

# Lecture Note on Terwilliger Algebra

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# About this lecturenote

## Setting

sudo This note is created by `bookdown` package on RStudio.

For `bookdown` See (Xie, 2015), (Xie, 2017), (Yihui Xie, 2018).

1. Log-in to my GitHub Account
2. Go to RStudio/bookdown-demo repository: <https://github.com/rstudio/bookdown-demo>
3. Use This Template
4. Input Repository Name
5. Select Public - default
6. Create repository from template
7. From Code download ZIP
8. Move the extracted folder into a favorite directory
9. Open RStudio Project in the folder
10. Use Terminal in the bottom left pane
  - confirm that the current directory is the home directory of the project by `pwd`
11. (failed to proceed by ssh)
12. Use Console
  1. `library(usethis)`
  2. `use_git()`
  3. `use_github()` — Error
  4. `gh_token_help()`
  5. `create_github_token()`: create a token in the github page. Copy the token
  6. `gitcreds::gitcreds_set()`: paste the token, the token is to be expired in 30 days
13. Use Terminal
  1. `git remote add origin https://github.com/icu-hsuzuki/t-algebra.git`
  2. `git push -u origin main`
  3. type in the password of the computer
14. Use GIT in R Studio

## Another Host

1. create a project by version control git
2. git init
3. git remote add origin git@github.com:/.git
4. git branch -r
5. git fetch
6. git pull origin main

# Chapter 1

## Subconstituent Algebra of a Graph

Wednesday, January 20, 1993

A graph (undirected, without loops or multiple edges) is a pair  $\Gamma = (X, E)$ , where

$$X = \text{finite set (of vertices)} \quad (1.1)$$

$$E = \text{set of (distinct) 2-element subsets of } X \text{ (= edges of } \Gamma). \quad (1.2)$$

vertices  $x$  and  $y \in X$  are adjacent if and only if  $xy \in E$ .

**Example 1.1.** Let  $\Gamma$  be a graph.  $X = \{a, b, c, d\}$ ,  $E = \{ab, ac, bc, bd\}$ .



Set  $n = |X|$ , the order of  $\Gamma$ .

Pick a field  $K$  ( $= \mathbb{R}$  or  $\mathbb{C}$ ). Then  $\text{Mat}_X(K)$  denotes the  $K$  algebra of all  $n \times n$  matrices with entries in  $K$ . (rows and columns are indexed by  $X$ )

*Adjacency matrix*  $A \in \text{Mat}_X(K)$  is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E \\ 0 & \text{else .} \end{cases} \quad (1.3)$$

**Example 1.2.** Let  $a, b, c, d$  be labels of rows and columns. Then

$$A = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

The subalgebra  $M$  of  $\text{Mat}_X(K)$  generated by  $A$  is called the *Bose-Mesner algebra* of  $\Gamma$ .

Set  $V = K^n$ , the set of  $n$ -dimensional column vectors, the coordinates are indexed by  $X$ .

Let  $\langle \cdot, \cdot \rangle$  denote the Hermitean inner product:

$$\langle u, v \rangle = u^\top \cdot v \quad (u, v \in V)$$

$V$  with  $\langle \cdot, \cdot \rangle$  is the *standard module* of  $\Gamma$ .

$M$  acts on  $V$ : For every  $x \in X$ , write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where 1 is at the  $x$  position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since  $A$  is a real symmetrix matrix,

$$V = V_0 + V_1 + \cdots + V_r \quad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal  $A$ -eigenspaces.

Let  $E_i \in \text{Mat}_X(K)$  denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then  $E_0, \dots, E_r$  are the primitive idempotents of  $M$ .

$$M = \text{Span}_K(E_0, \dots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all } i, j, \quad E_0 + \cdots + E_r = I.$$



Let  $\theta_i$  denote the eigenvalue of  $A$  for  $V_i$  in  $\mathbb{R}$ . Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \dots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \theta_i = \dim V_i = \text{rank } E_i.$$

Set

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \dots, & \theta_r \\ m_0, & m_1, & \dots, & m_r \end{pmatrix}.$$

**Problem.** What can we say about  $\Gamma$  when  $\text{Spec}(\Gamma)$  is given?

The following Lemma 1.1, is an example of Problem.

For every  $x \in X$ ,

$$k(x) \equiv \text{valency of } x \equiv \text{degree of } x \equiv |\{y \mid y \in X, xy \in E\}|.$$

**Definition 1.1.** The graph  $\Gamma$  is regular of valency  $k$  if  $k = k(x)$  for every  $x \in X$ .

**Lemma 1.1.** *With the above notation,*

- (i)  $\theta_0 \leq \max\{k(x) \mid x \in X\} = k^{\max}$ .
- (ii) *If  $\Gamma$  is regular of valency  $k$ , then  $\theta_0 = k$ .*

*Proof.*

(i) Without loss of generality we may assume that  $\theta_0 > 0$ , else done. Let  $v := \sum_{x \in X} \alpha_x \hat{x}$  denote the eivenvector for  $\theta_0$ .

Pick  $x \in X$  with  $|\alpha_x|$  maximal. Then  $|\alpha_x| \neq 0$ .

Since  $Av = \theta_0 v$ ,

$$\theta_0 \alpha_x = \sum_{y \in X, xy \in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \leq \sum_{y \in X, xy \in E} |\alpha_y| \leq k(x) |\alpha_x| \leq k^{\max} |\alpha_x|.$$

(ii) All 1's vector  $v = \sum_{x \in X} \hat{x}$  satisfies  $Av = kv$ .

□

### Subconstituent Algebra

Let  $x, y \in X$  and  $\ell \in \mathbb{Z}^{\geq 0}$ .

**Definition 1.2.** A path of length  $\ell$  connecting  $x, y$  is a sequence

$$x = x_0, x_1, \dots, x_\ell = y, \quad x_i \in X, \quad 0 \leq i \leq \ell$$

such that  $x_i x_{i+1} \in E$  for  $0 \leq i \leq \ell - 1$ .

**Definition 1.3.** The distance  $\partial(x, y)$  is the length of a shortest path connecting  $x$  and  $y$ .

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

**Definition 1.4.** The graph  $\Gamma$  is connected if and only if  $\partial(x, y) < \infty$  for all  $x, y \in X$ .

From now on, assume that  $\Gamma$  is connected with  $|X| \geq 2$ .

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \text{the diameter of } \Gamma.$$

Fix a ‘base’ vertex  $x \in X$ .

**Definition 1.5.**

$$d(x) = \text{the diameter with respect to } x = \max\{\partial(x, y) \mid y \in X\} \leq d.$$

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad (\text{orthogonal direct sum}),$$

where

$$V_i^* = \text{Span}_K(\hat{y} \mid \partial(x, y) = i) \equiv V_i * (x)$$

and  $V_i^* = V_i^*(x)$  is called the  $i$ -th subconstituent with respect to  $x$ .

Let  $E_i^* = E_i^*(x)$  denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View  $E_i^*(x) \in \text{Mat}_X(K)$ . So,  $E_i^*(x)$  is diagonal with  $yy$  entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{else,} \end{cases} \quad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \text{Span}_K(E_0^*(x), \dots, E_{d(x)}^*(x)).$$

Then  $M^*(x)$  is a commutative subalgebra of  $\text{Mat}_X(K)$  and is called the *dual Bose-Mesner algebra with respect to  $x$* .

**Definition 1.6** (Subconstituent Algebra). Let  $\Gamma = (X, E)$ ,  $x, M, M^*(x)$  be as above. Let  $T = T(x)$  denote the subalgebra of  $\text{Mat}_X(K)$  generated by  $M$  and  $M^*(x)$ .  $T$  is the *subconstituent algebra* of  $\Gamma$  with respect to  $x$ .

**Definition 1.7.** A  $T$ -module is any subspace  $W \subset V$  such that  $aw \in W$  for all  $a \in T$  and  $w \in W$ .

$T$ -module  $W$  is *irreducible* if and only if  $W \neq 0$  and  $W$  does not properly contain a nonzero  $T$ -module.

For any  $a \in \text{Mat}_X(K)$ , let  $a^*$  denote the conjugate transpose of  $a$ .

Observe that

$$\langle au, v \rangle = \langle u, a^*v \rangle \quad \text{for all } a \in \text{Mat}_X(K), \text{ and for all } u, v \in V.$$

**Lemma 1.2.** *Let  $\Gamma = (X, E)$ ,  $x \in X$  and  $T \equiv T(x)$  be as above.*

(i) *If  $a \in T$ , then  $a^* \in T$ .*

(ii) *For any  $T$ -module  $W \subset V$ ,*

$$W^\perp := \{v \in V \mid \langle w, v \rangle = 0, \text{ for all } w \in W\}$$

*is a  $T$ -module.*

(iii)  *$V$  decomposes as an orthogonal direct sum of irreducible  $T$ -modules.*

*Proof.*

(i) It is because  $T$  is generated by symmetric real matrices

$$A, E_0^*(x), E_1^*(x), \dots, E_{d(x)(x)}^*.$$

(ii) Pick  $v \in W^\perp$  and  $a \in T$ , it suffices to show that  $av \in W^\perp$ . For all  $w \in W$ ,

$$\langle w, av \rangle = \langle a^*w, v \rangle = 0$$

as  $a^* \in T$ .

(iii) This is proved by the induction on the dimension of  $T$ -modules. If  $W$  is an irreducible  $T$ -module of  $V$ , then

$$V = W + W^\perp \quad (\text{orthogonal direct sum}).$$

□

**Problem.** What does the structure of the  $T(x)$ -module tell us about  $\Gamma$ ?

Study those  $\Gamma$  whose modules take ‘simple’ form. The  $\Gamma$ ’s involved are highly regular.

*Remark.*

1. The subconstituent algebra  $T$  is semisimple as the left regular representation of  $T$  is completely reducible. See Curtis-Reiner 25.2 (Charles W. Curtis, 2006).
2. The inner product  $\langle a, b \rangle_T = \text{tr}(a^\top \bar{b})$  is nondegenerate on  $T$ .

3. In general,

$T$ : Semisimple and Artinian  $\Leftrightarrow T$ : Artinian with  $J(T) = 0$

$\Leftrightarrow T$ : Artinian with nonzero nilpotent element

$\Leftrightarrow T \subset \text{Mat}_X(K)$  such that for all  $a \in T$  is normal.

## Chapter 2

# Perron-Frobenius Theorem

Friday, January 22, 1993

In this lecture we use the Perron Frobenius theory of nonnegative matrices to obtain information on eigenvalues of a graph.

Let  $K = \mathbb{R}$ . For  $n \in \mathbb{Z}^{>0}$ , pick a symmetric matrix  $C \in \text{Mat}_n(\mathbb{R})$ .

**Definition 2.1.** The matrix  $C$  is *reducible* if and only if there is a bipartition  $\{1, 2, \dots, n\} = X^+ \cup X^-$  (disjoint union of nonempty sets) such that  $C_{ij} = 0$  for all  $i \in X^+$ , and for all  $j \in X^-$ , and for all  $i \in X^-$ , and for all  $j \in X^+$ , i.e.,

$$C \sim \begin{pmatrix} * & O \\ O & * \end{pmatrix}.$$

**Definition 2.2.** The matrix  $C$  is *bipartite* if and only if there is a bipartition  $\{1, 2, \dots, n\} = X^+ \cup X^-$  (disjoint union of nonempty sets) such that  $C_{ij} = 0$  for all  $i, j \in X^+$ , and for all  $i, j \in X^-$ , i.e.,

$$C \sim \begin{pmatrix} O & * \\ * & O \end{pmatrix}.$$

**Note.**

1. If  $C$  is bipartite, for every eigenvalue  $\theta$  of  $C$ ,  $-\theta$  is an eigenvalue of  $C$  such that  $\text{mult}(\theta) = \text{mult}(-\theta)$ .

Indeed, let  $C = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$ ,

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix},$$

where  $Ay = \theta x$  and  $Bx = \theta y$ .

2. If  $C$  is bipartite,  $C^2$  is reducible.
3. The matrix  $C$  is irreducible and  $C^2$  is reducible, if  $C_{ij} \geq 0$  for all  $i, j$  and  $C$  is reducible. (Exercise)

*Remark.* Note 1. Even if  $C$  is not symmetric

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

holds. So the geometrix multiplicities coincide. How about the algebraic multiplicities?

Note 3. Set  $x \sim y$  if and only if  $C_{xy} > 0$ . So the graph may have loops. Then

$$(C^2)_{xy} > 0 \Leftrightarrow \text{if there exists } z \in X \text{ such that } x \sim z \sim y.$$

Note that  $C$  is irreducible if and only if  $\Gamma(C)$  is connected. Let

$$X^+ = \{y \mid \text{there is a path of even length from } x \text{ to } y\} \quad (2.1)$$

$$X^- = \{y \mid \text{there is no path of even length from } x \text{ to } y\} \neq \emptyset. \quad (2.2)$$

If there is an edge  $y \sim z$  in  $X^+$  and  $w \in X^-$ . Then there would be a path from  $x$  to  $y$  of even length. So  $e(X^+, X^+) = e(X^-, X^-) = 0$ .

**Theorem 2.1** (Perron-Frobenius). *Given a matrix  $C$  in  $\text{Mat}_n(\mathbb{R})$  such that*

- (a)  $C$  is symmetric.
- (b)  $C$  is irreducible.
- (c)  $C_{ij} \geq 0$  for all  $i, j$ .

*Let  $\theta_0$  be the maximal eigenvalue of  $C$  with eigenspace  $V_0 \subseteq \mathbb{R}^n$ , and let  $\theta_r$  be the maximal eigenvalue of  $C$  with eigenspace  $V_r \subseteq \mathbb{R}^n$ . Then the following hold.*

$$(i) \text{ Suppose } 0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0. \text{ Then } \alpha_0 > 0 \text{ for all } i, \text{ or } \alpha_i < 0 \text{ for all } i.$$

$$(ii) \dim V_0 = 1.$$

$$(iii) \theta_r \geq -\theta_0.$$

$$(iv) \theta_r = \theta_0 \text{ if and only if } C \text{ is bipartite.}$$

First, we prove the following lemma.

**Lemma 2.1.** *Let  $\langle \cdot, \cdot \rangle$  be the dot product in  $V = \mathbb{R}^n$ . Pick a symmetric matrix  $B \in \text{Mat}_n(\mathbb{R})$ . Suppose all eigenvalues of  $B$  are nonnegative. (i.e.,  $B$  is positive semidefinite.) Then there exist vectors  $v_1, v_2, \dots, v_n \in V$  such that  $B_{ij} = \langle v_i, v_j \rangle$  for  $(1 \leq i, j \leq n)$ .*

*Proof.* By elementary linear algebra, there exists an orthonormal basis  $w_1, w_2, \dots, w_n$  of  $V$  consisting of eigenvectors of  $B$ . Set the  $i$ -th column of  $P$  is  $w_i$  and  $D = \text{diag}(\theta_1, \dots, \theta_n)$ . Then  $P^\top P = I$  and  $BP = PD$ .

Hence,

$$B = PDP^{-1} = PDP^\top = QQ^\top,$$

where

$$Q = P \cdot \text{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \dots, \sqrt{\theta_n}) \in \text{Mat}_n(\mathbb{R}).$$

Now, let  $v_i$  be the  $i$ -th column of  $Q^\top$ . Then

$$B_{ij} = v_i^\top \cdot v_j = \langle v_i, v_j \rangle.$$

□

Now we start the proof of Theorem 2.1.

*Proof of Theorem 2.1(i)*

Let  $\langle, \rangle$  denote the dot product on  $V = \mathbb{R}^n$ . Set

$$B = \theta I - C \tag{2.3}$$

$$= \text{symmetric matrix with eigenvalues } \theta_0 - \theta_i \geq 0 \tag{2.4}$$

$$= (\langle v_i, v_j \rangle)_{1 \leq i, j \leq n} \tag{2.5}$$

with the same  $v_1, \dots, v_n \in V$  by Lemma 2.1.

Observe:  $\sum_{i=1}^n \alpha_i v_i = 0$ .

*Pf.*

$$\left\| \sum_{i=1}^n \alpha_i v_i \right\|^2 = \left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \right\rangle \tag{2.6}$$

$$= (\alpha_1 \quad \dots \quad \alpha_n) B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{2.7}$$

$$= v^\top B v \tag{2.8}$$

$$= 0, \tag{2.9}$$

since  $Bv = (\theta_0 I - C)v = 0$ .

Now set

$$s = \text{the number of indices } i, \text{ where } \alpha_i > 0.$$

Replacing  $v$  by  $-v$  if necessary, without loss of generality we may assume that  $s \geq 1$ . We want to show  $s = n$ .

Assume  $s < n$ . Without loss of generality, we may assume that  $\alpha_i > 0$  for  $1 \leq i \leq s$  and  $\alpha_i = 0$  for  $s+1 \leq i \leq n$ . Set

$$\rho = \alpha_1 v_1 + \dots + \alpha_s v_s = -\alpha_{s+1} v_{s+1} - \dots - \alpha_n v_n.$$

Then, for  $i = 1, \dots, s$ ,

$$\langle v_i, \rho \rangle = \sum_{j=s+1}^n -\alpha_j \langle v_i, v_j \rangle \quad (\langle v_i, v_j \rangle = B_{ij}, B = \theta_0 I - C) \quad (2.10)$$

$$= \sum_{j=s+1}^n (-\alpha_j)(-C_{ij}) \quad (2.11)$$

$$\leq 0. \quad (2.12)$$

Hence

$$0 \leq \langle \rho, \rho \rangle = \sum_{i=1}^s \alpha_i \langle v_i, \rho \rangle \leq 0,$$

as  $\alpha > 0$  and  $\langle v_i, \rho \rangle \leq 0$ . Thus, we have  $\langle \rho, \rho \rangle = 0$  and  $\rho = 0$ . For  $j = s+1, \dots, n$ ,

$$0 = \langle \rho, v_j \rangle = \sum_{i=1}^s \alpha_i \langle v_i, v_j \rangle \leq 0,$$

as  $\langle v_i, v_j \rangle = -C_{ij}$ .

Therefore,

$$0 = \langle v_i, v_j \rangle = -C_{ij} \text{ for } 1 \leq i \leq s, s+1 \leq j \leq n.$$

Since  $C$  is symmetric,

$$C = \begin{pmatrix} * & O \\ O & * \end{pmatrix}$$

Thus  $C$  is reducible, which is not the case. Hence  $s = n$ .

*Proof of Theorem 2.1 (ii).*

Suppose  $\dim V_0 \geq 2$ . Then,

$$\dim \left( V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^\perp \right) \geq 1.$$

So, there is a vector

$$0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with  $\alpha_1 = 0$ . This contradicts 1.

Now pick

$$0 \neq w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r.$$

*Proof of Theorem 2.1 (iii).*



Suppose  $\theta_r < -\theta_0$ . Since the eigenvalues of  $C^2$  are the squares of those of  $C$ ,  $\theta_r^2$  is the maximal eigenvalue of  $C^2$ .

Also we have  $C^2 w = \theta_r^2 w$ .

Observe that  $C^2$  is irreducible. (As otherwise,  $C$  is bipartite by Note 3, and we must have  $\theta_r = -\theta_0$ .) Therefore,  $\beta_i > 0$  for all  $i$  or  $\beta_i < 0$  for all  $i$ . We have

$$\langle v, w \rangle = \sum_{i=1}^n \alpha_i \beta_i \neq 0.$$

This is a contradiction, as  $V_0 \perp V_r$ .

*Proof of Theorem 2.1 (iv)*

$\Rightarrow$ : Let  $\theta_r = -\theta_0$ . Then  $\theta = \theta_1^2 = \theta_0^2$  is the maximal eigenvalue of  $C^2$ , and  $v$  and  $w$  are linearly independent eigenvalues for  $\theta$  for  $C^2$ . Hence, for  $C^2$ ,  $\text{mult}(\theta) \geq 2$ .

Thus by 2,  $C^2$  must be reducible. Therefore,  $C$  is bipartite by Note 3.

$\Leftarrow$ : This is Note 1.  $\square$

Let  $\Gamma = (X, E)$  be any graph.

**Definition 2.3.**  $\Gamma$  is said to be *bipartite* if the adjacency matrix  $A$  is bipartite. That is,  $X$  can be written as a disjoint union of  $X^+$  and  $X^-$  such that  $X^+, X^-$  contain no edges of  $\Gamma$ .

**Corollary 2.1.** *For any (connected) graph  $\Gamma$  with*

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_1 & m_1 & \cdots & m_r \end{pmatrix} \quad \text{with } \theta_0 > \theta_1 > \cdots > \theta_r.$$

*Let  $V_i$  be the eigenspace of  $\theta_i$ . Then the following holds.*

1. *Supppose  $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \in \mathbb{R}^n$ . Then  $\alpha_i > 0$  for all  $i$  or  $\alpha_i < 0$  for all  $i$ .*
2.  $m_0 = 1$ .
3.  $\theta_r \geq -\theta_0$  *if and only if  $\Gamma$  is bipartite. In this case,*

$$-\theta_i = \theta_{r-i} \text{ and } m_i = m_{r-i} \quad (0 \leq i \leq r)$$

*Proof.* This is a direct consequences of Theorem 2.1 and Note 3.  $\square$



## Chapter 3

# Cayley Graphs

Monday, January 25, 1993

Given graphs  $\Gamma = (X, E)$  and  $\Gamma' = (X', E')$ .

**Definition 3.1.** A map  $\sigma : X \rightarrow X'$  is an *isomorphism* of graphs whenever;

- i.  $\sigma$  is one-to-one and onto,
- ii.  $xy \in E$  if and only if  $\sigma x \sigma y \in E'$  for all  $x, y \in X$ .

We do not distinguish between isomorphic graphs.

**Definition 3.2.** Suppose  $\Gamma = \Gamma'$ . Above isomorphism  $\sigma$  is called an *automorphism* of  $\Gamma$ . Then set  $\text{Aut}(\Gamma)$  of all automorphisms of  $\Gamma$  becomes a finite group under composition.

**Definition 3.3.** If  $\text{Aut}(\Gamma)$  acts transitive on  $X$ ,  $\Gamma$  is called *vertex transitive*.

**Example 3.1.** A Cayley graphs:

**Definition 3.4** (Cayley Graphs). Let  $G$  be any finite group, and  $\Delta$  any generating set for  $G$  such that  $1_G \notin \Delta$  and  $g \in \Delta \rightarrow g^{-1} \in \Delta$ . Then Cayley graph  $\Gamma = \Gamma(G, \Delta)$  is defined on the vertex set  $X = G$  with the edge set  $E$  defined by the following.

$$E = \{(h_1, h_2) \mid h_1, h_2 \in G, h_1^{-1}h_2 \in \Delta\} = \{(h, hg) \mid h \in G, g \in \Delta\}$$

**Example 3.2.**  $G = \langle a \mid a^6 = 1 \rangle$ ,  $\Delta = \{a, a^{-1}\}$ .



**Example 3.3.**  $G = \langle a \mid a^6 = 1 \rangle$ ,  $\Delta = \{a, a^{-1}, a^2, a^{-2}\}$ .



**Example 3.4.**  $G = \langle a, b \mid a^6 = 1, b^2 = 1, ab = ba \rangle$ ,  $\Delta = \{a, a^{-1}, b\}$ .



*Remark.*  $\text{Aut}(\Gamma) \simeq D_6 \times \mathbb{Z}_2$  contains two regular subgroups isomorphic to  $D_6$  and  $\mathbb{Z}_5 \times \mathbb{Z}_2$  and  $\Gamma$  is obtained as Cayley graphs in two ways.

Cayley graphs are vertex transitive, indeed.

**Theorem 3.1.** *The following hold.*

(i) *For any Cayley graph  $\Gamma = \Gamma(G, \Delta)$ , the map*

$$G \rightarrow \text{Aut}(\Gamma) \quad (g \mapsto \hat{g})$$

is an injective homomorphism of groups, where

$$\hat{g}(x) = gx \quad \text{for all } g \in G \text{ and for all } x \in X (= G).$$

Also, the image  $\hat{G}$  is regular on  $X$ . i.e., the image  $\hat{G}$  acts transitively on  $X$  with trivial vertex stabilizers.

(ii) For any graph  $\Gamma = (X, E)$ , suppose there exists a subgroup  $G \subseteq \text{Aut}(\Gamma)$  that is regular on  $X$ . Pick  $x \in X$ , and let

$$\Delta = \{g \in G \mid \langle x, g(x) \in E \rangle\}.$$

Then  $1 \notin \Delta$ ,  $g \in \Delta \rightarrow g^{-1} \in \Delta$ , and  $\Delta$  generates  $G$ . Moreover,  $\Gamma \simeq \Gamma(G, \Delta)$ .

*Proof.* (i) Let  $g \in G$ . We want to show that  $\hat{g} \in \text{Aut}(\Gamma)$ . Let  $h_1, h_2 \in X = G$ . Then,

$$(h_1, h_2) \in E \rightarrow h_1^{-1}h_2 \in \Delta \quad (3.1)$$

$$\rightarrow (gh_1)^{-1}(gh_2) \in \Delta \quad (3.2)$$

$$\rightarrow (gh_1, gh_2) \in E \quad (3.3)$$

$$\rightarrow (\hat{g}(h_1), \hat{g}(h_2)) \in E. \quad (3.4)$$

Hence,  $\hat{g} \in \text{Aut}(\Gamma)$ .

Observe:  $g \mapsto \hat{g}$  is a homomorphism of groups:

$$\hat{1}_G = 1, \widehat{g_1 g_2} = \widehat{g_1} \widehat{g_2}.$$

Observe:  $g \mapsto \hat{g}$  is one-to-one:

$$\widehat{g_1} = \widehat{g_2} \rightarrow g_1 = \widehat{g_1}(1_G) = \widehat{g_2}(1_G) = g_2.$$

Observe:  $\hat{G}$  is regular on  $X$ : Clear by construction.

(ii)  $1_G \notin \Delta$ : Since  $\Gamma$  has not loops,  $(x, 1_G x) \notin E$ .

$g \in \Delta \rightarrow g^{-1} \in \Delta$ :

$$g \in \Delta \rightarrow (x, g(x)) \in E \rightarrow E \ni (g^{-1}(x), g^{-1}(g(x))) = (g^{-1}(x), x).$$

$\Delta$  generates  $G$ : Suppose  $\langle \Delta \subsetneq G$ . Let  $\hat{X} = \{g(x) \mid g \in \langle \Delta \rangle\} \subsetneq X$ . ( $\hat{X} \subsetneq X$  as  $G$  acts regularly on  $X$ .)

Since  $\Gamma$  is connected, there exists  $y \in \hat{X}$  and  $z \in X \setminus \hat{X}$  with  $yz \in E$ .

Let  $y = g(x)$ ,  $g \in \langle \Delta \rangle$ ,  $z \in h(x)$ ,  $h \in G \setminus \langle \Delta \rangle$ . Then

$$(y, z) = (g(x), h(x)) \in E \rightarrow (x, g^{-1}h(x)) \in E \rightarrow g^{-1}h \in \langle \Delta \rangle \rightarrow h \in \langle \Delta \rangle.$$

This is a contradiction. Therefore,  $\Delta$  generates  $G$ .

Let  $\Gamma' = (X', E')$  denote  $\Gamma(G, \Delta)$ . We shall show that

$$\theta : X' \rightarrow X \ (g \mapsto g(x))$$

is an isomorphism of graphs.

$\theta$  is one-to-one: For  $h_1, h_2 \in X' = G$ ,

$$\theta(h_1) = \theta(h_2) \rightarrow h_1(x) = h_2(x) \rightarrow h_1^{-1}h_2(x) = x \rightarrow h_1^{-1}h_2 \in \text{Stab}_G(x) = \{1_G\} \rightarrow h_1 = h_2.$$

( $\text{Stab}_G = \{g \in G \mid g(x) = x\}$ .)

$\theta$  is onto: Since  $G$  is transitive,

$$X = \{g(x) \mid g \in G\} = \theta(X') = \theta(G).$$

$\theta$  respects adjacency: For  $h_1, h_2 \in X' = G$ ,

$$(h_1, h_2) \in E' \leftrightarrow h_1^{-1}h_2 \in \Delta \leftrightarrow (x, h_1^{-1}h_2(x)) \in E \leftrightarrow (h_1(x), h_2(x)) \in E \leftrightarrow (\theta(h_1), \theta(h_2)) \in E.$$

Therefore  $\theta$  is an isomorphism between graphs  $\Gamma(G, \Delta)$  and  $\Gamma(X, E)$ .  $\square$

How to compute the eigenvalues of the Cayley graph of an abelian group.

Let  $G$  be any finite abelian group. Let  $\mathbb{C}^*$  be the multiplicative group on  $\mathbb{C} \setminus \{0\}$ .

**Definition 3.5.** A (linear)  $G$ -character is any group homomorphism  $\theta : G \rightarrow \mathbb{C}^*$ .

**Example 3.5.**  $G = \langle a \mid a^3 = 1 \rangle$  has three characters,  $\theta_0, \theta_1, \theta_2$ .

$$\begin{array}{c|ccc} \theta_i(a^j) & 1 & a & a^2 \\ \hline \theta_0 & 1 & 1 & 1 \\ \theta_1 & 1 & \omega & \omega^2 \\ \theta_2 & 1 & \omega^2 & \omega \end{array}, \quad \text{with } \omega = \frac{-1 + \sqrt{-3}}{2}.$$

Here  $\omega$  is a primitive cube root of  $\omega$  in  $\mathbb{C}^*$ , i.e.,  $1 + \omega + \omega^2 = 0$ .

For arbitrary group  $G$ , let  $X(G)$  be the set of all characters of  $G$ .

Observe: For  $\theta_1, \theta_2 \in X(G)$ , one can define product  $\theta_1 \theta_2$ :

$$\theta_1 \theta_2(g) = \theta_1(g) \theta_2(g) \quad \text{for all } g \in G.$$

Then  $\theta_1 \theta_2 \in X(G)$ .

Observe:  $X(G)$  with this product is an (abelian) group.

**Lemma 3.1.** *The groups  $G$  and  $X(G)$  are isomorphic for all finite abelian groups  $G$ .*

*Proof.*  $G$  is a direct sum of cyclic groups;

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m, \quad \text{where } G_i = \langle a_i \mid a_i^{d_i} = 1 \rangle \quad (1 \leq i \leq m).$$

Pick any element  $\omega_i$  of order  $d_i$  in  $\mathbb{C}^*$ , i.e., a primitive  $d_i$ -th root of 1. Define

$$\theta_i : G \rightarrow \mathbb{C}^* \quad (a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \mapsto \omega_i^{\varepsilon_i} \quad \text{where } 0 \leq \varepsilon_i < d_i, 1 \leq i \leq m).$$

Then  $\theta_i \in X(G)$ . (Exercise)

Claim: There exists an isomorphism of groups  $G \rightarrow X(G)$  that sends  $a_i$  to  $\theta_i$ .

Observe:  $\theta_i^{d_i} = 1$ . For every  $g = a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \in G$ ,

$$\theta_i^{d_i}(g) = (\theta_i(g))^{d_i} = (\omega_i^{\varepsilon_i})^{d_i} = (\omega_i^{d_i})^{\varepsilon_i} = 1.$$

Observe: If  $\theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m} = 1$  for some  $0 \leq \varepsilon_i < d_i, 1 \leq i \leq m$ . Then  $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = 0$ .

*Pf.*  $1 = \theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m}(a_i) = \omega_i^{\varepsilon_i}$ , Since  $\omega_i$  is a primitive  $d_i$ -th root of 1,  $\varepsilon_i = 0$  for  $1 \leq i \leq m$ .

Observe:  $\theta_1, \dots, \theta_m$  generate  $X(G)$ . Pick  $\theta \in X(G)$ . Since  $a_i^{d_i} = 1$ ,  $1 = \theta(a_i^{d_i}) = \theta(a_i)^{d_i}$ .

Hence  $\theta(a_i) = \omega_i^{\varepsilon_i}$  for some  $\varepsilon_i$  with  $0 \leq \varepsilon_i < d_i$ .

Now  $\theta = \theta_1^{\varepsilon_1} \cdots \theta_m^{\varepsilon_m}$ , since these are both equal to  $\omega_i^{\varepsilon_i}$  at  $a_i$  for  $1 \leq i \leq m$ .

Therefore,

$$G \rightarrow X(G) \quad (a_i \mapsto \theta_i)$$

is an isomorphism of groups. □

**Note.** The correspondence above is clearly a group homomorphism.





## Chapter 4

# Examples

Wednesday, January 27, 1993

**Theorem 4.1.** *Given a Cayley graph  $\Gamma = \Gamma(G, \Delta)$ . View the standard module  $V \equiv \mathbb{C}G$  (the group algebra), so*

$$\left\langle \sum_{g \in G} \alpha_g g, \sum_{g \in G} \beta_g g \right\rangle = \sum_{g \in G} \alpha_g \overline{\beta_g}, \quad \text{with } \alpha_g, \beta_g \in \mathbb{C}.$$

For any  $\theta \in X(G)$ , write

$$\hat{\theta} = \sum_{g \in G} \theta(g^{-1})g.$$

Then the following hold.

(i)  $\langle \hat{\theta}_1, \hat{\theta}_2 \rangle = |G|$  if  $\theta_1 = \theta_2$  and 0 otherwise for  $\theta_1, \theta_2 \in X(G)$ . In particular,  $\{\hat{\theta} \mid \theta \in X(G)\}$  forms a basis for  $V$ .

(ii)  $A\hat{\theta} = \Delta_\theta \hat{\theta}$  for  $\theta \in X(G)$ , where  $A$  is the adjacency matrix and

$$\Delta_\theta = \sum_{g \in \Delta} \theta(g).$$

In particular, the eigenvalues of  $\Gamma$  are precisely

$$\Delta_\theta \mid \theta \in X(G)\}.$$

*Proof.*

(i) Claim: For every  $\theta \in X(G)$ , let

$$s := \sum_{g \in G} \theta(g^{-1}) = \begin{cases} |G| & \text{if } \theta = 1 \\ 0 & \text{if } \theta \neq 1. \end{cases}$$

*Pf.* Clear if  $\theta = 1$ .

Let  $\theta \neq 1$ . Then  $\theta(h) \neq 1$  for some  $h \in G$ .

$$s \cdot \theta(h) = \left( \sum_{g \in G} \theta(g^{-1}) \right) \theta(h) = \sum_{g \in G} \theta(g^{-1}h) = \sum_{g' \in G} \theta(g'^{-1}) = s.$$

Since  $\theta(h) \neq 1$ ,  $s = 0$ .

Claim.  $\theta(g^{-1}) = \overline{\theta(g)}$  for every  $\theta \in X(G)$  and every  $g \in G$ .

Since  $\theta(g) \in \mathbb{C}$  is a root of 1,

$$|\theta(g)|^2 = \theta(g)\overline{\theta(g)} = 1.$$

On the other hand, since  $\theta$  is a homomorphism,

$$\theta(g)\theta(g^{-1}) = \theta(1) = 1.$$

Hence  $\theta(g^{-1}) = \overline{\theta(g)}$ .

Now

$$\langle \widehat{\theta_1}, \widehat{\theta_2} \rangle = \sum_{g \in G} \theta_1(g^{-1}) \overline{\theta_2(g^{-1})} \quad (4.1)$$

$$= \sum_{g \in G} \theta_1(g^{-1}) \theta_2(g) \quad (4.2)$$

$$= \sum_{g \in G} \theta_1 \theta_2^{-1}(g^{-1}) \quad (4.3)$$

$$= \begin{cases} |G| & \text{if } \theta_1 \theta_2^{-1} = 1 \\ 0 & \text{if } \theta_1 \theta_2^{-1} \neq 1. \end{cases} \quad (4.4)$$

Since  $|G| = |X(G)|$  by Lemma 3.1, and  $\widehat{\theta_i}$ 's are orthogonal nonzero elements in  $V$ , they form a basis of  $V$ .

(ii) Let  $\Delta = \{g_1, \dots, g_r\}$ . Then

$$A\hat{\theta} = A \left( \sum_{g \in G} \theta(g^{-1}g) \right) \quad (4.5)$$

$$= \sum_{g \in G} \theta(g^{-1})(gg_1 + \cdots + gg_r) \quad (\Gamma(g) = \{gg_1, \dots, gg_r\}) \quad (4.6)$$

$$= \sum_{i=1}^r \left( \sum_{g \in G} \theta(g^{-1})(gg_i) \right) \quad (4.7)$$

$$= \sum_{i=1}^r \left( \sum_{g \in G} \theta(g_i g_i^{-1} g^{-1})(gg_i) \right) \quad (4.8)$$

$$= \sum_{i=1}^r \left( \sum_{g \in G} \theta(g_i) \theta((gg_i)^{-1}) gg_i \right) \quad (4.9)$$

$$= \sum_{i=1}^r \theta(g_i) \sum_{h \in G} \theta(h^{-1}) h \quad (4.10)$$

$$= \Delta_{\theta} \cdot \hat{\theta}. \quad (4.11)$$

Since  $\{\hat{\theta} \mid \theta \in X(G)\}$  forms a basis, the eigenvalues of  $\Gamma$  are precisely,

$$\{\Delta_{\theta} \mid \theta \in X(G)\}.$$

This completes the proof.

□

**Example 4.1.** Let  $G = \langle a \mid a^6 = 1 \rangle$ , and  $\Delta = \{a, a^{-1}\}$ . Pick a primitive 6-th root of 1,  $\omega$ . Then

$$X(G) = \{\theta^i \mid 0 \leq i \leq 5\} \quad \text{such that} \quad \theta(a) = \omega, \quad \omega + \omega^{-1} = 1.$$



$\varphi \in X(G)$	$\varphi(a)$	$\Delta_\varphi = \theta(a) + \theta(a)^{-1}$
1	1	2
$\theta$	$\omega$	$\omega + \omega^{-1} = 1$
$\theta^2$	$\omega^2$	-1
$\theta^3$	$\omega^3 = -1$	-2
$\theta^4$	$\omega^4$	-1
$\theta^5$	$\omega^5$	1

$$\text{Spec}(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

**Example 4.2.**  $D$ -cube,  $H(D, 2)$ . Let

$$X = \{(a_1, \dots, a_D) \mid a_i \in \{1, -1\}, 1 \leq i \leq D\},$$

$$E = \{xy \mid x, y \in X, x, y: \text{different in exactly one coordinate}\}.$$

Also  $H(D, 2)$  is a Cayley graph  $\Gamma(G, \Delta)$ , where

$$G = G_1 \oplus G_2 \oplus \dots \oplus G_D,$$

$$G_i = \langle a_i \mid a_i^2 = 1 \rangle, \quad \Delta = \{a_1, \dots, a_D\}.$$

**Homework:** The spectrum of  $H(D, 2)$  is

$$\begin{pmatrix} \theta_0 & \theta_1 & \dots & \theta_D \\ m_0 & m_1 & \dots & m_D \end{pmatrix},$$

where

$$\theta_i = D - 2i \quad (0 \leq i \leq D), \quad m_i = \binom{D}{i}.$$

*Remark.* Let  $\theta \in X(G)$ . Then  $\theta : X \rightarrow \{\pm 1\}$ . If

$$\nu(\theta) = |\{i \mid \theta(a_i) = -1\}|,$$

then  $\Delta_\theta = D - 2i$ . Since there are  $\binom{D}{i}$  such  $\theta$ , we have the assertion.

We want to compute the subconstituent algebra for  $H(D, 2)$ . First, we make a few observations about arbitrary graphs.

Let  $\Gamma = (X, E)$  be any graph,  $A$ , the adjacency matrix of  $\Gamma$ , and  $V$ , the standard module over  $K = \mathbb{C}$ .

Fix a base  $x \in X$ . Write  $E_i^* = E_i^*(x)$ , and

$$T \equiv T(x) = \text{the algebra generated by } A, E_0^*, E_1^*, \dots$$

**Definition 4.1.** Let  $W$  be any irreducible  $T$ -module ( $\subseteq V$ ). Then the endpoint  $r \equiv r(W)$  satisfied

$$r = \min\{i \mid E_i^* W \neq 0\}.$$

The diameter  $d = d(W)$  satisfied

$$d = |\{i \mid E_i^* W \neq 0\}| - 1.$$

**Lemma 4.1.** *With the above notation, let  $W$  be an irreducible  $T$ -module. Then*

- (i)  $E_i^* A E_j^* = 0$  if  $|i - j| = 1$ ,  $\neq 0$  if  $|i - j| = 1$ ,  $0 \leq i, j \leq d(x)$ .
- (ii)  $A E_j^* W \subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$ ,  $0 \leq j \leq d(x)$ . ( $E_i^* W = 0$  if  $i < j$  or  $i > d(x)$ .)
- (iii)  $E_j^* W \neq 0$  if  $r \leq j \leq r + d$ ,  $= 0$  if  $0 \leq j \leq r$  or  $r + d < j \leq d(x)$ .
- (iv)  $E_i^* A E_j^* W \neq 0$ , if  $|i - j| = 1$  ( $r \leq i, j \leq r + d$ ).

*Proof.*

(i) Pick  $y \in X$  with  $\partial(x, y) = j$ . We want to find  $E_i^* A E_j^* \hat{y}$ . Note,

$$E_j^* \hat{y} = \begin{cases} 0 & \text{if } \partial(x, y) \neq j \\ \hat{y} & \text{if } \partial(x, y) = j. \end{cases}$$

$$E_i^* A E_j^* \hat{y} = E_i^* A \hat{y} \tag{4.12}$$

$$= E_i^* \sum_{z \in X, yz \in E} \hat{z} \tag{4.13}$$

$$= \sum_{z \in X, yz \in E, \partial(x, z) = i} \hat{z} \quad (*) \tag{4.14}$$

$$= 0 \text{ if } |i - j| > 1 \text{ by triangle inequality.} \tag{4.15}$$

If  $|i - j| = 1$ , there exist  $y, y' \in X$  such that  $\partial(x, y) = j$ ,  $\partial(x, y') = i$ ,  $yy' \in E$  by connectivity of  $\Gamma$ . Hence  $(*)$  contains  $\widehat{yy'}$  and  $* \neq 0$

(ii) We have

$$A E_j^* W = \left( \sum_{i=0}^{d(x)} E_i^* \right) A E_j^* W \tag{4.16}$$

$$= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W \tag{4.17}$$

$$\subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W. \tag{4.18}$$

(iii) Suppose  $E_j^* W = 0$  for some  $j$  ( $r \leq j \leq r + d$ ). Then  $r < j$  by the definition of  $r$ . Set

$$\tilde{W} = E_r^*W + E_{r+1}^*W + \cdots + E_{j-1}^*W.$$

Observe  $0 \subsetneq \tilde{W} \subsetneq W$ . Also  $A\tilde{W} \subseteq \tilde{W}$  by (ii) and  $E_i^*\tilde{W} \subseteq \tilde{W}$  for every  $i$  by construction.

Thus  $T\tilde{W} \subseteq \tilde{W}$ , contradicting  $W$  being irreducible.

□

## Chapter 5

# $T$ -Modules of $H(D, 2)$ , I

**Friday, January 29, 1993**

Let  $\Gamma = (X, E)$  be a graph,  $A$  the adjacency matrix, and  $V$  the standard module over  $K = \mathbb{C}$ .

Fix a base  $x \in X$  and write  $E_i^* \equiv E_i^*(x)$ , and  $T \equiv T(x)$ .

Let  $W$  be an irreducible  $T$ -module with endpoint  $r := \min\{i \mid E_i^*W \neq 0\}$  and diameter  $d := |\{i \mid E_i^*W \neq 0\}| - 1$ .

We have

$$E_i^*W \neq 0 \quad r \leq i \leq r + d \quad (5.1)$$

$$= 0 \quad 0 \leq i < r \text{ or } r + d < i \leq d(x). \quad (5.2)$$

Claim:  $E_i^*AE_j^*W \neq 0$  if  $|i - j| = 1$  for  $r \leq i, j \leq r + d$ . (See Lemma 4.1.)

Suppose  $E_{j+1}^*AE_j^*W = 0$  for some  $j$  with  $r \leq j < r + d$ . Observe that

$$\tilde{W} = E_r^*W + \cdot E_j^*W$$

is  $T$ -invariant with

$$0 \subsetneq \tilde{W} \subsetneq W.$$

Becase  $A\tilde{W} \subseteq \tilde{W}$  since  $AE_j^*W \subseteq E_{j-1}^*W + E_j^*W$ ,

$$E_k^*\tilde{W} \subseteq \tilde{W} \quad \text{for all } k,$$

we have  $T\tilde{W} \subseteq \tilde{W}$ .

Suppose  $E_{i-1}^*AE_i^*W = 0$  for some  $i$  with  $r \leq i < r + d$ .

Similarly,

$$\tilde{W} = E_i^*W + \cdot E_{r+d}^*W$$

is a  $T$ -module with  $0 \subsetneq \tilde{W} \subsetneq W$ .

**Definition 5.1.** Let  $\Gamma$ ,  $E_i^*$ , and  $T$  be as above. Irreducible  $T$ -modules  $W$  and  $W'$  are isomorphic whenever there is an isomorphism  $\sigma : W \rightarrow W'$  of vector spaces such that  $a\sigma = \sigma a$  for all  $a \in T$ .

Recall that the standard module  $V$  is an orthogonal direct sum of irreducible  $T$ -modules  $W_1 \oplus W_2 \oplus \dots$ . Given  $W$  in this list, the multiplicity of  $W$  in  $V$  is

$$|\{j \mid W_j \simeq W\}|.$$

*Remark.* It is known that the multiplicity does not depend on the decomposition.

Now assume that  $\Gamma$  is the  $D$ -cube,  $H(D, 2)$  with  $D \geq 1$ . View

$$X = \{a_1 \cdots a_D \mid a_i \in \{1, -1\}, 1 \leq i \leq D\}, \quad (5.3)$$

$$E = \{xy \mid x, y \in X, x, y \text{ differ in exactly 1 coordinate.}\}. \quad (5.4)$$

Find  $T$ -modules.

Claim:  $H(D, 2)$  is bipartite with a partition  $X = X^+ \cup X^-$ , where

$$X^+ = \{a_1 \cdots a_D \in X \mid \prod a_i > 0\} \quad (5.5)$$

$$X^- = \{a_1 \cdots a_D \in X \mid \prod a_i < 0\} \quad (5.6)$$

Observe: for all  $y, z \in X$ ,

$$\partial(y, z) = i \Leftrightarrow y, z \text{ differ in exactly } i \text{ coordinates with } 0 \leq i \leq D.$$

Here, the diameter of  $H(D, 2) = D = d$  for all  $x \in X$ .

**Theorem 5.1.** Let  $\Gamma = H(D, 2)$  be as above. Fix  $x \in X$ , and write  $E_i^* = E_i^*(x)$ , and  $T = T(x)$ .

Let  $W$  be an irreducible  $T$ -module with endpoint  $r$ , and diameter  $d$  with  $0 \leq r \leq r + d \leq D$ .

(i)  $W$  has a basis  $w_0, w_1, \dots, w_d$  with  $w_i \in E_{i+r}^* W$  for  $0 \leq i \leq d$ . With respect to which the matrix representing  $A$  is

$$\begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & d-1 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 2 & 0 \\ 0 & 0 & 0 & \cdots & d-1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & d & 0 \end{pmatrix}$$



- (ii) \$d = D - 2r\$. In particular, \$0 \leq r \leq D/2\$.
- (iii) Let \$W'\$ denote an irreducible \$T\$-module with endpoint \$r'\$. Then \$W\$ and \$W'\$ are isomorphic as \$T\$-modules if and only if \$r = r'\$.
- (iv) The multiplicity of the irreducible \$T\$-module with endpoint \$r\$ is

$$\binom{D}{r} - \binom{D}{r-1} \quad \text{if } 1 \leq r \leq D/2,$$

and 1 if \$r = 0\$.

*Proof.* Recall that \$\Gamma\$ is vertex transitive. It is a Cayley graph.

Hence without loss of generality, we may assume that \$x = \overbrace{11 \cdots 1}^D\$.

Notation: Set \$\Omega = \{1, 2, \dots, D\}\$. For every subset \$S \subseteq \Omega\$, let

$$\hat{S} = a_1 \cdot a_d \in X \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S. \end{cases}$$

In particular, \$\text{emptyset} = x\$ and

$$|S| = i \Leftrightarrow \partial(x, \hat{S}) = i \Leftrightarrow \hat{S} \in E_i^* V.$$

For all \$S, T \subseteq \Omega\$, we say \$S\$ covers \$T\$ if and only if \$S \supseteq T\$ and \$|S| = |T| + 1\$.

Observe that \$\hat{S}, \hat{T}\$ are adjacent in \$\Gamma\$ if and only if either \$T\$ covers \$S\$ or \$S\$ covers \$T\$.

Define the ‘raising matrix’

$$R = \sum_{i=0}^D E_{i+1}^* A E_i^*.$$

Observe that

$$R E_i^* V \subseteq E_{i+1}^* V \quad \text{for } 0 \leq i \leq D, \quad \text{and } E_{D+1}^* V = 0.$$

Indeed for any \$S \subseteq \Omega\$ with \$|S| = i\$,

$$R \hat{S} = R E_i^* \hat{S} \tag{5.7}$$

$$= E_{i+1}^* A \hat{S} \tag{5.8}$$

$$= \sum_{T_1 \subseteq \Omega, S \text{ covers } T_1} E_{i+1}^* \hat{T}_1 + \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T} \tag{5.9}$$

$$= \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T}. \tag{5.10}$$

Define the ‘lowering matrix’

$$L = \sum_{i=0}^D E_{i-1}^* A E_i^*.$$

Observe that

$$L E_i^* V \subseteq E_{i-1}^* V \quad \text{for } 0 \leq i \leq D, \quad \text{and } E_{-1}^* V = 0.$$

Indeed for any  $S \subseteq \Omega$ ,

$$L \hat{S} = \sum_{T \subseteq \Omega, S \text{ covers } T} \hat{T}.$$

Observe that  $A = L + R$ .

For convenience, set

$$A^* = \sum_{i=0}^D (D - 2i) E_i^*.$$

Claim: The following hold.

- (a)  $LR - RL = A^*$ .
- (b)  $A * L - LA^* = 2L$ .
- (c)  $A^* R - RA^* = -2R$ .

In particular  $\text{Span}(R, L, A^*)$  is a ‘representation of Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$ .

*Remark* (Lie Algebra  $\mathfrak{sl}_2(\mathbb{C})$ ).

$$\mathfrak{sl}_2(\mathbb{C}) = \{X \mid \text{Mat}(\mathbb{C}) \mid \text{tr}(X) = 0\}.$$

For  $X, Y \in \mathfrak{sl}_2(\mathbb{C})$ , define a binary operation  $[X, Y] = XY - YX$ .

$$A^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then these satisfy the relations (a) - (c) above.

*Proof of Claim.* Apply both sides to  $\hat{S}$  ( $S \subseteq \Omega$ ). Say  $|S| = i$ .

*Proof of (a):*

$$(LR - RL)\hat{S} = L \left( \sum_{\substack{T \subseteq \Omega, T \text{ covers } S \\ (D-i \text{ of them})}} \hat{T} \right) - R \left( \sum_{\substack{U \subseteq \Omega, S \text{ covers } U \\ (i \text{ of them})}} \hat{T} \right) \quad (5.11)$$

$$= (D - i)\hat{S} + \sum_{V \subseteq \Omega, |V|=i, |S \cap V|=i-1} \hat{V} - \left( i\hat{S} + \sum_{V \subseteq \Omega, |V|=i, |S \cap V|=i-1} \hat{V} \right) \quad (5.12)$$

$$= (D - 2i)\hat{S} \quad (5.13)$$

$$= A^* \hat{S}. \quad (5.14)$$

*Proof of (b):*

$$(A^*L - LA^*)\hat{S} = (D - 2(i - 1))L\hat{S} - (D - 2i)L\hat{S} \quad (\text{since } L\hat{S} \in E_{i-1}^*V) \quad (5.15)$$

$$= 2L\hat{S}. \quad (5.16)$$

*Proof of (c):*

$$(A^*R - RA^*)\hat{S} = (D - 2(i + 1))R\hat{S} - (D - 2i)R\hat{S} \quad (\text{since } R\hat{S} \in E_{i+1}^*V) \quad (5.17)$$

$$= 2R\hat{S}. \quad (5.18)$$

Let  $W$  be an irreducible  $T$ -module with endpoint  $r$  and diameter  $d$  ( $0 \leq r \leq r + d \leq D$ ).

*Proof of (i) and (ii):*

Pick  $0 \neq w \in E_r^*W$ .

Claim:  $LRw = (D - 2r)w$ .

*Pf.*

$$LRw = (A^* + RL)w \quad (\text{by Claim (a)}) \quad (5.19)$$

$$= A^*w \quad (Lw \in E_{r-1}^*W = 0) \quad (5.20)$$

$$(D - 2r)w. \quad (5.21)$$

Define

$$w_i = \frac{1}{i!}R^i w \in E_{r+i}^*W \quad (0 \leq i \leq d).$$

Then,

$$Rw_i = (i + 1)w_{i+1} \quad (0 \leq i \leq d) \quad (5.22)$$

$$Rw_d = 0 \quad (\text{by definition of } d) \quad (5.23)$$

Claim:  $Lw_0 = 0$  and

$$Lw_i = (D - 2r - i + 1)w_{i-1} \quad (1 \leq i \leq d).$$

*Pf.* We prove by induction on  $i$ . The case  $i = 0$  is trivial, and the case  $i = 1$

follows from above claim. Let  $i \geq 2$ ,

$$Lw_i = \frac{1}{i}LRw_{i-1} = \frac{1}{i}(A^* + RL)w_{i-1} \quad (\text{by Claim (a)}) \quad (5.24)$$

$$(\text{by induction hypothesis}) \quad (5.25)$$

$$= \frac{1}{i}((D - 2(r + i - 1))w_{i-1} + (D - 2r - (i - 1) + 1)Rw_{i-2}) \quad (Rw_{i-2} = (i - 1)w_{i-1}) \quad (5.26)$$

$$= \frac{1}{i}i(D - 2r - i + 1)w_{i-1} \quad (5.27)$$

$$= (D - 2r - i + 1)w_{i-1}. \quad (5.28)$$

Claim:  $w_0, \dots, w_d$  is a basis for  $W$ .

*Pf.* Let  $W' = \text{Span}\{w_0, \dots, w_d\}$ . Then  $W'$  is  $R$  and  $L$  invariant. So it is  $A = R + L$  invariant.

Also it is  $E_i^*$ -invariant for every  $i$ .

Hence  $W'$  is a  $T$ -module.

Since  $W$  is irreducible,  $W' = W$ .

As  $w_i$ 's are orthogonal, they are linearly independent. Note that  $w_i \neq 0$  by the definition of  $d$  and Lemma 4.1 (iv).

Claim:  $d = D - 2r$ .

*Pf.* By (a),

$$0 = (LR - RL - A^*)w_d \quad (5.29)$$

$$= 0 - (D - 2r - d + 1)Rw_{d-1} - (D - 2(r + d))w_d \quad (5.30)$$

$$= -d(D - 2r - d + 1)w_d - (D - 2(r + d))w_d \quad (5.31)$$

$$= (-dD + 2rd + d^2 - d - D + 2r + 2d)w_d \quad (5.32)$$

$$= (d^2 + (2r - D + 1)d + 2r - D)w_d \quad (5.33)$$

$$= (d + 2r - D)(d + 1)w_d. \quad (5.34)$$

Hence  $d = D - 2r$ .

Therefore, with respect to a basis  $w_0, w_1, \dots, w_d$ ,  $A = L + R$ ,  $w_{-1} = w_{d+1} = 0$ ,

$$Lw_i = (d - i + 1)w_{i-1}, \quad Rw_i = (i + 1)w_{i+1}.$$

$$L = \begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0 \\ 0 & 0 & d-1 & \cdots & 0 & 0 \\ & & \cdots & \cdots & & \\ & & & & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & & \\ & & & & 0 & 1 \\ 0 & 0 & 0 & \cdots & d & 0 \end{pmatrix}.$$

This completes the proof of (i) and (ii).  $\square$

## Chapter 6

# $T$ -Modules of $H(D, 2)$ , II

Monday, February 1, 1993

*Proof of Theorem 5.1 Continued.*

(iii) Let  $r = r'$ ,

$w_0, \dots, w_d$ : a basis for  $W$  with  $w_i \in E_i^* W$ , and

$w'_0, \dots, w'_d$ : a basis for  $W'$  with  $w'_i \in E_i^* W'$ .

Then  $d = D - 2r = D - 2r' = d'$ , and

$$\sigma : W \rightarrow W' \quad (w_i \mapsto w'_i)$$

is an isomorphism of  $T$ -modules by (i).

If  $r \neq r'$ , then

$$d = D - 2r \neq D - 2r' = d',$$

hence,  $\dim W \neq \dim W'$ .

(iv) Let  $W_i$  be the irreducible  $T$ -module with endpoint  $i$ . Then

$$\dim E_r^* V = \binom{D}{r} = \sum_{i=0}^r \text{mult}(W_i).$$

Hence, we have that

$$\text{mult}(W_r) = \binom{D}{r} - \binom{D}{r-1}$$

by induction on  $r$ .

□

**Theorem 6.1.** *Let  $\Gamma = H(D, 2)$  with  $D \geq 1$ . Fix a vertex  $x \in X$  and write*

$$E_i^* \equiv E_i^*(x), \quad T = T(x), \text{ and } A^* \equiv \sum_{i=0}^D (D - 2i) E_i^*.$$

*Let  $W$  be an irreducible  $T$ -module with endpoint  $r$  with  $0 \leq r \leq D/2$ . Then,*

*(i)  $W$  has a basis*

$$w_0^*, w_1^*, \dots, w_d^* \quad (d = D - 2r), \quad \text{such that } w_i^* \in E_{i+r} W \quad (0 \leq i \leq d)$$

*with respect to which the matrix corresponding to  $A^*$  is*

$$\begin{pmatrix} 0 & d & 0 & & & & \\ 1 & 0 & d-1 & & & & \\ 0 & 2 & 0 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 0 & 2 & 0 \\ & & & & d-1 & 0 & 1 \\ & & & & 0 & d & 0 \end{pmatrix}.$$

*In particular, / (ii)  $E_i A^* E_j = 0$  if  $|i - j| \neq 1$  for  $0 \leq i, j \leq D$ .*

*Proof.* We use the notation,

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \quad (= -[\beta, \alpha]).$$

Recall that

- (a)  $[L, R] = A^*$ ,
- (b)  $[A^*, L] = wL$ ,
- (c)  $[A^*, R] = -2R$ ,

and  $A = L + R$ .

Write (a) – (c) in terms of  $A$  and  $A^*$ , we have,

$$[A, A^*] = [L, A^*] + [R, A^*] = 2(R - L).$$

$$\begin{cases} R + L &= A \\ R - L &= [A, A^*]/2. \end{cases}$$

Hence,

$$R = \frac{1}{4}(2A + [A, A^*]) \quad \text{and} \tag{6.1}$$

$$L = \frac{1}{4}(2A - [A, A^*]). \tag{6.2}$$

Now (a), (b) become

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0 \quad (6.3)$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0 \quad (6.4)$$

*Pf.* By (b),

$$2A - AA^* + A^*A = 4L \quad (6.5)$$

$$= 2[A^*, L] \quad (6.6)$$

$$= A^* \frac{2A - [A, A^*]}{2} - \frac{2A - [A, A^*]}{2} A^* \quad (6.7)$$

So we have ((6.4))

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0.$$

By (a),

$$-16A^* = [2A + [A, A^*], 2A - [A, A^*]] \quad (6.8)$$

$$(2A + [A, A^*])(2A - [A, A^*]) - (2A - [A, A^*])(2A + [A, A^*]) \quad (6.9)$$

$$= [4A^2 - 2A[A, A^*] + [A, A^*](2A) - [A, A^*]^2 \quad (6.10)$$

$$- 4A^2 - 2A[A, A^*] + [A, A^*](2A) + [A, A^*]^2 \quad (6.11)$$

$$= -4A^2A^* + 4AA^*A + 4AA^*A - 4A^*A^2. \quad (6.12)$$

So,

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0.$$

Claim:  $E_i^*A^*E_j = 0$  if  $|i - j| \neq 1$  for  $0 \leq i, j \leq D$ .

*Pf.* We have,

$$0 = E_i(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_j \quad (6.13)$$

$$= E_iA^*E_j(\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4) \quad (6.14)$$

$$(AE_j = \theta_jE_j, E_iA = (AE_j)^\top = (\theta_iE_i)^\top = \theta_iE_i) \quad (6.15)$$

$$= E_iA^*E_j(\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) \quad (6.16)$$

$$= E_iA^*E_j(D - 2i - (D - 2j) - 2)(D - 2i - (D - 2j) + 2) \quad (6.17)$$

$$(\theta_k = D - 2k) \quad (6.18)$$

$$= E_iA^*E_j \cdot 4(i - j + 1)(i - j - 1) \quad (6.19)$$

and  $i - j + 1 \neq 0, i - j - 1 \neq 0$ . Hence,  $E_i^*A^*E_j = 0$ .

Now define “dual raising matrix”,

$$R^* = \sum_{i=0}^D E_{i+1}A^*E_i.$$

So,

$$R^* E_i V \subseteq E_{i+1} V, \quad (0 \leq i \leq D, E_{D+1} V = 0).$$

Define “dual lowering matrix”

$$L^* = \sum_{i=0}^D E_{i-1} A^* E_i.$$

Then

$$L^* E_i V \subseteq E_{i-1} V \quad (0 \leq i \leq D, E_{-1} V = 0).$$

Observe that

$$A^* = \left( \sum_{i=0}^D E_i \right) A^* \left( \sum_{j=0}^D E_j \right) = L^* + R^*$$

by Claim 1.

Claim 2. We have | (a)  $[L^*, R^*] = A$ , | (b)  $[A, L^*] = 2L^*$ , | (c)  $[A, R^*] = -2R^*$ .

*Pf.* (b)

$$AL^* - L^* A = \sum_{i=0}^D (AE_{i-1} A^* E_i - E_{i-1} A^* E_i A) \quad (6.20)$$

$$= \sum_{i=0}^D E_{i-1} A^* E_i (\theta_{i-1} - \theta_i) \quad (6.21)$$

$$(\theta_k = D - 2k, \quad \theta_{i-1} - \theta_i = 2I - 2(i-1) = 2) \quad (6.22)$$

$$= 2L^*. \quad (6.23)$$

(c) Similar.

*Remark.*

$$AR^* - R^* A = \sum_{i=0}^D (AE_{i+1} A^* E_i - E_{i+1} A^* E_i A) \quad (6.24)$$

$$= \sum_{i=0}^D E_{i+1} A^* E_i (\theta_{i+1} - \theta_i) \quad (6.25)$$

$$= 2R^*. \quad (6.26)$$

(a) We have, by (b), (c)

$$[A, A^*] = [A, L^*] + [A, R^*] = 2(L^* - R^*). \quad (6.27)$$

Since  $A^* = L^* + R^*$ ,

$$R^* = \frac{2A^* + [A^*, A]}{4}, \quad L^* = \frac{2A^* - [A^*, A]}{4}.$$

Now (a) is seen to be equivalent to ((6.4)) upon evaluation. This proves Claim 2.



*Remark.*

$$[L^*, R^*] = \frac{1}{16}((2A^* - [A^*, A])(2A^* + [A^*, A]) - (2A^* + [A^*, A])(2A^* - [A, A^*])) \quad (6.28)$$

$$= \frac{1}{16}(4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* - [A^*, A]^2 - 4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* + [A^*, A]^2) \quad (6.29)$$

$$= \frac{1}{4}(A^{*2}A - 2A^*AA^* + AA^{*2}) \quad (6.30)$$

$$= A, \quad (6.31)$$

by ((6.4)).

Now apply same argument as for ((6.3)), ((6.4)) of Theorem 5.1 and observe  $A^*$  has  $D + 1$  distinct eigenvalues. So,

$$A^* = \sum_{i=0}^D (D - 2i)E_i^*$$

generates

$$M^* = \text{Span}(E_0^*, \dots, E_D^*).$$

Hence,  $E_0, \dots, E_D, A^*$  generates  $T$ .

Take an irreducible  $T$ -module  $W$  with endpoint  $r$  with  $0 \leq r \leq D/2$ . Set  $t = \min\{i \mid E_i W\}$ .

Pick  $0 \neq w_0^* \in E_t W$ . Set

$$w_i^* = \frac{1}{i!} R^{*i} w_0^* \in E_{t+i} W \quad \text{for all } i.$$

Then,

$$R^* w_i^* = (i + 1) w_{i+1}^* \quad \text{for all } i.$$

By (a), we get by induction,  $L^* w_i^* = (D - 2t - i + 1) w_{i-1}^*$ ,

$$L^* w_i^* = \frac{1}{i} L^* R^* w_{i-1}^* \quad (6.32)$$

$$= \frac{1}{i} (A + R^* L^*) w_{i-1}^* \quad (6.33)$$

$$= \frac{1}{i} ((D - 2(t + i - 1)) w_{i-1}^* + (i - 1)(D - 2t - i + 2) w_{i-1}^*) \quad (6.34)$$

$$= (D - 2t - i + 1) w_{i-1}^*. \quad (6.35)$$

So  $\text{Span}(w_0^*, w_1^*, \dots)$  is  $L^*, R^*, A^*$ -invariant. Hence,  $W = (\text{Span})(w_0^*, w_1^*, \dots, w_d^*)$ ,  $w_0^*, w_1^*, \dots, w_d^* \neq 0$ ,  $w_i^* = 0$  for every  $i > d$  by dimension.

Thus  $d = D - 2t$ .

*Pf.*

$$(D - 2(t + d))w_d^* = Aw_d^* \quad (6.36)$$

$$= (L^*R^* - R^*L^*)w_d^* \quad (6.37)$$

$$= -(D - 2t - d + 1)R^*w_{d-1}^* \quad (6.38)$$

$$= -(D - 2t - d + 1)dw_d^*. \quad (6.39)$$

Hence,

$$0 = d^2 + (2t - D - 1 + 2)d - (D - 2t) = (d - D + 2t)(d + 1)$$

So  $d = D - 2t$ . □

**Definition 6.1.** For any graph  $\Gamma = (X, E)$ , pick a vertex  $x \in X$  and set  $E_i^* \equiv E_i^*(x)$  and  $T \equiv T(x)$ .

- (i) an irreducible  $T$ -module  $W$  is thin if  $\dim E_i^*W \leq 1$  for every  $i$ ,
- (ii)  $\Gamma$  is thin with respect to  $x$ , if every irreducible  $T(x)$ -module is thin,
- (iii) an irreducible  $T$ -module  $W$  is dual thin if  $\dim E_iW \leq 1$  for every  $i$ ,
- (iv)  $\Gamma$  is dual thin with respect to  $x$ , if every irreducible  $T(x)$ -module is dual thin.

Observe:  $H(D, 2)$  is thin, dual thin with respect to each  $x \in X$ .

With above notation, write  $D \equiv D(x)$ .

- (i) an ordering  $E_0, E_1, \dots, E_R$  of primitive idempotents of  $\Gamma$  is restricted if  $E_0$  corresponds to the maximal eigenvalue.

Fix a restricted ordering,

- (ii)  $\Gamma$  is  $Q$ -polynomial with respect to  $x$ , above ordering if there exists  $A^* \equiv A^*(x)$  such that

$$(a) E_0^*V, \dots, E_D^*V \text{ are the maximal eigenspaces for } A^*.$$

$$(b) E_iA^*E_j = 0 \text{ if } |i - j| > 1 \text{ for } 0 \leq i, j \leq R.$$

Observe  $H(D, 2)$  is  $Q$ -polynomial with respect to the natural ordering of the idempotents and every vertex.

**Program.** Study graphs that are thin and  $Q$ -polynomial with respect to each vertex.

(In fact, thin with respect to  $x$  implies dual thin with respect to  $x$ .)

Get a situation like  $H(D, 2)$ , where  $T$  is generated by  $A, A^*$ . Except  $\mathfrak{sl}_s(\mathbb{C})$  is replaced by a quantum Lie algebra.

## Chapter 7

# The Johnson Graph $J(D, N)$

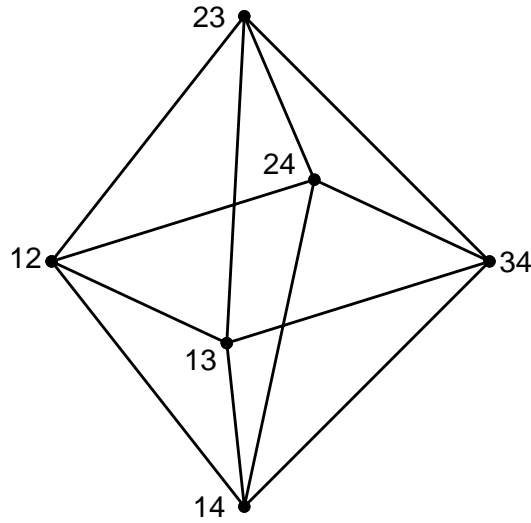
Wednesday, February 3, 1993

**Definition 7.1.** The Johnson graph,  $\Gamma = J(D, N)$  ( $1 \leq D \leq N - 1$ ) satisfies

$$X = \{S \mid S \subset \Omega, |S| = D\} \quad \text{where } \Omega = \{1, 2, \dots, N\} \quad (7.1)$$

$$E = \{ST \mid S, T \in X, |S \cap T| = D - 1\}. \quad (7.2)$$

**Example 7.1.**  $J(2, 4)$



**Note 1.** The symmetric group  $S_N$  acts on  $\Omega$ .  $S_N \subseteq \text{Aut}(\Gamma)$  acts vertex transitively on  $\Gamma$ .

**Note 2.**  $\Gamma = J(D, N)$  is isomorphic to  $\Gamma' = J(N - D, N)$ .

$$\Gamma = (X, E) \qquad \Gamma' = (X', E') \qquad (7.3)$$

$$X \ni S \quad \longrightarrow \quad \bar{S} = \Omega \quad S \in X' \qquad (7.4)$$

This correspondence induces an isomorphism of graphs.

*Pf.*

$$ST \in E \Leftrightarrow |S \cap T| = D - 1 \qquad (7.5)$$

$$\Leftrightarrow |\Omega - (S \cup T)| = N - D - 1 \qquad (7.6)$$

$$\Leftrightarrow |\bar{S} \cap \bar{T}| = N - D - 1 \qquad (7.7)$$

$$\Leftrightarrow \bar{S}\bar{T} \in E' \qquad (7.8)$$

Hence, without loss of generality, assume

$$D \leq N/2 \quad \text{for } J(D, N).$$

We still need the eigenvalues of  $J(D, N)$  for certain problem later in the course. We can get these eigenvalues from our study of  $H(D, 2)$ .

**Lemma 7.1.** *The eigenvalues for  $J(D, N)$  with  $1 \leq D \leq N/2$  are given by*

$$\theta_i = (N - D - i)(D - i) - i \quad (0 \leq i \leq D) \qquad (7.9)$$

$$m_i = \binom{D}{i} - \binom{N}{i-1}. \qquad (7.10)$$

*Proof.* Let

$$\Gamma_J \equiv J(D, N) = (X_J, E_J) \qquad (7.11)$$

$$\Gamma_H \equiv H(N, 2) = (X_H, E_H). \qquad (7.12)$$

Set  $x \equiv 11 \cdots 1 \in X_H$ .

Define  $\tilde{\Gamma} \equiv (\tilde{X}, \tilde{E})$ , where

$$\tilde{X} = \{y \in X_H \mid \partial_H(x, y) = D\} \quad \partial_H : \text{distance in } \Gamma_H \qquad (7.13)$$

$$\tilde{E} = \{yz \in X_H \mid \partial_H(y, z) = 2\}. \qquad (7.14)$$

Observe

$$X_J \rightarrow \tilde{X} \qquad (7.15)$$

$$S \mapsto \hat{S}, \qquad (7.16)$$

where

$$\hat{S} = a_1 \cdots a_N, \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$$

induces an isomorphism of graphs  $\Gamma_J \rightarrow \tilde{\Gamma}$ .

*Pf.*

$$ST \in E_J \Leftrightarrow |S \cap T| = D - 1 \quad (7.17)$$

$$\Leftrightarrow \partial_H(\hat{S}, \hat{T}) = 2 \quad (7.18)$$

$$\Leftrightarrow (\hat{S}, \hat{T}) \in \tilde{E}. \quad (7.19)$$

Identify,  $\Gamma_J$  with  $\tilde{\Gamma}$ . Then the standard module  $V_J$  of  $\Gamma_J$  becomes  $\tilde{V} = E_D^* V_H$ , where  $V_H$  is the standard module of  $\Gamma_H$ , and  $E_D^* \equiv E_D^*(x)$ .

Let  $R$  be the raising matrix with respect to  $x$  in  $\Gamma_H$ , and

let  $L$  be the lowering matrix with respect to  $x$  in  $\Gamma_H$ .

Recall

$$(RL - DE_D^*)|_{\tilde{V}}$$

is the adjacency map in  $\tilde{\Gamma}$ .

To find eigenvalues of  $\tilde{A}$ , pick any irreducible  $T(x)$ -module  $W$  with the endpoint  $r \leq D$ . Then by Theorem 5.1

$$\text{diam}(W) = N - 2r + 1.$$

Let  $w_0, w_1, \dots, w_{N-2r}$  denote a basis for  $W$  as in Theorem 5.1. Then,

$$w_{D-r} \in E_D^* W \subseteq \tilde{V}.$$

Observe:

$$\tilde{A}w_{D-r} = RLw_{D-r} - DE_D^*w_{D-r} \quad (7.20)$$

$$= R(N - 2r - D + r + 1)w_{D-r-1} - Dw_{D-r} \quad (7.21)$$

$$= ((N - D - r + 1)(D - r) - D)w_{D-r}. \quad (7.22)$$

Note that this is valid for  $D = r$  as well.

Hence,

$$\tilde{A}w_{D-r} = ((N - D - r)(D - r) - r)w_{D-r}.$$

Let

$$V_H = \sum W \quad (\text{direct sum of irreducible } T(x) \text{ - modules.})$$

Then,

$$V_J = E_D^* V_H \quad (7.23)$$

$$= \sum_{W: r(W) \leq D} E_D^* W \quad (7.24)$$

$$= \text{a direct sum of 1 dimensional eigenspaces for } \tilde{A}. \quad (7.25)$$

The eigenspace for eigenvalue

$$(N - D - r)(D - r) - r \quad (\text{monotonously decreasing with respect to } r)$$

appears with multiplicity

$$\binom{N}{r} - \binom{N}{r-1}$$

in this sum by Theorem 5.1 (iv).  $\square$

**Theorem 7.1.** *Let  $\Gamma = (X, E)$  be any graph. For a fixed vertex  $x \in X$ , let*

$$E_i^* \equiv E_i^*(x), \quad T \equiv T(x), \quad D \equiv D(x), \quad \text{and } K = \mathbb{C}.$$

*Then we have the following implications of conditions:*

$$TH \Leftrightarrow C \Leftarrow S \Leftarrow G.$$

*where*

*(TH)  $\Gamma$  is thinn with respect to  $x$ .*

*(C)  $E_i^*TE_i^*$  is commutative for every  $i$ ,  $(0 \leq i \leq D)$ .*

*(S)  $E_i^*TE_i^*$  is symmetric for every  $i$ ,  $(0 \leq i \leq D)$ .*

*(G) For every  $y, z \in X$  with  $\partial(x, y) = \partial(x, z)$ , there exists  $g \in \text{Aut}(\Gamma)$  such that*

$$gx = x, \quad gy = z, \quad gz = y.$$

*Proof.*

$(TH) \Rightarrow (C)$

Fix  $i$  with  $0 \leq i \leq D$ . Let

$V = \sum W$ . The standard module written as a direct sum of irreducible  $T$ -modules.

The,

$E_i^*V = \sum E_i^*W$ . The direct sum of 1-dimensional  $E_i^*TE_i^*$ -modules.

Since  $\dim E_i^*W = 1$ , for  $a, b \in E_i^*TE_i^*$ ,  $ab - ba|_{E_i^*W} = 0$ . Hence  $ab - ba = 0$ .

$(C) \Rightarrow (TH)$

Suppose  $\dim E_i^*W \geq 2$  for some irreducible  $T$ -module  $W$  with some  $i$  with  $1 \leq i \leq D$ .

Claim:  $E_i^*W$  is an irreducible  $E_i^*TE_i^*$ -module.

*Pf.* Suppose

$$0 \subsetneq U \subsetneq E_i^*W,$$

where  $U$  is a  $E_i^*TE_i^*$ -module. Then by the irreducibility,

$$TU = W.$$

So

$$U \supseteq E_i^*TE_i^*U = E_i^*TU = E_i^*W.$$

This is a contradiction.

Claim 2: Each irreducible  $S = E_i^*TE_i^*$ -module  $U$  has dimension 1. In particular,  $\Gamma$  is thin with respect to  $x$ .

*Pf.* Pick

$$0 \neq a \in E_i^*TE_i^*.$$

Since  $\mathbb{C}$  is algebraically closed,  $a$  has an eigenvector  $w \in U$  with eigenvalue  $\theta$ . Then,

$$(a - \theta I)U = (a - \theta I)Sw \tag{7.26}$$

$$= S(a - \theta I)w \tag{7.27}$$

$$= 0. \tag{7.28}$$

Hence,

$$a|_U = \theta I|_U \quad \text{for all } a \in S.$$

Thus each 1 dimensional subspace of  $U$  is an  $S$ -module. We have

$$\dim U = 1.$$

By Claim 1 and Claim 2, we have (TH).

□





## Chapter 8

# Thin Graphs

**Friday, February 5, 1993**

*Proof of Theorem 7.1 continued.*

(S)  $\Rightarrow$  (C)

Fix  $i$  and pick  $a, b \in E_i^* T E_i^*$ .

Since  $a$ ,  $b$  and  $ab$  are symmetric,

$$ab = (ab)^\top = b^\top a^\top = ba.$$

Hence  $E_i^* T E_i^*$  is commutative.

(G)  $\Rightarrow$  (S)

Fix  $i$  and pick  $a \in E_i^* T E_i^*$ . Pick vertices  $y, z \in X$ .

We want to show that

$$a_{yz} = a_{zy}.$$

We may assume that

$$\partial(x, y) = \partial(x, z) = i,$$

otherwise

$$a_{yz} = a_{zy} = 0.$$

By our assumption, there exists  $g \in G$  such that

$$g(y) = z, \quad g(z) = y, \quad g(x) = x.$$

Let  $\hat{g}$  denote the permutation matrix representing  $g$ , i.e.,

$$\hat{g}\hat{y} = \widehat{g(y)} \quad \text{for all } y \in X, \quad \hat{g} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow y.$$

If  $g \in \text{Aut}(\Gamma)$ , then

$$\hat{g}A = A\hat{g} \quad \text{Exercise.}$$

Also we have

$$\hat{g}E_j^* = E_j^*\hat{g} \quad (0 \leq j \leq D),$$

since

$$\partial(x, y) = \partial(g(x), g(y)) = \partial(x, g(y)).$$

Hence  $\hat{g}$  commutes with each element of  $T$ . We have

$$a_{yz} = (\hat{g}^{-1}a\hat{g})_{yz}, \quad (\hat{g})_{yz} = \begin{cases} 1 & g(z) = y \\ 0 & \text{else.} \end{cases} \quad (8.1)$$

$$= \sum_{y', z'} (\hat{g}^{-1})_{yy'} a_{y'z'} \hat{g}_{z'z} \quad (8.2)$$

$$(\text{zero except for } g^{-1}(y') = y, g(z) = z'.) \quad (8.3)$$

$$= a_{g(y)g(z)} \quad (8.4)$$

$$a_{zy}. \quad (8.5)$$

This proves Theorem 7.1. □

**Open Problem:** Find all the graphs that satisfy the condition (G) for every vertex  $x$ .

$H(N, 2)$  is one example, because

$$\text{Aut}\Gamma_{1\dots 1} \simeq S_\Omega, \quad x = (1 \dots 1), \Gamma_i(x) = \{\hat{S} \mid |S| = i\}.$$

Property (G) is clearly related to the distance-transitive property.

**Definition 8.1.** Let  $\Gamma = (X, E)$  be any graph.  $\Gamma$  with  $G \subseteq \text{Aut}(\Gamma)$  is said to be distance-transitive (or two-point homogeneous), whenever

$$\text{for all } x, x', y, y' \in X \text{ with } \partial(x, y) = \partial(x', y'),$$

there exists  $g \in G$  such that

$$g(x) = y, \quad g(x') = y'.$$

(This means  $G$  is as close to being doubly transitive as possible.)

**Lemma 8.1.** Suppose a graph  $\Gamma = (X, E)$  satisfies the property  $(G) = (G(x))$  for every  $x \in X$ . Then,

- (i) either
- (ia)  $\Gamma$  is vertex transitive; or
- (iia)  $\Gamma$  is bipartite ( $X = X^+ \cup X^-$ ) with  $X^+, X^-$  each an orbit of  $\text{Aut}(\Gamma)$ .
- (ii) if (ia) holds, then  $\Gamma$  is distance-transitive.

*Proof.* (i) Claim. Suppose  $y, z \in X$  are connected by a path of even length. Then  $y, z$  are in the same orbit of  $\text{Aut}(\Gamma)$ .

*Pf.* It suffices to assume that the path has length 2,  $y \sim w \sim z$ .

Now  $\partial(y, w) = \partial(w, z) = 1$ . So there exists  $g \in \text{Aut}(\Gamma)$  such that  $gw = w$ ,  $gy = z$ ,  $gz = y$ . This proves Claim.

Fix  $x \in X$ . Now suppose that  $\Gamma$  is not vertex transitive, and we shall show (ib).

Observe that  $X = X^+ \cup X^-$ , where

$$X^+ = \{y \in X \mid \text{there exists a path of even length connecting } x \text{ and } y\} \quad (8.6)$$

$$X^- = \{y \in X \mid \text{there exists a path of odd length connecting } x \text{ and } y\} \quad (8.7)$$

As  $X^+$  is contained in an orbit  $O^+$  of  $\text{Aut}(\Gamma)$ , and  $X^-$  is contained in an orbit  $O^-$  of  $\text{Aut}(\Gamma)$ .

Now  $O^+ \cap O^- = \emptyset$  (else  $O^+ = O^- = X$  and vertex transitive). So,  $X = O^+$ , and  $X^- = O^-$ .

Also  $X^+ \cup X^- = X$  is a bipartition by construction.

(ii) Fix  $x, y, x', y'$  with  $\partial(x, y) = \partial(x', y')$ .

By vertex transitivity, there exists an element

$$g_1 \in G \text{ such that } g_1x = x'.$$

Observe that

$$\partial(x', y') = \partial(x, y) = \partial(g_1x, g_1y) = \partial(x', g_1y).$$

Hence, there exists an element

$$g_2 \in G \text{ such that } g_1x' = x', g_2y' = g_1y', g_2g_1y = y'$$

by  $(G(x'))$  property.

Set  $g = g_2g_1$ . Then

$$gx = x', gy = y'$$

by construction. □

The following graphs  $\Gamma = (X, E)$  are vertex transitive, and satisfy the property  $(G(x))$  for all  $x \in X$ .

$$J(D, N), \quad H(D, r), \quad J_q(D, N),$$

where

$$H(D, r):$$

$$X = \{a_1 \cdots a_D \mid a_i \in F, 1 \leq i \leq D\} \quad (8.8)$$

$$F : \text{ any set of cardinality } r \quad (8.9)$$

$$E = \{xy \mid y, x \in X, x \text{ and } y \text{ differ in exactly one coordinate}\}. \quad (8.10)$$

$J_q(D, N)$ :

$X$  = the set of all  $D$ -dimensional subspaces of  $N$ -dimensional vector space over  $GF(q)$ .  
(8.11)

$$F : \text{ any set of cardinality } r \quad (8.12)$$

$$E = \{xy \mid y, x \in X, \dim(x \cap y) = D - 1\}. \quad (8.13)$$

The following graph is distance-transitive but does not satisfy  $(G(x))$  for any  $x \in G$ .

$H_q(D, N)$ :

$$X = \text{the set of all } D \times N \text{ matrices with entries in } GF(q). \quad (8.14)$$

$$E = \{xy \mid y, x \in X, \text{rank}(x - y) = 1\}. \quad (8.15)$$

*Remark.*

$$H(D, r): G = S_r \text{wr} S_D, G_x = S_{r-1} \text{wr} S_D,$$

For  $x, y \in X$  with  $\partial(x, y) = \partial(x, z) = i$ ,

$$Y = \{j \in \Omega \mid x_j \neq y_j\} \leftrightarrow Z = \{j \in \Omega \mid x_j \neq z_j\} \quad (8.16)$$

$$(y_{j_1}, \dots, y_{j_i}) \leftrightarrow (z_{\ell_1}, \dots, z_{\ell_i}) \quad (8.17)$$

$$J(D, N): G = S_N, G_x = S_D \times S_{N-D}.$$

$$X \cap Y \leftrightarrow X \cap Z \quad (8.18)$$

$$(\Omega - X) \cap Y \leftrightarrow (\Omega - X) \cap Z. \quad (8.19)$$

The following graph is distance-transitive but does not satisfy  $(G(x))$  for any  $x \in G$ .

$J_q(D, N)$ :

$$X \cap Y \leftrightarrow X \cap Z.$$

The theory of single thin irreducible  $T$ -module.

Let  $\Gamma = (X, E)$  be any graph.

$$M = \text{Bose-Mesner algebra over } K/\mathbb{C} \text{ generated by the adjacency matrix } A. \quad (8.20)$$

$$= \text{Span}(E_0, \dots, E_R). \quad (8.21)$$

$M$  acts on the standard module  $V = \mathbb{C}^{|X|}$ .

Fix  $x \in X$ , let  $D \equiv D(x)$  be the  $x$ -diameter, and  $k = k(x)$  be the valency of  $x$ .



## Chapter 9

# Thin $T$ -Module, I

**Monday, February 8, 1993**

Let  $\Gamma = (X, E)$  be any graph.

$M$ : Bose-Mesner algebra over  $K/\mathbb{C}$  generated by the adjacency matrix  $A$ .

$$M = \text{Span}(E_0, \dots, E_R).$$

$M$  acts on the standard module  $V = \mathbb{C}^{|X|}$ .

Fix  $x \in X$ , let  $D \equiv D(x)$  be the  $x$ -diameter, and  $k = k(x)$  be the valency of  $x$ .

**Definition 9.1.** Pick  $x \in X$  and write  $E_i^* \equiv E_i^*(x)$  and  $T \equiv T(x)$ .

Let  $W$  be an irreducible thin  $T$ -module with endpoint  $r$ , diameter  $d$ .

Let  $a_i = a_i(W) \in \mathbb{C}$  satisfying

$$E_{r+i}^* A E_{r+i}^* |_{E_{r+i}^* W} = a_i 1 |_{E_{r+i}^* W} \quad (0 \leq i \leq d).$$

Let  $x_i = x_i(W) \in \mathbb{C}$  satisfying

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* |_{E_{r+i}^* W} = x_i 1 |_{E_{r+i}^* W} \quad (0 \leq i \leq d).$$

**Lemma 9.1.** *With above notation, the following hold.*

(i)  $a_i \in \mathbb{R} \quad (0 \leq i \leq d)$ .

(ii)  $x_i \in \mathbb{R}^{>0} \quad (0 \leq i \leq d)$ .

(iii) Pick  $0 \neq w_0 \in E_r^* W$ . Set  $w_i = E_{r+i}^* A^i w_0$  for all  $i$ . Then

(iiia)  $w_0, w_1, \dots, w_d$  is a basis for  $W$ ,  $w_{-1} = w_{d+1} = 0$ .

(iiib)  $A w_i = w_{i+1} + a_i w_i + x_i w_{i-1} \quad (0 \leq i \leq d)$ .

(iv) Define  $p_0, p_1, \dots, p_{d+1} \in \mathbb{R}[\lambda]$  by

$$p_0 = 1, \quad \lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1} \quad (0 \leq i \leq d), \quad p_{-1} = 0.$$

(iva)  $p_i(A)w_0 = w_i$ ,  $(0 \leq i \leq d+1)$ .

(ivb)  $p_{d+1}$  is the minimal polynomial of  $A|_W$ .

*Proof.* (i)  $a_i$  is an eigenvalue of a real symmetric matrix  $E_{r+i}^* A E_{r+i}^*$ .

(ii)  $x_i$  is an eigenvalue of a real symmetric matrix  $B^\top B$ , where

$$B = E_{r+i}^* A E_{r+i-1}^*.$$

Hence,  $x_i \in \mathbb{R}$ .

Since  $B^\top B$  is positive semidefinite,

$$x_i \geq 0.$$

*Pf.* If  $B^\top B v = \sigma v$  for some  $\sigma \in \mathbb{R}$ ,  $v \in \mathbb{R}^m \setminus \{0\}$ , then

$$0 \leq \|Bv\|^2 = v^\top B^\top B v = \sigma v^\top v = \sigma \|v\|^2, \quad \|v\|^2 > 0.$$

Hence,  $\sigma \geq 0$ .

Moreover,  $x_i \neq 0$  by Lemma 4.1 (iv).

(iiia) Observe

$$w_i = E_{r+i}^* A E_{r+i-1}^* w_{i-1} \quad (1 \leq i \leq d).$$

So  $w_i \neq 0$   $(1 \leq i \leq d)$  by Lemma 4.1 (iv).

Hence,

$$W = \text{Span}(w_0, \dots, w_d)$$

by Lemma 4.1. (iii).

(iiib) We have that

$$Aw_i = E_{r+i+1}^* Aw_i + E_{r+i}^* Aw_i + E_{r+i-1}^* Aw_i \quad (9.1)$$

$$= w_{i+1} + E_{r+i}^* A E_{r+i}^* w_i + E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* w_{i-1} \quad (9.2)$$

$$= w_{i+1} + a_i w_i + x_i w_{i-1} \quad (9.3)$$

(iva) Clear for  $i = 0$ . Assume it is valid for  $0, \dots, i$ .

$$p_{i+1}(A)w_0 = (A - a_i I)w_i - x_i w_{i-1} = w_{i+1}.$$

(ivb) By definition,

$$p_{d+1}(A)w_0 = 0.$$



Moreover,  $p_{d+1}(A)W = 0$ . For every  $w \in W$ , write

$$w = \sum_{i=0}^d \alpha_i w_i \quad (9.4)$$

$$= \sum_{i=0}^d \alpha_i p_i(A)w_0 \quad \text{for some } \alpha_i \in \mathbb{C} \quad (9.5)$$

$$= p(A)w_0 \quad \text{for some } p \in \mathbb{C}[\lambda] \quad (9.6)$$

Hence,

$$p_{d+1}(A)w = p_{d+1}(A)p(A)w_0 \quad (9.7)$$

$$= p(A)p_{d+1}(A)w_0 \quad (9.8)$$

$$= 0. \quad (9.9)$$

Note that  $p_{d+1}$  is the minimal polynomial.

*Pf.* Suppose  $q(A)W = 0$  for some  $0 \neq q \in \mathbb{C}[\lambda]$  with  $\deg q < \deg p_{d+1} = d + 1$ . Then,

$$q = \sum_{i=0}^d \beta_i p_i \quad \text{for some } \beta_i \in \mathbb{C}.$$

We have,

$$0 = q(A)w_0 = \sum_{i=0}^d \beta_i w_i.$$

Hence  $\beta_0 = \dots = \beta_d = 0$  by (iiia). Thus  $q = 0$  and a contradiction.  $\square$

**Corollary 9.1.** *Let  $\Gamma$ ,  $W$ ,  $r$ ,  $d$  be as above. Then*

(i)  *$W$  is dual thin, that is,*

$$\dim E_i W \leq 1 \quad (1 \leq i \leq d).$$

(ii)  $d = |\{i \mid E_i W \neq 0\}| - 1$ .

*Proof.* (i) Set as in Lemma 9.1,

$$w_i = p_i(A)w_0 \in E_{r+i}^* W.$$

Then  $w_0, w_1, \dots, w_d$  is a basis for  $W$ . We have

$$W = Mw_0.$$

So,

$$E_i W = E_i M w_0 = \text{Span}(E_i w_0).$$

Thus,

$$\dim E_i W = \begin{cases} 1 & \text{if } E_i w_0 \neq 0 \\ 0 & \text{if } E_i w_0 = 0. \end{cases}$$

In particular,

$$\dim E_i^* W \leq 1.$$

(ii) Immediate as

$$\dim W = d + 1.$$

This proves the lemma.  $\square$

**Lemma 9.2.** *Given an irreducible  $T(x)$ -module  $W$  with endpoint  $r = r(W)$ , diameter  $d = d(W)$ . Write*

$$x_i = x_i(W) \ (0 \leq i \leq d), \quad w_i = p_i(A)w_0 \in E_{r+i}^* W \ (0 \leq i \leq d), \quad 0 \neq w_0 \in E_r^* W.$$

Then,

$$\frac{\|w_i\|^2}{\|w_0\|^2} = x_1 x_2 \cdots x_i \quad (1 \leq i \leq d).$$

*Proof.* It suffices to show that

$$\|w_i\|^2 = x_i \|w_i\|^2 \quad (1 \leq i \leq d).$$

Recall by Lemma 9.1 (iiib) that

$$Aw_j = w_{j+1} + a_j w_j + x_j w_{j-1} \quad (0 \leq j \leq d), \quad w_{-1} = w_{d+1} = 0.$$

Now observe,

$$\langle w_{i-1}, Aw_i \rangle = \langle w_{i-1}, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle \quad (9.10)$$

$$= \overline{x_i} \|w_{i-1}\|^2 \quad (9.11)$$

$$= x_i \|w_{i-1}\|^2. \quad (9.12)$$

by Lemma 9.1 (ii). Also,

$$\langle w_{i-1}, Aw_i \rangle = \langle Aw_{i-1}, w_i \rangle \quad (\text{since } \bar{A}^\top = A) \quad (9.13)$$

$$= \langle x_i + a_{i-1} w_{i-1} + x_{i-1} w_{i-2}, w_i \rangle \quad (9.14)$$

$$= \|w_i\|^2. \quad (9.15)$$

This proves the lemma.  $\square$

**Definition 9.2.** Let  $W$  be an irreducible thin  $T(x)$  module with endpoint  $r$ ,  $E_i^* \equiv E_i^*(x)$ .

The measure  $m = m_W$  is the function

$$m : \mathbb{R} \rightarrow \mathbb{R}$$

such that

$$m(\theta) = \begin{cases} \frac{\|E_i w\|^2}{\|w\|^2} & \text{where } 0 \neq w \in E_r^* W \\ & \text{if } \theta = \theta_i \text{ is an eigenvalue for } \Gamma \\ 0 & \text{if } \theta \text{ is not an eigenvalue for } \Gamma. \end{cases}$$



## Chapter 10

# Thin $T$ -Module, II

Wednesday, February 10, 1993

Let  $\Gamma = (X, E)$  be any graph.

Fix a vertex  $x \in X$ . Let  $E_i^* \equiv E_i^*(x)$ ,  $T \equiv T(x)$ , the subconstituent algebra over  $\mathbb{C}$ , and  $V = \mathbb{C}^{|X|}$  the standard module.

**Lemma 10.1.** *With above notation, let  $W$  denote a thin irreducible  $T(x)$ -module with endpoint  $r$  and diameter  $d$ . Let*

$$a_i = a_i(W) \quad (0 \leq i \leq d) \quad (10.1)$$

$$x_i = x_i(W) \quad (1 \leq i \leq d) \quad (10.2)$$

$$p_i = p_i(W) \quad (0 \leq i \leq d+1) \quad (10.3)$$

be from Lemma 9.1, and measure  $m = m_W$ . Then,

(i)  $p_0, \dots, p_{d+1}$  are orthogonal with respect to  $m$ , i.e.,

$$\sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \leq i, j \leq d+1) \text{ with } x_{d+1} = 0.$$

$$(ia) \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 m(\theta) = x_1 \cdots x_i \quad (0 \leq i \leq d).$$

$$(iia) \sum_{\theta \in \mathbb{R}} m(\theta) = 1.$$

$$(iiia) \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) = x_1 \cdots x_i a_i \quad (0 \leq i \leq d).$$

*Proof.* Pick  $0 \neq w_0 \in E_r^* W$ . Set

$$w_i = p_i(A) w_0 \in E_{r+i}^* W.$$

Since  $E_i^*W$  and  $E_j^*W$  are orthogonal if  $i \neq j$ ,

$$\delta_{ij}\|w_i\|^2 = \langle w_i, w_j \rangle \quad (10.4)$$

$$= \langle p_i(A)w_0, p_j(A)w_0 \rangle \quad (10.5)$$

$$= \left\langle p_i(A) \left( \sum_{\ell=0}^R E_\ell \right) w_0, p_j(A) \left( \sum_{\ell=0}^R E_\ell \right) w_0 \right\rangle \quad (10.6)$$

$$= \left\langle \sum_{\ell=0}^R p_i(\theta_\ell) E_\ell w_0, \sum_{\ell=0}^R p_j(\theta_\ell) E_\ell w_0 \right\rangle \quad (\text{as } AE_j = \theta_j E_j) \quad (10.7)$$

$$= \sum_{\ell=0}^R p_i(\theta_\ell) \overline{p_j(\theta_\ell)} \|E_\ell w_0\|^2 \quad (10.8)$$

$$(\text{as } p_j \in \mathbb{R}[\lambda], \quad \theta_\ell \in \mathbb{R}, \quad m(\theta_i)\|w_0\|^2 = \|E_i w_0\|^2) \quad (10.9)$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) \|w_0\|^2. \quad (10.10)$$

Now we are done by Lemma 9.2 as

$$\|w_i\|^2 = \|w_0\|^2 x_1 x_2 \dots x_i.$$

For (ia), set  $i = j$ , and for (ib), set  $i = j = 0$ .

(ii) We have

$$\langle w_i, Aw_i \rangle = \langle w_i, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle \quad (10.11)$$

$$= \overline{a_i} \|w_i\|^2 \quad (10.12)$$

$$= a_i x_1 \dots x_i \|w_0\|^2, \quad (10.13)$$

as  $a_i \in \mathbb{R}$  by Lemma 9.1.

Also,

$$\langle w_i, Aw_i \rangle = \langle p_i(A)w_0, Ap_i(A)w_0 \rangle \quad (10.14)$$

$$= \left\langle p_i(A) \left( \sum_{\ell=0}^R E_\ell \right) w_0, Ap_i(A) \left( \sum_{\ell=0}^R E_\ell \right) w_0 \right\rangle \quad (\text{as in (i)}) \quad (10.15)$$

$$= \sum_{\ell=0}^D p_i(\theta_\ell)^2 \theta_\ell \|E_\ell w_0\|^2 \quad (10.16)$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) \|w_0\|^2. \quad (10.17)$$

Thus, we have (ii).  $\square$

**Lemma 10.2.** *With above notation, let  $W$  be a thin irreducible  $T(x)$ -module with measure  $m$ . Then  $m$  determines diameter  $d(W)$ ,*

$$a_i = a_i(W) \quad (0 \leq i \leq d) \quad (10.18)$$

$$x_i = x_i(W) \quad (1 \leq i \leq d) \quad (10.19)$$

$$p_i = p_i(W) \quad (0 \leq i \leq d+1). \quad (10.20)$$

*Proof.* Note that  $d+1$  is the number of  $\theta \in \mathbb{R}$  such that  $m(\theta) \neq 0$ . Hence  $m$  determines  $d$ .

Apply (ia), (ii) of Lemma 10.1.

$$\sum_{\theta \in \mathbb{R}} m(\theta) = 1 \quad p_0 = 1. \quad (10.21)$$

$$\sum_{\theta \in \mathbb{R}} \theta m(\theta) = a_0 \quad p_1 = \lambda - a_0 \quad (10.22)$$

$$\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 m(\theta) = x_1 \quad (10.23)$$

$$\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 \theta m(\theta) = x_1 a \quad \rightarrow a_1 \quad (10.24)$$

$$p_2 = (\lambda - a_1)p_1 - x_1 p_0 \quad (10.25)$$

$$\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 m(\theta) = x_1 x_2 \quad \rightarrow x_2 \quad (10.26)$$

$$\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 \theta m(\theta) = x_1 x_2 a_2 \quad \rightarrow a_2 \quad (10.27)$$

$$p_3 = (\lambda - a_2)p_2 - x_2 p_1 \quad (10.28)$$

$$\vdots \quad (10.29)$$

$$\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 m(\theta) = x_1 x_2 \cdots x_d \quad \rightarrow x_d \quad (10.30)$$

$$\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 \theta m(\theta) = x_1 x_2 \cdots x_d a_d \quad \rightarrow a_d \quad (10.31)$$

$$p_{d+1} = (\lambda - a_d)p_d - x_d p_{d-1}. \quad (10.32)$$

$$(10.33)$$

This proves the assertions.  $\square$

**Corollary 10.1.** *With above notation, let  $W, W'$  denote thin irreducible  $T(x)$ -modules. The following are equivalent.*

(i)  $W, W'$  are isomorphic as  $T$ -modules.

(ii)  $r(W) = r(W')$  and  $m_W = m_{W'}$ .

(iii)  $r(W) = r(W')$ ,  $d(W) = d(W')$ ,  $a_i(W) = a_i(W')$  and  $x_i(W) = x_i(W')$  ( $0 \leq i \leq d$ ).

*Proof.* (i)  $\Rightarrow$  (iii) Write  $r \equiv r(W)$ ,  $r' \equiv r(W')$ ,  $d = d(W)$ ,  $d' = d(W')$ ,  $a_i = a_i(W)$ ,  $a'_i = a_i(W')$ ,  $x_i = x_i(W)$  and  $x'_i = x_i(W')$ .

Let  $\sigma : W \rightarrow W'$  denote an isomorphism of  $T$ -modules. (See Definition 5.1.)

For every  $i$ ,

$$\sigma E_i^* W = E_i^* \sigma W = E_i^* W'.$$

So,  $r = r'$  and  $d = d'$ .

To show  $a_i = a'_i$ , pick  $w \in E_{r+i}^* W \setminus \{0\}$ . Then,

$$E_{r+i}^* A E_{r+i}^* \sigma(W) = \sigma(E_{r+i}^* A E_{r+i}^* w) = \sigma(a_i w) = a_i \sigma(w),$$

and  $\sigma w \neq 0$ . So,

$$a_i = \text{eigenvalue of } E_{r+i}^* A E_{r+i}^* \text{ on } E_{r+i}^* W \quad (10.34)$$

$$= a'_i \quad (10.35)$$

It is similar to show  $x = x'$ .

*Remark.* Pick  $w \in E_{r+i-1}^* W \setminus \{0\}$

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* \sigma(W) = \sigma(E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* w) = x_i \sigma(w).$$

Hence,  $x_i$  is the eigenvalue of  $E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^*$  on  $E_{r+i-1}^* W = x'_i$ .

(iii)  $\Rightarrow$  (i)

Pick  $0 \neq w_0 \in E_r^* W$ ,  $0 \neq w'_0 \in E_r^* W'$ . Let  $p_i$  be in Lemma 9.1, and set

$$w_i = p_i(A)w_0 \in E_{r+i}^* W \quad (0 \leq i \leq d) \quad (10.36)$$

$$w'_i = p'_i(A)w'_0 \in E_{r+i}^* W' \quad (0 \leq i \leq d) \quad (10.37)$$

Define a linear transformation,

$$\sigma : W \rightarrow W' \quad (w_i \mapsto w'_i).$$

Since  $\{w_i\}$  and  $\{w'_i\}$  are bases with  $d = d'$ ,  $\sigma$  is an isomorphism of vector spaces.

We need to show

$$a\sigma = \sigma a \quad (\text{for all } a \in T).$$

Take  $a = E_j^*$  for some  $j$  ( $0 \leq j \leq d(x)$ ). Then for all  $i$ , we have

$$E_j^* \sigma w_i = E_j^* w'_i = \delta_{ij} w'_i,$$

$$\sigma E_j^* w_i = \delta_{ij} \sigma(w_i) = \delta_{ij} w'_i.$$

$$E_j^* \sigma w_i = \sigma E_j^* w_i?$$

Take an adjacency matrix  $A$  of  $a$ . Then,

$$A \sigma w_i = A w'_i = w'_{i+1} + a'_i w'_i + x'_i w'_{i-1} = \sigma(w_{i+1} + a_i w_i + x_i w_{i-1}) = \sigma A w_i.$$



(ii)  $\Rightarrow$  (iii) Lemma 10.2.

(iii)  $\Rightarrow$  (ii) Given  $d, a_i, x_i$ , we can compute the polynomial sequence

$$p_0, p_1, \dots, p_{d+1}$$

for  $W$ .

Show  $p_0, p_1, \dots, p_{d+1}$  determines  $m = m_W$ . Set

$$\Delta = \{\theta \in \mathbb{R} \mid p_{d+1}(\theta) = 0\}.$$

Observe:  $|\Delta| = d + 1$ . See ‘An Introcuction to Interlacing’.

$m(\theta) = 0$  if  $\theta \notin \Delta$  ( $\theta \in \mathbb{R}$ ). So it suffices to find  $m(\theta)$ ,  $\theta \in \Delta$ .

By Lemma 10.1 (i),

$$\begin{cases} \sum_{\theta \in \Delta} m(\theta) p_0(\theta) &= 1 \\ \sum_{\theta \in \Delta} m(\theta) p_1(\theta) &= 0 \\ \vdots & \\ \sum_{\theta \in \Delta} m(\theta) p_d(\theta) &= 0 \end{cases}$$

$d + 1$  linear equation with  $d + 1$  unknowns  $m(\theta)$  ( $\theta \in \Delta$ ).

But the coefficient matrix is essentially Vander Monde (since  $\deg p_i = i$ ). Hence the system is nonsingular and there are unique values for  $m(\theta)$  ( $\theta \in \Delta$ ).  $\square$

*Remark.*

$$\begin{pmatrix} \theta - a_0 & -1 & \cdots & 0 & 0 \\ -x_1 & \theta - a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta - a_{d-1} & -1 \\ 0 & 0 & \cdots & -x_d & \theta - a_d \end{pmatrix} \begin{pmatrix} p_0(\theta) \\ \vdots \\ \vdots \\ \vdots \\ p_d(\theta) \end{pmatrix} = 0,$$

where  $\theta$  is an eigenvalue of a diagonalizable matrix

$$L = \begin{pmatrix} a_0 & 1 & \cdots & 0 & 0 \\ x_1 & a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{d-1} & 1 \\ 0 & 0 & \cdots & x_d & \theta a_d \end{pmatrix}$$

with multiplicity  $\dim(\text{Ker}(\theta I - L)) = 1$ .



# Chapter 11

## Examples of $T$ -Module

**Friday, February 12, 1993**

Let  $\Gamma = (X, E)$  be a connected graph.

Let  $\theta_0$  be the maximal eigenvalue of  $\Gamma$ , and  $\delta$  its corresponding eigenvector.

$$\delta = \sum_{y \in X} \delta_y \hat{y}.$$

Without loss of generality, we may assume that  $\delta_y \in \mathbb{R}^*$  for all  $y \in X$ .

**Lemma 11.1.** *Fix a vertex  $x \in X$ . Write  $T \equiv T(x)$ ,  $E_i^* \equiv E_i^*(x)$ .*

- (i)  $T\delta = T\hat{x}$  is an irreducible  $T$ -module.
- (ii) *Given any irreducible  $T$ -module  $W$ , the following are equivalent:*
  - (iia)  $W = T\delta$ .
  - (iib) *The diameter  $d(W) = d(x)$ .*
  - (iic) *The endpoint  $r(W) = 0$ .*

*Proof.* (i) Observe: there exists an irreducible  $T$ -module  $W$  that contains  $\delta$ .

Let  $V = \sum_i W_i$  be a direct sum decomposition of the standard module. Then

$$\text{Span}(\delta) = E_0 V = \sum_i E_0 W_i.$$

So,  $E_0 W_i \neq 0$  for some  $i$ . Then,

$$\delta \in E_0 W_i \subseteq W_i.$$

Observe:  $T\delta$  is an irreducible  $T$ -module.

Since  $\delta \in W$ , where  $W$  is a  $T$ -module. As  $T\delta \subseteq W$  and  $W$  is irreducible,  $T\delta = W$ .

Observe:  $T\delta = T\hat{x}$ .

Since  $\hat{x} = \delta_x^{-1} E_0^* \delta \in T\delta$ ,  $T\hat{x} \subseteq T\delta$ . Since  $T\delta$  is irreducible,  $T\hat{x} = T\delta$ .

(ii) (a)  $\rightarrow$  (b):

$$E_i^* \delta = \sum_{y \in X, \partial(x,y)=i} \delta_y \hat{y} \neq 0, \quad (0 \leq i \leq d(x)),$$

because  $\delta_y > 0$  for every  $y \in X$ .

Hence,

$$E_i^* T\delta \neq 0, \quad (0 \leq i \leq d(x)).$$

Thus,  $d(x) = d(W)$ .

(b)  $\rightarrow$  (c): Immediate.

(c)  $\rightarrow$  (a): Since  $r(W) = 0$ ,  $E_0^* W \neq 0$ . Hence,  $\hat{x} \in W$  and  $T\hat{x} \subseteq W$ .

By the irreducibility, we have  $T\hat{x} = W$ . □

**Lemma 11.2.** Assume  $\Gamma$  is bipartite ( $X = X^+ \cup X^-$ ) ( $X^+$  and  $X^-$  are nonempty). Then the following are equivalent.

(i) There exist  $\alpha^+$  and  $\alpha^- \in \mathbb{R}$  such that

$$\delta_x = \begin{cases} \alpha^+ & \text{if } x \in X^+ \\ \alpha^- & \text{if } x \in X^-. \end{cases}$$

/(ii) There exist  $k^+$  and  $k^- \in \mathbb{Z}^{>0}$  such that

$$k(x) = \begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^-. \end{cases}$$

In this case,  $k^+ k^- = \theta_0^2$ , and  $\Gamma$  is called bi-regular.

*Proof.* (i)  $\rightarrow$  (ii)



$$A\delta = A \left( \alpha^+ \sum_{x \in X^+} \hat{x} + \alpha^- \sum_{y \in X^-} \hat{y} \right) \quad (11.1)$$

$$= \alpha^+ \sum_{y \in X^-} k(y) \hat{y} + \alpha^- \sum_{x \in X^+} k(x) \hat{x} \quad (11.2)$$

$$= \theta_0 \delta. \quad (11.3)$$

So,

$$k(x)\alpha^- = \theta_0\alpha^+, \quad k(y)\alpha^+ = \theta_0\alpha^-.$$

As  $\alpha^+ \neq 0$  and  $\alpha^- \neq 0$ ,

$$k^+ := k(x) \text{ is independent of the choice of } x \in X^+, \text{ and} \quad (11.4)$$

$$k^- := k(y) \text{ is independent of the choice of } y \in X^-. \quad (11.5)$$

Moreover,  $k^+k^- = \theta_0^2$ .

(ii)  $\rightarrow$  (i) Set

$$\delta' = \sum_{y \in X} \alpha_y \hat{y} \quad \text{where } \alpha = \begin{cases} 1/\sqrt{k^-} & \text{if } y \in X^+ \\ 1/\sqrt{k^+} & \text{if } y \in X^- \end{cases}.$$

Then one checks

$$A\delta' = A \left( \frac{1}{\sqrt{k^-}} \sum_{y \in X^+} \hat{y} + \frac{1}{\sqrt{k^+}} \sum_{y \in X^-} \hat{y} \right) \quad (11.6)$$

$$= \frac{k^-}{\sqrt{k^-}} \sum_{y \in X^-} \hat{y} + \frac{k^+}{\sqrt{k^+}} \sum_{y \in X^+} \hat{y} \quad (11.7)$$

$$= \sqrt{k^+k^-} \delta' \quad (11.8)$$

Since  $\delta' > 0$ ,  $\delta' \in \text{Span}(\delta)$ , and  $\theta_0 = \sqrt{k^+k^-}$ .  $\square$

**Definition 11.1.** For any graph  $\Gamma = (X, E)$ , fix a vertex  $x \in X$ . Set  $d = d(x)$ .

$\Gamma$  is distance-regular with respect to  $x$ , if for all  $i : (0 \leq i \leq d)$ , and all  $y \in X$  such that  $\partial(x, y) = i$ :

$$c_i(x) := |\{z \in X \mid \partial(x, z) = i-1, \partial(y, z) = 1\}| \quad (11.9)$$

$$a_i(x) := |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = 1\}| \quad (11.10)$$

$$b_i(x) := |\{z \in X \mid \partial(x, z) = i+1, \partial(y, z) = 1\}| \quad (11.11)$$

depends only on  $i$ ,  $x$ , and not on  $y$ .

(In this case,  $c_0(x) = a_0(x) = b_d(x) = 0$ ,  $c_1(x) = 1$ ,  $b_0(x) = k(x)$  is the valency of  $x$ .)

We call  $c_i(x)$ ,  $a_i(x)$  and  $b_i(x)$  the intersection numbers with respect to  $x$ .

**Example 11.1.**



$$c_0 = 1 \qquad c_1 = 1 \qquad c_2 = 1 \qquad (11.12)$$

$$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = 1 \qquad (11.13)$$

$$b_0 = 2 \qquad b_1 = 1 \qquad b_2 = 0 \qquad (11.14)$$

## Chapter 12

# Distance-Regular

Monday, February 15, 1993

**Lemma 12.1.** *For any connected graph  $\Gamma = (X, E)$ , the following are equivalent.*

(i) *The trivial  $T(x)$ -module is thin for all  $x \in X$ .*

(ii)  $\left\{ \sum_{y \in X, d(x,y)=i} \hat{y} \mid 0 \leq i \leq d(x) \right\}$  *is a basis for the trivial  $T(x)$ -module for every  $x \in X$ .*

(iii)  $\Gamma$  *is distance-regular with respect to  $x$  for all  $x \in X$ .*

**Note.** Let  $\Gamma = (X, E)$  be a graph, with  $X = \{x, y_1, y_2, y_3, z_1, z_2, z_3\}$ ,  $E = \{xy_1, xy_2, xy_3, y_1z_1, y_1z_2, y_2z_3, y_3z_3\}$ .



Then (i), (ii) are not equivalent for a single vertex  $x$ .

$$E_0^* T \hat{x} = \langle \hat{x} \rangle, \quad (12.1)$$

$$E_1^* T \hat{x} = \langle y_1 + y_2 + y_3 \rangle, \quad (12.2)$$

$$E_2^* T \hat{x} = \langle z_1 + z_2 + 2z_3 \rangle. \quad (12.3)$$

*Proof of Lemma 12.1.* (i)  $\rightarrow$  (ii) Let  $\delta = \sum_{y \in X} \delta_y \hat{y}$  be an eigenvector for the maximal eigenvalue  $\theta_0$ . Then,

$$\sum_{y \in X, \partial(x,y)=1} \hat{y} = A\hat{x} \in T(x)\hat{x} = T(x)\delta \ni E_1^* \delta \quad (12.4)$$

$$= \sum_{y \in X, \partial(x,y)=1} \delta_y \hat{y} \quad (12.5)$$

If the trivial  $T(x)$ -module is thin,

$$\delta_y = \delta_z \text{ for } y, z \in X, \partial(x, y) = \partial(x, z) = 1.$$

Hence,  $\delta_y = \delta_z$  if  $y$  and  $z$  in  $X$  are connected by a path of even length.

So,  $\Gamma$  is regular or bipartite biregular by Lemma 11.2.

In particular,  $\delta_y = \delta_z$  if  $\partial(x, y) = \partial(x, z)$ , as there is a path of length  $2 \cdot \partial(x, y)$ ;

$$y \sim \dots \sim x \sim \dots \sim z.$$

Hence,

$$E_i^* \delta \in \text{Span} \left( \sum_{y \in X, \partial(x,y)=i} \hat{y} \right).$$

Since  $E_0^* \delta, E_1^* \delta, \dots, E_d^* \delta$  forms a basis for  $T(x)\delta$ , we have (ii).

(ii)  $\rightarrow$  (iii) Fix  $x \in X$ , and let  $T \equiv T(x)$ ,  $E_i^* \equiv E_i^*(x)$ , and  $d \equiv d(x)$ .

$$A \sum_{y \in X, \partial(x,y)=i} \hat{y} = \sum_{z \in X} |\{y \in X \mid \partial(y, z) = 1, \partial(x, y) = i\}| \hat{z} \quad (12.6)$$

$$= \sum_{z \in X, \partial(x,y)=i-1} b_{i-1}(x, z) \hat{z} \quad (12.7)$$

$$+ \sum_{z \in X, \partial(x,y)=i} a_i(x, z) \hat{z} \quad (12.8)$$

$$+ \sum_{z \in X, \partial(x,y)=i+1} c_{i+1}(x, z) \hat{z} \quad (12.9)$$

$$\in \text{Span} \left\{ \sum_{z \in X, \partial(x,z)=j} \hat{z} \mid j = 0, 1, \dots, d \right\}. \quad (12.10)$$

Hence,  $b_{i-1}(x, z)$ ,  $a_i(x, z)$  and  $c_{i+1}(x, z)$  depend only on  $i$  and  $x$ , and not on  $z$ . Therefore,  $\Gamma$  is distance-regular with respect to  $x$ .



(iii)  $\rightarrow$  (i) Fix  $x \in X$ , and let  $T \equiv T(x)$ ,  $E_i^* \equiv E_i^*(x)$ , and  $d \equiv d(x)$ . By definition of distance-regularity, for every  $i$  ( $0 \leq i \leq d$ ),

$$A \left( \sum_{y \in X, \partial(x,y)=i} \hat{y} \right) = b_{i-1}(x) \sum_{y \in X, \partial(x,y)=i-1} \hat{y} \quad (12.11)$$

$$+ a_i(x) \sum_{y \in X, \partial(x,y)=i} \hat{y} \quad (12.12)$$

$$+ c_{i+1}(x) \sum_{y \in X, \partial(x,y)=i+1} \hat{y}. \quad (12.13)$$

Hence,

$$W = \left\{ \sum_{y \in X, \partial(x,y)=i} \hat{y} \mid 0 \leq i \leq d \right\}$$

is  $A$ -invariant and so  $T$ -invariant. Since  $\hat{x} \in W$ ,  $T\hat{x} = W$  is the trivial module and  $T\hat{x}$  is thin.  $\square$

Next, we show more is true if (i) – (iii) hold in Lemma 12.1.

In fact,  $d(x)$ ,  $a_i(x)$ ,  $c_i(x)$ , and  $b_i(x)$  are

$$\begin{cases} \text{independent of } X & \text{if } \Gamma \text{ is regular; or} \\ \text{constant over } X^+ \text{ and } X^- & \text{if } \Gamma \text{ is biregular.} \end{cases}$$

Let  $\Gamma = (X, E)$  be any (connected) graph. Pick vertices  $x, y \in X$ .

Let  $W$  be a thin, irreducible  $T(x)$ -module, and measure  $m : \mathbb{R} \rightarrow \mathbb{R}$  determined by  $W$ .

Let  $W'$  be a thin, irreducible  $T(y)$ -module, and measure  $m' : \mathbb{R} \rightarrow \mathbb{R}$  determined by  $W'$ .

Recall  $W, W'$  are orthogonal if

$$\langle w, w' \rangle = 0 \quad \text{for all } w \in W, w' \in W'.$$

We shall show if  $W$  and  $W'$  are not orthogonal, then  $m$  and  $m'$  are related:

$$m \cdot \text{poly}_1 = m' \cdot \text{poly}_2$$

for some polynomials with

$$\deg \text{poly}_1 + \deg \text{poly}_2 \leq 2 \cdot \partial(x, y).$$

**Notation.**  $V$ : standard module of  $\Gamma$ .

$H$ : any subspace of  $V$ .

$$V = H + H^\perp \quad \text{orthogonal direct sum,}$$

and for  $v = v_1 + v_2$   $\text{proj}_H : V \rightarrow H$  ( $v \mapsto v_1$ ): linear transformation.

Observe: For every  $v \in V$ ,

$$v - \text{proj}_H v \in H^\perp.$$

So,

$$\langle v - \text{proj}_H v, h \rangle = 0 \quad \text{for all } h \in H \text{ or,}$$

$$\langle v, h \rangle = \langle \text{proj}_H v, h \rangle \quad \text{for all } v \in V, \text{ and for all } h \in H.$$

**Theorem 12.1.** *Let  $\Gamma = (X, E)$  be any graph. Pick vertices  $x, y \in X$  and set  $\Delta = \partial(x, y)$ . Assume*

*$W$ : thin irreducible  $T(x)$ -module with endpoint  $r$ , diameter  $d$ , and measure  $m$ .*

*$W'$ : thin irreducible  $T(y)$ -module with endpoint  $r'$ , diameter  $d'$ , and measure  $m'$ .*

*$W$  and  $W'$  are not orthogonal.*

*Now pick*

$$0 \neq w \in E_r^*(x)W, \quad 0 \neq w' \in E_{r'}^*(x)W'.$$

*Then,*

$$(i) \quad \text{proj}_{W'} w = p(A) \frac{\|w\|}{\|w'\|} w'$$

*for some  $0 \neq p \in \mathbb{C}[\lambda]$  with  $\deg p \leq \Delta - r' + r, d'$ ,*

$$\text{proj}_W w' = p'(A) \frac{\|w'\|}{\|w\|} w$$

*for some  $0 \neq p' \in \mathbb{C}[\lambda]$  with  $\deg p' \leq \Delta - r + r', d$ .*

*(ii) For all eigenvalues  $\theta_i$  of  $\Gamma$ ,*

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = m(\theta_i) \overline{p'(\theta_i)}.$$

*(iii) For all eigenvalues  $\theta_i$  of  $\Gamma$ ,*

$$p(\theta_i) p'(\theta_i)$$

*is in a real number in interval  $[0, 1]$ .*

*Proof.* (i) Since  $W, W'$  are not orthogonal, there exist

$$v \in W, v' \in W' \text{ such that } \langle v, v' \rangle \neq 0.$$

Then there exists  $a \in M$  such that

$$v' = aw'.$$

(This is because  $w'_i = p'_i(A)w'_0$  and hence for every  $v' \in W'$ , there is a polynomial  $q \in \mathbb{C}[\lambda]$ ,  $q(A)w'_0 = v$ .)

We have

$$0 \neq \langle v', v \rangle = \langle aw', v \rangle = \langle w', a^*v \rangle$$

and  $a^*v \in W$ .

Hence,  $\text{proj}_W w' \neq 0$ .

Let  $p_0, \dots, p_d \in \mathbb{C}[\lambda]$  be from Lemma 9.1.

Then,  $w_i = p_i(A)w$  is a basis for  $E_{r+i}^*(x)W$  ( $0 \leq i \leq d$ ).

Hence,

$$\text{proj}_W w' = \alpha_0 w_0 + \dots + \alpha_d w_d \text{ for some } \alpha_j \in \mathbb{C}.$$

Set

$$p' := \frac{\|w\|}{\|w'\|} \sum_{i=0}^d \alpha_i p_i.$$

Then  $0 \neq p' \in \mathbb{C}[\lambda]$  and  $\deg p' \leq d$ .

Claim:  $\alpha_i = 0$  ( $\Delta - r + r' < i \leq d$ ).

In particular,  $\deg p' \leq \Delta - r + r'$ .

*Pf.* Observe:

$$w' \in E_{r'}^*(y)V, \quad w \in E_r^*(x)V,$$

for  $\partial(x, y) = \Delta$ .

$$E_{r'}^*(y)V \cap E_{r+i}^*(x)V = 0$$

by triangle inequality.

( $\Delta = \partial(x, y) < r + i - r'$  or  $\Delta + r' < r + i$  by our choice of  $i$ .)



Hence,

$$E_{r'}^*(y)V \perp E_{r+i}^*(x)V,$$

or

$$0 = \langle w', w_i \rangle \quad (12.14)$$

$$= \langle \text{proj}_W w', w_i \rangle \quad (12.15)$$

$$= \sum_{j=0}^d \alpha_j \langle w_j, w_i \rangle \quad (12.16)$$

$$= \alpha_i \|w_i\|^2. \quad (12.17)$$

Hence,  $\alpha_i = 0$ . Thus,

$$\text{proj}_W w' = \sum_{i=0}^{\Delta+r'-r} \alpha_i w_i \quad (12.18)$$

$$= \sum_{i=0}^{\Delta+r'-r} \alpha_i p_i(A) w_0 \quad (12.19)$$

$$= p'(A) \frac{\|w'\|}{\|w\|} w. \quad (12.20)$$

(ii) We have

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = \frac{\langle E_i w, w' \rangle}{\|w\| \|w'\|} \quad (12.21)$$

$$= \frac{\langle E_i w, \text{proj}_W w' \rangle}{\|w\| \|w'\|} \quad \text{as } \text{proj}_W w' = p'(A) \frac{\|w\|}{\|w'\|} w \quad (12.22)$$

$$= \frac{\langle E_i w, p'(A) w \rangle}{\|w\|^2} \quad (12.23)$$

$$= \frac{\langle E_i w, E_i p'(A) w \rangle}{\|w\|^2} \quad (12.24)$$

$$= \overline{p'(\theta_i)} \frac{\|E_i W\|^2}{\|w\|^2} \quad (12.25)$$

$$= \overline{p'(\theta_i)} m(\theta_i). \quad (12.26)$$

Moreover, as  $m(\theta_i), m'(\theta_i) \in \mathbb{R}$ ,

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = \frac{\overline{\langle E_i w, E_i w' \rangle}}{\|w'\| \|w\|} = \overline{p(\theta_i) m'(\theta_i)} = p(\theta_i) m'(\theta_i).$$

(iii) Since,

$$\frac{|\langle E_i w, E_i w' \rangle|^2}{\|w\|^2 \|w'\|^2} = p(\theta_i) p'(\theta_i) m(\theta_i) m'(\theta_i),$$

$$p(\theta_i)p'(\theta_i) = \frac{|\langle E_i w, E_i w' \rangle|^2}{m(\theta_i)m'(\theta_i)\|w\|^2\|w'\|^2} \in \mathbb{R} \quad (12.27)$$

$$= \frac{|\langle E_i w, E_i w' \rangle|^2}{\frac{\|E_i w\|^2}{\|w\|^2} \frac{\|E_i w'\|^2}{\|w'\|^2} \|w\|^2 \|w'\|^2}. \quad (12.28)$$

By Cauchy-Schwartz inequality,

$$(|\langle a, b \rangle| \leq \|a\| \|b\|, )$$

$$\frac{|\langle E_i w, E_i w' \rangle|^2}{\|E_i w\|^2 \|E_i w'\|^2} \leq 1.$$

Hence, we have the assertion.  $\square$



## Chapter 13

# Modules of a DRG

Wednesday, February 17, 1993





## Chapter 14

# Title of the Chapter

Wednesday, February 17, 1993 # Edit Date



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