

Lecture Note on Terwilliger Algebra

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About this lecturenote

Setting

This note is created by `bookdown` package on RStudio.

1. Log-in to my GitHub Account
2. Go to RStudio/bookdown-demo repository: <https://github.com/rstudio/bookdown-demo>
3. Use This Template
4. Input Repository Name
5. Select Public - default
6. Create repository from template
7. From Code download ZIP
8. Move the extracted folder into a favorite directory
9. Open RStudio Project in the folder
10. Use Terminal in the bottom left pane
 - confirm that the current directory is the home directory of the project by `pwd`
11. (failed to proceed by ssh)
12. Use Console
 1. `library(usethis)`
 2. `use_git()`
 3. `use_github()` — Error
 4. `gh_token_help()`
 5. `create_github_token()`: create a token in the github page. Copy the token
 6. `gitcreds::gitcreds_set()`: paste the token, the token is to be expired in 30 days
13. Use Terminal
 1. `git remote add origin https://github.com/icu-hsuzuki/t-algebra.git`
 2. `git push -u origin main`
 3. type in the password of the computer
14. Use GIT in R Studio

Another Host

1. `library(usethis)`
2. `use_git()`
3. `create_github_token()`
4. `gitcreds::gitcreds_set()`: Replace these credentials

Chapter 1

Lecture 1

Wednesday, January 20, 1993

A graph (undirected, without loops or multiple edges) is a pair $\Gamma = (X, E)$, where

$$X = \text{finite set (of vertices)} \quad (1.1)$$

$$E = \text{set of (distinct) 2-element subsets of } X \text{ (= edges of } \Gamma). \quad (1.2)$$

vertices x and $y \in X$ are adjacent if and only if $xy \in E$.

Example 1.1. Let Γ be a graph. $X = \{a, b, c, d\}$, $E = \{ab, ac, bc, bd\}$.

Set $n = |X|$, the order of Γ .

Pick a field K ($= \mathbb{R}$ or \mathbb{C}). Then $\text{Mat}_X(K)$ denotes the K algebra of all $n \times n$ matrices with entries in K . (rows and columns are indexed by X)

Adjacency matrix $A \in \text{Mat}_X(K)$ is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E \\ 0 & \text{else .} \end{cases} \quad (1.3)$$

Example 1.2. Let a, b, c, d be labels of rows and columns. Then

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The subalgebra M of $\text{Mat}_X(K)$ generated by A is called the *Bose-Mesner algebra* of Γ .

Set $V = K^n$, the set of n -dimensional column vectors, the coordinates are indexed by X .

Let $\langle \cdot, \cdot \rangle$ denote the Hermitean inner product:

$$\langle u, v \rangle = u^\top \cdot v \quad (u, v \in V)$$

V with $\langle \cdot, \cdot \rangle$ is the *standard module* of Γ .

M acts on V : For every $x \in X$, write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where 1 is at the x position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since A is a real symmetric matrix,

$$V = V_0 + V_1 + \cdots + V_r \quad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal A -eigenspaces.

Let $E_i \in \text{Mat}_X(K)$ denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then E_0, \dots, E_r are the primitive idempotents of M .

$$M = \text{Span}_K(E_0, \dots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all } i, j, \quad E_0 + \cdots + E_r = I.$$

Let θ_i denote the eigenvalue of A for V_i in \mathbb{R} . Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \cdots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \theta_i = \dim V_i = \text{rank } E_i.$$

Set

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \cdots, & \theta_r \\ m_0, & m_1, & \cdots, & m_r \end{pmatrix}.$$

Problem. What can we say about Γ when $\text{Spec}(\Gamma)$ is given?

The following Lemma 1.1, is an example of Problem.

For every $x \in X$,

$$k(x) \equiv \text{valency of } x \equiv \text{degree of } x \equiv |\{y \mid y \in X, xy \in E\}|.$$

Definition 1.1. The graph Γ is regular of valency k if $k = k(x)$ for every $x \in X$.

Lemma 1.1. *With the above notation,*

1. $\theta_0 \leq \max\{k(x) \mid x \in X\} = k^{\max}$.
2. *If Γ is regular of valency k , then $\theta_0 = k$.*

Proof. 1. Without loss of generality we may assume that $\theta_0 > 0$, else done.

Let $v := \sum_{x \in X} \alpha_x \hat{x}$ denote the eivenvector for θ_0 .

Pick $x \in X$ with $|\alpha_x|$ maximal. Then $|\alpha_x| \neq 0$.

Since $Av = \theta_0 v$,

$$\theta_0 \alpha_x = \sum_{y \in X, xy \in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \leq \sum_{y \in X, xy \in E} |\alpha_y| \leq k(x) |\alpha_x| \leq k^{\max} |\alpha_x|.$$

2. All 1's vector $v = \sum_{x \in X} \hat{x}$ satisfies $Av = kv$.

□

Subconstituent Algebra

Let $x, y \in X$ and $\ell \in \mathbb{Z}^{\geq 0}$.

Definition 1.2. A path of length ℓ connecting x, y is a sequence

$$x = x_0, x_1, \dots, x_\ell = y, \quad x_i \in X, \quad 0 \leq i \leq \ell$$

such that $x_i x_{i+1} \in E$ for $0 \leq i \leq \ell - 1$.

Definition 1.3. The distance $\partial(x, y)$ is the length of a shortest path connecting x and y .

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

Definition 1.4. The graph Γ is connected if and only if $\partial(x, y) < \infty$ for all $x, y \in X$.

From now on, assume that Γ is connected with $|X| \geq 2$.

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \text{the diameter of } \Gamma.$$

Fix a ‘base’ vertex $x \in X$.

Definition 1.5.

$d(x)$ = the diameter with respect to $x = \max\{\partial(x, y) \mid y \in X\} \leq d$.

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad (\text{orthogonal direct sum}),$$

where

$$V_i^* = \text{Span}_K(\hat{y} \mid \partial(x, y) = i) \equiv V_i^*(x)$$

and $V_i^* = V_i^*(x)$ is called the i -th subconstituent with respect to x .

Let $E_i^* = E_i^*(x)$ denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View $E_i^*(x) \in \text{Mat}_X(K)$. So, $E_i^*(x)$ is diagonal with yy entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{else,} \end{cases} \quad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \text{Span}_K(E_0^*(x), \dots, E_{d(x)}^*(x)).$$

Then $M^*(x)$ is a commutative subalgebra of $\text{Mat}_X(K)$ and is called the *dual Bose-Mesner algebra with respect to x* .

Definition 1.6 (Subconstituent Algebra). Let $\Gamma = (X, E)$, x , M , $M^*(x)$ be as above. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(K)$ generated by M and $M^*(x)$. T is the *subconstituent algebra* of Γ with respect to x .

Definition 1.7. A T -module is any subspace $W \subset V$ such that $aw \in W$ for all $a \in T$ and $w \in W$.

T -module W is *irreducible* if and only if $W \neq 0$ and W does not properly contain a nonzero T -module.

For any $a \in \text{Mat}_X(K)$, let a^* denote the conjugate transpose of a .

Observe that

$$\langle au, v \rangle = \langle u, a^*v \rangle \quad \text{for all } a \in \text{Mat}_X(K), \text{ and for all } u, v \in V.$$

Lemma 1.2. Let $\Gamma = (X, E)$, $x \in X$ and $T \equiv T(x)$ be as above.

1. If $a \in T$, then $a^* \in T$.

2. For any T -module $W \subset V$,

$$W^\perp := \{v \in V \mid \langle w, v \rangle = 0, \text{ for all } w \in W\}$$

is a T -module.

3. V decomposes as an orthogonal direct sum of irreducible T -modules.

Proof. 1. It is because T is generated by symmetric real matrices

$$A, E_0^*(x), E_1^*(x), \dots, E_{d(x)(x)}^*.$$

2. Pick $v \in W^\perp$ and $a \in T$, it suffices to show that $av \in W^\perp$. For all $w \in W$,

$$\langle w, av \rangle = \langle a^*w, v \rangle = 0$$

as $a^* \in T$.

3. This is proved by the induction on the dimension of T -modules. If W is an irreducible T -module of V , then

$$V = W + W^\perp \quad (\text{orthogonal direct sum}).$$

□

Problem. What does the structure of the $T(x)$ -module tell us about Γ ?

Study those Γ whose modules take ‘simple’ form. The Γ ’s involved are highly regular.

Remark. 1. The subconstituent algebra T is semisimple as the left regular representation of T is completely reducible. See Curtis-Reiner 25.2.
 2. The inner product $\langle a, b \rangle_T = \text{tr}(a^\top b)$ is nondegenerate on T .
 3. In general,

$$\begin{aligned} T: \text{Semisimple and Artinian} &\Leftrightarrow T: \text{Artinian with } J(T) = 0 \\ &\Leftrightarrow T: \text{Artinian with nonzero nilpotent element} \\ &\Leftrightarrow T \subset \text{Mat}_X(K) \text{ such that for all } a \in T \text{ is normal.} \end{aligned}$$

Chapter 2

Lecture 2

Friday, January 22, 1993

In this lecture we use the Perron Frobenius theory of nonnegative matrices to obtain information on eigenvalues of a graph.

Let $K = \mathbb{R}$. For $n \in \mathbb{Z}_{>0}$, pick a symmetric matrix $C \in \text{Mat}_n(\mathbb{R})$.

Definition 2.1. The matrix C is *reducible* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i \in X^+$, and for all $j \in X^-$, and for all $i \in X^-$, and for all $j \in X^+$, i.e.,

$$C \sim \begin{pmatrix} * & O \\ O & * \end{pmatrix}.$$

Definition 2.2. The matrix C is *bipartite* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i, j \in X^+$, and for all $i, j \in X^-$, i.e.,

$$C \sim \begin{pmatrix} O & * \\ * & O \end{pmatrix}.$$

Note.

1. If C is bipartite, for every eigenvalue θ of C , $-\theta$ is an eigenvalue of C such that $\text{mult}(\theta) = \text{mult}(-\theta)$.

Indeed, let $C = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$,

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix},$$

where $Ay = \theta x$ and $Bx = \theta y$.

2. If C is bipartite, C^2 is reducible.
3. The matrix C is irreducible and C^2 is reducible, if $C_{ij} \geq 0$ for all i, j and C is reducible. (Exercise)

Remark. Note 1. Even if C is not symmetric

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

holds. So the geometrix multiplicities coincide. How about the algebraic multiplicities?

Note 3. Set $x \sim y$ if and only if $C_{xy} > 0$. So the graph may have loops. Then

$$(C^2)_{xy} > 0 \Leftrightarrow \text{if there exists } z \in X \text{ such that } x \sim z \sim y.$$

Note that C is irreducible if and only if $\Gamma(C)$ is connected. Let

$$X^+ = \{y \mid \text{there is a path of even length from } x \text{ to } y\} \quad (2.1)$$

$$X^- = \{y \mid \text{there is no path of even length from } x \text{ to } y\} \neq \emptyset. \quad (2.2)$$

If there is an edge $y \sim z$ in X^+ and $w \in X^-$. Then there would be a path from x to y of even length. So $e(X^+, X^+) = e(X^-, X^-) = 0$.

Theorem 2.1 (Perron-Frobenius). *Given a matrix C in $\text{Mat}_n(\mathbb{R})$ such that*

- a. C is symmetric.
- b. C is irreducible.
- c. $C_{ij} \geq 0$ for all i, j . Let θ_0 be the maximal eigenvalue of C with eigenspace $V_0 \subseteq \mathbb{R}^n$, and let θ_r be the maximal eigenvalue of C with eigenspace $V_r \subseteq \mathbb{R}^n$. Then the following hold.

$$1. \text{ Suppose } 0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0. \text{ Then } \alpha_0 > 0 \text{ for all } i, \text{ or } \alpha_i < 0 \text{ for all } i.$$

$$2. \dim V_0 = 1.$$

$$3. \theta_r \geq -\theta_0.$$

$$4. \theta_r = \theta_0 \text{ if and only if } C \text{ is bipartite.}$$

First, we prove the following lemma.

Lemma 2.1. *Let \langle, \rangle be the dot product in $V = \mathbb{R}^n$. Pick a symmetric matrix $B \in \text{Mat}_n(\mathbb{R})$. Suppose all eigenvalues of B are nonnegative. (i.e., B is positive semidefinite.) Then there exist vectors $v_1, v_2, \dots, v_n \in V$ such that $B_{ij} = \langle v_i, v_j \rangle$ for $(1 \leq i, j \leq n)$.*

By elementary linear algebra, there exists an orthonormal basis w_1, w_2, \dots, w_n of V consisting of eigenvectors of B . Set the i -th column of P is w_i and $D = \text{diag}(\theta_1, \dots, \theta_n)$. Then $P^\top P = I$ and $BP = PD$.

Hence,

$$B = PDP^{-1} = PDP^\top = QQ^\top,$$

where

$$Q = P \cdot \text{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \dots, \sqrt{\theta_n}) \in \text{Mat}_n(\mathbb{R}).$$

Now, let v_i be the i -th column of Q^\top . Then

$$B_{ij} = v_i^\top \cdot v_j = \langle v_i, v_j \rangle.$$

Now we start the proof of Theorem 2.1.

Proof of Theorem 2.1

1. Let \langle, \rangle denote the dot product on $V = \mathbb{R}^n$. Set

$$B = \theta I - C \tag{2.3}$$

$$= \text{symmetric matrix with eigenvalues } \theta_0 - \theta_i \geq 0 \tag{2.4}$$

$$= (\langle v_i, v_j \rangle)_{1 \leq i, j \leq n} \tag{2.5}$$

with the same $v_1, \dots, v_n \in V$ by Lemma 2.1.

Observe: $\sum_{i=1}^n \alpha_i v_i = 0$.

Pf.

$$\left\| \sum_{i=1}^n \alpha_i v_i \right\|^2 = \left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \right\rangle \tag{2.6}$$

$$= (\alpha_1 \quad \dots \quad \alpha_n) B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{2.7}$$

$$= v^\top B v \tag{2.8}$$

$$= 0, \tag{2.9}$$

since $Bv = (\theta_0 I - C)v = 0$.

Now set

$s =$ the number of indices i , where $\alpha_i > 0$.

Replacing v by $-v$ if necessary, without loss of generality we may assume that $s \geq 1$. We want to show $s = n$.

Assume $s < n$. Without loss of generality, we may assume that $\alpha_i > 0$ for $1 \leq i \leq s$ and $\alpha_i = 0$ for $s+1 \leq i \leq n$. Set

$$\rho = \alpha_1 v_1 + \dots + \alpha_s v_s = -\alpha_{s+1} v_{s+1} - \dots - \alpha_n v_n.$$

Then, for $i = 1, \dots, s$,

$$\langle v_i, \rho \rangle = \sum_{j=s+1}^n -\alpha_j \langle v_i, v_j \rangle \quad (\langle v_i, v_j \rangle = B_{ij}, B = \theta_0 I - C) \quad (2.10)$$

$$= \sum_{j=s+1}^n (-\alpha_{ij})(-C_{ij}) \quad (2.11)$$

$$\leq 0. \quad (2.12)$$

Hence

$$0 \leq \langle \rho, \rho \rangle = \sum_{i=1}^s \alpha_i \langle v_i, \rho \rangle \leq 0,$$

as $\alpha > 0$ and $\langle v_i, \rho \rangle \leq 0$. Thus, we have $\langle \rho, \rho \rangle = 0$ and $\rho = 0$. For $j = s+1, \dots, n$,

$$0 = \langle \rho, v_j \rangle = \sum_{i=1}^s \alpha_i \langle v_i, v_j \rangle \leq 0,$$

as $\langle v_i, v_j \rangle = -C_{ij}$.

Therefore,

$$0 = \langle v_i, v_j \rangle = -C_{ij} \text{ for } 1 \leq i \leq s, s+1 \leq j \leq n.$$

Since C is symmetric,

$$C = \begin{pmatrix} * & O \\ O & * \end{pmatrix}$$

Thus C is reducible, which is not the case. Hence $s = n$.

Proof of Theorem 2.1 2.

Suppose $\dim V_0 \geq 2$. Then,

$$\dim \left(V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^\perp \right) \geq 1.$$

So, there is a vector

$$0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with $\alpha_1 = 0$. This contradicts 1.

Now pick

$$0 \neq w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r.$$

Proof of Theorem 2.1 3.

Suppose $\theta_r < -\theta_0$. Since the eigenvalues of C^2 are the squares of those of C , θ_r^2 is the maximal eigenvalue of C^2 .

Also we have $C^2 w = \theta_r^2 w$.

Observe that C^2 is irreducible. (As otherwise, C is bipartite by Note 3, and we must have $\theta_r = -\theta_0$.) Therefore, $\beta_i > 0$ for all i or $\beta_i < 0$ for all i . We have

$$\langle v, w \rangle = \sum_{i=1}^n \alpha_i \beta_i \neq 0.$$

This is a contradiction, as $V_0 \perp V_r$.

Proof of Theorem 2.1 4.

\Rightarrow : Let $\theta_r = -\theta_0$. Then $\theta = \theta_1^2 = \theta_0^2$ is the maximal eigenvalue of C^2 , and v and w are linearly independent eigenvalues for θ for C^2 . Hence, for C^2 , $\text{mult}(\theta) \geq 2$.

Thus by 2, C^2 must be reducible. Therefore, C is bipartite by Note 3.

\Leftarrow : This is Note 1. \square

Let $\Gamma = (X, E)$ be any graph.

Definition 2.3. Γ is said to be *bipartite* if the adjacency matrix A is bipartite. That is, X can be written as a disjoint union of X^+ and X^- such that X^+, X^- contain no edges of Γ .

Corollary 2.1. *For any (connected) graph Γ with*

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_1 & m_1 & \cdots & m_r \end{pmatrix} \quad \text{with } \theta_0 > \theta_1 > \cdots > \theta_r.$$

Let V_i be the eigenspace of θ_i . Then the following holds.

1. Suppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \in \mathbb{R}^n$. Then $\alpha_i > 0$ for all i or $\alpha_i < 0$ for all i .
2. $m_0 = 1$.
3. $\theta_r \geq -\theta_0$ if and only if Γ is bipartite. In this case,

$$-\theta_i = \theta_{r-i} \text{ and } m_i = m_{r-i} \quad (0 \leq i \leq r)$$

Proof. This is a direct consequences of Theorem 2.1 and Note 3. \square

Chapter 3

Lecture 3

Monday, January 25, 1993

Given graphs $\Gamma = (X, E)$ and $\Gamma' = (X', E')$.

Chapter 4

Applications

Some *significant* applications are demonstrated in this chapter.

4.1 Example one

4.2 Example two

Chapter 5

Final Words

We have finished a nice book.