Lecture Note on Terwilliger Algebra

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About this lecturenote

Setting

This note is created by `bookdown` package on RStudio.

1. Log-in to my GitHub Account
2. Go to RStudio/bookdown-demo repository: https://github.com/rstudio/bookdown-demo
3. Use This Template
4. Input Repository Name
5. Select Public - default
6. Create repository from template
7. From Code download ZIP
8. Move the extracted folder into a favorite directory
9. Open RStudio Project in the folder
10. Use Terminal in the buttom left pane

    - confirm that the current directory is the home directry of the project by pwd

11. (failed to proceed by ssh)
12. Use Console

    1. library(usethis)
    2. use_git()
    3. use_github() — Error
    4. gh_token_help()
    5. create_github_token(): create a token in the github page. Copy the token
    6. gitcreds::gitcreds_set(): paste the token, the token is to be expired in 30 days

13. Use Terminal

    1. git remote add origin https://github.com/icu-hsuzuki/t-alagebra.git
    2. git push -u origin main

    3. type in the password of the computer

14. Use GIT in R Studio

Another Host

1. library(usethis)
2. use_git()
3. create_github_token()
4. gitcreds::gitcreds_set(): Replace these credentials

Chapter 1

Lecture 1

Wednesday, January 20, 1993

A graph (undirected, without loops or multiple edges) is a pair $\Gamma = (X, E)$, where

$$X = \text{finite set (of vertices)} \tag{1.1}$$
$$E = \text{set of (distinct) 2-element subsets of } X \text{ (= edges of) } \Gamma. \tag{1.2}$$

vertices x and $y \in X$ are adjacent if and only if $xy \in E$.

Example 1.1. Let Γ be a graph. $X = \{a, b, c, d\}$, $E = \{ab, ac, bc, bd\}$.

Set $n = |X|$, the order of Γ.

Pick a field K $(= \mathbb{R}$ or $\mathbb{C})$. Then $\mathrm{Mat}_X(K)$ denotes the K algebra of all $n \times n$ matrices with entries in K. (rows and columns are indexed by X)

Adjacency matrix $A \in \mathrm{Mat}_X(K)$ is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } \ xy \in E \\ 0 & \text{else .} \end{cases} \tag{1.3}$$

Example 1.2. Let a, b, c, d be labels of rows and columns. Then

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The subalgebra M of $\mathrm{Mat}_X(K)$ generated by A is called the *Bose-Mesner algebra* of Γ.

Set $V = K^n$, the set of n-dimensional column vectors, the coorinates are indexed by X.

Let $\langle\ ,\ \rangle$ denote the Hermitean inner product:

$$\langle u, v\rangle = u^\top \cdot v \quad (u, v \in V)$$

V with $\langle\ ,\ \rangle$ is the *standard module* of Γ.

M acts on V: For every $x \in X$, write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where 1 is at the x position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since A is a real symmetrix matrix,

$$V = V_0 + V_1 + \cdots + V_r \qquad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal A-eigenspaces.

Let $E_i \in \mathrm{Mat}_X(K)$ denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then $E_0, \dots, E_r$ are the primitive idempotents of M.

$$M = \mathrm{Span}_K(E_0, \dots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all } \ i, j, \quad E_0 + \cdots + E_r = I.$$

Let θ_i denote the eigenvalue of A for V_i in $\mathbb{R}$. Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \cdots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \ \theta_i = \mathrm{dim} V_i = \mathrm{rank} E_i.$$

Set

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \cdots, & \theta_r \\ m_0, & m_1, & \cdots, & m_r \end{pmatrix}.$$

Problem. What can we say about Γ when $\mathrm{Spec}(\Gamma)$ is given?

The following Lemma1.1, is an example of Problem.

For every $x \in X$,

$$k(x) \equiv \text{ valency of } x \equiv \text{ degree of } x \equiv |\{y \mid y \in X,\ xy \in E\}|.$$

Definition 1.1. The graph Γ is regular of valency k if $k = k(x)$ for every $x \in X$.

Lemma 1.1. *With the above notation,*
1. $\theta_0 \leq \max\{k(x) \mid x \in X\} = k^{\max}$.
2. If Γ *is regular of valency* k*, then* $\theta_0 = k$.

Proof.

1. Without loss of generality we may assume that $\theta_0 > 0$, else done. Let $v := \sum_{x\in X} \alpha_x \hat{x}$ denote the eivenvector for θ_0.

Pick $x \in X$ with $|\alpha_x|$ maximal. Then $|\alpha_x| \neq 0$.

Since $Av = \theta_0 v$,

$$\theta_0 \alpha_x = \sum_{y\in X, xy\in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \leq \sum_{y\in X, xy\in E} |\alpha_y| \leq k(x)|\alpha_x| \leq k^{\max}|\alpha_x|.$$

2. All 1's vector $v = \sum_{x\in X} \hat{x}$ satisfies $Av = kv$.

$\square$

Subconstituent Algebra

Let $x, y \in X$ and $\ell \in \mathbb{Z}^{\geq 0}$.

Definition 1.2. A path of length ℓ connecting x, y is a sequence

$$x = x_0, x_1, \dots, x_\ell = y, \quad x_i \in X,\ 0 \leq i \leq \ell$$

such that $x_i x_{i+1} \in E$ for $0 \leq i \leq \ell - 1$.

Definition 1.3. The distance $\partial(x, y)$ is the length of a shortest path connecting x and y.

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

Definition 1.4. The graph Γ is connected if and only if $\partial(x, y) < \infty$ for all $x, y \in X$.

From now on, assume that Γ is connected with $|X| \geq 2$.

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \text{the diameter of } \Gamma.$$

Fix a 'base' vertex $x \in X$.

Definition 1.5.

$$d(x) = \text{the diameter with respect to } x = \max\{\partial(x,y) \mid y \in X\} \leq d.$$

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad \text{(orthogonal direct sum)},$$

where

$$V_i^* = \mathrm{Span}_K(\hat{y} \mid \partial(x,y) = i) \equiv V_i * (x)$$

and $V_i^* = V_i^*(x)$ is called the i-the subconstituent with respect to x.

Let $E_i^* = E_i^*(x)$ denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View $E_i^*(x) \in \mathrm{Mat}_X(K)$. So, $E_i^*(x)$ is diagonal with yy entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{else,} \end{cases} \qquad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \mathrm{Span}_K(E_0^*(x), \ldots, E_{d(x)}^*(x)).$$

Then $M^*(x)$ is a commutative subalgebra of $\mathrm{Mat}_X(K)$ and is calle the *dual Bose-Mesner algbara with respect to x.*

Definition 1.6 (Subconstituent Algebra)**.** Let $\Gamma = (X, E)$, x, M, $M^*(x)$ be as above. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(K)$ generated by M and $M^*(x)$. T is the *subconstituent algebra* of Γ with respect to x.

Definition 1.7. A T-module is any subspace $W \subset V$ such that $aw \in W$ for all $a \in T$ and $w \in W$.

T-module W is *irreducible* if and only if $W \neq 0$ and W does not properly contain a nonzero T-module.

For any $a \in \mathrm{Mat}_X(K)$, let a^* denbote the conjugate transpose of a.

Observe that

$$\langle au, v\rangle = \langle u, a^* v\rangle \quad \text{for all } \ a \in \mathrm{Mat}_X(K), \text{ and for all } \ u, v \in V.$$

Lemma 1.2. *Let $\Gamma = (X, E)$, $x \in X$ and $T \equiv T(x)$ be as above.*

1. *If $a \in T$, then $a^* \in T$.*
2. *For any T-module $W \subset V$,*

$$W^\perp := \{v \in V \mid \langle w, v\rangle = 0, \ \textit{for all } w \in W\}$$

is a T-module.

3. V decomposes as an orthogonal direct sum of irreducible T-modules.

Proof.

1. It is becase T is generated by symmetric real matrices
$$A, E_0^*(x), E_1^*(x), \dots, E_{d(x)(x)}^*.$$
2. Pick $v \in W^\perp$ and $a \in T$, it suffices to show that $av \in W^\perp$. For all $w \in W$,
$$\langle w, av\rangle = \langle a^* w, v\rangle = 0$$
as $a^* \in T$.
3. This is proved by the induction on the dimension of T-modules. If W is an irreducible T-module of V, then
$$V = W + W^\perp \quad \text{(orthogonal direct sum).}$$

$\square$

Problem. What does the structure of the $T(x)$-module tell us about Γ?

Study those Γ whose modules take 'simple' form. The Γ's involved are highly regular.

Remark.

1. The subconstituent algebra T is semisimple as the left regular representation of T is completely reducible. See Curtis-Reiner 25.2.
2. The inner product $\langle a, b\rangle_T = \mathrm{tr}(a^\top \bar{b})$ is nondegenerate on T.
3. In general,
$$\begin{aligned} T\text{: Semisimple and Artinian} &\Leftrightarrow T\text{: Artinian with } J(T) = 0 && (1.4)\\ &\Leftarrow T\text{: Artinian with nonzero nilpotent element} && (1.5)\\ &\Leftarrow T \subset \mathrm{Mat}_X(K) \text{ such that for all } a \in T \text{ is normal.} && (1.6) \end{aligned}$$

Chapter 2

Lecture 2

Here is a review of existing methods.

Chapter 3

Lecture 3

We describe our methods in this chapter.

Math can be added in body using usual syntax like this

3.1 math example

p is unknown but expected to be around 1/3. Standard error will be approximated

$$SE = \sqrt(\frac{p(1-p)}{n}) \approx \sqrt{\frac{1/3(1-1/3)}{300}} = 0.027$$

You can also use math in footnotes like this[1].

We will approximate standard error to 0.027[2]

[1] where we mention $p = \frac{a}{b}$

[2] p is unknown but expected to be around 1/3. Standard error will be approximated

$$SE = \sqrt(\frac{p(1-p)}{n}) \approx \sqrt{\frac{1/3(1-1/3)}{300}} = 0.027$$

Chapter 4

Applications

Some *significant* applications are demonstrated in this chapter.

4.1 Example one

4.2 Example two

Chapter 5

Final Words

We have finished a nice book.