

# Lecture Note on Terwilliger Algebra

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# About this lecturenote

## Setting

sudo This note is created by `bookdown` package on RStudio.

For `bookdown` See (Xie, 2015), (Xie, 2017), (Yihui Xie, 2018).

1. Log-in to my GitHub Account
2. Go to RStudio/bookdown-demo repository: <https://github.com/rstudio/bookdown-demo>
3. Use This Template
4. Input Repository Name
5. Select Public - default
6. Create repository from template
7. From Code download ZIP
8. Move the extracted folder into a favorite directory
9. Open RStudio Project in the folder
10. Use Terminal in the bottom left pane
  - confirm that the current directory is the home directory of the project by `pwd`
11. (failed to proceed by ssh)
12. Use Console
  1. `library(usethis)`
  2. `use_git()`
  3. `use_github()` — Error
  4. `gh_token_help()`
  5. `create_github_token()`: create a token in the github page. Copy the token
  6. `gitcreds::gitcreds_set()`: paste the token, the token is to be expired in 30 days
13. Use Terminal
  1. `git remote add origin https://github.com/icu-hsuzuki/t-algebra.git`
  2. `git push -u origin main`
  3. type in the password of the computer
14. Use GIT in R Studio

## Another Host

1. create a project by version control git
2. git init
3. git remote add origin git@github.com:/.git
4. git branch -r
5. git fetch
6. git pull origin main

# Chapter 1

## Subconstituent Algebra of a Graph

Wednesday, January 20, 1993

A graph (undirected, without loops or multiple edges) is a pair  $\Gamma = (X, E)$ , where

$$X = \text{finite set (of vertices)} \quad (1.1)$$

$$E = \text{set of (distinct) 2-element subsets of } X (= \text{edges of } \Gamma). \quad (1.2)$$

vertices  $x$  and  $y \in X$  are adjacent if and only if  $xy \in E$ .

**Example 1.1.** Let  $\Gamma$  be a graph.  $X = \{a, b, c, d\}$ ,  $E = \{ab, ac, bc, bd\}$ .



Set  $n = |X|$ , the order of  $\Gamma$ .

Pick a field  $K$  ( $= \mathbb{R}$  or  $\mathbb{C}$ ). Then  $\text{Mat}_X(K)$  denotes the  $K$  algebra of all  $n \times n$  matrices with entries in  $K$ . (rows and columns are indexed by  $X$ )

*Adjacency matrix*  $A \in \text{Mat}_X(K)$  is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E \\ 0 & \text{else .} \end{cases} \quad (1.3)$$

**Example 1.2.** Let  $a, b, c, d$  be labels of rows and columns. Then

$$A = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

The subalgebra  $M$  of  $\text{Mat}_X(K)$  generated by  $A$  is called the *Bose-Mesner algebra* of  $\Gamma$ .

Set  $V = K^n$ , the set of  $n$ -dimensional column vectors, the coordinates are indexed by  $X$ .

Let  $\langle \cdot, \cdot \rangle$  denote the Hermitean inner product:

$$\langle u, v \rangle = u^\top \cdot v \quad (u, v \in V)$$

$V$  with  $\langle \cdot, \cdot \rangle$  is the *standard module* of  $\Gamma$ .

$M$  acts on  $V$ : For every  $x \in X$ , write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where 1 is at the  $x$  position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since  $A$  is a real symmetrix matrix,

$$V = V_0 + V_1 + \cdots + V_r \quad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal  $A$ -eigenspaces.

Let  $E_i \in \text{Mat}_X(K)$  denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then  $E_0, \dots, E_r$  are the primitive idempotents of  $M$ .

$$M = \text{Span}_K(E_0, \dots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all } i, j, \quad E_0 + \cdots + E_r = I.$$



Let  $\theta_i$  denote the eigenvalue of  $A$  for  $V_i$  in  $\mathbb{R}$ . Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \dots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \theta_i = \dim V_i = \text{rank } E_i.$$

Set

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \dots, & \theta_r \\ m_0, & m_1, & \dots, & m_r \end{pmatrix}.$$

**Problem.** What can we say about  $\Gamma$  when  $\text{Spec}(\Gamma)$  is given?

The following Lemma 1.1, is an example of Problem.

For every  $x \in X$ ,

$$k(x) \equiv \text{valency of } x \equiv \text{degree of } x \equiv |\{y \mid y \in X, xy \in E\}|.$$

**Definition 1.1.** The graph  $\Gamma$  is regular of valency  $k$  if  $k = k(x)$  for every  $x \in X$ .

**Lemma 1.1.** *With the above notation,*

- (i)  $\theta_0 \leq \max\{k(x) \mid x \in X\} = k^{\max}$ .
- (ii) *If  $\Gamma$  is regular of valency  $k$ , then  $\theta_0 = k$ .*

*Proof.*

(i) Without loss of generality we may assume that  $\theta_0 > 0$ , else done. Let  $v := \sum_{x \in X} \alpha_x \hat{x}$  denote the eivenvector for  $\theta_0$ .

Pick  $x \in X$  with  $|\alpha_x|$  maximal. Then  $|\alpha_x| \neq 0$ .

Since  $Av = \theta_0 v$ ,

$$\theta_0 \alpha_x = \sum_{y \in X, xy \in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \leq \sum_{y \in X, xy \in E} |\alpha_y| \leq k(x) |\alpha_x| \leq k^{\max} |\alpha_x|.$$

(ii) All 1's vector  $v = \sum_{x \in X} \hat{x}$  satisfies  $Av = kv$ .

□

### Subconstituent Algebra

Let  $x, y \in X$  and  $\ell \in \mathbb{Z}^{\geq 0}$ .

**Definition 1.2.** A path of length  $\ell$  connecting  $x, y$  is a sequence

$$x = x_0, x_1, \dots, x_\ell = y, \quad x_i \in X, \quad 0 \leq i \leq \ell$$

such that  $x_i x_{i+1} \in E$  for  $0 \leq i \leq \ell - 1$ .

**Definition 1.3.** The distance  $\partial(x, y)$  is the length of a shortest path connecting  $x$  and  $y$ .

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

**Definition 1.4.** The graph  $\Gamma$  is connected if and only if  $\partial(x, y) < \infty$  for all  $x, y \in X$ .

From now on, assume that  $\Gamma$  is connected with  $|X| \geq 2$ .

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \text{the diameter of } \Gamma.$$

Fix a ‘base’ vertex  $x \in X$ .

**Definition 1.5.**

$$d(x) = \text{the diameter with respect to } x = \max\{\partial(x, y) \mid y \in X\} \leq d.$$

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad (\text{orthogonal direct sum}),$$

where

$$V_i^* = \text{Span}_K(\hat{y} \mid \partial(x, y) = i) \equiv V_i * (x)$$

and  $V_i^* = V_i^*(x)$  is called the  $i$ -th subconstituent with respect to  $x$ .

Let  $E_i^* = E_i^*(x)$  denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View  $E_i^*(x) \in \text{Mat}_X(K)$ . So,  $E_i^*(x)$  is diagonal with  $yy$  entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{else,} \end{cases} \quad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \text{Span}_K(E_0^*(x), \dots, E_{d(x)}^*(x)).$$

Then  $M^*(x)$  is a commutative subalgebra of  $\text{Mat}_X(K)$  and is called the *dual Bose-Mesner algebra with respect to  $x$* .

**Definition 1.6** (Subconstituent Algebra). Let  $\Gamma = (X, E)$ ,  $x, M, M^*(x)$  be as above. Let  $T = T(x)$  denote the subalgebra of  $\text{Mat}_X(K)$  generated by  $M$  and  $M^*(x)$ .  $T$  is the *subconstituent algebra* of  $\Gamma$  with respect to  $x$ .

**Definition 1.7.** A  $T$ -module is any subspace  $W \subset V$  such that  $aw \in W$  for all  $a \in T$  and  $w \in W$ .

$T$ -module  $W$  is *irreducible* if and only if  $W \neq 0$  and  $W$  does not properly contain a nonzero  $T$ -module.

For any  $a \in \text{Mat}_X(K)$ , let  $a^*$  denote the conjugate transpose of  $a$ .

Observe that

$$\langle au, v \rangle = \langle u, a^*v \rangle \quad \text{for all } a \in \text{Mat}_X(K), \text{ and for all } u, v \in V.$$

**Lemma 1.2.** *Let  $\Gamma = (X, E)$ ,  $x \in X$  and  $T \equiv T(x)$  be as above.*

(i) *If  $a \in T$ , then  $a^* \in T$ .*

(ii) *For any  $T$ -module  $W \subset V$ ,*

$$W^\perp := \{v \in V \mid \langle w, v \rangle = 0, \text{ for all } w \in W\}$$

*is a  $T$ -module.*

(iii)  *$V$  decomposes as an orthogonal direct sum of irreducible  $T$ -modules.*

*Proof.*

(i) It is because  $T$  is generated by symmetric real matrices

$$A, E_0^*(x), E_1^*(x), \dots, E_{d(x)(x)}^*.$$

(ii) Pick  $v \in W^\perp$  and  $a \in T$ , it suffices to show that  $av \in W^\perp$ . For all  $w \in W$ ,

$$\langle w, av \rangle = \langle a^*w, v \rangle = 0$$

as  $a^* \in T$ .

(iii) This is proved by the induction on the dimension of  $T$ -modules. If  $W$  is an irreducible  $T$ -module of  $V$ , then

$$V = W + W^\perp \quad (\text{orthogonal direct sum}).$$

□

**Problem.** What does the structure of the  $T(x)$ -module tell us about  $\Gamma$ ?

Study those  $\Gamma$  whose modules take ‘simple’ form. The  $\Gamma$ ’s involved are highly regular.

*Remark.*

1. The subconstituent algebra  $T$  is semisimple as the left regular representation of  $T$  is completely reducible. See Curtis-Reiner 25.2 (Charles W. Curtis, 2006).
2. The inner product  $\langle a, b \rangle_T = \text{tr}(a^\top \bar{b})$  is nondegenerate on  $T$ .

3. In general,

$T$ : Semisimple and Artinian  $\Leftrightarrow T$ : Artinian with  $J(T) = 0$

$\Leftrightarrow T$ : Artinian with nonzero nilpotent element

$\Leftrightarrow T \subset \text{Mat}_X(K)$  such that for all  $a \in T$  is normal.

## Chapter 2

# Perron-Frobenius Theorem

Friday, January 22, 1993

In this lecture we use the Perron Frobenius theory of nonnegative matrices to obtain information on eigenvalues of a graph.

Let  $K = \mathbb{R}$ . For  $n \in \mathbb{Z}^{>0}$ , pick a symmetric matrix  $C \in \text{Mat}_n(\mathbb{R})$ .

**Definition 2.1.** The matrix  $C$  is *reducible* if and only if there is a bipartition  $\{1, 2, \dots, n\} = X^+ \cup X^-$  (disjoint union of nonempty sets) such that  $C_{ij} = 0$  for all  $i \in X^+$ , and for all  $j \in X^-$ , and for all  $i \in X^-$ , and for all  $j \in X^+$ , i.e.,

$$C \sim \begin{pmatrix} * & O \\ O & * \end{pmatrix}.$$

**Definition 2.2.** The matrix  $C$  is *bipartite* if and only if there is a bipartition  $\{1, 2, \dots, n\} = X^+ \cup X^-$  (disjoint union of nonempty sets) such that  $C_{ij} = 0$  for all  $i, j \in X^+$ , and for all  $i, j \in X^-$ , i.e.,

$$C \sim \begin{pmatrix} O & * \\ * & O \end{pmatrix}.$$

**Note.**

1. If  $C$  is bipartite, for every eigenvalue  $\theta$  of  $C$ ,  $-\theta$  is an eigenvalue of  $C$  such that  $\text{mult}(\theta) = \text{mult}(-\theta)$ .

Indeed, let  $C = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$ ,

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix},$$

where  $Ay = \theta x$  and  $Bx = \theta y$ .

2. If  $C$  is bipartite,  $C^2$  is reducible.
3. The matrix  $C$  is irreducible and  $C^2$  is reducible, if  $C_{ij} \geq 0$  for all  $i, j$  and  $C$  is reducible. (Exercise)

*Remark.* Note 1. Even if  $C$  is not symmetric

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

holds. So the geometrix multiplicities coincide. How about the algebraic multiplicities?

Note 3. Set  $x \sim y$  if and only if  $C_{xy} > 0$ . So the graph may have loops. Then

$$(C^2)_{xy} > 0 \Leftrightarrow \text{if there exists } z \in X \text{ such that } x \sim z \sim y.$$

Note that  $C$  is irreducible if and only if  $\Gamma(C)$  is connected. Let

$$X^+ = \{y \mid \text{there is a path of even length from } x \text{ to } y\} \quad (2.1)$$

$$X^- = \{y \mid \text{there is no path of even length from } x \text{ to } y\} \neq \emptyset. \quad (2.2)$$

If there is an edge  $y \sim z$  in  $X^+$  and  $w \in X^-$ . Then there would be a path from  $x$  to  $y$  of even length. So  $e(X^+, X^+) = e(X^-, X^-) = 0$ .

**Theorem 2.1** (Perron-Frobenius). *Given a matrix  $C$  in  $\text{Mat}_n(\mathbb{R})$  such that*

- (a)  $C$  is symmetric.
- (b)  $C$  is irreducible.
- (c)  $C_{ij} \geq 0$  for all  $i, j$ .

*Let  $\theta_0$  be the maximal eigenvalue of  $C$  with eigenspace  $V_0 \subseteq \mathbb{R}^n$ , and let  $\theta_r$  be the maximal eigenvalue of  $C$  with eigenspace  $V_r \subseteq \mathbb{R}^n$ . Then the following hold.*

$$(i) \text{ Suppose } 0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0. \text{ Then } \alpha_0 > 0 \text{ for all } i, \text{ or } \alpha_i < 0 \text{ for all } i.$$

$$(ii) \dim V_0 = 1.$$

$$(iii) \theta_r \geq -\theta_0.$$

$$(iv) \theta_r = \theta_0 \text{ if and only if } C \text{ is bipartite.}$$

First, we prove the following lemma.

**Lemma 2.1.** *Let  $\langle \cdot, \cdot \rangle$  be the dot product in  $V = \mathbb{R}^n$ . Pick a symmetric matrix  $B \in \text{Mat}_n(\mathbb{R})$ . Suppose all eigenvalues of  $B$  are nonnegative. (i.e.,  $B$  is positive semidefinite.) Then there exist vectors  $v_1, v_2, \dots, v_n \in V$  such that  $B_{ij} = \langle v_i, v_j \rangle$  for  $(1 \leq i, j \leq n)$ .*

*Proof.* By elementary linear algebra, there exists an orthonormal basis  $w_1, w_2, \dots, w_n$  of  $V$  consisting of eigenvectors of  $B$ . Set the  $i$ -th column of  $P$  is  $w_i$  and  $D = \text{diag}(\theta_1, \dots, \theta_n)$ . Then  $P^\top P = I$  and  $BP = PD$ .

Hence,

$$B = PDP^{-1} = PDP^\top = QQ^\top,$$

where

$$Q = P \cdot \text{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \dots, \sqrt{\theta_n}) \in \text{Mat}_n(\mathbb{R}).$$

Now, let  $v_i$  be the  $i$ -th column of  $Q^\top$ . Then

$$B_{ij} = v_i^\top \cdot v_j = \langle v_i, v_j \rangle.$$

□

Now we start the proof of Theorem 2.1.

*Proof of Theorem 2.1(i)*

Let  $\langle, \rangle$  denote the dot product on  $V = \mathbb{R}^n$ . Set

$$B = \theta I - C \tag{2.3}$$

$$= \text{symmetric matrix with eigenvalues } \theta_0 - \theta_i \geq 0 \tag{2.4}$$

$$= (\langle v_i, v_j \rangle)_{1 \leq i, j \leq n} \tag{2.5}$$

with the same  $v_1, \dots, v_n \in V$  by Lemma 2.1.

Observe:  $\sum_{i=1}^n \alpha_i v_i = 0$ .

*Pf.*

$$\left\| \sum_{i=1}^n \alpha_i v_i \right\|^2 = \left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \right\rangle \tag{2.6}$$

$$= (\alpha_1 \quad \dots \quad \alpha_n) B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{2.7}$$

$$= v^\top B v \tag{2.8}$$

$$= 0, \tag{2.9}$$

since  $Bv = (\theta_0 I - C)v = 0$ .

Now set

$$s = \text{the number of indices } i, \text{ where } \alpha_i > 0.$$

Replacing  $v$  by  $-v$  if necessary, without loss of generality we may assume that  $s \geq 1$ . We want to show  $s = n$ .

Assume  $s < n$ . Without loss of generality, we may assume that  $\alpha_i > 0$  for  $1 \leq i \leq s$  and  $\alpha_i = 0$  for  $s+1 \leq i \leq n$ . Set

$$\rho = \alpha_1 v_1 + \dots + \alpha_s v_s = -\alpha_{s+1} v_{s+1} - \dots - \alpha_n v_n.$$

Then, for  $i = 1, \dots, s$ ,

$$\langle v_i, \rho \rangle = \sum_{j=s+1}^n -\alpha_j \langle v_i, v_j \rangle \quad (\langle v_i, v_j \rangle = B_{ij}, B = \theta_0 I - C) \quad (2.10)$$

$$= \sum_{j=s+1}^n (-\alpha_j)(-C_{ij}) \quad (2.11)$$

$$\leq 0. \quad (2.12)$$

Hence

$$0 \leq \langle \rho, \rho \rangle = \sum_{i=1}^s \alpha_i \langle v_i, \rho \rangle \leq 0,$$

as  $\alpha > 0$  and  $\langle v_i, \rho \rangle \leq 0$ . Thus, we have  $\langle \rho, \rho \rangle = 0$  and  $\rho = 0$ . For  $j = s+1, \dots, n$ ,

$$0 = \langle \rho, v_j \rangle = \sum_{i=1}^s \alpha_i \langle v_i, v_j \rangle \leq 0,$$

as  $\langle v_i, v_j \rangle = -C_{ij}$ .

Therefore,

$$0 = \langle v_i, v_j \rangle = -C_{ij} \text{ for } 1 \leq i \leq s, s+1 \leq j \leq n.$$

Since  $C$  is symmetric,

$$C = \begin{pmatrix} * & O \\ O & * \end{pmatrix}$$

Thus  $C$  is reducible, which is not the case. Hence  $s = n$ .

*Proof of Theorem 2.1 (ii).*

Suppose  $\dim V_0 \geq 2$ . Then,

$$\dim \left( V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^\perp \right) \geq 1.$$

So, there is a vector

$$0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with  $\alpha_1 = 0$ . This contradicts 1.

Now pick

$$0 \neq w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r.$$

*Proof of Theorem 2.1 (iii).*



Suppose  $\theta_r < -\theta_0$ . Since the eigenvalues of  $C^2$  are the squares of those of  $C$ ,  $\theta_r^2$  is the maximal eigenvalue of  $C^2$ .

Also we have  $C^2 w = \theta_r^2 w$ .

Observe that  $C^2$  is irreducible. (As otherwise,  $C$  is bipartite by Note 3, and we must have  $\theta_r = -\theta_0$ .) Therefore,  $\beta_i > 0$  for all  $i$  or  $\beta_i < 0$  for all  $i$ . We have

$$\langle v, w \rangle = \sum_{i=1}^n \alpha_i \beta_i \neq 0.$$

This is a contradiction, as  $V_0 \perp V_r$ .

*Proof of Theorem 2.1 (iv)*

$\Rightarrow$ : Let  $\theta_r = -\theta_0$ . Then  $\theta = \theta_1^2 = \theta_0^2$  is the maximal eigenvalue of  $C^2$ , and  $v$  and  $w$  are linearly independent eigenvalues for  $\theta$  for  $C^2$ . Hence, for  $C^2$ ,  $\text{mult}(\theta) \geq 2$ .

Thus by 2,  $C^2$  must be reducible. Therefore,  $C$  is bipartite by Note 3.

$\Leftarrow$ : This is Note 1.  $\square$

Let  $\Gamma = (X, E)$  be any graph.

**Definition 2.3.**  $\Gamma$  is said to be *bipartite* if the adjacency matrix  $A$  is bipartite. That is,  $X$  can be written as a disjoint union of  $X^+$  and  $X^-$  such that  $X^+, X^-$  contain no edges of  $\Gamma$ .

**Corollary 2.1.** *For any (connected) graph  $\Gamma$  with*

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_1 & m_1 & \cdots & m_r \end{pmatrix} \quad \text{with } \theta_0 > \theta_1 > \cdots > \theta_r.$$

*Let  $V_i$  be the eigenspace of  $\theta_i$ . Then the following holds.*

1. Suppose  $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \in \mathbb{R}^n$ . Then  $\alpha_i > 0$  for all  $i$  or  $\alpha_i < 0$  for all  $i$ .
2.  $m_0 = 1$ .
3.  $\theta_r \geq -\theta_0$  if and only if  $\Gamma$  is bipartite. In this case,

$$-\theta_i = \theta_{r-i} \text{ and } m_i = m_{r-i} \quad (0 \leq i \leq r)$$

*Proof.* This is a direct consequences of Theorem 2.1 and Note 3.  $\square$



## Chapter 3

# Cayley Graphs

Monday, January 25, 1993

Given graphs  $\Gamma = (X, E)$  and  $\Gamma' = (X', E')$ .

**Definition 3.1.** A map  $\sigma : X \rightarrow X'$  is an *isomorphism* of graphs whenever;

- i.  $\sigma$  is one-to-one and onto,
- ii.  $xy \in E$  if and only if  $\sigma x \sigma y \in E'$  for all  $x, y \in X$ .

We do not distinguish between isomorphic graphs.

**Definition 3.2.** Suppose  $\Gamma = \Gamma'$ . Above isomorphism  $\sigma$  is called an *automorphism* of  $\Gamma$ . Then set  $\text{Aut}(\Gamma)$  of all automorphisms of  $\Gamma$  becomes a finite group under composition.

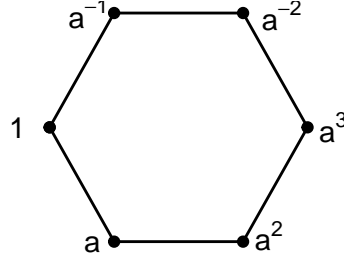
**Definition 3.3.** If  $\text{Aut}(\Gamma)$  acts transitive on  $X$ ,  $\Gamma$  is called *vertex transitive*.

**Example 3.1.** A Cayley graphs:

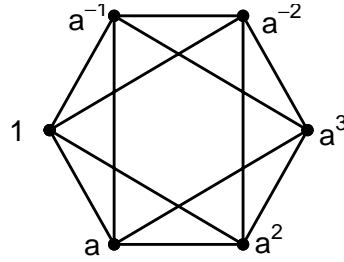
**Definition 3.4** (Cayley Graphs). Let  $G$  be any finite group, and  $\Delta$  any generating set for  $G$  such that  $1_G \notin \Delta$  and  $g \in \Delta \rightarrow g^{-1} \in \Delta$ . Then Cayley graph  $\Gamma = \Gamma(G, \Delta)$  is defined on the vertex set  $X = G$  with the edge set  $E$  defined by the following.

$$E = \{(h_1, h_2) \mid h_1, h_2 \in G, h_1^{-1}h_2 \in \Delta\} = \{(h, hg) \mid h \in G, g \in \Delta\}$$

**Example 3.2.**  $G = \langle a \mid a^6 = 1 \rangle$ ,  $\Delta = \{a, a^{-1}\}$ .



**Example 3.3.**  $G = \langle a \mid a^6 = 1 \rangle$ ,  $\Delta = \{a, a^{-1}, a^2, a^{-2}\}$ .



**Example 3.4.**  $G = \langle a, b \mid a^6 = 1, b^2 = 1, ab = ba \rangle$ ,  $\Delta = \{a, a^{-1}, b\}$ .



*Remark.*  $\text{Aut}(\Gamma) \simeq D_6 \times \mathbb{Z}_2$  contains two regular subgroups isomorphic to  $D_6$  and  $\mathbb{Z}_5 \times \mathbb{Z}_2$  and  $\Gamma$  is obtained as Cayley graphs in two ways.

Cayley graphs are vertex transitive, indeed.

**Theorem 3.1.** *The following hold.*

(i) *For any Cayley graph  $\Gamma = \Gamma(G, \Delta)$ , the map*

$$G \rightarrow \text{Aut}(\Gamma) \quad (g \mapsto \hat{g})$$

is an injective homomorphism of groups, where

$$\hat{g}(x) = gx \quad \text{for all } g \in G \text{ and for all } x \in X (= G).$$

Also, the image  $\hat{G}$  is regular on  $X$ . i.e., the image  $\hat{G}$  acts transitively on  $X$  with trivial vertex stabilizers.

(ii) For any graph  $\Gamma = (X, E)$ , suppose there exists a subgroup  $G \subseteq \text{Aut}(\Gamma)$  that is regular on  $X$ . Pick  $x \in X$ , and let

$$\Delta = \{g \in G \mid \langle x, g(x) \rangle \in E\}.$$

Then  $1 \notin \Delta$ ,  $g \in \Delta \rightarrow g^{-1} \in \Delta$ , and  $\Delta$  generates  $G$ . Moreover,  $\Gamma \simeq \Gamma(G, \Delta)$ .

*Proof.* (i) Let  $g \in G$ . We want to show that  $\hat{g} \in \text{Aut}(\Gamma)$ . Let  $h_1, h_2 \in X = G$ . Then,

$$(h_1, h_2) \in E \rightarrow h_1^{-1}h_2 \in \Delta \quad (3.1)$$

$$\rightarrow (gh_1)^{-1}(gh_2) \in \Delta \quad (3.2)$$

$$\rightarrow (gh_1, gh_2) \in E \quad (3.3)$$

$$\rightarrow (\hat{g}(h_1), \hat{g}(h_2)) \in E. \quad (3.4)$$

Hence,  $\hat{g} \in \text{Aut}(\Gamma)$ .

Observe:  $g \mapsto \hat{g}$  is a homomorphism of groups:

$$\hat{1}_G = 1, \widehat{g_1 g_2} = \widehat{g_1} \widehat{g_2}.$$

Observe:  $g \mapsto \hat{g}$  is one-to-one:

$$\widehat{g_1} = \widehat{g_2} \rightarrow g_1 = \widehat{g_1}(1_G) = \widehat{g_2}(1_G) = g_2.$$

Observe:  $\hat{G}$  is regular on  $X$ : Clear by construction.

(ii)  $1_G \notin \Delta$ : Since  $\Gamma$  has not loops,  $(x, 1_G x) \notin E$ .

$g \in \Delta \rightarrow g^{-1} \in \Delta$ :

$$g \in \Delta \rightarrow (x, g(x)) \in E \rightarrow E \ni (g^{-1}(x), g^{-1}(g(x))) = (g^{-1}(x), x).$$

$\Delta$  generates  $G$ : Suppose  $\langle \Delta \subsetneq G$ . Let  $\hat{X} = \{g(x) \mid g \in \langle \Delta \rangle\} \subsetneq X$ . ( $\hat{X} \subsetneq X$  as  $G$  acts regularly on  $X$ .)

Since  $\Gamma$  is connected, there exists  $y \in \hat{X}$  and  $z \in X \setminus \hat{X}$  with  $yz \in E$ .

Let  $y = g(x)$ ,  $g \in \langle \Delta \rangle$ ,  $z \in h(x)$ ,  $h \in G \setminus \langle \Delta \rangle$ . Then

$$(y, z) = (g(x), h(x)) \in E \rightarrow (x, g^{-1}h(x)) \in E \rightarrow g^{-1}h \in \langle \Delta \rangle \rightarrow h \in \langle \Delta \rangle.$$

This is a contradiction. Therefore,  $\Delta$  generates  $G$ .

Let  $\Gamma' = (X', E')$  denote  $\Gamma(G, \Delta)$ . We shall show that

$$\theta : X' \rightarrow X \ (g \mapsto g(x))$$

is an isomorphism of graphs.

$\theta$  is one-to-one: For  $h_1, h_2 \in X' = G$ ,

$$\theta(h_1) = \theta(h_2) \rightarrow h_1(x) = h_2(x) \rightarrow h_1^{-1}h_2(x) = x \rightarrow h_1^{-1}h_2 \in \text{Stab}_G(x) = \{1_G\} \rightarrow h_1 = h_2.$$

( $\text{Stab}_G = \{g \in G \mid g(x) = x\}$ .)

$\theta$  is onto: Since  $G$  is transitive,

$$X = \{g(x) \mid g \in G\} = \theta(X') = \theta(G).$$

$\theta$  respects adjacency: For  $h_1, h_2 \in X' = G$ ,

$$(h_1, h_2) \in E' \leftrightarrow h_1^{-1}h_2 \in \Delta \leftrightarrow (x, h_1^{-1}h_2(x)) \in E \leftrightarrow (h_1(x), h_2(x)) \in E \leftrightarrow (\theta(h_1), \theta(h_2)) \in E.$$

Therefore  $\theta$  is an isomorphism between graphs  $\Gamma(G, \Delta)$  and  $\Gamma(X, E)$ .  $\square$

How to compute the eigenvalues of the Cayley graph of an abelian group.

Let  $G$  be any finite abelian group. Let  $\mathbb{C}^*$  be the multiplicative group on  $\mathbb{C} \setminus \{0\}$ .

**Definition 3.5.** A (linear)  $G$ -character is any group homomorphism  $\theta : G \rightarrow \mathbb{C}^*$ .

**Example 3.5.**  $G = \langle a \mid a^3 = 1 \rangle$  has three characters,  $\theta_0, \theta_1, \theta_2$ .

$$\begin{array}{c|ccc} \theta_i(a^j) & 1 & a & a^2 \\ \hline \theta_0 & 1 & 1 & 1 \\ \theta_1 & 1 & \omega & \omega^2 \\ \theta_2 & 1 & \omega^2 & \omega \end{array}, \quad \text{with } \omega = \frac{-1 + \sqrt{-3}}{2}.$$

Here  $\omega$  is a primitive cube root of  $\omega$  in  $\mathbb{C}^*$ , i.e.,  $1 + \omega + \omega^2 = 0$ .

For arbitrary group  $G$ , let  $X(G)$  be the set of all characters of  $G$ .

Observe: For  $\theta_1, \theta_2 \in X(G)$ , one can define product  $\theta_1 \theta_2$ :

$$\theta_1 \theta_2(g) = \theta_1(g) \theta_2(g) \quad \text{for all } g \in G.$$

Then  $\theta_1 \theta_2 \in X(G)$ .

Observe:  $X(G)$  with this product is an (abelian) group.

**Lemma 3.1.** *The groups  $G$  and  $X(G)$  are isomorphic for all finite abelian groups  $G$ .*

*Proof.*  $G$  is a direct sum of cyclic groups;

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m, \quad \text{where } G_i = \langle a_i \mid a_i^{d_i} = 1 \rangle \quad (1 \leq i \leq m).$$

Pick any element  $\omega_i$  of order  $d_i$  in  $\mathbb{C}^*$ , i.e., a primitive  $d_i$ -th root of 1. Define

$$\theta_i : G \rightarrow \mathbb{C}^* \quad (a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \mapsto \omega_i^{\varepsilon_i} \quad \text{where } 0 \leq \varepsilon_i < d_i, 1 \leq i \leq m).$$

Then  $\theta_i \in X(G)$ . (Exercise)

Claim: There exists an isomorphism of groups  $G \rightarrow X(G)$  that sends  $a_i$  to  $\theta_i$ .

Observe:  $\theta_i^{d_i} = 1$ . For every  $g = a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \in G$ ,

$$\theta_i^{d_i}(g) = (\theta_i(g))^{d_i} = (\omega_i^{\varepsilon_i})^{d_i} = (\omega_i^{d_i})^{\varepsilon_i} = 1.$$

Observe: If  $\theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m} = 1$  for some  $0 \leq \varepsilon_i < d_i, 1 \leq i \leq m$ . Then  $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = 0$ .

*Pf.*  $1 = \theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m}(a_i) = \omega_i^{\varepsilon_i}$ , Since  $\omega_i$  is a primitive  $d_i$ -th root of 1,  $\varepsilon_i = 0$  for  $1 \leq i \leq m$ .

Observe:  $\theta_1, \dots, \theta_m$  generate  $X(G)$ . Pick  $\theta \in X(G)$ . Since  $a_i^{d_i} = 1$ ,  $1 = \theta(a_i^{d_i}) = \theta(a_i)^{d_i}$ .

Hence  $\theta(a_i) = \omega_i^{\varepsilon_i}$  for some  $\varepsilon_i$  with  $0 \leq \varepsilon_i < d_i$ .

Now  $\theta = \theta_1^{\varepsilon_1} \cdots \theta_m^{\varepsilon_m}$ , since these are both equal to  $\omega_i^{\varepsilon_i}$  at  $a_i$  for  $1 \leq i \leq m$ .

Therefore,

$$G \rightarrow X(G) \quad (a_i \mapsto \theta_i)$$

is an isomorphism of groups. □

**Note.** The correspondence above is clearly a group homomorphism.





## Chapter 4

# Examples

Wednesday, January 27, 1993

**Theorem 4.1.** *Given a Cayley graph  $\Gamma = \Gamma(G, \Delta)$ . View the standard module  $V \equiv \mathbb{C}G$  (the group algebra), so*

$$\left\langle \sum_{g \in G} \alpha_g g, \sum_{g \in G} \beta_g g \right\rangle = \sum_{g \in G} \alpha_g \overline{\beta_g}, \quad \text{with } \alpha_g, \beta_g \in \mathbb{C}.$$

For any  $\theta \in X(G)$ , write

$$\hat{\theta} = \sum_{g \in G} \theta(g^{-1})g.$$

Then the following hold.

(i)  $\langle \hat{\theta}_1, \hat{\theta}_2 \rangle = |G|$  if  $\theta_1 = \theta_2$  and 0 otherwise for  $\theta_1, \theta_2 \in X(G)$ . In particular,  $\{\hat{\theta} \mid \theta \in X(G)\}$  forms a basis for  $V$ .

(ii)  $A\hat{\theta} = \Delta_\theta \hat{\theta}$  for  $\theta \in X(G)$ , where  $A$  is the adjacency matrix and

$$\Delta_\theta = \sum_{g \in \Delta} \theta(g).$$

In particular, the eigenvalues of  $\Gamma$  are precisely

$$\Delta_\theta \mid \theta \in X(G)\}.$$

*Proof.*

(i) Claim: For every  $\theta \in X(G)$ , let

$$s := \sum_{g \in G} \theta(g^{-1}) = \begin{cases} |G| & \text{if } \theta = 1 \\ 0 & \text{if } \theta \neq 1. \end{cases}$$

*Pf.* Clear if  $\theta = 1$ .

Let  $\theta \neq 1$ . Then  $\theta(h) \neq 1$  for some  $h \in G$ .

$$s \cdot \theta(h) = \left( \sum_{g \in G} \theta(g^{-1}) \right) \theta(h) = \sum_{g \in G} \theta(g^{-1}h) = \sum_{g' \in G} \theta(g'^{-1}) = s.$$

Since  $\theta(h) \neq 1$ ,  $s = 0$ .

Claim.  $\theta(g^{-1}) = \overline{\theta(g)}$  for every  $\theta \in X(G)$  and every  $g \in G$ .

Since  $\theta(g) \in \mathbb{C}$  is a root of 1,

$$|\theta(g)|^2 = \theta(g)\overline{\theta(g)} = 1.$$

On the other hand, since  $\theta$  is a homomorphism,

$$\theta(g)\theta(g^{-1}) = \theta(1) = 1.$$

Hence  $\theta(g^{-1}) = \overline{\theta(g)}$ .

Now

$$\langle \widehat{\theta_1}, \widehat{\theta_2} \rangle = \sum_{g \in G} \theta_1(g^{-1}) \overline{\theta_2(g^{-1})} \quad (4.1)$$

$$= \sum_{g \in G} \theta_1(g^{-1}) \theta_2(g) \quad (4.2)$$

$$= \sum_{g \in G} \theta_1 \theta_2^{-1}(g^{-1}) \quad (4.3)$$

$$= \begin{cases} |G| & \text{if } \theta_1 \theta_2^{-1} = 1 \\ 0 & \text{if } \theta_1 \theta_2^{-1} \neq 1. \end{cases} \quad (4.4)$$

Since  $|G| = |X(G)|$  by Lemma 3.1, and  $\widehat{\theta_i}$ 's are orthogonal nonzero elements in  $V$ , they form a basis of  $V$ .

(ii) Let  $\Delta = \{g_1, \dots, g_r\}$ . Then

$$A\hat{\theta} = A \left( \sum_{g \in G} \theta(g^{-1}g) \right) \quad (4.5)$$

$$= \sum_{g \in G} \theta(g^{-1})(gg_1 + \cdots + gg_r) \quad (\Gamma(g) = \{gg_1, \dots, gg_r\}) \quad (4.6)$$

$$= \sum_{i=1}^r \left( \sum_{g \in G} \theta(g^{-1})(gg_i) \right) \quad (4.7)$$

$$= \sum_{i=1}^r \left( \sum_{g \in G} \theta(g_i g_i^{-1} g^{-1})(gg_i) \right) \quad (4.8)$$

$$= \sum_{i=1}^r \left( \sum_{g \in G} \theta(g_i) \theta((gg_i)^{-1}) gg_i \right) \quad (4.9)$$

$$= \sum_{i=1}^r \theta(g_i) \sum_{h \in G} \theta(h^{-1}) h \quad (4.10)$$

$$= \Delta_{\theta} \cdot \hat{\theta}. \quad (4.11)$$

Since  $\{\hat{\theta} \mid \theta \in X(G)\}$  forms a basis, the eigenvalues of  $\Gamma$  are precisely,

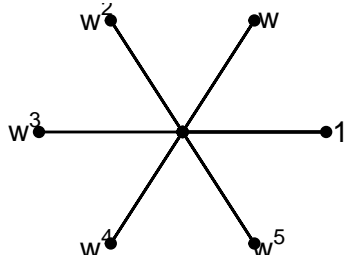
$$\{\Delta_{\theta} \mid \theta \in X(G)\}.$$

This completes the proof.

□

**Example 4.1.** Let  $G = \langle a \mid a^6 = 1 \rangle$ , and  $\Delta = \{a, a^{-1}\}$ . Pick a primitive 6-th root of 1,  $\omega$ . Then

$$X(G) = \{\theta^i \mid 0 \leq i \leq 5\} \quad \text{such that} \quad \theta(a) = \omega, \quad \omega + \omega^{-1} = 1.$$



$\varphi \in X(G)$	$\varphi(a)$	$\Delta_\varphi = \theta(a) + \theta(a)^{-1}$
1	1	2
$\theta$	$\omega$	$\omega + \omega^{-1} = 1$
$\theta^2$	$\omega^2$	-1
$\theta^3$	$\omega^3 = -1$	-2
$\theta^4$	$\omega^4$	-1
$\theta^5$	$\omega^5$	1

$$\text{Spec}(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

**Example 4.2.**  $D$ -cube,  $H(D, 2)$ . Let

$$X = \{(a_1, \dots, a_D) \mid a_i \in \{1, -1\}, 1 \leq i \leq D\},$$

$$E = \{xy \mid x, y \in X, x, y: \text{different in exactly one coordinate}\}.$$

Also  $H(D, 2)$  is a Cayley graph  $\Gamma(G, \Delta)$ , where

$$G = G_1 \oplus G_2 \oplus \dots \oplus G_D,$$

$$G_i = \langle a_i \mid a_i^2 = 1 \rangle, \quad \Delta = \{a_1, \dots, a_D\}.$$

**Homework:** The spectrum of  $H(D, 2)$  is

$$\begin{pmatrix} \theta_0 & \theta_1 & \dots & \theta_D \\ m_0 & m_1 & \dots & m_D \end{pmatrix},$$

where

$$\theta_i = D - 2i \quad (0 \leq i \leq D), \quad m_i = \binom{D}{i}.$$

*Remark.* Let  $\theta \in X(G)$ . Then  $\theta : X \rightarrow \{\pm 1\}$ . If

$$\nu(\theta) = |\{i \mid \theta(a_i) = -1\}|,$$

then  $\Delta_\theta = D - 2i$ . Since there are  $\binom{D}{i}$  such  $\theta$ , we have the assertion.

We want to compute the subconstituent algebra for  $H(D, 2)$ . First, we make a few observations about arbitrary graphs.

Let  $\Gamma = (X, E)$  be any graph,  $A$ , the adjacency matrix of  $\Gamma$ , and  $V$ , the standard module over  $K = \mathbb{C}$ .

Fix a base  $x \in X$ . Write  $E_i^* = E_i^*(x)$ , and

$$T \equiv T(x) = \text{the algebra generated by } A, E_0^*, E_1^*, \dots$$

**Definition 4.1.** Let  $W$  be any irreducible  $T$ -module ( $\subseteq V$ ). Then the endpoint  $r \equiv r(W)$  satisfied

$$r = \min\{i \mid E_i^* W \neq 0\}.$$

The diameter  $d = d(W)$  satisfied

$$d = |\{i \mid E_i^* W \neq 0\}| - 1.$$

**Lemma 4.1.** *With the above notation, let  $W$  be an irreducible  $T$ -module. Then*

- (i)  $E_i^* A E_j^* = 0$  if  $|i - j| = 1$ ,  $E_i^* A E_j^* \neq 0$  if  $|i - j| = 1$ ,  $0 \leq i, j \leq d(x)$ .
- (ii)  $A E_j^* W \subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$ ,  $0 \leq j \leq d(x)$ . ( $E_i^* W = 0$  if  $i < j$  or  $i > d(x)$ .)
- (iii)  $E_j^* W \neq 0$  if  $r \leq j \leq r + d$ ,  $= 0$  if  $0 \leq j \leq r$  or  $r + d < j \leq d(x)$ .
- (iv)  $E_i^* A E_j^* W \neq 0$ , if  $|i - j| = 1$  ( $r \leq i, j \leq r + d$ ).

*Proof.*

- (i) Pick  $y \in X$  with  $\partial(x, y) = j$ . We want to find  $E_i^* A E_j^* \hat{y}$ . Note,

$$E_j^* \hat{y} = \begin{cases} 0 & \text{if } \partial(x, y) \neq j \\ \hat{y} & \text{if } \partial(x, y) = j. \end{cases}$$

$$E_i^* A E_j^* \hat{y} = E_i^* A \hat{y} \tag{4.12}$$

$$= E_i^* \sum_{z \in X, yz \in E} \hat{z} \tag{4.13}$$

$$= \sum_{z \in X, yz \in E, \partial(x, z) = i} \hat{z} \tag{4.14}$$

$$= 0 \text{ if } |i - j| > 1 \quad \text{by triangle inequality.} \tag{4.15}$$

If  $|i - j| = 1$ , there exist  $y, y' \in X$  such that  $\partial(x, y) = j$ ,  $\partial(x, y') = i$ ,  $yy' \in E$  by connectivity of  $\Gamma$ . Hence (4.14) contains  $\widehat{yy'}$  and (4.14) is not equal to zero.

- (ii) We have

$$A E_j^* W = \left( \sum_{i=0}^{d(x)} E_i^* \right) A E_j^* W \tag{4.16}$$

$$= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W \tag{4.17}$$

$$\subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W. \tag{4.18}$$

- (iii) Suppose  $E_j^* W = 0$  for some  $j$  ( $r \leq j \leq r + d$ ). Then  $r < j$  by the definition of  $r$ . Set

$$\widetilde{W} = E_r^*W + E_{r+1}^*W + \cdots + E_{j-1}^*W.$$

Observe  $0 \subsetneq \widetilde{W} \subsetneq W$ . Also  $A\widetilde{W} \subseteq \widetilde{W}$  by (ii), and  $E_i^*\widetilde{W} \subseteq \widetilde{W}$  for every  $i$  by construction.

Thus,  $T\widetilde{W} \subseteq \widetilde{W}$ , contradicting  $W$  being irreducible.

□

## Chapter 5

# $T$ -Modules of $H(D, 2)$ , I

Friday, January 29, 1993

Let  $\Gamma = (X, E)$  be a graph,  $A$  the adjacency matrix, and  $V$  the standard module over  $K = \mathbb{C}$ .

Fix a base  $x \in X$  and write  $E_i^* \equiv E_i^*(x)$ , and  $T \equiv T(x)$ .

Let  $W$  be an irreducible  $T$ -module with endpoint  $r := \min\{i \mid E_i^*W \neq 0\}$  and diameter  $d := |\{i \mid E_i^*W \neq 0\}| - 1$ .

We have

$$E_i^*W \neq 0 \quad r \leq i \leq r + d \quad (5.1)$$

$$= 0 \quad 0 \leq i < r \text{ or } r + d < i \leq d(x). \quad (5.2)$$

Claim:  $E_i^*AE_j^*W \neq 0$  if  $|i - j| = 1$  for  $r \leq i, j \leq r + d$ . (See Lemma 4.1.)

Suppose  $E_{j+1}^*AE_j^*W = 0$  for some  $j$  with  $r \leq j < r + d$ . Observe that

$$\tilde{W} = E_r^*W + \cdots + E_j^*W$$

is  $T$ -invariant with

$$0 \subsetneq \tilde{W} \subsetneq W.$$

Because  $A\tilde{W} \subseteq \tilde{W}$  since  $AE_j^*W \subseteq E_{j-1}^*W + E_j^*W$ ,

$$E_k^*\tilde{W} \subseteq \tilde{W} \quad \text{for all } k,$$

we have  $T\tilde{W} \subseteq \tilde{W}$ .

Suppose  $E_{i-1}^*AE_i^*W = 0$  for some  $i$  with  $r \leq i < r + d$ .

Similarly,

$$\tilde{W} = E_i^*W + \cdots + E_{r+d}^*W$$

is a  $T$ -module with  $0 \subsetneq \tilde{W} \subsetneq W$ .

**Definition 5.1.** Let  $\Gamma$ ,  $E_i^*$ , and  $T$  be as above. Irreducible  $T$ -modules  $W$  and  $W'$  are isomorphic whenever there is an isomorphism  $\sigma : W \rightarrow W'$  of vector spaces such that  $a\sigma = \sigma a$  for all  $a \in T$ .

Recall that the standard module  $V$  is an orthogonal direct sum of irreducible  $T$ -modules  $W_1 \oplus W_2 \oplus \dots$ . Given  $W$  in this list, the multiplicity of  $W$  in  $V$  is

$$|\{j \mid W_j \simeq W\}|.$$

*Remark.* It is known that the multiplicity does not depend on the decomposition.

Now assume that  $\Gamma$  is the  $D$ -cube,  $H(D, 2)$  with  $D \geq 1$ . View

$$X = \{a_1 \cdots a_D \mid a_i \in \{1, -1\}, 1 \leq i \leq D\}, \quad (5.3)$$

$$E = \{xy \mid x, y \in X, x, y \text{ differ in exactly 1 coordinate.}\}. \quad (5.4)$$

Find  $T$ -modules.

Claim:  $H(D, 2)$  is bipartite with a partition  $X = X^+ \cup X^-$ , where

$$X^+ = \{a_1 \cdots a_D \in X \mid \prod a_i > 0\} \quad (5.5)$$

$$X^- = \{a_1 \cdots a_D \in X \mid \prod a_i < 0\} \quad (5.6)$$

Observe: for all  $y, z \in X$ ,

$$\partial(y, z) = i \Leftrightarrow y, z \text{ differ in exactly } i \text{ coordinates with } 0 \leq i \leq D.$$

Here, the diameter of  $H(D, 2) = D = d$  for all  $x \in X$ .

**Theorem 5.1.** Let  $\Gamma = H(D, 2)$  be as above. Fix  $x \in X$ , and write  $E_i^* = E_i^*(x)$ , and  $T = T(x)$ .

Let  $W$  be an irreducible  $T$ -module with endpoint  $r$ , and diameter  $d$  with  $0 \leq r \leq r + d \leq D$ .

(i)  $W$  has a basis  $w_0, w_1, \dots, w_d$  with  $w_i \in E_{i+r}^* W$  for  $0 \leq i \leq d$ . With respect to which the matrix representing  $A$  is

$$\begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & d-1 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \ddots & 0 & 2 & 0 \\ 0 & 0 & 0 & \cdots & d-1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & d & 0 \end{pmatrix}$$



(ii)  $d = D - 2r$ . In particular,  $0 \leq r \leq D/2$ .

(iii) Let  $W'$  denote an irreducible  $T$ -module with endpoint  $r'$ . Then  $W$  and  $W'$  are isomorphic as  $T$ -modules if and only if  $r = r'$ .

(iv) The multiplicity of the irreducible  $T$ -module with endpoint  $r$  is

$$\binom{D}{r} - \binom{D}{r-1} \quad \text{if } 1 \leq r \leq D/2,$$

and 1 if  $r = 0$ .

*Proof.* Recall that  $\Gamma$  is vertex transitive. It is a Cayley graph.

Hence without loss of generality, we may assume that  $x = \overbrace{11 \cdots 1}^D$ .

Notation: Set  $\Omega = \{1, 2, \dots, D\}$ . For every subset  $S \subseteq \Omega$ , let

$$\hat{S} = a_1 \cdots a_d \in X \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S. \end{cases}$$

In particular,  $\hat{\emptyset} = x$  and

$$|S| = i \Leftrightarrow \partial(x, \hat{S}) = i \Leftrightarrow \hat{S} \in E_i^* V.$$

For all  $S, T \subseteq \Omega$ , we say  $S$  covers  $T$  if and only if  $S \supseteq T$  and  $|S| = |T| + 1$ .

Observe that  $\hat{S}, \hat{T}$  are adjacent in  $\Gamma$  if and only if either  $T$  covers  $S$  or  $S$  covers  $T$ .

Define the ‘raising matrix’

$$R = \sum_{i=0}^D E_{i+1}^* A E_i^*.$$

Observe that

$$R E_i^* V \subseteq E_{i+1}^* V \quad \text{for } 0 \leq i \leq D, \quad \text{and } E_{D+1}^* V = 0.$$

Indeed for any  $S \subseteq \Omega$  with  $|S| = i$ ,

$$R \hat{S} = R E_i^* \hat{S} \tag{5.7}$$

$$= E_{i+1}^* A \hat{S} \tag{5.8}$$

$$= \sum_{T_1 \subseteq \Omega, S \text{ covers } T_1} E_{i+1}^* \hat{T}_1 + \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T} \tag{5.9}$$

$$= \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T}. \tag{5.10}$$

Define the ‘lowering matrix’

$$L = \sum_{i=0}^D E_{i-1}^* A E_i^*.$$

Observe that

$$L E_i^* V \subseteq E_{i-1}^* V \text{ for } 0 \leq i \leq D, \text{ and } E_{-1}^* V = 0.$$

Indeed for any  $S \subseteq \Omega$ ,

$$L \hat{S} = \sum_{T \subseteq \Omega, S \text{ covers } T} \hat{T}.$$

Observe that  $A = L + R$ .

For convenience, set

$$A^* = \sum_{i=0}^D (D - 2i) E_i^*.$$

Claim: The following hold.

- (a)  $LR - RL = A^*$ .
- (b)  $A^*L - LA^* = 2L$ .
- (c)  $A^*R - RA^* = -2R$ .

In particular  $\text{Span}(R, L, A^*)$  is a ‘representation of Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$ .

*Remark* (Lie Algebra  $\mathfrak{sl}_2(\mathbb{C})$ ).

$$\mathfrak{sl}_2(\mathbb{C}) = \{X \mid \text{Mat}(\mathbb{C}) \mid \text{tr}(X) = 0\}.$$

For  $X, Y \in \mathfrak{sl}_2(\mathbb{C})$ , define a binary operation  $[X, Y] = XY - YX$ .

$$A^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then these satisfy the relations (a) - (c) above.

*Proof of Claim.* Apply both sides to  $\hat{S}$  ( $S \subseteq \Omega$ ). Say  $|S| = i$ .

*Proof of (a):*

$$(LR - RL)\hat{S} = L \left( \sum_{\substack{T \subseteq \Omega, T \text{ covers } S \\ (D-i \text{ of them})}} \hat{T} \right) - R \left( \sum_{\substack{U \subseteq \Omega, S \text{ covers } U \\ (i \text{ of them})}} \hat{T} \right) \quad (5.11)$$

$$= (D - i)\hat{S} + \sum_{V \subseteq \Omega, |V|=i, |S \cap V|=i-1} \hat{V} - \left( i\hat{S} + \sum_{V \subseteq \Omega, |V|=i, |S \cap V|=i-1} \hat{V} \right) \quad (5.12)$$

$$= (D - 2i)\hat{S} \quad (5.13)$$

$$= A^*\hat{S}. \quad (5.14)$$

*Proof of (b):*

$$(A^*L - LA^*)\hat{S} = (D - 2(i - 1))L\hat{S} - (D - 2i)L\hat{S} \quad (\text{since } L\hat{S} \in E_{i-1}^*V) \quad (5.15)$$

$$= 2L\hat{S}. \quad (5.16)$$

*Proof of (c):*

$$(A^*R - RA^*)\hat{S} = (D - 2(i + 1))R\hat{S} - (D - 2i)R\hat{S} \quad (\text{since } R\hat{S} \in E_{i+1}^*V) \quad (5.17)$$

$$= 2R\hat{S}. \quad (5.18)$$

Let  $W$  be an irreducible  $T$ -module with endpoint  $r$  and diameter  $d$  ( $0 \leq r \leq r + d \leq D$ ).

*Proof of (i) and (ii):*

Pick  $0 \neq w \in E_r^*W$ .

Claim:  $LRw = (D - 2r)w$ .

*Pf.*

$$LRw = (A^* + RL)w \quad (\text{by Claim (a)}) \quad (5.19)$$

$$= A^*w \quad (Lw \in E_{r-1}^*W = 0) \quad (5.20)$$

$$(D - 2r)w. \quad (5.21)$$

Define

$$w_i = \frac{1}{i!}R^i w \in E_{r+i}^*W \quad (0 \leq i \leq d).$$

Then,

$$Rw_i = (i + 1)w_{i+1} \quad (0 \leq i \leq d) \quad (5.22)$$

$$Rw_d = 0 \quad (\text{by definition of } d) \quad (5.23)$$

Claim:  $Lw_0 = 0$  and

$$Lw_i = (D - 2r - i + 1)w_{i-1} \quad (1 \leq i \leq d).$$

*Pf.* We prove by induction on  $i$ . The case  $i = 0$  is trivial, and the case  $i = 1$  follows from above claim. Let  $i \geq 2$ ,

$$Lw_i = \frac{1}{i}LRw_{i-1} = \frac{1}{i}(A^* + RL)w_{i-1} \quad (\text{by Claim (a)}) \quad (5.24)$$

$$(\text{by induction hypothesis}) \quad (5.25)$$

$$= \frac{1}{i}((D - 2(r + i - 1))w_{i-1} + (D - 2r - (i - 1) + 1)Rw_{i-2}) \quad (Rw_{i-2} = (i - 1)w_{i-1}) \quad (5.26)$$

$$= \frac{1}{i}i(D - 2r - i + 1)w_{i-1} \quad (5.27)$$

$$= (D - 2r - i + 1)w_{i-1}. \quad (5.28)$$

Claim:  $w_0, \dots, w_d$  is a basis for  $W$ .

*Pf.* Let  $W' = \text{Span}\{w_0, \dots, w_d\}$ . Then  $W'$  is  $R$  and  $L$  invariant. So it is  $A = R + L$  invariant.

Also it is  $E_i^*$ -invariant for every  $i$ .

Hence  $W'$  is a  $T$ -module.

Since  $W$  is irreducible,  $W' = W$ .

As  $w_i$ 's are orthogonal, they are linearly independent. Note that  $w_i \neq 0$  by the definition of  $d$  and Lemma 4.1 (iv).

Claim:  $d = D - 2r$ .

*Pf.* By (a),

$$0 = (LR - RL - A^*)w_d \quad (5.29)$$

$$= 0 - (D - 2r - d + 1)Rw_{d-1} - (D - 2(r + d))w_d \quad (5.30)$$

$$= -d(D - 2r - d + 1)w_d - (D - 2(r + d))w_d \quad (5.31)$$

$$= (-dD + 2rd + d^2 - d - D + 2r + 2d)w_d \quad (5.32)$$

$$= (d^2 + (2r - D + 1)d + 2r - D)w_d \quad (5.33)$$

$$= (d + 2r - D)(d + 1)w_d. \quad (5.34)$$

Hence  $d = D - 2r$ .

Therefore, with respect to a basis  $w_0, w_1, \dots, w_d$ ,  $A = L + R$ ,  $w_{-1} = w_{d+1} = 0$ ,

$$Lw_i = (d - i + 1)w_{i-1}, \quad Rw_i = (i + 1)w_{i+1}.$$

$$L = \begin{pmatrix} 0 & d & 0 & \dots & 0 & 0 \\ 0 & 0 & d-1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \dots & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 1 \\ 0 & 0 & 0 & \dots & d & 0 \end{pmatrix}.$$

This completes the proof of (i) and (ii).  $\square$

## Chapter 6

# $T$ -Modules of $H(D, 2)$ , II

Monday, February 1, 1993

*Proof of Theorem 5.1 Continued.*

(iii) Let  $r = r'$ ,

$w_0, \dots, w_d$ : a basis for  $W$  with  $w_i \in E_i^*W$ , and

$w'_0, \dots, w'_d$ : a basis for  $W'$  with  $w'_i \in E_i^*W'$ .

Then  $d = D - 2r = D - 2r' = d'$ , and

$$\sigma : W \rightarrow W' \quad (w_i \mapsto w'_i)$$

is an isomorphism of  $T$ -modules by (i).

If  $r \neq r'$ , then

$$d = D - 2r \neq D - 2r' = d',$$

hence,  $\dim W \neq \dim W'$ .

(iv) Let  $W_i$  be the irreducible  $T$ -module with endpoint  $i$ . Then

$$\dim E_r^*V = \binom{D}{r} = \sum_{i=0}^r \text{mult}(W_i).$$

Hence, we have that

$$\text{mult}(W_r) = \binom{D}{r} - \binom{D}{r-1}$$

by induction on  $r$ .

□

**Theorem 6.1.** *Let  $\Gamma = H(D, 2)$  with  $D \geq 1$ . Fix a vertex  $x \in X$  and write*

$$E_i^* \equiv E_i^*(x), \quad T = T(x), \text{ and } A^* \equiv \sum_{i=0}^D (D - 2i) E_i^*.$$

*Let  $W$  be an irreducible  $T$ -module with endpoint  $r$  with  $0 \leq r \leq D/2$ . Then,*

*(i)  $W$  has a basis*

$$w_0^*, w_1^*, \dots, w_d^* \quad (d = D - 2r), \quad \text{such that } w_i^* \in E_{i+r} W \quad (0 \leq i \leq d)$$

*with respect to which the matrix corresponding to  $A^*$  is*

$$\begin{pmatrix} 0 & d & 0 & & & & \\ 1 & 0 & d-1 & & & & \\ 0 & 2 & 0 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 0 & 2 & 0 \\ & & & & d-1 & 0 & 1 \\ & & & & 0 & d & 0 \end{pmatrix}.$$

*In particular, / (ii)  $E_i A^* E_j = 0$  if  $|i - j| \neq 1$  for  $0 \leq i, j \leq D$ .*

*Proof.* We use the notation,

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \quad (= -[\beta, \alpha]).$$

Recall that

- (a)  $[L, R] = A^*$ ,
- (b)  $[A^*, L] = wL$ ,
- (c)  $[A^*, R] = -2R$ ,

and  $A = L + R$ .

Write (a) – (c) in terms of  $A$  and  $A^*$ , we have,

$$[A, A^*] = [L, A^*] + [R, A^*] = 2(R - L).$$

$$\begin{cases} R + L &= A \\ R - L &= [A, A^*]/2. \end{cases}$$

Hence,

$$R = \frac{1}{4}(2A + [A, A^*]) \quad \text{and} \tag{6.1}$$

$$L = \frac{1}{4}(2A - [A, A^*]). \tag{6.2}$$

Now (a), (b) become

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0 \quad (6.3)$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0 \quad (6.4)$$

*Pf.* By (b),

$$2A - AA^* + A^*A = 4L \quad (6.5)$$

$$= 2[A^*, L] \quad (6.6)$$

$$= A^* \frac{2A - [A, A^*]}{2} - \frac{2A - [A, A^*]}{2} A^* \quad (6.7)$$

So we have ((6.4))

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0.$$

By (a),

$$-16A^* = [2A + [A, A^*], 2A - [A, A^*]] \quad (6.8)$$

$$(2A + [A, A^*])(2A - [A, A^*]) - (2A - [A, A^*])(2A + [A, A^*]) \quad (6.9)$$

$$= [4A^2 - 2A[A, A^*] + [A, A^*](2A) - [A, A^*]^2] \quad (6.10)$$

$$- 4A^2 - 2A[A, A^*] + [A, A^*](2A) + [A, A^*]^2 \quad (6.11)$$

$$= -4A^2A^* + 4AA^*A + 4AA^*A - 4A^*A^2. \quad (6.12)$$

So,

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0.$$

Claim:  $E_i^*A^*E_j = 0$  if  $|i - j| \neq 1$  for  $0 \leq i, j \leq D$ .

*Pf.* We have,

$$0 = E_i(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_j \quad (6.13)$$

$$= E_iA^*E_j(\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4) \quad (6.14)$$

$$(AE_j = \theta_jE_j, E_iA = (AE_j)^\top = (\theta_iE_i)^\top = \theta_iE_i) \quad (6.15)$$

$$= E_iA^*E_j(\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) \quad (6.16)$$

$$= E_iA^*E_j(D - 2i - (D - 2j) - 2)(D - 2i - (D - 2j) + 2) \quad (6.17)$$

$$(\theta_k = D - 2k) \quad (6.18)$$

$$= E_iA^*E_j \cdot 4(i - j + 1)(i - j - 1) \quad (6.19)$$

and  $i - j + 1 \neq 0$ ,  $i - j - 1 \neq 0$ . Hence,  $E_i^*A^*E_j = 0$ .

Now define “dual raising matrix”,

$$R^* = \sum_{i=0}^D E_{i+1}A^*E_i.$$

So,

$$R^*E_iV \subseteq E_{i+1}V, \quad (0 \leq i \leq D, E_{D+1}V = 0).$$

Define “dual lowering matrix”

$$L^* = \sum_{i=0}^D E_{i-1}A^*E_i.$$

Then

$$L^*E_iV \subseteq E_{i-1}V \quad (0 \leq i \leq D, E_{-1}V = 0).$$

Observe that

$$A^* = \left( \sum_{i=0}^D E_i \right) A^* \left( \sum_{j=0}^D E_j \right) = L^* + R^*$$

by Claim 1.

Claim 2. We have | (a)  $[L^*, R^*] = A$ , | (b)  $[A, L^*] = 2L^*$ , | (c)  $[A, R^*] = -2R^*$ .

*Pf.* (b)

$$AL^* - L^*A = \sum_{i=0}^D (AE_{i-1}A^*E_i - E_{i-1}A^*E_iA) \quad (6.20)$$

$$= \sum_{i=0}^D E_{i-1}A^*E_i(\theta_{i-1} - \theta_i) \quad (6.21)$$

$$(\theta_k = D - 2k, \quad \theta_{i-1} - \theta_i = 2I - 2(i-1) = 2) \quad (6.22)$$

$$= 2L^*. \quad (6.23)$$

(c) Similar.

*Remark.*

$$AR^* - R^*A = \sum_{i=0}^D (AE_{i+1}A^*E_i - E_{i+1}A^*E_iA) \quad (6.24)$$

$$= \sum_{i=0}^D E_{i+1}A^*E_i(\theta_{i+1} - \theta_i) \quad (6.25)$$

$$= 2R^*. \quad (6.26)$$

(a) We have, by (b), (c)

$$[A, A^*] = [A, L^*] + [A, R^*] = 2(L^* - R^*). \quad (6.27)$$

Since  $A^* = L^* + R^*$ ,

$$R^* = \frac{2A^* + [A^*, A]}{4}, \quad L^* = \frac{2A^* - [A^*, A]}{4}.$$

Now (a) is seen to be equivalent to ((6.4)) upon evaluation. This proves Claim 2.



*Remark.*

$$[L^*, R^*] = \frac{1}{16}((2A^* - [A^*, A])(2A^* + [A^*, A]) - (2A^* + [A^*, A])(2A^* - [A, A^*])) \quad (6.28)$$

$$= \frac{1}{16}(4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* - [A^*, A]^2 - 4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* + [A^*, A]^2) \quad (6.29)$$

$$= \frac{1}{4}(A^{*2}A - 2A^*AA^* + AA^{*2}) \quad (6.30)$$

$$= A, \quad (6.31)$$

by ((6.4)).

Now apply same argument as for ((6.3)), ((6.4)) of Theorem 5.1 and observe  $A^*$  has  $D + 1$  distinct eigenvalues. So,

$$A^* = \sum_{i=0}^D (D - 2i)E_i^*$$

generates

$$M^* = \text{Span}(E_0^*, \dots, E_D^*).$$

Hence,  $E_0, \dots, E_D, A^*$  generates  $T$ .

Take an irreducible  $T$ -module  $W$  with endpoint  $r$  with  $0 \leq r \leq D/2$ . Set  $t = \min\{i \mid E_i W\}$ .

Pick  $0 \neq w_0^* \in E_t W$ . Set

$$w_i^* = \frac{1}{i!} R^{*i} w_0^* \in E_{t+i} W \quad \text{for all } i.$$

Then,

$$R^* w_i^* = (i + 1)w_{i+1}^* \quad \text{for all } i.$$

By (a), we get by induction,  $L^* w_i^* = (D - 2t - i + 1)w_{i-1}^*$ ,

$$L^* w_i^* = \frac{1}{i} L^* R^* w_{i-1}^* \quad (6.32)$$

$$= \frac{1}{i} (A + R^* L^*) w_{i-1}^* \quad (6.33)$$

$$= \frac{1}{i} ((D - 2(t + i - 1))w_{i-1}^* + (i - 1)(D - 2t - i + 2)w_{i-1}^*) \quad (6.34)$$

$$= (D - 2t - i + 1)w_{i-1}^*. \quad (6.35)$$

So  $\text{Span}(w_0^*, w_1^*, \dots)$  is  $L^*, R^*, A^*$ -invariant. Hence,  $W = (\text{Span})(w_0^*, w_1^*, \dots, w_d^*)$ ,  $w_0^*, w_1^*, \dots, w_d^* \neq 0$ ,  $w_i^* = 0$  for every  $i > d$  by dimension.

Thus  $d = D - 2t$ .

*Pf.*

$$(D - 2(t + d))w_d^* = Aw_d^* \quad (6.36)$$

$$= (L^*R^* - R^*L^*)w_d^* \quad (6.37)$$

$$= -(D - 2t - d + 1)R^*w_{d-1}^* \quad (6.38)$$

$$= -(D - 2t - d + 1)dw_d^*. \quad (6.39)$$

Hence,

$$0 = d^2 + (2t - D - 1 + 2)d - (D - 2t) = (d - D + 2t)(d + 1)$$

So  $d = D - 2t$ . □

**Definition 6.1.** For any graph  $\Gamma = (X, E)$ , pick a vertex  $x \in X$  and set  $E_i^* \equiv E_i^*(x)$  and  $T \equiv T(x)$ .

- (i) an irreducible  $T$ -module  $W$  is thin if  $\dim E_i^*W \leq 1$  for every  $i$ ,
- (ii)  $\Gamma$  is thin with respect to  $x$ , if every irreducible  $T(x)$ -module is thin,
- (iii) an irreducible  $T$ -module  $W$  is dual thin if  $\dim E_iW \leq 1$  for every  $i$ ,
- (iv)  $\Gamma$  is dual thin with respect to  $x$ , if every irreducible  $T(x)$ -module is dual thin.

Observe:  $H(D, 2)$  is thin, dual thin with respect to each  $x \in X$ .

**Definition 6.2.** With above notation, write  $D \equiv D(x)$ .

- (i) an ordering  $E_0, E_1, \dots, E_R$  of primitive idempotents of  $\Gamma$  is restricted if  $E_0$  corresponds to the maximal eigenvalue.

Fix a restricted ordering,

- (ii)  $\Gamma$  is  $Q$ -polynomial with respect to  $x$ , above ordering if there exists  $A^* \equiv A^*(x)$  such that

$$(a) E_0^*V, \dots, E_D^*V \text{ are the maximal eigenspaces for } A^*.$$

$$(b) E_iA^*E_j = 0 \text{ if } |i - j| > 1 \text{ for } 0 \leq i, j \leq R.$$

Observe  $H(D, 2)$  is  $Q$ -polynomial with respect to the natural ordering of the idempotents and every vertex.

**Program.** Study graphs that are thin and  $Q$ -polynomial with respect to each vertex.

(In fact, thin with respect to  $x$  implies dual thin with respect to  $x$ .)

Get a situation like  $H(D, 2)$ , where  $T$  is generated by  $A, A^*$ . Except  $\mathfrak{sl}_s(\mathbb{C})$  is replaced by a quantum Lie algebra.

## Chapter 7

# The Johnson Graph $J(D, N)$

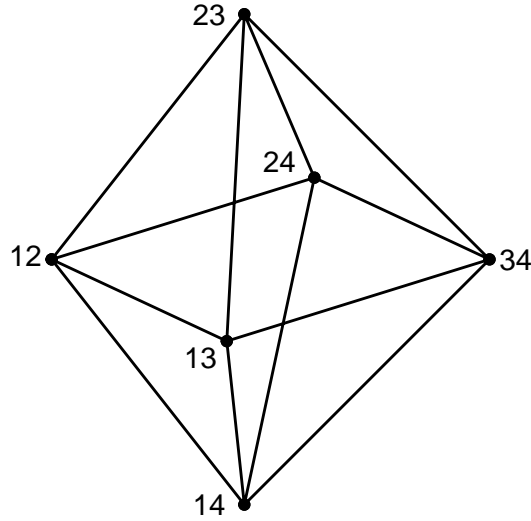
Wednesday, February 3, 1993

**Definition 7.1.** The Johnson graph,  $\Gamma = J(D, N)$  ( $1 \leq D \leq N - 1$ ) satisfies

$$X = \{S \mid S \subset \Omega, |S| = D\} \quad \text{where } \Omega = \{1, 2, \dots, N\} \quad (7.1)$$

$$E = \{ST \mid S, T \in X, |S \cap T| = D - 1\}. \quad (7.2)$$

**Example 7.1.**  $J(2, 4)$



**Note 1.** The symmetric group  $S_N$  acts on  $\Omega$ .  $S_N \subseteq \text{Aut}(\Gamma)$  acts vertex transitively on  $\Gamma$ .

**Note 2.**  $\Gamma = J(D, N)$  is isomorphic to  $\Gamma' = J(N - D, N)$ .

$$\Gamma = (X, E) \qquad \Gamma' = (X', E') \qquad (7.3)$$

$$X \ni S \quad \longrightarrow \quad \bar{S} = \Omega \quad S \in X' \qquad (7.4)$$

This correspondence induces an isomorphism of graphs.

*Pf.*

$$ST \in E \Leftrightarrow |S \cap T| = D - 1 \qquad (7.5)$$

$$\Leftrightarrow |\Omega - (S \cup T)| = N - D - 1 \qquad (7.6)$$

$$\Leftrightarrow |\bar{S} \cap \bar{T}| = N - D - 1 \qquad (7.7)$$

$$\Leftrightarrow \bar{S}\bar{T} \in E' \qquad (7.8)$$

Hence, without loss of generality, assume

$$D \leq N/2 \quad \text{for } J(D, N).$$

We still need the eigenvalues of  $J(D, N)$  for certain problem later in the course. We can get these eigenvalues from our study of  $H(D, 2)$ .

**Lemma 7.1.** *The eigenvalues for  $J(D, N)$  with  $1 \leq D \leq N/2$  are give by*

$$\theta_i = (N - D - i)(D - i) - i \quad (0 \leq i \leq D) \qquad (7.9)$$

$$m_i = \binom{D}{i} - \binom{N}{i-1}. \qquad (7.10)$$

*Proof.* Let

$$\Gamma_J \equiv J(D, N) = (X_J, E_J) \qquad (7.11)$$

$$\Gamma_H \equiv H(N, 2) = (X_H, E_H). \qquad (7.12)$$

Set  $x \equiv 11 \cdots 1 \in X_H$ .

Define  $\tilde{\Gamma} \equiv (\tilde{X}, \tilde{E})$ , where

$$\tilde{X} = \{y \in X_H \mid \partial_H(x, y) = D\} \quad \partial_H : \text{distance in } \Gamma_H \qquad (7.13)$$

$$\tilde{E} = \{yz \in X_H \mid \partial_H(y, z) = 2\}. \qquad (7.14)$$

Observe

$$X_J \rightarrow \tilde{X} \qquad (7.15)$$

$$S \mapsto \hat{S}, \qquad (7.16)$$

where

$$\hat{S} = a_1 \cdots a_N, \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$$

induces an isomorphism of graphs  $\Gamma_J \rightarrow \tilde{\Gamma}$ .

*Pf.*

$$ST \in E_J \Leftrightarrow |S \cap T| = D - 1 \quad (7.17)$$

$$\Leftrightarrow \partial_H(\hat{S}, \hat{T}) = 2 \quad (7.18)$$

$$\Leftrightarrow (\hat{S}, \hat{T}) \in \tilde{E}. \quad (7.19)$$

Identify,  $\Gamma_J$  with  $\tilde{\Gamma}$ . Then the standard module  $V_J$  of  $\Gamma_J$  becomes  $\tilde{V} = E_D^* V_H$ , where  $V_H$  is the standard module of  $\Gamma_H$ , and  $E_D^* \equiv E_D^*(x)$ .

Let  $R$  be the raising matrix with respect to  $x$  in  $\Gamma_H$ , and

let  $L$  be the lowering matrix with respect to  $x$  in  $\Gamma_H$ .

Recall

$$(RL - DE_D^*)|_{\tilde{V}}$$

is the adjacency map in  $\tilde{\Gamma}$ .

To find eigenvalues of  $\tilde{A}$ , pick any irreducible  $T(x)$ -module  $W$  with the endpoint  $r \leq D$ . Then by Theorem 5.1

$$\text{diam}(W) = N - 2r + 1.$$

Let  $w_0, w_1, \dots, w_{N-2r}$  denote a basis for  $W$  as in Theorem 5.1. Then,

$$w_{D-r} \in E_D^* W \subseteq \tilde{V}.$$

Observe:

$$\tilde{A}w_{D-r} = RLw_{D-r} - DE_D^*w_{D-r} \quad (7.20)$$

$$= R(N - 2r - D + r + 1)w_{D-r-1} - Dw_{D-r} \quad (7.21)$$

$$= ((N - D - r + 1)(D - r) - D)w_{D-r}. \quad (7.22)$$

Note that this is valid for  $D = r$  as well.

Hence,

$$\tilde{A}w_{D-r} = ((N - D - r)(D - r) - r)w_{D-r}.$$

Let

$$V_H = \sum W \quad (\text{direct sum of irreducible } T(x) \text{ - modules.})$$

Then,

$$V_J = E_D^* V_H \quad (7.23)$$

$$= \sum_{W: r(W) \leq D} E_D^* W \quad (7.24)$$

$$= \text{a direct sum of 1 dimensional eigenspaces for } \tilde{A}. \quad (7.25)$$

The eigenspace for eigenvalue

$$(N - D - r)(D - r) - r \quad (\text{monotonously decreasing with respect to } r)$$

appears with multiplicity

$$\binom{N}{r} - \binom{N}{r-1}$$

in this sum by Theorem 5.1 (iv).  $\square$

**Theorem 7.1.** *Let  $\Gamma = (X, E)$  be any graph. For a fixed vertex  $x \in X$ , let*

$$E_i^* \equiv E_i^*(x), \quad T \equiv T(x), \quad D \equiv D(x), \quad \text{and } K = \mathbb{C}.$$

*Then we have the following implications of conditions:*

$$TH \Leftrightarrow C \Leftarrow S \Leftarrow G.$$

*where*

*(TH)  $\Gamma$  is thinn with respect to  $x$ .*

*(C)  $E_i^*TE_i^*$  is commutative for every  $i$ ,  $(0 \leq i \leq D)$ .*

*(S)  $E_i^*TE_i^*$  is symmetric for every  $i$ ,  $(0 \leq i \leq D)$ .*

*(G) For every  $y, z \in X$  with  $\partial(x, y) = \partial(x, z)$ , there exists  $g \in \text{Aut}(\Gamma)$  such that*

$$gx = x, \quad gy = z, \quad gz = y.$$

*Proof.*

$(TH) \Rightarrow (C)$

Fix  $i$  with  $0 \leq i \leq D$ . Let

$V = \sum W$ . The standard module written as a direct sum of irreducible  $T$ -modules.

The,

$E_i^*V = \sum E_i^*W$ . The direct sum of 1-dimensional  $E_i^*TE_i^*$ -modules.

Since  $\dim E_i^*W = 1$ , for  $a, b \in E_i^*TE_i^*$ ,  $ab - ba|_{E_i^*W} = 0$ . Hence  $ab - ba = 0$ .

$(C) \Rightarrow (TH)$

Suppose  $\dim E_i^*W \geq 2$  for some irreducible  $T$ -module  $W$  with some  $i$  with  $1 \leq i \leq D$ .

Claim:  $E_i^*W$  is an irreducible  $E_i^*TE_i^*$ -module.

*Pf.* Suppose

$$0 \subsetneq U \subsetneq E_i^*W,$$

where  $U$  is a  $E_i^*TE_i^*$ -module. Then by the irreducibility,

$$TU = W.$$

So

$$U \supseteq E_i^*TE_i^*U = E_i^*TU = E_i^*W.$$

This is a contradiction.

Claim 2: Each irreducible  $S = E_i^*TE_i^*$ -module  $U$  has dimension 1. In particular,  $\Gamma$  is thin with respect to  $x$ .

*Pf.* Pick

$$0 \neq a \in E_i^*TE_i^*.$$

Since  $\mathbb{C}$  is algebraically closed,  $a$  has an eigenvector  $w \in U$  with eigenvalue  $\theta$ . Then,

$$(a - \theta I)U = (a - \theta I)Sw \tag{7.26}$$

$$= S(a - \theta I)w \tag{7.27}$$

$$= 0. \tag{7.28}$$

Hence,

$$a|_U = \theta I|_U \quad \text{for all } a \in S.$$

Thus each 1 dimensional subspace of  $U$  is an  $S$ -module. We have

$$\dim U = 1.$$

By Claim 1 and Claim 2, we have (TH).

□





## Chapter 8

# Thin Graphs

**Friday, February 5, 1993**

*Proof of Theorem 7.1 continued.*

(S)  $\Rightarrow$  (C)

Fix  $i$  and pick  $a, b \in E_i^* T E_i^*$ .

Since  $a$ ,  $b$  and  $ab$  are symmetric,

$$ab = (ab)^\top = b^\top a^\top = ba.$$

Hence  $E_i^* T E_i^*$  is commutative.

(G)  $\Rightarrow$  (S)

Fix  $i$  and pick  $a \in E_i^* T E_i^*$ . Pick vertices  $y, z \in X$ .

We want to show that

$$a_{yz} = a_{zy}.$$

We may assume that

$$\partial(x, y) = \partial(x, z) = i,$$

otherwise

$$a_{yz} = a_{zy} = 0.$$

By our assumption, there exists  $g \in G$  such that

$$g(y) = z, \quad g(z) = y, \quad g(x) = x.$$

Let  $\hat{g}$  denote the permutation matrix representing  $g$ , i.e.,

$$\hat{g}\hat{y} = \widehat{g(y)} \quad \text{for all } y \in X, \quad \hat{g} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow y.$$

If  $g \in \text{Aut}(\Gamma)$ , then

$$\hat{g}A = A\hat{g} \quad \text{Exercise.}$$

Also we have

$$\hat{g}E_j^* = E_j^*\hat{g} \quad (0 \leq j \leq D),$$

since

$$\partial(x, y) = \partial(g(x), g(y)) = \partial(x, g(y)).$$

Hence  $\hat{g}$  commutes with each element of  $T$ . We have

$$a_{yz} = (\hat{g}^{-1}a\hat{g})_{yz}, \quad (\hat{g})_{yz} = \begin{cases} 1 & g(z) = y \\ 0 & \text{else.} \end{cases} \quad (8.1)$$

$$= \sum_{y', z'} (\hat{g}^{-1})_{yy'} a_{y'z'} \hat{g}_{z'z} \quad (8.2)$$

$$(\text{zero except for } g^{-1}(y') = y, \ g(z) = z'.) \quad (8.3)$$

$$= a_{g(y)g(z)} \quad (8.4)$$

$$a_{zy}. \quad (8.5)$$

This proves Theorem 7.1. □

**Open Problem:** Find all the graphs that satisfy the condition (G) for every vertex  $x$ .

$H(N, 2)$  is one example, because

$$\text{Aut}\Gamma_{1\dots 1} \simeq S_\Omega, \quad x = (1 \dots 1), \Gamma_i(x) = \{\hat{S} \mid |S| = i\}.$$

Property (G) is clearly related to the distance-transitive property.

**Definition 8.1.** Let  $\Gamma = (X, E)$  be any graph.  $\Gamma$  with  $G \subseteq \text{Aut}(\Gamma)$  is said to be distance-transitive (or two-point homogeneous), whenever

$$\text{for all } x, x', y, y' \in X \text{ with } \partial(x, y) = \partial(x', y'),$$

there exists  $g \in G$  such that

$$g(x) = y, \quad g(x') = y'.$$

(This means  $G$  is as close to being doubly transitive as possible.)

**Lemma 8.1.** Suppose a graph  $\Gamma = (X, E)$  satisfies the property  $(G) = (G(x))$  for every  $x \in X$ . Then,

- (i) either
- (ia)  $\Gamma$  is vertex transitive; or
- (iia)  $\Gamma$  is bipartite ( $X = X^+ \cup X^-$ ) with  $X^+, X^-$  each an orbit of  $\text{Aut}(\Gamma)$ .
- (ii) if (ia) holds, then  $\Gamma$  is distance-transitive.

*Proof.* (i) Claim. Suppose  $y, z \in X$  are connected by a path of even length. Then  $y, z$  are in the same orbit of  $\text{Aut}(\Gamma)$ .

*Pf.* It suffices to assume that the path has length 2,  $y \sim w \sim z$ .

Now  $\partial(y, w) = \partial(w, z) = 1$ . So there exists  $g \in \text{Aut}(\Gamma)$  such that  $gw = w$ ,  $gy = z$ ,  $gz = y$ . This proves Claim.

Fix  $x \in X$ . Now suppose that  $\Gamma$  is not vertex transitive, and we shall show (ib).

Observe that  $X = X^+ \cup X^-$ , where

$$X^+ = \{y \in X \mid \text{there exists a path of even length connecting } x \text{ and } y\} \quad (8.6)$$

$$X^- = \{y \in X \mid \text{there exists a path of odd length connecting } x \text{ and } y\} \quad (8.7)$$

As  $X^+$  is contained in an orbit  $O^+$  of  $\text{Aut}(\Gamma)$ , and  $X^-$  is contained in an orbit  $O^-$  of  $\text{Aut}(\Gamma)$ .

Now  $O^+ \cap O^- = \emptyset$  (else  $O^+ = O^- = X$  and vertex transitive). So,  $X = O^+$ , and  $X^- = O^-$ .

Also  $X^+ \cup X^- = X$  is a bipartition by construction.

(ii) Fix  $x, y, x', y'$  with  $\partial(x, y) = \partial(x', y')$ .

By vertex transitivity, there exists an element

$$g_1 \in G \text{ such that } g_1x = x'.$$

Observe that

$$\partial(x', y') = \partial(x, y) = \partial(g_1x, g_1y) = \partial(x', g_1y).$$

Hence, there exists an element

$$g_2 \in G \text{ such that } g_1x' = x', g_2y' = g_1y', g_2g_1y = y'$$

by  $(G(x'))$  property.

Set  $g = g_2g_1$ . Then

$$gx = x', gy = y'$$

by construction. □

The following graphs  $\Gamma = (X, E)$  are vertex transitive, and satisfy the property  $(G(x))$  for all  $x \in X$ .

$$J(D, N), \quad H(D, r), \quad J_q(D, N),$$

where

$$H(D, r):$$

$$X = \{a_1 \cdots a_D \mid a_i \in F, 1 \leq i \leq D\} \quad (8.8)$$

$$F : \text{ any set of cardinality } r \quad (8.9)$$

$$E = \{xy \mid y, x \in X, x \text{ and } y \text{ differ in exactly one coordinate}\}. \quad (8.10)$$

$J_q(D, N)$ :

$X$  = the set of all  $D$ -dimensional subspaces of  $N$ -dimensional vector space over  $GF(q)$ .  
(8.11)

$$F : \text{ any set of cardinality } r \quad (8.12)$$

$$E = \{xy \mid y, x \in X, \dim(x \cap y) = D - 1\}. \quad (8.13)$$

The following graph is distance-transitive but does not satisfy  $(G(x))$  for any  $x \in G$ .

$H_q(D, N)$ :

$$X = \text{the set of all } D \times N \text{ matrices with entries in } GF(q). \quad (8.14)$$

$$E = \{xy \mid y, x \in X, \text{rank}(x - y) = 1.\}. \quad (8.15)$$

*Remark.*

$$H(D, r): G = S_r \text{wr} S_D, G_x = S_{r-1} \text{wr} S_D,$$

For  $x, y \in X$  with  $\partial(x, y) = \partial(x, z) = i$ ,

$$Y = \{j \in \Omega \mid x_j \neq y_j\} \leftrightarrow Z = \{j \in \Omega \mid x_j \neq z_j\} \quad (8.16)$$

$$(y_{j_1}, \dots, y_{j_i}) \leftrightarrow (z_{\ell_1}, \dots, z_{\ell_i}) \quad (8.17)$$

$$J(D, N): G = S_N, G_x = S_D \times S_{N-D}.$$

$$X \cap Y \leftrightarrow X \cap Z \quad (8.18)$$

$$(\Omega - X) \cap Y \leftrightarrow (\Omega - X) \cap Z. \quad (8.19)$$

The following graph is distance-transitive but does not satisfy  $(G(x))$  for any  $x \in G$ .

$J_q(D, N)$ :

$$X \cap Y \leftrightarrow X \cap Z.$$

The theory of single thin irreducible  $T$ -module.

Let  $\Gamma = (X, E)$  be any graph.

$$M = \text{Bose-Mesner algebra over } K/\mathbb{C} \text{ generated by the adjacency matrix } A. \quad (8.20)$$

$$= \text{Span}(E_0, \dots, E_R). \quad (8.21)$$

$M$  acts on the standard module  $V = \mathbb{C}^{|X|}$ .

Fix  $x \in X$ , let  $D \equiv D(x)$  be the  $x$ -diameter, and  $k = k(x)$  be the valency of  $x$ .



## Chapter 9

# Thin $T$ -Module, I

**Monday, February 8, 1993**

Let  $\Gamma = (X, E)$  be any graph.

$M$ : Bose-Mesner algebra over  $K/\mathbb{C}$  generated by the adjacency matrix  $A$ .

$$M = \text{Span}(E_0, \dots, E_R).$$

$M$  acts on the standard module  $V = \mathbb{C}^{|X|}$ .

Fix  $x \in X$ , let  $D \equiv D(x)$  be the  $x$ -diameter, and  $k = k(x)$  be the valency of  $x$ .

**Definition 9.1.** Pick  $x \in X$  and write  $E_i^* \equiv E_i^*(x)$  and  $T \equiv T(x)$ .

Let  $W$  be an irreducible thin  $T$ -module with endpoint  $r$ , diameter  $d$ .

Let  $a_i = a_i(W) \in \mathbb{C}$  satisfying

$$E_{r+i}^* A E_{r+i}^* |_{E_{r+i}^* W} = a_i 1 |_{E_{r+i}^* W} \quad (0 \leq i \leq d).$$

Let  $x_i = x_i(W) \in \mathbb{C}$  satisfying

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* |_{E_{r+i}^* W} = x_i 1 |_{E_{r+i}^* W} \quad (0 \leq i \leq d).$$

**Lemma 9.1.** *With above notation, the following hold.*

(i)  $a_i \in \mathbb{R} \quad (0 \leq i \leq d)$ .

(ii)  $x_i \in \mathbb{R}^{>0} \quad (0 \leq i \leq d)$ .

(iii) Pick  $0 \neq w_0 \in E_r^* W$ . Set  $w_i = E_{r+i}^* A^i w_0$  for all  $i$ . Then

(iiia)  $w_0, w_1, \dots, w_d$  is a basis for  $W$ ,  $w_{-1} = w_{d+1} = 0$ .

(iiib)  $A w_i = w_{i+1} + a_i w_i + x_i w_{i-1} \quad (0 \leq i \leq d)$ .

(iv) Define  $p_0, p_1, \dots, p_{d+1} \in \mathbb{R}[\lambda]$  by

$$p_0 = 1, \quad \lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1} \quad (0 \leq i \leq d), \quad p_{-1} = 0.$$

(iva)  $p_i(A)w_0 = w_i$ ,  $(0 \leq i \leq d+1)$ .

(ivb)  $p_{d+1}$  is the minimal polynomial of  $A|_W$ .

*Proof.* (i)  $a_i$  is an eigenvalue of a real symmetric matrix  $E_{r+i}^* A E_{r+i}^*$ .

(ii)  $x_i$  is an eigenvalue of a real symmetric matrix  $B^\top B$ , where

$$B = E_{r+i}^* A E_{r+i-1}^*.$$

Hence,  $x_i \in \mathbb{R}$ .

Since  $B^\top B$  is positive semidefinite,

$$x_i \geq 0.$$

*Pf.* If  $B^\top B v = \sigma v$  for some  $\sigma \in \mathbb{R}$ ,  $v \in \mathbb{R}^m \setminus \{0\}$ , then

$$0 \leq \|Bv\|^2 = v^\top B^\top B v = \sigma v^\top v = \sigma \|v\|^2, \quad \|v\|^2 > 0.$$

Hence,  $\sigma \geq 0$ .

Moreover,  $x_i \neq 0$  by Lemma 4.1 (iv).

(iiia) Observe

$$w_i = E_{r+i}^* A E_{r+i-1}^* w_{i-1} \quad (1 \leq i \leq d).$$

So  $w_i \neq 0$   $(1 \leq i \leq d)$  by Lemma 4.1 (iv).

Hence,

$$W = \text{Span}(w_0, \dots, w_d)$$

by Lemma 4.1. (iii).

(iiib) We have that

$$Aw_i = E_{r+i+1}^* Aw_i + E_{r+i}^* Aw_i + E_{r+i-1}^* Aw_i \quad (9.1)$$

$$= w_{i+1} + E_{r+i}^* A E_{r+i}^* w_i + E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* w_{i-1} \quad (9.2)$$

$$= w_{i+1} + a_i w_i + x_i w_{i-1} \quad (9.3)$$

(iva) Clear for  $i = 0$ . Assume it is valid for  $0, \dots, i$ .

$$p_{i+1}(A)w_0 = (A - a_i I)w_i - x_i w_{i-1} = w_{i+1}.$$

(ivb) By definition,

$$p_{d+1}(A)w_0 = 0.$$



Moreover,  $p_{d+1}(A)W = 0$ . For every  $w \in W$ , write

$$w = \sum_{i=0}^d \alpha_i w_i \quad (9.4)$$

$$= \sum_{i=0}^d \alpha_i p_i(A)w_0 \quad \text{for some } \alpha_i \in \mathbb{C} \quad (9.5)$$

$$= p(A)w_0 \quad \text{for some } p \in \mathbb{C}[\lambda] \quad (9.6)$$

Hence,

$$p_{d+1}(A)w = p_{d+1}(A)p(A)w_0 \quad (9.7)$$

$$= p(A)p_{d+1}(A)w_0 \quad (9.8)$$

$$= 0. \quad (9.9)$$

Note that  $p_{d+1}$  is the minimal polynomial.

*Pf.* Suppose  $q(A)W = 0$  for some  $0 \neq q \in \mathbb{C}[\lambda]$  with  $\deg q < \deg p_{d+1} = d + 1$ . Then,

$$q = \sum_{i=0}^d \beta_i p_i \quad \text{for some } \beta_i \in \mathbb{C}.$$

We have,

$$0 = q(A)w_0 = \sum_{i=0}^d \beta_i w_i.$$

Hence  $\beta_0 = \dots = \beta_d = 0$  by (iiia). Thus  $q = 0$  and a contradiction.  $\square$

**Corollary 9.1.** *Let  $\Gamma$ ,  $W$ ,  $r$ ,  $d$  be as above. Then*

(i)  *$W$  is dual thin, that is,*

$$\dim E_i W \leq 1 \quad (1 \leq i \leq d).$$

(ii)  $d = |\{i \mid E_i W \neq 0\}| - 1$ .

*Proof.* (i) Set as in Lemma 9.1,

$$w_i = p_i(A)w_0 \in E_{r+i}^* W.$$

Then  $w_0, w_1, \dots, w_d$  is a basis for  $W$ . We have

$$W = Mw_0.$$

So,

$$E_i W = E_i M w_0 = \text{Span}(E_i w_0).$$

Thus,

$$\dim E_i W = \begin{cases} 1 & \text{if } E_i w_0 \neq 0 \\ 0 & \text{if } E_i w_0 = 0. \end{cases}$$

In particular,

$$\dim E_i^* W \leq 1.$$

(ii) Immediate as

$$\dim W = d + 1.$$

This proves the lemma.  $\square$

**Lemma 9.2.** *Given an irreducible  $T(x)$ -module  $W$  with endpoint  $r = r(W)$ , diameter  $d = d(W)$ . Write*

$$x_i = x_i(W) \ (0 \leq i \leq d), \quad w_i = p_i(A)w_0 \in E_{r+i}^* W \ (0 \leq i \leq d), \quad 0 \neq w_0 \in E_r^* W.$$

Then,

$$\frac{\|w_i\|^2}{\|w_0\|^2} = x_1 x_2 \cdots x_i \quad (1 \leq i \leq d).$$

*Proof.* It suffices to show that

$$\|w_i\|^2 = x_i \|w_{i-1}\|^2 \quad (1 \leq i \leq d).$$

Recall by Lemma 9.1 (iiib) that

$$Aw_j = w_{j+1} + a_j w_j + x_j w_{j-1} \quad (0 \leq j \leq d), \quad w_{-1} = w_{d+1} = 0.$$

Now observe,

$$\langle w_{i-1}, Aw_i \rangle = \langle w_{i-1}, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle \quad (9.10)$$

$$= \overline{x_i} \|w_{i-1}\|^2 \quad (9.11)$$

$$= x_i \|w_{i-1}\|^2. \quad (9.12)$$

by Lemma 9.1 (ii). Also,

$$\langle w_{i-1}, Aw_i \rangle = \langle Aw_{i-1}, w_i \rangle \quad (\text{since } \overline{A}^\top = A) \quad (9.13)$$

$$= \langle x_i + a_{i-1} w_{i-1} + x_{i-1} w_{i-2}, w_i \rangle \quad (9.14)$$

$$= \|w_i\|^2. \quad (9.15)$$

This proves the lemma.  $\square$

**Definition 9.2.** Let  $W$  be an irreducible thin  $T(x)$  module with endpoint  $r$ ,  $E_i^* \equiv E_i^*(x)$ .

The measure  $m = m_W$  is the function

$$m : \mathbb{R} \rightarrow \mathbb{R}$$

such that

$$m(\theta) = \begin{cases} \frac{\|E_i w\|^2}{\|w\|^2} & \text{where } 0 \neq w \in E_r^* W \\ & \text{if } \theta = \theta_i \text{ is an eigenvalue for } \Gamma \\ 0 & \text{if } \theta \text{ is not an eigenvalue for } \Gamma. \end{cases}$$



## Chapter 10

# Thin $T$ -Module, II

Wednesday, February 10, 1993

Let  $\Gamma = (X, E)$  be any graph.

Fix a vertex  $x \in X$ . Let  $E_i^* \equiv E_i^*(x)$ ,  $T \equiv T(x)$ , the subconstituent algebra over  $\mathbb{C}$ , and  $V = \mathbb{C}^{|X|}$  the standard module.

**Lemma 10.1.** *With above notation, let  $W$  denote a thin irreducible  $T(x)$ -module with endpoint  $r$  and diameter  $d$ . Let*

$$a_i = a_i(W) \quad (0 \leq i \leq d) \quad (10.1)$$

$$x_i = x_i(W) \quad (1 \leq i \leq d) \quad (10.2)$$

$$p_i = p_i(W) \quad (0 \leq i \leq d+1) \quad (10.3)$$

be from Lemma 9.1, and measure  $m = m_W$ . Then,

(i)  $p_0, \dots, p_{d+1}$  are orthogonal with respect to  $m$ , i.e.,

$$\sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \leq i, j \leq d+1) \text{ with } x_{d+1} = 0.$$

$$(ia) \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 m(\theta) = x_1 \cdots x_i \quad (0 \leq i \leq d).$$

$$(iia) \sum_{\theta \in \mathbb{R}} m(\theta) = 1.$$

$$(iiia) \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) = x_1 \cdots x_i a_i \quad (0 \leq i \leq d).$$

*Proof.* Pick  $0 \neq w_0 \in E_r^* W$ . Set

$$w_i = p_i(A) w_0 \in E_{r+i}^* W.$$

Since  $E_i^*W$  and  $E_j^*W$  are orthogonal if  $i \neq j$ ,

$$\delta_{ij}\|w_i\|^2 = \langle w_i, w_j \rangle \quad (10.4)$$

$$= \langle p_i(A)w_0, p_j(A)w_0 \rangle \quad (10.5)$$

$$= \left\langle p_i(A) \left( \sum_{\ell=0}^R E_\ell \right) w_0, p_j(A) \left( \sum_{\ell=0}^R E_\ell \right) w_0 \right\rangle \quad (10.6)$$

$$= \left\langle \sum_{\ell=0}^R p_i(\theta_\ell) E_\ell w_0, \sum_{\ell=0}^R p_j(\theta_\ell) E_\ell w_0 \right\rangle \quad (\text{as } AE_j = \theta_j E_j) \quad (10.7)$$

$$= \sum_{\ell=0}^R p_i(\theta_\ell) \overline{p_j(\theta_\ell)} \|E_\ell w_0\|^2 \quad (10.8)$$

$$(\text{as } p_j \in \mathbb{R}[\lambda], \quad \theta_\ell \in \mathbb{R}, \quad m(\theta_i)\|w_0\|^2 = \|E_i w_0\|^2) \quad (10.9)$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) \|w_0\|^2. \quad (10.10)$$

Now we are done by Lemma 9.2 as

$$\|w_i\|^2 = \|w_0\|^2 x_1 x_2 \dots x_i.$$

For (ia), set  $i = j$ , and for (ib), set  $i = j = 0$ .

(ii) We have

$$\langle w_i, Aw_i \rangle = \langle w_i, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle \quad (10.11)$$

$$= \overline{a_i} \|w_i\|^2 \quad (10.12)$$

$$= a_i x_1 \dots x_i \|w_0\|^2, \quad (10.13)$$

as  $a_i \in \mathbb{R}$  by Lemma 9.1.

Also,

$$\langle w_i, Aw_i \rangle = \langle p_i(A)w_0, Ap_i(A)w_0 \rangle \quad (10.14)$$

$$= \left\langle p_i(A) \left( \sum_{\ell=0}^R E_\ell \right) w_0, Ap_i(A) \left( \sum_{\ell=0}^R E_\ell \right) w_0 \right\rangle \quad (\text{as in (i)}) \quad (10.15)$$

$$= \sum_{\ell=0}^D p_i(\theta_\ell)^2 \theta_\ell \|E_\ell w_0\|^2 \quad (10.16)$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) \|w_0\|^2. \quad (10.17)$$

Thus, we have (ii).  $\square$

**Lemma 10.2.** *With above notation, let  $W$  be a thin irreducible  $T(x)$ -module with measure  $m$ . Then  $m$  determines diameter  $d(W)$ ,*

$$a_i = a_i(W) \quad (0 \leq i \leq d) \quad (10.18)$$

$$x_i = x_i(W) \quad (1 \leq i \leq d) \quad (10.19)$$

$$p_i = p_i(W) \quad (0 \leq i \leq d+1). \quad (10.20)$$

*Proof.* Note that  $d+1$  is the number of  $\theta \in \mathbb{R}$  such that  $m(\theta) \neq 0$ . Hence  $m$  determines  $d$ .

Apply (ia), (ii) of Lemma 10.1.

$$\sum_{\theta \in \mathbb{R}} m(\theta) = 1 \quad p_0 = 1. \quad (10.21)$$

$$\sum_{\theta \in \mathbb{R}} \theta m(\theta) = a_0 \quad p_1 = \lambda - a_0 \quad (10.22)$$

$$\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 m(\theta) = x_1 \quad (10.23)$$

$$\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 \theta m(\theta) = x_1 a \quad \rightarrow a_1 \quad (10.24)$$

$$p_2 = (\lambda - a_1)p_1 - x_1 p_0 \quad (10.25)$$

$$\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 m(\theta) = x_1 x_2 \quad \rightarrow x_2 \quad (10.26)$$

$$\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 \theta m(\theta) = x_1 x_2 a_2 \quad \rightarrow a_2 \quad (10.27)$$

$$p_3 = (\lambda - a_2)p_2 - x_2 p_1 \quad (10.28)$$

$$\vdots \quad (10.29)$$

$$\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 m(\theta) = x_1 x_2 \cdots x_d \quad \rightarrow x_d \quad (10.30)$$

$$\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 \theta m(\theta) = x_1 x_2 \cdots x_d a_d \quad \rightarrow a_d \quad (10.31)$$

$$p_{d+1} = (\lambda - a_d)p_d - x_d p_{d-1}. \quad (10.32)$$

$$(10.33)$$

This proves the assertions.  $\square$

**Corollary 10.1.** *With above notation, let  $W, W'$  denote thin irreducible  $T(x)$ -modules. The following are equivalent.*

(i)  $W, W'$  are isomorphic as  $T$ -modules.

(ii)  $r(W) = r(W')$  and  $m_W = m_{W'}$ .

(iii)  $r(W) = r(W')$ ,  $d(W) = d(W')$ ,  $a_i(W) = a_i(W')$  and  $x_i(W) = x_i(W')$  ( $0 \leq i \leq d$ ).

*Proof.* (i)  $\Rightarrow$  (iii) Write  $r \equiv r(W)$ ,  $r' \equiv r(W')$ ,  $d = d(W)$ ,  $d' = d(W')$ ,  $a_i = a_i(W)$ ,  $a'_i = a_i(W')$ ,  $x_i = x_i(W)$  and  $x'_i = x_i(W')$ .

Let  $\sigma : W \rightarrow W'$  denote an isomorphism of  $T$ -modules. (See Definition 5.1.)

For every  $i$ ,

$$\sigma E_i^* W = E_i^* \sigma W = E_i^* W'.$$

So,  $r = r'$  and  $d = d'$ .

To show  $a_i = a'_i$ , pick  $w \in E_{r+i}^* W \setminus \{0\}$ . Then,

$$E_{r+i}^* A E_{r+i}^* \sigma(W) = \sigma(E_{r+i}^* A E_{r+i}^* w) = \sigma(a_i w) = a_i \sigma(w),$$

and  $\sigma w \neq 0$ . So,

$$a_i = \text{eigenvalue of } E_{r+i}^* A E_{r+i}^* \text{ on } E_{r+i}^* W \quad (10.34)$$

$$= a'_i \quad (10.35)$$

It is similar to show  $x = x'$ .

*Remark.* Pick  $w \in E_{r+i-1}^* W \setminus \{0\}$

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* \sigma(W) = \sigma(E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* w) = x_i \sigma(w).$$

Hence,  $x_i$  is the eigenvalue of  $E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^*$  on  $E_{r+i-1}^* W = x'_i$ .

(iii)  $\Rightarrow$  (i)

Pick  $0 \neq w_0 \in E_r^* W$ ,  $0 \neq w'_0 \in E_r^* W'$ . Let  $p_i$  be in Lemma 9.1, and set

$$w_i = p_i(A)w_0 \in E_{r+i}^* W \quad (0 \leq i \leq d) \quad (10.36)$$

$$w'_i = p'_i(A)w'_0 \in E_{r+i}^* W' \quad (0 \leq i \leq d) \quad (10.37)$$

Define a linear transformation,

$$\sigma : W \rightarrow W' \quad (w_i \mapsto w'_i).$$

Since  $\{w_i\}$  and  $\{w'_i\}$  are bases with  $d = d'$ ,  $\sigma$  is an isomorphism of vector spaces.

We need to show

$$a\sigma = \sigma a \quad (\text{for all } a \in T).$$

Take  $a = E_j^*$  for some  $j$  ( $0 \leq j \leq d(x)$ ). Then for all  $i$ , we have

$$E_j^* \sigma w_i = E_j^* w'_i = \delta_{ij} w'_i,$$

$$\sigma E_j^* w_i = \delta_{ij} \sigma(w_i) = \delta_{ij} w'_i.$$

$$E_j^* \sigma w_i = \sigma E_j^* w_i?$$

Take an adjacency matrix  $A$  of  $a$ . Then,

$$A \sigma w_i = A w'_i = w'_{i+1} + a'_i w'_i + x'_i w'_{i-1} = \sigma(w_{i+1} + a_i w_i + x_i w_{i-1}) = \sigma A w_i.$$



(ii)  $\Rightarrow$  (iii) Lemma 10.2.

(iii)  $\Rightarrow$  (ii) Given  $d, a_i, x_i$ , we can compute the polynomial sequence

$$p_0, p_1, \dots, p_{d+1}$$

for  $W$ .

Show  $p_0, p_1, \dots, p_{d+1}$  determines  $m = m_W$ . Set

$$\Delta = \{\theta \in \mathbb{R} \mid p_{d+1}(\theta) = 0\}.$$

Observe:  $|\Delta| = d + 1$ . See ‘An Introcuction to Interlacing’.

$m(\theta) = 0$  if  $\theta \notin \Delta$  ( $\theta \in \mathbb{R}$ ). So it suffices to find  $m(\theta)$ ,  $\theta \in \Delta$ .

By Lemma 10.1 (i),

$$\begin{cases} \sum_{\theta \in \Delta} m(\theta) p_0(\theta) &= 1 \\ \sum_{\theta \in \Delta} m(\theta) p_1(\theta) &= 0 \\ \vdots & \\ \sum_{\theta \in \Delta} m(\theta) p_d(\theta) &= 0 \end{cases}$$

$d + 1$  linear equation with  $d + 1$  unknowns  $m(\theta)$  ( $\theta \in \Delta$ ).

But the coefficient matrix is essentially Vander Monde (since  $\deg p_i = i$ ). Hence the system is nonsingular and there are unique values for  $m(\theta)$  ( $\theta \in \Delta$ ).  $\square$

*Remark.*

$$\begin{pmatrix} \theta - a_0 & -1 & \cdots & 0 & 0 \\ -x_1 & \theta - a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta - a_{d-1} & -1 \\ 0 & 0 & \cdots & -x_d & \theta - a_d \end{pmatrix} \begin{pmatrix} p_0(\theta) \\ \vdots \\ \vdots \\ \vdots \\ p_d(\theta) \end{pmatrix} = 0,$$

where  $\theta$  is an eigenvalue of a diagonalizable matrix

$$L = \begin{pmatrix} a_0 & 1 & \cdots & 0 & 0 \\ x_1 & a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{d-1} & 1 \\ 0 & 0 & \cdots & x_d & \theta a_d \end{pmatrix}$$

with multiplicity  $\dim(\text{Ker}(\theta I - L)) = 1$ .



# Chapter 11

## Examples of $T$ -Module

**Friday, February 12, 1993**

Let  $\Gamma = (X, E)$  be a connected graph.

Let  $\theta_0$  be the maximal eigenvalue of  $\Gamma$ , and  $\delta$  its corresponding eigenvector.

$$\delta = \sum_{y \in X} \delta_y \hat{y}.$$

Without loss of generality, we may assume that  $\delta_y \in \mathbb{R}^*$  for all  $y \in X$ .

**Lemma 11.1.** *Fix a vertex  $x \in X$ . Write  $T \equiv T(x)$ ,  $E_i^* \equiv E_i^*(x)$ .*

- (i)  $T\delta = T\hat{x}$  is an irreducible  $T$ -module.
- (ii) *Given any irreducible  $T$ -module  $W$ , the following are equivalent:*
  - (iia)  $W = T\delta$ .
  - (iib) *The diameter  $d(W) = d(x)$ .*
  - (iic) *The endpoint  $r(W) = 0$ .*

*Proof.* (i) Observe: there exists an irreducible  $T$ -module  $W$  that contains  $\delta$ .

Let  $V = \sum_i W_i$  be a direct sum decomposition of the standard module. Then

$$\text{Span}(\delta) = E_0 V = \sum_i E_0 W_i.$$

So,  $E_0 W_i \neq 0$  for some  $i$ . Then,

$$\delta \in E_0 W_i \subseteq W_i.$$

Observe:  $T\delta$  is an irreducible  $T$ -module.

Since  $\delta \in W$ , where  $W$  is a  $T$ -module. As  $T\delta \subseteq W$  and  $W$  is irreducible,  $T\delta = W$ .

Observe:  $T\delta = T\hat{x}$ .

Since  $\hat{x} = \delta_x^{-1} E_0^* \delta \in T\delta$ ,  $T\hat{x} \subseteq T\delta$ . Since  $T\delta$  is irreducible,  $T\hat{x} = T\delta$ .

(ii) (a)  $\rightarrow$  (b):

$$E_i^* \delta = \sum_{y \in X, \partial(x,y)=i} \delta_y \hat{y} \neq 0, \quad (0 \leq i \leq d(x)),$$

because  $\delta_y > 0$  for every  $y \in X$ .

Hence,

$$E_i^* T\delta \neq 0, \quad (0 \leq i \leq d(x)).$$

Thus,  $d(x) = d(W)$ .

(b)  $\rightarrow$  (c): Immediate.

(c)  $\rightarrow$  (a): Since  $r(W) = 0$ ,  $E_0^* W \neq 0$ . Hence,  $\hat{x} \in W$  and  $T\hat{x} \subseteq W$ .

By the irreducibility, we have  $T\hat{x} = W$ . □

**Lemma 11.2.** Assume  $\Gamma$  is bipartite ( $X = X^+ \cup X^-$ ) ( $X^+$  and  $X^-$  are nonempty). Then the following are equivalent.

(i) There exist  $\alpha^+$  and  $\alpha^- \in \mathbb{R}$  such that

$$\delta_x = \begin{cases} \alpha^+ & \text{if } x \in X^+ \\ \alpha^- & \text{if } x \in X^-. \end{cases}$$

/(ii) There exist  $k^+$  and  $k^- \in \mathbb{Z}^{>0}$  such that

$$k(x) = \begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^-. \end{cases}$$

In this case,  $k^+ k^- = \theta_0^2$ , and  $\Gamma$  is called bi-regular.

*Proof.* (i)  $\rightarrow$  (ii)



$$A\delta = A \left( \alpha^+ \sum_{x \in X^+} \hat{x} + \alpha^- \sum_{y \in X^-} \hat{y} \right) \quad (11.1)$$

$$= \alpha^+ \sum_{y \in X^-} k(y) \hat{y} + \alpha^- \sum_{x \in X^+} k(x) \hat{x} \quad (11.2)$$

$$= \theta_0 \delta. \quad (11.3)$$

So,

$$k(x)\alpha^- = \theta_0\alpha^+, \quad k(y)\alpha^+ = \theta_0\alpha^-.$$

As  $\alpha^+ \neq 0$  and  $\alpha^- \neq 0$ ,

$$k^+ := k(x) \text{ is independent of the choice of } x \in X^+, \text{ and} \quad (11.4)$$

$$k^- := k(y) \text{ is independent of the choice of } y \in X^-. \quad (11.5)$$

Moreover,  $k^+k^- = \theta_0^2$ .

(ii)  $\rightarrow$  (i) Set

$$\delta' = \sum_{y \in X} \alpha_y \hat{y} \quad \text{where } \alpha = \begin{cases} 1/\sqrt{k^-} & \text{if } y \in X^+ \\ 1/\sqrt{k^+} & \text{if } y \in X^- \end{cases}.$$

Then one checks

$$A\delta' = A \left( \frac{1}{\sqrt{k^-}} \sum_{y \in X^+} \hat{y} + \frac{1}{\sqrt{k^+}} \sum_{y \in X^-} \hat{y} \right) \quad (11.6)$$

$$= \frac{k^-}{\sqrt{k^-}} \sum_{y \in X^-} \hat{y} + \frac{k^+}{\sqrt{k^+}} \sum_{y \in X^+} \hat{y} \quad (11.7)$$

$$= \sqrt{k^+k^-} \delta' \quad (11.8)$$

Since  $\delta' > 0$ ,  $\delta' \in \text{Span}(\delta)$ , and  $\theta_0 = \sqrt{k^+k^-}$ .  $\square$

**Definition 11.1.** For any graph  $\Gamma = (X, E)$ , fix a vertex  $x \in X$ . Set  $d = d(x)$ .

$\Gamma$  is distance-regular with respect to  $x$ , if for all  $i : (0 \leq i \leq d)$ , and all  $y \in X$  such that  $\partial(x, y) = i$ :

$$c_i(x) := |\{z \in X \mid \partial(x, z) = i-1, \partial(y, z) = 1\}| \quad (11.9)$$

$$a_i(x) := |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = 1\}| \quad (11.10)$$

$$b_i(x) := |\{z \in X \mid \partial(x, z) = i+1, \partial(y, z) = 1\}| \quad (11.11)$$

depends only on  $i$ ,  $x$ , and not on  $y$ .

(In this case,  $c_0(x) = a_0(x) = b_d(x) = 0$ ,  $c_1(x) = 1$ ,  $b_0(x) = k(x)$  is the valency of  $x$ .)

We call  $c_i(x)$ ,  $a_i(x)$  and  $b_i(x)$  the intersection numbers with respect to  $x$ .

**Example 11.1.**



$$c_0 = 1 \qquad c_1 = 1 \qquad c_2 = 1 \qquad (11.12)$$

$$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = 1 \qquad (11.13)$$

$$b_0 = 2 \qquad b_1 = 1 \qquad b_2 = 0 \qquad (11.14)$$

## Chapter 12

# Distance-Regular

Monday, February 15, 1993

**Lemma 12.1.** *For any connected graph  $\Gamma = (X, E)$ , the following are equivalent.*

(i) *The trivial  $T(x)$ -module is thin for all  $x \in X$ .*

(ii)  $\left\{ \sum_{y \in X, d(x,y)=i} \hat{y} \mid 0 \leq i \leq d(x) \right\}$  *is a basis for the trivial  $T(x)$ -module for every  $x \in X$ .*

(iii)  $\Gamma$  *is distance-regular with respect to  $x$  for all  $x \in X$ .*

**Note.** Let  $\Gamma = (X, E)$  be a graph, with  $X = \{x, y_1, y_2, y_3, z_1, z_2, z_3\}$ ,  $E = \{xy_1, xy_2, xy_3, y_1z_1, y_1z_2, y_2z_3, y_3z_3\}$ .



Then (i), (ii) are not equivalent for a single vertex  $x$ .

$$E_0^* T \hat{x} = \langle \hat{x} \rangle, \quad (12.1)$$

$$E_1^* T \hat{x} = \langle y_1 + y_2 + y_3 \rangle, \quad (12.2)$$

$$E_2^* T \hat{x} = \langle z_1 + z_2 + 2z_3 \rangle. \quad (12.3)$$

*Proof of Lemma 12.1.* (i)  $\rightarrow$  (ii) Let  $\delta = \sum_{y \in X} \delta_y \hat{y}$  be an eigenvector for the maximal eigenvalue  $\theta_0$ . Then,

$$\sum_{y \in X, \partial(x,y)=1} \hat{y} = A\hat{x} \in T(x)\hat{x} = T(x)\delta \ni E_1^* \delta \quad (12.4)$$

$$= \sum_{y \in X, \partial(x,y)=1} \delta_y \hat{y} \quad (12.5)$$

If the trivial  $T(x)$ -module is thin,

$$\delta_y = \delta_z \text{ for } y, z \in X, \partial(x, y) = \partial(x, z) = 1.$$

Hence,  $\delta_y = \delta_z$  if  $y$  and  $z$  in  $X$  are connected by a path of even length.

So,  $\Gamma$  is regular or bipartite biregular by Lemma 11.2.

In particular,  $\delta_y = \delta_z$  if  $\partial(x, y) = \partial(x, z)$ , as there is a path of length  $2 \cdot \partial(x, y)$ ;

$$y \sim \dots \sim x \sim \dots \sim z.$$

Hence,

$$E_i^* \delta \in \text{Span} \left( \sum_{y \in X, \partial(x,y)=i} \hat{y} \right).$$

Since  $E_0^* \delta, E_1^* \delta, \dots, E_d^* \delta$  forms a basis for  $T(x)\delta$ , we have (ii).

(ii)  $\rightarrow$  (iii) Fix  $x \in X$ , and let  $T \equiv T(x)$ ,  $E_i^* \equiv E_i^*(x)$ , and  $d \equiv d(x)$ .

$$A \sum_{y \in X, \partial(x,y)=i} \hat{y} = \sum_{z \in X} |\{y \in X \mid \partial(y, z) = 1, \partial(x, y) = i\}| \hat{z} \quad (12.6)$$

$$= \sum_{z \in X, \partial(x,z)=i-1} b_{i-1}(x, z) \hat{z} \quad (12.7)$$

$$+ \sum_{z \in X, \partial(x,z)=i} a_i(x, z) \hat{z} \quad (12.8)$$

$$+ \sum_{z \in X, \partial(x,z)=i+1} c_{i+1}(x, z) \hat{z} \quad (12.9)$$

$$\in \text{Span} \left\{ \sum_{z \in X, \partial(x,z)=j} \hat{z} \mid j = 0, 1, \dots, d \right\}. \quad (12.10)$$

Hence,  $b_{i-1}(x, z)$ ,  $a_i(x, z)$  and  $c_{i+1}(x, z)$  depend only on  $i$  and  $x$ , and not on  $z$ . Therefore,  $\Gamma$  is distance-regular with respect to  $x$ .



(iii)  $\rightarrow$  (i) Fix  $x \in X$ , and let  $T \equiv T(x)$ ,  $E_i^* \equiv E_i^*(x)$ , and  $d \equiv d(x)$ . By definition of distance-regularity, for every  $i$  ( $0 \leq i \leq d$ ),

$$A \left( \sum_{y \in X, \partial(x,y)=i} \hat{y} \right) = b_{i-1}(x) \sum_{y \in X, \partial(x,y)=i-1} \hat{y} \quad (12.11)$$

$$+ a_i(x) \sum_{y \in X, \partial(x,y)=i} \hat{y} \quad (12.12)$$

$$+ c_{i+1}(x) \sum_{y \in X, \partial(x,y)=i+1} \hat{y}. \quad (12.13)$$

Hence,

$$W = \left\{ \sum_{y \in X, \partial(x,y)=i} \hat{y} \mid 0 \leq i \leq d \right\}$$

is  $A$ -invariant and so  $T$ -invariant. Since  $\hat{x} \in W$ ,  $T\hat{x} = W$  is the trivial module and  $T\hat{x}$  is thin.  $\square$

Next, we show more is true if (i) – (iii) hold in Lemma 12.1.

In fact,  $d(x)$ ,  $a_i(x)$ ,  $c_i(x)$ , and  $b_i(x)$  are

$$\begin{cases} \text{independent of } X & \text{if } \Gamma \text{ is regular; or} \\ \text{constant over } X^+ \text{ and } X^- & \text{if } \Gamma \text{ is biregular.} \end{cases}$$

Let  $\Gamma = (X, E)$  be any (connected) graph. Pick vertices  $x, y \in X$ .

Let  $W$  be a thin, irreducible  $T(x)$ -module, and measure  $m : \mathbb{R} \rightarrow \mathbb{R}$  determined by  $W$ .

Let  $W'$  be a thin, irreducible  $T(y)$ -module, and measure  $m' : \mathbb{R} \rightarrow \mathbb{R}$  determined by  $W'$ .

Recall  $W, W'$  are orthogonal if

$$\langle w, w' \rangle = 0 \quad \text{for all } w \in W, w' \in W'.$$

We shall show if  $W$  and  $W'$  are not orthogonal, then  $m$  and  $m'$  are related:

$$m \cdot \text{poly}_1 = m' \cdot \text{poly}_2$$

for some polynomials with

$$\deg \text{poly}_1 + \deg \text{poly}_2 \leq 2 \cdot \partial(x, y).$$

**Notation.**  $V$ : standard module of  $\Gamma$ .

$H$ : any subspace of  $V$ .

$$V = H + H^\perp \quad \text{orthogonal direct sum,}$$

and for  $v = v_1 + v_2$   $\text{proj}_H : V \rightarrow H$  ( $v \mapsto v_1$ ): linear transformation.

Observe: For every  $v \in V$ ,

$$v - \text{proj}_H v \in H^\perp.$$

So,

$$\langle v - \text{proj}_H v, h \rangle = 0 \quad \text{for all } h \in H \text{ or,}$$

$$\langle v, h \rangle = \langle \text{proj}_H v, h \rangle \quad \text{for all } v \in V, \text{ and for all } h \in H.$$

**Theorem 12.1.** *Let  $\Gamma = (X, E)$  be any graph. Pick vertices  $x, y \in X$  and set  $\Delta = \partial(x, y)$ . Assume*

*$W$ : thin irreducible  $T(x)$ -module with endpoint  $r$ , diameter  $d$ , and measure  $m$ .*

*$W'$ : thin irreducible  $T(y)$ -module with endpoint  $r'$ , diameter  $d'$ , and measure  $m'$ .*

*$W$  and  $W'$  are not orthogonal.*

*Now pick*

$$0 \neq w \in E_r^*(x)W, \quad 0 \neq w' \in E_{r'}^*(x)W'.$$

*Then,*

$$(i) \quad \text{proj}_{W'} w = p(A) \frac{\|w\|}{\|w'\|} w'$$

*for some  $0 \neq p \in \mathbb{C}[\lambda]$  with  $\deg p \leq \Delta - r' + r, d'$ ,*

$$\text{proj}_W w' = p'(A) \frac{\|w'\|}{\|w\|} w$$

*for some  $0 \neq p' \in \mathbb{C}[\lambda]$  with  $\deg p' \leq \Delta - r + r', d$ .*

*(ii) For all eigenvalues  $\theta_i$  of  $\Gamma$ ,*

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = m(\theta_i) \overline{p'(\theta_i)}.$$

*(iii) For all eigenvalues  $\theta_i$  of  $\Gamma$ ,*

$$p(\theta_i) p'(\theta_i)$$

*is in a real number in interval  $[0, 1]$ .*

*Proof.* (i) Since  $W, W'$  are not orthogonal, there exist

$$v \in W, v' \in W' \text{ such that } \langle v, v' \rangle \neq 0.$$

Then there exists  $a \in M$  such that

$$v' = aw'.$$

(This is because  $w'_i = p'_i(A)w'_0$  and hence for every  $v' \in W'$ , there is a polynomial  $q \in \mathbb{C}[\lambda]$ ,  $q(A)w'_0 = v$ .)

We have

$$0 \neq \langle v', v \rangle = \langle aw', v \rangle = \langle w', a^*v \rangle$$

and  $a^*v \in W$ .

Hence,  $\text{proj}_W w' \neq 0$ .

Let  $p_0, \dots, p_d \in \mathbb{C}[\lambda]$  be from Lemma 9.1.

Then,  $w_i = p_i(A)w$  is a basis for  $E_{r+i}^*(x)W$  ( $0 \leq i \leq d$ ).

Hence,

$$\text{proj}_W w' = \alpha_0 w_0 + \dots + \alpha_d w_d \text{ for some } \alpha_j \in \mathbb{C}.$$

Set

$$p' := \frac{\|w\|}{\|w'\|} \sum_{i=0}^d \alpha_i p_i.$$

Then  $0 \neq p' \in \mathbb{C}[\lambda]$  and  $\deg p' \leq d$ .

Claim:  $\alpha_i = 0$  ( $\Delta - r + r' < i \leq d$ ).

In particular,  $\deg p' \leq \Delta - r + r'$ .

*Pf.* Observe:

$$w' \in E_{r'}^*(y)V, \quad w \in E_r^*(x)V,$$

for  $\partial(x, y) = \Delta$ .

$$E_{r'}^*(y)V \cap E_{r+i}^*(x)V = 0$$

by triangle inequality.

( $\Delta = \partial(x, y) < r + i - r'$  or  $\Delta + r' < r + i$  by our choice of  $i$ .)



Hence,

$$E_{r'}^*(y)V \perp E_{r+i}^*(x)V,$$

or

$$0 = \langle w', w_i \rangle \quad (12.14)$$

$$= \langle \text{proj}_W w', w_i \rangle \quad (12.15)$$

$$= \sum_{j=0}^d \alpha_j \langle w_j, w_i \rangle \quad (12.16)$$

$$= \alpha_i \|w_i\|^2. \quad (12.17)$$

Hence,  $\alpha_i = 0$ . Thus,

$$\text{proj}_W w' = \sum_{i=0}^{\Delta+r'-r} \alpha_i w_i \quad (12.18)$$

$$= \sum_{i=0}^{\Delta+r'-r} \alpha_i p_i(A) w_0 \quad (12.19)$$

$$= p'(A) \frac{\|w'\|}{\|w\|} w. \quad (12.20)$$

(ii) We have

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = \frac{\langle E_i w, w' \rangle}{\|w\| \|w'\|} \quad (12.21)$$

$$= \frac{\langle E_i w, \text{proj}_W w' \rangle}{\|w\| \|w'\|} \quad \text{as } \text{proj}_W w' = p'(A) \frac{\|w\|}{\|w'\|} w \quad (12.22)$$

$$= \frac{\langle E_i w, p'(A) w \rangle}{\|w\|^2} \quad (12.23)$$

$$= \frac{\langle E_i w, E_i p'(A) w \rangle}{\|w\|^2} \quad (12.24)$$

$$= \overline{p'(\theta_i)} \frac{\|E_i W\|^2}{\|w\|^2} \quad (12.25)$$

$$= \overline{p'(\theta_i)} m(\theta_i). \quad (12.26)$$

Moreover, as  $m(\theta_i), m'(\theta_i) \in \mathbb{R}$ ,

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = \frac{\overline{\langle E_i w, E_i w' \rangle}}{\|w'\| \|w\|} = \overline{p(\theta_i) m'(\theta_i)} = p(\theta_i) m'(\theta_i).$$

(iii) Since,

$$\frac{|\langle E_i w, E_i w' \rangle|^2}{\|w\|^2 \|w'\|^2} = p(\theta_i) p'(\theta_i) m(\theta_i) m'(\theta_i),$$

$$p(\theta_i)p'(\theta_i) = \frac{|\langle E_i w, E_i w' \rangle|^2}{m(\theta_i)m'(\theta_i)\|w\|^2\|w'\|^2} \in \mathbb{R} \quad (12.27)$$

$$= \frac{|\langle E_i w, E_i w' \rangle|^2}{\frac{\|E_i w\|^2}{\|w\|^2} \frac{\|E_i w'\|^2}{\|w'\|^2} \|w\|^2 \|w'\|^2}. \quad (12.28)$$

By Cauchy-Schwartz inequality,

$$(|\langle a, b \rangle| \leq \|a\| \|b\|, )$$

$$\frac{|\langle E_i w, E_i w' \rangle|^2}{\|E_i w\|^2 \|E_i w'\|^2} \leq 1.$$

Hence, we have the assertion.  $\square$



## Chapter 13

# Modules of a DRG

Wednesday, February 17, 1993

**Lemma 13.1.** *Let  $\Gamma = (X, E)$  be any graph. Pick an edge  $xy \in E$ .*

*Assume the trivial  $T(x)$ -module  $T(x)\delta$  is thin with measure  $m_x$ ,*

*and the trivial  $T(y)$ -module  $T(y)\delta$  is thin with measure  $m_y$ .*

*Then,*

$$(ia) \quad \frac{m_x(\theta)}{k_x} = \frac{m_y(\theta)}{k_y} \text{ for all } \theta \in \mathbb{R} \setminus \{0\}.$$

$$(ib) \quad \frac{m_x(0) - 1}{k_x} = \frac{m_y(0) - 1}{k_y} \text{ for all } \theta \in \mathbb{R} \setminus \{0\}.$$

$$(\delta = \sum_{y \in X} \delta_y \hat{y} \quad \text{eigenvector corresponding to the maximal eigenvalue})$$

*Proof.* Apply Theorem 12.1,

$$W = T(x)\delta \quad r = 0, \quad d = d(x) \tag{13.1}$$

$$W' = T(y)\delta \quad r' = 0, \quad d' = d(y). \tag{13.2}$$

Take  $w = \hat{x}$ ,  $w' = \hat{y}$ .

Claim.  $\text{proj}_{T(y)\delta} \hat{x} = k_y^{-1} A \hat{y}$ .

*Pf.* Since

$$\hat{y} \in T(y)\delta, \quad A\hat{y} \in T(y)\delta.$$

Show

$$(\hat{x} - k_y^{-1} A \hat{y}) \perp (T(y)\delta).$$

Recall

$$A\hat{y} = \sum_{z \in X, yz \in E} \hat{z}.$$

$$\hat{x} - k_y^{-1}Ay \in E_1^*(y)V.$$

So,

$$\hat{x} - \frac{1}{k_y}A\hat{y} \perp E_j^*(y)T(y)\delta \quad \text{if } j \neq 1 \quad (0 \leq j \leq k(y)).$$

And we have,

$$\left\langle \hat{x} - \frac{1}{k_y}A\hat{y}, A\hat{y} \right\rangle = \left\langle \hat{x}, \sum_{z \in X, yz \in E} \hat{z} \right\rangle - \frac{1}{k_y} \left\| \sum_{z \in X, yz \in E} \hat{z} \right\|^2 \quad (13.3)$$

$$= 1 - 1 \quad (13.4)$$

$$= 0 \quad (13.5)$$

This proves Claim.

Similarly,

$$\text{prof}_{T(x)\delta} \hat{y} = k_x^{-1}A\hat{x}.$$

Hence, the polynomials  $p, p' \in \mathbb{C}[\lambda]$  from Theorem 12.1 equal

$$\frac{\lambda}{k_y} \quad \text{and} \quad \frac{\lambda}{k_x}$$

respectively.

By Theorem 12.1,

$$\frac{m_x(\theta)\theta}{k_x} = m_x(\theta)\overline{p'(\theta)} = m_y(\theta)\overline{p(\theta)} = \frac{m_y(\theta)\theta}{k_y}.$$

If  $\theta \neq 0$ , we have (ia).

Also,

$$\frac{1 - m_x(0)}{k_x} = \left( \sum_{\theta \in \mathbb{R} \setminus \{0\}} m_x(0) \right) \frac{1}{k_x} \quad \text{by (ia)} \quad (13.6)$$

$$= \left( \sum_{\theta \in \mathbb{R} \setminus \{0\}} m_y(0) \right) \frac{1}{k_y} \quad (13.7)$$

$$= \frac{1 - m_y(0)}{k_y} \quad (13.8)$$

Hence, we have (ib). □



**Theorem 13.1.** *Suppose any graph  $\Gamma = (X, E)$  is distance-regular with respect to every vertex  $x \in X$ . (So  $\Gamma$  is regular or biregular by Lemma 12.1.)*

*Then,*

*Case  $\Gamma$  is regular: the diameter  $d(x)$  and the intersection numbers  $a_i(x)$ ,  $b_i(x)$ ,  $c_i(x)$  ( $0 \leq i \leq d(x)$ ) are independent of  $x \in X$ .*

*(And  $\Gamma$  is called distance-regular.)*

*Case  $\Gamma$  is biregular: ( $X = X^+ \cup X^-$ )*

*$d(x)$  and  $a_i(x)$ ,  $b_i(x)$ ,  $c_i(x)$  ( $0 \leq i \leq d(x)$ ) are constant over  $X^+$  and  $X^-$ . (And  $\Gamma$  is called distance-biregular.)*

*Proof.* We apply Lemma 13.1.

Case  $\Gamma$ : regular.

Then  $m_x = m_y$  for all  $xy \in E$ . Hence, the measure of the trivial  $T(x)$ -module is independent of  $x \in X$ .

Case  $\Gamma$  is biregular.

Then  $m_x = m_{x'}$  for all  $x, x' \in X$  with  $\partial(x, x') = 2$ .

Hence, the measure of the trivial  $T(x)$ -module is constant over  $x \in X^+$ ,  $X^-$ .

Fix  $x \in X$ . Write  $T \equiv T(x)$ ,  $E_i^* \equiv E_i^*(x)$ ,  $W = T\delta$  with measure  $m$ , diameter  $d = d(x)$ .

We know by Corollary 10.1 that  $m$  determines

$$d, \quad a_i(W) \ (0 \leq i \leq d), \quad x_i(W) \ (1 \leq i \leq d)$$

(as  $d = D(x) = d(W)$  by Lemma 11.1.)

We shall show that  $m$  determines

$$a_i(x), \ c_i(x), \ b_i(x) \quad (0 \leq i \leq d).$$

Observe:

$$a_i(W) = a_i(x) \quad (0 \leq i \leq d) \tag{13.9}$$

$$x_i(W) = b_{i-1}c_i(x) \quad (1 \leq i \leq d) \tag{13.10}$$

*Remark.*  $a_i = a_i(W)$  is an eigenvalue of

$$E_i^* A E_i^* \text{ on } E_i^* W = \langle \sum_{y \in \Gamma_i(x)} \hat{y} \rangle.$$

(See Lemma 12.1.)

$x_i = x_i(W)$  is an eigenvalue of

$$E_{i-1}^* A E_i^* A E_{i-1}^* \text{ on } E_{i-1}^* W,$$

and

$$A \sum_{y \in X, \partial(x,y)} \hat{y} = b_{i-1}(x) \sum_{y \in X, \partial(x,y)=i-1} \hat{y} \quad (13.11)$$

$$+ a_i(x) \sum_{y \in X, \partial(x,y)=i} \hat{y} \quad (13.12)$$

$$+ c_{i+1} \sum_{y \in X, \partial(x,y)=i+1} \hat{y} \quad (13.13)$$

So  $x_i = b_{i-1}(x)c_i(x)$ .

Set  $k^+ = k_x$ . Define

$$k^- = \frac{\theta_0^2}{k^+},$$

where  $\theta_0$  is the maximal eigenvalue. (See Lemma 11.1.)

(So,  $k^+ = k^-$  is the valency, if  $\Gamma$  is regular.)

For every  $i$  ( $0 \leq i \leq d$ ) and for every  $z \in X$  with  $\partial(x, z) = i$ ,

$$k_z = c_i(x) + a_i(x) + b_i(x) \quad (13.14)$$

$$= \begin{cases} k^+ & \text{if } i \text{ is even,} \\ k^- & \text{if } i \text{ is odd.} \end{cases} \quad (13.15)$$

Now  $m$  determines

$$c_0(x) = a_0(x) = 0, \quad c_1(x) = 1,$$

$$b_0(x) = b_0(x)c_1(x) = x_1(W).$$

$$k^+ = b_0(x) \quad (13.16)$$

$$k^- = \theta_0^2 / k^+ \quad (13.17)$$

$$c_i(x) = x_i(W) / b_{i-1}(x) \quad (1 \leq i \leq d) \quad (13.18)$$

$$b_i(x) = \begin{cases} k^+ - a_i(x) - c_i(x) & i: \text{ even,} \\ k^- - a_i(x) - c_i(x) & i: \text{ odd.} \end{cases} \quad (13.19)$$

This proves the assertions.  $\square$

**Proposition 13.1.** *Under the assumption of Theorem 13.1, the following hold.*

*Case  $\Gamma$ : regular.*

- (i)  $\dim E_i V = |X| m(\theta_i)$ .
- (ii)  $\Gamma$  has exactly  $d + 1$  distinct eigenvalues
- ( $d = \text{diam} \Gamma = d(x)$ , for all  $x \in X$ ).

Case  $\Gamma$ : biregular.

- (i)  $\dim E_V = |X^+| m^+(\theta_i) + |X^-| m^-(\theta_i)$ .
- (ii)  $\Gamma$  has exactly  $d^+ + 1$  distinct eigenvalues ( $d^+ \geq d^-$ ).
- (iii) If  $d^+$  is odd, the  $\Gamma$  is regular.
- (iv)  $d^+ = d^-$ , or  $d^+ = d^- + 1$  is even.
- (v)  $a_i(x) = 0$  for all  $i$  and for all  $x$ .

*Proof.* (i) Suppose  $\Gamma$  is regular.

Let  $m_x$  be the measure of the trivial  $T(x)$ -module,

$$m_x(\theta_i) = \|E_i \hat{x}\|^2, \quad \text{as } \|\hat{x}\| = 1.$$

Now,

$$|X| m_x(\theta_i) = \sum_{x \in X} m_x(\theta_i) \tag{13.20}$$

$$= \sum_{x \in X} \|E_i \hat{x}\|^2 \tag{13.21}$$

$$= \sum_{y, z \in X} |(E_i)_{yz}|^2 \tag{13.22}$$

$$= \text{trace} E_i \overline{E_i}^\top. \tag{13.23}$$

Since  $A$  is real symmetric and

$$E_i \overline{E_i}^\top = E_i^2 = E_i$$

with  $E_i$  symmetric

$$E_i \sim \begin{pmatrix} I & O \\ O & I \end{pmatrix}.$$

$$\text{trace} E_i = \text{rank} E_i = \dim E_i V.$$

Thus, we have the assertion in this case.

Suppose  $\Gamma$  is biregular.

Then, same except,

$$\sum_{x \in X} m_x(\theta_i) = |X^+| m^+(\theta_i) + |X^-| m^-(\theta_i).$$

- (ii)  $\Gamma$ : regular. Immediately, if  $\theta$  is an eigenvalue of  $\Gamma$ , then  $m(\theta) \neq 0$ .

$\Gamma$ : biregular. For each  $\theta = \theta_i \in \mathbb{R} \setminus \{0\}$ ,

$$m^-(\theta) \neq 0 \Leftrightarrow m^+(\theta) \neq 0 \quad (13.24)$$

$$\Leftrightarrow \theta \text{ is an eigenvalue of } \Gamma \quad (13.25)$$

$$\left( \frac{m^+(\theta)}{k^+} = \frac{m^-(\theta)}{k^-} \right) \quad (13.26)$$

(iv) and (v) are clear.

*Remark.* (iii) If  $d^+$  is odd,  $d^+ = d^-$  and  $\Gamma$  has even number of eigenvalues, i.e., 0 is not an eigenvalue. So  $A$  is nonsingular, and  $\Gamma$  is regular.

□

# Chapter 14

## Parameters of Thin Modules, I

Friday, February 19, 1993

Summary.

**Definition 14.1.** Assume  $\Gamma = (X, E)$  is distance-regular with respect to every vertex  $x \in X$ .

Notation: Let  $x \in X$ . The data of the trivial  $T(x)$ -module.

	Case DR	Case DBR
valency $k_x$	$k$	$\begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^- \end{cases}$
$x$ -diameter $D_x$	$D$	$\begin{cases} D^+ & \text{if } x \in X^+ \\ D^- & \text{if } x \in X^- \end{cases}$
measure $m_x$	$m$	$\begin{cases} m^+ & \text{if } x \in X^+ \\ m^- & \text{if } x \in X^- \end{cases}$
int. number $c_i(x)$	$c_i$	$\begin{cases} c_i^+ & \text{if } x \in X^+ \\ c_i^- & \text{if } x \in X^- \end{cases}$
int. number $b_i(x)$	$b_i$	$\begin{cases} b_i^+ & \text{if } x \in X^+ \\ b_i^- & \text{if } x \in X^- \end{cases}$
int. number $a_i(x)$	$a_i$	$\begin{cases} a_i^+ & \text{if } x \in X^+ \\ a_i^- & \text{if } x \in X^- \end{cases}$

Call  $m, m^{\pm 1}$  the measure of  $\Gamma$ .

Assume  $\Gamma = (X, E)$  is distance-regular.

To what extent do  $a_i$ 's,  $b_i$ 's and  $c_i$ 's determine the structure of irreducible  $T(x)$ -modules? In general the following hold.

**Lemma 14.1.** *Assume  $\Gamma = (X, E)$  is distance-regular. Pick  $x \in X$ . Let  $X$  be a thin irreducible  $T(x)$ -module with endpoint  $r$ , diameter  $d$  and measure  $m_W$ .*

(i) *There is a unique polynomial  $f_W \in \mathbb{C}[\lambda]$  with the following properties.*

(ia)  $\deg f_W \leq D$  (diameter of  $\Gamma$ ).

(ib)  $m_W(\theta) = m(\theta)f_W(\theta)$  for every  $\theta \in \mathbb{R}$ , where  $m$  is the measure of  $\Gamma$ .

Moreover,  $f_W \in \mathbb{R}[\lambda]$ , and

(ii)  $\deg f_W \leq 2r$ .

(iii) *For all eigenvalues  $\theta_i$  of  $\Gamma$ ,  $\lambda - \theta_i$  is a factor of  $f_W$  whenever,  $E_i W = 0$ .*

In particular,  $2r - D + d \geq 0$ .

*Proof.* Let  $\theta_0, \dots, \theta_D$  denote distinct eigenvalues of  $\Gamma$ . Then  $m(\theta_i) \neq 0$  ( $0 \leq i \leq D$ ) by Proposition 13.1.

There exists a unique  $f_W \in \mathbb{C}[\lambda]$  with  $\deg f_W \leq D$  such that

$$f_W(\theta_i) = \frac{m_W(\theta_i)}{m(\theta_i)} \quad (0 \leq i \leq D)$$

by polynomial interpolation.

$f_W \in \mathbb{R}[\lambda]$  since

$$\theta_0, \dots, \theta_D \in \mathbb{R} \quad \text{and} \quad f_W(\theta_0), \dots, f_W(\theta_D) \in \mathbb{R}.$$

(ii) Without loss of generality, we may assume  $r < D/2$ , else trivial.

Pick  $0 \neq w \in E_r^*(x)W$ .

$$w = \sum_{y \in W, \partial(x,y)=r} \alpha_y \hat{y} \quad \text{some } \alpha_y \in \mathbb{C}.$$

Pick  $y \in X$  such that  $\alpha_y \neq 0$ .

Set  $W'$  be the trivial  $T(y)$ -module. ( $\langle w, \hat{y} \rangle \neq 0$ , as  $W \not\perp W'$ .)

$$r' = 0, \quad m' = m, \quad \Delta = r.$$

Apply Theorem 12.1, we have

$$\deg p \leq \Delta - r' + r = 2r, \quad p \neq 0 \tag{14.1}$$

$$\deg p' \leq \Delta - r + r' = 0, \quad p' \neq 0. \tag{14.2}$$

$$m_W(\theta)\overline{p'(\theta)} = m(\theta)p(\theta) \quad (\text{for all } \theta \in \mathbb{R}).$$

So,

$$\deg p/\bar{p}' \leq 2r,$$

and  $p/\bar{p}'$  satisfies the conditions of  $f_W$ .

$$\left( \frac{p(\theta)}{\bar{p}'(\theta)} = \frac{m_W(\theta)}{m(\theta)} \right)$$

(iii)

$$E_i W = 0 \rightarrow m_W(\theta_i) = 0 \rightarrow f_W(\theta_i) = 0.$$

that is,  $E_i W = 0$ . Hence  $\theta_i$  is a root of  $f_W(\lambda) = 0$ . So,

$$2r \geq \deg f_W \geq |\{\theta_i \mid E_i W = 0\}| = D - d.$$

Hence,

$$2r - D + d \geq 0.$$

This proves the assertions.  $\square$

**Lemma 14.2.** *Let  $\Gamma = (X, E)$  be any distance-regular graph with valency  $k$ , diameter  $D$  ( $d \geq 2$ ), measure  $m$ , and eigenvalues*

$$k = \theta_0 > \theta_1 > \dots > \theta_D.$$

*Pick  $x \in X$ . Let  $W$  be a thin irreducible  $T(x)$ -module with endpoint  $r = 1$ , diameter  $D$  and measure  $m_W = mf_W$ . Then one fo the following cases (i)–(iv) occurs.*

Case	$d$	$f_W(\lambda)$	$a_0(W)$
(i)	$D - 2$	$\frac{(\lambda-k)(\lambda-\theta_1)}{k(\theta_1+1)}$	$-\frac{b_1}{\theta_1+1} - 1$
(ii)	$D - 2$	$\frac{(\lambda-k)(\lambda-\theta_D)}{k(\theta_D+1)}$	$-\frac{b_1}{\theta_1+1} - 1$
(iii)	$D - 1$	$\frac{k-\lambda}{k}$	$-1$
(iv)	$D - 1$	$\frac{(\lambda-k)(\lambda-\beta)}{k(\beta+1)}$	$-\frac{b_1}{\beta+1} - 1$

*for some  $\beta \in \mathbb{R}$  with  $\beta \in (-\infty, \theta_D) \cup (\theta_1, \infty)$ . Moreover, the isomorphism class of  $W$  is determined by  $a_0(W)$ .*

**Note.** By (iii), the possible “shapes” of a thin irreducible  $T(x)$ -modules are:

$$r = 0 \quad d = D \tag{14.3}$$

$$r = 1 \quad d = D - 1 \tag{14.4}$$

$$r = 1 \quad d = D - 2 \tag{14.5}$$





## Chapter 15

# Parameters of Thin Modules, II

**Monday, February 22, 1993**

*Proof of Lemma 14.2 Continued.*

We have  $\deg f_W \leq 2$  by Lemma 14.1 (ii).

Also by Lemma 11.1,  $E_0 W = 0$ .

(As otherwise  $\langle \delta \rangle = E_0 V \subseteq W$  and  $r(W) = 0$ .)

Hence,  $\lambda - \theta_0 = \lambda - k$  is a factor of  $f_W$  by Lemma 14.1 (iii).

Let  $p_0, p_1, \dots, p_D$  denote the polynomials for the trivial  $T(x)$ -module from Lemma 9.1.

Recall,

$$\sum_{\theta \in \mathbb{R}} m(\theta) p_i(\theta) p_j(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \leq i, j \leq D) \quad (15.1)$$

$$= \delta_{ij} b_0 b_1 \cdots b_{i-1} c_1 c_2 \cdots c_i. \quad (15.2)$$

Note that  $x_i = b_{i-1} c_i$  is in the proof of Theorem 7.1.

By construction,

$$p_0(\lambda) = 1.p_1(\lambda) \quad \quad \quad = \lambda.p_2(\lambda)\lambda^2 - a_1\lambda - k. \quad (15.3)$$

Apparently,

$$f_W = \sigma_0 p_0 + \sigma_1 p_1 + \sigma_2 p_2$$

for some  $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{C}$ .

Claim:

$$\sigma_0 = 1, \quad (15.4)$$

$$\sigma_1 = \frac{a_0(W)}{k}, \quad (15.5)$$

$$\sigma_2 = \frac{1 + a_0(W)}{kb_1}. \quad (15.6)$$

*Pf of Claim.*

$$1 = \sum_{\theta \in \mathbb{R}} m_W(\theta) \quad (15.7)$$

$$= \sum_{\theta \in \mathbb{R}} m(\theta) f_W(\theta) \quad (15.8)$$

$$= \sum_{j=0}^2 \sigma_j \left( \sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) \right) \quad (15.9)$$

$$= \sigma_0. \quad (15.10)$$

We applied Lemma 10.1 (ib), Lemma 14.1 (ib), and Lemma 10.1 (i) in this order.

Next by Lemma 10.1 (ii), and  $p_1(\theta) = \theta$ ,

$$a_0(W) = \sum_{\theta \in \mathbb{R}} m_W(\theta) \theta \quad (15.11)$$

$$= \sum_{\theta \in \mathbb{R}} f_W(\theta) \theta \quad (15.12)$$

$$= \sum_{j=0}^2 \sigma_j \sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) p_1(\theta) \quad (15.13)$$

$$= \sigma_1 x_1(T\delta) \quad (15.14)$$

$$= \sigma_1 b_0 c_1 \quad (15.15)$$

$$= \sigma_1 k. \quad (15.16)$$

So for,

$$f_W(\lambda) = 1 + \frac{a_0(W)}{k} \lambda + \sigma_2 (\lambda^2 - a_1 \lambda - k).$$

But,

$$0 = f_W(k) \quad (15.17)$$

$$= 1 + a_0(W) + \sigma_2 k(k - a_1 - 1) \quad (15.18)$$

$$1 + a_0(W) + \sigma_2 kb_1. \quad (15.19)$$

Thus,

$$\sigma_2 = -\frac{1 + a_0(W)}{kb_1}.$$

This proves Claim.

Case:  $a_0(W) = -1$ .

Here,  $\sigma_2 = 0$  and

$$f_W(\lambda) = 1 + \frac{a_0(W)\lambda}{k} = 1 - \frac{\lambda}{k}.$$

Also,

$$d + 1 = |\{\theta \mid \theta \text{ is an eigenvalue of } \Gamma, f_W(\theta) \neq 0\}| = D.$$

Case:  $a_0(W) \neq -1$ .

Here,  $\sigma_2 \neq 0$ , and  $\deg f_W = 2$ . So,

$$f_W(\lambda) = (\lambda - k)(\lambda - \beta)\alpha$$

for some  $\alpha, \beta \in \mathbb{C}$ ,  $\alpha \neq 0$ .

Comparing the coefficients in

$$(\lambda - k)(\lambda - \beta)\alpha = 1 + \frac{a_0(W)}{k}\lambda - \frac{a_0(W) + 1}{kb_1}(\lambda^2 - a_1\lambda - k),$$

we find

$$\alpha = -\frac{a_0(W) + 1}{kb_1}, \quad (15.20)$$

$$-(k + \beta)\alpha = \frac{a_0(W)}{k} + \frac{a_0(W) + 1}{kb_1}a_1, \quad (15.21)$$

$$k\beta\alpha = 1 + \frac{a_0(W) + 1}{b_1}. \quad (15.22)$$

Hence,

$$-\beta(a_0(W) + 1) = b_1 + (a_0(W) + 1).$$

Thus, we have

$$(1 + a_0(W))(1 + \beta) = -b_1. \quad (15.23)$$

In particular,  $\beta \neq -1$ , and

$$\alpha = -\frac{1 + a_0(W)}{kb_1} = \frac{1}{k(\beta + 1)}.$$

Also, by Definition 9.2,

$$0 \leq m_W(\theta) \quad (15.24)$$

$$= m(\theta)f_W(\theta) \quad (\text{for all } \theta \in \mathbb{R}). \quad (15.25)$$

But if  $\theta$  is an eigenvalue of  $\Gamma$ ,

$$0 < m(\theta).$$

So,

$$0 \leq f_W(\theta) \quad (15.26)$$

$$= \frac{(\theta - k)(\theta - \beta)}{k(\beta + 1)}. \quad (15.27)$$

Either

$$\beta + 1 > 0 \rightarrow \theta - \beta \leq 0 \text{ or } \beta \geq \theta_1,$$

or

$$\beta + 1 < 0 \rightarrow \theta - \beta \geq 0 \text{ or } \beta \leq \theta_D.$$

If  $\beta = \theta_1$ ,

$$a_0(W) = -\frac{b_1}{\beta + 1} - 1 = -\frac{b_1}{\theta_1 + 1} - 1 \quad (15.28)$$

$$f_W(\lambda) = \frac{(\lambda - k)(\lambda - \theta_1)}{k(\theta_1 + 1)}, \quad (15.29)$$

and we have (i).

If  $\beta = \theta_D$ ,

$$a_0(W) = -\frac{b_1}{\theta_D + 1} - 1 \quad (15.30)$$

$$f_W(\lambda) = \frac{(\lambda - k)(\lambda - \theta_D)}{k(\theta_D + 1)}, \quad (15.31)$$

and we have (ii).

If  $\beta \notin \{\theta_1, \theta_2\}$ ,

$$\theta \in (-\infty, \theta_D) \cup (\theta_1, \infty),$$

we have (iv).

Note using (15.23), we have (iv).

□

**Note.** Using (15.23),

$$a_0(W) \rightarrow \beta \rightarrow f_W \rightarrow m_W \rightarrow \text{isomorphism class of } W.$$

**Note on Lemma 14.2.** In fact,  $\theta_1 > -1$ ,  $\theta_D < -1$  if  $D \geq 2$ .

**Definition 15.1.** The complete graph  $K_n$  has  $n$  vertices and diameter  $D = 1$ , i.e.,  $xy \in E$  for all vertices  $x, t$ .

$K_n$  is distance-regular with valency  $k = n - 1$  and  $a_1 = n - 2$ ,  $D = 1$ . Moreover, it has two distance eigenvalues  $\theta_0, \theta_1$ .

Recall,  $\theta_0, \dots, \theta_D$  are roots of  $p_{D+1}$ , i.e.,  $D + 1$  st polynomial for the trivial module/

$$p_0 = 1 \quad (15.32)$$

$$p_1 = \lambda \quad (15.33)$$

$$p_2 = \lambda^2 - a_1\lambda - k \quad (15.34)$$

$$= \lambda^2 - (n - 2)\lambda - (n - 1) \quad (15.35)$$

$$= (\lambda - (n - 1))(\lambda + 1). \quad (15.36)$$

The roots are  $\theta_0 = n - 1 = k$  and  $\theta_1 = -1$ .

**Lemma 15.1.** *Let  $\Gamma = (X, E)$  be distance-regular of diameter  $D \geq 1$  with distinct eigenvalues*

$$k = \theta_0 > \theta_1 > \dots > \theta_D.$$

(i)  $\theta_D \leq -1$  with equality if and only if  $D = 1$ .

(ii)  $\theta_1 \geq -1$  with equality if and only if  $D = 1$ .

*Proof.* (i) Suppose  $\theta_D \geq -1$ .

Then  $I + A$  is positive semi-definite.

By Lemma 2.1, there exists vectors  $\{v_x \mid x \in X\}$  in a Euclidean space such that

$$\langle v_x, v_y \rangle = (I + A)_{xy} \quad (15.37)$$

$$= \begin{cases} 1 & \text{if } x = y \text{ or } xy \in E, \\ 0 & \text{otherwise.} \end{cases} \quad (15.38)$$

For every  $xy \in E$ ,

$$\langle v_x, v_y \rangle = \|v_x\| \|v_y\| = 1.$$

Hence,  $v_x = v_y$ , and  $v_x$  is independent of  $x \in X$ .

Shus  $\langle v_x, v_y \rangle = 1$  for all  $x, y \in X$ .

We have  $I + A = J$ , (all 1's matrix), and  $D = 1$ .

(ii) Let  $m$  be the trivial measure. Then,

$$1 = \sum_{\theta \in \mathbb{R}} m(\theta) + \sum_{\theta \in \mathbb{R}} m(\theta)\theta \quad (15.39)$$

$$= \sum_{\theta \in \mathbb{R}} m(\theta)(\theta + 1) \quad (15.40)$$

$$= m(k)(k + 1) + \sum_{\theta \neq k} m(\theta)(\theta + 1) \quad (15.41)$$

$$\leq (k + 1)|X|^{-1}. \quad (15.42)$$

Note that  $m(k) = |X|^{-1} \dim d_0 V = |X|^{-1}$ .

So  $k+1 \geq |X|$  or  $k = |X| - 1$ . Thus,  $xy \in E$  for every  $x, y \in X$ , and  $D = 1$ .  $\square$

**Note.** Lemma 15.1 does not require distance-regular assumption.

## Chapter 16

# Thin Modoles of a DRG

Wednesday, February 24, 1993

Let  $\Gamma = (X, E)$  denote any graph of diameter  $D$ .

**Definition 16.1.** For all integer  $i$ , the  $i$ -th incidence matrix  $A_i \in \text{Mat}_X(\mathbb{C})$  satisfies

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \quad (x, y \in X).$$

Observe,

$$A_0 = I \quad (\text{identity}) \quad (16.1)$$

$$A_1 = A \quad (\text{adjacency matrix}) \quad (16.2)$$

$$A_0 + A_1 + \cdots + A_D = J \quad (\text{all 1's matrix}). \quad (16.3)$$

In general,  $A_i$  may not belong to Bose-Mesner algebra.

**Lemma 16.1.** Assume  $\Gamma = (X, E)$  is distance-regular with diameter  $D \geq 1$  and intersection numbers  $c_i, a_i, b_i$ .

(i)

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1}, \quad (0 \leq i \leq D, A_{-1} = A_{D+1} = O).$$

(ii)  $A_i = \frac{p_i(A)}{c_1 c_2 \cdots c_i}$ ,  $(0 \leq i \leq D)$ , where  $p_0, p_1, \dots, p_D$  are polynomials for the trivial module from Lemma 9.1.

(iii)  $A_0, A_1, \dots, A_D$  form a basis for Bose-Mesner algebra  $M$ .

(iv) For all distances  $h, i, j$   $(0 \leq i, j, h \leq D)$ , and for all vertices  $x, y \in X$  with  $\partial(x, y) = h$ , the constant

$$p_{i,j}^h = |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|$$

depends only on  $h, i, j$  and not on  $x, y$ .

$$(v) \ E_0 = \frac{1}{|X|} J.$$

*Proof.*

(i) Pick  $x \in X$ . Apply each side to  $\hat{x}$ , we want to show that

$$AA_i\hat{x} = c_{i+1}A_{i+1}\hat{x} + a_iA_i\hat{x} + b_{i-1}A_{i-1}\hat{x}.$$

$$\text{LHS} = A \left( \sum_{y \in X, \partial(x, y) = i} \hat{y} \right) \tag{16.4}$$

$$= c_{i+1} \left( \sum_{z \in X, \partial(x, z) = i+1} \hat{z} \right) + a_i \left( \sum_{z \in X, \partial(x, z) = i} \hat{z} \right) + b_{i-1} \left( \sum_{z \in X, \partial(x, z) = i-1} \hat{z} \right) \tag{16.5}$$

$$= \text{RHS}. \tag{16.6}$$

(ii) Recall (Lemma 9.1)

$$Ap_i(A) = p_{i+1}(A) + a_i p_i(A) + b_{i-1} c_i p_{i-1}(A) \quad (0 \leq i \leq D).$$

Dividing by  $c_1 c_2 \cdots c_i$ , we have

$$A \frac{p_i(A)}{c_1 c_2 \cdots c_i} = c_{i+1} \frac{p_{i+1}(A)}{c_1 c_2 \cdots c_{i+1}} + a_i \frac{p_i(A)}{c_1 c_2 \cdots c_i} + b_{i-1} \frac{p_{i-1}(A)}{c_1 c_2 \cdots c_i}.$$

So,  $A_i, p_i(A)/(c_1 c_2 \cdots c_i)$  satisfy the same recurrence.

Also boundary condition,

$$A_0 = p_0(A) = I.$$

Hence,

$$A_i = \frac{p_i(A)}{c_1 c_2 \cdots c_i} \quad (0 \leq i \leq D).$$

(iii) Since  $E_0, E_1, \dots, E_D$  form a basis for  $M$ ,  $\dim M = D + 1$ .

Observe  $A_0, A_1, \dots, A_D \in M$  by (ii),  $A_0, A_1, \dots, A_D$  are linearly independent, since  $p_0, p_1, \dots, p_D$  are linearly independent.

Thus,  $A_0, A_1, \dots, A_D$  form a basis for  $M$ .

(iv)  $A_0, A_1, \dots, A_D$  form a basis for an algebra  $M$ ,



$$A_i A_j = \sum_{\ell=0}^D p_{ij}^{\ell} A_{\ell} \quad \text{for some } p_{ij}^{\ell} \in \mathbb{C}. \quad (16.7)$$

Fix  $h$  ( $0 \leq h \leq D$ ). Pick  $x, y \in X$  with  $\partial(x, y) = h$ .

Compute  $x, y$  entry in (16.7),

$$(A_i A_j)_{xy} = \sum_{z \in X} (A_i)_{xz} (A_j)_{zy} \quad (16.8)$$

$$= \sum_{z \in X, \partial(x, z)=i, \partial(y, z)=j} 1 \cdot 1 \quad (16.9)$$

$$= |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|. \quad (16.10)$$

On the other hand,

$$\left( \sum_{\ell=0}^D p_{ij}^{\ell} A_{\ell} \right)_{xy} = p_{ij}^h (A_h)_{xy} = p_{ij}^h.$$

(v)  $\frac{1}{|X|}J$  is the orthogonal projection onto  $\text{Span}(\delta) = E_0 V$ . Hence,

$$\frac{1}{|X|} = E_0.$$

This proves the assertions. □

**Theorem 16.1.** *Let  $\Gamma = (X, E)$  be distance-regular with diameter  $D \geq 2$  and intersection numbers  $c_i, a_i, b_i$ . Pick a vertex  $x \in X$ . Let  $W$  be a thin irreducible  $T(x)$ -module with endpoint  $r = 1$  and diameter  $d$  ( $d = D - 2$  or  $D - 1$ ). Set  $\gamma_0 = a_0(W) + 1$ .*

(i) *The scalars*

$$\gamma_i := \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_1(W) x_2(W) \cdots x_i(W)} \quad (0 \leq i \leq d) \quad (16.11)$$

*$a_i(W), x_i(W)$  are algebraic integers in  $\mathbb{Q}[\gamma_0]$ . In particular, if  $\gamma_0 \in \mathbb{Q}$ , then  $\gamma_i, a_i(W)$  and  $x_i(W)$  are integers for all  $i$ .*

(ii) *The numbers,  $\gamma_i, a_i(W), x_i(W)$  can all be determined from  $\gamma_0$  and the intersection numbers of  $\Gamma$  in order*

$$x_1(W), \gamma_1, a_1(W), x_2(W), \gamma_2, a_2(W), \dots$$

*using (i),*

$$x_i(W) = c_i b_i + \gamma_{i-1} (a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad (1 \leq i \leq D - 1), \quad (16.12)$$

and

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad (1 \leq i \leq D). \quad (16.13)$$

**Note.**

$$p_i = p_1^W + \gamma_{i-1}p_{i-1}^W - c_i(p_{i-1}^W + \gamma_{i-2}^W), \quad (\gamma_{-1} = -\gamma_{-2} = 0, \quad 0 \leq i \leq d+1).$$

*Proof.* Set

$$\tilde{A}_i = A_0 + A_1 + \cdots + A_i \quad (0 \leq i \leq D).$$

$$\text{Claim 1. } A\tilde{A}_i = c_{i+1}\tilde{A}_{i+1} + (a_i - c_{i+1} + c_i)\tilde{A}_i + b_i\tilde{A}_{i-1} \quad (0 \leq i \leq D-1).$$

*Proof of Claim 1.*

$$\text{LHS} = \sum_{j=0}^i AA_j \quad (16.14)$$

$$= \sum_{j=0}^i (c_{j+1}A_{j+1} + a_jA_j + b_{j-1}A_{j-1}) \quad (16.15)$$

$$= \sum_{j=0}^{i-1} A_j(c_j + a_j + b_j) + A_i(c_i + a_i) + A_{i+1}c_{i+1} \quad (16.16)$$

$$= k(A_0 + \cdots + A_{i-1}) + (a_i + c_i)A_i + c_{i+1}A_{i+1}. \quad (16.17)$$

$$\text{RHS} = c_{i+1}(A_0 + A_1 + \cdots + A_{i-1} + A_i + A_{i+1}) \quad (16.18)$$

$$+ (a_i - c_{i+1} + c_i)(A_0 + A_1 + \cdots + A_{i-1} + A_i) \quad (16.19)$$

$$+ b_i(A_0 + A_1 + \cdots + A_{i-1}) \quad (16.20)$$

$$= k(A_0 + \cdots + A_{i-1}) + A_i(a_i + c_i) + A_{i+1}c_{i+1}. \quad (16.21)$$

This proves Claim 1.

Now pick  $0 \neq w \in E_1^*(x)W$  and let

$$w = \sum_{z \in X, \partial(x,z)=1} \alpha_z \hat{z}.$$

Pick  $y$ , where  $\alpha_y \neq 0$ .

For  $i$  ( $0 \leq i \leq D$ ), define

$$B_i = \tilde{A}_i(\hat{x} - \hat{y}) \quad (16.22)$$

$$= \sum_{z \in X, \partial(x,z) \leq i} \hat{z} - \sum_{z \in X, \partial(y,z) \leq i} \hat{z} \quad (16.23)$$

$$= \sum_{z \in X, \partial(x,z)=i, \partial(y,z)=i+1} \hat{z} - \sum_{z \in X, \partial(y,z)=i+1, \partial(y,z)=i} \hat{z}. \quad (16.24)$$

Note that  $B_D = O$ ,  $B_0 = \hat{x} - \hat{y}$ , and

$$\langle B_0, w_0 \rangle = -\alpha_y \neq 0.$$

From Claim 1,

$$AB_i = c_{i+1}B_{i+1} + (a_i - c_{i+1} + c_i)B_i + b_iB_{i-1} \quad (0 \leq i \leq D), \quad B_{-1} = O.$$

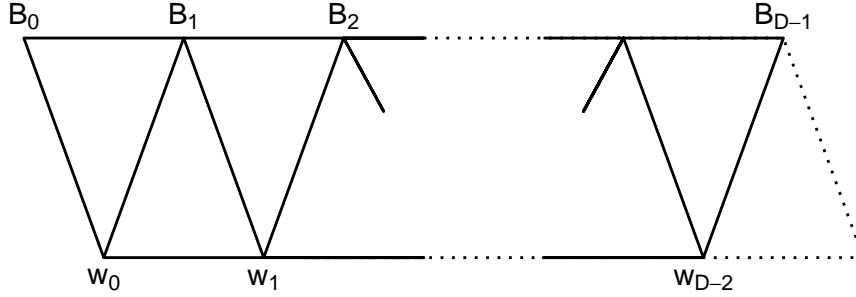
Let  $p_0^W, \dots, p_d^W$  denote polynomials for  $W$  from Lemma 9.1. So,

$$w_i = p_i^W(A)w \in E_{1+i}^*(x)W, \quad (0 \leq i \leq d).$$

Claim 2.  $\langle w_i, B_j \rangle = 0$  if  $j \notin \{i, i+1\}$ ,  $(0 \leq i \leq d, 0 \leq j \leq D)$ .

*Proof of Claim 2.*

$$w_i \in E_{1+i}^*W, \quad B_j \in E_j^*(x)W + E_{j+1}^*(x)W.$$



Vertical lines indicate possible non-orthogonality.

Compute

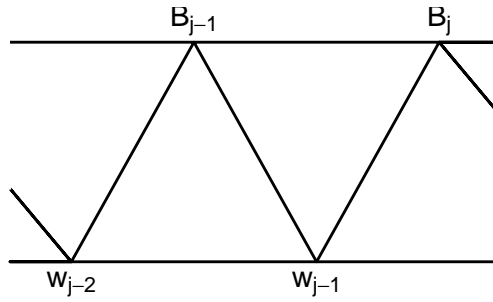
$$\langle Aw_i, B_j \rangle = \langle w_i, AB_j \rangle, \quad quad(0 \leq i \leq D, 0 \leq j \leq D-1). \quad (16.25)$$

$$\text{LHS} = \langle w_{i+1}, B_j \rangle + a_i(W)\langle w_i, B_j \rangle + x_i(W)\langle w_{i-1}, B_j \rangle \quad (16.26)$$

$$\text{RHD} = b_j\langle w_i, B_{j-1} \rangle + (a_j - c_{j+1} + c_j)\langle w_i, B_j \rangle + c_{j+1}\langle w_i, B_{j+1} \rangle. \quad (16.27)$$

Evaluate for  $i = j-2, j-1, j, j+1$ .

Set  $i = j-2$ .



Then (16.25) becomes

$$\langle w_{j-1}, B_j \rangle = b_j \langle w_{j-2}, B_{j-1} \rangle \quad (2 \leq j \leq D-1).$$

By induction,

$$\langle w_{j-1}, B_j \rangle = b_2 b_3 \cdots b_j \langle w_0, B_1 \rangle \quad (1 \leq j \leq D-1).$$

Define

$$\gamma_0 = \frac{\langle w_0, B_1 \rangle}{\langle w_0, B_0 \rangle}.$$

(We will show  $\gamma_0 = 1 + a_0(W)$ .)

Then,

$$\langle w_{j-1}, B_j \rangle = b_2 b_3 \cdots b_j \gamma_0 \langle w_0, B_0 \rangle. \quad (16.28)$$

Set  $i = j + 1$ . Then (16.25) becomes

$$x_{j+1}(W) \langle w_j, B_j \rangle = c_{j+1} \langle w_0, B_{j+1} \rangle \quad (0 \leq j \leq d).$$

Hence,

$$\langle w_j, B_j \rangle = \frac{x_1(W) \cdots x_j(W)}{c_1 c_2 \cdots c_j} \langle w_0, B_0 \rangle \quad (0 \leq j \leq d). \quad (16.29)$$

Set  $i = j - 1$ . Then (16.25) becomes

$$\langle w_j, B_j \rangle + a_{j-1}(W) \langle w_{j-1}, B_j \rangle = (a_j - c_{j+1} + c_j) \langle w_{j-1}, B_j \rangle + b_j \langle w_{j-1}, B_{j-1} \rangle.$$

Evaluate this using (16.28) and (16.29). ( $\langle w_0, B_0 \rangle \neq 0$ ). Then we have

$$\frac{w_1(W) \cdots x_j(W)}{c_1 \cdots c_j} + (a_{j-1}(W) - a_j + c_{j+1} - c_j) b_2 \cdots b_j \gamma_0 = b_j \frac{x_1(W) \cdots x_{j-1}(W)}{c_1 \cdots c_{j-1}},$$

$$\left( \gamma_i := \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_0(W) x_2(W) \cdots x_i(W)} \right).$$

$$\frac{x_j(W)}{c_j} = b_j + \frac{c_1 c_3 \cdots c_{j-1} b_2 b_3 \cdots b_j \gamma_0}{x_0(W) x_2(W) \cdots x_{j-1}(W)} (a_j + c_j - c_{j+1} - a_{j-1}).$$

So,

$$x_j(W) = c_j b_j + \gamma_{j-1} (a_j + c_j - c_{j+1} - a_{j-1}(W)).$$

This proves (16.12).

Set  $i = j$ . Then (16.25) becomes

$$a_j(W) \langle w_j, B_j \rangle + x_j(W) \langle w_{j-1}, B_j \rangle = (a_j - c_{j+1} + c_j) \langle w_j, B_j \rangle + c_{j+1} \langle w_j, B_{j+1} \rangle.$$

$$(a_j(W) - (a_j - c_{j+1} + c_j)) \frac{x_1(W) \cdots x_j(W)}{c_1 \cdots c_j} x_j(W) b_2 \cdots b_j \gamma_0 - c_{j+1} b_2 \cdots b_{j+1} \gamma_0 = 0.$$

Thus,

$$a_j(W) - (a_j - c_{j+1} + c_j) + \frac{c_1 \cdots c_j b_2 \cdots b_j \gamma_0}{x_1(W) \cdots x_{j-1}(W)} - \frac{c_1 \cdots c_j c_{j+1} b_2 \cdots b_{j+1} \gamma_0}{x_1(W) \cdots x_j(W)} = 0,$$

or

$$a_j(W) = a_j + c_j - c_{j+1} - \gamma_{j-1} + \gamma_j.$$

This proves (16.13).

Also by setting  $i = j = 0$ , we have

$$a_0(W) \langle w_0, B_0 \rangle = (a_0 - c_1 + c_0) \langle w_0, B_0 \rangle + c_1 \langle w_0, B_1 \rangle \quad (16.30)$$

$$= -\langle w_0, B_0 \rangle + \gamma_0 \langle w_0, B_0 \rangle. \quad (16.31)$$

Hence,

$$\gamma_0 = 1 + a_0(W).$$

Both  $a_i(W)$  and  $x_i(W)$  are algebraic integers, since they are eigenvalues of matrices with integer entries, namely,

$$E_{i+1}^*(x) A E_{i+1}^*(x) \quad \text{and} \quad E_i^*(x) A E_{i+1}^*(x) A E_i^*(x).$$

Also  $\gamma_0 = 1 + a_0(W)$  is an algebraic integer, and  $\gamma_i - \gamma_{i-1}$  is an algebraic integer by (16.12).

Hence,  $\gamma_i$  is an algebraic integer by induction.

This completes the proof of Theorem 16.1.  $\square$

**Example 16.1** ( $D=2$ ).

$$D = 2 \Leftrightarrow \text{strongly regular.}$$

Free parameters are  $k, a_1, c_2$ . Let  $W$  be an irreducible module of endpoint 1. The matrix representation of  $A|_W$  is

$$\begin{pmatrix} a_0(W) & x_1(W) \\ 1 & a_1(W) \end{pmatrix}.$$

$a_0(W)$ : free.

$$x_1(W) = c_1 b_1 + (a_0(W) + 1)(a_1 + c_1 - c_2 - a_0(W)) \quad (16.32)$$

$$= k - a_1 - 1 + a_1 a_0(W) + a_0(W) - c_2 a_0(W) - a_0(W)^2 + a_1 + a - c_2 - a_0(W) \quad (16.33)$$

$$= a_1 a_0(W) - c_2 a_0(W) + k - c_2 - a_0(W)^2, \quad (16.34)$$

$$\gamma_1 = 0, \quad (16.35)$$

$$a_1(W) = -(a_0(W) + 1) + a_1 + c_1 - c_2 \quad (16.36)$$

$$= -a_0(W) + a_1 - c_2. \quad (16.37)$$

Then the matrix has eigenvalues  $\theta, \theta_1$ . There is one feasible condition:  $a_0(W)$  is an algebraic integer.

**Example 16.2** (D=3). Free parameters  $c_2, c_3, k, a_1, a_2$ . The matrix representation becomes

$$A|_W = \begin{pmatrix} a_0(W) & x_1(W) & 0 \\ 1 & a_1(W) & x_2(W) \\ 0 & 1 & a_2(W) \end{pmatrix}.$$

Here,  $a_0(W)$  is free ( $= \gamma - 1$ )

$$x_1(W) = k - 1 - a_1 + \gamma_0(a_1 + 1 - c_2 - a_0(W)) \quad (16.38)$$

$$= \gamma_0(a_1 - c_2 - a_0(W)) + k - a_1 + a_0(W). \quad (16.39)$$

Set

$$\gamma_1(W) = \frac{c_2 b_2 \gamma_0}{x_1(W)}.$$

$$a_1(W) = \gamma_1 - \gamma_0 + a_1 + 1 - c_2 \quad (16.40)$$

$$x_2(W) = \gamma_1(a_2 - c_3 - a_1(W)) + c_2(\gamma_0 + b_1 - a_2 + a_1(W)) \quad (16.41)$$

$$a_2(W) = -\gamma_1 + a_2 + c_2 - c_2. \quad (16.42)$$

The matrix has eigenvalues,  $\theta, \theta_2, \theta_3$ .

There are two feasibility conditions;  $\gamma_0, \gamma_1$  are algebraic integers.

For arbitrary  $D$ , there are  $D - 1$  feasibility conditions;  $\gamma_0, \gamma_1, \dots, \gamma_{D-1}$  are algebraic integers.

**Lemma 16.2.** *With the notation of Theorem 16.1, suppose*

$$f_W = \frac{k - \lambda}{k} \quad (\text{so, } a_0(W) = -1).$$

Then,

$$a_i(W) = a_i + c_i - c_{i+1} \quad (0 \leq i \leq D - 1) \quad (16.43)$$

$$x_i(W) = b_i c_i \quad (1 \leq i \leq D - 1) \quad (16.44)$$

$$\gamma_i(W) = 0. \quad (16.45)$$

*Proof.* Since  $\gamma_0 = a_0(W) = 1$ ,  $\gamma_i = 0$ . □

## Chapter 17

# Association Schemes

Monday, March 1, 1993

### Review

Let  $\Gamma = (X, E)$  be a distance-regular graph of diameter  $D \geq 2$ . Pick a vertex  $x \in X$ .

Let  $W$  be a thin irreducible  $T(x)$ -module with endpoint  $r = 1$ , diameter  $d = D - 1$  or  $D - 2$ , and  $r_0 = a(W) + 1$ .

Show

$$\gamma_i = \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_1(W) \cdots x_i(W)},$$

$a_i(W)$  and  $x_i(W)$  are all algebraic integers in  $\mathbb{Q}[\gamma_0]$ , where

$$x_i(W) = c_i b_i + \gamma_{i-1}(a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad (1 \leq i \leq d) \quad (17.1)$$

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad (1 \leq i \leq d) \quad (17.2)$$

Certainly,  $x_i(W)$ ,  $\gamma_i$ , and  $a_i(W)$  are in  $\mathbb{Q}[\gamma_0]$  by the above lines and so on.

$$\gamma_0 \rightarrow a_0(W) \rightarrow x_1(W) \rightarrow \gamma_1 \rightarrow a_1(W) \rightarrow x_1(W) \rightarrow \cdots$$

Recall some  $B \in \text{Mat}_n(\mathbb{C})$  is integral whenever

$$B \in \text{Mat}_n(\mathbb{Z}).$$

In this case, the characteristic polynomial

$$\det(\lambda I - B) = \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_0, \quad \text{some } \alpha_0, \dots, \alpha_{n-1} \in \mathbb{Z}.$$

Hence, eigenvalues of  $B$  are algebraic integers. But  $a_i(W)$  is an eigenvalue of an integral matrices,

$$B = E_{i+1}^*(x) A E_{i+1}^*(x).$$

Hence,  $a_i(W)$  is an algebraic integer.

Also,  $x_i(W)$  is an eigenvalue of an integral matrix

$$B = E_i^*(x)AE_{i+1}^*(x)AE_i^*(x).$$

So  $x_i(W)$  is an algebraic integer.

$$\gamma_i - \gamma_{i-1} = a_i(W) - a_i - c_i + c_{i+1}$$

is an algebraic integer.

Since  $\gamma_0 = a_0(W) + 1$  is an algebraic integer, we find  $\gamma$  is an algebraic integer for all  $i$ .

**Definition 17.1.** A (commutative) association scheme is a configuration  $Y = (X, \{R_i\}_{0 \leq i \leq D})$ , where  $X$  is a finite nonempty set (of vertices),  $R_0, R_1, \dots, R_D$  are nonempty subsets of  $X \times X$  such that

- (i)  $R_0 = \{(x, x) \mid x \in X\}$ ,
- (ii)  $R_0 \cup \dots \cup R_D = X \times X$  (disjoint union),
- (iii) for every  $i$ ,  $R_i^\top = \{(y, x) \mid xy \in R\} = R_{i'}$  some  $i' \in \{0, 1, \dots, D\}$ ,
- (iv) for every  $h, i, j$  ( $0 \leq h, i, j \leq D$ ), and every  $x, y \in X$  such that  $(x, y) \in R_h$ ,

$$p_{ij}^h = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|$$

depends only on  $h, i, j$  and not on  $x, y$ ; and

- (v)  $p_{ij}^h = p_{ji}^h$  for all  $h, i, j$ .

If  $i' = i$  for all  $i$ , we say  $Y$  is symmetric. We call  $D$  the class of scheme and  $R_i$ , the  $i$ th relation of  $Y$ . We say vertices  $x, y \in X$  are  $i$ -related, or ‘at distance  $i$ ’, whenever  $(x, y) \in R_i$ .

We always assume that a ‘scheme’ is a commutative association scheme.

Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be an association scheme.

**Definition 17.2.** The  $i$ -th association matrix  $A_i \in \text{Mat}_X(\mathbb{C})$

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i, \end{cases} \quad (x, y \in X, 0 \leq i \leq D) \quad (17.3)$$

Then,

$$(i') \quad A_0 = I.$$

$$(ii') \quad A_0 + A_1 + \dots + A_D = J \text{ (= all 1's matrix).}$$



$$(iii') \quad A_i^\top = A_{i'} \quad (0 \leq i \leq D).$$

$$(iv') \quad A_i A_j = \sum_{h=0}^D p_{ij}^h A_h \quad (0 \leq i, j \leq D).$$

$$(v') \quad A_i A_j = A_j A_i.$$

$M := \text{Span}_{\mathbb{C}}(A_0, \dots, A_D)$  (Bose-Mesner algebra of  $Y$ ) is a commutative  $\mathbb{C}$ -algebra of dimension  $D + 1$ .

Observe:

$$Y \text{ is symmetric} \leftrightarrow A_i^\top = A_i \text{ for all } i \leftrightarrow M \text{ is symmetric.}$$

**Example 17.1.** Let  $\Gamma = (X, E)$  be distance-regular of diameter  $D$ . Set

$$R_i = \{(x, y) \mid \partial(x, y) = i\} \quad (0 \leq i \leq D). \quad (17.4)$$

Then,

$$Y = (X, \{R_i\}_{0 \leq i \leq D})$$

is a symmetric scheme.

$$i\text{-th association matrix} = i\text{-th distance matrix} \quad \text{for all } i.$$

**Example 17.2.** Suppose a group  $G$  acts transitively on a set  $X$ . Assume  $G$  is generously transitive, i.e.,

$$\text{for all } x, y \in X, \text{ there exists } g \in G \text{ such that } gx = y, gy = x.$$

Then  $G$  acts on  $X \times X$  by rule;

$$g(x, y) = (gx, gy), \quad \text{for all } g \in G, \text{ and for all } x, y \in X.$$

Let  $R_0, \dots, R_D$  denote orbits of  $G$  on  $X \times X$ .

Observe that  $R_i^\top = R_i$  for all  $i$  by generous transitivity, and

$$Y = (X, \{R_i\}_{0 \leq i \leq D})$$

is a symmetric scheme.

**Exercise 17.1.** In Example Example 17.2, Bose-Mesner algebra

$$M = \{B \in \text{Mat}_X(\mathbb{C}) \mid Bg = gB, \text{ for all } g \in G\} \quad (17.5)$$

$$= \text{the commuting algebra of } G \text{ on } X. \quad (17.6)$$

Here, we view each  $g \in G$  as a permutation matrix in  $\text{Mat}_X(\mathbb{C})$  satisfying

$$g\hat{x} = \widehat{gx}, \quad \text{for all } x \in G.$$

**Example 17.3.** Let  $G$  be any finite group.  $G$  acts on  $X = G$  by conjugation.

$$G \times X \rightarrow X, \quad (g, x) \mapsto gxg^{-1}.$$

Let  $C_0, C_1, \dots, C_D$  denote orbits (i.e., conjugacy classes), and let  $C_0 = \{1_G\}$ . Claim that  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  is a commutative scheme (not symmetric in general).

(i)  $R_0 = \{xx \mid x \in X\}$  as  $C_0 = \{1_G\}$ .

(ii)  $R_0, \dots, R_D$  is a partition of  $X \times X$  since  $C_0, \dots, C_D$  is a partition of  $X = G$ .

(iii)  $R_i^\top = R_{i'}$ , where  $C_{i'} = \{g^{-1} \mid g \in C_i\}$ .

(iv) Set  $H = G \oplus G$ , the direct sum. Then  $H$  acts on  $X = G$ :

$$\text{for all } h = (g, gz), \text{ for all } x \in X, \quad h(x) = gx(gx)^{-1} = gxz^{-1}g^{-1}.$$

$$R_i = \{(x, y) \mid x^{-1}y \in C_i\}, \quad h_i \in C_i, \quad x^{-1}y = gh_i g^{-1}.$$

$$(x, y) = (x, xgh_i g^{-1}) \tag{17.7}$$

$$= (xgg^{-1}, xgh_i g^{-1}) \tag{17.8}$$

$$= (xg, g)(1, h_i). \tag{17.9}$$

So,  $R_0, \dots, R_D$  are the orbits of  $H$  on  $X \times X$ .

(v)  $p_{ij}^h = p_{ji}^h$ ?

Fix  $i, j, h$  and  $x, y \in X$  with  $(x, y) \in R_h$ . Set

$$S = \{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\} \tag{17.10}$$

$$T = \{z \in X \mid (x, z) \in R_j, (z, y) \in R_i\}. \tag{17.11}$$

Show  $|S| = |T|$ .

For all  $z \in S$ , set  $\hat{z} = xz^{-1}y$ .

Observe,  $\hat{z} \in T$ .

$$x^{-1}z \in C_i x^{-1}\hat{z} = x^{-1}xz^{-1}y \in C_j \tag{17.12}$$

$$z^{-1}y \in C_j \hat{z}^{-1}y = y^{-1}zx^{-1}x^{-1}y = y^{-1}x(x^{-1}z)x^{-1}y \in C_i. \tag{17.13}$$

Observe

$$S \rightarrow T \quad (z \mapsto z^{-1}) \quad \text{is one-to-one and onto.}$$

# Chapter 18

## Polynomial Schemes

Wednesday, March 3, 1993

**Lemma 18.1.** *Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  denote the symmetric scheme with associated matrices  $A_0, A_1, \dots, A_D$ . Then the following are equivalent.*

(i) *The graph  $\Gamma = (X, R_1)$  is distance-regular, and  $R_0, \dots, R_D$  are labelled so that*

$$R_i = \{xy \mid \partial(x, y) = i\}.$$

(ii) *There exists  $f_i \in \mathbb{C}[\lambda]$ ,  $\deg f_i = i$  such that  $f_i(A_1) = A_i$  for all  $i$  with  $0 \leq i \leq D$ .*

(iii) *The parameter  $p_{ij}^h$*

$$\begin{cases} = 0 & \text{if one of } h, i, j \text{ is larger than the sum of the other two} \\ \neq 0 & \text{if one of } h, i, j \text{ is equal to the sum of the other two.} \end{cases}$$

*Proof.*

(i)  $\Rightarrow$  (ii): Lemma 16.1.

(ii)  $\Rightarrow$  (iii): Define

$$k_i \equiv p_{ii}^0 = |\{z \mid z \text{ in } X, \partial(x, z) = i \text{ } ((x, z) \in R_i)\}|$$

for any  $x \in X$ . Then  $k_i \neq 0$  ( $0 \leq i \leq D$ ),  $k_0 = 1$ .

(By symmetricity,  $(x, y) \in R_i$  if and only if  $(y, x) \in R_i$ .)

Claim.

$$k_h p_{ij}^h = k_i p_{hj}^i = k_j p_{ih}^j \quad (18.1)$$

$$= |X|^{-1} |\{xyz \in X^3 \mid \partial(x, y) = h, \partial(x, z) = i, \partial(y, z) = j\}|. \quad (18.2)$$

*Pf.* The number of  $xyz \in X^3$ ,  $\partial(x, y) = h, \partial(x, z) = i, \partial(y, z) = j$  is equal to

$$|X| k_h p_{ij}^h = |X| k_i p_{hj}^i = k_j p_{ih}^j.$$

In particular,

$$p_{ij}^h = 0 \leftrightarrow p_{hj}^i = 0 \leftrightarrow p_{ih}^j = 0.$$

Hence, it suffices to show

$$\begin{cases} p_{ij}^h = 0 & \text{if } h > i + j \\ p_{ij}^h \neq 0 & \text{if } h = i + j. \end{cases}$$

Fix  $i, j$ . Without loss of generality, we may assume that  $i + j \leq D$  as trivial otherwise.

$$f_i(A) f_j(A) = A_i A_j = \sum_{\ell=0}^D p_{ij}^{\ell} A_{\ell} = \sum_{\ell=0}^D p_{ij}^{\ell} f_{\ell}(A).$$

$$i + j = \deg \text{LHS} \quad (18.3)$$

$$= \deg \text{RHS} \quad (18.4)$$

$$= \max\{\ell \mid p_{ij}^{\ell} \neq 0\}. \quad (18.5)$$

(iii)  $\Rightarrow$  (i)

Let  $A = A_1$ , and consider a graph  $\Gamma$  with adjacency matrix  $A$ .

$$A A_j = \sum_h p_{1j}^h A_h \quad (18.6)$$

$$= p_{1j}^{j+1} A_{j+1} + p_{1j}^j A_j + p_{1j}^{j-1} A_{j-1}. \quad (18.7)$$

Then,  $p_{1j}^{j+1} \neq 0 \neq p_{1j}^{j-1}$ .

Fix a vertex  $x \in X$ , and set  $R_i(x) = \{y \mid (x, y) \in R_i\}$ .

Then each  $y \in R_i(x)$  is adjacent in  $\Gamma$  to exactly

$$p_{1,i+1}^i \neq 0 \quad \text{vertices in } R_i(x), \quad (18.8)$$

$$p_{1i}^i \quad \text{vertices in } R_{i+1}(x), \quad (18.9)$$

$$p_{1,i-1}^i \neq 0 \quad \text{vertices in } R_{i-1}(x). \quad (18.10)$$

Hence, by induction,

$$R_i(x) = \{y \mid \partial(x, y) = i \text{ in } \Gamma\} \quad (0 \leq i \leq D), \quad (18.11)$$

and  $\Gamma$  is distance regular.

□

## Chapter 19

# Commutative Association Schemes

Friday, March 5, 1993

**Lemma 19.1.** *Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a commutative scheme with Bose-Mesner algebra  $M$ .*

*Then there exists a basis  $E_0, E_1, \dots, E_D$  for  $M$  such that*

- (i)  $E_0 = |X|^{-1}J$ .
- (ii)  $E_i E_j = E_j E_i = \delta_{ij} E_i \quad (0 \leq i, j \leq D)$ .
- (iii)  $E_0 + E_1 + \dots + E_D = I$ .
- (iv)  $E_i^\top = \overline{E_i} = E_{\hat{i}}$  for some  $\hat{i} \in \{0, 1, \dots, D\}$ .

*Proof.*  $M$  acts on Hermitean space  $V = \mathbb{C}^n$  ( $n = |X|$ ).

If  $W$  is an  $M$ -module, so is  $W^\perp$ .

Each irreducible  $M$ -module is 1 dimensional by commutativity of  $M$ . So  $V$  is orthognal direct sum of 1-dimensional  $M$ -modules.

Let  $v_1, \dots, v_n$  be an orthonormal basis for  $V$  consisiting of eigenvectors for all  $m \in M$ .

Set  $P \in \text{Mat}_X(\mathbb{C})$  so that the  $i$ -th column of  $P$  is equal to  $v_i$ . So,

$$\bar{P}^\top P = I = P \bar{P}^\top = \bar{P} P^\top,$$

and  $P$  is unitary.

Also, for all  $m \in M$ ,

$$P^{-1}mP = \text{diagonal} \quad (19.1)$$

$$= \text{diag}(\theta_1(m), \dots, \theta_n(m)). \quad (19.2)$$

for some functions

$$\theta_i : M \longrightarrow \mathbb{C}.$$

Observe: each  $\theta = \theta_i$  is a character of  $M$ , i.e.,

$$\theta : M \longrightarrow \mathbb{C}$$

is a  $\mathbb{C}$ -algebra homomorphism.

Observe: the  $\theta_1, \dots, \theta_n$  are not all distinct.

Let  $\sigma_0, \dots, \sigma_r$  denote distinct elements of

$$\theta_1, \dots, \theta_n.$$

Say  $\sigma_i$  appears  $m_i$  times. Without loss of generality, we may assume that

$$P^{-1}mP = \begin{pmatrix} \sigma_0(m)I_{m_0} & O & O & O \\ O & \sigma_1(m)I_{m_1} & O & O \\ O & O & \ddots & O \\ O & O & O & \sigma_r(m)I_{m_r} \end{pmatrix}.$$

Set

$$E_i = P \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix} P^{-1},$$

where  $I_{m_i}$  is in the  $i$ -th block.

Then,

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq r),$$

$$E_0 + E_1 + \dots + E_r = I.$$

Hence for all  $m \in M$ ,

$$m = \sum_{i=0}^r \sigma_i(m) E_i \in \text{Span}(E_0, \dots, E_r).$$

So,

$$M \subseteq \text{Span}(E_0, \dots, E_r).$$

Since  $E_0, \dots, E_r$  are linearly independent,  $r \geq D$ .

Show  $E_i \in M$ .

Claim 1. For all distinct  $i, j$  ( $0 \leq i, j \leq D$ ), there exists  $m \in M$  such that  $\sigma_i(m) \neq 0$ ,  $\sigma_j(m) = 0$ .

*Pf of Claim 1.*  $\sigma_i \neq \sigma_j$  implies that there exists  $m' \in M$  such that  $\sigma_i(m') \neq \sigma_j(m')$ .

Set  $m = m' - \sigma_j(m')I$ . Then,

$$\sigma_j(m)\sigma_j(m') - \sigma_j(m') = 0, \quad (19.3)$$

$$\sigma_i(m)\sigma_i(m') - \sigma_j(m') \neq 0. \quad (19.4)$$

Claim 2.  $E_i \in M$  ( $0 \leq i \leq D$ ).

*Pf of Claim 2.* Fix a vertex  $x \in X$ . For all  $j \neq i$ , there exists  $m_j \in M$  such that  $\sigma_i(m_j) \neq 0$ ,  $\sigma_j(m_j) = 0$ ,  $i \neq j$ . Observe

$$s = \sigma_i \left( \prod_{\ell \neq i} m_\ell \right) \neq 0.$$

Set

$$m^* = \sigma_i \left( \prod_{\ell \neq i} m_\ell \right) s^{-1}.$$

Observe

$$\sigma_i(m^*) = 1, \quad \sigma_j(m^*) = 0, \quad \text{for all } j \neq i \quad (0 \leq j \leq D).$$

So

$$P^{-1}m^*P = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}.$$

We have

$$E_i = m^* \in M.$$

Now  $r = D$ ,  $M = \text{Span}(E_0, \dots, E_D)$  and  $E_0, \dots, E_D$  is a basis for  $M$ .

Observe

$$P^{-1}E_iP = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}$$

implies

$$P^{-1}\overline{E_i}^\top P = \overline{P}^\top \overline{E_i}^\top \overline{P^{-1}}^\top = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}^\top = P^{-1}E_iP.$$

Hence,

$$\overline{E_i}^\top = E_i.$$

$E_0^\top, \dots, E_D^\top$  are nonzero matrices satisfying

$$E_i^\top E_j^\top = \delta_{ij} E_i^\top,$$

$$E_0^\top + E_1^\top + \cdots + E_D^\top = I.$$

Each  $E_i^\top$  is a linear combination of  $E_0, \dots, E_D$  with coefficientss that are 0 or 1, and for no two  $E_i$ 's are coefficients of any  $E_j$  both 1's.

So,  $E_0^\top, \dots, E_D^\top$  is a permutation of  $E_0, \dots, E_D$ .

Observe  $J = A_0 + \cdots + A_D \in M$ .

The matrix  $|X|^{-1}J$  is an idempotent of rank 1.

So, without loss of generality we may assume that

$$E_0 = \frac{1}{|X|}J.$$

We have the assertions. □

Define entry-wise product  $\circ$  on  $\text{Mat}_X(\mathbb{C})$ .

$$A_i \circ A_j = \delta_{ij}A_i.$$

So,  $M$  is closed under  $\circ$ .

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h.$$

The numbers  $q_{ij}^h$  is called Krein parameters of  $Y$ .

Claim.  $q_{ij}^h \in \mathbb{R}$ .

*Pf.*

$$\frac{1}{|X|} \sum_{h=0}^D \overline{q_{ij}^h} E_h = \frac{1}{|X|} \sum_{h=0}^D \overline{q_{ij}^h} \overline{E_h}^\top \quad (19.5)$$

$$= (\overline{E_i} \circ \overline{E_j})^\top \quad (19.6)$$

$$= E_i \circ E_j \quad (19.7)$$

$$= \frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h. \quad (19.8)$$

Hence,  $q_{ij}^h = \overline{q_{ij}^h}$ .

Observe  $A_0, \dots, A_D, E_0, \dots, E_D$  are bases for  $M$ . Hence, there exist  $p_i(j), q_i(j) \in \mathbb{C}$  such that

$$A_i = \sum_{j=0}^D p_i(j) E_j \quad (19.9)$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j. \quad (19.10)$$



Taking transpose and conjugate we find,

$$\overline{p_i(j)} = p_i(j) = p_{i'}(\hat{j}) \quad (0 \leq i, j \leq D) \quad (19.11)$$

$$\overline{q_i(j)} = q_i(j) = q_{\hat{i}}(j') \quad (0 \leq i, j \leq D). \quad (19.12)$$

Fix a vertex  $x \in X$ . Define

$$E_i^* \equiv E_i^*(x) \in \text{Mat}_X(\mathbb{C})$$

to be a diagonal matrix such that

$$(E_i^*)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases} \quad (0 \leq i \leq D, y \in X.)$$

Then,

$$\begin{aligned} E_i^* E_j^* &= \delta_{ij} E_i^*, \\ E_0^* + \cdots + E_D^* &= I, \\ (E_i^*)^\top &= \overline{E_i^*} = E_i^*. \end{aligned}$$

**Definition 19.1.** Dual Bose-Mesner algebra  $M^* \equiv M^*(x)$  with respect to  $x$  is

$$\text{Span}(E_0^*, \dots, E_D^*).$$

Define dual associate matrices  $A_0^*, \dots, A_D^*$ . Indeed  $A_i^* \equiv A_i^*(x) \in \text{Mat}_X(\mathbb{C})$  is a diagonal matrix with

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad (y \in X).$$

$A_i^*$  is a diagonal matrix having the row  $x$  of  $E_i^*$  on the diagonal.

Observe

$$A_i^* = \sum_{j=0}^D q_i(j) E_j^* \quad \left( E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j \right) \quad (19.13)$$

$$E_i^* = \frac{1}{|X|} \sum_{j=0}^D p_i(j) A_j^* \quad \left( A_i = \sum_{j=0}^D p_i(j) E_j \right). \quad (19.14)$$

So,  $A_0^*, \dots, A_D^*$  form a basis for  $M^*$ .

Also,

$$A_i^* E_j^* = q_i(j) E_j^*.$$

$$\left( A_i^* E_j^* = \sum_{h=0}^D q_i(h) E_h^* E_j^* = q_i(j) E_j^* \right)$$

So,  $q_i(j)$  are dual eigenvalues of  $A_i^*$ .

Observe,

$$A_0^* = I, \quad A_0^* + \cdots + A_D^* = |X|E_0^*, \quad \overline{A_i^*} = A_i^*,$$

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* \quad (0 \leq i, j \leq D).$$

*Remark. Proof.*

$$(A_0^*)_{yy} = |X|(E_0)_{xy} = (J)_{xy} = 1.$$

$$A_0^* + \cdots + A_D^* = \sum_{i=0}^D \sum_{j=0}^D q_i(j) E_j^* = |X|E_0^*.$$

Note that

$$I = E_0 + \cdots + E_D = \frac{1}{|X|} \sum_{i=0}^D \sum_{j=0}^D q_i(j) A_j.$$

$$\sum_{i=0}^D q_i(j) = \delta_{j0}|X|.$$

$$\overline{A_i^*} = \sum_{j=0}^D \overline{q_i(j) E_j^*} = \sum_{j=0}^D q_i(j) E_j^* = A_i^*.$$

$$(A_i^* A_j^*)_{yy} = |X|^2 (E_i)_{xy} (E_j)_{xy} \tag{19.15}$$

$$= |X|^2 (E_i \circ E_j)_{xy} \tag{19.16}$$

$$= |X| \sum_{h=0}^D q_{ij}^h (E_h)_{xy} \tag{19.17}$$

$$= \sum_{h=0}^D q_{ij}^h (A_h^*)_{yy}. \tag{19.18}$$

The following statements will be proved after a couple of lemmas in the next lecture.

**Lemma.** Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a commutative scheme. Fix a vertex  $x \in X$ , and set  $E^* \equiv E_i^*(x)$  and  $A_i^* \equiv A^*(x)$ . Then the following hold.

(i)  $E_i^* A_j E_k^* = O$  if and only if  $p_{ij}^k = 0$  for  $0 \leq i, j, k \leq D$ .

(ii)  $E_i A_j^* E_k = O$  if and only if  $q_{ij}^k = 0$  for  $0 \leq i, j, k \leq D$ .

## Chapter 20

# Vanishing Conditions

**Monday, March 15, 1993** (Monday after Spring break)

**Lemma 20.1.** *Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a commutative scheme.*

(i)  $p_0(i) = 1$ .

(ii)  $p_i(0) = k_i$ , where

$$k_i = p_{ii'}^0 = |\{y \in X \mid (x, y) \in R_i\}|.$$

(iii)  $q_0(i) = 1$ .

(iv)  $q_i(0) = m_i$ , where

$$m_i = \text{rank} E_i.$$

*Proof.*

(i) Since  $A_0 = I$  and

$$A_0 = p_0(0)E_0 + p_0(1)E_1 + \cdots + p_0(D)E_D \quad (20.1)$$

$$I = E_0 + E_1 + \cdots + E_D, \quad (20.2)$$

$p_0(i) = 1$  for all  $i$ .

(ii) Since

$$A_i = p_i(0)E_0 + p_i(1)E_1 + \cdots + p_i(D)E_D,$$

$A_i E_0 = p_i(0) E_0$ , and

$$k_i J = A_i J = p_i(0) J$$

as there are  $k_i$  1's in each row of  $A_i$ , we have  $k_i = p_i(0)$ .

(iii) Since  $E_0 = |X|^{-1} J$  and

$$E_0 = |X|^{-1} (q_0(0) A_0 + q_0(1) A_1 + \cdots + q_0(D) A_D) \quad (20.3)$$

$$|X|^{-1} J = |X|^{-1} (A_0 + A_1 + \cdots + A_D), \quad (20.4)$$

$q_0(i) = 1$  for all  $i$ .

(iv)  $E_i = |X|^{-1} (q_i(0) A_0 + q_i(1) A_1 + \cdots + q_i(D) A_D)$ ,  $E_i^2 = E_i$ , and  $E_i$  is similar to a matrix

$$\begin{pmatrix} I_{m_i} & O \\ O & O \end{pmatrix}.$$

So,

$$m_i = \text{rank} E_i = \text{trace} E_i = \sum_{x \in X} (E_i)_{xx} = |X| |X|^{-1} q_i(0) = q_i(0).$$

Note that as

$$E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j \rightarrow (E_i)_{xx} = \frac{1}{|X|} q_i(0) (A_0)_{xx}.$$

Hence, we have all formulas. □

**Lemma 20.2.** *With the above notation*

$$(i) \ p_{ij}^h = p_{j'i'}^{h'}.$$

$$(ii) \ k_h p_{ij}^h = k_j p_{i'h}^j = k_{hj'}^i.$$

$$(iii) \ q_{ij}^h = q_{ji}^{\hat{h}}.$$

$$(iv) \ m_h q_{ij}^h = m_j q_{ih}^j = m_i q_{hj}^i.$$

*Proof.*

(i) We have

$$\sum_{h=0}^D p_{ij}^h A_{h'} \left( \sum_{h=0}^D p_{ij}^h A_h \right)^\top \quad (20.5)$$

$$= (A_i A_j)^\top \quad (20.6)$$

$$= A_j^\top A_i^\top \quad (20.7)$$

$$= A_{j'} A_{i'} \quad (20.8)$$

$$= \sum_{h=0}^D p_{j'i'}^{h'} A_h'. \quad (20.9)$$

(ii) Count the following number,

$$|\{xyz \in X^3 \mid (x, y) \in R_h, (x, z) \in R_i, (z, y) \in R_j\}| \quad (20.10)$$

$$= |X| k_h p_{ij}^h = |X| k_j p_{i'h}^j = |X| k_{hj'}^i. \quad (20.11)$$

(iii)

$$\frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_{\hat{h}} = \left( \frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h \right)^\top \quad (20.12)$$

$$= (E_i \circ E_j)^\top \quad (20.13)$$

$$= E_j^\top \circ E_i^\top \quad (20.14)$$

$$= E_{\hat{j}} E_{\hat{i}} \quad (20.15)$$

$$= \frac{1}{|X|} \sum_{h=0}^D q_{\hat{j}\hat{i}}^{\hat{h}} E_{\hat{h}}. \quad (20.16)$$

(iv) Let  $\tau(B)$  denote the sum of the entries in the matrix  $B$ .

Observe:  $\tau(B \circ C) = \text{trace}(BC^\top)$ .

Observe

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \tau((E_i \circ E_j \circ E_{\hat{k}})^\top) = \tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}).$$

Compute each one.

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \text{trace}((E_i \circ E_j)E_k) = \text{trace} \left( \left( \frac{1}{|X|} \sum_h q_{ij}^h E_h \right) E_k \right) \quad (20.17)$$

$$= \text{trace} \left( \frac{1}{|X|} q_{ij}^k E_k \right) = \frac{1}{|X|} m_k q_{ij}^k, \quad (20.18)$$

$$\tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \text{trace}((E_{\hat{i}} \circ E_k)E_{\hat{j}}) = \text{trace} \left( \left( \frac{1}{|X|} \sum_h q_{ik}^h E_h \right) E_{\hat{j}} \right) \quad (20.19)$$

$$= \text{trace} \left( \frac{1}{|X|} q_{ik}^j E_k \right) = \frac{1}{|X|} m_j q_{ik}^j, \quad (20.20)$$

$$\tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}) = \text{trace}((E_k \circ E_{\hat{j}})E_i) = \text{trace} \left( \left( \frac{1}{|X|} \sum_h q_{kj}^h E_h \right) E_i \right) \quad (20.21)$$

$$= \text{trace} \left( \frac{1}{|X|} q_{kj}^i E_i \right) = \frac{1}{|X|} m_i q_{kj}^i. \quad (20.22)$$

Hence, we have (iv). □

**Lemma 20.3.** *Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a commutative scheme. Fix a vertex  $x \in X$ , and set  $E^* \equiv E_i^*(x)$  and  $A_i^* \equiv A^*(x)$ . Then the following hold.*

(i)  $E_i^* A_j E_k^* = O$  if and only if  $p_{ij}^k = 0$  for  $0 \leq i, j, k \leq D$ .

(ii)  $E_i A_j^* E_k = O$  if and only if  $q_{ij}^k = 0$  for  $0 \leq i, j, k \leq D$ .

*Proof.*

(i) Partition rows and columns by  $R_0(x), R_1(x), \dots, R_D(x)$ . Then,

$$E_i^*(x) A_j E_h^*(x)$$

is the  $(i, h)$  block of  $A_j$ .

Hence this submatrix is zero if and only if there exists no  $y, z \in X$  such that  $(x, y) \in R_i$ ,  $(x, z) \in R_h$  and  $(y, z) \in R_j$ . This is exactly when  $p_{ij}^h = 0$ .

(ii) The sum of the squares of norms of entries in  $E_i A_j^* E_k$

$$= \tau((E_i A_j^* E_k) \circ (\overline{E_j A_j^* E_k})) \quad (20.23)$$

$$= \text{trace}(E_i A_j^* E_k (\overline{E_j A_j^* E_k})^\top) \quad (20.24)$$

$$= \text{trace}(E_i A_j^* E_k A_j^* E_i) \quad (20.25)$$

$$= \text{trace}(E_i A_j^* E_k A_j^*) \quad \text{as } \text{trace}(XY) = \text{trace}(YX) \quad (20.26)$$

$$= \sum_{y \in X} (E_i A_j^* E_k A_j^*)_{yy} \quad (20.27)$$

$$= \sum_{y \in X} \left( \sum_{z \in X} (E_i)_{yz} (A_j^*)_{zz} (E_k)_{zy} (A_j^*)_{yy} \right) \quad (20.28)$$

$$= \sum_{y \in X} \left( \sum_{z \in X} (E_i)_{zy} (|X| (E_j)_{xz}) (E_k)_{zy} (|X| (E_j)_{yx}) \right) \quad (20.29)$$

$$= |X|^2 (E_j (E_i \circ E_k) E_j)_{xx} \quad (20.30)$$

$$= |X| q_{ik}^j (E_j)_{xx} \quad (20.31)$$

$$= q_{ik}^j m_j \quad (20.32)$$

$$= m_k q_{ij}^k. \quad (20.33)$$

Note that since  $|X|E_j = q_j(0)A_0 + q_j(1)A_1 + \cdots q_j(D)A_D$ ,

$$(E_j)_{xx} = \frac{1}{|X|} q_j(0) = \frac{m_j}{|X|}.$$

Thus, we have (ii). □

**Corollary 20.1** (Krein Condition). *For any commutative scheme  $Y = (X, \{R_i\}_{0 \leq i \leq D})$ ,  $q_{ij}^h$  is a non-negative real number for  $0 \leq h, i, j \leq D$ .*

*Proof.* Since  $q_{ij}^h m_h$  is a non-negative real by the proof of Lemma 20.3 (ii).

Note that  $m_h$  is a positive integer. □

An interpretation of the Krein parameters.

Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a commutative scheme with standard module  $V$ .

Pick a vector  $v \in V$  with

$$v = \sum_{x \in X} \alpha_x \hat{x}.$$

View  $v$  as a function

$$v : X \longrightarrow \mathbb{C} \quad (x \mapsto \alpha_x).$$

View  $V$  as the set of all functions  $V \rightarrow \mathbb{C}$ . Then the vector space  $V$  together with product of functions is a  $\mathbb{C}$ -algebra.

For

$$v = \sum_{x \in X} \alpha_x \hat{x}, \quad w = \sum_{x \in X} \beta_x \hat{x} \in V,$$

write

$$v \circ w = \sum_{x \in X} \alpha_x \beta_x \hat{x}$$

to represent the product of  $v$  and  $w$  viewed as functions.

**Lemma 20.4.** *With the above notation,*

- (i)  $A_j^*(x)v = |X|(E_j \hat{x} \circ v)$  for all  $v \in V$  and for all  $x \in X$ .
- (ii)  $E_i V \circ E_j V \subseteq \sum_{h: q_{ij}^h \neq 0} E_h V$  for all  $0 \leq i, j \leq D$ .
- (iii)  $E_h(E_i \circ E_j V) = E_h V$  if  $q_{ij}^h \neq 0$  for all  $0 \leq h, i, j \leq D$ .



## Chapter 21

# Norton Algebras

Wednesday, March 17, 1993

*Proof of Lemma 20.4.*

(i) Suppose

$$v = \sum_{x \in X} \alpha_x \hat{x}.$$

Pick a vertex  $z \in X$  and compare  $z$ -coordinate of each side in (i).

$$(A_j^*(x)v)_z = (A_j^*(x))_{zz}v_z = |X|(E_j)_{xz}\alpha_z. \quad (21.1)$$

$$|X|(E_{\hat{j}}\hat{x} \circ v)_z = |X|(E_{\hat{j}}\hat{x})_z \cdot \alpha_z = |X|(E_j)_{xz}\alpha_z. \quad (21.2)$$

Note that  $E_{\hat{j}}\hat{x}$  is the column  $x$  of  $E_{\hat{j}}$  is the row  $x$  of  $E_j$ .

(ii) Fix  $i, j, h$  such that  $q_{ij}^h = 0$ .

Claim.  $E_h(E_i V \circ E_j V) = 0$ .

$$E_h(E_i V \circ E_j V) = E_h(\text{Span}(v \circ w \mid v \in E_i V, w \in E_j V)) \quad (21.3)$$

$$= E_h(\text{Span}(E_i \hat{y} \circ E_j \hat{z} \mid y, z \in X)) \quad (21.4)$$

$$= \text{Span}(E_h(E_j \hat{z} \circ E_i \hat{y}) \mid y, z \in X) \quad (21.5)$$

$$= \text{Span}((E_h A_j^*(z) E_i) \hat{y} \mid y, z \in X) \quad \text{by (i)} \quad (21.6)$$

But  $q_{ij}^h = 0$  implies  $q_{ji}^{\hat{h}} = 0$ .

So, by Lemma 20.3 (ii),

$$0 = (E_{\hat{i}} A_j^* E_{\hat{h}})^\top = E_h A_j^* E_i.$$

Hence,  $E_h(E_i V \circ E_j V) = 0$ .

(iii) Fix  $i, j, h$  such that  $q_{ij}^h \neq 0$ . Then,

$$E_h(E_i V \circ E_j V) \subseteq E_h V$$

is clear. We show the other inclusion. Since

$$E_i \hat{y} \circ E_j \hat{y} = (\text{column } y \text{ of } E_i \circ \text{column } y \text{ of } E_j) \quad (21.7)$$

$$= \text{column } y \text{ of } E_i \circ E_j \quad (21.8)$$

$$= (E_i \circ E_j) \hat{y} \quad (21.9)$$

$$= \left( \frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h \right) \hat{y}, \quad (21.10)$$

we have,

$$E_h(E_i V \circ E_j V) = E_h \text{Span}(E_i \hat{y} \circ E_j \hat{z} \mid y, z \in X) \quad (21.11)$$

$$\supseteq E_h \text{Span}(E_i \hat{y} \circ E_j \hat{y} \mid y \in X) \quad (21.12)$$

$$= \text{Span}(q_{ij}^h E_h \hat{y} \mid y \in X) \quad (21.13)$$

$$= \text{Span}(E_h \hat{y} \mid y \in X) \quad \text{since } q_{ij}^h \neq 0 \quad (21.14)$$

$$= E_h V. \quad (21.15)$$

This proves the assertion. □

**Lemma 21.1.** *Given a commutative scheme  $Y = (X, \{R_i\}_{0 \leq i \leq D})$ , fix  $j$  ( $0 \leq j \leq D$ ). Define binary multiplication:*

$$E_j V \times E_j V \longrightarrow E_j V \quad ((v, w) \mapsto v * w = E_j(v \circ w)).$$

Then,

(i)  $v * w = w * v$ , for all  $v, w \in E_j V$ ,

(ii)  $v * (w + w') = v * w + v * w'$  for all  $v, w, w' \in E_j V$ , and

(iii)  $(\alpha v) * w = \alpha(v * w)$  for all  $\alpha \in \mathbb{C}$ .

In particular, the vector space  $E_j V$  together with  $*$  is a commutative  $\mathbb{C}$ -algebra, (not associative in general).

$(N_j : (E_j V, *))$  is called the Norton algebra on  $E_j V$ .

(iv)  $v * w = 0$  for all  $v, w \in E_j V$  if and only if  $q_{jj}^j = 0$ .

*Proof.*

(i) – (iii) Immediate.

(iv) Immediate from Lemma 20.4 (ii), (iii).

□

Let  $Y, j, N_j$  be as in Lemma 21.1, and  $M$  Bose-Mesner algebra of  $Y$ . Let

$$\text{Aut}Y = \{\sigma \in \text{Mat}_X(\mathbb{C}) \mid \sigma : \text{permutation matrix}, \sigma \cdot m = m \cdot \sigma \text{ for all } m \in M\} \quad (21.16)$$

$$= \{\sigma \in \text{Mat}_X(\mathbb{C}) \mid \sigma : \text{permutation matrix}, \quad (21.17)$$

$$(x, y) \in R_i \rightarrow (\sigma x, \sigma y) \in R_i, \text{ for all } i, \text{ and for all } x, y \in X\} \quad (21.18)$$

$$\text{Aut}(N_j) = \{\sigma : E_j V \rightarrow E_j V \mid \sigma \text{ is } \mathbb{C}\text{-algebra isomorphisms, i.e.,} \quad (21.19)$$

$$\sigma(v * w) = \sigma(v) * \sigma(w) \text{ for all } v, w \in E_j V\}. \quad (21.20)$$

**Lemma 21.2.** *Let  $Y, j, *$  be as in Lemma 21.1.*

(i)  $E_j V$  is a module for  $\text{Aut}(Y)$ .

(ii)  $\sigma|_{E_j V} \in \text{Aut}(N_j)$  for all  $\sigma \in \text{Aut}(Y)$ .

(iii)  $\text{Aut}Y \longrightarrow \text{Aut}(N_j)$ ,  $(\sigma \mapsto \sigma|_{E_j})$  is a homomorphism of groups,

(i.e., a representation of  $\text{Aut}(Y)$ ).

(iv) Suppose  $R_0, \dots, R_D$  are orbits of  $\text{Aut}(Y)$  acting on  $X \times X$ , (so, we are in Example 17.2) then above representation is irreducible.

*Proof.*

(i) Pick  $\sigma \in \text{Aut}Y$  and  $v \in V$ . Then,

$$\sigma E_j v = E_j \sigma v,$$

since  $\sigma$  commutes with each element of  $M$ .

(ii)  $\sigma|_{E_j V} : E_j V \rightarrow E_j V$  is an isomorphism of a vector space. Since  $\sigma$  is invertible, for all  $v, w \in E_j V$ ,

$$\sigma(v * w) = \sigma(E_j(E_j v \circ E_j w)) = E_j \sigma(E_j v \circ E_j w) = E_j(E_j \sigma v \circ E_j \sigma w) = \sigma(v) * \sigma(w).$$

(iii) Immediate from (i) and (ii).

(iv) Here Bose-Mesner algebra  $M$  is the full commuting algebra, i.e.,

$$M = \{m \in \text{Mat}_X(\mathbb{C}) \mid \sigma \cdot m = m \cdot \sigma, \text{ for all } \sigma \in \text{Aut}(Y)\}.$$

Suppose there is a nonzero proper subspace  $0 \neq W \subsetneq E_j V$  that is  $\text{Aut}(Y)$ -invariant.

Set

$$W^\perp = \{v \in E_j V \mid \langle w, v \rangle = 0, \text{ for all } w \in W\}.$$

Then,  $W^\perp$  is a module for  $\text{Aut}(Y)$ , since  $\text{Aut}(Y)$  is closed under transpose conjugate.

Let  $e : V \rightarrow W$  and  $f : V \rightarrow W^\perp$  be orthogonal projection such that  $e + f = E_j$ ,

$$e^e = e, f^e = f, ef = fe = 0, eE_h = 0, \text{ if } h \neq j.$$

Since  $e$  commutes with all  $\sigma \in \text{Aut}(Y)$ ,  $e \in M$  and

$$e = \sum_{i=0}^D \alpha_i E_i.$$

If  $h \neq j$ , then  $0 = eE_h$  and  $\alpha_h = 0$ . Thus,  $e = \alpha_j E_j$ , i.e.,  $e = 0$  or  $f = 0$ .

A contradiction.

□

Norton algebras were used in original construction of Monster, a finite simple group  $G$ .

Compute character table of  $G$ ,

- $p_{ij}^h, q_{ij}^h$  of group scheme on  $G$ ,
- find  $j$  where  $m_j = \dim E_j V$  is small and  $q_{jj}^j \neq 0$ ,
- guess abstract structure of  $N_j$  using the knowledge of  $p_{ij}^h$ 's and  $q_{ij}^h$ 's,
- compute  $\text{Aut}(N_j)$ ,
- $G$ .

## Chapter 22

# Q-Polynomial Schemes

Friday, March 19, 1993

**Lemma 22.1.** *Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a commutative scheme.*

$$(i) \ p_{0j}^h = p_{j0}^h = \delta_{jh}.$$

$$(ii) \ p_{ij}^0 = \delta_{ij} k_i.$$

$$(iii) \ q_{0j}^h = q_{j0}^h = \delta_{jh}.$$

$$(iv) \ q_{ij}^0 = \delta_{ij} m_i.$$

$$(v) \ \sum_{j=0}^D p_{ij}^h = k_i.$$

$$(vi) \ \sum_{j=0}^D q_{ij}^h = m_i.$$

*Proof.*

(i), (ii) These are trivial.

(iii) We have

$$|X|^{-1} \sum_{\ell=0}^D q_{0j}^{\ell} E_{\ell} = E_0 \circ E_j = |X|^{-1} J \circ E_j = |X|^{-1} E_j.$$

(iv) Recall from Lemma 20.2

$$|X|^{-1} m_h q_{ij}^h = \tau(E_i \circ E_j \circ E_{\hat{h}}),$$

(where  $\tau(B)$  is the sum of entries in matrix  $B$ .)

$$|X|^{-1}m_0q_{ij}^0 = \tau(E_i \circ E_j \circ E_0) \quad (22.1)$$

$$= |X|^{-1}\tau(E_i \circ E_j) \quad (E_0 = |X|^{-1}J) \quad (22.2)$$

$$= |X|^{-1}\text{trace}(E_i E_j) \quad (22.3)$$

$$= |X|^{-1}\delta_{ij}\text{trace}E_i \quad (22.4)$$

$$= |X|^{-1}\delta_{ij}m_i. \quad (22.5)$$

(v) Pick  $x, y \in X$  with  $(x, y) \in R_h$ . Then,

$$\{j=0\} \wedge D \quad p \wedge h\{ij\} \quad \& \quad = \quad |\{z \in X \mid (x,z) \in R_{\perp i}, \quad ; \quad (z,y) \in R_{\perp j} \quad ; \quad \text{for some } j\}| \quad \& \quad = \\ |\{z \in X \mid (x,z) \in R_{\perp i}\}| \quad \& \quad k_{\perp i}. \quad \backslash \text{end}\{\text{align}\}$$

(vi)

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h.$$

So,

$$\sum_{j=0}^D E_i \circ E_j = |X|^{-1} \sum_{h=0}^D \left( \sum_{j=0}^D q_{ij}^h \right) E_h \quad (22.6)$$

$$= E_i \circ \sum_{j=0}^D E_j \quad (22.7)$$

$$= E_i \circ I \quad (22.8)$$

$$= |X|^{-1}(q_i(0)A_0 + q_i(1)A_1 + \cdots + q_i(D)A_D) \circ I \quad (22.9)$$

$$= |X|^{-1}q_i(0)I \quad (22.10)$$

$$= |X|^{-1}m_i(E_0 + E_1 + \cdots + E_D). \quad (22.11)$$

This proves the assertions. □

**Definition 22.1.** Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a commutative scheme.

$Y$  is  $Q$ -polynomial with respect to ordering  $E_0, E_1, \dots, E_D$  of primitive idempotents, if

$$q_{ij}^h \begin{cases} = 0 & \text{if one of } h, i, j \text{ is greater than the sum of the other two} \\ \neq 0 & \text{if one of } h, i, j \text{ is equal to the sum of the other two.} \end{cases}$$

In this case, set

$$c_i^* = q_{1,i-1}^i, \quad a_i^* = q_{1,i}^i, \quad b_i^* = q_{1,i+1}^i \quad (0 \leq i \leq D), \quad (c_0^* = b_D^* = 0).$$

Observe:  $Q$ -polynomial  $\rightarrow Y$  is symmetric.

Suppose  $i \neq \hat{i}$  for some  $i$ . Then, by the condition in Definition 22.1,

$$0 = q_{i\hat{i}}^0 = m_i (\neq 0)$$

by Lemma 22.1 (iv). This is a contradiction.

Hence,  $E_i^\top = E_{\hat{i}} = E_i$  for all  $i$ .

Therefore  $M$  is symmetric and  $Y$  is symmetric.

Observe: If  $Y$  is  $Q$ -polynomial,

$$c_i^* + a_i^* + b_i^* = m_1 \quad (0 \leq i \leq D)$$

(just as  $c_i + a_i + b_i = k$  for  $P$ -polynomial.)

By Lemma 22.1 (iv),

$$m_1 = q_{10}^i + q_{11}^i + \cdots + q_{1,i-1}^i + q_{1i}^i + q_{1,i+1}^i + \cdots$$

and  $q_{10}^i = q_{11}^i = 0$ ,  $q_{1,i-1}^i = c_i^*$ ,  $q_{1i}^i = a_i^*$ , and  $q_{1,i+1}^i = b_i^*$ .

**Lemma 22.2.** Assume  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  is a symmetric scheme. Pick  $x \in X$ , and set  $E_i^* \equiv E_i^*(x)$ ,  $A^* \equiv A^*(x)$ . Then the following are equivalent.

- (i)  $\Gamma$  is  $Q$ -polynomial with respect to  $E_0, \dots, E_D$ .
- (ii)  $\$q \wedge h \_ \{1j\} \setminus \begin{cases} \text{cases} \end{cases} = 0 \ \& \text{if} \ |h-j| > 0 \setminus$   
 $0 \ \& \text{if} \ |h-j| = 1. \setminus \end{cases} \quad (0 \leq h, j \leq D).$
- (iii) There exists  $f_i^* \in \mathbb{C}[\lambda]$ ,  $\deg f_i^* = i$ , and

$$A_i^* = f_i^*(A_1^*) \quad (0 \leq i \leq D).$$

(iv)  $E_0^*V, \dots, E_D^*V$  are maximal eigenspaces of  $A_1^*$ , and

$$E_i A_1^* E_j = O \quad \text{if } |i-j| > 0, \quad (0 \leq i, j \leq D).$$

(Compare (iv) with the definition of  $Q$ -polynomial in Definition 6.2.)

*Proof.*

(i)  $\rightarrow$  (ii) Clear.

(ii)  $\rightarrow$  (iii)  $A_0^* = I$ ,

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* \quad (22.12)$$

$$A_1^* A_j^* = q_{1j}^{j-1} A_{j-1}^* + q_{1j}^j A_j^* + q_{1j}^{j+1} A_{j+1}^* \quad (q_{1j}^{j+1} \neq 0, 1 \leq j \leq D-1). \quad (22.13)$$

Hence  $A_j^*$  is a polynomial of degree exactly  $j$  in  $A_1^*$  by induction on  $j$ .

$$\lambda f_j^*(\lambda) = b_{j-1}^* f_{j-1}^*(\lambda) + a_j^* f_j^*(\lambda) + c_{j+1}^* f_{j+1}^*(\lambda) \quad \text{with } c_{j+1}^* \neq 0,$$

and  $f_{-1}^* = 0$ ,  $f_0^*(\lambda) = 1$ .

(iii)  $\rightarrow$  (i) Pick  $i, j, h$  with  $0 \leq i, j, h \leq D$  and  $h \geq i + j$ . Since

$$m_h q_{ij}^h = m_j q_{ih}^j = m_i q_{hj}^i$$

by Lemma 20.2, it suffices to show that

$$q_{ij}^h \begin{cases} = 0 & \text{if } h > i + j \\ \neq 0 & \text{if } h = i + j. \end{cases}$$

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* \quad (22.14)$$

$$f_i^*(A_1) f_j^*(A_1) = \sum_{h=0}^D q_{ij}^h f_h^*(A_1). \quad (22.15)$$

Hence,

$$f_i^*(\lambda) f_j^*(\lambda) = \sum_{h=0}^D q_{ij}^h f_h^*(\lambda).$$

Note that since  $A_0^*, A_1^*, \dots, A_D^*$  are linearly independent,  $f(A_1^*) = 0$  implies  $\deg f > D$ .

$$\deg \text{LHS} = i + j \rightarrow q_{ij}^{i+j} \neq 0, q_{ij}^h = 0, \text{ if } h > i + j.$$

(iii)  $\rightarrow$  (iv) Recall

$$A_1^* = q_1(0)E_0^* + q_1(1)E_1^* + \dots.$$

Each  $A_i^*$  is a polynomial in  $A_1^*$ . Then  $A_1^*$  generates the dual Bose-Mesner algebra. So,  $q_1(0), q_1(1), \dots, q_1(D)$  are distinct.

So,  $E_0^* V, \dots, E_D^* V$  are maximal eigenspaces.

Also,  $|i - j| > 1$  implies  $q_{11}^j = 0$ .



Thus,  $E_i A_1^* E_j = 0$  by Lemma 20.3 (ii).

(iv)  $\rightarrow$  (ii)  $q_{1j}^i = 0$  if  $|i - j| > 1$ . since in this case,

$E_i A_1^* E_j = 0$  implies  $q_{1j}^i = 0$  by Lemma 20.3 (ii).

Suppose  $q_{1j}^{j+1} = 0$  for some  $j$  ( $0 \leq j \leq D - 1$ ).

Without loss of generalith, choose  $j$  minimum. Then  $A_h^*$  is a polynomial of degree  $h$  in  $A_1^*$  ( $0 \leq h \leq j$ ), and

$$A_1^* A_j^* - q_{1j}^{j-1} A_{j-1}^* - q_{1j}^j A_j^* = 0.$$

the left hand side is a polynomial in  $A_1^*$  of degree  $j + 1$ .

Hence, the minimal polynomial of  $A_1^*$  has degree less than or equal to  $j + 1 \leq D$ . But  $A_1^*$  has  $D + 1$  distinct eigenvalues.

This is a contradiction.

□



## Chapter 23

# Representation of a Scheme

Monday, March 22, 1993

**Theorem 23.1.** *Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a symmetric scheme. (View the standard module  $V$  as an algebra of functions from  $X$  to  $\mathbb{C}$ .) Then the following are equivalent.*

- (i)  $Y$  is  $Q$ -polynomial with respect to ordering  $E_0, E_1, \dots, E_D$  of primitive idempotents.
- (ii) For all  $i$  ( $0 \leq i \leq D$ ),

$$E_0V + E_1V + (E_1V)^2 + \dots + (E_1V)^i = E_0V + E_1V + \dots + E_iV.$$

*Proof.*

By Lemma 20.4 (ii), (iii).

$$E_h(E_iV \circ E_jV) = 0 \text{ if and only if } q_{ij}^h = 0 \quad (0 \leq i, j, h \leq D).$$

(i)  $\rightarrow$  (ii) By our assumption,

$$q_{1j}^h = 0 \text{ if } |h - j| > 1, \text{ and } q_{1j}^{j+1} \neq 0.$$

So,

$$E_1V \circ E_jV \subseteq E_{j-1}V + E_jV + E_{j+1}V \quad (0 \leq j \leq D), \quad (23.1)$$

$$E_{j+1}(E_1V \circ E_jV) = E_{j+1}V \quad (0 \leq j \leq D-1), \quad (23.2)$$

by Lemma 20.4.

Also  $E_0V \subseteq \text{Span}(\delta)$ , where  $\delta$  is all 1's vector, i.e., 1 as a function  $X \rightarrow \mathbb{C}$ . So,

$$E_0 \circ E_j V = E_j V \quad (0 \leq j \leq D). \quad (23.3)$$

Show (ii) by induction on  $i$ .

The cases  $i = 0, 1$  are trivial.

$i > 1$ :  $\subseteq$ .

$$E_0V + E_1V + (E_1V)^2 + \cdots + (E_1V)^i \quad (23.4)$$

$$= E_0V + E_1V \circ (E_0V + E_1V + \cdots + (E_1V)^{i-1}) \quad (23.5)$$

$$= E_0V + E_1V \circ (E_0V + E_1V + \cdots + E_{i-1}V) \quad (23.6)$$

$$\subseteq E_0V + E_1V + \cdots + E_iV \quad (23.7)$$

by (23.1).

$\supseteq$ .

Claim.  $E_i \subseteq E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V \quad (2 \leq i \leq D)$ .

*Proof of Claim.* By (23.2),

$$E_i(E_1V \circ E_{i-1}V) = E_iV.$$

For all  $v \in E_iV$ , there exists  $u \in E_1V \circ E_{i-1}V$  such that  $E_iu = v$ .

On the other hand, by (23.1),

$$E_1V \circ E_{i-1}V \subseteq E_{i-2}V + E_{i-1}V + E_{i-2}V.$$

So,  $u = w + v$ , where  $w \in E_{i-2}V + E_{i-1}V$ . We have,

$$w = u - v \in E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V$$

as desired.

*Remark.*

$$E_iV \circ E_jV = \text{Span}(u \circ v \mid u \in E_iV, v \in E_jV).$$

By claim,

$$E_0V + E_1V + \cdots + E_iV \quad (23.8)$$

$$\subseteq E_0V + E_1V + \cdots + E_iV + E_1V \circ E_{i-1}V \quad (23.9)$$

$$\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + E_1V(E_0V + E_1V + \cdots + (E_1V)^{i-1}) \quad (23.10)$$

$$\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + (E_1V)^i. \quad (23.11)$$

(ii)  $\rightarrow$  (i)

Claim 1. Pick  $i, j$  ( $0 \leq i, j \leq D$ ) with  $j > i + 1$ . Then  $q_{1i}^j = 0$ .

*Proof of Claim 1.*

$$E_j(E_1 \circ E_j V) \subseteq E_j(E_1 V \circ (E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^i)) \quad (23.12)$$

$$\subseteq E_j(E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^{i+1}) \quad (23.13)$$

$$= E_j(E_0 V + E_1 V + \cdots + E_{i+1} V) \quad (23.14)$$

$$= 0. \quad (23.15)$$

So  $q_{1i}^j = 0$  by Lemma 20.4.

Claim 2.  $q_{1i}^{i+1} \neq 0$  ( $0 \leq i < D$ ).

*Proof of Claim 2.*

$$E_0 V + E_1 V + \cdots + E_{i+1} V \quad (23.16)$$

$$= E_0 V + E_1 V + \cdots + (E_1 V)^{i+1} \quad (23.17)$$

$$= E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + (E_1 V)^i) \quad (23.18)$$

$$= E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + E_i V) \quad (23.19)$$

$$= E_0 V + E_1 V \circ (E_0 V + \cdots + E_i V). \quad (23.20)$$

So,

$$E_{i+1} V = E_{i+1} (E_1 V \circ (E_0 V + \cdots + E_i V)) \quad (23.21)$$

$$= E_{i+1} (E_1 V \circ E_i V) \quad (23.22)$$

by Claim 1 and Lemma 20.4.

Hence,  $q_{1i}^{i+1} \neq 0$  by Lemma 20.4.

□

Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a commutative scheme with standard module  $V$ .

**Definition 23.1.** A representation of  $Y$  is a pair  $(\rho, H)$ , where  $H$  is a non-zero Hermitean space (with inner product  $\langle \cdot, \cdot \rangle$ ) and  $\rho : X \rightarrow H$  is a map satisfying the following.

R1.  $H = \text{Span}(\rho(x) \mid x \in X)$ .

R2.  $\langle \rho(x), \rho(y) \rangle$  depends only on  $i$  for which  $(x, y) \in R_i$  ( $x, y \in X$ ).

R3. For every  $x \in X$  and for all  $i$  ( $0 \leq i \leq D$ ),

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) \in \text{Span}(\rho(x)).$$

Above representation is nondegenerate if  $\{\rho(x) \mid x \in X\}$  are distinct.

**Example 23.1.**  $Y = H(D, 2)$ ,  $X = \{a_1 \cdots a_D \mid a_i \in \{1, -1\}, 1 \leq i \leq D\}$ . Let  $H = \mathbb{C}^D$  and  $\langle \cdot, \cdot \rangle$  usual Hermitean dot product.

For a vertex  $x = a_1 \cdots a_D \in X$ , define

$$\rho(x) = a_1 \cdots a_D \in H.$$

Then, R1 – R3 hold.

*Remark.* R1, R2 are obvious. For R3, we may assume that  $x = 1 \cdots 1$ . Restrict

$$\sum_{y \in X, (y, x) \in R_i} \rho(y)$$

on the first coordinate. Then,

$$-1 \text{ appers } \binom{D-1}{i-1} \text{ times} \quad (23.23)$$

$$1 \text{ appers } \binom{D-1}{i} \text{ times.} \quad (23.24)$$

Hence,

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) = \left( \binom{D-1}{i} - \binom{D-1}{i-1} \right) \rho(x).$$

Let  $(\rho, H)$  be a representation of arbitrary commutative scheme  $Y$ . Set

$$E = (\langle \rho(x), \rho(y) \rangle)_{x, y \in X}$$

Gram matrix of the representation.

**Definition 23.2.** Representations  $(\rho, H)$ ,  $(\rho', H')$  of  $Y$  are equivalent, whenever, Gram matrices are related by

$$E' \in \text{Span} E.$$

We do not distinguish between equivalent representations.

**Note.** Suppose  $(\rho, H)$  is a representation of a symmetric scheme  $Y$ . Pick  $x, y \in X$  with  $(x, y) \in R_j$ .

Then  $(y, x) \in R_j$ . So, by R2,

$$\langle \rho(x), \rho(y) \rangle = \langle \rho(y), \rho(x) \rangle = \overline{\langle \rho(x), \rho(y) \rangle},$$

since  $\langle \cdot, \cdot \rangle$  is Hermitean.

Hence, the Gram matrix  $E$  of  $\rho$  is real symmetirc. Without loss of generality, we can view  $H$  as a real Euclidean space in this case.

**Lemma 23.1.** *Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a commutative scheme and  $V$  a standard module.*

*Let  $E_j$  be any primitive idempotent of  $Y$ .*

*(i)  $(\rho, H)$  is a representation of  $Y$ , where  $H = E_j V$  (with inner product inherited from  $Y$ ).*

$$\rho : X \rightarrow H \quad (x \mapsto E_j \hat{x})$$

*(i.e.,  $\rho(x)$  is the  $x$ -th column of  $E_j$ .)*

*(ii)  $\langle \rho(x), \rho(y) \rangle = |X|^{-1} q_j(i)$ , if  $(x, y) \in R_i$ ,  $(x, y \in X)$ .*

*(iii) For  $0 \leq i \leq D$  and  $x, y \in X$ ,*

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) = p_i(j) \rho(x).$$

*(iv)  $(\rho, H)$  is nondegenerate if and only if  $q_j(i) \neq q_j(0)$  for all  $i$ ,  $(0 \leq i \leq D)$ .*

*(v) Every representation of  $Y$  is equivalent to a representation of the above type for some  $j$  ( $0 \leq j \leq D$ ), and  $j$  is unique.*

*Proof.*

*(i) – (iii).*

R1:  $\text{Span}(\rho X)$  is the column space of  $E_j$  which is equal to  $H$ .

R2:

$$\langle \rho(x), \rho(y) \rangle = \langle E_j \hat{x}, E_j \hat{y} \rangle \quad (23.25)$$

$$= (\overline{E_j \hat{x}})^\top E_j \hat{y} \quad (23.26)$$

$$= \hat{x}^\top \overline{E_j}^\top E_j \hat{y} \quad (23.27)$$

$$= \hat{x}^\top E_j \hat{y} \quad (23.28)$$

$$(E_j)_{xy}. \quad (23.29)$$

Note that  $\overline{E_j}^\top = E_j$  by Lemma 19.1.

Recall

$$E_j = |X|^{-1} (q_j(0)A_0 + \cdots + q_j(D)A_D).$$

So,

$$(E_j)_{xy} = |X|^{-1} q_j(i), \quad \text{where } (x, y) \in R_i.$$

R2: Recall

$$A_i = p_i(0)E_0 + \cdots + p_i(D)E_D.$$

So,  $E_j A_i = p_i(j) E_j$ , and

$$p_i(j) \rho(x) = p_i(j) E_j \hat{x} = E_j A_i \hat{x} = E_j \sum_{y \in X, (y, x) \in R_i} \hat{y} = \sum_{y \in X, (y, x) \in R_i} \rho(y).$$

**Note.**

$$A_i \hat{x} = \sum_{y \in X, (x, y) \in R_{i'}} \hat{y}.$$

*Pf.*

$$z \text{ entry of LHS} = (A_i \hat{x})_z \quad (23.30)$$

$$= \sum_{w \in X} (A_i)_{zw} \hat{x}_w \quad (23.31)$$

$$= (A_i)_{zx} \quad (23.32)$$

$$= \begin{cases} 1 & \text{if } (x, z) \in R_{i'} \\ 0 & \text{else.} \end{cases} \quad (23.33)$$

$$z \text{ entry of RHS} = \sum_{y \in X, (x, y) \in R_{i'}, z=y} 1 \quad (23.34)$$

$$= \begin{cases} 1 & \text{if } (x, z) \in R_{i'} \\ 0 & \text{else.} \end{cases} \quad (23.35)$$

(iv) By (ii),

$$\|\rho(x)\|^2 = \langle \rho(x), \rho(y) \rangle \quad (23.36)$$

$$|X|^{-1} q_j(0) \quad (23.37)$$

$$|X|^{-1} m_j, \quad (23.38)$$

as  $m_j = \dim E_j V$ , and is independent of  $x \in X$ .

Pick distinct  $x, y \in X$  such that  $(x, y) \in R_i$  with  $i \neq 0$ .

Then,

$$\rho(x) = \rho(y) \Leftrightarrow \langle \rho(x), \rho(y) \rangle = \|\rho(x)\|^2 = |X|^{-1} q_j(0) \quad (23.39)$$

$$\Leftrightarrow |X|^{-1} q_j(i) = |X|^{-1} q_j(0) \quad (23.40)$$

$$\Leftrightarrow q_j(i) = q_j(0). \quad (23.41)$$

Hence, we have (iv). To be continued.

□



## Chapter 24

# Balanced Conditions, I

Wednesday, March 23, 1993

No Class on Friday (another conference).

*Proof of Lemma 23.1 continued.* Let  $E_j$  be a primitive idempotent,  $H = E_j V$  and

$$\rho : X \rightarrow H \quad (x \mapsto E_j \hat{x}).$$

(v) Every representation  $(\rho, H)$  of  $Y$  is equivalent to a representation of above type, for some  $j$  ( $0 \leq j \leq D$ ) and  $j$  is unique.

Let  $E := (\langle \rho(x), \rho(y) \rangle)_{x, y \in X}$ .

By R2,

$$E = \sum_{i=0}^D \sigma_i A_i, \quad \text{some } \sigma_0, \sigma_1, \dots, \sigma_D \in \mathbb{C}.$$

Hence,  $E$  belongs to the Bose-Mesner algebra  $\mathcal{M}$  of  $Y$ .

We want to show that  $E$  is a scalar multiple of a primitive idempotent.

Fix  $x \in X$  and fix  $i$  ( $0 \leq i \leq D$ ).

By R3,

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) = \alpha \rho(x), \quad \text{some } \alpha \in \mathbb{C}. \quad (24.1)$$

So,

$$k_i \bar{\sigma}_i = \left\langle \sum_{y \in X, (y, x) \in R_i} \rho(y), \rho(x) \right\rangle = \bar{\alpha} \langle \rho(x), \rho(x) \rangle = \bar{\alpha} \sigma_0.$$

Hence,  $\alpha$  is independent of  $x$ . In maatrix form (24.1) becomes

$$EA_i \hat{x} = \alpha E \hat{x}.$$

*Remark.*

$$Eu = Ev \Leftrightarrow \langle z, Eu \rangle = \langle z, Ev \rangle \text{ for all } z \in X \Leftrightarrow (Eu)_z = (Ev)_z \text{ for all } z \in X.$$

$$(EA_i \hat{x})_z = \left\langle \rho(z), \sum_{y \in X, (y, x) \in R_i} \rho(y) \right\rangle \quad (24.2)$$

$$= \alpha \langle \rho(z), \rho(x) \rangle \quad (24.3)$$

$$= (\alpha E \hat{x})_z. \quad (24.4)$$

Hence,

$$EA_i \hat{x} = \alpha E \hat{x}.$$

Since  $x$  is arbitrary,

$$EA_i = \alpha E.$$

So,

$$EA_i \in \text{Span} E \text{ and } EM = \text{Span} E.$$

We have  $E \in E_j$  for unique  $j$  ( $0 \leq j \leq D$ ). □

*Remark.*

$$E = \tau_0 E_0 + \cdots + \tau_D E_D, \quad \tau_j \in \mathbb{C} \quad (0 \leq j \leq D).$$

And, at least one of  $\tau_j$  is nonzero, and

$$\tau_j E_j = EE_j \in \text{Span} E.$$

So,

$$\tau_j E_j = E$$

as  $E_0, \dots, E_D$  are linearly independent.

Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a symmetric scheme, and let  $E$  be a primitive idempotent.

**Definition 24.1.**  $Y$  is  $Q$ -polynomial with respect to  $E$ , if and only if  $Y$  is  $Q$ -polynomial with respect to some ordering  $E_0, E_1, \dots, E_D$  of primitive idempotents, where  $E_0 = |X|^{-1}J$ , and  $E_1 = E$ .

**Theorem 24.1.** Assume  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  is  $P$ -polynomial (i.e.,  $(X, R_1)$  is distance-regular). Let  $E$  be any primitive idempotent of  $Y$ . Let  $(\rho, H)$  be the corresponding representation.

(i) The following are equivalent.

(ia)  $Y$  is  $Q$ -polynomial with respect to  $E$ .

(ib)  $(\rho, H)$  is nondegenerate and for all  $x, y \in X$ , and for all  $i, j$  ( $0 \leq i, j \leq D$ ),

$$\sum_{z \in X, (x, z) \in R_i, (y, z) \in R_j} \rho(z) - \sum_{z' \in X, (x, z') \in R_j, (y, z') \in R_i} \rho(z') \in \text{Span}(\rho(x) - \rho(y)).$$

(ic)  $(\rho, H)$  is nondegenerate and for all  $x, y \in X$ ,

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') \in \text{Span}(\rho(x) - \rho(y)).$$

(ii) Wirte

$$E = |X|^{-1} \sum_{j=0}^D \theta_j^* A_j,$$

and suppose (ia) – (ic) hold. Then the coefficient in (ib) is

$$p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \quad (1 \leq h \leq D, 0 \leq i, j \leq D).$$

*Proof.*

(ia)  $\rightarrow$  (ib) Without loss of generality, assume  $E \equiv E_1$ , and  $Y$  is  $Q$ -polynomial with respect to  $E$ .

Then by Lemma 22.2,  $\theta_0^*, \dots, \theta_D^*$  are distinct. So  $\theta_h^* \neq \theta_0^*$  for all  $h \in \{1, 2, \dots, D\}$ , and  $(\rho, H)$  is nondegenerate.

Fix  $x \in X$ , write  $E_i^* \equiv E_i^*(x)$ ,  $A_i^* \equiv A_i^*(x)$ ,  $A^* \equiv A_1^*$ .

Let  $M$  be the Bose-Mesner algebra. Set

$$L = \{mA^*n - nA^*m \mid m, n \in M\}.$$

Claim 1.  $\dim L \leq D$ .

*Proof of Claim 1.*

$$L = \text{Span}(E_i A^* E_j - E_j A^* E_i \mid 0 \leq i < j \leq D) \quad (24.5)$$

$$= \text{Span}(E_i A^* E_{i+1} - E_{i+1} A^* E_i \mid 0 \leq i \leq D-1). \quad (24.6)$$

Since  $E_i A^* E_j = O$  if  $q_{ij}^1 = 0$  by Lemma 20.2 and Lemma 20.3, and this occurs if  $|i - j| > 1$  by  $Q$ -polynomial property.

Hence,  $\dim L \leq D$ .

Claim 2. (i)  $\{A^* A_h - A_h A^* \mid 1 \leq h \leq D\}$  is a basis for  $L$ . In particular,

(ii) there exist  $r_{ij}^h \in \mathbb{C}$  ( $1 \leq h \leq D, 0 \leq i, j \leq D$ ) such that

$$A_i A^* A_j - A_j A^* A_i = \sum_{h=1}^D r_{ij}^h (A^* A_h - A_h A^*).$$

*Proof of Claim 2.*

(i) The column  $x$  of  $A^* A_h - A_h A^*$  is a nonzero scalar  $\theta_h^* - \theta_0^*$  times the column  $x$  of  $A_h$ .

*Remark.*

$$((A^* A_h - A_h A^*) \hat{x})_y = E_{xy}(A_h)_{yx} - (A_h)_{yx} E_{xx} = (\theta_h^* - \theta_0^*)(A_h)_{yz}.$$

Also the column  $x$  of  $A_0, A_1, \dots, A_D$  are linearly independent.

Hence, the matrices given are linearly independent.

They are in  $L$  by construction, so they form a basis for  $L$  by Claim 1.

(ii) This is immediate since

$$A_i A^* A_j - A_j A^* A_i \in L, \quad \text{for all } i, j.$$

Claim 3.

$$r_{ij}^\ell = p_{ij}^\ell \left( \frac{\theta^* - \theta_j^*}{\theta_0^* - \theta_\ell^*} \right) \quad (1 \leq \ell \leq D, 0 \leq i, j \leq D).$$

*Proof of Claim 3.* Fix  $i, j$ ,

$$A_i A^* A_j - A_j A^* A_i - \sum_{h=1}^D r_{ij}^h (A^* A_h - A_h A^*) = 0.$$

Pick  $\ell$  ( $1 \leq \ell \leq D$ ). Pick  $y \in X$  such that  $(x, y) \in R_\ell$ .

$$(A_i A^* A_j)_{xy} = \sum_{z \in X} (A_i)_{xz} (A^*)_{zz} (A_j)_{zy} \quad (24.7)$$

$$= \sum_{z \in X, (x, z) \in R_i, (y, z) \in R_j} (A^*)_{zz} \quad (24.8)$$

$$= |X|^{-1} p_{ij}^\ell \theta_i^*. \quad (24.9)$$

Similarly,

$$(A_j A^* A_i)_{xy} = |X|^{-1} p_{ij}^\ell \theta_j^*.$$

$$(A^*A_h - A_hA^*)_{xy} = (A_0A^*A_h - A_hA^*A_0)_{xy} \quad (24.10)$$

$$= |X|^{-1}p_{0h}^\ell(\theta_0^* - \theta_h^*) \quad (24.11)$$

$$= \begin{cases} 0 & \text{if } \ell \neq h \\ |X|^{-1}(\theta_0^* - \theta_h^*) & \text{if } \ell = h. \end{cases} \quad (24.12)$$

Hence,

$$\sum_{h=1}^D r_{ij}^h (A^*A_h - A_hA^*)_{xy} = |X|^{-1}r_{ij}^\ell(\theta_0^* - \theta_\ell^*).$$

Comparing terms, we have

$$p_{ij}^\ell(\theta_i^* - \theta_j^*) - r_{ij}^\ell(\theta_0^* - \theta_\ell^*) = 0.$$

Claim 4. For all  $h$  ( $1 \leq h \leq D$ ), for all  $i, j$  ( $0 \leq i, j \leq D$ ), for all  $w, y \in X$ ,  $(w, y) \in R_h$ ,

$$\sum_{z \in X, (w, z) \in R_i, (y, z) \in R_j} \rho(z) - \sum_{z' \in X, (w, z') \in R_j, (y, z') \in R_i} \rho(z') - r_{ij}^h(\rho(w) - \rho(y)) = 0. \quad (24.13)$$

*Proof of Claim 4.* Set  $L = \langle \text{LHS of (24.13)}, \rho(x) \rangle$ . It suffices to show that  $L = 0$ .

Note that since  $x$  is arbitrary, if LHS of (24.13) is zero.

$$L = \sum_{z \in X, (w, z) \in R_i, (y, z) \in R_j} \langle \rho(z), \rho(x) \rangle - \sum_{z' \in X, (w, z') \in R_j, (y, z') \in R_i} \langle \rho(z'), \rho(x) \rangle \quad (24.14)$$

$$- r_{ij}^h \langle \rho(w) - \rho(y), \rho(x) \rangle \quad (24.15)$$

$$= |X|^{-1}(A_iA^*A_j)_{wy} - |X|^{-1}(A_jA^*A_i)_{wy} - |X|^{-1} \sum_{\ell=1}^D r_{ij}^\ell (A^*A_\ell - A_\ell A^*)_{wy} \quad (24.16)$$

$$= |X|^{-1} \text{times } wy \text{ entry of a matrix known to be zero by Claim 2} \quad (24.17)$$

$$= 0. \quad (24.18)$$

□

*Remark.*

$$|X|^{-1} \sum_{\ell=1}^D r_{ij}^\ell (A^*A_\ell - A_\ell A^*)_{wy} = |X|^{-1} r_{ij}^h (A^*A_h - A_h A^*)_{wy} \quad (24.19)$$

$$= r_{ij}^h (\langle \rho(x), \rho(w) \rangle - \langle \rho(x), \rho(y) \rangle) \quad (24.20)$$



## Chapter 25

# Balanced Conditions, II

Monday, March 29, 1993

*Proof of Theorem 24.1 continued.*

(ib)  $\rightarrow$  (ic) Obvious.

(ic)  $\rightarrow$  (ia) Without loss of generality, we may assume  $D \geq 3$ , else trivial.

*Remark.* The case  $D = 2$  should be treated somewhere, but the assumption  $D \geq 3$  is not used.

Fix  $w \in X$ , and write  $E_i^* \equiv E_i^*(w)$ ,  $A_i^* \equiv A_i^*(w)$ ,  $A^* \equiv A_1^*$ , and  $A_i$ ,  $i$ -th distance matrix. Set

$$E \equiv E_1 = |X|^{-1} \sum_{i=0}^D \theta_i^* A_i.$$

Since  $(\rho, H)$  is nondegenerate,

$$\theta_0^* \neq \theta_h^*, \text{ for all } h \in \{1, 2, \dots, D\}$$

See Lemma 23.1 (iv).

Claim 1. Pick  $h$  ( $1 \leq h \leq D$ ), and  $x, y$  with  $(x, y) \in R_h$ . Then

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') = r_{12}^h (\rho(x) - \rho(y)),$$

where

$$r_{12}^h = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}.$$

*Proof of Claim 1.* By our assumption,

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') = \alpha (\rho(x) - \rho(y)).$$

Hence,

$$|X|^{-1}p_{12}^h(\theta_1^* - \theta_2^*) = \left\langle \sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z'), \rho(x) \right\rangle \quad (25.1)$$

$$= \alpha \langle \rho(x) - \rho(y), \rho(x) \rangle \quad (25.2)$$

$$= \alpha |X|^{-1}(\theta_0^* - \theta_h^*). \quad (25.3)$$

We have

$$\alpha = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}.$$

$$\text{Claim 2. } A_1 A^* A_2 - A_2 A^* A_1 = \sum_{h=1}^D r_{12}^h (A^* A_h - A_h A^*).$$

*Proof of Claim 2.* The  $xy$  entry of the LHS – RHS is

$$|X| \left\langle \sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z') - r_{12}^h (\rho(x) - \rho(y)), \rho(w) \right\rangle,$$

where  $(x, y) \in R_h$ ,  $h = 1, 2, \dots, D$ , and the  $xy$  entry of the LHS – RHS is 0 if  $x = y$ .

But the vector on the left in the above inner product is 0 by Claim 1, so the inner product is 0.

Thus, the  $xy$  entry of the LHS – RHS is always 0, and we have Claim 2.

$$\text{Claim 3. } A^* A_3 - A_3 A^* \in \text{Span}(AA^* A_2 - A_2 A^* A, A^* A_2 - A_2 A^*, A^* A - AA^*).$$

*Proof of Claim 3.* Since  $p_{12}^h = 0$ , if  $h > 3$ , and  $p_{12}^h \neq 0$ , if  $h = 3$ , we have  $r_{12}^h = 0$  if  $h > 0$ , and  $r_{12}^h \neq 0$ , if  $h = 3$ . Note that  $\theta_1^* \neq \theta_2^*$ .

Now we are done by Claim 2.

Claim 4. There exist  $\beta, \gamma, \delta \in \mathbb{R}$  such that

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(AA^* + A^* A) - \delta A^*] \quad (25.4)$$

$$= A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A). \quad (25.5)$$

*Proof of Claim 4.* There exists  $f_i \in \mathbb{R}[\lambda]$ ,  $\deg f_i = i$  such that  $A_i = f_i(A_1)$ .

Writing  $A_2, A_3$  as polynomials in  $A$  in Claim 3 and simplifying, we find

$$A^3 A^* - A^* A^3 \in \text{Span}(A^2 A^* A - AA^* A^2, A^2 A^* - A^* A^2, AA^* - A^* A).$$



*Remark.* Let  $A_3 = \beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I$  with  $\beta_3 \neq 0$ , and  $A_2 = \gamma_2 A^2 + \gamma_1 A + \gamma_0 I$ , with  $\gamma_2 \neq 0$ . Then

$$A^* A_3 - A_3 A^* = A^*(\beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I) - (\beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I)A^*. \quad (25.6)$$

$$A^3 A^* - A^* A^3 \in \text{Span}(A^* A_3 - A_3 A^*, A^2 A^* - A^* A^2, AA^* - A^* A) \quad (25.7)$$

$$\subseteq \text{Span}(AA^* A_2 - A_2 A^* A, A^* A_2 - A_2 A^*, A^2 A^* - A^* A^2, AA^* - A^* A) \quad (25.8)$$

$$A^* A_2 - A_2 A^* = A^*(\gamma_2 A^2 + \gamma_1 A + \gamma_0 I) - (\gamma_2 A^2 + \gamma_1 A + \gamma_0 I)A^* \quad (25.9)$$

$$AA^* A_2 - A_2 A^* A = AA^*(\gamma_2 A^2 + \gamma_1 A + \gamma_0 I) - (\gamma_2 A^2 + \gamma_1 A + \gamma_0 I)A^* A \quad (25.10)$$

$$A^* A_2 - A_2 A^* \in \text{Span}(A^2 A^* - A^* A^2, AA^* - AA^*) \quad (25.11)$$

$$AA^* A_2 - A_2 A^* A \in \text{Span}(A^2 A^* A - AA^* A^2, AA^* - AA^*) \quad (25.12)$$

$$A^3 A^* - A^* A^3 \in \text{Span}(A^2 A^* A - AA^* A^2, A^2 A^* - A^* A^2, AA^* - A^* A). \quad (25.13)$$

Hence, we can find  $\delta, \gamma, \delta$  satisfying

$$0 = A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A).$$

On the other hand,

$$[A, A^2 A^* - \beta AA^* A + A^* A^2 - \gamma(AA^* + A^* A) - \delta A^*] \quad (25.14)$$

$$= A^3 A^* - A^2 A^* A - \beta A^2 A^* A + \beta AA^* A^2 + AA^* A^2 - A^* A^3 \quad (25.15)$$

$$- \gamma A^2 A^* - \gamma AA^* A + \gamma AA^* A + \gamma A^* A^2 - \delta AA^* + \delta A^* A \quad (25.16)$$

$$= A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A). \quad (25.17)$$

Thus we have (i) and (ii).

Define a diagram  $D_E$  on nodes  $0, 1, \dots, D$ .

Connect distinct nodes, by undirected arc if  $q_{ij}^1 \neq 0$ . (Note  $q_{ij}^1 = q_{ji}^1$ ).

Since  $q_{0j}^1 = \delta_{1j}$ , the 0-node is adjacent to the 1-node and no other node.

$Y$  is  $Q$ -polynomial with respect to  $E$  if and only if  $E_E$  is a path.

Claim 5.  $D_E$  is connected.

*Proof of Claim 5.* Suppose there exists  $\Delta \subseteq \{0, 1, \dots, D\}$  such that  $i, j$  not connected for every  $i \in \Delta$  and  $j \in \{0, 1, \dots, D\} \setminus \Delta$ .

Set

$$f = \sum_{i \in \Delta} E_i.$$

Observe

$$fA^* = \sum_{i \in \Delta} E_i A^* \left( \sum_{j=0}^D E_j \right) \quad (25.18)$$

$$= \sum_{i \in \Delta, j \in \Delta} E_i A^* E_j \quad (\text{since } E_i A^* E_j = O \text{ if } q_{ij}^1 = 0) \quad (25.19)$$

$$= fA^* f. \quad (25.20)$$

Also,  $A^* f = fA^* f$ .

Hence,  $f$  commutes with  $A^*$ .

But  $f$  is an element of the Bose-Mesner algebra

$$f = \sum_{i=0}^D \alpha_i A_i \quad \text{for some } \alpha_0, \dots, \alpha_D \in \mathbb{C}.$$

We have

$$0 = fA^* - A^* f = \sum_{i=1}^D \alpha_i (A_i A^* - A^* A_i).$$

But  $\{A_h A^* - A^* A_h \mid 1 \leq h \leq D\}$  are linearly independent. (The column  $w$  of  $A_h A^* - A^* A_h$  is  $\theta_h^* - \theta_0^*$  times the column  $w$  of  $A_h$ .)

Hence,  $\alpha_1 = \dots = \alpha_D = 0$ , and  $f = \alpha_0 I$ . Since  $f^2 = f$ ,  $\alpha_0$  is 0 or 1.

If  $\alpha_0 = 0$ ,  $f = O$  and  $\Delta = \emptyset$ .

If  $\alpha_0 = 1$ ,  $f = I$  and  $\Delta = \{0, 1, \dots, D\}$ .

This proves Claim 5. □

*Remark.* Claim 5 proves the following in general.

Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  be a symmetric association scheme. Fix a vertex  $x \in X$ , and let

$$E = \frac{1}{|X|} \sum_{j=0}^D \theta_j^* A_j \quad (\theta_j^* = q_1(j) \text{ if } E = E_1)$$

be a primitive idempotent and  $E_j^* \equiv E_j^*(x)$ .

$$A^* = \sum_{j=0}^D \theta_j^* E_j^*.$$

If  $\theta_0 = \theta_h^*$ ,  $h = 1, \dots, D$ , then the following hold.

- (i)  $\{A_h A^* - A^* A_h \mid 1 \leq h \leq D\}$  are linearly independent.
- (ii) The diagram  $D_E$  on nodes  $0, 1, \dots, D$  defined by

$$i \sim j \Leftrightarrow E(E_i \circ E_j) \neq O$$

is connected.

(iii)  $C_M(A^*) = \{L \in M \mid LA^* = A^*L\} = \text{Span}(I)$ .

*Proof.* | (i) The column  $x$  of  $A_h A^* - A^*(A_h)$  is  $\theta_0^* - \theta_h^*$  times the column  $x$  of  $A_h$ .

(iii)  $0 = [\sum_{h=0}^D \alpha_h A_h, A^*] = \sum_{h=1}^D \alpha_h (A_h A^* - A^* A_h)$ . Hence,  $\alpha_0 = \dots = \alpha_D = 0$ .

(ii)  $\Delta$  is a connected component. Let  $f = \sum_{i \in \Delta} E_i$ , then  $f \in C_M(A^*)$ .

Let  $Y = (X, \{R_i\}_{0 \leq i \leq 2})$  be a symmetric association scheme with  $D = 2$ . Let

$$E = \frac{1}{|X|} \sum_{j=0}^2 \theta_j^* A_j$$

be a primitive idempotent. If  $\theta_0^*, \theta_1^*, \theta_2^*$ .

Then  $Y$  is  $Q$ -polynomial with respect to  $E$ .

*Proof.* By the previous lemma,  $D_E$  is connected.

**Note.** It seems  $\theta_1^* \neq \theta_2^*$  is necessary. Clarify the condition  $\theta_1^* = \theta_2^*$ .

Terwilliger claims that  $\theta_1^* = \theta_2^*$  does not occur under the assumption (ic).  
(March 7, 1995)



## Chapter 26

# Representation Diagrams

Wednesday, March 31, 1993

*Proof of Theorem 24.1 continued.* Assume  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  is  $P$ -polynomial. Let  $E$  be a primitive idempotent of  $Y$  such that the corresponding representation  $(\rho, H)$  is nondegenerate.

Show for all  $x, y \in X$ ,

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') \in \text{Span}(\rho(x) - \rho(y))$$

implies that  $Y$  is  $Q$ -polynomial with respect to  $E$ .

Define a diagram  $D_E$  on nodes  $0, 1, \dots, D$ , for  $i \neq j$ ,

$$i \frown j \leftrightarrow q_{ij}^1 \neq 0$$

by setting  $E = E_1$ .

We showed that  $0 \frown j \leftrightarrow j = 1$  ( $1 \leq j \leq D$ ) and  $D_E$  is connected.

Now it is sufficient to show the following.

Claim 6. Let  $i$  be a node in  $D_E$ . Then  $i$  is adjacent to at most 2 arcs.

*Proof of Claim 6.* Suppose the node  $j$  is adjacent to  $i$  in  $D_E$ . By claim 4,

$$0 = E_i(A^3A^* - A^*A^3 - (\beta + 1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A) - \delta(AA^* - A^*A))E_j \quad (26.1)$$

$$= E_iA^*E_j(\theta_i^3 - \theta_j^3 - (\beta + 1)(\theta_i^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j)) \quad (26.2)$$

$$= E_iA^*A_j(\theta_i - \theta_j)p(\theta_i, \theta_j), \quad (26.3)$$

where

$$p(s, t) = s^2 - \beta st + t^2 - \gamma(s + t) - \delta.$$

*Remark.*

$$(\theta_i - \theta_j)(\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \delta) \quad (26.4)$$

$$= \theta_i^3 - \theta_j^3 - (\beta + 1)(\theta_i^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j) \quad (26.5)$$

Since  $i$  is adjacent to  $j$ ,  $q_{ij}^1 \neq 0$  and

$$E_i A^* E_j \neq O$$

by Lemma 20.3 (ii). Since  $Y$  is  $P$ -polynomial,

$$\theta_i \neq \theta_j \quad \text{if } i \neq j.$$

Hence  $p(\theta_i, \theta_j) = 0$ . But  $p$  is quadratic in  $t$ . So  $p(\theta_i, t) = 0$  has at most two solutions for  $\theta_j$ .

Now  $D_E$  is a pth, and  $\Gamma$  is  $Q$ -polynomial with respect to  $E$ .

This proves Theorem 24.1. □

## Chapter 27

# Title of the Chapter

Wednesday, February 17, 1993 # Edit Date





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