Lecture Note on Terwilliger Algebra

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2022-11-20

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About this lecturenote

Setting

sudo This note is created by `bookdown` package on RStudio.

For `bookdown` See (Xie, 2015), (Xie, 2017), (Yihui Xie, 2018).

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3. Use This Template
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Chapter 1

Subconstituent Algebra of a Graph

Wednesday, January 20, 1993

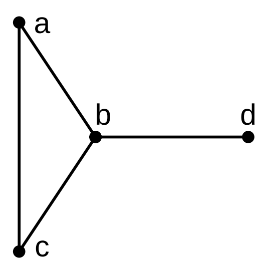
A graph (undirected, without loops or multiple edges) is a pair $\Gamma = (X, E)$, where

$$X = \text{finite set (of vertices)} \tag{1.1}$$

$$E = \text{set of (distinct) 2-element subsets of } X \text{ (= edges of) } \Gamma. \tag{1.2}$$

vertices x and $y \in X$ are adjacent if and only if $xy \in E$.

Example 1.1. Let Γ be a graph. $X = \{a, b, c, d\}$, $E = \{ab, ac, bc, bd\}$.



Set $n = |X|$, the order of Γ.

Pick a field K (= $\mathbb{R}$ or $\mathbb{C}$). Then $\mathrm{Mat}_X(K)$ denotes the K algebra of all $n \times n$ matrices with entries in K. (rows and columns are indexed by X)

Adjacency matrix $A \in \mathrm{Mat}_X(K)$ is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E \\ 0 & \text{else .} \end{cases} \tag{1.3}$$

Example 1.2. Let a, b, c, d be labels of rows and columns. Then

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The subalgebra M of $\mathrm{Mat}_X(K)$ generated by A is called the *Bose-Mesner algebra* of Γ.

Set $V = K^n$, the set of n-dimensional column vectors, the coorinates are indexed by X.

Let $\langle\ ,\ \rangle$ denote the Hermitean inner product:

$$\langle u, v\rangle = u^\top \cdot v \quad (u, v \in V)$$

V with $\langle\ ,\ \rangle$ is the *standard module* of Γ.

M acts on V: For every $x \in X$, write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where 1 is at the x position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since A is a real symmetrix matrix,

$$V = V_0 + V_1 + \cdots + V_r \qquad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal A-eigenspaces.

Let $E_i \in \mathrm{Mat}_X(K)$ denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then $E_0, \dots, E_r$ are the primitive idempotents of M.

$$M = \mathrm{Span}_K(E_0, \dots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all } \ i, j, \quad E_0 + \cdots + E_r = I.$$

Let θ_i denote the eigenvalue of A for V_i in $\mathbb{R}$. Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \cdots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \ \theta_i = \dim V_i = \text{rank} E_i.$$

Set

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \cdots, & \theta_r \\ m_0, & m_1, & \cdots, & m_r \end{pmatrix}.$$

Problem. What can we say about Γ when Spec(Γ) is given?

The following Lemma1.1, is an example of Problem.

For every $x \in X$,

$$k(x) \equiv \text{ valency of } x \equiv \text{ degree of } x \equiv |\{y \mid y \in X, \ xy \in E\}|.$$

Definition 1.1. The graph Γ is regular of valency k if $k = k(x)$ for every $x \in X$.

Lemma 1.1. *With the above notation,*

(i) $\theta_0 \le \max\{k(x) \mid x \in X\} = k^{\max}$.
(ii) If Γ *is regular of valency* k*, then* $\theta_0 = k$.

Proof. (i) Without loss of generality we may assume that $\theta_0 > 0$, else done. Let $v := \sum_{x \in X} \alpha_x \hat{x}$ denote the eivenvector for θ_0.

Pick $x \in X$ with $|\alpha_x|$ maximal. Then $|\alpha_x| \neq 0$.

Since $Av = \theta_0 v$,

$$\theta_0 \alpha_x = \sum_{y \in X, xy \in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \le \sum_{y \in X, xy \in E} |\alpha_y| \le k(x)|\alpha_x| \le k^{\max}|\alpha_x|.$$

(ii) All 1's vector $v = \sum_{x \in X} \hat{x}$ satisfies $Av = kv$.

□

Subconstituent Algebra

Let $x, y \in X$ and $\ell \in \mathbb{Z}^{\ge 0}$.

Definition 1.2. A path of length ℓ connecting x, y is a sequence

$$x = x_0, x_1, \ldots, x_\ell = y, \quad x_i \in X, \ 0 \le i \le \ell$$

such that $x_i x_{i+1} \in E$ for $0 \le i \le \ell - 1$.

Definition 1.3. The distance $\partial(x, y)$ is the length of a shortest path connecting x and y.

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

Definition 1.4. The graph Γ is connected if and only if $\partial(x, y) < \infty$ for all $x, y \in X$.

From now on, assume that Γ is connected with $|X| \geq 2$.

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \text{the diameter of } \Gamma.$$

Fix a 'base' vertex $x \in X$.

Definition 1.5.

$$d(x) = \text{the diameter with respect to } x = \max\{\partial(x, y) \mid y \in X\} \leq d.$$

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad \text{(orthogonal direct sum)},$$

where

$$V_i^* = \mathrm{Span}_K(\hat{y} \mid \partial(x, y) = i) \equiv V_i * (x)$$

and $V_i^* = V_i^*(x)$ is called the i-the subconstituent with respect to x.

Let $E_i^* = E_i^*(x)$ denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View $E_i^*(x) \in \mathrm{Mat}_X(K)$. So, $E_i^*(x)$ is diagonal with yy entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{else,} \end{cases} \qquad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \mathrm{Span}_K(E_0^*(x), \ldots, E_{d(x)}^*(x)).$$

Then $M^*(x)$ is a commutative subalgebra of $\mathrm{Mat}_X(K)$ and is calle the *dual Bose-Mesner algbara with respect to* x.

Definition 1.6 (Subconstituent Algebra)**.** Let $\Gamma = (X, E)$, x, M, $M^*(x)$ be as above. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(K)$ generated by M and $M^*(x)$. T is the *subconstituent algebra* of Γ with respect to x.

Definition 1.7. A T-module is any subspace $W \subset V$ such that $aw \in W$ for all $a \in T$ and $w \in W$.

T-module W is *irreducible* if and only if $W \neq 0$ and W does not properly contain a nonzero T-module.

For any $a \in \mathrm{Mat}_X(K)$, let a^* denbote the conjugate transpose of a.

Observe that

$$\langle au, v\rangle = \langle u, a^* v\rangle \quad \text{for all } \ a \in \mathrm{Mat}_X(K), \ \text{and for all } \ u, v \in V.$$

Lemma 1.2. *Let $\Gamma = (X, E)$, $x \in X$ and $T \equiv T(x)$ be as above.*

(i) If $a \in T$, then $a^* \in T$.

(ii) For any T-module $W \subset V$,

$$W^{\perp} := \{v \in V \mid \langle w, v\rangle = 0, \ \textit{for all } w \in W\}$$

is a T-module.

(iii) V decomposes as an orthogonal direct sum of irreducible T-modules.

Proof. (i) It is becase T is generated by symmetric real matrices

$$A, E_0^*(x), E_1^*(x), \dots, E_{d(x)(x)}^*.$$

(ii) Pick $v \in W^{\perp}$ and $a \in T$, it suffices to show that $av \in W^{\perp}$. For all $w \in W$,

$$\langle w, av\rangle = \langle a^* w, v\rangle = 0$$

as $a^* \in T$.

(iii) This is proved by the induction on the dimension of T-modules. If W is an irreducible T-module of V, then

$$V = W + W^{\perp} \quad \text{(orthogonal direct sum).}$$

$\square$

Problem. What does the structure of the $T(x)$-module tell us about Γ?

Study those Γ whose modules take ‘simple’ form. The Γ’s involved are highly regular.

Remark.

1. The subconstituent algebra T is semisimple as the left regular representation of T is completely reducible. See Curtis-Reiner 25.2 (Charles W. Curtis, 2006).
2. The inner product $\langle a, b\rangle_T = \mathrm{tr}(a^{\top}\bar{b})$ is nondegenerate on T.
3. In general,

$$\begin{aligned} T\text{: Semisimple and Artinian} &\Leftrightarrow T\text{: Artinian with } J(T) = 0 \\ &\Leftarrow T\text{: Artinian with nonzero nilpotent element} \\ &\Leftarrow T \subset \mathrm{Mat}_X(K) \text{ such that for all } a \in T \text{ is normal.} \end{aligned}$$

Chapter 2

Perron-Frobenius Theorem

Friday, January 22, 1993

In this lecture we use the Perron Frobenius theory of nonnegative matrices to obtain informaiton on eigenvalues of a graph.

Let $K = \mathbb{R}$. For $n \in \mathbb{Z}^{>0}$, pick a symmetrix matrix $C \in \mathrm{Mat}_n(\mathbb{R})$.

Definition 2.1. The matrix C is *reducible* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i \in X^+$, and for all $j \in X^-$, and for all $i \in X^-$, and for all $j \in X^+$,i.e.,

$$C \sim \begin{pmatrix} * & O \\ O & * \end{pmatrix}.$$

Definition 2.2. The matrix C is *bipartite* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i, j \in X^+$, and for all $i, j \in X^-$, i.e.,

$$C \sim \begin{pmatrix} O & * \\ * & O \end{pmatrix}.$$

Note.

1. If C is bipatite, for every eigenvalue θ of C, $-\theta$ is an eigenvalue of C such that $\mathrm{mult}(\theta) = \mathrm{mult}(-\theta)$.

Indeed, let $C = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$,

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix},$$

where $Ay = \theta x$ and $Bx = \theta y$.

2. If C is bipartite, C^2 is reducible.

3. The matrix C is irreducible and C^2 is reducible, if $C_{ij} \geq 0$ for all i, j and C is reducible. (Exercise)

Remark. Note 1. Even if C is not symmetric

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

holds. So the geometrix multiplicities coincide. How about the algebraic multiplicities?

Note 3. Set $x \sim y$ if and only if $C_{xy} > 0$. So the graph may have loops. Then

$$(C^2)_{xy} > 0 \Leftrightarrow \text{ if there exists } z \in X \text{ such that } x \sim z \sim y.$$

Note that C is irreducible if and only if $\Gamma(C)$ is connected. Let

$$X^+ = \{y \mid \text{there is a path of even length from } x \text{ to } y\} \tag{2.1}$$
$$X^- = \{y \mid \text{there is no path of even length from } x \text{ to } y\} \neq \emptyset. \tag{2.2}$$

If there is an edge $y \sim z$ in X^+ and $w \in X^-$. Then there would be a path from x to y of even length. So $\mathrm{e}(X^+, X^+) = \mathrm{e}(X^-, X^-) = 0$..

Theorem 2.1 (Perron-Frobenius)**.** *Given a matrix C in $\mathrm{Mat}_n(\mathbb{R})$ such that*

(a) C is symmetric.

(b) C is irreducible.

(c) $C_{ij} \geq 0$ for all i, j.

Let θ_0 be the maximal eigenvalue of C with eigenspace $V_0 \subseteq \mathbb{R}^n$, and let θ_r be the maximal eigenvalue of C with eigenspace $V_r \subseteq \mathbb{R}^n$. Then the following hold.

(i) Suppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$. *Then $\alpha_0 > 0$ for all i, or $\alpha_i < 0$ for all i.*

(ii) $\dim V_0 = 1$.

(iii) $\theta_r \geq -\theta_0$.

(iv) $\theta_r = \theta_0$ if and only if C is bipartite.

First, we prove the following lemma.

Lemma 2.1. *Let $\langle\ ,\ \rangle$ be the dot product in $V = \mathbb{R}^n$. Pick a symmetric matrix $B \in \mathrm{Mat}_n(\mathbb{R})$. Suppose all eigenvalues of B are nonnegative. (i.e., B is positive semidefinite.) Then there exist vectors $v_1, v_2, \dots, v_n \in V$ such that $B_{ij} = \langle v_i, v_j \rangle$ for $(1 \leq i, j \leq n)$.*

Proof. By elementary linear algebra, there exists an orthonormal basis $w_1, w_2, \dots, w_n$ of V consisting of eigenvectors of B. Set the i-th column of P is w_i and $D = \mathrm{diag}(\theta_1, \dots, \theta_n)$. Then $P^\top P = I$ and $BP = PD$.

Hence,

$$B = PDP^{-1} = PDP^\top = QQ^\top,$$

where

$$Q = P \cdot \mathrm{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \dots, \sqrt{\theta_n}) \in \mathrm{Mat}_n(\mathbb{R}).$$

Now, let v_i be the i-th column of $Q^\top$. Then

$$B_{ij} = v_i^\top \cdot v_j^- = \langle v_i, v_j \rangle.$$

□

Now we start the proof of Theorem 2.1.

Proof of Theorem 2.1(i)

Let $\langle , \rangle$ denote the dot product on $V = \mathbb{R}^n$. Set

$$\begin{aligned} B &= \theta I - C && (2.3) \\ &= \text{symmetric matrix with eigenvalues } \theta_0 - \theta_i \geq 0 && (2.4) \\ &= (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n} && (2.5) \end{aligned}$$

with the same $v_1, \dots, v_n \in V$ by Lemma 2.1.

Observe: $\sum_{i=1}^n \alpha_i v_i = 0$.

Pf.

$$\begin{aligned} \| \sum_{i=1}^n \alpha_i v_i \|^2 &= \langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \rangle && (2.6) \\ &= \begin{pmatrix} \alpha_1 & \dots & \alpha_n \end{pmatrix} B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} && (2.7) \\ &= v^\top B v && (2.8) \\ &= 0, && (2.9) \end{aligned}$$

since $Bv = (\theta_0 I - C)v = 0$.

Now set

$$s = \text{the number of indices} i, \text{ where } \alpha_i > 0.$$

Replacing v by $-v$ if necessary, without loss of generality we may assume that $s \geq 1$. We want to show $s = n$.

Assume $s < n$. Without loss of generality, we may assume that $\alpha_i > 0$ for $1 \leq s \leq s$ and $\alpha_i = 0$ for $s + 1 \leq i \leq n$. Set

$$\rho = \alpha_1 v_1 + \dots + \alpha_s v_s = -\alpha_{s+1} v_{s+1} - \dots - \alpha_n v_n.$$

Then, for $i = 1, \ldots, s$,

$$\begin{aligned}
\langle v_i, \rho \rangle &= \sum_{j=s+1}^{n} -\alpha_j \langle v_i, v_j \rangle \quad (\langle v_i, v_j \rangle = B_{ij}, B = \theta_0 I - C) && (2.10)\\
&= \sum_{j=s+1}^{n} (-\alpha_{ij})(-C_{ij}) && (2.11)\\
&\leq 0. && (2.12)
\end{aligned}$$

Hence

$$0 \leq \langle \rho, \rho \rangle = \sum_{i=1}^{s} \alpha_i \langle v_i, \rho \rangle \leq 0,$$

as $\alpha > 0$ and $\langle v_i, \rho \rangle \leq 0$. Thus, we have $\langle, \rho, \rho \rangle = 0$ and $\rho = 0$. For $j = s+1, \ldots, n$,

$$0 = \langle \rho, v_j \rangle = \sum_{i=1}^{s} \alpha_i \langle v_i, v_j \rangle \leq 0,$$

as $\langle v_i, v_j \rangle = -C_{ij}$.

Therefore,

$$0 = \langle v_i, v_j \rangle = -C_{ij} \text{ for } 1 \leq i, \leq s,\ s+1 \leq j \leq n.$$

Since C is symmetric,

$$C = \begin{pmatrix} * & O \\ O & * \end{pmatrix}$$

Thus C is reducible, which is not the case. Hence $s = n$.

Proof of Theorem 2.1 (*ii*).

Suppose $\dim V_0 \geq 2$. Then,

$$\dim \left(V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{\perp} \right) \geq 1.$$

So, there is a vector

$$0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with $\alpha_1 = 0$. This contradicts 1.

Now pick

$$0 \neq w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r.$$

Proof of Theorem 2.1 (*iii*).

Suppose $\theta_r < -\theta_0$. Since the eigenvalues of C^2 are the squares of those of C, θ_r^2 is the maximal eigenvalue of C^2.

Also we have $C^2 w = \theta_r^2 w$.

Observe that C^2 is irreducible. (As otherwise, C is bipartite by Note 3, and we must have $\theta_r = -\theta_0$.) Therefore, $\beta_i > 0$ for all i or $\beta_i < 0$ for all i. We have

$$\langle v, w\rangle = \sum_{i=1}^{n} \alpha_i \beta_j \neq 0.$$

This is a contradiction, as $V_0 \perp V_r$.

Proof of Theorem 2.1 (iv)

$\Rightarrow$: Let $\theta_r = -\theta_0$. Then $\theta = \theta_1^2 = \theta_0^2$ is the maximal eigenvalue of C^2, and v and w are linearly independent eigenvalues for θ for C^2. Hence, for C^2, $\mathrm{mult}(\theta) \geq 2$.

Thus by 2, C^2 must be reducible. Therefore, C is bipartite by Note 3.

$\Leftarrow$: This is Note 1. $\square$

Let $\Gamma = (X, E)$ be any graph.

Definition 2.3. Γ is said to be *bipartite* if the adjacency matrix A is bipartite. That is, X can be written as a disjoint union of X^+ and X^- such that X^+, X^- contain no edges of Γ.

Corollary 2.1. *For any (connected) graph* Γ *with*

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_1 & m_1 & \cdots & m_r \end{pmatrix} \quad \textit{with } \theta_0 > \theta_1 > \cdots > \theta_r.$$

Let V_i be the eigenspace of θ_i. Then the following holds.

1. *Supppose* $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \in \mathbb{R}^n$. *Then* $\alpha_i > 0$ *for all* i *or* $\alpha_i < 0$ *for all* i.
2. $m_0 = 1$.
3. $\theta_r \geq -\theta_0$ *if and only if* Γ *is bipartite. In this case,*

$$-\theta_i = \theta_{r-i} \textit{ and } m_i = m_{r-i} \quad (0 \leq i \leq r)$$

Proof. This is a direct consequences of Theorem 2.1 and Note 3. $\square$

Chapter 3

Cayley Graphs

Monday, January 25, 1993

Given graphs $\Gamma = (X, E)$ and $\Gamma' = (X', E')$.

Definition 3.1. A map $\sigma : X \to X'$ is an *isomorphism*\index{isomorophism of graphs whenever;

i. σ is one-to-one and onto,
ii. $xy \in E$ if and only if $\sigma x \sigma y \in E'$ for all $x, y \in X$.

We do not distinguish between isomorphic graphs.

Definition 3.2. Suppose $\Gamma = \Gamma'$. Above isomorphism σ is called an *automorphism* of Γ. Then set $\mathrm{Aut}(\Gamma)$ of all automorphisms of Γ becomes a finite group under composition.

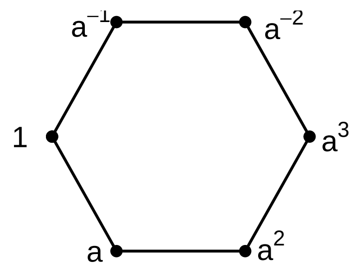
Definition 3.3. If $\mathrm{Aut}(\Gamma)$ acts transitive on X, Γ is called *vertex transitive*.

Example 3.1. A Cayley graphs:

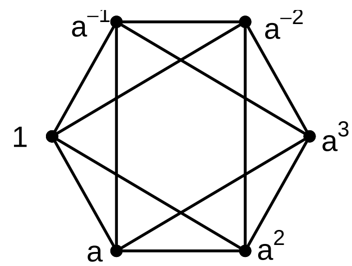
Definition 3.4 (Cayley Graphs)**.** Let G be any finite group, and Δ any generating set for G such that $1_G \notin \Delta$ and $g \in \Delta \to g^{-1} \in \Delta$. Then Cayley graph $\Gamma = \Gamma(G, \Delta)$ is defined on the vetex set $X = G$ with the edge set E define by the following.

$$E = \{(h_1, h_2) \mid h_1, h_2 \in G, h_1^{-1} h_2 \in \Delta\} = \{(h, hg) \mid h \in G, g \in \Delta\}$$

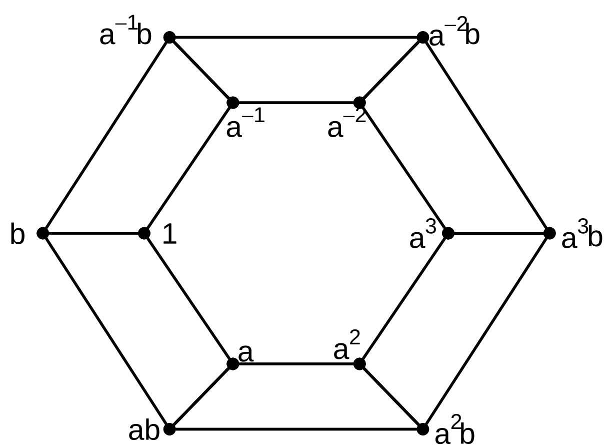
Example 3.2. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}\}$.



Example 3.3. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}, a^2, a^{-2}\}$.



Example 3.4. $G = \langle a, b \mid a^6 = 1, b^2, ab = ba \rangle$, $\Delta = \{a, a^{-1}, b\}$.



Remark. $\mathrm{Aut}(\Gamma) \simeq D_6 \times \mathbb{Z}_2$ contains two regular subgroups isomorphic to D_6 and $\mathbb{Z}_5 \times \mathbb{Z}_2$ and Γ is obtained as Cayley graphs in two ways.

Cayley graphs are vertex transitive, indeed.

Theorem 3.1. *The following hold.*

(i) *For any Cayley graph* $\Gamma = \Gamma(G, \Delta)$*, the map*

$$G \to \mathrm{Aut}(\Gamma)\ (g \mapsto \hat{g})$$

is an injective homomorphism of groups, where

$$\hat{g}(x) = gx \quad \textit{for all } g \in G \textit{ and for all } x \in X(= G).$$

Also, the image $\hat{G}$ is regular on X. i.e., the image $\hat{G}$ acts transitively on X with trivial vertex stabilizers.

(ii) For any graph $\Gamma = (X, E)$, suppose there exists a subgroup $G \subseteq \mathrm{Aut}(\Gamma)$ that is regular on X. Pick $x \in X$, and let

$$\Delta = \{g \in G \mid \langle x, g(x) \in E\}.$$

Then $1 \notin \Delta$, $g \in \Delta \to g^{-1} \in \Delta$, and Δ generates G. Moreover, $\Gamma \simeq \Gamma(G, \Delta)$.

Proof. (i) Let $g \in G$. We want to show that $\hat{g} \in \mathrm{Aut}(\Gamma)$. Let $h_1, h_2 \in X = G$. Then,

$$\begin{aligned} (h_1, h_2) \in E &\to h_1^{-1} h_2 \in \Delta && (3.1)\\ &\to (gh_1)^{-1}(gh_2) \in \Delta && (3.2)\\ &\to (gh_1, gh_2) \in E && (3.3)\\ &\to (\hat{g}(h_1), \hat{g}(h_2)) \in E. && (3.4) \end{aligned}$$

Hence, $\hat{g} \in \mathrm{Aut}(\Gamma)$.

Observe: $g \mapsto \hat{g}$ is a homomorphism of groups:

$$\hat{1}_G = 1, \; \widehat{g_1 g_2} = \widehat{g_1}\widehat{g_2}.$$

Observe: $g \mapsto \hat{g}$ is one-to-one:

$$\widehat{g_1} = \widehat{g_2} \to g_1 = \widehat{g_1}(1_G) = \widehat{g_2}(1_G) = g_2.$$

Observe: $\hat{G}$ is regular on X: Clear by construction.

(ii) $1_G \notin \Delta$: Since Γ has not loops, $(x, 1_G x) \notin E$.

$g \in \Delta \to g^{-1} \in \Delta$:

$$g \in \Delta \to (x, g(x)) \in E \to E \ni (g^{-1}(x), g^{-1}(g(x))) = (g^{-1}(x), x).$$

Δ generates G: Suppose $\langle \Delta \subsetneq G$. Let $\hat{X} = \{g(x) \mid g \in \langle \Delta \rangle\} \subsetneq X$. ($\hat{X} \subsetneq X$ as G acts regularly on X.)

Since Γ is connected, there exists $y \in \hat{X}$ and $z \in X \quad \hat{X}$ with $yz \in E$.

Let $y = g(x)$, $g \in \langle \Delta \rangle$, $z \in h(x)$, $h \in G \quad \langle \Delta \rangle$. Then

$$(y, z) = (g(x), h(x)) \in E \to (x, g^{-1}h(x)) \in E \to g^{-1}h \in \langle \Delta \rangle \to h \in \langle \Delta \rangle.$$

This is a contradition. Therefore, Δ generates G.

Let $\Gamma' = (X', E')$ denote $\Gamma(G, \Delta)$. We shall show that

$$\theta : X' \to X \ (g \mapsto g(x))$$

is an isomorphism of graphs.

θ is one-to-one: For $h_1, h_2 \in X' = G$,

$\theta(h_1) = \theta(h_2) \to h_1(x) = h_2(x) \to h_2^{-1}h_1(x) = x \to h_2^{-1}h_1 \in \mathrm{Stab}_G(x) = \{1_G\} \to h_1 = h_2$.

$(\mathrm{Stab}_G = \{g \in G \mid g(x) = x\}.)$

θ is onto: Since G is transitive,

$$X = \{g(x) \mid g \in G\} = \theta(X') = \theta(G).$$

θ respects adjacency: For $h_1, h_2 \in X' = G$,

$(h_1, h_2) \in E' \leftrightarrow h_1^{-1}h_2 \in \Delta \leftrightarrow (x, h_1^{-1}h_2(x)) \in E \leftrightarrow (h_1(x), h_2(x)) \in E \leftrightarrow (\theta(h_1), \theta(h_2)) \in E$.

Therefore θ is an isomorphism between graphs $\Gamma(G, \Delta)$ and $\Gamma(X, E)$. □

How to compute the eigenvalues of the Cayley graph of and abelian group.

Let G be any finite abelian group. Let $\mathbb{C}^*$ be the multiplicative group on $\mathbb{C}\ \ \{0\}$.

Definition 3.5. A (linear) G-character is any group homomorphism $\theta : G \to \mathbb{C}^*$.

Example 3.5. $G = \langle a \mid a^3 = 1\rangle$ has three characters, $\theta_0, \theta_1, \theta_2$.

$$\begin{array}{c|ccc} \theta_i(a^j) & 1 & a & a^2 \\ \hline \theta_0 & 1 & 1 & 1 \\ \theta_1 & 1 & \omega & \omega^2 \\ \theta_2 & 1 & \omega^2 & \omega \end{array}, \quad \text{with} \ \ \omega = \frac{-1+\sqrt{-3}}{2}.$$

Here ω is a primitive cube root pf q in $\mathbb{C}^*$, i.e., $1 + \omega + \omega^2 = 0$.

For arbitraty group G, let $X(G)$ be the set of all characters of G.

Observe: For $\theta_1, \theta_2 \in X(G)$, one can define product \$ _1 _2:

$$\theta_1\theta_2(g) = \theta_1(g)\theta_2(g) \quad \text{for all} \ \ g \in G.$$

Then $\theta_1\theta_2 \in X(G)$.

Observe: $X(G)$ with this product is an (abelian) group.

Lemma 3.1. *The groups G and $X(G)$ are isomorphic for all finite abelian groups G.*

Proof. G is a direct sum of cyclic groups;

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m, \quad \text{where } \; G_i = \langle a_i \mid a_i^{d_i} = 1 \rangle \quad (1 \leq i \leq m).$$

Pick any alement ω_i of order d_i in $\mathbb{C}^*$, i.e., a primitive d_i-the root of 1. Define

$$\theta_i : G \to \mathbb{C}^* \quad (a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \mapsto \omega_i^{\varepsilon_i} \quad \text{where } \; 0 \leq \varepsilon_i < d_i, 1 \leq i \leq m).$$

Then $\theta_i \in X(G)$. (Exercise)

Claim: There exists an isomorphism of groups $G \to X(G)$ that sends a_i to θ_i.

Observe: $\theta_i^{d_i} = 1$. For every $g = a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \in G$,

$$\theta_i^{d_i}(g) = (\theta_i(g))^{d_i} = (\omega_i^{\varepsilon_i})^{d_i} = (\omega_i^{d_i})^{\varepsilon_i} = 1.$$

Observe: If $\theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m} = 1$ for some $0 \leq \varepsilon_i < d_i, 1 \leq i \leq m$. Then $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = 0$.

Pf. $1 = \theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m}(a_i) = \omega_i^{\varepsilon_i}$, Since ω_i is a primitive d_i-th root of 1, $\varepsilon_i = 0$ for $1 \leq i \leq m$.

Observe: $\theta_1, \dots, \theta_m$ generate $X(G)$. Pick $\theta \in X(G)$. Since $a_i^{d_i} = 1$, $1 = \theta(a_i^{d_i}) = \theta(a_i)^{d_i}$.

Hence $\theta(a_i) = \omega^{\varepsilon_i}$ for some ε_i with $0 \leq \varepsilon_i < d_i$.

Now $\theta = \theta_1^{\varepsilon_1} \cdots \theta_m^{\varepsilon_m}$, since these are both equal to $\omega_i^{\varepsilon_i}$ at a_i for $1 \leq i \leq m$.

Therefore,

$$G \to X(G) \quad (a_i \mapsto \theta_i)$$

is an isomorphism of groups. $\square$

Note. The correspondence above is clearly a group homomorphism.

Chapter 4

Examples

Wednesday, January 27, 1993

Theorem 4.1. *Given a Cayley graph* $\Gamma = \Gamma(G, \Delta)$*. View the standard module* $V \equiv \mathbb{C}G$ *(the group algebra), so*

$$\left\langle \sum_{g \in G} \alpha_g g,\ \sum_{g \in G} \beta_g g \right\rangle = \sum_{g \in G} \alpha_g \overline{\beta_g}, \quad \textit{with } \alpha_g, \beta_g \in \mathbb{C}.$$

For any $\theta \in X(G)$*, write*

$$\hat{\theta} = \sum_{g \in G} \theta(g^{-1}) g.$$

Then the following hold.

(i) $\langle \hat{\theta}_1, \hat{\theta}_2 \rangle = |G|$ *if* $\theta_1 = \theta_2$ *and* 0 *othewise for* $\theta_1, \theta_2 \in X(G)$*. In particular,* $\{\hat{\theta} \mid \theta \in X(G)\}$ *forms a basis for* V*.*

(ii) $A\hat{\theta} = \Delta_\theta \hat{\theta}$ *for* $\theta \in X(G)$*, where* A *is the adjacency matrix and*

$$\Delta_\theta = \sum_{g \in \Delta} \theta(g).$$

In particular, the eigenvalues of Γ *are precisely*

$$\Delta_\theta \mid \theta \in X(G)\}.$$

Proof. (i) Claim: For every $\theta \in X(G)$, let

$$s := \sum_{g \in G} \theta(g^{-1}) = \begin{cases} |G| & \text{if } \theta = 1 \\ 0 & \text{if } \theta \neq 1. \end{cases}$$

Pf. Clear if $\theta = 1$.

Let $\theta \neq 1$. Then $\theta(h) \neq 1$ for some $h \in G$.

$$s \cdot \theta(h) = \left(\sum_{g \in G} \theta(g^{-1}) \right) \theta(h) = \sum_{g \in G} \theta(g^{-1}h) = \sum_{g' \in G} \theta(g'^{-1}) = s.$$

Since $\theta(h) \neq 1$, $s = 0$.

Claim. $\theta(g^{-1}) = \overline{\theta(g)}$ for every $\theta \in X(G)$ and every $g \in G$.

Since $\theta(g) \in \mathbb{C}$ is a root of 1,

$$|\theta(g)|^2 = \theta(g)\overline{\theta(g)} = 1.$$

On the other hand, since θ is a homomorphism,

$$\theta(g)\theta(g^{-1}) = \theta(1) = 1.$$

Hence $\theta(g^1) = \overline{\theta(g)}$.

Now

$$\begin{aligned}
\langle \widehat{\theta_1}, \widehat{\theta_2} \rangle &= \sum_{g \in G} \theta_1(g^{-1})\overline{\theta_2(g^{-1})} && (4.1)\\
&= \sum_{g \in G} \theta_1(g^{-1})\theta_2(g) && (4.2)\\
&= \sum_{g \in G} \theta_1\theta_2^{-1}(g^{-1}) && (4.3)\\
&= \begin{cases} |G| & \text{if} \quad \theta_1\theta_2^{-1} = 1 \\ 0 & \text{if} \quad \theta_1\theta_2^{-1} \neq 1. \end{cases} && (4.4)
\end{aligned}$$

Since $|G| = |X(G)|$ by Lemma 3.1, and $\widehat{\theta_i}$'s are orthogonal nonzero elements in V, thet form a basis of V.

(ii) Let $\Delta = \{g_1, \dots, g_r\}$. Then

$$
\begin{aligned}
A\hat{\theta} &= A\left(\sum_{g\in G}\theta(g^{-1}g)\right) && (4.5)\\
&= \sum_{g\in G}\theta(g^{-1})(gg_1+\cdots+gg_r) \quad (\Gamma(g)=\{gg_1,\ldots,gg_r\}) && (4.6)\\
&= \sum_{i=1}^{r}\left(\sum_{g\in G}\theta(g^{-1})(gg_i)\right) && (4.7)\\
&= \sum_{i=1}^{r}\left(\sum_{g\in G}\theta(g_i g_i^{-1}g^{-1})(gg_i)\right) && (4.8)\\
&= \sum_{i=1}^{r}\left(\sum_{g\in G}\theta(g_i)\theta((gg_i)^{-1})gg_i\right) && (4.9)\\
&= \sum_{i=1}^{r}\theta(g_i)\sum_{h\in G}\theta(h^{-1})h && (4.10)\\
&= \Delta_\theta\cdot\hat{\theta}. && (4.11)
\end{aligned}
$$

Since $\{\hat{\theta} \mid \theta\in X(G)\}$ forms a basis, the eigenvalues of Γ are precisely,

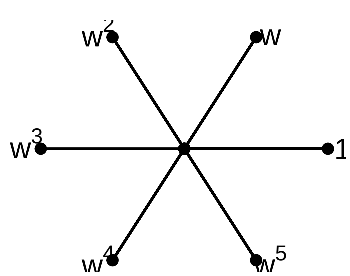
$$\{\Delta_\theta \mid \theta\in X(G)\}.$$

This completes the proof.

□

Example 4.1. Let $G=\langle a \mid a^6=1\rangle$, and $\Delta=\{a,a^{-1}\}$. Pick a primitive 6-th root of 1, ω. Then

$$X(G)=\{\theta^i \mid 0\le i\le 5\} \quad \text{such that} \quad \theta(a)=\omega,\ \omega+\omega^{-1}=1.$$



$\varphi \in X(G)$	$\varphi(a)$	$\Delta_\varphi = \theta(a) + \theta(a)^{-1}$
1	1	2
θ	ω	$\omega + \omega^{-1} = 1$
θ^2	ω^2	-1
θ^3	$\omega^3 = -1$	-2
θ^4	ω^4	-1
θ^5	ω^5	1

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

Example 4.2. *D*-cube, $H(D,2)$. Let

$$X = \{(a_1, \dots, a_D) \mid a_i \in \{1, -1\},\ 1 \le i \le D\},$$

$$E = \{xy \mid x, y \in X,\ x, y\text{: different in exactly one coordinate}\}.$$

Also $H(D,2)$ is a Cayley graph $\Gamma(G, \Delta)$, where

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_D,$$

$$G_i = \langle a_i \mid a_i^2 = 1 \rangle, \quad \Delta = \{a_1, \dots, a_D\}.$$

Homework: The spectrum of $H(D,2)$ is

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_D \\ m_0 & m_1 & \cdots & m_D \end{pmatrix},$$

where

$$\theta_i = D - 2i \quad (0 \le i \le D), \quad m_i = \binom{D}{i}.$$

Remark. Let $\theta \in X(G)$. Then $\theta : X \to \{\pm 1\}$. If

$$\nu(\theta) = |\{i \mid \theta(a_i) = -1\}|,$$

then $\Delta_\theta = D - 2i$. Since there are $\binom{D}{i}$ such θ, we have te assertion.

We want to compute the subconstituent algebra for $H(D,2)$. First, we make a few observations about arbitrary graphs.

Let $\Gamma = (X, E)$ be any graph, A, the adjacemcy matrix of Γ, and V, the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$. Write $E_i^* = E_i^*(x)$, and

$$T \equiv T(x) = \text{the algebra generated by } A, E_0^*, E_1^*, \dots.$$

Definition 4.1. Let W be any orreducible T-module ($\subseteq V$). Then the endpoint $r \equiv r(W)$ satisfied
$$r = \min\{i \mid E_i^* W \neq 0\}.$$
The diameter $d = d(W)$ satisfied
$$d = |\{i \mid E_i^* W \neq 0\}| - 1.$$

Lemma 4.1. *With the above notation, let W be an irreducible T-module. Then*

(i) $E_i^* A E_j^* = 0$ *if* $|i-j| = 1, \quad \neq 0$ *if* $|i-j| = 1, \quad 0 \leq i, j \leq d(x)$.
(ii) $A E_j^* W \subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$, $0 \leq j \leq d(x)$. ($E_i^* W = 0$ *if* $i < j$ *or* $i > d(x)$.)
(iii) $E_j^* W \neq 0$ *if* $r \leq j \leq r+d$, $= 0$ *if* $0 \leq j \leq r$ *or* $r + d < j \leq d(x)$.
(iv) $E_i^* A E_j^* W \neq 0$, *if* $|i-j| = 1$ $(r \leq i, j \leq r+d)$.

Proof. (i) Pick $y \in X$ with $\partial(x, y) = j$. We want to find $E_i^* A E_j^* \hat{y}$. Note,

$$E_j^* \hat{y} = \begin{cases} 0 & \text{if } \partial(x.y) \neq j \\ \hat{y} & \text{if } \partial(x,y) = j. \end{cases}.$$

$$\begin{aligned} E_i^* A E_j^* \hat{y} &= E_i^* A \hat{y} && (4.12) \\ &= E_i^* \sum_{z \in X, yz \in E} \hat{z} && (4.13) \\ &= \sum_{z \in X, yz \in E, \partial(x,z) = i} \hat{z} \qquad (*) && (4.14) \\ &= 0 \text{ if } |i-j| > 1 \text{ by triangle inequality.} && (4.15) \end{aligned}$$

If $|i-j| = 1$, there exist $y, y' \in X$ such that $\partial(x,y) = j$, $\partial(x, y') = i$, $yy' \in E$ by connectivity of Γ. Hence (*) contains $\widehat{y'}$ and $* \neq 0$

(ii) We have

$$\begin{aligned} A E_j^* W &= \left(\sum_{i=0}^{d(x)} E_i^* \right) A E_j^* W && (4.16) \\ &= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W && (4.17) \\ &\subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W. && (4.18) \end{aligned}$$

(iii) Suppose $E_j^* W = 0$ for some j $(r \leq j \leq r+d)$. Then $r < j$ by the definition of r. Set

$$\tilde{W} = E_r^* W + E_{r+1}^* W + \cdots + E_{j-1}^* W.$$

Observe $0 \subsetneq \tilde{W} \subsetneq W$. Also $A\tilde{W} \subseteq \tilde{W}$ by (ii) and $E_i^*\tilde{W} \subseteq \tilde{W}$ for every i by construction.

Thus $T\tilde{W} \subseteq \tilde{W}$, contradicting W beging irreducible.

$\square$

Chapter 5

T-Modules of $H(D,2)$, I

Friday, January 29, 1993

Let $\Gamma = (X, E)$ be a graph, A the adjacency matrix, and V the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$ and write $E_I^* \equiv E_i^*(x)$, and $T \equiv T(x)$.

Let W be an irreducible T-module with endpoint $r := \min\{i \mid E_i^* W \neq 0\}$ and diameter $d := |\{i \mid E_i^* W \neq 0\}| - 1$.

We have

$$
\begin{aligned}
E_i^* W &\neq 0 & r \leq i \leq r+d \qquad (5.1)\\
&= 0 & 0 \leq i < r \ \text{ or } \ r+d < i \leq d(x). \qquad (5.2)
\end{aligned}
$$

Claim: $E_i^* A E_j^* W \neq 0$ if $|i-j| = 1$ for $r \leq i, j \leq r+d$. (See Lemma 4.1.)

Suppose $E_{j+1}^* A E_j^* W = 0$ for some j with $r \leq j < r+d$. Observe that

$$\tilde{W} = E_r^* W + \cdot E_j^* W$$

is T-invariant with

$$0 \subsetneq \tilde{W} \subsetneq W.$$

Becase $A\tilde{W} \subseteq \tilde{W}$ since $AE_j^* W \subseteq E_{j-1}^* W + E_j^* W$,

$$E_k^* \tilde{W} \subseteq \tilde{W} \quad \text{for all} \ \ k,$$

we have $T\tilde{W} \subseteq W$.

Suppose $E_{i-1}^* A E_i^* W = 0$ for some i with $r \leq i < r+d$.

Similarly,

$$\tilde{W} = E_i^* W + \cdot E_{r+d}^* . W$$

is a T-module with $0 \subsetneq \tilde{W} \subsetneq W$.

Definition 5.1. Let Γ, E_i^*, and T be as above. Irreducible T-modules W and W' are isomorphic whenever there is an isomorphism $\sigma : W \to W'$ of vector spaces such that $a\sigma = \sigma a$ for all $a \in T$.

Recall that the standard module V is an orthogonal direct sum of irreducible T-modules $W_1 \oplus W_2 \oplus \cdots$. Given W in this list, the multiplicity of W in V is

$$|\{j \mid W_j \simeq W\}|.$$

Remark. It is known that the multiplicity does not depend on the decomposition.

Now assume that Γ is the D-cube, $H(D,2)$ with $D \geq 1$. Vew

$$X = \{a_1 \cdots a_D \mid a_i \in \{1,-1\}, 1 \leq i \leq D\}, \tag{5.3}$$
$$E = \{xy \mid x,y \in X,\ x,y \text{ differ in exactly 1 coordinate.}\}. \tag{5.4}$$

Find T-modules.

Claim: $H(D,2)$ is bipartite with a partition $X = X^+ \cup X^-$, where

$$X^+ = \{a_1 \cdots a_D \in X \mid \prod a_i > 0\} \tag{5.5}$$
$$X^- = \{a_1 \cdots a_D \in X \mid \prod a_i < 0\} \tag{5.6}$$

Observe: for all $y, z \in X$,

$$\partial(y,z) = i \Leftrightarrow y,z \text{ differ in exactly in } i \text{ coorinates with } 0 \leq i \leq D.$$

Here, the diameter of $H(D,2) = D = d$ for all $x \in X$.

Theorem 5.1. *Let $\Gamma = H(D,2)$ be as above. Fix $x \in X$, and write $E_i^* = E_i^*(x)$, and $T = T(x)$.*

Let W be an irreducible T-module with endpoint r, and diameter d with $0 \leq r \leq r + d \leq D$.

(i) W has a basis $w_0, w_1, \dots, w_d$ with $w_i \in E_{i+r}^* W$ for $0 \leq i \leq d$. With respect to which the matrix representing A is

$$\begin{pmatrix}
0 & d & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & d-1 & \cdots & 0 & 0 & 0 \\
0 & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 3 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 2 & 0 \\
0 & 0 & 0 & \cdots & d-1 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & d & 0
\end{pmatrix}$$

(ii) \$ d= D - 2r\$. In particular, $0 \leq r \leq D/2$.

(iii) Let W' *denote an irreducible* T*-module with endpoint* r'*. Then* W *and* W' *are isormorphic as* T*-modules if and only if* $r = r'$*.*

(iv) The multiplicity of the irreducible T*-module with endpoint* r *is*

$$\binom{D}{r} - \binom{D}{r-1} \quad \textit{if } 1 \leq r \leq R/2,$$

and 1 *if* $r = 0$.

Proof. Recall that Γ is vertex transitive. It is a Cayley graph.

Hence without loss of generality, we may assume that $x = \overbrace{11 \cdots 1}^{D}$.

Notation: Set $\Omega = \{1, 2, \ldots, D\}$. For every subset $S \subseteq \Omega$, let

$$\hat{S} = a_1 \cdot a_d \in X \quad a_i = \begin{cases} -1 & \text{if } \ i \in S \\ 1 & \text{if } i \notin S. \end{cases}$$

In particular, $\hat{emptyset} = x$ and

$$|S| = i \Leftrightarrow \partial(x, \hat{S}) = i \Leftrightarrow \hat{S} \in E_i^* V.$$

For all $S, T \subseteq \Omega$, we say S covers T\$ if and only if $S \supseteq T$ and $|S| = |T| + 1$.

Observe that $\hat{S}, \hat{T}$ are adjacent in Γ if and only if either T coverse S or S coverr T.

Define the 'raising matrix'

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*.$$

Observe that

$$R E_i^* V \subseteq E_{i+1}^* V \ \text{ for } \ 0 \leq i \leq D, \ \text{ and } E_{D+1}^* V = 0.$$

Indeed for any $S \subseteq \Omega$ with $|S| = i$,

$$R\hat{S} = R E_i^* \hat{S} \tag{5.7}$$

$$= E_{i+1}^* A \hat{S} \tag{5.8}$$

$$= \sum_{T_1 \subseteq \Omega, S \text{ covers } T_1} E_{i+1}^* \widehat{T_1} + \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T} \tag{5.9}$$

$$= \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T}. \tag{5.10}$$

Define the 'lowering matrix'

$$L = \sum_{i=0}^{D} E^*_{i-1} A E^*_i.$$

Observe that

$$LE^*_i V \subseteq E^*_{i-1V} \text{ for } 0 \le i \le D, \text{ and } E^*_{-1}V = 0.$$

Indeed for any $S \subseteq \Omega$,

$$L\hat{S} = \sum_{T \subseteq \Omega, S \text{ covers } T} \hat{T}.$$

Observe that $A = L + R$.

For convenience, set

$$A^* = \sum_{i=0}^{D} (D - 2i) E^*_i.$$

Claim: The following hold.

(a) $LR - RL = A^*$.
(b) $A * L - LA^* = 2L$.
(c) $A^* R - RA^* = -2R$.

In particular Span(R, L, A^*) is a 'representation of Lie algebra $\mathrm{sl}_2(\mathbb{C})$.

Remark (Lie Algebra sl2(C)).

$$\mathrm{sl}_2(\mathbb{C}) = \{X \mid \mathrm{Mat}(\mathbb{C} \mid \mathrm{tr}(X) = 0\}.$$

For $X, Y \in \mathrm{sl}_2(\mathbb{C})$, define a binary operation $[X, Y] = XY - YX$.

$$A^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then these satisfy the relations (a) - (c) above.

Proof of Claim. Apply both sides to $\hat{S} \quad (S \subseteq \Omega)$. Say $|S| = i$.

Proof of (a):

$$(LR - RL)\hat{S} = L\left(\sum_{\substack{T \subseteq \Omega, T \text{ covers } S \\ (D-i \text{ of them})}} \hat{T}\right) - R\left(\sum_{\substack{U \subseteq \Omega, S \text{ covers } U \\ (i \text{ of them})}} \hat{T}\right) \tag{5.11}$$

$$= (D - i)\hat{S} + \sum_{V \subseteq \Omega, |V| = i, |S \cap V| = i-1} \hat{V} - \left(i\hat{S} + \sum_{V \subseteq \Omega, |V| = i, |S \cap V| = i-1} \hat{V}\right) \tag{5.12}$$

$$= (D - 2i)\hat{S} \tag{5.13}$$

$$= A^*\hat{S}. \tag{5.14}$$

Proof of (b)*:*

$$(A^*L - LA^*)\hat{S} = (D-2(i-1))L\hat{S} - (D-2i)L\hat{S} \quad (\text{since } L\hat{S} \in E^*_{i-1}V) \tag{5.15}$$

$$= 2L\hat{S}. \tag{5.16}$$

Proof of (c)*:*

$$(A^*R - RA^*)\hat{S} = (D-2(i+1))R\hat{S} - (D-2i)R\hat{S} \quad (\text{since } R\hat{S} \in E^*_{i+1}V) \tag{5.17}$$

$$= 2R\hat{S}. \tag{5.18}$$

Let W be an irreducible T-module with endpoint r and diameter d $(0 \leq r \leq r+d \leq D)$.

Proof of (i) *and* (ii)*:*

Pick $0 \neq w \in E^*_rW$.

Claim: $LRw = (D-2r)w$.

Pf.

$$LRw = (A^* + RL)w \quad (\text{by Claim } (a)) \tag{5.19}$$

$$= A^*w \quad (Lw \in E^*_{r-1}W = 0) \tag{5.20}$$

$$(D-2r)w. \tag{5.21}$$

Define

$$w_i = \frac{1}{i!}R^iw \in E^*_{r+i}W \quad (0 \leq i \leq d).$$

Then,

$$Rw_i = (i+1)w_{i+1} \quad (0 \leq i \leq d) \tag{5.22}$$

$$Rw_d = 0 \quad (\text{by definition of } d) \tag{5.23}$$

Claim: $Lw_0 = 0$ and

$$Lw_i = (D-2r-i+1)w_{i-1} \quad (1 \leq i \leq d).$$

Pf. We prove by induction on i. The case $i=0$ is trivial, and the case $i=1$

follows from above claim. Let $i \geq 2$,

$$
\begin{aligned}
Lw_i &= \frac{1}{i} LRw_{i-1} = \frac{1}{i}(A^* + RL)w_{i-1} \quad \text{(by Claim (a))} && (5.24)\\
&\quad \text{(by induction hypothesis)} && (5.25)\\
&= \frac{1}{i}((D-2(r+i-1))w_{i-1} + (D-2r-(i-1)+1)Rw_{i-2} \quad (Rw_{i-2} = (i-1)w_{i-1}) && (5.26)\\
&= \frac{1}{i} i(D-2r-i+1)w_{i-1} && (5.27)\\
&= (D-2r-i+1)w_{i-1}. && (5.28)
\end{aligned}
$$

Claim: $w_0, \dots, w_d$ is a basis for W.

Pf. Let $W' = \mathrm{Span}\{w_0, \dots, w_d\}$. Then W' is R and L invariant. So it is $A = R + L$ invariant.

Also it is E_i^*-invariant for every i.

Hence W' is a T-module.

Since W is irreducible, $W' = W$.

As w_i's are orthogonal, theyy are linearly independent. Note that $w_i \neq 0$ by the definition of d and Lemma 4.1 (iv).

Claim: $d = D - 2r$.

Pf. By (a),

$$
\begin{aligned}
0 &= (LR - RL - A^*)w_d && (5.29)\\
&= 0 - (D-2r-d+1)Rw_{d-1} - (D-2(r+d))w_d && (5.30)\\
&= -d(D-2r-d+1)w_d - (D-2(r+d))w_d && (5.31)\\
&= (-dD + 2rd + d^2 - d - D + 2r + 2d)w_d && (5.32)\\
&= (d^2 + (2r-D+1)d + 2r - D)w_d && (5.33)\\
&= (d+2r-D)(d+1)w_d. && (5.34)
\end{aligned}
$$

Hence $d = D - 2r$.

Therefore, with respect to a bais $w_0, w_1, \dots, w_d$, $A = L + R$, $w_{-1} = w_{d+1} = 0$,

$$Lw_i = (d-i+1)w_{i-1}, \quad Rw_i = (i+1)w_{i+1}.$$

$$
L = \begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0\\ 0 & 0 & d-1 & \cdots & 0 & 0\\ & & \cdots & \cdots & & \\ & & & & 0 & 1\\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 2 & 0 & \cdots & & \\ & & & & 0 & 1\\ 0 & 0 & 0 & \cdots & d & 0 \end{pmatrix}.
$$

This completes the proof of (i) and (ii). □

Chapter 6

T-Modules of $H(D,2)$, II

Monday, February 1, 1993

Proof of Theorem 5.1 Continued. (iii) Let $r = r'$,

$w_0, \dots, w_d$: a basis for W with $w_i \in E_i^* W$, and

$w_0', \dots, w_d'$: a basis for W' with $w_i' \in E_i^* W'$.

Then $d = D - 2r = D - 2r' = d'$, and

$$\sigma : W \to W' \quad (w_i \mapsto w_i')$$

is an isomorsphism of T-modules by (i).

If $r \neq r'$, then

$$d = D - 2r \neq D - 2r' = d',$$

hence, $\dim W \neq \dim W'$.

(iv) Let W_i be the irreducible T-module with endpoint i. Then

$$\dim E_r^* V = \binom{D}{r} = \sum_{i=0}^{r} \mathrm{mult}(W_i).$$

Hence, we have that

$$\mathrm{mult}(W_r) = \binom{D}{r} - \binom{D}{r-1}$$

by induction on r.

$\square$

Theorem 6.1. *Let $\Gamma = H(D,2)$ with $D \geq 1$. Fix a vertex $x \in X$ and write*

$$E_i^* \equiv E_i^*(x), \quad T = T(x), and A^* \equiv \sum_{i=0}^{D}(D-2i)E_i^*.$$

Let W be an irreducible T-module with endpoint r with $0 \leq r \leq D/2$. Then,
(i) W has a basis

$$w_0^*, w_1^*, \dots, w_d^* \quad (d = D-2r), \quad \textit{such that} \;\; w_i^* \in E_{i+r}W \; (0 \leq i \leq d)$$

with respect to which the matrix corresponding to A^* is

$$\begin{pmatrix} 0 & d & 0 & & & & \\ 1 & 0 & d-1 & & & & \\ 0 & 2 & 0 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 0 & 2 & 0 \\ & & & & d-1 & 0 & 1 \\ & & & & 0 & d & 0 \end{pmatrix}.$$

In particular, / (ii) $E_iA^*E_j = 0$ if $|i-j| \neq 1$ for $0 \leq i,j \leq D$.

Proof. We use the notation,

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \; (= -[\beta, \alpha]).$$

Recall that

(a) $[L, R] = A^*$,
(b) $[A^*, L] = wL$,
(c) $[A^*, R] = -2R$,

and $A = L + R$.

Write $(a) - (c)$ in terms of A and A^*, we have,

$$[A, A^*] = [L, A^*] + [R, A^*] = 2(R-L).$$

$$\begin{cases} R + L &= A \\ R - L &= [A, A^*]/2. \end{cases}.$$

Hence,

$$R = \frac{1}{4}(2A + [A, A^*]) \quad \text{and} \tag{6.1}$$

$$L = \frac{1}{4}(2A - [A, A^*]). \tag{6.2}$$

Now (a), (b) become

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0 \tag{6.3}$$
$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0 \tag{6.4}$$

Pf. By (b),

$$\begin{aligned}
2A - AA^* + A^*A &= 4L && (6.5)\\
&= 2[A^*, L] && (6.6)\\
&= A^*\frac{2A - [A, A^*]}{2} - \frac{2A - [A, A^*]}{2}A^* && (6.7)
\end{aligned}$$

So we have ((6.4))

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0.$$

By (a),

$$\begin{aligned}
-16A^* &= [2A + [A, A^*], 2A - [A, A^*]] && (6.8)\\
&(2A + [A, A^*])(2A - [A, A^*]) - (2A - [A, A^*])(2A + [A, A^*]) && (6.9)\\
&= [4A^2 - 2A[A, A^*] + [A, A^*](2A) - [A, A^*]^2 && (6.10)\\
&\quad - 4A^2 - 2A[A, A^*] + [A, A^*](2A) + [A, A^*]^2 && (6.11)\\
&= -4A^2A^* + 4AA^*A + 4AA^*A - 4A^*A^2. && (6.12)
\end{aligned}$$

So,

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0.$$

Claim: $E_i^*A^*E_j = 0$ if $|i - j| \neq 1$ for $0 \leq i, j \leq D$.

Pf. We have,

$$\begin{aligned}
0 &= E_i(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_j && (6.13)\\
&= E_iA^*E_j(\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4) && (6.14)\\
&\quad (AE_j = \theta_jE_j,\ E_iA = (AE_j)^\top = (\theta_iE_i)^\top = \theta_iE_i) && (6.15)\\
&= E_iA^*E_j(\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) && (6.16)\\
&= E_iA^*E_j(D - 2i - (D - 2j) - 2)(D - 2i - (D - 2j) + 2) && (6.17)\\
&\quad (\theta_k = D - 2k) && (6.18)\\
&= E_iA^*E_j \cdot 4(i - j + 1)(i - j - 1) && (6.19)
\end{aligned}$$

and $i - j + 1 \neq 0$, $i - j - 1 \neq 0$. Hence, $E_i^*A^*E_j = 0$.

Now define "dual raising matrix",

$$R^* = \sum_{i=0}^{D} E_{i+1}A^*E_i.$$

So,

$$R^*E_iV \subseteq E_{i+1}V, \quad (0 \le i \le D,\ E_{D+1}V = 0).$$

Define "dual lowering matrix"

$$L^* = \sum_{i=0}^{D} E_{i-1}A^*E_i.$$

Then

$$L^*E_iV \subseteq E_{i-1}V \quad (0 \le i \le D,\ E_{-1}V = 0).$$

Observe that

$$A^* = \left(\sum_{i=0}^{D} E_i\right) A^* \left(\sum_{j=0}^{D} E_j\right) = L^* + R^*$$

by Claim 1.

Claim 2. We have | (a) $[L^*, R^*] = A$, | (b) $[A, L^*] = 2L^*$, | (c) $[A, R^*] = -2R^*$.

Pf. (b)

$$AL^* - L^*A = \sum_{i=0}^{D}(AE_{i-1}A^*E_i - E_{i-1}A^*E_iA) \tag{6.20}$$

$$= \sum_{i=0}^{D} E_{i-1}A^*E_i(\theta_{i-1} - \theta_i) \tag{6.21}$$

$$(\theta_k = D - 2k, \quad \theta_{i-1} - \theta_i = 2I - 2(i-1) = 2 \tag{6.22}$$

$$= 2L^*. \tag{6.23}$$

(c) Similar.

Remark.

$$AR^* - R^*A = \sum_{i=0}^{D}(AE_{i+1}A^*E_i - E_{i+1}A^*E_iA) \tag{6.24}$$

$$= \sum_{i=0}^{D} E_{i+1}A^*E_i(\theta_{i+1} - \theta_i) \tag{6.25}$$

$$= 2R^*. \tag{6.26}$$

(a) We have, by (b), (c)

$$[A, A^*] = [A, L^*] + [A, R^*] = 2(L^* - R^*). \tag{6.27}$$

Since $A^* = L^* + R^*$,

$$R^* = \frac{2A^* + [A^*, A]}{4}, \quad L^* = \frac{2A^* - [A^* - A]}{4}.$$

Now (a) is seen to be equivalent to ((6.4)) upon evaluation. This proves Claim 2.

Remark.

$$[L^*, R^*] = \frac{1}{16}((2A^* - [A^*, A])(2A^* + [A^*, A]) - (2A^* + [A^*, A])(2A^* - [A, A^*])) \tag{6.28}$$

$$= \frac{1}{16}(4{A^*}^2 + 2A^*[A^*, A] - [A^*, A]2A^* - [A^*, A]^2 - 4{A^*}^2 + 2A^*[A^*, A] - [A^*, A]2A^* + [A^*, A]^2) \tag{6.29}$$

$$= \frac{1}{4}({A^*}^2 A - 2A^*AA^* + A{A^*}^2) \tag{6.30}$$

$$= A, \tag{6.31}$$

by ((6.4)).

Now apply same argument as for ((6.3)), ((6.4)) of Theorem 5.1 and observe A^* has $D+1$ distinct eigenvalues. So,

$$A^* = \sum_{i=0}^{D}(D - 2i)E_i^*$$

generates

$$M^* = \mathrm{Span}(E_0^*, \dots, E_D^*).$$

Hence, $E_0, \dots, E_D,\ A^*$ generates T.

Take an irreducible T-module W with endpoint r with $0 \le r \le D/2$. Set $t = \min\{i \mid E_i W\}$.

Pick $0 \neq w_0^* \in E_t W$. Set

$$w_i^* = \frac{1}{i!}{R^*}^i w_0^* \in E_{t+i}W \quad \text{for all } i.$$

Then,

$$R^* w_i^* = (i+1)w_{i+1}^* \quad \text{for all } i.$$

By (a), we get by induction, $L^* w_i^* = (D - 2t - i + 1)w_{i-1}^*$,

$$L^* w_i^* = \frac{1}{i} L^* R^* w_{i-1}^* \tag{6.32}$$

$$= \frac{1}{i}(A + R^*L^*)w_{i-1}^* \tag{6.33}$$

$$= \frac{1}{i}((D - 2(t + i - 1))w_{i-1}^* + (i-1)(D - 2t - i + 2)w_{i-1}^*) \tag{6.34}$$

$$= (D - 2t - i + 1)w_{i-1}^*. \tag{6.35}$$

So $\mathrm{Span}(w_0^*, w_1^*, \dots)$ is L^*, R^*, A^*-invariant. Hence, $W = (Span)(w_0^*, w_1^*, \dots, w_d^*)$, $w_0^*, w_1^*, \dots, w_d^* \neq 0$, $w_i^* = 0$ for every $i > d$ by dimension.

Thus $d = D - 2t$.

Pf.

$$\begin{aligned}(D-2(t+d))w_d^* &= Aw_d^* && (6.36)\\ &= (L^*R^*-R^*L^*)w_d^* && (6.37)\\ &= -(D-2t-d+1)R^*w_{d-1}^* && (6.38)\\ &= -(D-2t-d+1)dw_d^*. && (6.39)\end{aligned}$$

Hence,

$$0 = d^2+(2t-D-1+2)d-(D-2t) = (d-D+2t)(d+1)$$

So $d = D-2t$. □

Definition 6.1. For any graph $\Gamma = (X, E)$, pick a vertex $x \in X$ and set $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

(i) an irreducible T-module W is thin if $\dim E_i^*W \leq 1$ for every i,

(ii) Γ is thin with respet to x, if every irreducible $T(x)$-module is thin,

(iii) an irreducible T-module W is dual thin if $\dim E_iW \leq 1$ for every i,

(iv) Γ is dual thin with respect to x, if every irreducible $T(x)$-module is dual thin.

Observe: $H(D,2)$ is thin, dual thin with respect to each $x \in X$.

With above notation, write $D \equiv D(x)$.

(i) an ordering $E_0, E_1, \ldots, E_R$ of primitive idempotents of Γ is restricted if E_0 corresponds to the maximal eigenvalue.

Fix a restricted ordering,

(ii) Γ is Q-polynomial with respect to x, above ordering if there exists $A^* \equiv A^*(x)$ such that

(a) $E_0^*V, \ldots, E_D^*V$ are the maximal eigenspaces for A^*.

(b) $E_iA^*E_j = 0$ if $|i-j| > 1$ for $0 \leq i,j \leq R$.

Observe $H(D,2)$ is Q-polynomial with respect to the natural ordering of the idempotents and every vetex.

Program. Study graphs that are thin and Q-polynomial with respect to each vertex.

(In fact, thin with respect to x implies dual thin with respect to x.)

Get a situation like $H(D,2)$, where T is generated by A, A^*. Except $\mathrm{sl}_s(\mathbb{C})$ is repalaced by a quantum Lie algebra.

Chapter 7

The Johnson Graph $J(D, N)$

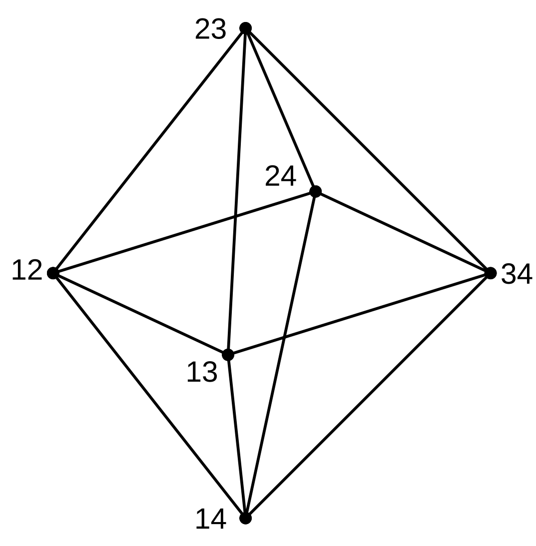
Wednesday, February 3, 1993

Definition 7.1. The Johnson graph, $\Gamma = J(D, N)$ $(1 \leq D \leq N-1)$ satisfies

$$X = \{S \mid S \subset \Omega,\ |S| = D\} \quad \text{where} \quad \Omega = \{1, 2, \dots, N\} \tag{7.1}$$

$$E = \{ST \mid S, T \in X, \quad |S \cap T| = D - 1\}. \tag{7.2}$$

Example 7.1. $J(2, 4)$



Note 1. The symmetric group S_N acts on Ω. $S_N \subseteq \operatorname{Aut}(\Gamma)$ acts vertex transitively on Γ.

Note 2. $\Gamma = J(D,N)$ is isomorphic to $\Gamma' = J(N-D,N)$.

$$\Gamma = (X,E) \qquad\qquad \Gamma' = (X',E') \tag{7.3}$$
$$X \ni S \quad \longrightarrow \quad \bar{S} = \Omega \quad S \in X' \tag{7.4}$$

This correspondence induces an isomorphism of graphs.

Pf.

$$ST \in E \Leftrightarrow |S \cap T| = D-1 \tag{7.5}$$
$$\Leftrightarrow |\Omega - (S \cup T)| = N-D-1 \tag{7.6}$$
$$\Leftrightarrow |\bar{S} \cap \bar{T}| = N-D-1 \tag{7.7}$$
$$\Leftrightarrow \bar{S}\bar{T} \in E' \tag{7.8}$$

Hence, without loss of generality, assume

$$D \le N/2 \quad \text{for } J(D,N).$$

We sill need the eigenvalues of $J(D,N)$ for certain problem later in the course. We can get these eigenvalues from our study of $H(D,2)$.

Lemma 7.1. *The eigenvalues for $J(D,N)$ with $1 \le D \le N/2$ are give by*

$$\theta_i = (N-D-i)(D-i)-i \quad (0 \le i \le D) \tag{7.9}$$
$$m_i = \binom{D}{i} - \binom{N}{i-1}. \tag{7.10}$$

Proof. Let

$$\Gamma_J \equiv J(D,N) = (X_J, E_J) \tag{7.11}$$
$$\Gamma_H \equiv H(N,2) = (X_H, E_H). \tag{7.12}$$

Set $x \equiv 11\cdots 1 \in X_H$.

Define $\tilde{\Gamma} \equiv (\tilde{X}, \tilde{E})$, where

$$\tilde{X} = \{y \in X_H \mid \partial_H(x,y) = D\} \quad \partial_H : \text{distance in } \Gamma_H \tag{7.13}$$
$$\tilde{E} = \{yz \in X_H \mid \partial_H(y,z) = 2\}. \tag{7.14}$$

Observe

$$X_J \to \qquad \tilde{X} \tag{7.15}$$
$$S \mapsto \qquad \hat{S}, \tag{7.16}$$

where

$$\hat{S} = a_1 \cdots a_N, \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$$

induces an isomorphism of graphs $\Gamma_J \to \tilde{\Gamma}$.

Pf.

$$\begin{aligned} ST \in E_J &\Leftrightarrow |S \cap T| = D - 1 && (7.17)\\ &\Leftrightarrow \partial_H(\hat{S}, \hat{T}) = 2 && (7.18)\\ &\Leftrightarrow (\hat{S}, \hat{T}) \in \tilde{E}. && (7.19)\end{aligned}$$

Identify, Γ_J with $\tilde{\Gamma}$. Then the standard module V_J of Γ_J becomes $\tilde{V} = E_D^* V_H$, where V_H is the standard module of Γ_H, and $E_D^* \equiv E_D^*(x)$.

Let R be the raising matrix with respect to x in Γ_H, and

let L be the lowering matrix with respect to x in Γ_H.

Recall

$$(RL - DE_D^*)|_{\tilde{V}}$$

is the adjacency map in $\tilde{\Gamma}$.

To find eigenvalues of $\tilde{A}$, pick any irreducible $T(x)$-module W with the endpoint $r \leq D$. Then by Theorem 5.1

$$\text{diam}(W) = N - 2r + 1.$$

Let $w_0, w_1, \dots, w_{N-2r}$ denote a basis for W as in Theorem 5.1. Then,

$$w_{D-r} \in E_D^* W \subseteq \tilde{V}.$$

Observe:

$$\begin{aligned} \tilde{A} w_{D-r} &= RL w_{D-r} - DE_D^* w_{D-r} && (7.20)\\ &= R(N - 2r - D + r + 1) w_{D-r-1} - D w_{D-r} && (7.21)\\ &= ((N - D - r + 1)(D - r) - D) w_{D-r}. && (7.22)\end{aligned}$$

Note that this is valid for $D = r$ as well.

Hence,

$$\tilde{A} w_{D-r} = ((N - D - r)(D - r) - r) w_{D-r}.$$

Let

$$V_H = \sum W \quad \text{(direct sum of irreducible } T(x) - \text{modules.)}$$

Then,

$$\begin{aligned} V_J &= E_D^* V_H && (7.23)\\ &= \sum_{W : r(W) \leq D} E_D^* W && (7.24)\\ &= \text{a direct sum of 1 dimensional eigenspaces for } \tilde{A}. && (7.25)\end{aligned}$$

The eigenspace for eigenvalue

$$(N-D-r)(D-r)-r \quad \text{(monotonously decreasing with respec to } r\text{)}$$

appears with multiplicity

$$\binom{N}{r} - \binom{N}{r-1}$$

in this sum by Theorem 5.1 (*iv*). □

Theorem 7.1. *Let $\Gamma = (X,E)$ be any graph. For a fixed vertex $x \in X$, let*

$$E_i^* \equiv E_i^*(x), \quad T \equiv T(x), \quad D \equiv D(x), \text{ and } K = \mathbb{C}.$$

Then we have the following implications of conditions:

$$TH \Leftrightarrow C \Leftarrow S \Leftarrow G.$$

where

(TH) Γ *is thinn with respect to x.*

(C) $E_i^*TE_i^*$ is commutative for every i, $(0 \le i \le D)$.

(S) $E_i^*TE_i^*$ is symmetric for every i, $(0 \le i \le D)$.

(G) For every $y,z \in X$ with $\partial(x,y) = \partial(x,z)$, there exists $g \in \mathrm{Aut}(\Gamma)$ such that

$$gx = x,\ gy = z,\ gz = y.$$

Proof. (TH) ⇒ (C)

Fix i with $0 \le i \le D$. Let

$V = \sum W$. The standard module written as a direct sum of irreducible T-modules.

The,

$$E_i^*V = \sum E_i^*W. \text{ The direct sum of 1-dimensional } E_i^*TE_i^*\text{-modules.}$$

Since $\dim E_i^*W = 1$, for $a,b \in E_i^*TE_i^*$, $ab - ba_{|E_i^*W} = 0$. Hence $ab - ba = 0$.

(C) ⇒ (TH)

Suppose $\dim E_i^*W \ge 2$ for some irreducible T-module W with some i with $1 \le i \le D$.

Claim: E_i^*W is an irreducible $E_i^*TE_i^*$-module.

Pf. Suppose

$$0 \subsetneq U \subsetneq E_i^*W,$$

where U is a $E_i^*TE_i^*$-module. Then by the irreducibility,

$$TU = W.$$

So

$$U \supseteq E_i^*TE_i^*U = E_i^*TU = E_i^*W.$$

This is a contradiction.

Claim 2: Each irreducible $S = E_i^*TE_i^*$-module U has dimension 1. In particular, Γ is thin with respect to x.

Pf. Pick

$$0 \neq a \in E_i^*TE_i^*.$$

Since $\mathbb{C}$ is algebraicallt closed, a has an eigenvector $w \in U$ with eigenvalue θ. Then,

$$\begin{aligned} (a-\theta I)U &= (a-\theta I)Sw && (7.26)\\ &= S(a-\theta I)w && (7.27)\\ &= 0. && (7.28) \end{aligned}$$

Hence,

$$a_{|U} = \theta I_{|U} \quad \text{for all} \quad a \in S.$$

Thus each 1 dimensional subspace of U is an S-module. We have

$$\dim U = 1.$$

By Claim 1 and Claim 2, we hat (TH).

□

Chapter 8

Thin Graphs

Friday, February 5, 1993

Proof of Theorem 7.1 continued. (S) $\Rightarrow$ (C)

Fix i and pick $a, b \in E_i^* T E_i^*$.

Since a, b and ab are symmetric,

$$ab = (ab)^\top = b^\top a^\top = ba.$$

Hence $E_i^* T E_i^*$ is commutative.

(G) $\Rightarrow$ (S)

Fix i and pick $a \in E_i^* T E_i^*$. Pick vertices $y, z \in X$.

We want to show that

$$a_{yz} = a_{zy}.$$

We may assume that

$$\partial(x, y) = \partial(x, z) = i,$$

othewise

$$a_{yz} = a_{zy} = 0.$$

By our assumption, there exists $g \in G$ such that

$$g(y) = z, \quad g(z) = y, \quad g(x) = x.$$

Let $\hat{g}$ denote the permutation matrix representing g, i.e.,

$$\hat{g}\hat{y} = \widehat{g(y)} \quad \text{for all} \quad y \in X, \quad \hat{y} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow y.$$

If $g \in \mathrm{Aut}(\Gamma)$, then

$$\hat{g}A = A\hat{g} \quad \text{Exercise.}$$

Also we have

$$\hat{g}E_j^* = E_j^*\hat{g} \quad (0 \leq j \leq D),$$

since

$$\partial(x, y) = \partial(g(x), g(y)) = \partial(x, g(y)).$$

Hence $\hat{g}$ commutes with each element of T. We have

$$a_{yz} = (\hat{g}^{-1}a\hat{g})_{yz}, \quad (\hat{g})_{yz} = \begin{cases} 1 & g(z) = y \\ 0 & \text{else.} \end{cases} \tag{8.1}$$

$$= \sum_{y',z'} (\hat{g}^{-1})_{yy'} a_{y'z'} \hat{g}_{z'z} \tag{8.2}$$

$$\text{(zero except for } g^{-1}(y') = y,\ g(z) = z'.) \tag{8.3}$$

$$= a_{g(y)g(z)} \tag{8.4}$$

$$a_{zy}. \tag{8.5}$$

This proves Theorem 7.1.

□

Open Problem: Find all the graphs that satisfy the condition (G) for every vertex x.

$H(N, 2)$ is one example, because

$$\mathrm{Aut}\Gamma_{1\cdots 1} \simeq S_\Omega, \quad x = (1 \cdots 1), \Gamma_i(x) = \{\hat{S} \mid |S| = i\}.$$

Property (G) is clearly related to the distance-transitive property.

Definition 8.1. Let $\Gamma = (X, E)$ be any graph. Γ with $G \subseteq \mathrm{Aut}(\Gamma)$ is said to be distance-transitive (or two-point homogeneous), whenever

$$\text{for all } x, x', y, y' \in X \text{ with } \partial(x, y) = \partial(x', y'),$$

there exists $g \in G$ such that

$$g(x) = y, \quad g(x') = y'.$$

(This means G is as close to being doubly transitive as possible.)

Lemma 8.1. *Suppose a graph $\Gamma = (X, E)$ satisfies the property (G) = (G(x)) for every $x \in X$. Then,*

(i) *either*
(ia) Γ *is vertex transitive; or*
(iia) Γ *is bipartite* $(X = X^+ \cup X^-)$ *with* X^+, X^- *each an orbit of* $\mathrm{Aut}(\Gamma)$.

(ii) *if* (ia) *holds, then* Γ *is distance-transitive.*

Proof. (i) Claim. Suppose $y, z \in X$ are conneced by a path of even length. Then y, z are in the same orbit of $\mathrm{Aut}(\Gamma)$.

Pf. It suffices to assume that the path has lenght 2, $y \sim w \sim z$.

Now $\partial(y, w) = \partial(w, z) = 1$. So there exits $g \in \mathrm{Aut}(\Gamma)$ such that \$$gw = w, \quad gy = z, \quad gz = y$. This proves Claim.

Fix $x \in X$. Now suppose that Γ is not vertex transitive, and we shall show (ib).

Observe that $X = X^+ \cup X^-$, where

$$X^+ = \{y \in X \mid \text{there exists a path of even length connecting } x \text{ and } y\} \quad (8.6)$$
$$X^- = \{y \in X \mid \text{there exists a path of odd length connecting } x \text{ and } y\} \quad (8.7)$$

Asi X^+ is contained in an orbit O^+ of $\mathrm{Aut}(\Gamma)$, and X^- is contained in an orbit O^- of $\mathrm{Aut}(\Gamma)$.

Now $O^+ \cap O^- = \emptyset$ (else $O^+ = O^- = X$ and vertex transitive). So, $X = O^+$, and $X^- = O^-$.

Also $X^+ \cup X^- = X$ is a bipartition by construction.

(ii) Fix x, y, x', y' with $\partial(x, y) = \partial(x', y')$.

By vertex transitivity, there exists an element

$$g_1 \in G \text{ such that } g_1 x = x'.$$

Observe that

$$\partial(x', y') = \partial(x, y) = \partial(g_1 x, g_1 y) = \partial(x', g_1 y).$$

Hence, there exisits an element

$$g_2 \in G \text{ such that } g_1 x' = x', g_2 y' = g_1 y', g_2 g_1 y = y'$$

by (G(x')) property.

Set $g = g_2 g_1$. Then
$$gx = x', gy = y'$$
by construction. □

The following graphs $\Gamma = (X, E)$ are vertex transitive, and satisfy the property (G(x)) for all $x \in X$.

$$J(D, N), \quad H(D, r), \quad J_q(D, N),$$

where

$H(D, r)$:

$$
\begin{aligned}
X &= \{a_1 \cdots a_D \mid a_i \in F, 1 \leq i \leq D\} && (8.8)\\
F &: \text{ any set of cardinality } r && (8.9)\\
E &= \{xy \mid y, x \in X,\ x \text{ and } y \text{ differ in exactly one coordiate}\}. && (8.10)
\end{aligned}
$$

$J_q(D, N)$:

$$
X = \text{the set of all } D\text{-dimensional subspaces of } N\text{-dimensional vector space over } GF(q). \tag{8.11}
$$

$$
\begin{aligned}
F &: \text{ any set of cardinality } r && (8.12)\\
E &= \{xy \mid y, x \in X,\ \dim(x \cap y) = D - 1\}. && (8.13)
\end{aligned}
$$

The following graph is distance-transitive but does not satisify (G(x)) for any $x \in G$.

$H_q(D, N)$:

$$
\begin{aligned}
X &= \text{the set of all } D \times N \text{ matrices with entries in } GF(q). && (8.14)\\
E &= \{xy \mid y, x \in X,\ \operatorname{rank}(x - y) = 1.\}. && (8.15)
\end{aligned}
$$

Remark. $H(D, r)$: $G = S_r \mathrm{wr} S_D$, $G_x = S_{r-1} \mathrm{wr} S_D$,

For $x, y \in X$ with $\partial(x, y) = \partial(x, z) = i$,

$$
\begin{aligned}
Y = \{j \in \Omega \mid x_j \neq y_j\} &\leftrightarrow Z = \{j \in \Omega \mid x_j \neq z_j\} && (8.16)\\
(y_{j_1}, \dots, y_{j_i}) &\leftrightarrow (z_{\ell_1}, \dots, z_{\ell_i}) && (8.17)
\end{aligned}
$$

$J(D, N)$: $G = S_N$, $G_x = S_D \times S_{N-D}$.

$$
\begin{aligned}
X \cap Y &\leftrightarrow X \cap Z && (8.18)\\
(\Omega \quad X) \cap Y &\leftrightarrow (\Omega \quad X) \cap Z. && (8.19)
\end{aligned}
$$

The following graph is distance-transitive but does not satisify (G(x)) for any $x \in G$.

$J_q(D, N)$:

$$
X \cap Y \leftrightarrow X \cap Z.
$$

The theory of single thin irreducible T-module.

Let $\Gamma = (X, E)$ be any graph.

$$M = \text{Bose-Mesner algebra over } K/\mathbb{C} \text{ generated by the adjacency matrix } A. \tag{8.20}$$

$$= \mathrm{Span}(E_0, \dots, E_R). \tag{8.21}$$

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix \$x X%, let $D \equiv D(x)$ be the x-diameter, and $k = k(x)$ be the valenct of x.

Chapter 9

Thin T-Module, I

Monday, February 8, 1993

Let $\Gamma = (X, E)$ be any graph.

Chapter 10

Thin T-Module, II

Wednesday, February 10, 1993

Let $\Gamma = (X, E)$ be any graph.

Chapter 11

Examples of T-Module

Friday, February 12, 1993

Let $\Gamma = (X, E)$ be any graph.

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