \in X$ with $\partial(x, z) = i$,

$$k_z = c_i(x) + a_i(x) + b_i(x) \quad (13.14)$$

$$= \begin{cases} k^+ & \text{if } i \text{ is even,} \\ k^- & \text{if } i \text{ is odd.} \end{cases} \quad (13.15)$$

Now m determines

$$c_0(x) = a_0(x) = 0, \quad c_1(x) = 1,$$

$$b_0(x) = b_0(x)c_1(x) = x_1(W).$$

$$k^+ = b_0(x) \quad (13.16)$$

$$k^- = \theta_0^2 / k^+ \quad (13.17)$$

$$c_i(x) = x_i(W) / b_{i-1}(x) \quad (1 \leq i \leq d) \quad (13.18)$$

$$b_i(x) = \begin{cases} k^+ - a_i(x) - c_i(x) & i: \text{ even,} \\ k^- - a_i(x) - c_i(x) & i: \text{ odd.} \end{cases} \quad (13.19)$$

This proves the assertions. \square

Proposition 13.1. *Under the assumption of Theorem 13.1, the following hold.*

Case Γ : regular.

- (i) $\dim E_i V = |X| m(\theta_i)$.
- (ii) Γ has exactly $d + 1$ distinct eigenvalues
- ($d = \text{diam} \Gamma = d(x)$, for all $x \in X$).

Case Γ : biregular.

- (i) $\dim E_V = |X^+| m^+(\theta_i) + |X^-| m^-(\theta_i)$.
- (ii) Γ has exactly $d^+ + 1$ distinct eigenvalues ($d^+ \geq d^-$).
- (iii) If d^+ is odd, the Γ is regular.
- (iv) $d^+ = d^-$, or $d^+ = d^- + 1$ is even.
- (v) $a_i(x) = 0$ for all i and for all x .

Proof. (i) Suppose Γ is regular.

Let m_x be the measure of the trivial $T(x)$ -module,

$$m_x(\theta_i) = \|E_i \hat{x}\|^2, \quad \text{as } \|\hat{x}\| = 1.$$

Now,

$$|X| m_x(\theta_i) = \sum_{x \in X} m_x(\theta_i) \tag{13.20}$$

$$= \sum_{x \in X} \|E_i \hat{x}\|^2 \tag{13.21}$$

$$= \sum_{y, z \in X} |(E_i)_{yz}|^2 \tag{13.22}$$

$$= \text{trace} E_i \overline{E_i}^\top. \tag{13.23}$$

Since A is real symmetric and

$$E_i \overline{E_i}^\top = E_i^2 = E_i$$

with E_i symmetric

$$E_i \sim \begin{pmatrix} I & O \\ O & I \end{pmatrix}.$$

$$\text{trace} E_i = \text{rank} E_i = \dim E_i V.$$

Thus, we have the assertion in this case.

Suppose Γ is biregular.

Then, same except,

$$\sum_{x \in X} m_x(\theta_i) = |X^+| m^+(\theta_i) + |X^-| m^-(\theta_i).$$

- (ii) Γ : regular. Immediately, if θ is an eigenvalue of Γ , then $m(\theta) \neq 0$.

Γ : biregular. For each $\theta = \theta_i \in \mathbb{R} \setminus \{0\}$,

$$m^-(\theta) \neq 0 \Leftrightarrow m^+(\theta) \neq 0 \quad (13.24)$$

$$\Leftrightarrow \theta \text{ is an eigenvalue of } \Gamma \quad (13.25)$$

$$\left(\frac{m^+(\theta)}{k^+} = \frac{m^-(\theta)}{k^-} \right) \quad (13.26)$$

(iv) and (v) are clear.

Remark. (iii) If d^+ is odd, $d^+ = d^-$ and Γ has even number of eigenvalues, i.e., 0 is not an eigenvalue. So A is nonsingular, and Γ is regular.

□

Chapter 14

Parameters of Thin Modules, I

Friday, February 19, 1993

Summary.

Definition 14.1. Assume $\Gamma = (X, E)$ is distance-regular with respect to every vertex $x \in X$.

Notation: Let $x \in X$. The data of the trivial $T(x)$ -module.

	Case DR	Case DBR
valency k_x	k	$\begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^- \end{cases}$
x -diameter D_x	D	$\begin{cases} D^+ & \text{if } x \in X^+ \\ D^- & \text{if } x \in X^- \end{cases}$
measure m_x	m	$\begin{cases} m^+ & \text{if } x \in X^+ \\ m^- & \text{if } x \in X^- \end{cases}$
int. number $c_i(x)$	c_i	$\begin{cases} c_i^+ & \text{if } x \in X^+ \\ c_i^- & \text{if } x \in X^- \end{cases}$
int. number $b_i(x)$	b_i	$\begin{cases} b_i^+ & \text{if } x \in X^+ \\ b_i^- & \text{if } x \in X^- \end{cases}$
int. number $a_i(x)$	a_i	$\begin{cases} a_i^+ & \text{if } x \in X^+ \\ a_i^- & \text{if } x \in X^- \end{cases}$

Call $m, m^{\pm 1}$ the measure of Γ .

Assume $\Gamma = (X, E)$ is distance-regular.

To what extent do a_i 's, b_i 's and c_i 's determine the structure of irreducible $T(x)$ -modules? In general the following hold.

Lemma 14.1. *Assume $\Gamma = (X, E)$ is distance-regular. Pick $x \in X$. Let X be a thin irreducible $T(x)$ -module with endpoint r , diameter d and measure m_W .*

(i) *There is a unique polynomial $f_W \in \mathbb{C}[\lambda]$ with the following properties.*

(ia) $\deg f_W \leq D$ (diameter of Γ).

(ib) $m_W(\theta) = m(\theta)f_W(\theta)$ for every $\theta \in \mathbb{R}$, where m is the measure of Γ .

Moreover, $f_W \in \mathbb{R}[\lambda]$, and

(ii) $\deg f_W \leq 2r$.

(iii) *For all eigenvalues θ_i of Γ , $\lambda - \theta_i$ is a factor of f_W whenever, $E_i W = 0$.*

In particular, $2r - D + d \geq 0$.

Proof. Let $\theta_0, \dots, \theta_D$ denote distinct eigenvalues of Γ . Then $m(\theta_i) \neq 0$ ($0 \leq i \leq D$) by Proposition 13.1.

There exists a unique $f_W \in \mathbb{C}[\lambda]$ with $\deg f_W \leq D$ such that

$$f_W(\theta_i) = \frac{m_W(\theta_i)}{m(\theta_i)} \quad (0 \leq i \leq D)$$

by polynomial interpolation.

$f_W \in \mathbb{R}[\lambda]$ since

$$\theta_0, \dots, \theta_D \in \mathbb{R} \quad \text{and} \quad f_W(\theta_0), \dots, f_W(\theta_D) \in \mathbb{R}.$$

(ii) Without loss of generality, we may assume $r < D/2$, else trivial.

Pick $0 \neq w \in E_r^*(x)W$.

$$w = \sum_{y \in W, \partial(x, y) = r} \alpha_y \hat{y} \quad \text{some } \alpha_y \in \mathbb{C}.$$

Pick $y \in X$ such that $\alpha_y \neq 0$.

Set W' be the trivial $T(y)$ -module. ($\langle w, \hat{y} \rangle \neq 0$, as $W \not\perp W'$.)

$$r' = 0, \quad m' = m, \quad \Delta = r.$$

Apply Theorem 12.1, we have

$$\deg p \leq \Delta - r' + r = 2r, \quad p \neq 0 \tag{14.1}$$

$$\deg p' \leq \Delta - r + r' = 0, \quad p' \neq 0. \tag{14.2}$$

$$m_W(\theta)\overline{p'(\theta)} = m(\theta)p(\theta) \quad (\text{for all } \theta \in \mathbb{R}).$$

So,

$$\deg p/\bar{p}' \leq 2r,$$

and p/\bar{p}' satisfies the conditions of f_W .

$$\left(\frac{p(\theta)}{\bar{p}'(\theta)} = \frac{m_W(\theta)}{m(\theta)} \right)$$

(iii)

$$E_i W = 0 \rightarrow m_W(\theta_i) = 0 \rightarrow f_W(\theta_i) = 0.$$

that is, $E_i W = 0$. Hence θ_i is a root of $f_W(\lambda) = 0$. So,

$$2r \geq \deg f_W \geq |\{\theta_i \mid E_i W = 0\}| = D - d.$$

Hence,

$$2r - D + d \geq 0.$$

This proves the assertions. \square

Lemma 14.2. *Let $\Gamma = (X, E)$ be any distance-regular graph with valency k , diameter D ($d \geq 2$), measure m , and eigenvalues*

$$k = \theta_0 > \theta_1 > \dots > \theta_D.$$

Pick $x \in X$. Let W be a thin irreducible $T(x)$ -module with endpoint $r = 1$, diameter D and measure $m_W = mf_W$. Then one fo the following cases (i)–(iv) occurs.

Case	d	$f_W(\lambda)$	$a_0(W)$
(i)	$D - 2$	$\frac{(\lambda-k)(\lambda-\theta_1)}{k(\theta_1+1)}$	$-\frac{b_1}{\theta_1+1} - 1$
(ii)	$D - 2$	$\frac{(\lambda-k)(\lambda-\theta_D)}{k(\theta_D+1)}$	$-\frac{b_1}{\theta_1+1} - 1$
(iii)	$D - 1$	$\frac{k-\lambda}{k}$	-1
(iv)	$D - 1$	$\frac{(\lambda-k)(\lambda-\beta)}{k(\beta+1)}$	$-\frac{b_1}{\beta+1} - 1$

for some $\beta \in \mathbb{R}$ with $\beta \in (-\infty, \theta_D) \cup (\theta_1, \infty)$. Moreover, the isomorphism class of W is determined by $a_0(W)$.

Note. By (iii), the possible “shapes” of a thin irreducible $T(x)$ -modules are:

$$r = 0 \quad d = D \tag{14.3}$$

$$r = 1 \quad d = D - 1 \tag{14.4}$$

$$r = 1 \quad d = D - 2 \tag{14.5}$$

Chapter 15

Parameters of Thin Modules, II

Monday, February 22, 1993

Proof of Lemma 14.2 Continued.

We have $\deg f_W \leq 2$ by Lemma 14.1 (ii).

Also by Lemma 11.1, $E_0 W = 0$.

(As otherwise $\langle \delta \rangle = E_0 V \subseteq W$ and $r(W) = 0$.)

Hence, $\lambda - \theta_0 = \lambda - k$ is a factor of f_W by Lemma 14.1 (iii).

Let p_0, p_1, \dots, p_D denote the polynomials for the trivial $T(x)$ -module from Lemma 9.1.

Recall,

$$\sum_{\theta \in \mathbb{R}} m(\theta) p_i(\theta) p_j(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \leq i, j \leq D) \quad (15.1)$$

$$= \delta_{ij} b_0 b_1 \cdots b_{i-1} c_1 c_2 \cdots c_i. \quad (15.2)$$

Note that $x_i = b_{i-1} c_i$ is in the proof of Theorem 7.1.

By construction,

$$p_0(\lambda) = 1.p_1(\lambda) \quad \quad \quad = \lambda.p_2(\lambda)\lambda^2 - a_1\lambda - k. \quad (15.3)$$

Apparently,

$$f_W = \sigma_0 p_0 + \sigma_1 p_1 + \sigma_2 p_2$$

for some $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{C}$.

Claim:

$$\sigma_0 = 1, \quad (15.4)$$

$$\sigma_1 = \frac{a_0(W)}{k}, \quad (15.5)$$

$$\sigma_2 = \frac{1 + a_0(W)}{kb_1}. \quad (15.6)$$

Pf of Claim.

$$1 = \sum_{\theta \in \mathbb{R}} m_W(\theta) \quad (15.7)$$

$$= \sum_{\theta \in \mathbb{R}} m(\theta) f_W(\theta) \quad (15.8)$$

$$= \sum_{j=0}^2 \sigma_j \left(\sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) \right) \quad (15.9)$$

$$= \sigma_0. \quad (15.10)$$

We applied Lemma 10.1 (ib), Lemma 14.1 (ib), and Lemma 10.1 (i) in this order.

Next by Lemma 10.1 (ii), and $p_1(\theta) = \theta$,

$$a_0(W) = \sum_{\theta \in \mathbb{R}} m_W(\theta) \theta \quad (15.11)$$

$$= \sum_{\theta \in \mathbb{R}} f_W(\theta) \theta \quad (15.12)$$

$$= \sum_{j=0}^2 \sigma_j \sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) p_1(\theta) \quad (15.13)$$

$$= \sigma_1 x_1(T\delta) \quad (15.14)$$

$$= \sigma_1 b_0 c_1 \quad (15.15)$$

$$= \sigma_1 k. \quad (15.16)$$

So for,

$$f_W(\lambda) = 1 + \frac{a_0(W)}{k} \lambda + \sigma_2 (\lambda^2 - a_1 \lambda - k).$$

But,

$$0 = f_W(k) \quad (15.17)$$

$$= 1 + a_0(W) + \sigma_2 k(k - a_1 - 1) \quad (15.18)$$

$$1 + a_0(W) + \sigma_2 kb_1. \quad (15.19)$$

Thus,

$$\sigma_2 = -\frac{1 + a_0(W)}{kb_1}.$$

This proves Claim.

Case: $a_0(W) = -1$.

Here, $\sigma_2 = 0$ and

$$f_W(\lambda) = 1 + \frac{a_0(W)\lambda}{k} = 1 - \frac{\lambda}{k}.$$

Also,

$$d + 1 = |\{\theta \mid \theta \text{ is an eigenvalue of } \Gamma, f_W(\theta) \neq 0\}| = D.$$

Case: $a_0(W) \neq -1$.

Here, $\sigma_2 \neq 0$, and $\deg f_W = 2$. So,

$$f_W(\lambda) = (\lambda - k)(\lambda - \beta)\alpha$$

for some $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$.

Comparing the coefficients in

$$(\lambda - k)(\lambda - \beta)\alpha = 1 + \frac{a_0(W)}{k}\lambda - \frac{a_0(W) + 1}{kb_1}(\lambda^2 - a_1\lambda - k),$$

we find

$$\alpha = -\frac{a_0(W) + 1}{kb_1}, \quad (15.20)$$

$$-(k + \beta)\alpha = \frac{a_0(W)}{k} + \frac{a_0(W) + 1}{kb_1}a_1, \quad (15.21)$$

$$k\beta\alpha = 1 + \frac{a_0(W) + 1}{b_1}. \quad (15.22)$$

Hence,

$$-\beta(a_0(W) + 1) = b_1 + (a_0(W) + 1).$$

Thus, we have

$$(1 + a_0(W))(1 + \beta) = -b_1. \quad (15.23)$$

In particular, $\beta \neq -1$, and

$$\alpha = -\frac{1 + a_0(W)}{kb_1} = \frac{1}{k(\beta + 1)}.$$

Also, by Definition 9.2,

$$0 \leq m_W(\theta) \quad (15.24)$$

$$= m(\theta)f_W(\theta) \quad (\text{for all } \theta \in \mathbb{R}). \quad (15.25)$$

But if θ is an eigenvalue of Γ ,

$$0 < m(\theta).$$

So,

$$0 \leq f_W(\theta) \quad (15.26)$$

$$= \frac{(\theta - k)(\theta - \beta)}{k(\beta + 1)}. \quad (15.27)$$

Either

$$\beta + 1 > 0 \rightarrow \theta - \beta \leq 0 \text{ or } \beta \geq \theta_1,$$

or

$$\beta + 1 < 0 \rightarrow \theta - \beta \geq 0 \text{ or } \beta \leq \theta_D.$$

If $\beta = \theta_1$,

$$a_0(W) = -\frac{b_1}{\beta + 1} - 1 = -\frac{b_1}{\theta_1 + 1} - 1 \quad (15.28)$$

$$f_W(\lambda) = \frac{(\lambda - k)(\lambda - \theta_1)}{k(\theta_1 + 1)}, \quad (15.29)$$

and we have (i).

If $\beta = \theta_D$,

$$a_0(W) = -\frac{b_1}{\theta_D + 1} - 1 \quad (15.30)$$

$$f_W(\lambda) = \frac{(\lambda - k)(\lambda - \theta_D)}{k(\theta_D + 1)}, \quad (15.31)$$

and we have (ii).

If $\beta \notin \{\theta_1, \theta_2\}$,

$$\theta \in (-\infty, \theta_D) \cup (\theta_1, \infty),$$

we have (iv).

Note using (15.23), we have (iv).

□

Note. Using (15.23),

$$a_0(W) \rightarrow \beta \rightarrow f_W \rightarrow m_W \rightarrow \text{isomorphism class of } W.$$

Note on Lemma 14.2. In fact, $\theta_1 > -1$, $\theta_D < -1$ if $D \geq 2$.

Definition 15.1. The complete graph K_n has n vertices and diameter $D = 1$, i.e., $xy \in E$ for all vertices x, t .

K_n is distance-regular with valency $k = n - 1$ and $a_1 = n - 2$, $D = 1$. Moreover, it has two distance eigenvalues θ_0, θ_1 .

Recall, $\theta_0, \dots, \theta_D$ are roots of p_{D+1} , i.e., $D + 1$ st polynomial for the trivial module/

$$p_0 = 1 \quad (15.32)$$

$$p_1 = \lambda \quad (15.33)$$

$$p_2 = \lambda^2 - a_1\lambda - k \quad (15.34)$$

$$= \lambda^2 - (n - 2)\lambda - (n - 1) \quad (15.35)$$

$$= (\lambda - (n - 1))(\lambda + 1). \quad (15.36)$$

The roots are $\theta_0 = n - 1 = k$ and $\theta_1 = -1$.

Lemma 15.1. *Let $\Gamma = (X, E)$ be distance-regular of diameter $D \geq 1$ with distinct eigenvalues*

$$k = \theta_0 > \theta_1 > \dots > \theta_D.$$

(i) $\theta_D \leq -1$ with equality if and only if $D = 1$.

(ii) $\theta_1 \geq -1$ with equality if and only if $D = 1$.

Proof. (i) Suppose $\theta_D \geq -1$.

Then $I + A$ is positive semi-definite.

By Lemma 2.1, there exists vectors $\{v_x \mid x \in X\}$ in a Euclidean space such that

$$\langle v_x, v_y \rangle = (I + A)_{xy} \quad (15.37)$$

$$= \begin{cases} 1 & \text{if } x = y \text{ or } xy \in E, \\ 0 & \text{otherwise.} \end{cases} \quad (15.38)$$

For every $xy \in E$,

$$\langle v_x, v_y \rangle = \|v_x\| \|v_y\| = 1.$$

Hence, $v_x = v_y$, and v_x is independent of $x \in X$.

Shus $\langle v_x, v_y \rangle = 1$ for all $x, y \in X$.

We have $I + A = J$, (all 1's matrix), and $D = 1$.

(ii) Let m be the trivial measure. Then,

$$1 = \sum_{\theta \in \mathbb{R}} m(\theta) + \sum_{\theta \in \mathbb{R}} m(\theta)\theta \quad (15.39)$$

$$= \sum_{\theta \in \mathbb{R}} m(\theta)(\theta + 1) \quad (15.40)$$

$$= m(k)(k + 1) + \sum_{\theta \neq k} m(\theta)(\theta + 1) \quad (15.41)$$

$$\leq (k + 1)|X|^{-1}. \quad (15.42)$$

Note that $m(k) = |X|^{-1} \dim d_0 V = |X|^{-1}$.

So $k+1 \geq |X|$ or $k = |X| - 1$. Thus, $xy \in E$ for every $x, y \in X$, and $D = 1$. \square

Note. Lemma 15.1 does not require distance-regular assumption.

Chapter 16

Thin Modoles of a DRG

Wednesday, February 24, 1993

Let $\Gamma = (X, E)$ denote any graph of diameter D .

Definition 16.1. For all integer i , the i -th incidence matrix $A_i \in \text{Mat}_X(\mathbb{C})$ satisfies

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \quad (x, y \in X).$$

Observe,

$$A_0 = I \quad (\text{identity}) \quad (16.1)$$

$$A_1 = A \quad (\text{adjacency matrix}) \quad (16.2)$$

$$A_0 + A_1 + \cdots + A_D = J \quad (\text{all 1's matrix}). \quad (16.3)$$

In general, A_i may not belong to Bose-Mesner algebra.

Lemma 16.1. Assume $\Gamma = (X, E)$ is distance-regular with diameter $D \geq 1$ and intersection numbers c_i, a_i, b_i .

(i)

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1}, \quad (0 \leq i \leq D, A_{-1} = A_{D+1} = O).$$

(ii) $A_i = \frac{p_i(A)}{c_1 c_2 \cdots c_i}$, $(0 \leq i \leq D)$, where p_0, p_1, \dots, p_D are polynomials for the trivial module from Lemma 9.1.

(iii) A_0, A_1, \dots, A_D form a basis for Bose-Mesner algebra M .

(iv) For all distances h, i, j $(0 \leq i, j, h \leq D)$, and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the constant

$$p_{i,j}^h = |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|$$

depends only on h, i, j and not on x, y .

$$(v) \ E_0 = \frac{1}{|X|} J.$$

Proof.

(i) Pick $x \in X$. Apply each side to \hat{x} , we want to show that

$$AA_i\hat{x} = c_{i+1}A_{i+1}\hat{x} + a_iA_i\hat{x} + b_{i-1}A_{i-1}\hat{x}.$$

$$\text{LHS} = A \left(\sum_{y \in X, \partial(x, y) = i} \hat{y} \right) \tag{16.4}$$

$$= c_{i+1} \left(\sum_{z \in X, \partial(x, z) = i+1} \hat{z} \right) + a_i \left(\sum_{z \in X, \partial(x, z) = i} \hat{z} \right) + b_{i-1} \left(\sum_{z \in X, \partial(x, z) = i-1} \hat{z} \right) \tag{16.5}$$

$$= \text{RHS}. \tag{16.6}$$

(ii) Recall (Lemma 9.1)

$$Ap_i(A) = p_{i+1}(A) + a_i p_i(A) + b_{i-1} c_i p_{i-1}(A) \quad (0 \leq i \leq D).$$

Dividing by $c_1 c_2 \cdots c_i$, we have

$$A \frac{p_i(A)}{c_1 c_2 \cdots c_i} = c_{i+1} \frac{p_{i+1}(A)}{c_1 c_2 \cdots c_{i+1}} + a_i \frac{p_i(A)}{c_1 c_2 \cdots c_i} + b_{i-1} \frac{p_{i-1}(A)}{c_1 c_2 \cdots c_i}.$$

So, $A_i, p_i(A)/(c_1 c_2 \cdots c_i)$ satisfy the same recurrence.

Also boundary condition,

$$A_0 = p_0(A) = I.$$

Hence,

$$A_i = \frac{p_i(A)}{c_1 c_2 \cdots c_i} \quad (0 \leq i \leq D).$$

(iii) Since E_0, E_1, \dots, E_D form a basis for M , $\dim M = D + 1$.

Observe $A_0, A_1, \dots, A_D \in M$ by (ii), A_0, A_1, \dots, A_D are linearly independent, since p_0, p_1, \dots, p_D are linearly independent.

Thus, A_0, A_1, \dots, A_D form a basis for M .

(iv) A_0, A_1, \dots, A_D form a basis for an algebra M ,

$$A_i A_j = \sum_{\ell=0}^D p_{ij}^{\ell} A_{\ell} \quad \text{for some } p_{ij}^{\ell} \in \mathbb{C}. \quad (16.7)$$

Fix h ($0 \leq h \leq D$). Pick $x, y \in X$ with $\partial(x, y) = h$.

Compute x, y entry in (16.7),

$$(A_i A_j)_{xy} = \sum_{z \in X} (A_i)_{xz} (A_j)_{zy} \quad (16.8)$$

$$= \sum_{z \in X, \partial(x, z)=i, \partial(y, z)=j} 1 \cdot 1 \quad (16.9)$$

$$= |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|. \quad (16.10)$$

On the other hand,

$$\left(\sum_{\ell=0}^D p_{ij}^{\ell} A_{\ell} \right)_{xy} = p_{ij}^h (A_h)_{xy} = p_{ij}^h.$$

(v) $\frac{1}{|X|}J$ is the orthogonal projection onto $\text{Span}(\delta) = E_0 V$. Hence,

$$\frac{1}{|X|} = E_0.$$

This proves the assertions. □

Theorem 16.1. *Let $\Gamma = (X, E)$ be distance-regular with diameter $D \geq 2$ and intersection numbers c_i, a_i, b_i . Pick a vertex $x \in X$. Let W be a thin irreducible $T(x)$ -module with endpoint $r = 1$ and diameter d ($d = D - 2$ or $D - 1$). Set $\gamma_0 = a_0(W) + 1$.*

(i) *The scalars*

$$\gamma_i := \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_1(W) x_2(W) \cdots x_i(W)} \quad (0 \leq i \leq d) \quad (16.11)$$

$a_i(W), x_i(W)$ are algebraic integers in $\mathbb{Q}[\gamma_0]$. In particular, if $\gamma_0 \in \mathbb{Q}$, then $\gamma_i, a_i(W)$ and $x_i(W)$ are integers for all i .

(ii) *The numbers, $\gamma_i, a_i(W), x_i(W)$ can all be determined from γ_0 and the intersection numbers of Γ in order*

$$x_1(W), \gamma_1, a_1(W), x_2(W), \gamma_2, a_2(W), \dots$$

using (i),

$$x_i(W) = c_i b_i + \gamma_{i-1} (a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad (1 \leq i \leq D - 1), \quad (16.12)$$

and

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad (1 \leq i \leq D). \quad (16.13)$$

Note.

$$p_i = p_1^W + \gamma_{i-1}p_{i-1}^W - c_i(p_{i-1}^W + \gamma_{i-2}^W), \quad (\gamma_{-1} = -\gamma_{-2} = 0, \quad 0 \leq i \leq d+1).$$

Proof. Set

$$\tilde{A}_i = A_0 + A_1 + \cdots + A_i \quad (0 \leq i \leq D).$$

$$\text{Claim 1. } A\tilde{A}_i = c_{i+1}\tilde{A}_{i+1} + (a_i - c_{i+1} + c_i)\tilde{A}_i + b_i\tilde{A}_{i-1} \quad (0 \leq i \leq D-1).$$

Proof of Claim 1.

$$\text{LHS} = \sum_{j=0}^i AA_j \quad (16.14)$$

$$= \sum_{j=0}^i (c_{j+1}A_{j+1} + a_jA_j + b_{j-1}A_{j-1}) \quad (16.15)$$

$$= \sum_{j=0}^{i-1} A_j(c_j + a_j + b_j) + A_i(c_i + a_i) + A_{i+1}c_{i+1} \quad (16.16)$$

$$= k(A_0 + \cdots + A_{i-1}) + (a_i + c_i)A_i + c_{i+1}A_{i+1}. \quad (16.17)$$

$$\text{RHS} = c_{i+1}(A_0 + A_1 + \cdots + A_{i-1} + A_i + A_{i+1}) \quad (16.18)$$

$$+ (a_i - c_{i+1} + c_i)(A_0 + A_1 + \cdots + A_{i-1} + A_i) \quad (16.19)$$

$$+ b_i(A_0 + A_1 + \cdots + A_{i-1}) \quad (16.20)$$

$$= k(A_0 + \cdots + A_{i-1}) + A_i(a_i + c_i) + A_{i+1}c_{i+1}. \quad (16.21)$$

This proves Claim 1.

Now pick $0 \neq w \in E_1^*(x)W$ and let

$$w = \sum_{z \in X, \partial(x,z)=1} \alpha_z \hat{z}.$$

Pick y , where $\alpha_y \neq 0$.

For i ($0 \leq i \leq D$), define

$$B_i = \tilde{A}_i(\hat{x} - \hat{y}) \quad (16.22)$$

$$= \sum_{z \in X, \partial(x,z) \leq i} \hat{z} - \sum_{z \in X, \partial(y,z) \leq i} \hat{z} \quad (16.23)$$

$$= \sum_{z \in X, \partial(x,z)=i, \partial(y,z)=i+1} \hat{z} - \sum_{z \in X, \partial(y,z)=i+1, \partial(y,z)=i} \hat{z}. \quad (16.24)$$

Note that $B_D = O$, $B_0 = \hat{x} - \hat{y}$, and

$$\langle B_0, w_0 \rangle = -\alpha_y \neq 0.$$

From Claim 1,

$$AB_i = c_{i+1}B_{i+1} + (a_i - c_{i+1} + c_i)B_i + b_iB_{i-1} \quad (0 \leq i \leq D), \quad B_{-1} = O.$$

Let p_0^W, \dots, p_d^W denote polynomials for W from Lemma 9.1. So,

$$w_i = p_i^W(A)w \in E_{1+i}^*(x)W, \quad (0 \leq i \leq d).$$

Claim 2. $\langle w_i, B_j \rangle = 0$ if $j \notin \{i, i+1\}$, $(0 \leq i \leq d, 0 \leq j \leq D)$.

Proof of Claim 2.

$$w_i \in E_{1+i}^*W, \quad B_j \in E_j^*(x)W + E_{j+1}^*(x)W.$$



Vertical lines indicate possible non-orthogonality.

Compute

$$\langle Aw_i, B_j \rangle = \langle w_i, AB_j \rangle, \quad quad(0 \leq i \leq D, 0 \leq j \leq D-1). \quad (16.25)$$

$$\text{LHS} = \langle w_{i+1}, B_j \rangle + a_i(W)\langle w_i, B_j \rangle + x_i(W)\langle w_{i-1}, B_j \rangle \quad (16.26)$$

$$\text{RHD} = b_j\langle w_i, B_{j-1} \rangle + (a_j - c_{j+1} + c_j)\langle w_i, B_j \rangle + c_{j+1}\langle w_i, B_{j+1} \rangle. \quad (16.27)$$

Evaluate for $i = j-2, j-1, j, j+1$.

Set $i = j-2$.



Then (16.25) becomes

$$\langle w_{j-1}, B_j \rangle = b_j \langle w_{j-2}, B_{j-1} \rangle \quad (2 \leq j \leq D-1).$$

By induction,

$$\langle w_{j-1}, B_j \rangle = b_2 b_3 \cdots b_j \langle w_0, B_1 \rangle \quad (1 \leq j \leq D-1).$$

Define

$$\gamma_0 = \frac{\langle w_0, B_1 \rangle}{\langle w_0, B_0 \rangle}.$$

(We will show $\gamma_0 = 1 + a_0(W)$.)

Then,

$$\langle w_{j-1}, B_j \rangle = b_2 b_3 \cdots b_j \gamma_0 \langle w_0, B_0 \rangle. \quad (16.28)$$

Set $i = j + 1$. Then (16.25) becomes

$$x_{j+1}(W) \langle w_j, B_j \rangle = c_{j+1} \langle w_0, B_{j+1} \rangle \quad (0 \leq j \leq d).$$

Hence,

$$\langle w_j, B_j \rangle = \frac{x_1(W) \cdots x_j(W)}{c_1 c_2 \cdots c_j} \langle w_0, B_0 \rangle \quad (0 \leq j \leq d). \quad (16.29)$$

Set $i = j - 1$. Then (16.25) becomes

$$\langle w_j, B_j \rangle + a_{j-1}(W) \langle w_{j-1}, B_j \rangle = (a_j - c_{j+1} + c_j) \langle w_{j-1}, B_j \rangle + b_j \langle w_{j-1}, B_{j-1} \rangle.$$

Evaluate this using (16.28) and (16.29). ($\langle w_0, B_0 \rangle \neq 0$). Then we have

$$\frac{w_1(W) \cdots x_j(W)}{c_1 \cdots c_j} + (a_{j-1}(W) - a_j + c_{j+1} - c_j) b_2 \cdots b_j \gamma_0 = b_j \frac{x_1(W) \cdots x_{j-1}(W)}{c_1 \cdots c_{j-1}},$$

$$\left(\gamma_i := \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_0(W) x_2(W) \cdots x_i(W)} \right).$$

$$\frac{x_j(W)}{c_j} = b_j + \frac{c_1 c_3 \cdots c_{j-1} b_2 b_3 \cdots b_j \gamma_0}{x_0(W) x_2(W) \cdots x_{j-1}(W)} (a_j + c_j - c_{j+1} - a_{j-1}).$$

So,

$$x_j(W) = c_j b_j + \gamma_{j-1} (a_j + c_j - c_{j+1} - a_{j-1}(W)).$$

This proves (16.12).

Set $i = j$. Then (16.25) becomes

$$a_j(W) \langle w_j, B_j \rangle + x_j(W) \langle w_{j-1}, B_j \rangle = (a_j - c_{j+1} + c_j) \langle w_j, B_j \rangle + c_{j+1} \langle w_j, B_{j+1} \rangle.$$

$$(a_j(W) - (a_j - c_{j+1} + c_j)) \frac{x_1(W) \cdots x_j(W)}{c_1 \cdots c_j} x_j(W) b_2 \cdots b_j \gamma_0 - c_{j+1} b_2 \cdots b_{j+1} \gamma_0 = 0.$$

Thus,

$$a_j(W) - (a_j - c_{j+1} + c_j) + \frac{c_1 \cdots c_j b_2 \cdots b_j \gamma_0}{x_1(W) \cdots x_{j-1}(W)} - \frac{c_1 \cdots c_j c_{j+1} b_2 \cdots b_{j+1} \gamma_0}{x_1(W) \cdots x_j(W)} = 0,$$

or

$$a_j(W) = a_j + c_j - c_{j+1} - \gamma_{j-1} + \gamma_j.$$

This proves (16.13).

Also by setting $i = j = 0$, we have

$$a_0(W) \langle w_0, B_0 \rangle = (a_0 - c_1 + c_0) \langle w_0, B_0 \rangle + c_1 \langle w_0, B_1 \rangle \quad (16.30)$$

$$= -\langle w_0, B_0 \rangle + \gamma_0 \langle w_0, B_0 \rangle. \quad (16.31)$$

Hence,

$$\gamma_0 = 1 + a_0(W).$$

Both $a_i(W)$ and $x_i(W)$ are algebraic integers, since they are eigenvalues of matrices with integer entries, namely,

$$E_{i+1}^*(x) A E_{i+1}^*(x) \quad \text{and} \quad E_i^*(x) A E_{i+1}^*(x) A E_i^*(x).$$

Also $\gamma_0 = 1 + a_0(W)$ is an algebraic integer, and $\gamma_i - \gamma_{i-1}$ is an algebraic integer by (16.12).

Hence, γ_i is an algebraic integer by induction.

This completes the proof of Theorem 16.1. \square

Example 16.1 ($D=2$).

$$D = 2 \Leftrightarrow \text{strongly regular.}$$

Free parameters are k, a_1, c_2 . Let W be an irreducible module of endpoint 1. The matrix representation of $A|_W$ is

$$\begin{pmatrix} a_0(W) & x_1(W) \\ 1 & a_1(W) \end{pmatrix}.$$

$a_0(W)$: free.

$$x_1(W) = c_1 b_1 + (a_0(W) + 1)(a_1 + c_1 - c_2 - a_0(W)) \quad (16.32)$$

$$= k - a_1 - 1 + a_1 a_0(W) + a_0(W) - c_2 a_0(W) - a_0(W)^2 + a_1 + a - c_2 - a_0(W) \quad (16.33)$$

$$= a_1 a_0(W) - c_2 a_0(W) + k - c_2 - a_0(W)^2, \quad (16.34)$$

$$\gamma_1 = 0, \quad (16.35)$$

$$a_1(W) = -(a_0(W) + 1) + a_1 + c_1 - c_2 \quad (16.36)$$

$$= -a_0(W) + a_1 - c_2. \quad (16.37)$$

Then the matrix has eigenvalues θ, θ_1 . There is one feasible condition: $a_0(W)$ is an algebraic integer.

Example 16.2 (D=3). Free parameters c_2, c_3, k, a_1, a_2 . The matrix representation becomes

$$A|_W = \begin{pmatrix} a_0(W) & x_1(W) & 0 \\ 1 & a_1(W) & x_2(W) \\ 0 & 1 & a_2(W) \end{pmatrix}.$$

Here, $a_0(W)$ is free ($= \gamma - 1$)

$$x_1(W) = k - 1 - a_1 + \gamma_0(a_1 + 1 - c_2 - a_0(W)) \quad (16.38)$$

$$= \gamma_0(a_1 - c_2 - a_0(W)) + k - a_1 + a_0(W). \quad (16.39)$$

Set

$$\gamma_1(W) = \frac{c_2 b_2 \gamma_0}{x_1(W)}.$$

$$a_1(W) = \gamma_1 - \gamma_0 + a_1 + 1 - c_2 \quad (16.40)$$

$$x_2(W) = \gamma_1(a_2 - c_3 - a_1(W)) + c_2(\gamma_0 + b_1 - a_2 + a_1(W)) \quad (16.41)$$

$$a_2(W) = -\gamma_1 + a_2 + c_2 - c_2. \quad (16.42)$$

The matrix has eigenvalues, $\theta, \theta_2, \theta_3$.

There are two feasibility conditions; γ_0, γ_1 are algebraic integers.

For arbitrary D , there are $D - 1$ feasibility conditions; $\gamma_0, \gamma_1, \dots, \gamma_{D-1}$ are algebraic integers.

Lemma 16.2. *With the notation of Theorem 16.1, suppose*

$$f_W = \frac{k - \lambda}{k} \quad (\text{so, } a_0(W) = -1).$$

Then,

$$a_i(W) = a_i + c_i - c_{i+1} \quad (0 \leq i \leq D - 1) \quad (16.43)$$

$$x_i(W) = b_i c_i \quad (1 \leq i \leq D - 1) \quad (16.44)$$

$$\gamma_i(W) = 0. \quad (16.45)$$

Proof. Since $\gamma_0 = a_0(W) = 1$, $\gamma_i = 0$. □

Chapter 17

Association Schemes

Monday, March 1, 1993

Review

Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 2$. Pick a vertex $x \in X$.

Let W be a thin irreducible $T(x)$ -module with endpoint $r = 1$, diameter $d = D - 1$ or $D - 2$, and $r_0 = a(W) + 1$.

Show

$$\gamma_i = \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_1(W) \cdots x_i(W)},$$

$a_i(W)$ and $x_i(W)$ are all algebraic integers in $\mathbb{Q}[\gamma_0]$, where

$$x_i(W) = c_i b_i + \gamma_{i-1}(a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad (1 \leq i \leq d) \quad (17.1)$$

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad (1 \leq i \leq d) \quad (17.2)$$

Certainly, $x_i(W)$, γ_i , and $a_i(W)$ are in $\mathbb{Q}[\gamma_0]$ by the above lines and so on.

$$\gamma_0 \rightarrow a_0(W) \rightarrow x_1(W) \rightarrow \gamma_1 \rightarrow a_1(W) \rightarrow x_1(W) \rightarrow \cdots.$$

Recall some $B \in \text{Mat}_n(\mathbb{C})$ is integral whenever

$$B \in \text{Mat}_n(\mathbb{Z}).$$

In this case, the characteristic polynomial

$$\det(\lambda I - B) = \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_0, \quad \text{some } \alpha_0, \dots, \alpha_{n-1} \in \mathbb{Z}.$$

Hence, eigenvalues of B are algebraic integers. But $a_i(W)$ is an eigenvalue of an integral matrices,

$$B = E_{i+1}^*(x) A E_{i+1}^*(x).$$

Hence, $a_i(W)$ is an algebraic integer.

Also, $x_i(W)$ is an eigenvalue of an integral matrix

$$B = E_i^*(x)AE_{i+1}^*(x)AE_i^*(x).$$

So $x_i(W)$ is an algebraic integer.

$$\gamma_i - \gamma_{i-1} = a_i(W) - a_i - c_i + c_{i+1}$$

is an algebraic integer.

Since $\gamma_0 = a_0(W) + 1$ is an algebraic integer, we find γ is an algebraic integer for all i .

Definition 17.1. A (commutative) association scheme is a configuration $Y = (X, \{R_i\}_{0 \leq i \leq D})$, where X is a finite nonempty set (of vertices), R_0, R_1, \dots, R_D are nonempty subsets of $X \times X$ such that

- (i) $R_0 = \{(x, x) \mid x \in X\}$,
- (ii) $R_0 \cup \dots \cup R_D = X \times X$ (disjoint union),
- (iii) for every i , $R_i^\top = \{(y, x) \mid xy \in R\} = R_{i'}$ some $i' \in \{0, 1, \dots, D\}$,
- (iv) for every h, i, j ($0 \leq h, i, j \leq D$), and every $x, y \in X$ such that $(x, y) \in R_h$,

$$p_{ij}^h = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|$$

depends only on h, i, j and not on x, y ; and

- (v) $p_{ij}^h = p_{ji}^h$ for all h, i, j .

If $i' = i$ for all i , we say Y is symmetric. We call D the class of scheme and R_i , the i th relation of Y . We say vertices $x, y \in X$ are i -related, or ‘at distance i ’, whenever $(x, y) \in R_i$.

We always assume that a ‘scheme’ is a commutative association scheme.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be an association scheme.

Definition 17.2. The i -th association matrix $A_i \in \text{Mat}_X(\mathbb{C})$

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i, \end{cases} \quad (x, y \in X, 0 \leq i \leq D) \quad (17.3)$$

Then,

$$(i') A_0 = I.$$

$$(ii') A_0 + A_1 + \dots + A_D = J \text{ (= all 1's matrix).}$$

$$(iii') \quad A_i^\top = A_{i'} \quad (0 \leq i \leq D).$$

$$(iv') \quad A_i A_j = \sum_{h=0}^D p_{ij}^h A_h \quad (0 \leq i, j \leq D).$$

$$(v') \quad A_i A_j = A_j A_i.$$

$M := \text{Span}_{\mathbb{C}}(A_0, \dots, A_D)$ (Bose-Mesner algebra of Y) is a commutative \mathbb{C} -algebra of dimension $D + 1$.

Observe:

$$Y \text{ is symmetric} \leftrightarrow A_i^\top = A_i \text{ for all } i \leftrightarrow M \text{ is symmetric.}$$

Example 17.1. Let $\Gamma = (X, E)$ be distance-regular of diameter D . Set

$$R_i = \{(x, y) \mid \partial(x, y) = i\} \quad (0 \leq i \leq D). \quad (17.4)$$

Then,

$$Y = (X, \{R_i\}_{0 \leq i \leq D})$$

is a symmetric scheme.

$$i\text{-th association matrix} = i\text{-th distance matrix} \quad \text{for all } i.$$

Example 17.2. Suppose a group G acts transitively on a set X . Assume G is generously transitive, i.e.,

$$\text{for all } x, y \in X, \text{ there exists } g \in G \text{ such that } gx = y, gy = x.$$

Then G acts on $X \times X$ by rule;

$$g(x, y) = (gx, gy), \quad \text{for all } g \in G, \text{ and for all } x, y \in X.$$

Let R_0, \dots, R_D denote orbits of G on $X \times X$.

Observe that $R_i^\top = R_i$ for all i by generous transitivity, and

$$Y = (X, \{R_i\}_{0 \leq i \leq D})$$

is a symmetric scheme.

Exercise 17.1. In Example 17.2, Bose-Mesner algebra

$$M = \{B \in \text{Mat}_X(\mathbb{C}) \mid Bg = gB, \text{ for all } g \in G\} \quad (17.5)$$

$$= \text{the commuting algebra of } G \text{ on } X. \quad (17.6)$$

Here, we view each $g \in G$ as a permutation matrix in $\text{Mat}_X(\mathbb{C})$ satisfying

$$g\hat{x} = \widehat{gx}, \quad \text{for all } x \in G.$$

Example 17.3. Let G be any finite group. G acts on $X = G$ by conjugation.

$$G \times X \rightarrow X, \quad (g, x) \mapsto gxg^{-1}.$$

Let C_0, C_1, \dots, C_D denote orbits (i.e., conjugacy classes), and let $C_0 = \{1_G\}$. Claim that $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is a commutative scheme (not symmetric in general).

(i) $R_0 = \{xx \mid x \in X\}$ as $C_0 = \{1_G\}$.

(ii) R_0, \dots, R_D is a partition of $X \times X$ since C_0, \dots, C_D is a partition of $X = G$.

(iii) $R_i^\top = R_{i'}$, where $C_{i'} = \{g^{-1} \mid g \in C_i\}$.

(iv) Set $H = G \oplus G$, the direct sum. Then H acts on $X = G$:

$$\text{for all } h = (g, gz), \text{ for all } x \in X, \quad h(x) = gx(gx)^{-1} = gxz^{-1}g^{-1}.$$

$$R_i = \{(x, y) \mid x^{-1}y \in C_i\}, \quad h_i \in C_i, \quad x^{-1}y = gh_i g^{-1}.$$

$$(x, y) = (x, xgh_i g^{-1}) \tag{17.7}$$

$$= (xgg^{-1}, xgh_i g^{-1}) \tag{17.8}$$

$$= (xg, g)(1, h_i). \tag{17.9}$$

So, R_0, \dots, R_D are the orbits of H on $X \times X$.

(v) $p_{ij}^h = p_{ji}^h$?

Fix i, j, h and $x, y \in X$ with $(x, y) \in R_h$. Set

$$S = \{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\} \tag{17.10}$$

$$T = \{z \in X \mid (x, z) \in R_j, (z, y) \in R_i\}. \tag{17.11}$$

Show $|S| = |T|$.

For all $z \in S$, set $\hat{z} = xz^{-1}y$.

Observe, $\hat{z} \in T$.

$$x^{-1}z \in C_i x^{-1}\hat{z} = x^{-1}xz^{-1}y \in C_j \tag{17.12}$$

$$z^{-1}y \in C_j \hat{z}^{-1}y = y^{-1}zx^{-1}x^{-1}y = y^{-1}x(x^{-1}z)x^{-1}y \in C_i. \tag{17.13}$$

Observe

$$S \rightarrow T \quad (z \mapsto z^{-1}) \quad \text{is one-to-one and onto.}$$

Chapter 18

Polynomial Schemes

Wednesday, March 3, 1993

Lemma 18.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote the symmetric scheme with associated matrices A_0, A_1, \dots, A_D . Then the following are equivalent.*

(i) *The graph $\Gamma = (X, R_1)$ is distance-regular, and R_0, \dots, R_D are labelled so that*

$$R_i = \{xy \mid \partial(x, y) = i\}.$$

(ii) *There exists $f_i \in \mathbb{C}[\lambda]$, $\deg f_i = i$ such that $f_i(A_1) = A_i$ for all i with $0 \leq i \leq D$.*

(iii) *The parameter p_{ij}^h*

$$\begin{cases} = 0 & \text{if one of } h, i, j \text{ is larger than the sum of the other two} \\ \neq 0 & \text{if one of } h, i, j \text{ is equal to the sum of the other two.} \end{cases}$$

Proof.

(i) \Rightarrow (ii): Lemma 16.1.

(ii) \Rightarrow (iii): Define

$$k_i \equiv p_{ii}^0 = |\{z \mid z \text{ in } X, \partial(x, z) = i \text{ } ((x, z) \in R_i)\}|$$

for any $x \in X$. Then $k_i \neq 0$ ($0 \leq i \leq D$), $k_0 = 1$.

(By symmetry, $(x, y) \in R_i$ if and only if $(y, x) \in R_i$.)

Claim.

$$k_h p_{ij}^h = k_i p_{hj}^i = k_j p_{ih}^j \quad (18.1)$$

$$= |X|^{-1} |\{xyz \in X^3 \mid \partial(x, y) = h, \partial(x, z) = i, \partial(y, z) = j\}|. \quad (18.2)$$

Pf. The number of $xyz \in X^3$, $\partial(x, y) = h, \partial(x, z) = i, \partial(y, z) = j$ is equal to

$$|X| k_h p_{ij}^h = |X| k_i p_{hj}^i = k_j p_{ih}^j.$$

In particular,

$$p_{ij}^h = 0 \leftrightarrow p_{hj}^i = 0 \leftrightarrow p_{ih}^j = 0.$$

Hence, it suffices to show

$$\begin{cases} p_{ij}^h = 0 & \text{if } h > i + j \\ p_{ij}^h \neq 0 & \text{if } h = i + j. \end{cases}$$

Fix i, j . Without loss of generality, we may assume that $i + j \leq D$ as trivial otherwise.

$$f_i(A) f_j(A) = A_i A_j = \sum_{\ell=0}^D p_{ij}^{\ell} A_{\ell} = \sum_{\ell=0}^D p_{ij}^{\ell} f_{\ell}(A).$$

$$i + j = \deg \text{LHS} \quad (18.3)$$

$$= \deg \text{RHS} \quad (18.4)$$

$$= \max\{\ell \mid p_{ij}^{\ell} \neq 0\}. \quad (18.5)$$

(iii) \Rightarrow (i)

Let $A = A_1$, and consider a graph Γ with adjacency matrix A .

$$A A_j = \sum_h p_{1j}^h A_h \quad (18.6)$$

$$= p_{1j}^{j+1} A_{j+1} + p_{1j}^j A_j + p_{1j}^{j-1} A_{j-1}. \quad (18.7)$$

Then, $p_{1j}^{j+1} \neq 0 \neq p_{1j}^{j-1}$.

Fix a vertex $x \in X$, and set $R_i(x) = \{y \mid (x, y) \in R_i\}$.

Then each $y \in R_i(x)$ is adjacent in Γ to exactly

$$p_{1,i+1}^i \neq 0 \quad \text{vertices in } R_i(x), \quad (18.8)$$

$$p_{1i}^i \quad \text{vertices in } R_{i+1}(x), \quad (18.9)$$

$$p_{1,i-1}^i \neq 0 \quad \text{vertices in } R_{i-1}(x). \quad (18.10)$$

Hence, by induction,

$$R_i(x) = \{y \mid \partial(x, y) = i \text{ in } \Gamma\} \quad (0 \leq i \leq D), \quad (18.11)$$

and Γ is distance regular.

□

Chapter 19

Commutative Association Schemes

Friday, March 5, 1993

Lemma 19.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme with Bose-Mesner algebra M .*

Then there exists a basis E_0, E_1, \dots, E_D for M such that

- (i) $E_0 = |X|^{-1}J$.
- (ii) $E_i E_j = E_j E_i = \delta_{ij} E_i \quad (0 \leq i, j \leq D)$.
- (iii) $E_0 + E_1 + \dots + E_D = I$.
- (iv) $E_i^\top = \overline{E_i} = E_{\hat{i}}$ for some $\hat{i} \in \{0, 1, \dots, D\}$.

Proof. M acts on Hermitean space $V = \mathbb{C}^n$ ($n = |X|$).

If W is an M -module, so is W^\perp .

Each irreducible M -module is 1 dimensional by commutativity of M . So V is orthognal direct sum of 1-dimensional M -modules.

Let v_1, \dots, v_n be an orthonormal basis for V consisiting of eigenvectors for all $m \in M$.

Set $P \in \text{Mat}_X(\mathbb{C})$ so that the i -th column of P is equal to v_i . So,

$$\bar{P}^\top P = I = P \bar{P}^\top = \bar{P} P^\top,$$

and P is unitary.

Also, for all $m \in M$,

$$P^{-1}mP = \text{diagonal} \quad (19.1)$$

$$= \text{diag}(\theta_1(m), \dots, \theta_n(m)). \quad (19.2)$$

for some functions

$$\theta_i : M \longrightarrow \mathbb{C}.$$

Observe: each $\theta = \theta_i$ is a character of M , i.e.,

$$\theta : M \longrightarrow \mathbb{C}$$

is a \mathbb{C} -algebra homomorphism.

Observe: the $\theta_1, \dots, \theta_n$ are not all distinct.

Let $\sigma_0, \dots, \sigma_r$ denote distinct elements of

$$\theta_1, \dots, \theta_n.$$

Say σ_i appears m_i times. Without loss of generality, we may assume that

$$P^{-1}mP = \begin{pmatrix} \sigma_0(m)I_{m_0} & O & O & O \\ O & \sigma_1(m)I_{m_1} & O & O \\ O & O & \ddots & O \\ O & O & O & \sigma_r(m)I_{m_r} \end{pmatrix}.$$

Set

$$E_i = P \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix} P^{-1},$$

where I_{m_i} is in the i -th block.

Then,

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq r),$$

$$E_0 + E_1 + \dots + E_r = I.$$

Hence for all $m \in M$,

$$m = \sum_{i=0}^r \sigma_i(m) E_i \in \text{Span}(E_0, \dots, E_r).$$

So,

$$M \subseteq \text{Span}(E_0, \dots, E_r).$$

Since E_0, \dots, E_r are linearly independent, $r \geq D$.

Show $E_i \in M$.

Claim 1. For all distinct i, j ($0 \leq i, j \leq D$), there exists $m \in M$ such that $\sigma_i(m) \neq 0$, $\sigma_j(m) = 0$.

Pf of Claim 1. $\sigma_i \neq \sigma_j$ implies that there exists $m' \in M$ such that $\sigma_i(m') \neq \sigma_j(m')$.

Set $m = m' - \sigma_j(m')I$. Then,

$$\sigma_j(m)\sigma_j(m') - \sigma_j(m') = 0, \quad (19.3)$$

$$\sigma_i(m)\sigma_i(m') - \sigma_j(m') \neq 0. \quad (19.4)$$

Claim 2. $E_i \in M$ ($0 \leq i \leq D$).

Pf of Claim 2. Fix a vertex $x \in X$. For all $j \neq i$, there exists $m_j \in M$ such that $\sigma_i(m_j) \neq 0$, $\sigma_j(m_j) = 0$, $i \neq j$. Observe

$$s = \sigma_i \left(\prod_{\ell \neq i} m_\ell \right) \neq 0.$$

Set

$$m^* = \sigma_i \left(\prod_{\ell \neq i} m_\ell \right) s^{-1}.$$

Observe

$$\sigma_i(m^*) = 1, \quad \sigma_j(m^*) = 0, \quad \text{for all } j \neq i \quad (0 \leq j \leq D).$$

So

$$P^{-1}m^*P = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}.$$

We have

$$E_i = m^* \in M.$$

Now $r = D$, $M = \text{Span}(E_0, \dots, E_D)$ and E_0, \dots, E_D is a basis for M .

Observe

$$P^{-1}E_iP = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}$$

implies

$$P^{-1}\overline{E_i}^\top P = \overline{P}^\top \overline{E_i}^\top \overline{P^{-1}}^\top = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}^\top = P^{-1}E_iP.$$

Hence,

$$\overline{E_i}^\top = E_i.$$

$E_0^\top, \dots, E_D^\top$ are nonzero matrices satisfying

$$E_i^\top E_j^\top = \delta_{ij} E_i^\top,$$

$$E_0^\top + E_1^\top + \cdots + E_D^\top = I.$$

Each E_i^\top is a linear combination of E_0, \dots, E_D with coefficientss that are 0 or 1, and for no two E_i 's are coefficients of any E_j both 1's.

So, $E_0^\top, \dots, E_D^\top$ is a permutation of E_0, \dots, E_D .

Observe $J = A_0 + \cdots + A_D \in M$.

The matrix $|X|^{-1}J$ is an idempotent of rank 1.

So, without loss of generality we may assume that

$$E_0 = \frac{1}{|X|}J.$$

We have the assertions. □

Define entry-wise product \circ on $\text{Mat}_X(\mathbb{C})$.

$$A_i \circ A_j = \delta_{ij}A_i.$$

So, M is closed under \circ .

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h.$$

The numbers q_{ij}^h is called Krein parameters of Y .

Claim. $q_{ij}^h \in \mathbb{R}$.

Pf.

$$\frac{1}{|X|} \sum_{h=0}^D \overline{q_{ij}^h} E_h = \frac{1}{|X|} \sum_{h=0}^D \overline{q_{ij}^h} \overline{E_h}^\top \quad (19.5)$$

$$= (\overline{E_i} \circ \overline{E_j})^\top \quad (19.6)$$

$$= E_i \circ E_j \quad (19.7)$$

$$= \frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h. \quad (19.8)$$

Hence, $q_{ij}^h = \overline{q_{ij}^h}$.

Observe $A_0, \dots, A_D, E_0, \dots, E_D$ are bases for M . Hence, there exist $p_i(j), q_i(j) \in \mathbb{C}$ such that

$$A_i = \sum_{j=0}^D p_i(j) E_j \quad (19.9)$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j. \quad (19.10)$$

Taking transpose and conjugate we find,

$$\overline{p_i(j)} = p_i(j) = p_{i'}(\hat{j}) \quad (0 \leq i, j \leq D) \quad (19.11)$$

$$\overline{q_i(j)} = q_i(j) = q_{\hat{i}}(j') \quad (0 \leq i, j \leq D). \quad (19.12)$$

Fix a vertex $x \in X$. Define

$$E_i^* \equiv E_i^*(x) \in \text{Mat}_X(\mathbb{C})$$

to be a diagonal matrix such that

$$(E_i^*)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases} \quad (0 \leq i \leq D, y \in X.)$$

Then,

$$\begin{aligned} E_i^* E_j^* &= \delta_{ij} E_i^*, \\ E_0^* + \cdots + E_D^* &= I, \\ (E_i^*)^\top &= \overline{E_i^*} = E_i^*. \end{aligned}$$

Definition 19.1. Dual Bose-Mesner algebra $M^* \equiv M^*(x)$ with respect to x is

$$\text{Span}(E_0^*, \dots, E_D^*).$$

Define dual associate matrices A_0^*, \dots, A_D^* . Indeed $A_i^* \equiv A_i^*(x) \in \text{Mat}_X(\mathbb{C})$ is a diagonal matrix with

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad (y \in X).$$

A_i^* is a diagonal matrix having the row x of E_i^* on the diagonal.

Observe

$$A_i^* = \sum_{j=0}^D q_i(j) E_j^* \quad \left(E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j \right) \quad (19.13)$$

$$E_i^* = \frac{1}{|X|} \sum_{j=0}^D p_i(j) A_j^* \quad \left(A_i = \sum_{j=0}^D p_i(j) E_j \right). \quad (19.14)$$

So, A_0^*, \dots, A_D^* form a basis for M^* .

Also,

$$A_i^* E_j^* = q_i(j) E_j^*.$$

$$\left(A_i^* E_j^* = \sum_{h=0}^D q_i(h) E_h^* E_j^* = q_i(j) E_j^* \right)$$

So, $q_i(j)$ are dual eigenvalues of A_i^* .

Observe,

$$A_0^* = I, \quad A_0^* + \cdots + A_D^* = |X|E_0^*, \quad \overline{A_i^*} = A_i^*,$$

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* \quad (0 \leq i, j \leq D).$$

Remark. Proof.

$$(A_0^*)_{yy} = |X|(E_0)_{xy} = (J)_{xy} = 1.$$

$$A_0^* + \cdots + A_D^* = \sum_{i=0}^D \sum_{j=0}^D q_i(j) E_j^* = |X|E_0^*.$$

Note that

$$I = E_0 + \cdots + E_D = \frac{1}{|X|} \sum_{i=0}^D \sum_{j=0}^D q_i(j) A_j.$$

$$\sum_{i=0}^D q_i(j) = \delta_{j0}|X|.$$

$$\overline{A_i^*} = \sum_{j=0}^D \overline{q_i(j) E_j^*} = \sum_{j=0}^D q_i(j) E_j^* = A_i^*.$$

$$(A_i^* A_j^*)_{yy} = |X|^2 (E_i)_{xy} (E_j)_{xy} \tag{19.15}$$

$$= |X|^2 (E_i \circ E_j)_{xy} \tag{19.16}$$

$$= |X| \sum_{h=0}^D q_{ij}^h (E_h)_{xy} \tag{19.17}$$

$$= \sum_{h=0}^D q_{ij}^h (A_h^*)_{yy}. \tag{19.18}$$

The following statements will be proved after a couple of lemmas in the next lecture.

Lemma. Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme. Fix a vertex $x \in X$, and set $E^* \equiv E_i^*(x)$ and $A_i^* \equiv A^*(x)$. Then the following hold.

(i) $E_i^* A_j E_k^* = O$ if and only if $p_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

(ii) $E_i A_j^* E_k = O$ if and only if $q_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

Chapter 20

Vanishing Conditions

Monday, March 15, 1993 (Monday after Spring break)

Lemma 20.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme.*

(i) $p_0(i) = 1$.

(ii) $p_i(0) = k_i$, where

$$k_i = p_{ii'}^0 = |\{y \in X \mid (x, y) \in R_i\}|.$$

(iii) $q_0(i) = 1$.

(iv) $q_i(0) = m_i$, where

$$m_i = \text{rank} E_i.$$

Proof.

(i) Since $A_0 = I$ and

$$A_0 = p_0(0)E_0 + p_0(1)E_1 + \cdots + p_0(D)E_D \quad (20.1)$$

$$I = E_0 + E_1 + \cdots + E_D, \quad (20.2)$$

$p_0(i) = 1$ for all i .

(ii) Since

$$A_i = p_i(0)E_0 + p_i(1)E_1 + \cdots + p_i(D)E_D,$$

$A_i E_0 = p_i(0) E_0$, and

$$k_i J = A_i J = p_i(0) J$$

as there are k_i 1's in each row of A_i , we have $k_i = p_i(0)$.

(iii) Since $E_0 = |X|^{-1} J$ and

$$E_0 = |X|^{-1} (q_0(0) A_0 + q_0(1) A_1 + \cdots + q_0(D) A_D) \quad (20.3)$$

$$|X|^{-1} J = |X|^{-1} (A_0 + A_1 + \cdots + A_D), \quad (20.4)$$

$q_0(i) = 1$ for all i .

(iv) $E_i = |X|^{-1} (q_i(0) A_0 + q_i(1) A_1 + \cdots + q_i(D) A_D)$, $E_i^2 = E_i$, and E_i is similar to a matrix

$$\begin{pmatrix} I_{m_i} & O \\ O & O \end{pmatrix}.$$

So,

$$m_i = \text{rank} E_i = \text{trace} E_i = \sum_{x \in X} (E_i)_{xx} = |X| |X|^{-1} q_i(0) = q_i(0).$$

Note that as

$$E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j \rightarrow (E_i)_{xx} = \frac{1}{|X|} q_i(0) (A_0)_{xx}.$$

Hence, we have all formulas.

□

Lemma 20.2. *With the above notation*

$$(i) \ p_{ij}^h = p_{j'i'}^{h'}.$$

$$(ii) \ k_h p_{ij}^h = k_j p_{i'h}^j = k_{hj'}^i.$$

$$(iii) \ q_{ij}^h = q_{ji}^{\hat{h}}.$$

$$(iv) \ m_h q_{ij}^h = m_j q_{ih}^j = m_i q_{hj}^i.$$

Proof.

(i) We have

$$\sum_{h=0}^D p_{ij}^h A_{h'} \left(\sum_{h=0}^D p_{ij}^h A_h \right)^\top \quad (20.5)$$

$$= (A_i A_j)^\top \quad (20.6)$$

$$= A_j^\top A_i^\top \quad (20.7)$$

$$= A_{j'} A_{i'} \quad (20.8)$$

$$= \sum_{h=0}^D p_{j'i'}^{h'} A_h'. \quad (20.9)$$

(ii) Count the following number,

$$|\{xyz \in X^3 \mid (x, y) \in R_h, (x, z) \in R_i, (z, y) \in R_j\}| \quad (20.10)$$

$$= |X| k_h p_{ij}^h = |X| k_j p_{i'h}^j = |X| k_{hj'}^i. \quad (20.11)$$

(iii)

$$\frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_{\hat{h}} = \left(\frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h \right)^\top \quad (20.12)$$

$$= (E_i \circ E_j)^\top \quad (20.13)$$

$$= E_j^\top \circ E_i^\top \quad (20.14)$$

$$= E_{\hat{j}} E_{\hat{i}} \quad (20.15)$$

$$= \frac{1}{|X|} \sum_{h=0}^D q_{\hat{j}\hat{i}}^{\hat{h}} E_{\hat{h}}. \quad (20.16)$$

(iv) Let $\tau(B)$ denote the sum of the entries in the matrix B .

Observe: $\tau(B \circ C) = \text{trace}(BC^\top)$.

Observe

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \tau((E_i \circ E_j \circ E_{\hat{k}})^\top) = \tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}).$$

Compute each one.

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \text{trace}((E_i \circ E_j)E_k) = \text{trace} \left(\left(\frac{1}{|X|} \sum_h q_{ij}^h E_h \right) E_k \right) \quad (20.17)$$

$$= \text{trace} \left(\frac{1}{|X|} q_{ij}^k E_k \right) = \frac{1}{|X|} m_k q_{ij}^k, \quad (20.18)$$

$$\tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \text{trace}((E_{\hat{i}} \circ E_k)E_{\hat{j}}) = \text{trace} \left(\left(\frac{1}{|X|} \sum_h q_{ik}^h E_h \right) E_{\hat{j}} \right) \quad (20.19)$$

$$= \text{trace} \left(\frac{1}{|X|} q_{ik}^j E_k \right) = \frac{1}{|X|} m_j q_{ik}^j, \quad (20.20)$$

$$\tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}) = \text{trace}((E_k \circ E_{\hat{j}})E_i) = \text{trace} \left(\left(\frac{1}{|X|} \sum_h q_{k\hat{j}}^h E_h \right) E_i \right) \quad (20.21)$$

$$= \text{trace} \left(\frac{1}{|X|} q_{k\hat{j}}^i E_i \right) = \frac{1}{|X|} m_i q_{k\hat{j}}^i. \quad (20.22)$$

Hence, we have (iv). □

Lemma 20.3. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme. Fix a vertex $x \in X$, and set $E^* \equiv E_i^*(x)$ and $A_i^* \equiv A^*(x)$. Then the following hold.*

(i) $E_i^* A_j E_k^* = O$ if and only if $p_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

(ii) $E_i A_j^* E_k = O$ if and only if $q_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

Proof.

(i) Partition rows and columns by $R_0(x), R_1(x), \dots, R_D(x)$. Then,

$$E_i^*(x) A_j E_h^*(x)$$

is the (i, h) block of A_j .

Hence this submatrix is zero if and only if there exists no $y, z \in X$ such that $(x, y) \in R_i$, $(x, z) \in R_h$ and $(y, z) \in R_j$. This is exactly when $p_{ij}^h = 0$.

(ii) The sum of the squares of norms of entries in $E_i A_j^* E_k$

$$= \tau((E_i A_j^* E_k) \circ (\overline{E_j A_j^* E_k})) \quad (20.23)$$

$$= \text{trace}(E_i A_j^* E_k (\overline{E_j A_j^* E_k})^\top) \quad (20.24)$$

$$= \text{trace}(E_i A_j^* E_k A_j^* E_i) \quad (20.25)$$

$$= \text{trace}(E_i A_j^* E_k A_j^*) \quad \text{as } \text{trace}(XY) = \text{trace}(YX) \quad (20.26)$$

$$= \sum_{y \in X} (E_i A_j^* E_k A_j^*)_{yy} \quad (20.27)$$

$$= \sum_{y \in X} \left(\sum_{z \in X} (E_i)_{yz} (A_j^*)_{zz} (E_k)_{zy} (A_j^*)_{yy} \right) \quad (20.28)$$

$$= \sum_{y \in X} \left(\sum_{z \in X} (E_i)_{zy} (|X| (E_j)_{xz}) (E_k)_{zy} (|X| (E_j)_{yx}) \right) \quad (20.29)$$

$$= |X|^2 (E_j (E_i \circ E_k))_{xx} \quad (20.30)$$

$$= |X| q_{ik}^j (E_j)_{xx} \quad (20.31)$$

$$= q_{ik}^j m_j \quad (20.32)$$

$$= m_k q_{ij}^k. \quad (20.33)$$

Note that since $|X|E_j = q_j(0)A_0 + q_j(1)A_1 + \cdots q_j(D)A_D$,

$$(E_j)_{xx} = \frac{1}{|X|} q_j(0) = \frac{m_j}{|X|}.$$

Thus, we have (ii). □

Corollary 20.1 (Krein Condition). *For any commutative scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$, q_{ij}^h is a non-negative real number for $0 \leq h, i, j \leq D$.*

Proof. Since $q_{ij}^h m_h$ is a non-negative real by the proof of Lemma 20.3 (ii).

Note that m_h is a positive integer. □

An interpretation of the Krein parameters.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme with standard module V .

Pick a vector $v \in V$ with

$$v = \sum_{x \in X} \alpha_x \hat{x}.$$

View v as a function

$$v : X \longrightarrow \mathbb{C} \quad (x \mapsto \alpha_x).$$

View V as the set of all functions $V \rightarrow \mathbb{C}$. Then the vector space V together with product of functions is a \mathbb{C} -algebra.

For

$$v = \sum_{x \in X} \alpha_x \hat{x}, \quad w = \sum_{x \in X} \beta_x \hat{x} \in V,$$

write

$$v \circ w = \sum_{x \in X} \alpha_x \beta_x \hat{x}$$

to represent the product of v and w viewed as functions.

Lemma 20.4. *With the above notation,*

- (i) $A_j^*(x)v = |X|(E_j \hat{x} \circ v)$ for all $v \in V$ and for all $x \in X$.
- (ii) $E_i V \circ E_j V \subseteq \sum_{h: q_{ij}^h \neq 0} E_h V$ for all $0 \leq i, j \leq D$.
- (iii) $E_h(E_i \circ E_j V) = E_h V$ if $q_{ij}^h \neq 0$ for all $0 \leq h, i, j \leq D$.

Chapter 21

Norton Algebras

Wednesday, March 17, 1993

Proof of Lemma 20.4.

(i) Suppose

$$v = \sum_{x \in X} \alpha_x \hat{x}.$$

Pick a vertex $z \in X$ and compare z -coordinate of each side in (i).

$$(A_j^*(x)v)_z = (A_j^*(x))_{zz}v_z = |X|(E_j)_{xz}\alpha_z. \quad (21.1)$$

$$|X|(E_{\hat{j}}\hat{x} \circ v)_z = |X|(E_{\hat{j}}\hat{x})_z \cdot \alpha_z = |X|(E_j)_{xz}\alpha_z. \quad (21.2)$$

Note that $E_{\hat{j}}\hat{x}$ is the column x of $E_{\hat{j}}$ is the row x of E_j .

(ii) Fix i, j, h such that $q_{ij}^h = 0$.

Claim. $E_h(E_i V \circ E_j V) = 0$.

$$E_h(E_i V \circ E_j V) = E_h(\text{Span}(v \circ w \mid v \in E_i V, w \in E_j V)) \quad (21.3)$$

$$= E_h(\text{Span}(E_i \hat{y} \circ E_j \hat{z} \mid y, z \in X)) \quad (21.4)$$

$$= \text{Span}(E_h(E_j \hat{z} \circ E_i \hat{y}) \mid y, z \in X) \quad (21.5)$$

$$= \text{Span}((E_h A_j^*(z) E_i) \hat{y} \mid y, z \in X) \quad \text{by (i)} \quad (21.6)$$

But $q_{ij}^h = 0$ implies $q_{ji}^{\hat{h}} = 0$.

So, by Lemma 20.3 (ii),

$$0 = (E_{\hat{i}} A_j^* E_{\hat{h}})^\top = E_h A_j^* E_i.$$

Hence, $E_h(E_i V \circ E_j V) = 0$.

(iii) Fix i, j, h such that $q_{ij}^h \neq 0$. Then,

$$E_h(E_i V \circ E_j V) \subseteq E_h V$$

is clear. We show the other inclusion. Since

$$E_i \hat{y} \circ E_j \hat{y} = (\text{column } y \text{ of } E_i \circ \text{column } y \text{ of } E_j) \quad (21.7)$$

$$= \text{column } y \text{ of } E_i \circ E_j \quad (21.8)$$

$$= (E_i \circ E_j) \hat{y} \quad (21.9)$$

$$= \left(\frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h \right) \hat{y}, \quad (21.10)$$

we have,

$$E_h(E_i V \circ E_j V) = E_h \text{Span}(E_i \hat{y} \circ E_j \hat{z} \mid y, z \in X) \quad (21.11)$$

$$\supseteq E_h \text{Span}(E_i \hat{y} \circ E_j \hat{y} \mid y \in X) \quad (21.12)$$

$$= \text{Span}(q_{ij}^h E_h \hat{y} \mid y \in X) \quad (21.13)$$

$$= \text{Span}(E_h \hat{y} \mid y \in X) \quad \text{since } q_{ij}^h \neq 0 \quad (21.14)$$

$$= E_h V. \quad (21.15)$$

This proves the assertion. □

Lemma 21.1. *Given a commutative scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$, fix j ($0 \leq j \leq D$). Define binary multiplication:*

$$E_j V \times E_j V \longrightarrow E_j V \quad ((v, w) \mapsto v * w = E_j(v \circ w)).$$

Then,

(i) $v * w = w * v$, for all $v, w \in E_j V$,

(ii) $v * (w + w') = v * w + v * w'$ for all $v, w, w' \in E_j V$, and

(iii) $(\alpha v) * w = \alpha(v * w)$ for all $\alpha \in \mathbb{C}$.

In particular, the vector space $E_j V$ together with $*$ is a commutative \mathbb{C} -algebra, (not associative in general).

$(N_j : (E_j V, *))$ is called the Norton algebra on $E_j V$.

(iv) $v * w = 0$ for all $v, w \in E_j V$ if and only if $q_{jj}^j = 0$.

Proof.

(i) – (iii) Immediate.

(iv) Immediate from Lemma 20.4 (ii), (iii).

□

Let Y, j, N_j be as in Lemma 21.1, and M Bose-Mesner algebra of Y . Let

$$\text{Aut}Y = \{\sigma \in \text{Mat}_X(\mathbb{C}) \mid \sigma : \text{permutation matrix}, \sigma \cdot m = m \cdot \sigma \text{ for all } m \in M\} \quad (21.16)$$

$$= \{\sigma \in \text{Mat}_X(\mathbb{C}) \mid \sigma : \text{permutation matrix}, \quad (21.17)$$

$$(x, y) \in R_i \rightarrow (\sigma x, \sigma y) \in R_i, \text{ for all } i, \text{ and for all } x, y \in X\} \quad (21.18)$$

$$\text{Aut}(N_j) = \{\sigma : E_j V \rightarrow E_j V \mid \sigma \text{ is } \mathbb{C}\text{-algebra isomorphisms, i.e.,} \quad (21.19)$$

$$\sigma(v * w) = \sigma(v) * \sigma(w) \text{ for all } v, w \in E_j V\}. \quad (21.20)$$

Lemma 21.2. *Let $Y, j, *$ be as in Lemma 21.1.*

(i) $E_j V$ is a module for $\text{Aut}(Y)$.

(ii) $\sigma|_{E_j V} \in \text{Aut}(N_j)$ for all $\sigma \in \text{Aut}(Y)$.

(iii) $\text{Aut}Y \rightarrow \text{Aut}(N_j)$, $(\sigma \mapsto \sigma|_{E_j})$ is a homomorphism of groups,

(i.e., a representation of $\text{Aut}(Y)$).

(iv) Suppose R_0, \dots, R_D are orbits of $\text{Aut}(Y)$ acting on $X \times X$, (so, we are in Example 17.2) then above representation is irreducible.

Proof.

(i) Pick $\sigma \in \text{Aut}Y$ and $v \in V$. Then,

$$\sigma E_j v = E_j \sigma v,$$

since σ commutes with each element of M .

(ii) $\sigma|_{E_j V} : E_j V \rightarrow E_j V$ is an isomorphism of a vector space. Since σ is invertible, for all $v, w \in E_j V$,

$$\sigma(v * w) = \sigma(E_j(E_j v \circ E_j w)) = E_j \sigma(E_j v \circ E_j w) = E_j(E_j \sigma v \circ E_j \sigma w) = \sigma(v) * \sigma(w).$$

(iii) Immediate from (i) and (ii).

(iv) Here Bose-Mesner algebra M is the full commuting algebra, i.e.,

$$M = \{m \in \text{Mat}_X(\mathbb{C}) \mid \sigma \cdot m = m \cdot \sigma, \text{ for all } \sigma \in \text{Aut}(Y)\}.$$

Suppose there is a nonzero proper subspace $0 \neq W \subsetneq E_j V$ that is $\text{Aut}(Y)$ -invariant.

Set

$$W^\perp = \{v \in E_j V \mid \langle w, v \rangle = 0, \text{ for all } w \in W\}.$$

Then, W^\perp is a module for $\text{Aut}(Y)$, since $\text{Aut}(Y)$ is closed under transpose conjugate.

Let $e : V \rightarrow W$ and $f : V \rightarrow W^\perp$ be orthogonal projection such that $e + f = E_j$,

$$e^e = e, f^e = f, ef = fe = 0, eE_h = 0, \text{ if } h \neq j.$$

Since e commutes with all $\sigma \in \text{Aut}(Y)$, $e \in M$ and

$$e = \sum_{i=0}^D \alpha_i E_i.$$

If $h \neq j$, then $0 = eE_h$ and $\alpha_h = 0$. Thus, $e = \alpha_j E_j$, i.e., $e = 0$ or $f = 0$.

A contradiction.

□

Norton algebras were used in original construction of Monster, a finite simple group G .

Compute character table of G ,

- p_{ij}^h, q_{ij}^h of group scheme on G ,
- find j where $m_j = \dim E_j V$ is small and $q_{jj}^j \neq 0$,
- guess abstract structure of N_j using the knowledge of p_{ij}^h 's and q_{ij}^h 's,
- compute $\text{Aut}(N_j)$,
- G .

Chapter 22

Q-Polynomial Schemes

Friday, March 19, 1993

Lemma 22.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme.*

$$(i) \ p_{0j}^h = p_{j0}^h = \delta_{jh}.$$

$$(ii) \ p_{ij}^0 = \delta_{ij} k_i.$$

$$(iii) \ q_{0j}^h = q_{j0}^h = \delta_{jh}.$$

$$(iv) \ q_{ij}^0 = \delta_{ij} m_i.$$

$$(v) \ \sum_{j=0}^D p_{ij}^h = k_i.$$

$$(vi) \ \sum_{j=0}^D q_{ij}^h = m_i.$$

Proof.

(i), (ii) These are trivial.

(iii) We have

$$|X|^{-1} \sum_{\ell=0}^D q_{0j}^{\ell} E_{\ell} = E_0 \circ E_j = |X|^{-1} J \circ E_j = |X|^{-1} E_j.$$

(iv) Recall from Lemma 20.2

$$|X|^{-1} m_h q_{ij}^h = \tau(E_i \circ E_j \circ E_{\hat{h}}),$$

(where $\tau(B)$ is the sum of entries in matrix B .)

$$|X|^{-1}m_0q_{ij}^0 = \tau(E_i \circ E_j \circ E_0) \quad (22.1)$$

$$= |X|^{-1}\tau(E_i \circ E_j) \quad (E_0 = |X|^{-1}J) \quad (22.2)$$

$$= |X|^{-1}\text{trace}(E_i E_j) \quad (22.3)$$

$$= |X|^{-1}\delta_{ij}\text{trace}E_i \quad (22.4)$$

$$= |X|^{-1}\delta_{ij}m_i. \quad (22.5)$$

(v) Pick $x, y \in X$ with $(x, y) \in R_h$. Then,

$$\{j=0\} \wedge D \quad p \wedge h\{ij\} \quad \& \quad = \quad |\{z \in X \mid (x,z) \in R_{\perp i}, \quad ; \quad (z,y) \in R_{\perp j} \quad ; \quad \text{for some } j\}| \quad \& \quad = \\ |\{z \in X \mid (x,z) \in R_{\perp i}\}| \quad \& \quad k_{\perp i}. \quad \backslash \text{end}\{\text{align}\}$$

(vi)

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h.$$

So,

$$\sum_{j=0}^D E_i \circ E_j = |X|^{-1} \sum_{h=0}^D \left(\sum_{j=0}^D q_{ij}^h \right) E_h \quad (22.6)$$

$$= E_i \circ \sum_{j=0}^D E_j \quad (22.7)$$

$$= E_i \circ I \quad (22.8)$$

$$= |X|^{-1}(q_i(0)A_0 + q_i(1)A_1 + \cdots + q_i(D)A_D) \circ I \quad (22.9)$$

$$= |X|^{-1}q_i(0)I \quad (22.10)$$

$$= |X|^{-1}m_i(E_0 + E_1 + \cdots + E_D). \quad (22.11)$$

This proves the assertions. □

Definition 22.1. Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme.

Y is Q -polynomial with respect to ordering E_0, E_1, \dots, E_D of primitive idempotents, if

$$q_{ij}^h \begin{cases} = 0 & \text{if one of } h, i, j \text{ is greater than the sum of the other two} \\ \neq 0 & \text{if one of } h, i, j \text{ is equal to the sum of the other two.} \end{cases}$$

In this case, set

$$c_i^* = q_{1,i-1}^i, \quad a_i^* = q_{1,i}^i, \quad b_i^* = q_{1,i+1}^i \quad (0 \leq i \leq D), \quad (c_0^* = b_D^* = 0).$$

Observe: Q -polynomial $\rightarrow Y$ is symmetric.

Suppose $i \neq \hat{i}$ for some i . Then, by the condition in Definition 22.1,

$$0 = q_{i\hat{i}}^0 = m_i \ (\neq 0)$$

by Lemma 22.1 (iv). This is a contradiction.

Hence, $E_i^\top = E_{\hat{i}} = E_i$ for all i .

Therefore M is symmetric and Y is symmetric.

Observe: If Y is Q -polynomial,

$$c_i^* + a_i^* + b_i^* = m_1 \quad (0 \leq i \leq D)$$

(just as $c_i + a_i + b_i = k$ for P -polynomial.)

By Lemma 22.1 (iv),

$$m_1 = q_{10}^i + q_{11}^i + \cdots + q_{1,i-1}^i + q_{1i}^i + q_{1,i+1}^i + \cdots$$

and $q_{10}^i = q_{11}^i = 0$, $q_{1,i-1}^i = c_i^*$, $q_{1i}^i = a_i^*$, and $q_{1,i+1}^i = b_i^*$.

Lemma 22.2. Assume $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is a symmetric scheme. Pick $x \in X$, and set $E_i^* \equiv E_i^*(x)$, $A^* \equiv A^*(x)$. Then the following are equivalent.

- (i) Γ is Q -polynomial with respect to E_0, \dots, E_D .
- (ii) $\begin{aligned} & \text{\$} q_{h-j} \text{\$} \begin{cases} \text{\$} \text{\$} q_{h-j} \text{\$} \end{cases} \begin{cases} \text{\$} \text{\$} q_{h-j} \text{\$} \end{cases} = 0 \text{ \& if } |h-j| > 0 \text{ \& } \\ & 0 \text{ \& if } |h-j| = 1. \end{cases} \quad (0 \leq h, j \leq D).$
- (iii) There exists $f_i^* \in \mathbb{C}[\lambda]$, $\deg f_i^* = i$, and

$$A_i^* = f_i^*(A_1^*) \quad (0 \leq i \leq D).$$

(iv) E_0^*V, \dots, E_D^*V are maximal eigenspaces of A_1^* , and

$$E_i A_1^* E_j = 0 \quad \text{if } |i-j| > 0, \quad (0 \leq i, j \leq D).$$

(Compare (iv) with the definition of Q -polynomial in Definition 6.2.)

Proof.

(i) \rightarrow (ii) Clear.

(ii) \rightarrow (iii) $A_0^* = I$,

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* \quad (22.12)$$

$$A_1^* A_j^* = q_{1j}^{j-1} A_{j-1}^* + q_{1j}^j A_j^* + q_{1j}^{j+1} A_{j+1}^* \quad (q_{1j}^{j+1} \neq 0, 1 \leq j \leq D-1). \quad (22.13)$$

Hence A_j^* is a polynomial of degree exactly j in A_1^* by induction on j .

$$\lambda f_j^*(\lambda) = b_{j-1}^* f_{j-1}^*(\lambda) + a_j^* f_j^*(\lambda) + c_{j+1}^* f_{j+1}^*(\lambda) \quad \text{with } c_{j+1}^* \neq 0,$$

and $f_{-1}^* = 0$, $f_0^*(\lambda) = 1$.

(iii) \rightarrow (i) Pick i, j, h with $0 \leq i, j, h \leq D$ and $h \geq i + j$. Since

$$m_h q_{ij}^h = m_j q_{ih}^j = m_i q_{hj}^i$$

by Lemma 20.2, it suffices to show that

$$q_{ij}^h \begin{cases} = 0 & \text{if } h > i + j \\ \neq 0 & \text{if } h = i + j. \end{cases}$$

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* \quad (22.14)$$

$$f_i^*(A_1) f_j^*(A_1) = \sum_{h=0}^D q_{ij}^h f_h^*(A_1). \quad (22.15)$$

Hence,

$$f_i^*(\lambda) f_j^*(\lambda) = \sum_{h=0}^D q_{ij}^h f_h^*(\lambda).$$

Note that since $A_0^*, A_1^*, \dots, A_D^*$ are linearly independent, $f(A_1^*) = 0$ implies $\deg f > D$.

$$\deg \text{LHS} = i + j \rightarrow q_{ij}^{i+j} \neq 0, \quad q_{ij}^h = 0, \quad \text{if } h > i + j.$$

(iii) \rightarrow (iv) Recall

$$A_1^* = q_1(0)E_0^* + q_1(1)E_1^* + \dots.$$

Each A_i^* is a polynomial in A_1^* . Then A_1^* generates the dual Bose-Mesner algebra. So, $q_1(0), q_1(1), \dots, q_1(D)$ are distinct.

So, $E_0^* V, \dots, E_D^* V$ are maximal eigenspaces.

Also, $|i - j| > 1$ implies $q_{11}^j = 0$.

Thus, $E_i A_1^* E_j = 0$ by Lemma 20.3 (ii).

(iv) \rightarrow (ii) $q_{1j}^i = 0$ if $|i - j| > 1$. since in this case,

$E_i A_1^* E_j = 0$ implies $q_{1j}^i = 0$ by Lemma 20.3 (ii).

Suppose $q_{1j}^{j+1} = 0$ for some j ($0 \leq j \leq D - 1$).

Without loss of generalith, choose j minimum. Then A_h^* is a polynomial of degree h in A_1^* ($0 \leq h \leq j$), and

$$A_1^* A_j^* - q_{1j}^{j-1} A_{j-1}^* - q_{1j}^j A_j^* = 0.$$

the left hand side is a polynomial in A_1^* of degree $j + 1$.

Hence, the minimal polynomial of A_1^* has degree less than or equal to $j + 1 \leq D$. But A_1^* has $D + 1$ distinct eigenvalues.

This is a contradiction.

□

Chapter 23

Representation of a Scheme

Monday, March 22, 1993

Theorem 23.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a symmetric scheme. (View the standard module V as an algebra of functions from X to \mathbb{C} .) Then the following are equivalent.*

- (i) Y is Q -polynomial with respect to ordering E_0, E_1, \dots, E_D of primitive idempotents.
- (ii) For all i ($0 \leq i \leq D$),

$$E_0V + E_1V + (E_1V)^2 + \dots + (E_1V)^i = E_0V + E_1V + \dots + E_iV.$$

Proof.

By Lemma 20.4 (ii), (iii).

$$E_h(E_iV \circ E_jV) = 0 \text{ if and only if } q_{ij}^h = 0 \quad (0 \leq i, j, h \leq D).$$

(i) \rightarrow (ii) By our assumption,

$$q_{1j}^h = 0 \text{ if } |h - j| > 1, \text{ and } q_{1j}^{j+1} \neq 0.$$

So,

$$E_1V \circ E_jV \subseteq E_{j-1}V + E_jV + E_{j+1}V \quad (0 \leq j \leq D), \quad (23.1)$$

$$E_{j+1}(E_1V \circ E_jV) = E_{j+1}V \quad (0 \leq j \leq D-1), \quad (23.2)$$

by Lemma 20.4.

Also $E_0V \subseteq \text{Span}(\delta)$, where δ is all 1's vector, i.e., 1 as a function $X \rightarrow \mathbb{C}$. So,

$$E_0 \circ E_j V = E_j V \quad (0 \leq j \leq D). \quad (23.3)$$

Show (ii) by induction on i .

The cases $i = 0, 1$ are trivial.

$i > 1$: \subseteq .

$$E_0V + E_1V + (E_1V)^2 + \cdots + (E_1V)^i \quad (23.4)$$

$$= E_0V + E_1V \circ (E_0V + E_1V + \cdots + (E_1V)^{i-1}) \quad (23.5)$$

$$= E_0V + E_1V \circ (E_0V + E_1V + \cdots + E_{i-1}V) \quad (23.6)$$

$$\subseteq E_0V + E_1V + \cdots + E_iV \quad (23.7)$$

by (23.1).

\supseteq .

Claim. $E_i \subseteq E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V \quad (2 \leq i \leq D)$.

Proof of Claim. By (23.2),

$$E_i(E_1V \circ E_{i-1}V) = E_iV.$$

For all $v \in E_iV$, there exists $u \in E_1V \circ E_{i-1}V$ such that $E_iu = v$.

On the other hand, by (23.1),

$$E_1V \circ E_{i-1}V \subseteq E_{i-2}V + E_{i-1}V + E_{i-2}V.$$

So, $u = w + v$, where $w \in E_{i-2}V + E_{i-1}V$. We have,

$$w = u - v \in E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V$$

as desired.

Remark.

$$E_iV \circ E_jV = \text{Span}(u \circ v \mid u \in E_iV, v \in E_jV).$$

By claim,

$$E_0V + E_1V + \cdots + E_iV \quad (23.8)$$

$$\subseteq E_0V + E_1V + \cdots + E_iV + E_1V \circ E_{i-1}V \quad (23.9)$$

$$\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + E_1V(E_0V + E_1V + \cdots + (E_1V)^{i-1}) \quad (23.10)$$

$$\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + (E_1V)^i. \quad (23.11)$$

(ii) \rightarrow (i)

Claim 1. Pick i, j ($0 \leq i, j \leq D$) with $j > i + 1$. Then $q_{1i}^j = 0$.

Proof of Claim 1.

$$E_j(E_1 \circ E_j V) \subseteq E_j(E_1 V \circ (E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^i)) \quad (23.12)$$

$$\subseteq E_j(E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^{i+1}) \quad (23.13)$$

$$= E_j(E_0 V + E_1 V + \cdots + E_{i+1} V) \quad (23.14)$$

$$= 0. \quad (23.15)$$

So $q_{1i}^j = 0$ by Lemma 20.4.

Claim 2. $q_{1i}^{i+1} \neq 0$ ($0 \leq i < D$).

Proof of Claim 2.

$$E_0 V + E_1 V + \cdots + E_{i+1} V \quad (23.16)$$

$$= E_0 V + E_1 V + \cdots + (E_1 V)^{i+1} \quad (23.17)$$

$$= E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + (E_1 V)^i) \quad (23.18)$$

$$= E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + E_i V) \quad (23.19)$$

$$= E_0 V + E_1 V \circ (E_0 V + \cdots + E_i V). \quad (23.20)$$

So,

$$E_{i+1} V = E_{i+1}(E_1 V \circ (E_0 V + \cdots + E_i V)) \quad (23.21)$$

$$= E_{i+1}(E_1 V \circ E_i V) \quad (23.22)$$

by Claim 1 and Lemma 20.4.

Hence, $q_{1i}^{i+1} \neq 0$ by Lemma 20.4.

□

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme with standard module V .

Definition 23.1. A representation of Y is a pair (ρ, H) , where H is a non-zero Hermitean space (with inner product $\langle \cdot, \cdot \rangle$) and $\rho : X \rightarrow H$ is a map satisfying the following.

R1. $H = \text{Span}(\rho(x) \mid x \in X)$.

R2. $\langle \rho(x), \rho(y) \rangle$ depends only on i for which $(x, y) \in R_i$ ($x, y \in X$).

R3. For every $x \in X$ and for all i ($0 \leq i \leq D$),

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) \in \text{Span}(\rho(x)).$$

Above representation is nondegenerate if $\{\rho(x) \mid x \in X\}$ are distinct.

Example 23.1. $Y = H(D, 2)$, $X = \{a_1 \cdots a_D \mid a_i \in \{1, -1\}, 1 \leq i \leq D\}$. Let $H = \mathbb{C}^D$ and $\langle \cdot, \cdot \rangle$ usual Hermitean dot product.

For a vertex $x = a_1 \cdots a_D \in X$, define

$$\rho(x) = a_1 \cdots a_D \in H.$$

Then, R1 – R3 hold.

Remark. R1, R2 are obvious. For R3, we may assume that $x = 1 \cdots 1$. Restrict

$$\sum_{y \in X, (y, x) \in R_i} \rho(y)$$

on the first coordinate. Then,

$$-1 \text{ appers } \binom{D-1}{i-1} \text{ times} \quad (23.23)$$

$$1 \text{ appers } \binom{D-1}{i} \text{ times.} \quad (23.24)$$

Hence,

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) = \left(\binom{D-1}{i} - \binom{D-1}{i-1} \right) \rho(x).$$

Let (ρ, H) be a representation of arbitrary commutative scheme Y . Set

$$E = (\langle \rho(x), \rho(y) \rangle)_{x, y \in X}$$

Gram matrix of the representation.

Definition 23.2. Representations (ρ, H) , (ρ', H') of Y are equivalent, whenever, Gram matrices are related by

$$E' \in \text{Span} E.$$

We do not distinguish between equivalent representations.

Note. Suppose (ρ, H) is a representation of a symmetric scheme Y . Pick $x, y \in X$ with $(x, y) \in R_j$.

Then $(y, x) \in R_j$. So, by R2,

$$\langle \rho(x), \rho(y) \rangle = \langle \rho(y), \rho(x) \rangle = \overline{\langle \rho(x), \rho(y) \rangle},$$

since $\langle \cdot, \cdot \rangle$ is Hermitean.

Hence, the Gram matrix E of ρ is real symmetirc. Without loss of generality, we can view H as a real Euclidean space in this case.

Lemma 23.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme and V a standard module.*

Let E_j be any primitive idempotent of Y .

(i) (ρ, H) is a representation of Y , where $H = E_j V$ (with inner product inherited from Y).

$$\rho : X \rightarrow H \quad (x \mapsto E_j \hat{x})$$

(i.e., $\rho(x)$ is the x -th column of E_j .)

(ii) $\langle \rho(x), \rho(y) \rangle = |X|^{-1} q_j(i)$, if $(x, y) \in R_i$, $(x, y \in X)$.

(iii) For $0 \leq i \leq D$ and $x, y \in X$,

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) = p_i(j) \rho(x).$$

(iv) (ρ, H) is nondegenerate if and only if $q_j(i) \neq q_j(0)$ for all i , $(0 \leq i \leq D)$.

(v) Every representation of Y is equivalent to a representation of the above type for some j ($0 \leq j \leq D$), and j is unique.

Proof.

(i) – (iii).

R1: $\text{Span}(\rho X)$ is the column space of E_j which is equal to H .

R2:

$$\langle \rho(x), \rho(y) \rangle = \langle E_j \hat{x}, E_j \hat{y} \rangle \quad (23.25)$$

$$= (\overline{E_j \hat{x}})^\top E_j \hat{y} \quad (23.26)$$

$$= \hat{x}^\top \overline{E_j}^\top E_j \hat{y} \quad (23.27)$$

$$= \hat{x}^\top E_j \hat{y} \quad (23.28)$$

$$(E_j)_{xy}. \quad (23.29)$$

Note that $\overline{E_j}^\top = E_j$ by Lemma 19.1.

Recall

$$E_j = |X|^{-1} (q_j(0)A_0 + \cdots + q_j(D)A_D).$$

So,

$$(E_j)_{xy} = |X|^{-1} q_j(i), \quad \text{where } (x, y) \in R_i.$$

R2: Recall

$$A_i = p_i(0)E_0 + \cdots + p_i(D)E_D.$$

So, $E_j A_i = p_i(j) E_j$, and

$$p_i(j) \rho(x) = p_i(j) E_j \hat{x} = E_j A_i \hat{x} = E_j \sum_{y \in X, (y, x) \in R_i} \hat{y} = \sum_{y \in X, (y, x) \in R_i} \rho(y).$$

Note.

$$A_i \hat{x} = \sum_{y \in X, (x, y) \in R_{i'}} \hat{y}.$$

Pf.

$$z \text{ entry of LHS} = (A_i \hat{x})_z \quad (23.30)$$

$$= \sum_{w \in X} (A_i)_{zw} \hat{x}_w \quad (23.31)$$

$$= (A_i)_{zx} \quad (23.32)$$

$$= \begin{cases} 1 & \text{if } (x, z) \in R_{i'} \\ 0 & \text{else.} \end{cases} \quad (23.33)$$

$$z \text{ entry of RHS} = \sum_{y \in X, (x, y) \in R_{i'}, z=y} 1 \quad (23.34)$$

$$= \begin{cases} 1 & \text{if } (x, z) \in R_{i'} \\ 0 & \text{else.} \end{cases} \quad (23.35)$$

(iv) By (ii),

$$\|\rho(x)\|^2 = \langle \rho(x), \rho(y) \rangle \quad (23.36)$$

$$|X|^{-1} q_j(0) \quad (23.37)$$

$$|X|^{-1} m_j, \quad (23.38)$$

as $m_j = \dim E_j V$, and is independent of $x \in X$.

Pick distinct $x, y \in X$ such that $(x, y) \in R_i$ with $i \neq 0$.

Then,

$$\rho(x) = \rho(y) \Leftrightarrow \langle \rho(x), \rho(y) \rangle = \|\rho(x)\|^2 = |X|^{-1} q_j(0) \quad (23.39)$$

$$\Leftrightarrow |X|^{-1} q_j(i) = |X|^{-1} q_j(0) \quad (23.40)$$

$$\Leftrightarrow q_j(i) = q_j(0). \quad (23.41)$$

Hence, we have (iv). To be continued.

□

Chapter 24

Balanced Conditions, I

Wednesday, March 23, 1993

No Class on Friday (another conference).

Proof of Lemma 23.1 continued. Let E_j be a primitive idempotent, $H = E_j V$ and

$$\rho : X \rightarrow H \quad (x \mapsto E_j \hat{x}).$$

(v) Every representation (ρ, H) of Y is equivalent to a representation of above type, for some j ($0 \leq j \leq D$) and j is unique.

Let $E := (\langle \rho(x), \rho(y) \rangle)_{x, y \in X}$.

By R2,

$$E = \sum_{i=0}^D \sigma_i A_i, \quad \text{some } \sigma_0, \sigma_1, \dots, \sigma_D \in \mathbb{C}.$$

Hence, E belongs to the Bose-Mesner algebra \mathcal{M} of Y .

We want to show that E is a scalar multiple of a primitive idempotent.

Fix $x \in X$ and fix i ($0 \leq i \leq D$).

By R3,

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) = \alpha \rho(x), \quad \text{some } \alpha \in \mathbb{C}. \quad (24.1)$$

So,

$$k_i \bar{\sigma}_i = \left\langle \sum_{y \in X, (y, x) \in R_i} \rho(y), \rho(x) \right\rangle = \bar{\alpha} \langle \rho(x), \rho(x) \rangle = \bar{\alpha} \sigma_0.$$

Hence, α is independent of x . In maatrix form (24.1) becomes

$$EA_i \hat{x} = \alpha E \hat{x}.$$

Remark.

$$Eu = Ev \Leftrightarrow \langle z, Eu \rangle = \langle z, Ev \rangle \text{ for all } z \in X \Leftrightarrow (Eu)_z = (Ev)_z \text{ for all } z \in X.$$

$$(EA_i \hat{x})_z = \left\langle \rho(z), \sum_{y \in X, (y, x) \in R_i} \rho(y) \right\rangle \quad (24.2)$$

$$= \alpha \langle \rho(z), \rho(x) \rangle \quad (24.3)$$

$$= (\alpha E \hat{x})_z. \quad (24.4)$$

Hence,

$$EA_i \hat{x} = \alpha E \hat{x}.$$

Since x is arbitrary,

$$EA_i = \alpha E.$$

So,

$$EA_i \in \text{Span} E \text{ and } EM = \text{Span} E.$$

We have $E \in E_j$ for unique j ($0 \leq j \leq D$). □

Remark.

$$E = \tau_0 E_0 + \cdots + \tau_D E_D, \quad \tau_j \in \mathbb{C} \quad (0 \leq j \leq D).$$

And, at least one of τ_j is nonzero, and

$$\tau_j E_j = EE_j \in \text{Span} E.$$

So,

$$\tau_j E_j = E$$

as E_0, \dots, E_D are linearly independent.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a symmetric scheme, and let E be a primitive idempotent.

Definition 24.1. Y is Q -polynomial with respect to E , if and only if Y is Q -polynomial with respect to some ordering E_0, E_1, \dots, E_D of primitive idempotents, where $E_0 = |X|^{-1}J$, and $E_1 = E$.

Theorem 24.1. Assume $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is P -polynomial (i.e., (X, R_1) is distance-regular). Let E be any primitive idempotent of Y . Let (ρ, H) be the corresponding representation.

(i) The following are equivalent.

(ia) Y is Q -polynomial with respect to E .

(ib) (ρ, H) is nondegenerate and for all $x, y \in X$, and for all i, j ($0 \leq i, j \leq D$),

$$\sum_{z \in X, (x, z) \in R_i, (y, z) \in R_j} \rho(z) - \sum_{z' \in X, (x, z') \in R_j, (y, z') \in R_i} \rho(z') \in \text{Span}(\rho(x) - \rho(y)).$$

(ic) (ρ, H) is nondegenerate and for all $x, y \in X$,

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') \in \text{Span}(\rho(x) - \rho(y)).$$

(ii) Wirte

$$E = |X|^{-1} \sum_{j=0}^D \theta_j^* A_j,$$

and suppose (ia) – (ic) hold. Then the coefficient in (ib) is

$$p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \quad (1 \leq h \leq D, 0 \leq i, j \leq D).$$

Proof.

(ia) \rightarrow (ib) Without loss of generality, assume $E \equiv E_1$, and Y is Q -polynomial with respect to E .

Then by Lemma 22.2, $\theta_0^*, \dots, \theta_D^*$ are distinct. So $\theta_h^* \neq \theta_0^*$ for all $h \in \{1, 2, \dots, D\}$, and (ρ, H) is nondegenerate.

Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $A_i^* \equiv A_i^*(x)$, $A^* \equiv A_1^*$.

Let M be the Bose-Mesner algebra. Set

$$L = \{mA^*n - nA^*m \mid m, n \in M\}.$$

Claim 1. $\dim L \leq D$.

Proof of Claim 1.

$$L = \text{Span}(E_i A^* E_j - E_j A^* E_i \mid 0 \leq i < j \leq D) \quad (24.5)$$

$$= \text{Span}(E_i A^* E_{i+1} - E_{i+1} A^* E_i \mid 0 \leq i \leq D-1). \quad (24.6)$$

Since $E_i A^* E_j = O$ if $q_{ij}^1 = 0$ by Lemma 20.2 and Lemma 20.3, and this occurs if $|i - j| > 1$ by Q -polynomial property.

Hence, $\dim L \leq D$.

Claim 2. (i) $\{A^* A_h - A_h A^* \mid 1 \leq h \leq D\}$ is a basis for L . In particular,

(ii) there exist $r_{ij}^h \in \mathbb{C}$ ($1 \leq h \leq D, 0 \leq i, j \leq D$) such that

$$A_i A^* A_j - A_j A^* A_i = \sum_{h=1}^D r_{ij}^h (A^* A_h - A_h A^*).$$

Proof of Claim 2.

(i) The column x of $A^* A_h - A_h A^*$ is a nonzero scalar $\theta_h^* - \theta_0^*$ times the column x of A_h .

Remark.

$$((A^* A_h - A_h A^*) \hat{x})_y = E_{xy}(A_h)_{yx} - (A_h)_{yx} E_{xx} = (\theta_h^* - \theta_0^*)(A_h)_{yz}.$$

Also the column x of A_0, A_1, \dots, A_D are linearly independent.

Hence, the matrices given are linearly independent.

They are in L by construction, so they form a basis for L by Claim 1.

(ii) This is immediate since

$$A_i A^* A_j - A_j A^* A_i \in L, \quad \text{for all } i, j.$$

Claim 3.

$$r_{ij}^\ell = p_{ij}^\ell \left(\frac{\theta^* - \theta_j^*}{\theta_0^* - \theta_\ell^*} \right) \quad (1 \leq \ell \leq D, 0 \leq i, j \leq D).$$

Proof of Claim 3. Fix i, j ,

$$A_i A^* A_j - A_j A^* A_i - \sum_{h=1}^D r_{ij}^h (A^* A_h - A_h A^*) = 0.$$

Pick ℓ ($1 \leq \ell \leq D$). Pick $y \in X$ such that $(x, y) \in R_\ell$.

$$(A_i A^* A_j)_{xy} = \sum_{z \in X} (A_i)_{xz} (A^*)_{zz} (A_j)_{zy} \quad (24.7)$$

$$= \sum_{z \in X, (x, z) \in R_i, (y, z) \in R_j} (A^*)_{zz} \quad (24.8)$$

$$= |X|^{-1} p_{ij}^\ell \theta_i^*. \quad (24.9)$$

Similarly,

$$(A_j A^* A_i)_{xy} = |X|^{-1} p_{ij}^\ell \theta_j^*.$$

$$(A^*A_h - A_hA^*)_{xy} = (A_0A^*A_h - A_hA^*A_0)_{xy} \quad (24.10)$$

$$= |X|^{-1}p_{0h}^\ell(\theta_0^* - \theta_h^*) \quad (24.11)$$

$$= \begin{cases} 0 & \text{if } \ell \neq h \\ |X|^{-1}(\theta_0^* - \theta_h^*) & \text{if } \ell = h. \end{cases} \quad (24.12)$$

Hence,

$$\sum_{h=1}^D r_{ij}^h (A^*A_h - A_hA^*)_{xy} = |X|^{-1}r_{ij}^\ell(\theta_0^* - \theta_\ell^*).$$

Comparing terms, we have

$$p_{ij}^\ell(\theta_i^* - \theta_j^*) - r_{ij}^\ell(\theta_0^* - \theta_\ell^*) = 0.$$

Claim 4. For all h ($1 \leq h \leq D$), for all i, j ($0 \leq i, j \leq D$), for all $w, y \in X$, $(w, y) \in R_h$,

$$\sum_{z \in X, (w, z) \in R_i, (y, z) \in R_j} \rho(z) - \sum_{z' \in X, (w, z') \in R_j, (y, z') \in R_i} \rho(z') - r_{ij}^h(\rho(w) - \rho(y)) = 0. \quad (24.13)$$

Proof of Claim 4. Set $L = \langle \text{LHS of (24.13)}, \rho(x) \rangle$. It suffices to show that $L = 0$.

Note that since x is arbitrary, if LHS of (24.13) is zero.

$$L = \sum_{z \in X, (w, z) \in R_i, (y, z) \in R_j} \langle \rho(z), \rho(x) \rangle - \sum_{z' \in X, (w, z') \in R_j, (y, z') \in R_i} \langle \rho(z'), \rho(x) \rangle \quad (24.14)$$

$$- r_{ij}^h \langle \rho(w) - \rho(y), \rho(x) \rangle \quad (24.15)$$

$$= |X|^{-1}(A_iA^*A_j)_{wy} - |X|^{-1}(A_jA^*A_i)_{wy} - |X|^{-1} \sum_{\ell=1}^D r_{ij}^\ell (A^*A_\ell - A_\ell A^*)_{wy} \quad (24.16)$$

$$= |X|^{-1} \text{times } wy \text{ entry of a matrix known to be zero by Claim 2} \quad (24.17)$$

$$= 0. \quad (24.18)$$

□

Remark.

$$|X|^{-1} \sum_{\ell=1}^D r_{ij}^\ell (A^*A_\ell - A_\ell A^*)_{wy} = |X|^{-1} r_{ij}^h (A^*A_h - A_h A^*)_{wy} \quad (24.19)$$

$$= r_{ij}^h (\langle \rho(x), \rho(w) \rangle - \langle \rho(x), \rho(y) \rangle) \quad (24.20)$$

Chapter 25

Balanced Conditions, II

Monday, March 29, 1993

Proof of Theorem 24.1 continued.

(ib) \rightarrow (ic) Obvious.

(ic) \rightarrow (ia) Without loss of generality, we may assume $D \geq 3$, else trivial.

Remark. The case $D = 2$ should be treated somewhere, but the assumption $D \geq 3$ is not used.

Fix $w \in X$, and write $E_i^* \equiv E_i^*(w)$, $A_i^* \equiv A_i^*(w)$, $A^* \equiv A_1^*$, and A_i , i -th distance matrix. Set

$$E \equiv E_1 = |X|^{-1} \sum_{i=0}^D \theta_i^* A_i.$$

Since (ρ, H) is nondegenerate,

$$\theta_0^* \neq \theta_h^*, \text{ for all } h \in \{1, 2, \dots, D\}$$

See Lemma 23.1 (iv).

Claim 1. Pick h ($1 \leq h \leq D$), and x, y with $(x, y) \in R_h$. Then

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') = r_{12}^h (\rho(x) - \rho(y)),$$

where

$$r_{12}^h = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}.$$

Proof of Claim 1. By our assumption,

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') = \alpha (\rho(x) - \rho(y)).$$

Hence,

$$|X|^{-1}p_{12}^h(\theta_1^* - \theta_2^*) = \left\langle \sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z'), \rho(x) \right\rangle \quad (25.1)$$

$$= \alpha \langle \rho(x) - \rho(y), \rho(x) \rangle \quad (25.2)$$

$$= \alpha |X|^{-1}(\theta_0^* - \theta_h^*). \quad (25.3)$$

We have

$$\alpha = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}.$$

$$\text{Claim 2. } A_1 A^* A_2 - A_2 A^* A_1 = \sum_{h=1}^D r_{12}^h (A^* A_h - A_h A^*).$$

Proof of Claim 2. The xy entry of the LHS – RHS is

$$|X| \left\langle \sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z') - r_{12}^h (\rho(x) - \rho(y)), \rho(w) \right\rangle,$$

where $(x, y) \in R_h$, $h = 1, 2, \dots, D$, and the xy entry of the LHS – RHS is 0 if $x = y$.

But the vector on the left in the above inner product is 0 by Claim 1, so the inner product is 0.

Thus, the xy entry of the LHS – RHS is always 0, and we have Claim 2.

$$\text{Claim 3. } A^* A_3 - A_3 A^* \in \text{Span}(AA^* A_2 - A_2 A^* A, A^* A_2 - A_2 A^*, A^* A - AA^*).$$

Proof of Claim 3. Since $p_{12}^h = 0$, if $h > 3$, and $p_{12}^h \neq 0$, if $h = 3$, we have $r_{12}^h = 0$ if $h > 0$, and $r_{12}^h \neq 0$, if $h = 3$. Note that $\theta_1^* \neq \theta_2^*$.

Now we are done by Claim 2.

Claim 4. There exist $\beta, \gamma, \delta \in \mathbb{R}$ such that

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(AA^* + A^* A) - \delta A^*] \quad (25.4)$$

$$= A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A). \quad (25.5)$$

Proof of Claim 4. There exists $f_i \in \mathbb{R}[\lambda]$, $\deg f_i = i$ such that $A_i = f_i(A_1)$.

Writing A_2, A_3 as polynomials in A in Claim 3 and simplifying, we find

$$A^3 A^* - A^* A^3 \in \text{Span}(A^2 A^* A - AA^* A^2, A^2 A^* - A^* A^2, AA^* - A^* A).$$

Remark. Let $A_3 = \beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I$ with $\beta_3 \neq 0$, and $A_2 = \gamma_2 A^2 + \gamma_1 A + \gamma_0 I$, with $\gamma_2 \neq 0$. Then

$$A^* A_3 - A_3 A^* = A^*(\beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I) - (\beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I)A^*. \quad (25.6)$$

$$A^3 A^* - A^* A^3 \in \text{Span}(A^* A_3 - A_3 A^*, A^2 A^* - A^* A^2, AA^* - A^* A) \quad (25.7)$$

$$\subseteq \text{Span}(AA^* A_2 - A_2 A^* A, A^* A_2 - A_2 A^*, A^2 A^* - A^* A^2, AA^* - A^* A) \quad (25.8)$$

$$A^* A_2 - A_2 A^* = A^*(\gamma_2 A^2 + \gamma_1 A + \gamma_0 I) - (\gamma_2 A^2 + \gamma_1 A + \gamma_0 I)A^* \quad (25.9)$$

$$AA^* A_2 - A_2 A^* A = AA^*(\gamma_2 A^2 + \gamma_1 A + \gamma_0 I) - (\gamma_2 A^2 + \gamma_1 A + \gamma_0 I)A^* A \quad (25.10)$$

$$A^* A_2 - A_2 A^* \in \text{Span}(A^2 A^* - A^* A^2, AA^* - AA^*) \quad (25.11)$$

$$AA^* A_2 - A_2 A^* A \in \text{Span}(A^2 A^* A - AA^* A^2, AA^* - AA^*) \quad (25.12)$$

$$A^3 A^* - A^* A^3 \in \text{Span}(A^2 A^* A - AA^* A^2, A^2 A^* - A^* A^2, AA^* - A^* A). \quad (25.13)$$

Hence, we can find δ, γ, δ satisfying

$$0 = A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A).$$

On the other hand,

$$[A, A^2 A^* - \beta AA^* A + A^* A^2 - \gamma(AA^* + A^* A) - \delta A^*] \quad (25.14)$$

$$= A^3 A^* - A^2 A^* A - \beta A^2 A^* A + \beta AA^* A^2 + AA^* A^2 - A^* A^3 \quad (25.15)$$

$$- \gamma A^2 A^* - \gamma AA^* A + \gamma AA^* A + \gamma A^* A^2 - \delta AA^* + \delta A^* A \quad (25.16)$$

$$= A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A). \quad (25.17)$$

Thus we have (i) and (ii).

Define a diagram D_E on nodes $0, 1, \dots, D$.

Connect distinct nodes, by undirected arc if $q_{ij}^1 \neq 0$. (Note $q_{ij}^1 = q_{ji}^1$).

Since $q_{0j}^1 = \delta_{1j}$, the 0-node is adjacent to the 1-node and no other node.

Y is Q -polynomial with respect to E if and only if E_E is a path.

Claim 5. D_E is connected.

Proof of Claim 5. Suppose there exists $\Delta \subseteq \{0, 1, \dots, D\}$ such that i, j not connected for every $i \in \Delta$ and $j \in \{0, 1, \dots, D\} \setminus \Delta$.

Set

$$f = \sum_{i \in \Delta} E_i.$$

Observe

$$fA^* = \sum_{i \in \Delta} E_i A^* \left(\sum_{j=0}^D E_j \right) \quad (25.18)$$

$$= \sum_{i \in \Delta, j \in \Delta} E_i A^* E_j \quad (\text{since } E_i A^* E_j = O \text{ if } q_{ij}^1 = 0) \quad (25.19)$$

$$= fA^* f. \quad (25.20)$$

Also, $A^* f = fA^* f$.

Hence, f commutes with A^* .

But f is an element of the Bose-Mesner algebra

$$f = \sum_{i=0}^D \alpha_i A_i \quad \text{for some } \alpha_0, \dots, \alpha_D \in \mathbb{C}.$$

We have

$$0 = fA^* - A^* f = \sum_{i=1}^D \alpha_i (A_i A^* - A^* A_i).$$

But $\{A_h A^* - A^* A_h \mid 1 \leq h \leq D\}$ are linearly independent. (The column w of $A_h A^* - A^* A_h$ is $\theta_h^* - \theta_0^*$ times the column w of A_h .)

Hence, $\alpha_1 = \dots = \alpha_D = 0$, and $f = \alpha_0 I$. Since $f^2 = f$, α_0 is 0 or 1.

If $\alpha_0 = 0$, $f = O$ and $\Delta = \emptyset$.

If $\alpha_0 = 1$, $f = I$ and $\Delta = \{0, 1, \dots, D\}$.

This proves Claim 5. □

Remark. Claim 5 proves the following in general.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a symmetric association scheme. Fix a vertex $x \in X$, and let

$$E = \frac{1}{|X|} \sum_{j=0}^D \theta_j^* A_j \quad (\theta_j^* = q_1(j) \text{ if } E = E_1)$$

be a primitive idempotent and $E_j^* \equiv E_j^*(x)$.

$$A^* = \sum_{j=0}^D \theta_j^* E_j^*.$$

If $\theta_0 = \theta_h^*$, $h = 1, \dots, D$, then the following hold.

- (i) $\{A_h A^* - A^* A_h \mid 1 \leq h \leq D\}$ are linearly independent.
- (ii) The diagram D_E on nodes $0, 1, \dots, D$ defined by

$$i \sim j \Leftrightarrow E(E_i \circ E_j) \neq O$$

is connected.

$$(iii) C_M(A^*) = \{L \in M \mid LA^* = A^*L\} = \text{Span}(I).$$

Proof. | (i) The column x of $A_h A^* - A^*(A_h)$ is $\theta_0^* - \theta_h^*$ times the column x of A_h .

$$(iii) 0 = [\sum_{h=0}^D \alpha_h A_h, A^*] = \sum_{h=1}^D \alpha_h (A_h A^* - A^* A_h). \text{ Hence, } \alpha_0 = \dots = \alpha_D = 0.$$

(ii) Δ is a connected component. Let $f = \sum_{i \in \Delta} E_i$, then $f \in C_M(A^*)$.

Let $Y = (X, \{R_i\}_{0 \leq i \leq 2})$ be a symmetric association scheme with $D = 2$. Let

$$E = \frac{1}{|X|} \sum_{j=0}^2 \theta_j^* A_j$$

be a primitive idempotent. If $\theta_0^*, \theta_1^*, \theta_2^*$.

Then Y is Q -polynomial with respect to E .

Proof. By the previous lemma, D_E is connected.

Note. It seems $\theta_1^* \neq \theta_2^*$ is necessary. Clarify the condition $\theta_1^* = \theta_2^*$.

Terwilliger claims that $\theta_1^* = \theta_2^*$ does not occur under the assumption (ic).
(March 7, 1995)

Chapter 26

Representation Diagrams

Wednesday, March 31, 1993

Proof of Theorem 24.1 continued. Assume $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is P -polynomial. Let E be a primitive idempotent of Y such that the corresponding representation (ρ, H) is nondegenerate.

Show for all $x, y \in X$,

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') \in \text{Span}(\rho(x) - \rho(y))$$

implies that Y is Q -polynomial with respect to E .

Define a diagram D_E on nodes $0, 1, \dots, D$, for $i \neq j$,

$$i \frown j \leftrightarrow q_{ij}^1 \neq 0$$

by setting $E = E_1$.

We showed that $0 \frown j \leftrightarrow j = 1$ ($1 \leq j \leq D$) and D_E is connected.

Now it is sufficient to show the following.

Claim 6. Let i be a node in D_E . Then i is adjacent to at most 2 arcs.

Proof of Claim 6. Suppose the node j is adjacent to i in D_E . By claim 4,

$$0 = E_i(A^3A^* - A^*A^3 - (\beta + 1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A) - \delta(AA^* - A^*A))E_j \quad (26.1)$$

$$= E_iA^*E_j(\theta_i^3 - \theta_j^3 - (\beta + 1)(\theta_i^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j)) \quad (26.2)$$

$$= E_iA^*A_j(\theta_i - \theta_j)p(\theta_i, \theta_j), \quad (26.3)$$

where

$$p(s, t) = s^2 - \beta st + t^2 - \gamma(s + t) - \delta.$$

Remark.

$$(\theta_i - \theta_j)(\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \delta) \quad (26.4)$$

$$= \theta_i^3 - \theta_j^3 - (\beta + 1)(\theta_i^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j) \quad (26.5)$$

Since i is adjacent to j , $q_{ij}^1 \neq 0$ and

$$E_i A^* E_j \neq O$$

by Lemma 20.3 (ii). Since Y is P -polynomial,

$$\theta_i \neq \theta_j \quad \text{if } i \neq j.$$

Hence $p(\theta_i, \theta_j) = 0$. But p is quadratic in t . So $p(\theta_i, t) = 0$ has at most two solutions for θ_j .

Now D_E is a pth, and Γ is Q -polynomial with respect to E .

This proves Theorem 24.1. □

Corollary 26.1. *Assume $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is P -polynomial, and Q -polynomial with respect to a primitive idempotent*

$$E = \frac{1}{|X|} \sum_{i=0}^D \theta_i^* A_i.$$

Then,

$$\beta = \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*}$$

is independent of i ($0 \leq i \leq D-3$).

Proof. Fix i . Without loss of generality, $D \geq 3$, else vacuous.

Pick $x, y \in X$ with $(x, y) \in R_3$.

Let (ρ, H) be the representation for E .

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') = \frac{p_{12}^3(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_3^*} (\rho(x) - \rho(y)), \quad (26.6)$$

and $p_{12}^3 = c_3$.

Since $p_{i, i+3}^3 \neq 0$, there exists $w \in X$ such that $(x, w) \in R_{i+3}$, $(y, w) \in R_i$.

Take inner product of (26.6) with $\rho(w)$. We have

$$P_{12}^3(x, y) \subseteq P_{1, i+2}^{i+3}(x, w) \cap P_{2, i+2}^i(y, w) \quad (26.7)$$

$$P_{21}^3(x, y) \subseteq P_{2, i+1}^{i+3}(x, w) \cap P_{2, i+1}^i(y, w). \quad (26.8)$$

Hence,

$$\left\langle \sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z'), \rho(w) \right\rangle = |X|^{-1} c_3 (\theta_{i+2}^* - \theta_{i+1}^*),$$

$$\left\langle \frac{c_3(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_3^*} (\rho(x) - \rho(y)), \rho(w) \right\rangle = \frac{c_3(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_3^*} |X|^{-1} (\theta_{i+3}^* - \theta_{i+1}^*).$$

We have,

$$\sigma = \frac{\theta_{i+1}^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+3}^*} = \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_3^*}.$$

Remark. Note that since Y is P and Q with respect to A_1 and E_1 , $\theta_0^*, \theta_1^*, \dots, \theta_D^*$, $\theta_0, \theta_1, \dots, \theta_D$ are all distinct.

So

$$\beta = \frac{1}{\sigma} - 1 = \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} = \frac{\theta_0^* - \theta_1^* + \theta_2^* - \theta_3^*}{\theta_1^* - \theta_2^*}.$$

We have the assertion. \square

Given the intersection number of a distance-regular graph Γ . The following 2 lemmas give an efficient method to determine if Γ is Q -polynomial with respect to some primitive idempotent.

Lemma 26.1. *Let Γ be a distance-regular graph of diameter $D \geq 1$. Pick $\theta, \theta_0^*, \theta_1^*, \dots, \theta_D^* \in \mathbb{R}$ such that $\theta_0^* \neq 0$, and set*

$$E = \frac{1}{|X|} \sum_{i=0}^D \theta_i^* A_i.$$

(i) *The following are equivalent.*

(ia) *θ is an eigenvalue of Γ , and E is a corresponding primitive idempotent.*

(ib)

$$\begin{pmatrix} a_0 & b_0 & 0 & \cdots & \cdots & 0 \\ c_1 & a_1 & b_1 & 0 & \cdots & 0 \\ 0 & c_2 & a_2 & b_2 & \ddots & \vdots \\ \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_{D-1} & a_{D-1} & b_{D-1} \\ 0 & \cdots & \cdots & 0 & c_D & a_D \end{pmatrix} \begin{pmatrix} \theta_0^* \\ \theta_1^* \\ \vdots \\ \vdots \\ \vdots \\ \theta_D^* \end{pmatrix} = \theta \cdots \begin{pmatrix} \theta_0^* \\ \theta_1^* \\ \vdots \\ \vdots \\ \vdots \\ \theta_D^* \end{pmatrix},$$

and $\theta_0^* = \text{rank } E$.

(ii) *Suppose (ia), (ib) hold. Then,*

$$\frac{\theta_1^*}{\theta_0^*}, \dots, \frac{\theta_D^*}{\theta_0^*}$$

can be computed from θ using

$$\frac{\theta_i^*}{\theta_0^*} = \frac{p_i(\theta)}{kb_1 \cdots b_{i-1}}, \quad (1 \leq i \leq D),$$

where $p_0 = 1$, $p_1(\lambda) = \lambda$, and

$$\lambda p_i(\lambda) = p_{i+1}(\lambda) + a_i p_i(\lambda) + b_{i-1} c_i p_{i-1}(\lambda) \quad (0 \leq i \leq D).$$

Proof.

(i) We have

$$(ia) \leftrightarrow (A - \theta I)E = O \text{ and } E^2 = E \quad (26.9)$$

$$\leftrightarrow 0 = \sum_{i=0}^D (A - \theta I) \theta_i^* A_i \text{ and } \text{rank} E = \text{trace} E = \theta_0^* \quad (26.10)$$

$$= \sum_{i=0}^D \theta_i^* (c_{i+1} A_{i+1} + a_i A_i + b_{i-1} A_{i-1} - \theta A_i) \quad (26.11)$$

$$= \sum_{j=0}^D A_j (c_j \theta_{j-1}^* + a_j \theta_j^* + b_j \theta_{j+1}^* - \theta \theta_j^*) \quad (26.12)$$

$$\leftrightarrow c_j \theta_{j-1}^* + a_j \theta_j^* + b_j \theta_{j+1}^* = \theta \theta_j^* \quad (0 \leq j \leq D) \text{ and } \text{rank} E = \theta_0^* \quad (26.13)$$

$$\leftrightarrow (ib). \quad (26.14)$$

Remark. The first \leftrightarrow . \rightarrow is clear.

\leftarrow : By the first condition, $AE = \theta E$. So E is a scalar multiple of the primitive idempotent corresponding to θ . Hence, $\text{rank} E = \text{trace} E$ implies E is the primitive idempotent.

(ii) We prove by induction on i .

$i = 0$ is trivial.

$i = 1$: Set $j = 0$ above $c_0 = 0, a_0 = 0, b_0 = k$. We have

$$k\theta_1^* = \theta\theta_0^*.$$

So

$$\frac{\theta_1^*}{\theta_0^*} = \frac{\theta}{k} = \frac{p_1(\theta)}{k}.$$

$i \geq 2$: Set $j = i - 1$ above. We have

$$c_{i-2}\theta_{i-2}^* + a_{i-1}\theta_{i-1}^* + b_{i-1}\theta_i^* = \theta\theta_{i-1}^*.$$

So,

$$\frac{\theta_i^*}{\theta_0^*} = \frac{\theta\theta_{i-1}^* - a_{i-1}\theta_{i-1}^* - c_{i-1}\theta_{i-2}^*}{b_{i-1}\theta_0^*} \quad (26.15)$$

$$= \left((\theta - a_{i-1}) \frac{\theta_{i-1}^*}{\theta_0^*} - c_{i-1} \frac{\theta_{i-2}^*}{\theta_0^*} \right) \frac{1}{b_{i-1}} \quad (26.16)$$

$$= \left((\theta - a_{i-1}) \frac{p_{i-1}(\theta)}{kb_1 \cdots b_{i-2}} - c_{i-1} \frac{p_{i-2}(\theta)}{kb_1 \cdots b_{i-3}} \right) \frac{1}{b_{i-1}} \quad (26.17)$$

$$= \frac{p_i(\theta)}{kb_1 \cdots b_{i-2}b_{i-1}}, \quad (26.18)$$

as desired.

□

Chapter 27

P -and Q -Polynomial Schemes

Friday, April 2, 1993

Theorem 27.1. *Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$.*

Let θ denote an eigenvalue of Γ with associated primitive idempotent

$$E = \frac{1}{|X|} \sum_{i=0}^D \theta_i^* A_i.$$

Then the following are equivalent.

- (i) Γ is Q -polynomial with respect to E .
- (ii) $\theta_0^* \neq \theta_h^*$ for all $h \in \{1, 2, \dots, D\}$ and for $i \in \{3, \dots, D\}$,

$$c_i \left(\theta_2^* - \theta_i^* - \frac{(\theta_1^* - \theta_{i-1}^*)^2}{\theta_0^* - \theta_i^*} \right) + b_{i-1} \left(\theta_2^* - \theta_{i-1}^* - \frac{(\theta_1^* - \theta_i^*)^2}{\theta_0^* - \theta_{i-1}^*} \right) \quad (27.1)$$

$$= (k - \theta)(\theta_1^* + \theta_2^* - \theta_{i-1}^* - \theta_i^*) - (\theta + 1)(\theta_0^* - \theta_2^*) \quad (27.2)$$

- (iii) $\theta_0^* \neq \theta_h^*$ for all $h \in \{1, 2, \dots, D\}$ and (27.2) holds for $i = 3$.

Remark. Note (27.2) is trivial for $i = 1, 2$.

$i = 1$:

$$\text{LHS} = \left(\theta_2^* - \theta_1^* - \frac{\theta_1^* - \theta_0^*}{\theta^* - \theta_1^*} \right) + k(\theta^* - \theta_0^*) \quad (27.3)$$

$$= \theta_2^* - \theta_1^* - \theta_0^* + \theta_1^* + k(\theta_2^* - \theta_0^*) \quad (27.4)$$

$$= (k+1)(\theta_2^* - \theta_0^*) \quad (27.5)$$

$$\text{RHS} = (k-\theta)(\theta_1^* + \theta_2^* - \theta_0^* - \theta_1^*) - (\theta+1)(\theta_0^* - \theta_2^*) \quad (27.6)$$

$$= (k+1)(\theta_2^* - \theta_0^*). \quad (27.7)$$

$i = 2$:

$$\text{LHS} = b_1 \left(\theta_2^* - \theta_1^* - \frac{\theta_1^* - \theta_0^*}{\theta^* - \theta_1^*} \right) \quad (27.8)$$

$$= b_1 \frac{(\theta_2^* - \theta_1^*)(\theta_0^* - \theta_1^* - \theta_2^* + \theta_1^*)}{\theta_0^* - \theta_1^*} \quad (27.9)$$

$$= b_1 \frac{(\theta_2^* - \theta_1^*)(\theta_0^* - \theta_2^*)}{\theta_0^* - \theta_1^*} \quad (27.10)$$

$$\text{RHS} = (\theta+1)(\theta_0^* - \theta_2^*). \quad (27.11)$$

Hence,

$$\text{LHS} = \text{RHS} \leftrightarrow b_1 \frac{\theta_2^* - \theta_1^*}{\theta_0^* - \theta_1^*} + (\theta+1) = 0 \quad (27.12)$$

$$= b_1(\theta_2^* - \theta_1^*) + (\theta+1)(\theta_0^* - \theta_1^*) = 0. \quad (27.13)$$

On the other hand,

$$b_1\theta_2^* + a_1\theta_1^* + c_1\theta_0^* = \theta\theta_1^* \quad (27.14)$$

$$b_1\theta_1^* + a_1\theta_1^* + c_1\theta_1^* = k\theta_1^*, \quad (27.15)$$

as $\theta\theta_0^* = k\theta_1^*$ We have

$$b_1(\theta_2^* - \theta_1^*) + (\theta_0^* - \theta_1^*) = \theta(\theta_1^* - \theta_0^*).$$

Proof. Immediate from the proof of Theorem 2.1 in ‘A new inequality for distance-regular graphs’ (Terwilliger, 1995) and Theorem 24.1. \square

Note. Suppose (i) – (iii) hold. In particular, $\theta_0^*, \theta_1^*, \dots, \theta_D^*$ are distinct. Then,

$$c_i + a_i + b_i = k \quad (0 \leq i \leq D).$$

$$c_i\theta_{i-1}^* + a_i\theta_i^* + b_i\theta_{i+1}^* = \theta\theta_j^* \quad (0 \leq i \leq D).$$

$$\frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i-3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} \quad \text{is independent of } i \quad (0 \leq i \leq D-3).$$

$$c_i \left(\theta_2^* - \theta_i^* - \frac{(\theta_1^* - \theta_{i-1}^*)^2}{\theta_0^* - \theta_i^*} \right) + b_{i-1} \left(\theta_2^* - \theta_{i-1}^* - \frac{(\theta_1^* - \theta_i^*)^2}{\theta_0^* - \theta_{i-1}^*} \right) \quad (27.16)$$

$$= (k - \theta)(\theta_1^* + \theta_2^* - \theta_{i-1}^* - \theta_i^*) - (\theta + 1)(\theta_0^* - \theta_2^*). \quad (27.17)$$

Furthermore, we can show for $c_1, \dots, c_D, a_1, \dots, a_D, b_0, b_1, \dots, b_{D-1}$ in terms of 5 parameters.

In general, we can take the 5 parameters to be

$$D, q, s^*, r_1, r_2$$

and get

$$b_i = \frac{h(1 - q^{i-D})(1 - s^*q^{i+1})(1 - r_1q^{i+1})(1 - r_2q^{i+1})}{(1 - s^*q^{2i+1})(1 - s^*q^{2i+2})} \quad (0 \leq i \leq D), \quad (27.18)$$

$$c_i = \frac{h(1 - q^i)(1 - s^*q^{D+i+1})(r_1 - s^*q^i)(r_2 - s^*q^i)}{s^*q^D(1 - s^*q^{2i})(1 - s^*q^{2i+1})} \quad (0 \leq i \leq D), \quad (27.19)$$

$$a_i = b_0 - c_i - b_i \quad (0 \leq i \leq D), \quad (27.20)$$

where h variable is chosen so that $c_1 = 1$.

(We must also consider limiting cases $h \rightarrow 0, s^* \rightarrow 0, q^* \rightarrow \pm 1$.)

See Theorem 2.1 in “The subconstituent algebra of an association scheme, I, II, III, (Terwilliger, 1992), (Terwilliger, 1993a), (Terwilliger, 1993b).

Definition 27.1. Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$. Choose $q \in \mathbb{R} \setminus \{0, -1\}$, set

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = 1 + q + \dots + q^{i-1} = \begin{cases} \frac{q^i - 1}{q - 1} & q \neq 1 \\ i & q = 1. \end{cases}$$

Definition 27.2. Γ has classical parameters if

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \quad (27.21)$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (27.22)$$

for some $\sigma, \alpha \in \mathbb{R}$.

(This happens for essentially all known families of distance-regular graphs with unbounded diameter, and is essentially equivalent to $s^* = 0$.)

Lemma 27.1. *With above notation, suppose (27.21), (27.22) hold. Then,*

(i) $\theta = \frac{b_1}{q} - 1$ is an eigenvalue of Γ with $\theta \neq k$.

(ii) Let $E = |X|^{-1} \sum_{i=0}^D \theta_i^* A_i$ be associated primitive idempotent. Then

$$\frac{\theta_i^*}{\theta_0^*} = 1 + \left(\frac{\theta}{k} - 1 \right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \quad (0 \leq i \leq D).$$

In particular, $\theta_i^* \neq \theta_0^*$ for all $i \in \{1, 2, \dots, D\}$.

(iii) Γ is Q -polynomial with respect to E .

Proof.

(i), (ii). Need to check

$$c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta \theta_i^* \quad (0 \leq i \leq D),$$

where $a_i = k - c_i - b_i \quad (0 \leq i \leq D)$.

(equivalently: check

$$c_i(\theta_{i-1}^* - \theta_i^*) + b_i(\theta_i^* - \theta_{i+1}^*) = (\theta - k)\theta_i^* \quad (0 \leq i \leq D), \quad (27.23)$$

where $c_i, b_i, \theta_i^*, \theta$ are as given.)

Remark.

$$\theta = \frac{b_1}{q} - 1, \quad \frac{\theta_i^*}{\theta_0^*} = 1 + \left(\frac{\theta}{k} - 1 \right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i}, \quad b_0 = \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma.$$

$i = 0$.

$$\frac{\theta_i^*}{\theta_0^*} = \frac{\theta}{k}, \quad -k \left(1 - \frac{\theta_1^*}{\theta_0^*} \right) = -k \left(1 - \frac{\theta}{k} \right) = \theta - k.$$

$$\frac{\theta_{i-1}^* - \theta_i^*}{\theta_0^*} = \left(\frac{\theta}{k} - 1 \right) \left(\begin{bmatrix} i-1 \\ 1 \end{bmatrix} q^{2-i} - \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) = - \left(\frac{\theta}{k} - 1 \right) q^{1-i}.$$

$$\theta - k = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - 1 \right) (\sigma - \alpha)/q - 1 \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma = \begin{bmatrix} D-1 \\ 1 \end{bmatrix} (\sigma - \alpha) - 1 - \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma.$$

$$(c_i(\theta_{i-1}^* - \theta_i^*) + b_i(\theta_i^* - \theta_{i+1}^*) - (\theta - k)\theta_i^*)/\theta_0^* \quad (27.24)$$

$$= - \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \left(\frac{\theta}{k} - 1 \right) q^{1-i} + \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\frac{\theta}{k} - 1 \right) q^{-i} \quad (27.25)$$

$$- (\theta - k) \left(1 + \left(\frac{\theta}{k} - 1 \right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) \quad (27.26)$$

$$= \left(\frac{\theta}{k} - 1 \right) \left(- \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) q^{1-i} + \begin{bmatrix} D-i \\ 1 \end{bmatrix} \left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \right) \quad (27.27)$$

$$- \left(\begin{bmatrix} D \\ 1 \end{bmatrix} \sigma + \left(\begin{bmatrix} D-1 \\ 1 \end{bmatrix} (\sigma - \alpha) - 1 - \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma \right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) \quad (27.28)$$

$$= \left(\frac{\theta}{k} - 1 \right) \left(- \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} - \alpha \left(\begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} q^{1-i} + \begin{bmatrix} D-i \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) \right) \quad (27.29)$$

$$+ \sigma \left(\begin{bmatrix} D-i \\ 1 \end{bmatrix} - \begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} + \begin{bmatrix} D \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) + \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \quad (27.30)$$

Check $\theta \neq k$. Suppose $\theta = k$. Then

$$\frac{b_1}{q} - 1 = k, \quad \text{and} \quad q > 0.$$

By (27.21), (27.22),

$$qc_i - b_i - q(qc_{i-1} - b_{i-1}) = (k - \theta)q \quad (1 \leq i \leq D) \quad (27.31)$$

$$= 0. \quad (27.32)$$

Remark. With the notation of Lemma 27.1, we have the above equality in

general.

$$qc_i - b_i - q(qc_{i-1} - b_{i-1}) \quad (27.33)$$

$$= q \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) - \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (27.34)$$

$$- q \left(q \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-2 \\ 1 \end{bmatrix} \right) - \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \right) \quad (27.35)$$

$$= \left(q \begin{bmatrix} i \\ 1 \end{bmatrix} - q^2 \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \quad (27.36)$$

$$+ \alpha \left(q \begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} + \begin{bmatrix} D \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (27.37)$$

$$- q^2 \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \begin{bmatrix} i-2 \\ 1 \end{bmatrix} - q \begin{bmatrix} D \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} + q \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \quad (27.38)$$

$$+ \sigma \left(- \begin{bmatrix} D \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ 1 \end{bmatrix} + q \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \quad (27.39)$$

$$= q + \alpha \left(- \begin{bmatrix} i \\ 1 \end{bmatrix} + \begin{bmatrix} D \\ 1 \end{bmatrix} + q \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) + \sigma(q^D - 1 + 1) \quad (27.40)$$

$$= q \left(1 + \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \alpha + q^{D-1} \sigma \right) \quad (27.41)$$

$$= q \left(\begin{bmatrix} D \\ 1 \end{bmatrix} \sigma - \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \sigma + \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \alpha + 1 \right) \quad (27.42)$$

$$= q \left(k - \frac{\begin{bmatrix} D \\ 1 \end{bmatrix} - 1}{q} (\sigma - \alpha) + 1 \right) \quad (27.43)$$

$$= q(k - \theta). \quad (27.44)$$

Hence,

$$qc_i - b_i = q(qc_{i-1} - b_{i-1}) \quad (1 \leq i \leq D) \quad (27.45)$$

$$= q^i(qc_0 - b_0) \quad (27.46)$$

$$= -q^i k. \quad (27.47)$$

If $i = D$, $qc_D = -q^D k$, $c_D = -q^{D-1} k < 0$, a contradiction.

(iii) Check the equation (ii) of Theorem 27.1 holds for $i = 3$.

□

Remark. $\theta_0^* \neq \theta_h^*$ for all $h \in \{1, 2, \dots, D\}$ and

$$c_3 \left(\theta_2^* - \theta_3^* - \frac{(\theta_1^* - \theta_2^*)^2}{\theta_0^* - \theta_3^*} \right) - b_2 \frac{(\theta_1^* - \theta_3^*)^2}{\theta_0^* - \theta_2^*} = (k - \theta)(\theta_1^* - \theta_3^*) - (\theta + 1)(\theta_0^* - \theta_2^*).$$

Pf.

$$\frac{\text{LHS}}{\theta_0^*} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \left(1 - \frac{\theta}{k} \right) \left(q^{-2} - \frac{q^{-2}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix} q^{-2}} \right) \quad (27.48)$$

$$- \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \left(1 - \frac{\theta}{k} \right) \frac{\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} q^{1-3} - 1 \right)^2}{\begin{bmatrix} 2 \\ 1 \end{bmatrix} q^{-1}} \quad (27.49)$$

$$= \left(1 - \frac{\theta}{k} \right) \left(\left(1 + \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) q^{-2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 2 \\ 1 \end{bmatrix} q^{-3} \right) \quad (27.50)$$

$$= \left(1 - \frac{\theta}{k} \right) \left(q^{-2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha \left(q^{-2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + q^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} D-2 \\ 1 \end{bmatrix} \right) \quad (27.51)$$

$$- q^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} D-2 \\ 1 \end{bmatrix} \sigma \right) \quad (27.52)$$

$$\frac{\text{RHS}}{\theta_0^*} = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} \sigma - \begin{bmatrix} D-1 \\ 1 \end{bmatrix} (\sigma - \alpha) + 1 \right) \left(1 - \frac{\theta}{k} \right) \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} q^{-2} - 1 \right) \quad (27.53)$$

$$\begin{bmatrix} D-1 \\ 1 \end{bmatrix} (\sigma - \alpha) \left(1 - \frac{\theta}{k} \right) \begin{bmatrix} 2 \\ 1 \end{bmatrix} q^{-1} \quad (27.54)$$

$$= \left(1 - \frac{\theta}{k} \right) \left(q^{-2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} q^{-1} \sigma \left(q^{D-2} - \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \right) \right) \quad (27.55)$$

$$+ \begin{bmatrix} 2 \\ 1 \end{bmatrix} q^{-2} \alpha \left(\begin{bmatrix} D-1 \\ 1 \end{bmatrix} + q \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \right) \right) \quad (27.56)$$

$$= \left(1 - \frac{\theta}{k} \right) \left(q^{-2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sigma q^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} D-2 \\ 1 \end{bmatrix} + \alpha q^{-2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \right) \quad (27.57)$$

Example 27.1. Q -polynomial distance-regular graphs with classical parameters.

D -cube: $c_i = i$, $b_i = D - i$

has classical parameters: $(q, \alpha, \sigma) = (1, 0, 1)$.

Johnson graph $J(D, N)$ ($N \geq 2D$):

$c_i = i^2$, $b_i = (D-i)(N-D-i)$ has classical parameters $(q, \alpha, \sigma) = (1, 1, N-D)$.

q -analogue of Johnson graph $J_q(D, N)$ ($D \geq 2D$):

$$c_i = \left(\frac{q^i - 1}{q - 1} \right)^2 = \begin{bmatrix} i \\ 1 \end{bmatrix}^2, \quad b_i = \frac{q(q^D - q^i)(q^{N-D} - q^i)}{(q - 1)^2}$$

has classical parameters

$$(q, \alpha, \sigma) = \left(q, q, \left(\frac{q^{N-D+1} - 1}{q - 1} \right) - 1 \right) = \left(q, q, \begin{bmatrix} N - D + 1 \\ 1 \end{bmatrix} - 1 \right).$$

Remark.

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\begin{bmatrix} N - D + 1 \\ 1 \end{bmatrix} - 1 - q \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (27.58)$$

$$= \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\begin{bmatrix} N - D + 1 \\ 1 \end{bmatrix} - \begin{bmatrix} i + 1 \\ 1 \end{bmatrix} \right) \quad (27.59)$$

$$= \frac{q(q^D - q^i)(q^{N-D} - q^i)}{(q - 1)^2}. \quad (27.60)$$

Chapter 28

The First Eigenspace of a Q -DRG

Monday, April 5, 1993

Lemma 28.1. *Let $\Gamma = (X, E)$ be distance-regular of diameter $D \geq 3$ with standard module V . Suppose Γ is Q -polynomial with respect to a primitive idempotent E_1 . Pick a vertex $x \in X$. Then*

$$E_1 V = \text{Span}\{E_1 \hat{y} \mid \partial(x, y) \leq 2\}.$$

In particular,

$$\dim E_1 V \leq 1 + k_1 + k_2.$$

Proof. Let $\Delta = \{E_1 \hat{y} \mid \partial(x, y) \leq 2\}$.

$E_1 V \supseteq \text{Span}\Delta$: clear.

$E_1 V \subseteq \text{Span}\Delta$: Pick a vertex $y \in X$. Show that $E_1 \hat{y} \in \text{Span}\Delta$.

Induction on $h = \partial(x, y)$.

Case $h \leq 2$.

$E_1 \hat{y} \in \text{Span}\Delta$ follows from construction.

Case $h \geq 3$.

Pick a vertex $x' \in X$ such that

$$\partial(x, x') = h - 3, \quad \partial(x', y) = 3.$$

By Theorem 24.1.

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} E_1 \hat{z} - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} E_1 \hat{z}' = r_{12}^3 (E_1 \hat{x} - E_1 \hat{y}),$$

$$r_{12}^3 = \frac{c_3(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_3^*} \neq 0.$$

So, $E_1 \hat{y}$ in $\text{Span}\{f, g, E_1 \hat{x}'\}$, where

$$f = \sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} E_1 \hat{z}, \quad g = \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} E_1 \hat{z}'.$$

Observe that each z in the f -sum satisfies $\partial(x, z) = h - 2$.

So, by induction hypothesis

$$E_1 \hat{z} \in \text{Span} \Delta, \quad \text{or } f \in \text{Span} \Delta.$$

Observe that each z' in the g -sum satisfies $\partial(x, z') = h - 1$.

So by induction hypothesis

$$E_1 \hat{z}' \in \text{Span} \Delta, \quad \text{or } g \in \text{Span} \Delta.$$

Also $\partial(x, x') = h - 3$ implies $E_1 \hat{x}' \in \text{Span} \Delta$.

Therefore $E_1 \hat{y} \in \text{Span} \Delta$. □

Note. Let Γ , E_1 , x be as in Lemma 28.1.

Assume $D \geq 4$.

Observe that there are many linear dependences among

$$\{E_1 \hat{y} \mid y \in \Delta\},$$

where $\Delta = \{y \in X \mid \partial(x, y) \leq 2\}$.

Take any $y \in X$ such that $\partial(x, y) \geq 4$.

More than one choice for x' in the proof of Lemma 28.1 implies

“more than one way to put $E_1 \hat{y} \in \text{Span} E_1 \Delta$.”

Open Problem:

(i) Give a precise description of the linear dependences among

$$\{E_1 \hat{y} \mid y \in \Delta\}.$$

(ii) Find a subset $\Delta' \subseteq \Delta$ such that

$$\{E_1 \hat{y} \mid y \in \Delta'\}$$

is a basis for $E_1 V$, (or find some other ‘nice’ basis for $E_1 V$).

Conjecture 28.1. *Let Γ , E_1 , x be as in Lemma 28.1. Set*

$$\widetilde{X} = \{y \in X \mid \partial(x, y) \leq 2\}, \quad (28.1)$$

$$\widetilde{\partial} = \text{the restriction of the distance function } \partial \text{ to } \widetilde{X}. \quad (28.2)$$

Then Γ is determined by \widetilde{X} and $\widetilde{\partial}$.

(There should be some canonical way to reconstruct Γ from \widetilde{X} and $\widetilde{\partial}$.)

Chapter 29

Tridiagonal Pair A, A^*

Wednesday, April 7, 1993

Introduction to Theorem 29.1

Let $\Gamma = (X, E)$ be distance-regular with diameter $D \geq 3$.

Assume Γ is Q -polynomial with respect to E_1 .

Fix a vertex $x \in X$. Write $E_i^* \equiv E_i^*(x)$, $A_i^* \equiv A_i^*(x)$, $A^* = A_1^*$.

We know for h, i, j ($0 \leq h, i, j \leq D$),

$$E_i^* A_h E_j^* = O \leftrightarrow p_{ij}^h = 0 \quad (29.1)$$

$$E_i A_h^* E_j = O \leftrightarrow q_{ij}^h = 0. \quad (29.2)$$

Also, for h, i, j ($0 \leq h, i, j \leq D$),

$$h < |i - j| \rightarrow p_{ij}^h = 0, \quad q_{ij}^h = 0 \quad (29.3)$$

$$h = |i - j| \rightarrow p_{ij}^h \neq 0, \quad q_{ij}^h \neq 0. \quad (29.4)$$

Some A_h (resp. A_h^*) is a polynomial of degree exactly h in A (resp. A^*), it follows, for h, i, j ($0 \leq h, i, j \leq D$),

$$E_i^* A^h E_j^*, \quad E_i A^{*h} E_j \quad \begin{cases} = 0 & \text{if } h < |i - j| \\ \neq 0 & \text{if } h = |i - j|. \end{cases}$$

We saw that there exist $\beta, \gamma, \delta \in \mathbb{R}$ such that

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \delta A^*].$$

In fact, there exist $\beta, \gamma^*, \delta^* \in \mathbb{R}$ such that

$$0 = [A^*, A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^*(A^* A + A A^*) - \delta^* A]$$

as well as will now show.

Let K denote any field. Let V denote any vector space over K of finite positive dimension. Let $\text{End}_K(V)$ denote the K -algebra of all K -linear transformations $V \rightarrow V$.

Theorem 29.1. *Given semi-simple elements $A, A^* \in \text{End}_K(V)$, suppose*

$$E_i(A^*)^h E_j : \begin{cases} = 0 & \text{if } h < |i - j| \\ \neq 0 & \text{if } h = |i - j|. \end{cases} \quad (0 \leq h, i, j \leq D) \quad (29.5)$$

$$E_i^* A^h E_j^* : \begin{cases} = 0 & \text{if } h < |i - j| \\ \neq 0 & \text{if } h = |i - j|. \end{cases} \quad (0 \leq h, i, j \leq R) \quad (29.6)$$

for some ordering E_0, E_1, \dots, E_D of the primitive idempotents for A , and some ordering $E_0^*, E_1^*, \dots, E_R^*$ of primitive idempotents for A^* . Then

(i) $R = D$.

(ii) There exist $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{K}$ such that

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \delta A^*] \quad (29.7)$$

$$= A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - A A^* A^2) \quad (29.8)$$

$$- \gamma(A^2 A^* - A^* A^2) - \delta(A A^* - A^* A) \quad (29.9)$$

$$0 = [A^*, A^{*2} A - \beta^* A^* A A^* + A A^{*2} - \gamma^*(A^* A + A A^*) - \delta^* A] \quad (29.10)$$

$$= A^{*3} A - A A^{*3} - (\beta + 1)(A^{*2} A A^* - A^* A A^{*2}) \quad (29.11)$$

$$- \gamma^*(A^{*2} A - A A^{*2}) - \delta^*(A^* A - A A^*). \quad (29.12)$$

(iii) Let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). Then,

$$\beta = \frac{\theta_i - \theta_{i+1} + \theta_{i+2} - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}} \quad (0 \leq i \leq D-3) \quad (29.13)$$

$$= \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} \quad (0 \leq i \leq D-3) \quad (29.14)$$

$$\gamma = \theta_i - \beta \theta_{i+1} + \theta_{i+2} \quad (0 \leq i \leq D-2) \quad (29.15)$$

$$\gamma^* = \theta_i^* - \beta \theta_{i+1}^* + \theta_{i+2}^* \quad (0 \leq i \leq D-2) \quad (29.16)$$

$$\delta = \theta_i^2 - \beta \theta_i \theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) \quad (0 \leq i \leq D-1) \quad (29.17)$$

$$\delta^* = \theta_i^{*2} - \beta \theta_i^* \theta_{i+1}^* + \theta_{i+1}^{*2} - \gamma^*(\theta_i^* + \theta_{i+1}^*) \quad (0 \leq i \leq D-1) \quad (29.18)$$

In particular, $\beta, \gamma, \gamma^*, \delta, \delta^*$ are uniquely determined by A, A^* and the above ordering of their primitive idempotents, whenever $D \geq 3$.

∴{.proof}

(i) By symmetry, it suffices to show $D \geq R$. Suppose $R > D$.

Since A is semisimple with exactly $D + 1$ distinct eigenvalues, the minimal polynomial of A has degree $D + 1$.

Since $R \geq D + 1$,

$$A^R \in \text{Span}\{A^j \mid 0 \leq j \leq D\}.$$

Multiplying each term on the left by E_R^* and on the right by E_0^* , we find

$$E_R^* A^R E_0^* \in \text{Span}\{E_R^* A^j E_0^* \mid 0 \leq j \leq D\}. \quad (29.19)$$

But by (29.6), the left side of (29.19) is nonzero and the right side of (29.19) is 0, a contradiction.

Hence $D \geq R$.

(ii), (iii)

Recalling the definitions, we have

$$A = \sum_{i=0}^D \theta_i E_i, \quad (29.20)$$

$$A^* = \sum_{i=0}^D \theta_i^* E_i^*, \quad (29.21)$$

$$A E_i = E_i A = \theta_i E_i \quad (0 \leq i \leq D), \quad (29.22)$$

$$A^* E_i^* = E_i^* A^* = \theta_i^* E_i^* \quad (0 \leq i \leq D). \quad (29.23)$$

Claim 1. For all integers i, j, k, ℓ ($0 \leq i, j, k, \ell \leq D$) such that $j + k \leq i - \ell$,

$$E_i^* A^j A^* A^k E_\ell^* = \begin{cases} \theta_{\ell+k}^* E_i^* A^{j+k} E_\ell^* & \text{if } j + k = i - \ell, \\ 0 & \text{if } j + k < i - \ell. \end{cases} \quad (29.24)$$

Proof of Claim 1. The product (29.24) eqia;s

$$E_i^* A^j \left(\sum_{h=0}^D \theta_h^* E_h^* \right) A^k E_\ell^* = \sum_{h=0}^D \theta_h^* E_i^* A^j E_h^* A^k E_\ell^*.$$

Now pick any h ($0 \leq h \leq D$), where

$$E_i^* A^j E_h^* A^k E_\ell^* \neq O.$$

Tehn by (29.6), $j \geq |i - h|$, otherwise

$$E_i^* A^j E_h^* = O$$

and by (29.5), $k \geq |h - \ell|$ otherwise

$$E_h^* A^k E_\ell^* = O.$$

Hence

$$j + k \geq |i - h| + |h - \ell| \geq |i - \ell| \geq i - \ell.$$

Now if $j + k < i - \ell$, we see there is no such h , so (29.24) holds.

(Pf. Suppose $i = j + k + \ell$ with $0 \leq i, j, k, \ell, h \leq D$.

Then $i \geq j, k, \ell$. Since $k = |h - \ell|$, if $h \neq \ell + k$, $h = \ell - k$ and $j - i - h$, $\ell - h + i - h = i - \ell$ implies $h = \ell$, $k = 0$ and $h = \ell + k$.)

This proves Claim 1.

Let M denote the subalgebra of $\text{End}_K(V)$ generated by A . Observe that M has a basis E_0, \dots, E_D as a vector space over K . Set

$$L := \text{Span}\{mA^*m - nA^*m \mid m, n \in M\}.$$

Claim 2. $\dim L \leq D$.

Proof of Claim 2. Since E_0, \dots, E_D span M ,

$$L = \text{Span}\{E_i A^* E_j - E_j A^* E_i \mid 0 \leq i < j \leq D\} \quad (29.25)$$

$$= \text{Span}\{E_{j-1} A^* E_j - E_j A^* E_{j-1} \mid 1 \leq j \leq D\} \quad (29.26)$$

by (29.5).

In particular, L has a spanning set of order D .

So, Claim 2 holds.

Claim 3. $\{A^i A^* - A^* A^i \mid 1 \leq i \leq D\}$ is a basis for L .

Proof of Claim 3. Since

$$A^i A^* - A^* A^i = A^i A^* I - i A^* A^i$$

is contained in L ($1 \leq i \leq D$), and since $\dim L \leq D$, it suffices to show the given elements are linearly independent.

Suppose they are dependent. Then there exists an integer i ($1 \leq i \leq D$) such that

$$A^i A^* - A^* A^i \in \text{Span}(A^j A^* - A^* A^j \mid 1 \leq j < i). \quad (29.27)$$

Multiplying each term in (29.27) on the left by E_i^* , and on the left by E_0^* , and simplifying using

$$E_i^*(A^\ell A^* - A^* A^\ell)E_0^* = (\theta_0^* - \theta_i^*)E_i^* A^\ell E_0^*,$$

we find

$$E_i^* A^\ell E_0^* \in \text{Span}(E_i^* A^j E_0^* \mid 1 \leq j < i). \quad (29.28)$$

But the left side of (29.28) is nonzero.

A contradiction.

Since $A^2 A^* A - A A^* A^2$ is contained in L , we find by Claim 2,

$$A^2 A^* A - A A^* A^2 = \sum_{i=1}^D \alpha_i (A^i A^* - A^* A^i) \quad (29.29)$$

for some $\alpha_0, \dots, \alpha_D \in K$.

Claim 4. $\alpha_i = 0$ ($3 < i \leq D$).

Proof of Claim 4. Suppose not, and set

$$t = \max\{i \mid 3 < i \leq D, \alpha_i \neq 0\}.$$

Then by (29.29), and Claim 1,

$$0 = E_t^* \left(A^2 A^* A - A A^* A^2 - \sum_{i=1}^D \alpha_i (A^i A^* - A^* A^i) \right) E_0^* \quad (29.30)$$

$$= \alpha_t (\theta_t^* - \theta_0^*) E_t^* A^t E_0^* \quad (29.31)$$

$$\neq 0. \quad (29.32)$$

(Since $\alpha_i = 0$ if $i > t$,

$$E_t^* A^2 A^* A E_0^* = E_t^* A A^* A^2 E_0^* = 0 \quad (\text{as } 2 + 1 < t - 0) \quad (29.33)$$

$$E_t^* A^i A^* E_0^* = E_t^* A^* A^i E_0^* = 0 \quad (29.34)$$

$$E_t^* A^t A^* E_0^* = \theta_0^* E_t^* A^t E_0^*, \quad (29.35)$$

$$E_t^* A^* A^t E_0^* = \theta_t^* E_t^* A^* A^t E_0^*. \quad (29.36)$$

$$(29.37)$$

A contradiction. This proves Claim 4.

Claim 5. Suppose $D \geq 3$. Then

$$\alpha_3 = \frac{\theta_{i+1}^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+3}^*} \quad \text{for all } i, (0 \leq i \leq D-3). \quad (29.38)$$

In particular, $\alpha \neq 0$.

Proof of Claim 5. Fix an integer i ($0 \leq i \leq D-3$). Then by (29.24) and (29.29),

$$0 = E_{i+3}^* \left(A^2 A^* A - A A^* A^2 - \sum_{j=1}^3 \alpha_j (A^j A^* - A^* A^j) \right) E_i^* \quad (29.39)$$

$$= (\theta_{i+1}^* - \theta_{i+2}^* - \alpha_3 (\theta_i^* - \theta_{i+3}^*)) E_{i+3}^* A^3 E_i^*. \quad (29.40)$$

But $E_{i+3}^* A^3 E_i^* \neq O$ by (29.6), so (29.38) holds.

This proves Claim 5.

Claim 6. Lines (29.7), (29.9), (29.14) hold.

Proof of Claim 6. First suppose $D \geq 3$. Then by (29.29), Claims 4, and 5,

$$A^2 2A^* A - AA^* A^2 = \alpha_3(A^3 A^* - A^* A^3) + \alpha_2(A^2 A^* - A^* A^2) + \alpha_1(AA^* - A^* A), \quad (29.41)$$

where $\alpha_3 \neq 0$. Hence

$$A^3 A^* - A^* A^3 - \frac{1}{\alpha_3}(A^2 A^* A - AA^* A^2) + \frac{\alpha_2}{\alpha_3}(A^2 A^* - A^* A^2) + \frac{\alpha_1}{\alpha_3}(AA^* - A^* A) = O.$$

Now (29.9) is immediate, where

$$\beta = \frac{1}{\alpha_3} - 1, \quad (29.42)$$

$$\gamma = -\frac{\alpha_2}{\alpha_3}, \quad (29.43)$$

$$\delta = -\frac{\alpha_1}{\alpha_3}. \quad (29.44)$$

The line (29.7) follows from the definition of $[\cdot, \cdot]$.

The line (29.14) is immediate from (29.38) and (29.42).

Now suppose $D < 3$. Then the line (29.14) is vacuously true, so consider (29.9).

Let α_3 denote any nonzero element of K .

Then $A^2 A^* - A^* A^2, AA^* - A^* A$ certainly span L by Claim 3.

So, (29.41) holds for appropriate α_1 and $\alpha_2 \in K$.

Now, (29.9) holds, where β, γ, δ are given by (29.42), (29.43), (29.44).

Claim 7. Lines (29.13), (16.11), (29.17) hold.

Proof of Claim 7. Pick an integer i ($0 \leq i \leq D-1$).

By (29.9), we have

$$O = E_i(A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A))E_{i+1} \quad (29.45)$$

$$= E_i A^* E_{i+1}(\theta_i^3 - \theta^3 - (\beta + 1)(\theta_i^* \theta_{i+1} - \theta_i \theta^2) - \gamma(\theta_i^2 - \theta_{i+1}^2) - \delta(\theta_i - \theta_{i+1})) \quad (29.46)$$

$$= E_i A^* E_{i+1}(\theta_i - \theta_{i+1})(\theta_i^2 + \theta_i \theta_{i+1} + \theta_{i+1}^2 - (\beta + 1)\theta_i \theta_{i+1} - \gamma(\theta_i + \theta_{i+1}) - \delta) \quad (29.47)$$

$$= E_i A^* E_{i+1}(\theta_i - \theta_{i+1})(\theta_i^2 - \beta \theta_i \theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \delta). \quad (29.48)$$

But $E_i A^* E_{i+1} \neq O$ by (29.5), and of course, $\theta_i \neq \theta_{i+1}$, so

$$0 = \theta_i^2 - \beta \theta_i \theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \delta.$$

This proves (29.17).

To obtain (16.11), pick any integer i ($0 \leq i \leq D-2$). Then by (29.17),

$$0 = \theta_i^2 - \beta \theta_i \theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \delta \quad (29.49)$$

$$- (\theta_{i+1}^2 - \beta \theta_{i+1} \theta_{i+2} + \theta_{i+2}^2 - \gamma(\theta_{i+1} + \theta_{i+2}) - \delta) \quad (29.50)$$

$$= \theta_i^2 - \beta \theta_i \theta_{i+1} - \gamma \theta_i + \beta \theta_{i+1} \theta_{i+2} - \theta_{i+2}^2 + \gamma \theta_{i+2} \quad (29.51)$$

$$= (\theta_i - \theta_{i+2})(\theta_i - \beta \theta_{i+1} + \theta_{i+2} - \gamma). \quad (29.52)$$

So $0 = \theta_i - \beta \theta_{i+1} + \theta_{i+2} - \gamma$.

This gives (16.11).

To see (29.13), pick an integer i ($0 \leq i \leq D-3$).

Then by (16.11),

$$0 = (\theta_i - \beta \theta_{i+1} + \theta_{i+2} - \gamma) - (\theta_{i+1} - \beta \theta_{i+2} + \theta_{i+3} - \gamma) \quad (29.53)$$

$$= \theta_i - (\beta + 1)\theta_{i+1} + (\beta + 1)\theta_{i+2} - \theta_{i+3}. \quad (29.54)$$

We have

$$\beta = \frac{\theta_i - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}} - 1 = \frac{\theta_i - \theta_{i+1} + \theta_{i+2} - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}},$$

as desired.

This proves Claim 7.

We have now proved (29.7), (29.9), (29.13), (29.14), (29.15), (29.17).

Interchanging the roles of A and A^* , we obtain (29.10), (29.12), (29.16), (29.18).

Chapter 30

R, F, L Matrices

Monday, April 12, 1993 # Edit Date

Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$ with standard module V .

Assume Γ is Q -polynomial with respect to the ordering

$$E_0, E_1, \dots, E_D$$

of primitive idempotents. Let A_i be an i -th adjacency matrix, and $A = A_1$.

$$A = \sum_{i=0}^D \theta_i A_i, \quad E_i = |X|^{-1} \sum_{i=0}^D \theta_i^* A_i.$$

Fix a vertex $x \in X$, write

$$E_i^* \equiv E_i^*(x), \quad A^* \equiv A^*(x), \quad A^* \equiv A_1^*, \quad T \equiv T(x).$$

Then

$$A^* = \sum_{i=0}^D \theta_i^* E_i^*.$$

By Theorem 29.1, there exist $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{R}$ such that

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \delta A^*] \quad (30.1)$$

$$0 = [A^*, A^{*2} A - \beta^* A^* A A^* + A A^{*2} - \gamma^*(A^* A + A A^*) - \delta^* A] \quad (30.2)$$

Recall raising matrix

$$R = \sum_{i=0}^D E_{i+1}^* A E_i^*$$

satisfies

$$R(E_i^*V) \subseteq E_{i+1}^*V \quad (0 \leq i \leq D), \quad E_{D+1}^*V = 0,$$

lowering matrix

$$L = \sum_{i=0}^D E_{i-1}^* A E_i^*$$

satisfies

$$L(E_i^*V) \subseteq E_{i-1}^*V \quad (0 \leq i \leq D), \quad E_{-1}^*V = 0,$$

and flat matrix

$$F = \sum_{i=0}^D E_i^* A E_i^*$$

satisfies

$$F(E_i^*V) \subseteq E_i^*V \quad (0 \leq i \leq D).$$

Also,

$$A = R + F + L.$$

Theorem 30.1. *With the above notation and assumptions,*

(i) *For all i ($2 \leq i \leq D$),*

$$g_i^- FL^2 + LFL + g_i^+ L^2 F - \gamma L^2) E_i^* = O,$$

where

$$g_i^+ = \frac{\theta_{i-2}^* - (\beta + 1)\theta_{i-1}^* + \beta\theta_i^*}{\theta_{i-2}^* - \theta_i^*} \quad (30.3)$$

$$g_i^- = \frac{\theta_{i-2}^* + (\beta + 1)\theta_{i-1}^* - \theta_i^*}{\theta_{i-2}^* - \theta_i^*}. \quad (30.4)$$

(ii) *For all i ($0 \leq i \leq D$),*

$$[F, LR - h_i RL] E_i^* = O,$$

where

$$h_i = \frac{\theta_{i-1}^* - \theta_i^*}{\theta_i^* - \theta_{i+1}^*} \quad (1 \leq i \leq D-1), \quad (30.5)$$

and h_0, h_D are indeterminants.

(iii) *For all i ($1 \leq i \leq D$),*

$$(e^- e_i RL^2 + (\beta + 2)LRL + e_i^+ L^2 R + LF^2 - \beta FLF + F^2 L - \gamma(LF + FL) - \delta L) E_i^* = O,$$

where

$$e_i^+ = \frac{\theta_{i-1}^* - (\beta + 2)\theta_i^* + (\beta + 1)\theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \quad (1 \leq i \leq D) \quad (30.6)$$

$$e_i^- = \frac{-(\beta + 1)\theta_{i-2}^* + (\beta + 2)\theta_{i-1}^* - \theta_i^*}{\theta_{i-1}^* - \theta_i^*} \quad (2 \leq i \leq D), \quad (30.7)$$

and e_0^+, e_1^- are indeterminants.

Proof. WE have

$$O = A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - A A^* A^2) \gamma (A^2 A^* - A^* A^2) - \delta (A A^* - A^* A).$$

(i) Fix i ($2 \leq i \leq D$), and multiply above on the left by E_{i-2}^* , and on the right by E_i^* . Now reduce.

For example,

$$E_{i-2}^* A^3 A^* E_i^* = \theta_i^* E_{i-2}^* A^3 E_i^*,$$

where

$$E_{i-2}^* A^3 E_i^* = E_{i-2}^* A \left(\sum_{r=0}^D E_r^* \right) A \left(\sum_{s=0}^D E_s^* \right) A E_i^* \quad (30.8)$$

$$= \sum_{r,s} E_{i-2}^* A E_r^* A E_s^* A E_i^* \quad (30.9)$$

$$= \sum_{r,s, |i-2-r| \leq 1, |r-s| \leq 1, |s-i| \leq 1} E_{i-2}^* A E_r^* A E_s^* A E_i^* \quad (30.10)$$

$$= E_{i-2}^* A E_{i-2}^* A E_{i-1}^* A E_i^* + E_{i-2}^* A E_{i-1}^* A E_{i-1}^* A E_i^* + E_{i-2}^* A E_{i-1}^* A E_i^* A E_i^* \quad (30.11)$$

$$= (FL^2 + LFL + L^2 F) E_i^*. \quad (30.12)$$

Reducing the other terms in a similar manner, and simplifying, we obtain (i).

Remark.

$$E_{i-2}^* A^3 A^* E_i^* = \theta_{i-2}^* E_{i-2}^* A^3 E_i^* \quad (30.13)$$

$$= \theta_{i-2}^* (FL^2 + LFL + L^2 F) E_i^* \quad (30.14)$$

$$E_{i-2}^* A^2 A^* A E_i^* = (\theta_{i-1}^* (FL^2 + LFL) + \theta_i^* L^2 F) E_i^* \quad (30.15)$$

$$E_{i-2}^* A A^* A^2 E_i^* = (\theta_{i-2}^* FL^2 + \theta_{i-1}^* (LFL + L^2 F)) E_i^* \quad (30.16)$$

$$E_{i-2}^* (A^2 A^* - A^* A^2) E_i^* = (\theta_i^* - \theta_{i-2}^*) L^2 E_i^* \quad (30.17)$$

$$E_{i-2}^* (A A^* - A^* A) E_i^* = O. \quad (30.18)$$

Then we have

$$O = ((\theta_i^* - \theta_{i-2}^*)(FL^2 + LFL + L^2F)) \quad (30.19)$$

$$- (\beta + 1)(\theta_{i-1}^*(FL^2 + LFL) + \theta^*L^2F - \theta_{i-2}^*FL^2 - \theta_{i-1}^*(LFL + L^2F)) \quad (30.20)$$

$$- \gamma(\theta_i^* - \theta_{i-2}^*)L^2)E_i^* \quad (30.21)$$

$$= ((\theta_i^* - \theta_{i-2}^* - (\beta + 1)(\theta_{i-1}^* - \theta_{i-2}^*))FL^2 + (\theta_i^* - \theta_{i-2}^*)LFL \quad (30.22)$$

$$+ (\theta_i^* - \theta_{i-2}^* - (\beta + 1)(\theta_i^* - \theta_{i-1}^*))L^2F - \gamma(\theta_i^* - \theta_{i-2}^*)L^2)E_i^* \quad (30.23)$$

$$= -(\theta_{i-2}^* - \theta_i^*) \left(\left(\frac{-\beta\theta_{i-2}^* + (\beta + 1)\theta_{i-1}^* - \theta_i^*}{\theta_{i-2}^* - \theta_i^*} \right) FL^2 + LFL \quad (30.24)$$

$$+ \left(\frac{\theta_{i-2}^* - (\beta + 1)\theta_{i-1}^* + \beta\theta_i^*}{\theta_{i-2}^* - \theta_i^*} \right) L^2F - \gamma L^2 \right) E_i^* \quad (30.25)$$

$$= (\theta_i^* - \theta_{i-2}^*)(g_i^- FL^2 + LFL + g_i^+ L^2F - \gamma L^2)E_i^*. \quad (30.26)$$

(ii), (iii) are obtained in a similar manner replacing $i - 2$ by i (resp. $i - 1$).

□

Remark.

(ii) We have

$$O = E_i^*(A^3A^* - A^*A^3 - (\beta + 1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*E))E_i^*.$$

Since $\beta + 1 \neq 0$, by (29.42) if $D \geq 3$,

$$O = E_i^*(A^2A^*A - AA^*A^2)E_i^* \quad (30.27)$$

$$= ((\theta_i^* - \theta_{i-1}^*)RLF + (\theta_i^* - \theta_{i+1}^*)LRF) + (\theta_{i-1}^* - \theta_i^*)FRL + (\theta_{i+1}^* - \theta_i^*)FLR)E_i^* \quad (30.28)$$

$$= [F, (\theta_{i-1}^* - \theta_i^*)RL - (\theta_i^* - \theta_{i+1}^*)LR]E_i^* \quad (30.29)$$

$$= (\theta_{i+1}^* - \theta_i^*) \left[F, LR - \frac{\theta_{i-1}^* - \theta_i^*}{\theta_i^* - \theta_{i+1}^*} RL \right] E_i^* \quad (30.30)$$

$$= (\theta_{i+1}^* - \theta_i^*)[F, LR - h_i RL]E_i^*. \quad (30.31)$$

(iii) We have

$$O = E_{i-1}^*(A^3A^* - A^*A^3 - (\beta + 1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*E))E_i^* \quad (30.32)$$

$$= ((\theta_i^* - \theta_{i-1}^*)(RL^2 + LRL + L^2R + LF^2 + FLF + F^2L)) \quad (30.33)$$

$$- (\beta + 1)((\theta_{i-1}^* - \theta_{i-2}^*)RL^2 + (\theta_{i-1}^* - \theta_i^*)LRL + (\theta_{i+1}^* - \theta_i^*)L^2R \quad (30.34)$$

$$+ (\theta_i^* - \theta_{i-1}^*)FLF \quad (30.35)$$

$$- \gamma(\theta_i^* - \theta_{i-1}^*)(LF + FL) \quad (30.36)$$

$$- \delta(\theta_i^* - \theta_{i-1}^*)E_i^* \quad (30.37)$$

$$= ((\theta_i^* - \theta_{i-1}^*) - (\beta + 1)(\theta_{i-1}^* - \theta_{i-2}^*))RL^2 \quad (30.38)$$

$$+ (\theta_i^* - \theta_{i-1}^*) - (\beta + 1)(\theta_{i-1}^* - \theta_i^*))LRL \quad (30.39)$$

$$+ (\theta_i^* - \theta_{i-1}^*) - (\beta + 1)(\theta_{i+1}^* - \theta_i^*))L^2R \quad (30.40)$$

$$+ (\theta_i^* - \theta_{i-1}^*)LF^2 + (\theta_i^* - \theta_{i-1}^*)F^2L \quad (30.41)$$

$$+ (\theta_i^* - \theta_{i-1}^* - (\beta + 1)(\theta_i^* - \theta_{i-1}^*))FLF \quad (30.42)$$

$$- \gamma(\theta_i^* - \theta_{i-1}^*)(LF + FL) \quad (30.43)$$

$$- \delta(\theta_i^* - \theta_{i-1}^*)E_i^* \quad (30.44)$$

$$= (\theta_i^* - \theta_{i-1}^*) \left(\frac{-(\beta + 1)\theta_{i-2}^* + (\beta + 2)\theta_{i-2}^* - \theta_i^*}{\theta_{i-1}^* - \theta_i^*} RL^2 + (\beta + 2)LRL \quad (30.45)$$

$$+ \frac{\theta_{i-1}^* - (\beta + 2)\theta_i^* + (\beta + 1)\theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} L^2R + LF^2 - \beta FLF + F^2L \quad (30.46)$$

$$- \gamma(LF + FL) - \delta L \Big) E_i^* \quad (30.47)$$

$$= (e_i^- RL^2 + (\beta + 2)LRL + e_i^+ L^2R + LF^2 - \beta FLF + F^2L - \gamma(LF + FL) - \delta L)E_i^*. \quad (30.48)$$

Lemma 30.1. *With the notation of Theorem 30.1,*

$$e_i^+ = \frac{\theta_i^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i-1}^*} \quad (1 \leq i \leq D - 2) \quad (30.49)$$

$$e_i^- = \frac{\theta_{i-1}^* - \theta_{i-3}^*}{\theta_{i-1}^* - \theta_i^*} \quad (3 \leq i \leq D) \quad (30.50)$$

$$g_i^+ = \frac{\theta_i^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i-2}^*} \quad (2 \leq i \leq D - 1) \quad (30.51)$$

$$g_i^- = \frac{\theta_{i-2}^* - \theta_{i-3}^*}{\theta_{i-2}^* - \theta_i^*} \quad (3 \leq i \leq D). \quad (30.52)$$

In particular, e_i^\pm, g_i^\pm are non-zero for the range of i given above.

Proof. In each case, equate the above expression with the corresponding expression in Theorem 30.1. The resulting equation is equal to (29.13). \square

Remark. By Corollary 26.1 and Theorem 29.1,

$$e_i^+ = \frac{\theta_{i-1}^* - (\beta + 2)\theta_i^* + (\beta + 1)\theta_{i+1}^*}{\beta_{i-1}^* - \theta_i^*},$$

and

$$\beta + 1 = \frac{\theta_{j-1}^* - \theta_j^* + \theta_{j+1}^* - \theta_{j+2}^*}{\theta_j^* - \theta_{j+1}^*} + 1 = \frac{\theta_{j-1}^* - \theta_{j+2}^*}{\theta_j^* - \theta_{j+1}^*}.$$

Hence,

$$e_i^+ = \frac{1}{\theta_{i-1}^* - \theta_i^*}(\theta_{i-1}^* - \theta_i^* - (\beta + 1)(\theta_i^* - \theta_{i+1}^*)) \quad (30.53)$$

$$= \frac{1}{\theta_{i-1}^* - \theta_i^*}(\theta_{i-1}^* - \theta_i^* - (\beta + 1)(\theta_i^* - \theta_{i+2}^*)) \quad (30.54)$$

$$= \frac{\theta_i^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i-1}^*}, \quad (30.55)$$

$$e_i^- = \frac{1}{\theta_{i-1}^* - \theta_i^*}(-(\beta + 1)\theta_{i-2}^* + (\beta + 2)\theta_{i-1}^* - \theta_i^*) \quad (30.56)$$

$$= \frac{1}{\theta_{i-1}^* - \theta_i^*}(\theta_{i-1}^* - \theta_i^* - \theta_{i-3}^* + \theta_i^*) \quad (30.57)$$

$$= \frac{\theta_{i-1}^* - \theta_{i-3}^*}{\theta_{i-1}^* - \theta_i^*}, \quad (30.58)$$

$$g_i^+ = \frac{1}{\theta_{i-2}^* - \theta_i^*}(\theta_{i-2}^* - (\beta + 1)\theta_{i-1}^* + \beta\theta_i^*) \quad (30.59)$$

$$= \frac{1}{\theta_i^* - \theta_{i-2}^*}(\theta_i^* - \theta_{i-2}^* + \theta_{i-2}^* - \theta_{i+1}^*) \quad (30.60)$$

$$= \frac{\theta_i^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i-3}^*}, \quad (30.61)$$

$$g_i^- = \frac{1}{\theta_{i-2}^* - \theta_i^*}(-\beta\theta_{i-2}^* + (\beta + 1)\theta_{i-1}^* - \theta_i^*) \quad (30.62)$$

$$= \frac{1}{\theta_{i-2}^* - \theta_i^*}(\theta_{i-2}^* - \theta_i^* + \theta_i^* - \theta_{i-3}^*) \quad (30.63)$$

$$= \frac{\theta_{i-2}^* - \theta_{i-3}^*}{\theta_{i-2}^* - \theta_i^*}. \quad (30.64)$$

Corollary 30.1. *Let $\Gamma = (X, E)$ be distance-regular of diameter $D \geq 3$, Q -polynomial with respect to E_0, E_1, \dots, E_D . Fix a vertex $x \in X$, write $E_i^* \equiv E_i^*(x)$, $R \equiv R(x)$, $L \equiv L(x)$, $F \equiv F(x)$. Then the following hold.*

(i) $FR^2E_j^* \in \text{Span}(RFRE_j^*, R^2FE_j^*, R^2E_j^*)$, $(0 \leq j \leq D - 3)$.

(ii) $R^2FE_j^* \in \text{Span}(RFRE_j^*, FR^2E_j^*, R^2E_j^*)$, $(1 \leq j \leq D - 2)$.

(iii) $LR^2E_j^* \in \text{Span}(RLRE_j^*, R^2LE_j^*, F^2RE_j^*, FRFE_j^*, RF^2E_j^*, RFE_j^*, FRE_j^*, RE_j^*)$, $(0 \leq j \leq D - 3)$.

(iv) $R^2LE_j^* \in \text{Span}(RLRE_j^*, LR^2E_j^*, F^2RE_j^*, FRFE_j^*, RF^2E^*j, RFE_j^*, FRE_j^*, RE_j^*),$
 $(1 \leq j \leq D).$

Proof. Immediate from Theorem 30.1, and Lemma 30.1. \square

Remark. By Theorem 30.1, and Lemma 30.1, we have the following, but similarly we can obtain above.

(i) $FL^2E_j^* \in \text{Span}(LFLE_j^*, L^2FE_j^*, L^2E_j^*), (3 \leq j \leq D).$

(ii) $L^2FE_j^* \in \text{Span}(LFLE_j^*, FL^2E_j^*, L^2E_j^*), (2 \leq j \leq D-1).$

(iii) $RL^2E_j^* \in \text{Span}(LRLE_j^*, L^2RE_j^*, F^2LE_j^*, FLFE_j^*, LF^2E^*j, LFE_j^*, FLE_j^*, LE_j^*),$
 $(3 \leq j \leq D).$

(iv) $L^2RE_j^* \in \text{Span}(LRLE_j^*, RL^2E_j^*, F^2LE_j^*, FLFE_j^*, LF^2E^*j, LFE_j^*, FLE_j^*, LE_j^*),$
 $(2 \leq j \leq D).$

Chapter 31

The “Inverse” of R

Wednesday, April 14, 1993

Let $\Gamma = (X, E)$ be any graph of diameter $D \geq 2$. Fix a vertex $x \in X$. Let $E_i^* \equiv E_i^*(x)$, and $T \equiv T(x)$.

Recall adjacency matrix

$$A = R + L + F \quad (31.1)$$

$$R = \sum_{i=0}^D E_{i+1}^* A E_i^*, \quad (31.2)$$

$$L = \sum_{i=0}^D E_{i-1}^* A E_i^*, \quad (31.3)$$

$$F = \sum_{i=0}^D E_i^* A E_i^*. \quad (31.4)$$

Observe R is not invertible (indeed RE .) So, R^{-1} does not exist.

Below we find a matrix “ R^{-1} ” $\in T(x)$ such that $R^{-1}Rv = v$ for “almost all” $v \in V$.

Lemma 31.1. *Let $\Gamma = (X, E)$ denote any graph, and the standard module V over \mathbb{C} .*

Fix a vertex $x \in X$, write

$$R \equiv R(x), \quad L \equiv L(x), \quad E_i^* \equiv E_i^*(x) \quad \text{for all } i.$$

Then,

(i) There exists unique “ R^{-1} ” $\in \text{Mat}_X(\mathbb{C})$ such that;

- (ia) $R^{-1}v = 0$ if $Lv = 0$ for $v \in V$.
- (ib) $R^{-1}RLv = Lv$ for all $v \in V$.
- (ii) $R^{-1}(E_i^*V) \subseteq E_{i-1}^*V$ ($0 \leq i \leq D$), $E_{-1}^*V = 0$.
- (iii) $R^{-1} \in \text{Mat}_X(\mathbb{Q})$.
- (iv) $R^{-1} \in T(x)$.

Proof.

(i) Consider the orthogonal direct sum.

$$V = (\text{Ker}L) + (\text{Ker}L)^\perp.$$

Claim 1. $RL(\text{Ker}L)^\perp \subseteq (\text{Ker}L)^\perp$.

Proof of Claim 1. Pick $v \in (\text{Ker}L)^\perp$, and $w \in \text{Ker}L$. Show

$$\langle RLv, w \rangle = 0.$$

But

$$\bar{R}^\top = R^\top = \left(\sum_{i=0}^D E_{i+1}^* A E_i^* \right)^\top = \sum_{i=0}^D E_i^* A E_{i+1}^* = L.$$

So,

$$\langle RLv, w \rangle = \langle Lv, \bar{R}^\top w \rangle = \langle Lv, Lw \rangle = 0.$$

Claim 2. $RL : (\text{Ker}L)^\perp \rightarrow (\text{Ker}L)^\perp$ is an isomorphism of vector spaces.

Proof of Claim 2. It suffices to show above map is one-to-one.

Suppose there is a vector $v \in (\text{Ker}L)^\perp$ such that $RLv = 0$.

Then,

$$0 = \langle RLv, v \rangle = \langle Lv, \bar{R}^\top v \rangle = \|Lv\|^2.$$

So $Lv = 0$.

Hence $v \in \text{Ker}L \cap (\text{Ker}L)^\perp = 0$.

This proves Claim 2.

Now “ R^{-1} ” denote the unique matrix in $\text{Mat}(\mathbb{C})$ such that

$$R^{-1}v = \begin{cases} 0 & \text{if } v \in \text{Ker}L \\ L(RL)^{-1}v & \text{if } v \in (\text{Ker}L)^\perp. \end{cases} \quad (31.5)$$

Observe that $(RL)^{-1} : (\text{Ker}L)^\perp \rightarrow (\text{Ker}L)^\perp$ exists by Claim 2.

Observe R^{-1} satisfies (ia) by (31.5).

Claim 3. R^{-1} satisfies (ib).

Proof of Claim 3. It suffices to check

$$R^{-1}RLv = Lv$$

for $v \in \text{Ker}L$ and $v \in (\text{Ker}L)^\perp$.

The case $v \in \text{Ker}L$ is clear. So assume $v \in (\text{Ker}L)^\perp$ by Claim 1. So,

$$R^{-1}(RLv) = L(RL)^{-1}RLv = Lv$$

as desired.

Uniqueness: Suppose a matrix $\hat{R}^{-1} \in \text{Mat}_X(\mathbb{C})$ satisfies (ia), (ib). Then, \hat{R}^{-1} satisfies (31.5) above.

(Pf. The first part is clear. Let $v \in (\text{Ker}L)^\perp$. By Claim 2, there exists $w \in (\text{Ker}L)^\perp$ such that $v \in RLw$. So $\hat{R}^{-1}v = \hat{R}^{-1}RLw = Lw = L(RL)^{-1}v$.)

Therefore, \hat{R}^{-1} agrees with R^{-1} on a basis for V , and $\hat{R}^{-1} = R^{-1}$.

(ii) Pick $v \in E_i^*V$. Show $R^{-1}v \in E_{i-1}^*V$.

Without loss of generality we may assume that $v \in \text{Ker}L$ or $v \in (\text{Ker}L)^\perp$.

If $v \in \text{Ker}L$, then $R^{-1}v = 0 \in E_{i-1}^*V$.

If $v \in (\text{Ker}L)^\perp$, then

$$R^{-1}v = L(RL)^{-1}v \in LE_i^*V \subseteq E_{i-1}^*V.$$

(iii) Observe $R, L \in (\text{Mat})_X(\mathbb{Q})$.

So $V, \text{Ker}L$, each has basis consisting of vectors in $\mathbb{Q}^{|X|}$.

Replacing the construction of R^{-1} with the base field replaced by \mathbb{Q} , we find a matrix $\tilde{R}^{-1} \in \text{Mat}_X(\mathbb{Q})$ satisfying (ia), (ib).

Now R^{-1} and \tilde{R}^{-1} agree on a basis, and hence $R^{-1} = \tilde{R}^{-1}$.

(iv) $RL = \bar{L}^\top L$ is a real symmetric matrix. So it is diagonalizable.

Let θ be any eigenvalue of RL . Let V_θ denote the corresponding maximal eigenspace in V . Then

$$V = \sum_{\theta: \text{eigenvalue for } RL} V_\theta \quad (\text{orthogonal direct sum}).$$

Let $E_\theta : V \rightarrow V_\theta$ denote the orthogonal projection. Then E_θ is a complex polynomial in RL .

Thus $E_\theta \in T(x)$.

E_θ is real. Since RL is an integral matrix, every eigenvalue of RL is an algebraic integer.

Claim 4. We have

$$R^{-1} = \sum_{\theta: \text{eigenvalue of } RL} \theta^{-1} L E_\theta. \quad (31.6)$$

In particular, $R^{-1} \in T(x)$.

Proof of Claim 4. Show two sides of (31.6) agree, when applied to arbitrary $v \in V$.

Without loss of generality, we may assume that $v \in V_\theta$ for some eigenvalue θ of RL .

Let θ' denote any eigenvalue of RL .

$$E_{\theta'} v = \begin{cases} 0 & \text{if } \theta' \neq \theta, \\ v & \text{if } \theta' = \theta. \end{cases}$$

RHS of (31.6) applied to v equals

$$\begin{cases} 0 & \text{if } \theta = 0, \\ \theta^{-1} L v & \text{if } \theta \neq 0. \end{cases}$$

Show this equals $R^{-1}v$.

Case $\theta = 0$: Since $RLv = 0$,

$$0 = \langle v, RLv \rangle = \|Lv\|^2.$$

Hence $Lv = 0$, or $v \in \text{Ker } L$. By (ia), $R^{-1}v = 0$.

Case $\theta \neq 0$: Since $RLv = \theta v$, $v = \theta^{-1}RLv$. Hence,

$$R^{-1}v = \theta^{-1}R^{-1}RLv = \theta^{-1}Lv$$

by (ib).

□

Chapter 32

Irreducible Modules of Endpoint i

Monday, April 19, 1993

Lemma 32.1. *Let $\Gamma = (X, E)$ be any graph. With the notation of Lemma 31.1, the following hold.*

(i) *Let W denote a thin irreducible T -module with endpoint r , diameter d . Pick i ($0 \leq i \leq d$), and pick $v \in E_{r+i}^* W$. Then,*

$$R^{-1}Rv = \begin{cases} v & \text{if } i < d, \\ 0 & \text{if } i = d. \end{cases}$$

(ii) *Assume Γ is distance regular and thin with respect to x . Pick t ($0 \leq t \leq D/2$), and pick $v \in E_t^* V$. Then*

$$R^{-1}R^i v = R^{i-1}v \quad (1 \leq i \leq D - 2t).$$

In particular, $R^{-1}Rv = v$.

(iii) *Assume Γ is distance regular and thin with respect to x . Then*

$$R : E_i^* V \rightarrow E_{i+1}^* V \quad (0 \leq i < D/2)$$

is one-to-one.

Proof.

(i) Let w_0, w_1, \dots, w_d be a basis for W and $w_i \in E_{r+i}^* W$,

$$Rw_i = w_{i+1} \quad (0 \leq i < d), \quad Lw_i = x_i(W)w_{i-1} \quad (1 \leq i \leq d).$$

So,

$$RLw_i = x_i(W)w_i \quad (1 \leq i \leq d).$$

(See Lemma 9.1.)

We want to find $R^{-1}Rw_i$.

If $i = d$, $R^{-1}Rw_d = 0$.

If $0 \leq i < d$,

$$R^{-1}Rw_i = R^{-1}w_{i+1} \tag{32.1}$$

$$= x_{i+1}(W)^{-1}R^{-1}RLw_{i+1} \tag{32.2}$$

$$= x_{i+1}(W)^{-1}Lw_{i+1} \tag{32.3}$$

$$= x_{i+1}(W)^{-1}x_{i+1}(W)w_i \tag{32.4}$$

$$= w_i. \tag{32.5}$$

Thus, we have (i).

$$RLw_i = Rx_i(W)w_{i-1} = x_i(W)w_i, \quad LRw_i = Lw_{i+1} = x_{i+1}(W)w_i \tag{32.6}$$

$$[L, R]w_i = (x_{i+1}(W) - x_i(W))w_i, \quad (0 \leq i \leq d), \tag{32.7}$$

$$x_0(W) = 0, \quad x_{d+1}(W) = 0, \tag{32.8}$$

$$[L, R]|_W = \sum_{i=0}^d (x_{i+1} - x_i(W))E_{r+i}^*|_W. \tag{32.9}$$

(ii) Let

$$V = \sum W \quad \text{orthogonal direct sum of thin irreducible } T\text{-modules.}$$

Then,

$$E_t^*V = \sum_{r(W) \leq t} E_t^*W \quad (\text{orthogonal direct sum}).$$

Without loss of generality, we may assume

$$v \in E_t^*W$$

for some thin irreducible T -module with endpoint at most t .

Now if $i \leq D - 2t$, then

$$t + i \leq D - t \quad (32.10)$$

$$\leq D - r(W) \quad (32.11)$$

$$\leq r(W) + d(W) \quad (D \leq 2r + d), \quad (32.12)$$

by Lemma 14.1 (iii).

So

$$t + i - 1 \leq r(W) + d(W) - 1.$$

Hence,

$$R^{-1}R^i v = R^{-1}R(R^{i-1}v) \quad (R^{i-1}v \in E_{t+i-1}^* W) \quad (32.13)$$

$$= R^{i-1}v \quad \text{by (i)}. \quad (32.14)$$

(iii) Suppose $Rv = 0$ for some $v \in E_i^* V$ ($0 \leq i < D/2$). Then

$$0 = R^{-1}Rv = v,$$

by (ii) with $t = i$ and $i = 1$.

□

Definition 32.1. Let $\Gamma = (X, E)$ denote any graph with the standard module V . Fix a vertex $x \in X$. Write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$, $L \equiv L(x)$.

1. For every i ($0 \leq i \leq D$), define subspace $V_i := V_i(x) \subseteq V$ by

$$V_i = \sum W,$$

where the sum begin over irreducible T -modules W with endpoint i .

Observe:

$$V = V_0 + V_1 + \cdots + V_D \quad (\text{orthogonal direct sum.})$$

V_0 is the trivial T -module.

2. $(E_i^* V)_{\text{new}} \equiv E_i^* V_i \quad (0 \leq i \leq D)$.

In general,

$$(E_i^* V)_{\text{new}} \subseteq \text{Ker} L \cap E^* i V \subseteq \text{Ker} L \cap E_i^* V \subseteq \text{Ker}(L E_i^*).$$

If each irreducible T -module with endpoint strictly less than i is thin,

$$(E_i^* V)_{\text{new}} = \text{Ker} L \cap E_i^* V \subseteq \text{Ker}(L \cdot E_i^*).$$

We have the assertion.

Remark.

$$E_i^*V = \sum_{j < i} V_j + V_i.$$

For V_j part, take $w_{i-j} \in W$ irreducible with endpoint $j < i$. Then,

$$Lw_{i-j} = x_{i-j}(W)w_{i-j-1} \neq 0,$$

and

$$L|_{\sum_{j < i} E_i^*V_j} : \sum_{j < i} E_i^*V_j \rightarrow V$$

is one to one.

Lemma 32.2. *Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$. Fix a vertex $x \in X$, $R \equiv R(x)$, $L \equiv L(x)$, $F \equiv F(x)$. Pick $v \in (E_1^*V)_{new}$. Then,*

- (i) $RE_i^*A_{i-1}v = c_iE_{i+1}^*A_i v \quad (1 \leq i \leq D)$.
- (ii) $FE_i^*A_{i-1}v = RE_{i-1}^*A_i V + (a_{i-1} - c_i + c_{i-1})E_i^*A_{i-1}v + c_iE_i^*A_{i+1}v \quad (1 \leq i \leq D)$.
- (iii) $LE_i^*A_{i-1}v = FE_{i-1}^*A_i V + (a_{i-1} - c_i + c_{i-1})E_{i-1}^*A_i v + b_{i-1}E_{i-1}^*A_{i-2}v \quad (2 \leq i \leq D)$.
- (iv) $LE_i^*A_{i+1}v = b_iE_{i-1}^*A_i v \quad (1 \leq i \leq D-1)$.

Proof.

(i) Let

$$v = \sum_{y \in X, \partial(x,y)=1} \alpha_y \hat{y} \quad \text{for some } \{\alpha_y\} \subseteq \mathbb{C}.$$

Then

$$Lv = \left(\sum_{y \in X, \partial(x,y)=1} \alpha_y \right) \hat{x} = 0.$$

So,

$$\sum_{y \in X, \partial(x,y)=1} \alpha_y = 0.$$

Thus,

$$v = \sum_{y \in X, \partial(x,y)=1} \alpha_y (\hat{y} - \hat{x}).$$

Let

$$\tilde{A}_i = A_0 + A_1 + \cdots + A_i \quad (0 \leq i \leq D).$$

Then

$$\tilde{A}_i v = \sum_{y \in X, \partial(x, y)=1} \alpha_y \tilde{A}_i (\hat{y} - \hat{x}) \quad (32.15)$$

$$= \sum_{y \in X, \partial(x, y)=1} \alpha_y \left(\sum_{z \in X, \partial(y, z)=i, \partial(x, z)=i+1} \hat{z} - \sum_{z' \in X, \partial(y, z')=i+1, \partial(x, z')=i} \hat{z}' \right) \quad (32.16)$$

$$= \sum_{y \in X, \partial(x, y)=1} \alpha_y (E_{i+1}^* A_i \hat{y} - E_i^* A_{i+1} \hat{y}) \quad (32.17)$$

$$= E_{i+1}^* A_i v - E_{i+1}^* A_{i+1} v. \quad (32.18)$$

Recall (Claim 1 in the proof of Theorem 16.1.)

$$A \tilde{A}_i = c_{i+1} \tilde{A}_{i+1} + (a_i - c_{i+1} + c_i) \tilde{A}_i + b_i \tilde{A}_{i-1} \quad (0 \leq i \leq D-1).$$

(This is valid for $i = 0$ as $A \tilde{A}_0 = AI = c_1 \tilde{A} - \tilde{A}_0 = A$ by setting $\tilde{A}_{i-1} = O$.)

Now (i)–(iv) are obtained by applying this to v on the right and multiplied by E_j^* ($0 \leq j \leq D$) on the left.

□

Remark. $A \tilde{A}_{i-1} v = A E_i^* A_{i-1} v - A E_{i-1}^* A_i v$. For $1 \leq i \leq D$,

$$(c_i \tilde{A}_i + (a_{i-1} - c_i + c_{i-1}) \tilde{A}_{i-1} + b_{i-1} \tilde{A}_{i-2}) v \quad (32.19)$$

$$= c_i E_{i+1}^* A_i v - c_i E_i^* A_{i+1} v \quad (32.20)$$

$$+ (a_{i-1} - c_i + c_{i-1}) E_i^* A_{i-1} v - (a_{i-1} - c_i + c_{i-1}) E_{i-1}^* A_i v \quad (32.21)$$

$$+ b_{i-1} E_{i-1}^* A_{i-2} v - b_{i-1} E_{i-2}^* A_{i-1} v. \quad (32.22)$$

(i) $R E_i^* A_{i-1} v = E_{i+1}^* A E_i^* A_{i-1} v = c_i E_{i+1}^* A_i v$ ($1 \leq i \leq D$).

(ii) For $1 \leq i \leq D$,

$$F E_i^* A_{i-1} v = E_i^* A E_i^* A_{i-1} v \quad (32.23)$$

$$= R E_{i-1}^* A_i v - c_i E_i^* A_{i+1} v + (a_{i-1} - c_i + c_{i-1}) E_i^* A_{i-1} v. \quad (32.24)$$

(iii) For $2 \leq i \leq D$,

$$L E_i^* A_{i-1} v = E_{i-1}^* A E_i^* A_{i-1} v \quad (32.25)$$

$$= F E_{i-1}^* A_i v - (a_{i-1} - c_i + c_{i-1}) E_{i-1}^* A_i v + b_{i-1} E_{i-1}^* A_{i-2} v. \quad (32.26)$$

(Even if $i = 1$, this is valid by setting $A_{i-2} = O$.)

(iv) For $1 \leq i \leq D-1$, $L E_i^* A_{i+1} v = E_{i-1}^* A E_i^* A_{i+1} v = b_i E_{i-1}^* A_i v$.

Lemma 32.3. *Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$. Fix a vertex $x \in X$, $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, $R = R(x)$, $F = F(x)$, $L = L(x)$.*

*For every $v \in (E_1^*V)_{\text{new}}$, the following are equivalent.*

- (i) $E_i^*A_{i-1}v, E_i^*A_{i+1}v$ are linearly dependent for every i ($1 \leq i \leq D-1$).
 - (ii) *There exists a thin irreducible T -module W with endpoint 1 that contains v .*
- If (i), (ii) hold then*

$$W = \text{Span}(E_1^*, E_2^*A_1v, \dots, E_D^*A_{i-1}v).$$

Proof. (ii) \rightarrow (i). Clear as

$$E_i^*A_{i-1}v, E_i^*A_{i+1}v \in E_i^*W = \text{Span}(w_{i-1}).$$

(i) \rightarrow (ii) Consider the sequence

$$E_1^*A_{i-1}v, E_2^*A_1v, E_3^*A_2v, \dots, E_{D+1}^*A_Dv.$$

The first term is nonzero and the last term is 0. So there exists

$$n := \min\{i \mid 1 \leq i \leq D, E_{i+1}^*A_i v = 0\}.$$

Now

$$E_{j+1}^*A_j v = 0 \quad (n \leq j \leq D). \quad (32.27)$$

Remark. Use induction and Lemma 32.2 (i),

$$E_{j+1}^*A_j v \in \text{Span}(RE_j^*A_{j-1}v) \quad (j \geq 1).$$

By our assumption (i), and the definition of n ,

$$E_j^*A_{j+1}v \in \text{Span}(E_j^*A_{j-1}v) \neq 0 \quad (1 \leq j \leq n).$$

By Lemma 32.2 (i),

$$RE_j^*A_{j-1}v \in \text{Span}(E_{j+1}^*A_j v) \quad (1 \leq j \leq n).$$

By Lemma 32.2 (ii),

$$FE_j^*A_{j-1}v \in \text{Span}(RE_{j-1}^*A_j v, E_j^*A_{j-1}v, E_j^*A_{j+1}v) \quad (32.28)$$

$$\subseteq \text{Span}(RE_{j-1}^*A_{j-2}v, E_j^*A_{j-1}v) \quad (32.29)$$

$$\text{Span}(E_{j-1}^*A_{j-1}v) \quad (1 \leq j \leq n). \quad (32.30)$$

By Lemma 32.2 (iii),

$$FE_j^*A_{j-1}v \in \text{Span}(FE_{j-1}^*A_jv, E_{j-1}^*A_jv, E_{j-1}^*A_{j-2}v) \quad (32.31)$$

$$\subseteq \text{Span}(FE_{j-1}^*A_{j-2}v, E_{j-1}^*A_{j-2}v) \quad (32.32)$$

$$\text{Span}(E_{j-1}^*A_{j-2}v) \quad (2 \leq j \leq n). \quad (32.33)$$

Hence,

$$W = \text{Span}(E_1^*A_0v, E_2^*A_1v, \dots, E_n^*A_{n-1}v).$$

is R , F , L invariant.

Therefore W is a thin T -module with endpoint 1 that contains v . □

Chapter 33

Title of the Chapter

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