

Lecture Note on Terwilliger Algebra

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Contents

About this lecturenote	5
Setting	5
Another Host	6
1 Subconstituent Algebra of a Graph	7
2 Perron-Frobenius Theorem	13
3 Cayley Graphs	19
4 Examples	25
5 T-Modules of $H(D, 2)$, I	31
6 T-Modules of $H(D, 2)$, II	37
7 The Johnson Graph $J(D, N)$	43
8 Thin Graphs	49
9 Thin T-Module, I	55
10 Thin T-Module, II	61
11 Examples of T-Module	67
12 Distance-Regular	71
13 Modules of a DRG	79
14 Parameters of Thin Modules, I	85
15 Parameters of Thin Modules, II	89
16 Thin Modoles of a DRG	95

17 Association Schemes	103
18 Polynomial Schemes	107
19 Commutative Association Schemes	109
20 Vanishing Conditions	115
21 Norton Algebras	121
22 Q-Polynomial Schemes	125
23 Representation of a Scheme	131
24 Balanced Conditions, I	137
25 Balanced Conditions, II	143
26 Representation Diagrams	149
27 P-and Q-Polynomial Schemes	155
28 Title of the Chapter	161

About this lecturenote

Setting

sudo This note is created by `bookdown` package on RStudio.

For `bookdown` See (Xie, 2015), (Xie, 2017), (Yihui Xie, 2018).

1. Log-in to my GitHub Account
2. Go to RStudio/bookdown-demo repository: <https://github.com/rstudio/bookdown-demo>
3. Use This Template
4. Input Repository Name
5. Select Public - default
6. Create repository from template
7. From Code download ZIP
8. Move the extracted folder into a favorite directory
9. Open RStudio Project in the folder
10. Use Terminal in the bottom left pane
 - confirm that the current directory is the home directory of the project by `pwd`
11. (failed to proceed by ssh)
12. Use Console
 1. `library(usethis)`
 2. `use_git()`
 3. `use_github()` — Error
 4. `gh_token_help()`
 5. `create_github_token()`: create a token in the github page. Copy the token
 6. `gitcreds::gitcreds_set()`: paste the token, the token is to be expired in 30 days
13. Use Terminal
 1. `git remote add origin https://github.com/icu-hsuzuki/t-algebra.git`
 2. `git push -u origin main`
 3. type in the password of the computer
14. Use GIT in R Studio

Another Host

1. create a project by version control git
2. git init
3. git remote add origin git@github.com:/.git
4. git branch -r
5. git fetch
6. git pull origin main

Chapter 1

Subconstituent Algebra of a Graph

Wednesday, January 20, 1993

A graph (undirected, without loops or multiple edges) is a pair $\Gamma = (X, E)$, where

$$X = \text{finite set (of vertices)} \quad (1.1)$$

$$E = \text{set of (distinct) 2-element subsets of } X \text{ (= edges of } \Gamma). \quad (1.2)$$

vertices x and $y \in X$ are adjacent if and only if $xy \in E$.

Example 1.1. Let Γ be a graph. $X = \{a, b, c, d\}$, $E = \{ab, ac, bc, bd\}$.



Set $n = |X|$, the order of Γ .

Pick a field K ($= \mathbb{R}$ or \mathbb{C}). Then $\text{Mat}_X(K)$ denotes the K algebra of all $n \times n$ matrices with entries in K . (rows and columns are indexed by X)

Adjacency matrix $A \in \text{Mat}_X(K)$ is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E \\ 0 & \text{else .} \end{cases} \quad (1.3)$$

Example 1.2. Let a, b, c, d be labels of rows and columns. Then

$$A = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

The subalgebra M of $\text{Mat}_X(K)$ generated by A is called the *Bose-Mesner algebra* of Γ .

Set $V = K^n$, the set of n -dimensional column vectors, the coordinates are indexed by X .

Let $\langle \cdot, \cdot \rangle$ denote the Hermitean inner product:

$$\langle u, v \rangle = u^\top \cdot v \quad (u, v \in V)$$

V with $\langle \cdot, \cdot \rangle$ is the *standard module* of Γ .

M acts on V : For every $x \in X$, write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where 1 is at the x position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since A is a real symmetrix matrix,

$$V = V_0 + V_1 + \cdots + V_r \quad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal A -eigenspaces.

Let $E_i \in \text{Mat}_X(K)$ denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then E_0, \dots, E_r are the primitive idempotents of M .

$$M = \text{Span}_K(E_0, \dots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all } i, j, \quad E_0 + \cdots + E_r = I.$$

Let θ_i denote the eigenvalue of A for V_i in \mathbb{R} . Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \dots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \theta_i = \dim V_i = \text{rank } E_i.$$

Set

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \dots, & \theta_r \\ m_0, & m_1, & \dots, & m_r \end{pmatrix}.$$

Problem. What can we say about Γ when $\text{Spec}(\Gamma)$ is given?

The following Lemma 1.1, is an example of Problem.

For every $x \in X$,

$$k(x) \equiv \text{valency of } x \equiv \text{degree of } x \equiv |\{y \mid y \in X, xy \in E\}|.$$

Definition 1.1. The graph Γ is regular of valency k if $k = k(x)$ for every $x \in X$.

Lemma 1.1. *With the above notation,*

- (i) $\theta_0 \leq \max\{k(x) \mid x \in X\} = k^{\max}$.
- (ii) *If Γ is regular of valency k , then $\theta_0 = k$.*

Proof.

(i) Without loss of generality we may assume that $\theta_0 > 0$, else done. Let $v := \sum_{x \in X} \alpha_x \hat{x}$ denote the eivenvector for θ_0 .

Pick $x \in X$ with $|\alpha_x|$ maximal. Then $|\alpha_x| \neq 0$.

Since $Av = \theta_0 v$,

$$\theta_0 \alpha_x = \sum_{y \in X, xy \in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \leq \sum_{y \in X, xy \in E} |\alpha_y| \leq k(x) |\alpha_x| \leq k^{\max} |\alpha_x|.$$

(ii) All 1's vector $v = \sum_{x \in X} \hat{x}$ satisfies $Av = kv$.

□

Subconstituent Algebra

Let $x, y \in X$ and $\ell \in \mathbb{Z}^{\geq 0}$.

Definition 1.2. A path of length ℓ connecting x, y is a sequence

$$x = x_0, x_1, \dots, x_\ell = y, \quad x_i \in X, \quad 0 \leq i \leq \ell$$

such that $x_i x_{i+1} \in E$ for $0 \leq i \leq \ell - 1$.

Definition 1.3. The distance $\partial(x, y)$ is the length of a shortest path connecting x and y .

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

Definition 1.4. The graph Γ is connected if and only if $\partial(x, y) < \infty$ for all $x, y \in X$.

From now on, assume that Γ is connected with $|X| \geq 2$.

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \text{the diameter of } \Gamma.$$

Fix a ‘base’ vertex $x \in X$.

Definition 1.5.

$$d(x) = \text{the diameter with respect to } x = \max\{\partial(x, y) \mid y \in X\} \leq d.$$

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad (\text{orthogonal direct sum}),$$

where

$$V_i^* = \text{Span}_K(\hat{y} \mid \partial(x, y) = i) \equiv V_i * (x)$$

and $V_i^* = V_i^*(x)$ is called the i -th subconstituent with respect to x .

Let $E_i^* = E_i^*(x)$ denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View $E_i^*(x) \in \text{Mat}_X(K)$. So, $E_i^*(x)$ is diagonal with yy entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{else,} \end{cases} \quad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \text{Span}_K(E_0^*(x), \dots, E_{d(x)}^*(x)).$$

Then $M^*(x)$ is a commutative subalgebra of $\text{Mat}_X(K)$ and is called the *dual Bose-Mesner algebra with respect to x* .

Definition 1.6 (Subconstituent Algebra). Let $\Gamma = (X, E)$, $x, M, M^*(x)$ be as above. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(K)$ generated by M and $M^*(x)$. T is the *subconstituent algebra* of Γ with respect to x .

Definition 1.7. A T -module is any subspace $W \subset V$ such that $aw \in W$ for all $a \in T$ and $w \in W$.

T -module W is *irreducible* if and only if $W \neq 0$ and W does not properly contain a nonzero T -module.

For any $a \in \text{Mat}_X(K)$, let a^* denote the conjugate transpose of a .

Observe that

$$\langle au, v \rangle = \langle u, a^*v \rangle \quad \text{for all } a \in \text{Mat}_X(K), \text{ and for all } u, v \in V.$$

Lemma 1.2. *Let $\Gamma = (X, E)$, $x \in X$ and $T \equiv T(x)$ be as above.*

(i) *If $a \in T$, then $a^* \in T$.*

(ii) *For any T -module $W \subset V$,*

$$W^\perp := \{v \in V \mid \langle w, v \rangle = 0, \text{ for all } w \in W\}$$

is a T -module.

(iii) *V decomposes as an orthogonal direct sum of irreducible T -modules.*

Proof.

(i) It is because T is generated by symmetric real matrices

$$A, E_0^*(x), E_1^*(x), \dots, E_{d(x)(x)}^*.$$

(ii) Pick $v \in W^\perp$ and $a \in T$, it suffices to show that $av \in W^\perp$. For all $w \in W$,

$$\langle w, av \rangle = \langle a^*w, v \rangle = 0$$

as $a^* \in T$.

(iii) This is proved by the induction on the dimension of T -modules. If W is an irreducible T -module of V , then

$$V = W + W^\perp \quad (\text{orthogonal direct sum}).$$

□

Problem. What does the structure of the $T(x)$ -module tell us about Γ ?

Study those Γ whose modules take ‘simple’ form. The Γ ’s involved are highly regular.

Remark.

1. The subconstituent algebra T is semisimple as the left regular representation of T is completely reducible. See Curtis-Reiner 25.2 (Charles W. Curtis, 2006).
2. The inner product $\langle a, b \rangle_T = \text{tr}(a^\top \bar{b})$ is nondegenerate on T .

3. In general,

T : Semisimple and Artinian $\Leftrightarrow T$: Artinian with $J(T) = 0$

$\Leftrightarrow T$: Artinian with nonzero nilpotent element

$\Leftrightarrow T \subset \text{Mat}_X(K)$ such that for all $a \in T$ is normal.

Chapter 2

Perron-Frobenius Theorem

Friday, January 22, 1993

In this lecture we use the Perron Frobenius theory of nonnegative matrices to obtain information on eigenvalues of a graph.

Let $K = \mathbb{R}$. For $n \in \mathbb{Z}^{>0}$, pick a symmetric matrix $C \in \text{Mat}_n(\mathbb{R})$.

Definition 2.1. The matrix C is *reducible* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i \in X^+$, and for all $j \in X^-$, and for all $i \in X^-$, and for all $j \in X^+$, i.e.,

$$C \sim \begin{pmatrix} * & O \\ O & * \end{pmatrix}.$$

Definition 2.2. The matrix C is *bipartite* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i, j \in X^+$, and for all $i, j \in X^-$, i.e.,

$$C \sim \begin{pmatrix} O & * \\ * & O \end{pmatrix}.$$

Note.

1. If C is bipartite, for every eigenvalue θ of C , $-\theta$ is an eigenvalue of C such that $\text{mult}(\theta) = \text{mult}(-\theta)$.

Indeed, let $C = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$,

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix},$$

where $Ay = \theta x$ and $Bx = \theta y$.

2. If C is bipartite, C^2 is reducible.
3. The matrix C is irreducible and C^2 is reducible, if $C_{ij} \geq 0$ for all i, j and C is reducible. (Exercise)

Remark. Note 1. Even if C is not symmetric

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

holds. So the geometrix multiplicities coincide. How about the algebraic multiplicities?

Note 3. Set $x \sim y$ if and only if $C_{xy} > 0$. So the graph may have loops. Then

$$(C^2)_{xy} > 0 \Leftrightarrow \text{if there exists } z \in X \text{ such that } x \sim z \sim y.$$

Note that C is irreducible if and only if $\Gamma(C)$ is connected. Let

$$X^+ = \{y \mid \text{there is a path of even length from } x \text{ to } y\} \quad (2.1)$$

$$X^- = \{y \mid \text{there is no path of even length from } x \text{ to } y\} \neq \emptyset. \quad (2.2)$$

If there is an edge $y \sim z$ in X^+ and $w \in X^-$. Then there would be a path from x to y of even length. So $e(X^+, X^+) = e(X^-, X^-) = 0$.

Theorem 2.1 (Perron-Frobenius). *Given a matrix C in $\text{Mat}_n(\mathbb{R})$ such that*

- (a) C is symmetric.
- (b) C is irreducible.
- (c) $C_{ij} \geq 0$ for all i, j .

Let θ_0 be the maximal eigenvalue of C with eigenspace $V_0 \subseteq \mathbb{R}^n$, and let θ_r be the maximal eigenvalue of C with eigenspace $V_r \subseteq \mathbb{R}^n$. Then the following hold.

$$(i) \text{ Suppose } 0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0. \text{ Then } \alpha_0 > 0 \text{ for all } i, \text{ or } \alpha_i < 0 \text{ for all } i.$$

$$(ii) \dim V_0 = 1.$$

$$(iii) \theta_r \geq -\theta_0.$$

$$(iv) \theta_r = \theta_0 \text{ if and only if } C \text{ is bipartite.}$$

First, we prove the following lemma.

Lemma 2.1. *Let $\langle \cdot, \cdot \rangle$ be the dot product in $V = \mathbb{R}^n$. Pick a symmetric matrix $B \in \text{Mat}_n(\mathbb{R})$. Suppose all eigenvalues of B are nonnegative. (i.e., B is positive semidefinite.) Then there exist vectors $v_1, v_2, \dots, v_n \in V$ such that $B_{ij} = \langle v_i, v_j \rangle$ for $(1 \leq i, j \leq n)$.*

Proof. By elementary linear algebra, there exists an orthonormal basis w_1, w_2, \dots, w_n of V consisting of eigenvectors of B . Set the i -th column of P is w_i and $D = \text{diag}(\theta_1, \dots, \theta_n)$. Then $P^\top P = I$ and $BP = PD$.

Hence,

$$B = PDP^{-1} = PDP^\top = QQ^\top,$$

where

$$Q = P \cdot \text{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \dots, \sqrt{\theta_n}) \in \text{Mat}_n(\mathbb{R}).$$

Now, let v_i be the i -th column of Q^\top . Then

$$B_{ij} = v_i^\top \cdot v_j = \langle v_i, v_j \rangle.$$

□

Now we start the proof of Theorem 2.1.

Proof of Theorem 2.1(i)

Let \langle, \rangle denote the dot product on $V = \mathbb{R}^n$. Set

$$B = \theta I - C \tag{2.3}$$

$$= \text{symmetric matrix with eigenvalues } \theta_0 - \theta_i \geq 0 \tag{2.4}$$

$$= (\langle v_i, v_j \rangle)_{1 \leq i, j \leq n} \tag{2.5}$$

with the same $v_1, \dots, v_n \in V$ by Lemma 2.1.

Observe: $\sum_{i=1}^n \alpha_i v_i = 0$.

Pf.

$$\left\| \sum_{i=1}^n \alpha_i v_i \right\|^2 = \left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \right\rangle \tag{2.6}$$

$$= (\alpha_1 \quad \dots \quad \alpha_n) B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{2.7}$$

$$= v^\top B v \tag{2.8}$$

$$= 0, \tag{2.9}$$

since $Bv = (\theta_0 I - C)v = 0$.

Now set

$$s = \text{the number of indices } i, \text{ where } \alpha_i > 0.$$

Replacing v by $-v$ if necessary, without loss of generality we may assume that $s \geq 1$. We want to show $s = n$.

Assume $s < n$. Without loss of generality, we may assume that $\alpha_i > 0$ for $1 \leq i \leq s$ and $\alpha_i = 0$ for $s+1 \leq i \leq n$. Set

$$\rho = \alpha_1 v_1 + \dots + \alpha_s v_s = -\alpha_{s+1} v_{s+1} - \dots - \alpha_n v_n.$$

Then, for $i = 1, \dots, s$,

$$\langle v_i, \rho \rangle = \sum_{j=s+1}^n -\alpha_j \langle v_i, v_j \rangle \quad (\langle v_i, v_j \rangle = B_{ij}, B = \theta_0 I - C) \quad (2.10)$$

$$= \sum_{j=s+1}^n (-\alpha_j)(-C_{ij}) \quad (2.11)$$

$$\leq 0. \quad (2.12)$$

Hence

$$0 \leq \langle \rho, \rho \rangle = \sum_{i=1}^s \alpha_i \langle v_i, \rho \rangle \leq 0,$$

as $\alpha > 0$ and $\langle v_i, \rho \rangle \leq 0$. Thus, we have $\langle \rho, \rho \rangle = 0$ and $\rho = 0$. For $j = s+1, \dots, n$,

$$0 = \langle \rho, v_j \rangle = \sum_{i=1}^s \alpha_i \langle v_i, v_j \rangle \leq 0,$$

as $\langle v_i, v_j \rangle = -C_{ij}$.

Therefore,

$$0 = \langle v_i, v_j \rangle = -C_{ij} \text{ for } 1 \leq i \leq s, s+1 \leq j \leq n.$$

Since C is symmetric,

$$C = \begin{pmatrix} * & O \\ O & * \end{pmatrix}$$

Thus C is reducible, which is not the case. Hence $s = n$.

Proof of Theorem 2.1 (ii).

Suppose $\dim V_0 \geq 2$. Then,

$$\dim \left(V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^\perp \right) \geq 1.$$

So, there is a vector

$$0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with $\alpha_1 = 0$. This contradicts 1.

Now pick

$$0 \neq w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r.$$

Proof of Theorem 2.1 (iii).

Suppose $\theta_r < -\theta_0$. Since the eigenvalues of C^2 are the squares of those of C , θ_r^2 is the maximal eigenvalue of C^2 .

Also we have $C^2 w = \theta_r^2 w$.

Observe that C^2 is irreducible. (As otherwise, C is bipartite by Note 3, and we must have $\theta_r = -\theta_0$.) Therefore, $\beta_i > 0$ for all i or $\beta_i < 0$ for all i . We have

$$\langle v, w \rangle = \sum_{i=1}^n \alpha_i \beta_i \neq 0.$$

This is a contradiction, as $V_0 \perp V_r$.

Proof of Theorem 2.1 (iv)

\Rightarrow : Let $\theta_r = -\theta_0$. Then $\theta = \theta_1^2 = \theta_0^2$ is the maximal eigenvalue of C^2 , and v and w are linearly independent eigenvalues for θ for C^2 . Hence, for C^2 , $\text{mult}(\theta) \geq 2$.

Thus by 2, C^2 must be reducible. Therefore, C is bipartite by Note 3.

\Leftarrow : This is Note 1. \square

Let $\Gamma = (X, E)$ be any graph.

Definition 2.3. Γ is said to be *bipartite* if the adjacency matrix A is bipartite. That is, X can be written as a disjoint union of X^+ and X^- such that X^+, X^- contain no edges of Γ .

Corollary 2.1. *For any (connected) graph Γ with*

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_1 & m_1 & \cdots & m_r \end{pmatrix} \quad \text{with } \theta_0 > \theta_1 > \cdots > \theta_r.$$

Let V_i be the eigenspace of θ_i . Then the following holds.

1. *Supppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \in \mathbb{R}^n$. Then $\alpha_i > 0$ for all i or $\alpha_i < 0$ for all i .*
2. $m_0 = 1$.
3. $\theta_r \geq -\theta_0$ *if and only if Γ is bipartite. In this case,*

$$-\theta_i = \theta_{r-i} \text{ and } m_i = m_{r-i} \quad (0 \leq i \leq r)$$

Proof. This is a direct consequences of Theorem 2.1 and Note 3. \square

Chapter 3

Cayley Graphs

Monday, January 25, 1993

Given graphs $\Gamma = (X, E)$ and $\Gamma' = (X', E')$.

Definition 3.1. A map $\sigma : X \rightarrow X'$ is an *isomorphism* of graphs whenever;

- i. σ is one-to-one and onto,
- ii. $xy \in E$ if and only if $\sigma x \sigma y \in E'$ for all $x, y \in X$.

We do not distinguish between isomorphic graphs.

Definition 3.2. Suppose $\Gamma = \Gamma'$. Above isomorphism σ is called an *automorphism* of Γ . Then set $\text{Aut}(\Gamma)$ of all automorphisms of Γ becomes a finite group under composition.

Definition 3.3. If $\text{Aut}(\Gamma)$ acts transitive on X , Γ is called *vertex transitive*.

Example 3.1. A Cayley graphs:

Definition 3.4 (Cayley Graphs). Let G be any finite group, and Δ any generating set for G such that $1_G \notin \Delta$ and $g \in \Delta \rightarrow g^{-1} \in \Delta$. Then Cayley graph $\Gamma = \Gamma(G, \Delta)$ is defined on the vertex set $X = G$ with the edge set E defined by the following.

$$E = \{(h_1, h_2) \mid h_1, h_2 \in G, h_1^{-1}h_2 \in \Delta\} = \{(h, hg) \mid h \in G, g \in \Delta\}$$

Example 3.2. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}\}$.



Example 3.3. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}, a^2, a^{-2}\}$.



Example 3.4. $G = \langle a, b \mid a^6 = 1, b^2 = 1, ab = ba \rangle$, $\Delta = \{a, a^{-1}, b\}$.



Remark. $\text{Aut}(\Gamma) \simeq D_6 \times \mathbb{Z}_2$ contains two regular subgroups isomorphic to D_6 and $\mathbb{Z}_5 \times \mathbb{Z}_2$ and Γ is obtained as Cayley graphs in two ways.

Cayley graphs are vertex transitive, indeed.

Theorem 3.1. *The following hold.*

(i) *For any Cayley graph $\Gamma = \Gamma(G, \Delta)$, the map*

$$G \rightarrow \text{Aut}(\Gamma) \quad (g \mapsto \hat{g})$$

is an injective homomorphism of groups, where

$$\hat{g}(x) = gx \quad \text{for all } g \in G \text{ and for all } x \in X (= G).$$

Also, the image \hat{G} is regular on X . i.e., the image \hat{G} acts transitively on X with trivial vertex stabilizers.

(ii) For any graph $\Gamma = (X, E)$, suppose there exists a subgroup $G \subseteq \text{Aut}(\Gamma)$ that is regular on X . Pick $x \in X$, and let

$$\Delta = \{g \in G \mid \langle x, g(x) \rangle \in E\}.$$

Then $1 \notin \Delta$, $g \in \Delta \rightarrow g^{-1} \in \Delta$, and Δ generates G . Moreover, $\Gamma \simeq \Gamma(G, \Delta)$.

Proof. (i) Let $g \in G$. We want to show that $\hat{g} \in \text{Aut}(\Gamma)$. Let $h_1, h_2 \in X = G$. Then,

$$(h_1, h_2) \in E \rightarrow h_1^{-1}h_2 \in \Delta \quad (3.1)$$

$$\rightarrow (gh_1)^{-1}(gh_2) \in \Delta \quad (3.2)$$

$$\rightarrow (gh_1, gh_2) \in E \quad (3.3)$$

$$\rightarrow (\hat{g}(h_1), \hat{g}(h_2)) \in E. \quad (3.4)$$

Hence, $\hat{g} \in \text{Aut}(\Gamma)$.

Observe: $g \mapsto \hat{g}$ is a homomorphism of groups:

$$\hat{1}_G = 1, \widehat{g_1 g_2} = \widehat{g_1} \widehat{g_2}.$$

Observe: $g \mapsto \hat{g}$ is one-to-one:

$$\widehat{g_1} = \widehat{g_2} \rightarrow g_1 = \widehat{g_1}(1_G) = \widehat{g_2}(1_G) = g_2.$$

Observe: \hat{G} is regular on X : Clear by construction.

(ii) $1_G \notin \Delta$: Since Γ has not loops, $(x, 1_G x) \notin E$.

$g \in \Delta \rightarrow g^{-1} \in \Delta$:

$$g \in \Delta \rightarrow (x, g(x)) \in E \rightarrow E \ni (g^{-1}(x), g^{-1}(g(x))) = (g^{-1}(x), x).$$

Δ generates G : Suppose $\langle \Delta \subsetneq G$. Let $\hat{X} = \{g(x) \mid g \in \langle \Delta \rangle\} \subsetneq X$. ($\hat{X} \subsetneq X$ as G acts regularly on X .)

Since Γ is connected, there exists $y \in \hat{X}$ and $z \in X \setminus \hat{X}$ with $yz \in E$.

Let $y = g(x)$, $g \in \langle \Delta \rangle$, $z \in h(x)$, $h \in G \setminus \langle \Delta \rangle$. Then

$$(y, z) = (g(x), h(x)) \in E \rightarrow (x, g^{-1}h(x)) \in E \rightarrow g^{-1}h \in \langle \Delta \rangle \rightarrow h \in \langle \Delta \rangle.$$

This is a contradiction. Therefore, Δ generates G .

Let $\Gamma' = (X', E')$ denote $\Gamma(G, \Delta)$. We shall show that

$$\theta : X' \rightarrow X \ (g \mapsto g(x))$$

is an isomorphism of graphs.

θ is one-to-one: For $h_1, h_2 \in X' = G$,

$$\theta(h_1) = \theta(h_2) \rightarrow h_1(x) = h_2(x) \rightarrow h_1^{-1}h_2(x) = x \rightarrow h_1^{-1}h_2 \in \text{Stab}_G(x) = \{1_G\} \rightarrow h_1 = h_2.$$

($\text{Stab}_G = \{g \in G \mid g(x) = x\}$.)

θ is onto: Since G is transitive,

$$X = \{g(x) \mid g \in G\} = \theta(X') = \theta(G).$$

θ respects adjacency: For $h_1, h_2 \in X' = G$,

$$(h_1, h_2) \in E' \leftrightarrow h_1^{-1}h_2 \in \Delta \leftrightarrow (x, h_1^{-1}h_2(x)) \in E \leftrightarrow (h_1(x), h_2(x)) \in E \leftrightarrow (\theta(h_1), \theta(h_2)) \in E.$$

Therefore θ is an isomorphism between graphs $\Gamma(G, \Delta)$ and $\Gamma(X, E)$. \square

How to compute the eigenvalues of the Cayley graph of an abelian group.

Let G be any finite abelian group. Let \mathbb{C}^* be the multiplicative group on $\mathbb{C} \setminus \{0\}$.

Definition 3.5. A (linear) G -character is any group homomorphism $\theta : G \rightarrow \mathbb{C}^*$.

Example 3.5. $G = \langle a \mid a^3 = 1 \rangle$ has three characters, $\theta_0, \theta_1, \theta_2$.

$$\begin{array}{c|ccc} \theta_i(a^j) & 1 & a & a^2 \\ \hline \theta_0 & 1 & 1 & 1 \\ \theta_1 & 1 & \omega & \omega^2 \\ \theta_2 & 1 & \omega^2 & \omega \end{array}, \quad \text{with } \omega = \frac{-1 + \sqrt{-3}}{2}.$$

Here ω is a primitive cube root of q in \mathbb{C}^* , i.e., $1 + \omega + \omega^2 = 0$.

For arbitrary group G , let $X(G)$ be the set of all characters of G .

Observe: For $\theta_1, \theta_2 \in X(G)$, one can define product $\theta_1 \theta_2$:

$$\theta_1 \theta_2(g) = \theta_1(g) \theta_2(g) \quad \text{for all } g \in G.$$

Then $\theta_1 \theta_2 \in X(G)$.

Observe: $X(G)$ with this product is an (abelian) group.

Lemma 3.1. *The groups G and $X(G)$ are isomorphic for all finite abelian groups G .*

Proof. G is a direct sum of cyclic groups;

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m, \quad \text{where } G_i = \langle a_i \mid a_i^{d_i} = 1 \rangle \quad (1 \leq i \leq m).$$

Pick any element ω_i of order d_i in \mathbb{C}^* , i.e., a primitive d_i -th root of 1. Define

$$\theta_i : G \rightarrow \mathbb{C}^* \quad (a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \mapsto \omega_i^{\varepsilon_i} \quad \text{where } 0 \leq \varepsilon_i < d_i, 1 \leq i \leq m).$$

Then $\theta_i \in X(G)$. (Exercise)

Claim: There exists an isomorphism of groups $G \rightarrow X(G)$ that sends a_i to θ_i .

Observe: $\theta_i^{d_i} = 1$. For every $g = a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \in G$,

$$\theta_i^{d_i}(g) = (\theta_i(g))^{d_i} = (\omega_i^{\varepsilon_i})^{d_i} = (\omega_i^{d_i})^{\varepsilon_i} = 1.$$

Observe: If $\theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m} = 1$ for some $0 \leq \varepsilon_i < d_i, 1 \leq i \leq m$. Then $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = 0$.

Pf. $1 = \theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m}(a_i) = \omega_i^{\varepsilon_i}$, Since ω_i is a primitive d_i -th root of 1, $\varepsilon_i = 0$ for $1 \leq i \leq m$.

Observe: $\theta_1, \dots, \theta_m$ generate $X(G)$. Pick $\theta \in X(G)$. Since $a_i^{d_i} = 1$, $1 = \theta(a_i^{d_i}) = \theta(a_i)^{d_i}$.

Hence $\theta(a_i) = \omega_i^{\varepsilon_i}$ for some ε_i with $0 \leq \varepsilon_i < d_i$.

Now $\theta = \theta_1^{\varepsilon_1} \cdots \theta_m^{\varepsilon_m}$, since these are both equal to $\omega_i^{\varepsilon_i}$ at a_i for $1 \leq i \leq m$.

Therefore,

$$G \rightarrow X(G) \quad (a_i \mapsto \theta_i)$$

is an isomorphism of groups. □

Note. The correspondence above is clearly a group homomorphism.

Chapter 4

Examples

Wednesday, January 27, 1993

Theorem 4.1. *Given a Cayley graph $\Gamma = \Gamma(G, \Delta)$. View the standard module $V \equiv \mathbb{C}G$ (the group algebra), so*

$$\left\langle \sum_{g \in G} \alpha_g g, \sum_{g \in G} \beta_g g \right\rangle = \sum_{g \in G} \alpha_g \overline{\beta_g}, \quad \text{with } \alpha_g, \beta_g \in \mathbb{C}.$$

For any $\theta \in X(G)$, write

$$\hat{\theta} = \sum_{g \in G} \theta(g^{-1})g.$$

Then the following hold.

(i) $\langle \hat{\theta}_1, \hat{\theta}_2 \rangle = |G|$ if $\theta_1 = \theta_2$ and 0 otherwise for $\theta_1, \theta_2 \in X(G)$. In particular, $\{\hat{\theta} \mid \theta \in X(G)\}$ forms a basis for V .

(ii) $A\hat{\theta} = \Delta_\theta \hat{\theta}$ for $\theta \in X(G)$, where A is the adjacency matrix and

$$\Delta_\theta = \sum_{g \in \Delta} \theta(g).$$

In particular, the eigenvalues of Γ are precisely

$$\Delta_\theta \mid \theta \in X(G)\}.$$

Proof.

(i) Claim: For every $\theta \in X(G)$, let

$$s := \sum_{g \in G} \theta(g^{-1}) = \begin{cases} |G| & \text{if } \theta = 1 \\ 0 & \text{if } \theta \neq 1. \end{cases}$$

Pf. Clear if $\theta = 1$.

Let $\theta \neq 1$. Then $\theta(h) \neq 1$ for some $h \in G$.

$$s \cdot \theta(h) = \left(\sum_{g \in G} \theta(g^{-1}) \right) \theta(h) = \sum_{g \in G} \theta(g^{-1}h) = \sum_{g' \in G} \theta(g'^{-1}) = s.$$

Since $\theta(h) \neq 1$, $s = 0$.

Claim. $\theta(g^{-1}) = \overline{\theta(g)}$ for every $\theta \in X(G)$ and every $g \in G$.

Since $\theta(g) \in \mathbb{C}$ is a root of 1,

$$|\theta(g)|^2 = \theta(g)\overline{\theta(g)} = 1.$$

On the other hand, since θ is a homomorphism,

$$\theta(g)\theta(g^{-1}) = \theta(1) = 1.$$

Hence $\theta(g^{-1}) = \overline{\theta(g)}$.

Now

$$\langle \widehat{\theta_1}, \widehat{\theta_2} \rangle = \sum_{g \in G} \theta_1(g^{-1}) \overline{\theta_2(g^{-1})} \quad (4.1)$$

$$= \sum_{g \in G} \theta_1(g^{-1}) \theta_2(g) \quad (4.2)$$

$$= \sum_{g \in G} \theta_1 \theta_2^{-1}(g^{-1}) \quad (4.3)$$

$$= \begin{cases} |G| & \text{if } \theta_1 \theta_2^{-1} = 1 \\ 0 & \text{if } \theta_1 \theta_2^{-1} \neq 1. \end{cases} \quad (4.4)$$

Since $|G| = |X(G)|$ by Lemma 3.1, and $\widehat{\theta_i}$'s are orthogonal nonzero elements in V , they form a basis of V .

(ii) Let $\Delta = \{g_1, \dots, g_r\}$. Then

$$A\hat{\theta} = A \left(\sum_{g \in G} \theta(g^{-1}g) \right) \quad (4.5)$$

$$= \sum_{g \in G} \theta(g^{-1})(gg_1 + \cdots + gg_r) \quad (\Gamma(g) = \{gg_1, \dots, gg_r\}) \quad (4.6)$$

$$= \sum_{i=1}^r \left(\sum_{g \in G} \theta(g^{-1})(gg_i) \right) \quad (4.7)$$

$$= \sum_{i=1}^r \left(\sum_{g \in G} \theta(g_i g_i^{-1} g^{-1})(gg_i) \right) \quad (4.8)$$

$$= \sum_{i=1}^r \left(\sum_{g \in G} \theta(g_i) \theta((gg_i)^{-1}) gg_i \right) \quad (4.9)$$

$$= \sum_{i=1}^r \theta(g_i) \sum_{h \in G} \theta(h^{-1}) h \quad (4.10)$$

$$= \Delta_{\theta} \cdot \hat{\theta}. \quad (4.11)$$

Since $\{\hat{\theta} \mid \theta \in X(G)\}$ forms a basis, the eigenvalues of Γ are precisely,

$$\{\Delta_{\theta} \mid \theta \in X(G)\}.$$

This completes the proof.

□

Example 4.1. Let $G = \langle a \mid a^6 = 1 \rangle$, and $\Delta = \{a, a^{-1}\}$. Pick a primitive 6-th root of 1, ω . Then

$$X(G) = \{\theta^i \mid 0 \leq i \leq 5\} \quad \text{such that} \quad \theta(a) = \omega, \quad \omega + \omega^{-1} = 1.$$



$\varphi \in X(G)$	$\varphi(a)$	$\Delta_\varphi = \theta(a) + \theta(a)^{-1}$
1	1	2
θ	ω	$\omega + \omega^{-1} = 1$
θ^2	ω^2	-1
θ^3	$\omega^3 = -1$	-2
θ^4	ω^4	-1
θ^5	ω^5	1

$$\text{Spec}(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

Example 4.2. D -cube, $H(D, 2)$. Let

$$X = \{(a_1, \dots, a_D) \mid a_i \in \{1, -1\}, 1 \leq i \leq D\},$$

$$E = \{xy \mid x, y \in X, x, y: \text{different in exactly one coordinate}\}.$$

Also $H(D, 2)$ is a Cayley graph $\Gamma(G, \Delta)$, where

$$G = G_1 \oplus G_2 \oplus \dots \oplus G_D,$$

$$G_i = \langle a_i \mid a_i^2 = 1 \rangle, \quad \Delta = \{a_1, \dots, a_D\}.$$

Homework: The spectrum of $H(D, 2)$ is

$$\begin{pmatrix} \theta_0 & \theta_1 & \dots & \theta_D \\ m_0 & m_1 & \dots & m_D \end{pmatrix},$$

where

$$\theta_i = D - 2i \quad (0 \leq i \leq D), \quad m_i = \binom{D}{i}.$$

Remark. Let $\theta \in X(G)$. Then $\theta : X \rightarrow \{\pm 1\}$. If

$$\nu(\theta) = |\{i \mid \theta(a_i) = -1\}|,$$

then $\Delta_\theta = D - 2i$. Since there are $\binom{D}{i}$ such θ , we have the assertion.

We want to compute the subconstituent algebra for $H(D, 2)$. First, we make a few observations about arbitrary graphs.

Let $\Gamma = (X, E)$ be any graph, A , the adjacency matrix of Γ , and V , the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$. Write $E_i^* = E_i^*(x)$, and

$$T \equiv T(x) = \text{the algebra generated by } A, E_0^*, E_1^*, \dots$$

Definition 4.1. Let W be any irreducible T -module ($\subseteq V$). Then the endpoint $r \equiv r(W)$ satisfied

$$r = \min\{i \mid E_i^* W \neq 0\}.$$

The diameter $d = d(W)$ satisfied

$$d = |\{i \mid E_i^* W \neq 0\}| - 1.$$

Lemma 4.1. *With the above notation, let W be an irreducible T -module. Then*

- (i) $E_i^* A E_j^* = 0$ if $|i - j| = 1$, $E_i^* A E_j^* \neq 0$ if $|i - j| = 1$, $0 \leq i, j \leq d(x)$.
- (ii) $A E_j^* W \subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$, $0 \leq j \leq d(x)$. ($E_i^* W = 0$ if $i < j$ or $i > d(x)$.)
- (iii) $E_j^* W \neq 0$ if $r \leq j \leq r + d$, $= 0$ if $0 \leq j \leq r$ or $r + d < j \leq d(x)$.
- (iv) $E_i^* A E_j^* W \neq 0$, if $|i - j| = 1$ ($r \leq i, j \leq r + d$).

Proof.

(i) Pick $y \in X$ with $\partial(x, y) = j$. We want to find $E_i^* A E_j^* \hat{y}$. Note,

$$E_j^* \hat{y} = \begin{cases} 0 & \text{if } \partial(x, y) \neq j \\ \hat{y} & \text{if } \partial(x, y) = j. \end{cases}$$

$$E_i^* A E_j^* \hat{y} = E_i^* A \hat{y} \tag{4.12}$$

$$= E_i^* \sum_{z \in X, yz \in E} \hat{z} \tag{4.13}$$

$$= \sum_{z \in X, yz \in E, \partial(x, z) = i} \hat{z} \tag{4.14}$$

$$= 0 \text{ if } |i - j| > 1 \quad \text{by triangle inequality.} \tag{4.15}$$

If $|i - j| = 1$, there exist $y, y' \in X$ such that $\partial(x, y) = j$, $\partial(x, y') = i$, $yy' \in E$ by connectivity of Γ . Hence (4.14) contains $\widehat{yy'}$ and (4.14) is not equal to zero.

(ii) We have

$$A E_j^* W = \left(\sum_{i=0}^{d(x)} E_i^* \right) A E_j^* W \tag{4.16}$$

$$= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W \tag{4.17}$$

$$\subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W. \tag{4.18}$$

(iii) Suppose $E_j^* W = 0$ for some j ($r \leq j \leq r + d$). Then $r < j$ by the definition of r . Set

$$\widetilde{W} = E_r^*W + E_{r+1}^*W + \cdots + E_{j-1}^*W.$$

Observe $0 \subsetneq \widetilde{W} \subsetneq W$. Also $A\widetilde{W} \subseteq \widetilde{W}$ by (ii), and $E_i^*\widetilde{W} \subseteq \widetilde{W}$ for every i by construction.

Thus, $T\widetilde{W} \subseteq \widetilde{W}$, contradicting W being irreducible.

□

Chapter 5

T -Modules of $H(D, 2)$, I

Friday, January 29, 1993

Let $\Gamma = (X, E)$ be a graph, A the adjacency matrix, and V the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$ and write $E_i^* \equiv E_i^*(x)$, and $T \equiv T(x)$.

Let W be an irreducible T -module with endpoint $r := \min\{i \mid E_i^*W \neq 0\}$ and diameter $d := |\{i \mid E_i^*W \neq 0\}| - 1$.

We have

$$E_i^*W \neq 0 \quad r \leq i \leq r + d \quad (5.1)$$

$$= 0 \quad 0 \leq i < r \text{ or } r + d < i \leq d(x). \quad (5.2)$$

Claim: $E_i^*AE_j^*W \neq 0$ if $|i - j| = 1$ for $r \leq i, j \leq r + d$. (See Lemma 4.1.)

Suppose $E_{j+1}^*AE_j^*W = 0$ for some j with $r \leq j < r + d$. Observe that

$$\tilde{W} = E_r^*W + \cdots + E_j^*W$$

is T -invariant with

$$0 \subsetneq \tilde{W} \subsetneq W.$$

Because $A\tilde{W} \subseteq \tilde{W}$ since $AE_j^*W \subseteq E_{j-1}^*W + E_j^*W$,

$$E_k^*\tilde{W} \subseteq \tilde{W} \quad \text{for all } k,$$

we have $T\tilde{W} \subseteq \tilde{W}$.

Suppose $E_{i-1}^*AE_i^*W = 0$ for some i with $r \leq i < r + d$.

Similarly,

$$\tilde{W} = E_i^*W + \cdots + E_{r+d}^*W$$

is a T -module with $0 \subsetneq \tilde{W} \subsetneq W$.

Definition 5.1. Let Γ , E_i^* , and T be as above. Irreducible T -modules W and W' are isomorphic whenever there is an isomorphism $\sigma : W \rightarrow W'$ of vector spaces such that $a\sigma = \sigma a$ for all $a \in T$.

Recall that the standard module V is an orthogonal direct sum of irreducible T -modules $W_1 \oplus W_2 \oplus \dots$. Given W in this list, the multiplicity of W in V is

$$|\{j \mid W_j \simeq W\}|.$$

Remark. It is known that the multiplicity does not depend on the decomposition.

Now assume that Γ is the D -cube, $H(D, 2)$ with $D \geq 1$. View

$$X = \{a_1 \cdots a_D \mid a_i \in \{1, -1\}, 1 \leq i \leq D\}, \quad (5.3)$$

$$E = \{xy \mid x, y \in X, x, y \text{ differ in exactly 1 coordinate.}\}. \quad (5.4)$$

Find T -modules.

Claim: $H(D, 2)$ is bipartite with a partition $X = X^+ \cup X^-$, where

$$X^+ = \{a_1 \cdots a_D \in X \mid \prod a_i > 0\} \quad (5.5)$$

$$X^- = \{a_1 \cdots a_D \in X \mid \prod a_i < 0\} \quad (5.6)$$

Observe: for all $y, z \in X$,

$$\partial(y, z) = i \Leftrightarrow y, z \text{ differ in exactly } i \text{ coordinates with } 0 \leq i \leq D.$$

Here, the diameter of $H(D, 2) = D = d$ for all $x \in X$.

Theorem 5.1. Let $\Gamma = H(D, 2)$ be as above. Fix $x \in X$, and write $E_i^* = E_i^*(x)$, and $T = T(x)$.

Let W be an irreducible T -module with endpoint r , and diameter d with $0 \leq r \leq r + d \leq D$.

(i) W has a basis w_0, w_1, \dots, w_d with $w_i \in E_{i+r}^* W$ for $0 \leq i \leq d$. With respect to which the matrix representing A is

$$\begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & d-1 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \ddots & 0 & 2 & 0 \\ 0 & 0 & 0 & \cdots & d-1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & d & 0 \end{pmatrix}$$

(ii) $d = D - 2r$. In particular, $0 \leq r \leq D/2$.

(iii) Let W' denote an irreducible T -module with endpoint r' . Then W and W' are isomorphic as T -modules if and only if $r = r'$.

(iv) The multiplicity of the irreducible T -module with endpoint r is

$$\binom{D}{r} - \binom{D}{r-1} \quad \text{if } 1 \leq r \leq D/2,$$

and 1 if $r = 0$.

Proof. Recall that Γ is vertex transitive. It is a Cayley graph.

Hence without loss of generality, we may assume that $x = \overbrace{11 \cdots 1}^D$.

Notation: Set $\Omega = \{1, 2, \dots, D\}$. For every subset $S \subseteq \Omega$, let

$$\hat{S} = a_1 \cdots a_d \in X \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S. \end{cases}$$

In particular, $\hat{\emptyset} = x$ and

$$|S| = i \Leftrightarrow \partial(x, \hat{S}) = i \Leftrightarrow \hat{S} \in E_i^* V.$$

For all $S, T \subseteq \Omega$, we say S covers T if and only if $S \supseteq T$ and $|S| = |T| + 1$.

Observe that \hat{S}, \hat{T} are adjacent in Γ if and only if either T covers S or S covers T .

Define the ‘raising matrix’

$$R = \sum_{i=0}^D E_{i+1}^* A E_i^*.$$

Observe that

$$R E_i^* V \subseteq E_{i+1}^* V \quad \text{for } 0 \leq i \leq D, \quad \text{and } E_{D+1}^* V = 0.$$

Indeed for any $S \subseteq \Omega$ with $|S| = i$,

$$R \hat{S} = R E_i^* \hat{S} \tag{5.7}$$

$$= E_{i+1}^* A \hat{S} \tag{5.8}$$

$$= \sum_{T_1 \subseteq \Omega, S \text{ covers } T_1} E_{i+1}^* \hat{T}_1 + \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T} \tag{5.9}$$

$$= \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T}. \tag{5.10}$$

Define the ‘lowering matrix’

$$L = \sum_{i=0}^D E_{i-1}^* A E_i^*.$$

Observe that

$$L E_i^* V \subseteq E_{i-1}^* V \text{ for } 0 \leq i \leq D, \text{ and } E_{-1}^* V = 0.$$

Indeed for any $S \subseteq \Omega$,

$$L \hat{S} = \sum_{T \subseteq \Omega, S \text{ covers } T} \hat{T}.$$

Observe that $A = L + R$.

For convenience, set

$$A^* = \sum_{i=0}^D (D - 2i) E_i^*.$$

Claim: The following hold.

- (a) $LR - RL = A^*$.
- (b) $A^*L - LA^* = 2L$.
- (c) $A^*R - RA^* = -2R$.

In particular $\text{Span}(R, L, A^*)$ is a ‘representation of Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Remark (Lie Algebra $\mathfrak{sl}_2(\mathbb{C})$).

$$\mathfrak{sl}_2(\mathbb{C}) = \{X \mid \text{Mat}(\mathbb{C}) \mid \text{tr}(X) = 0\}.$$

For $X, Y \in \mathfrak{sl}_2(\mathbb{C})$, define a binary operation $[X, Y] = XY - YX$.

$$A^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then these satisfy the relations (a) - (c) above.

Proof of Claim. Apply both sides to \hat{S} ($S \subseteq \Omega$). Say $|S| = i$.

Proof of (a):

$$(LR - RL)\hat{S} = L \left(\sum_{\substack{T \subseteq \Omega, T \text{ covers } S \\ (D-i \text{ of them})}} \hat{T} \right) - R \left(\sum_{\substack{U \subseteq \Omega, S \text{ covers } U \\ (i \text{ of them})}} \hat{T} \right) \quad (5.11)$$

$$= (D - i)\hat{S} + \sum_{V \subseteq \Omega, |V|=i, |S \cap V|=i-1} \hat{V} - \left(i\hat{S} + \sum_{V \subseteq \Omega, |V|=i, |S \cap V|=i-1} \hat{V} \right) \quad (5.12)$$

$$= (D - 2i)\hat{S} \quad (5.13)$$

$$= A^*\hat{S}. \quad (5.14)$$

Proof of (b):

$$(A^*L - LA^*)\hat{S} = (D - 2(i - 1))L\hat{S} - (D - 2i)L\hat{S} \quad (\text{since } L\hat{S} \in E_{i-1}^*V) \quad (5.15)$$

$$= 2L\hat{S}. \quad (5.16)$$

Proof of (c):

$$(A^*R - RA^*)\hat{S} = (D - 2(i + 1))R\hat{S} - (D - 2i)R\hat{S} \quad (\text{since } R\hat{S} \in E_{i+1}^*V) \quad (5.17)$$

$$= 2R\hat{S}. \quad (5.18)$$

Let W be an irreducible T -module with endpoint r and diameter d ($0 \leq r \leq r + d \leq D$).

Proof of (i) and (ii):

Pick $0 \neq w \in E_r^*W$.

Claim: $LRw = (D - 2r)w$.

Pf.

$$LRw = (A^* + RL)w \quad (\text{by Claim (a)}) \quad (5.19)$$

$$= A^*w \quad (Lw \in E_{r-1}^*W = 0) \quad (5.20)$$

$$(D - 2r)w. \quad (5.21)$$

Define

$$w_i = \frac{1}{i!}R^i w \in E_{r+i}^*W \quad (0 \leq i \leq d).$$

Then,

$$Rw_i = (i + 1)w_{i+1} \quad (0 \leq i \leq d) \quad (5.22)$$

$$Rw_d = 0 \quad (\text{by definition of } d) \quad (5.23)$$

Claim: $Lw_0 = 0$ and

$$Lw_i = (D - 2r - i + 1)w_{i-1} \quad (1 \leq i \leq d).$$

Pf. We prove by induction on i . The case $i = 0$ is trivial, and the case $i = 1$

follows from above claim. Let $i \geq 2$,

$$Lw_i = \frac{1}{i}LRw_{i-1} = \frac{1}{i}(A^* + RL)w_{i-1} \quad (\text{by Claim (a)}) \quad (5.24)$$

$$(\text{by induction hypothesis}) \quad (5.25)$$

$$= \frac{1}{i}((D - 2(r + i - 1))w_{i-1} + (D - 2r - (i - 1) + 1)Rw_{i-2}) \quad (Rw_{i-2} = (i - 1)w_{i-1}) \quad (5.26)$$

$$= \frac{1}{i}i(D - 2r - i + 1)w_{i-1} \quad (5.27)$$

$$= (D - 2r - i + 1)w_{i-1}. \quad (5.28)$$

Claim: w_0, \dots, w_d is a basis for W .

Pf. Let $W' = \text{Span}\{w_0, \dots, w_d\}$. Then W' is R and L invariant. So it is $A = R + L$ invariant.

Also it is E_i^* -invariant for every i .

Hence W' is a T -module.

Since W is irreducible, $W' = W$.

As w_i 's are orthogonal, they are linearly independent. Note that $w_i \neq 0$ by the definition of d and Lemma 4.1 (iv).

Claim: $d = D - 2r$.

Pf. By (a),

$$0 = (LR - RL - A^*)w_d \quad (5.29)$$

$$= 0 - (D - 2r - d + 1)Rw_{d-1} - (D - 2(r + d))w_d \quad (5.30)$$

$$= -d(D - 2r - d + 1)w_d - (D - 2(r + d))w_d \quad (5.31)$$

$$= (-dD + 2rd + d^2 - d - D + 2r + 2d)w_d \quad (5.32)$$

$$= (d^2 + (2r - D + 1)d + 2r - D)w_d \quad (5.33)$$

$$= (d + 2r - D)(d + 1)w_d. \quad (5.34)$$

Hence $d = D - 2r$.

Therefore, with respect to a basis w_0, w_1, \dots, w_d , $A = L + R$, $w_{-1} = w_{d+1} = 0$,

$$Lw_i = (d - i + 1)w_{i-1}, \quad Rw_i = (i + 1)w_{i+1}.$$

$$L = \begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0 \\ 0 & 0 & d-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & d & 0 \end{pmatrix}.$$

This completes the proof of (i) and (ii). \square

Chapter 6

T -Modules of $H(D, 2)$, II

Monday, February 1, 1993

Proof of Theorem 5.1 Continued.

(iii) Let $r = r'$,

w_0, \dots, w_d : a basis for W with $w_i \in E_i^*W$, and

w'_0, \dots, w'_d : a basis for W' with $w'_i \in E_i^*W'$.

Then $d = D - 2r = D - 2r' = d'$, and

$$\sigma : W \rightarrow W' \quad (w_i \mapsto w'_i)$$

is an isomorphism of T -modules by (i).

If $r \neq r'$, then

$$d = D - 2r \neq D - 2r' = d',$$

hence, $\dim W \neq \dim W'$.

(iv) Let W_i be the irreducible T -module with endpoint i . Then

$$\dim E_r^*V = \binom{D}{r} = \sum_{i=0}^r \text{mult}(W_i).$$

Hence, we have that

$$\text{mult}(W_r) = \binom{D}{r} - \binom{D}{r-1}$$

by induction on r .

□

Theorem 6.1. *Let $\Gamma = H(D, 2)$ with $D \geq 1$. Fix a vertex $x \in X$ and write*

$$E_i^* \equiv E_i^*(x), \quad T = T(x), \text{ and } A^* \equiv \sum_{i=0}^D (D - 2i) E_i^*.$$

Let W be an irreducible T -module with endpoint r with $0 \leq r \leq D/2$. Then,

(i) W has a basis

$$w_0^*, w_1^*, \dots, w_d^* \quad (d = D - 2r), \quad \text{such that } w_i^* \in E_{i+r} W \quad (0 \leq i \leq d)$$

with respect to which the matrix corresponding to A^ is*

$$\begin{pmatrix} 0 & d & 0 & & & & \\ 1 & 0 & d-1 & & & & \\ 0 & 2 & 0 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & 2 & 0 & \\ & & & d-1 & 0 & 1 & \\ & & & 0 & d & 0 & \end{pmatrix}.$$

In particular, / (ii) $E_i A^ E_j = 0$ if $|i - j| \neq 1$ for $0 \leq i, j \leq D$.*

Proof. We use the notation,

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \quad (= -[\beta, \alpha]).$$

Recall that

- (a) $[L, R] = A^*$,
- (b) $[A^*, L] = wL$,
- (c) $[A^*, R] = -2R$,

and $A = L + R$.

Write (a) – (c) in terms of A and A^* , we have,

$$[A, A^*] = [L, A^*] + [R, A^*] = 2(R - L).$$

$$\begin{cases} R + L &= A \\ R - L &= [A, A^*]/2. \end{cases}$$

Hence,

$$R = \frac{1}{4}(2A + [A, A^*]) \quad \text{and} \tag{6.1}$$

$$L = \frac{1}{4}(2A - [A, A^*]). \tag{6.2}$$

Now (a), (b) become

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0 \quad (6.3)$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0 \quad (6.4)$$

Pf. By (b),

$$2A - AA^* + A^*A = 4L \quad (6.5)$$

$$= 2[A^*, L] \quad (6.6)$$

$$= A^* \frac{2A - [A, A^*]}{2} - \frac{2A - [A, A^*]}{2} A^* \quad (6.7)$$

So we have ((6.4))

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0.$$

By (a),

$$-16A^* = [2A + [A, A^*], 2A - [A, A^*]] \quad (6.8)$$

$$(2A + [A, A^*])(2A - [A, A^*]) - (2A - [A, A^*])(2A + [A, A^*]) \quad (6.9)$$

$$= [4A^2 - 2A[A, A^*] + [A, A^*](2A) - [A, A^*]^2 \quad (6.10)$$

$$- 4A^2 - 2A[A, A^*] + [A, A^*](2A) + [A, A^*]^2 \quad (6.11)$$

$$= -4A^2A^* + 4AA^*A + 4AA^*A - 4A^*A^2. \quad (6.12)$$

So,

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0.$$

Claim: $E_i^*A^*E_j = 0$ if $|i - j| \neq 1$ for $0 \leq i, j \leq D$.

Pf. We have,

$$0 = E_i(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_j \quad (6.13)$$

$$= E_iA^*E_j(\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4) \quad (6.14)$$

$$(AE_j = \theta_jE_j, E_iA = (AE_j)^\top = (\theta_iE_i)^\top = \theta_iE_i) \quad (6.15)$$

$$= E_iA^*E_j(\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) \quad (6.16)$$

$$= E_iA^*E_j(D - 2i - (D - 2j) - 2)(D - 2i - (D - 2j) + 2) \quad (6.17)$$

$$(\theta_k = D - 2k) \quad (6.18)$$

$$= E_iA^*E_j \cdot 4(i - j + 1)(i - j - 1) \quad (6.19)$$

and $i - j + 1 \neq 0$, $i - j - 1 \neq 0$. Hence, $E_i^*A^*E_j = 0$.

Now define “dual raising matrix”,

$$R^* = \sum_{i=0}^D E_{i+1}A^*E_i.$$

So,

$$R^* E_i V \subseteq E_{i+1} V, \quad (0 \leq i \leq D, E_{D+1} V = 0).$$

Define “dual lowering matrix”

$$L^* = \sum_{i=0}^D E_{i-1} A^* E_i.$$

Then

$$L^* E_i V \subseteq E_{i-1} V \quad (0 \leq i \leq D, E_{-1} V = 0).$$

Observe that

$$A^* = \left(\sum_{i=0}^D E_i \right) A^* \left(\sum_{j=0}^D E_j \right) = L^* + R^*$$

by Claim 1.

Claim 2. We have | (a) $[L^*, R^*] = A$, | (b) $[A, L^*] = 2L^*$, | (c) $[A, R^*] = -2R^*$.

Pf. (b)

$$AL^* - L^* A = \sum_{i=0}^D (AE_{i-1} A^* E_i - E_{i-1} A^* E_i A) \quad (6.20)$$

$$= \sum_{i=0}^D E_{i-1} A^* E_i (\theta_{i-1} - \theta_i) \quad (6.21)$$

$$(\theta_k = D - 2k, \quad \theta_{i-1} - \theta_i = 2I - 2(i-1) = 2) \quad (6.22)$$

$$= 2L^*. \quad (6.23)$$

(c) Similar.

Remark.

$$AR^* - R^* A = \sum_{i=0}^D (AE_{i+1} A^* E_i - E_{i+1} A^* E_i A) \quad (6.24)$$

$$= \sum_{i=0}^D E_{i+1} A^* E_i (\theta_{i+1} - \theta_i) \quad (6.25)$$

$$= 2R^*. \quad (6.26)$$

(a) We have, by (b), (c)

$$[A, A^*] = [A, L^*] + [A, R^*] = 2(L^* - R^*). \quad (6.27)$$

Since $A^* = L^* + R^*$,

$$R^* = \frac{2A^* + [A^*, A]}{4}, \quad L^* = \frac{2A^* - [A^*, A]}{4}.$$

Now (a) is seen to be equivalent to ((6.4)) upon evaluation. This proves Claim 2.

Remark.

$$[L^*, R^*] = \frac{1}{16}((2A^* - [A^*, A])(2A^* + [A^*, A]) - (2A^* + [A^*, A])(2A^* - [A, A^*])) \quad (6.28)$$

$$= \frac{1}{16}(4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* - [A^*, A]^2 - 4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* + [A^*, A]^2) \quad (6.29)$$

$$= \frac{1}{4}(A^{*2}A - 2A^*AA^* + AA^{*2}) \quad (6.30)$$

$$= A, \quad (6.31)$$

by ((6.4)).

Now apply same argument as for ((6.3)), ((6.4)) of Theorem 5.1 and observe A^* has $D + 1$ distinct eigenvalues. So,

$$A^* = \sum_{i=0}^D (D - 2i)E_i^*$$

generates

$$M^* = \text{Span}(E_0^*, \dots, E_D^*).$$

Hence, E_0, \dots, E_D, A^* generates T .

Take an irreducible T -module W with endpoint r with $0 \leq r \leq D/2$. Set $t = \min\{i \mid E_i W\}$.

Pick $0 \neq w_0^* \in E_t W$. Set

$$w_i^* = \frac{1}{i!} R^{*i} w_0^* \in E_{t+i} W \quad \text{for all } i.$$

Then,

$$R^* w_i^* = (i + 1) w_{i+1}^* \quad \text{for all } i.$$

By (a), we get by induction, $L^* w_i^* = (D - 2t - i + 1) w_{i-1}^*$,

$$L^* w_i^* = \frac{1}{i} L^* R^* w_{i-1}^* \quad (6.32)$$

$$= \frac{1}{i} (A + R^* L^*) w_{i-1}^* \quad (6.33)$$

$$= \frac{1}{i} ((D - 2(t + i - 1)) w_{i-1}^* + (i - 1)(D - 2t - i + 2) w_{i-1}^*) \quad (6.34)$$

$$= (D - 2t - i + 1) w_{i-1}^*. \quad (6.35)$$

So $\text{Span}(w_0^*, w_1^*, \dots)$ is L^*, R^*, A^* -invariant. Hence, $W = (\text{Span})(w_0^*, w_1^*, \dots, w_d^*)$, $w_0^*, w_1^*, \dots, w_d^* \neq 0$, $w_i^* = 0$ for every $i > d$ by dimension.

Thus $d = D - 2t$.

Pf.

$$(D - 2(t + d))w_d^* = Aw_d^* \quad (6.36)$$

$$= (L^*R^* - R^*L^*)w_d^* \quad (6.37)$$

$$= -(D - 2t - d + 1)R^*w_{d-1}^* \quad (6.38)$$

$$= -(D - 2t - d + 1)dw_d^*. \quad (6.39)$$

Hence,

$$0 = d^2 + (2t - D - 1 + 2)d - (D - 2t) = (d - D + 2t)(d + 1)$$

So $d = D - 2t$. □

Definition 6.1. For any graph $\Gamma = (X, E)$, pick a vertex $x \in X$ and set $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

- (i) an irreducible T -module W is thin if $\dim E_i^*W \leq 1$ for every i ,
- (ii) Γ is thin with respect to x , if every irreducible $T(x)$ -module is thin,
- (iii) an irreducible T -module W is dual thin if $\dim E_iW \leq 1$ for every i ,
- (iv) Γ is dual thin with respect to x , if every irreducible $T(x)$ -module is dual thin.

Observe: $H(D, 2)$ is thin, dual thin with respect to each $x \in X$.

Definition 6.2. With above notation, write $D \equiv D(x)$.

- (i) an ordering E_0, E_1, \dots, E_R of primitive idempotents of Γ is restricted if E_0 corresponds to the maximal eigenvalue.

Fix a restricted ordering,

- (ii) Γ is Q -polynomial with respect to x , above ordering if there exists $A^* \equiv A^*(x)$ such that

$$(a) E_0^*V, \dots, E_D^*V \text{ are the maximal eigenspaces for } A^*.$$

$$(b) E_iA^*E_j = 0 \text{ if } |i - j| > 1 \text{ for } 0 \leq i, j \leq R.$$

Observe $H(D, 2)$ is Q -polynomial with respect to the natural ordering of the idempotents and every vertex.

Program. Study graphs that are thin and Q -polynomial with respect to each vertex.

(In fact, thin with respect to x implies dual thin with respect to x .)

Get a situation like $H(D, 2)$, where T is generated by A, A^* . Except $\mathfrak{sl}_s(\mathbb{C})$ is replaced by a quantum Lie algebra.

Chapter 7

The Johnson Graph $J(D, N)$

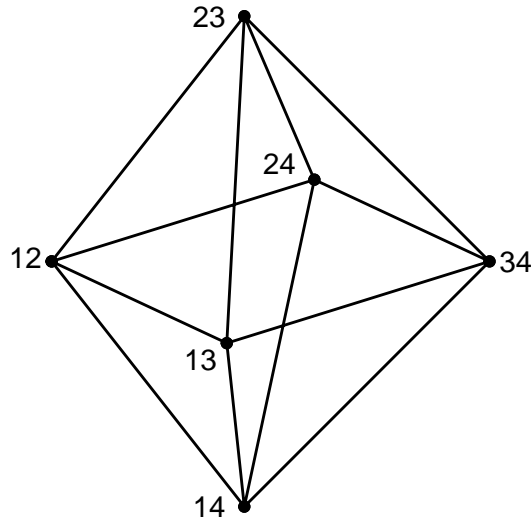
Wednesday, February 3, 1993

Definition 7.1. The Johnson graph, $\Gamma = J(D, N)$ ($1 \leq D \leq N - 1$) satisfies

$$X = \{S \mid S \subset \Omega, |S| = D\} \quad \text{where } \Omega = \{1, 2, \dots, N\} \quad (7.1)$$

$$E = \{ST \mid S, T \in X, |S \cap T| = D - 1\}. \quad (7.2)$$

Example 7.1. $J(2, 4)$



Note 1. The symmetric group S_N acts on Ω . $S_N \subseteq \text{Aut}(\Gamma)$ acts vertex transitively on Γ .

Note 2. $\Gamma = J(D, N)$ is isomorphic to $\Gamma' = J(N - D, N)$.

$$\Gamma = (X, E) \qquad \Gamma' = (X', E') \qquad (7.3)$$

$$X \ni S \quad \longrightarrow \quad \bar{S} = \Omega \quad S \in X' \qquad (7.4)$$

This correspondence induces an isomorphism of graphs.

Pf.

$$ST \in E \Leftrightarrow |S \cap T| = D - 1 \qquad (7.5)$$

$$\Leftrightarrow |\Omega - (S \cup T)| = N - D - 1 \qquad (7.6)$$

$$\Leftrightarrow |\bar{S} \cap \bar{T}| = N - D - 1 \qquad (7.7)$$

$$\Leftrightarrow \bar{S}\bar{T} \in E' \qquad (7.8)$$

Hence, without loss of generality, assume

$$D \leq N/2 \quad \text{for } J(D, N).$$

We still need the eigenvalues of $J(D, N)$ for certain problem later in the course. We can get these eigenvalues from our study of $H(D, 2)$.

Lemma 7.1. *The eigenvalues for $J(D, N)$ with $1 \leq D \leq N/2$ are given by*

$$\theta_i = (N - D - i)(D - i) - i \quad (0 \leq i \leq D) \qquad (7.9)$$

$$m_i = \binom{D}{i} - \binom{N}{i-1}. \qquad (7.10)$$

Proof. Let

$$\Gamma_J \equiv J(D, N) = (X_J, E_J) \qquad (7.11)$$

$$\Gamma_H \equiv H(N, 2) = (X_H, E_H). \qquad (7.12)$$

Set $x \equiv 11 \cdots 1 \in X_H$.

Define $\tilde{\Gamma} \equiv (\tilde{X}, \tilde{E})$, where

$$\tilde{X} = \{y \in X_H \mid \partial_H(x, y) = D\} \quad \partial_H : \text{distance in } \Gamma_H \qquad (7.13)$$

$$\tilde{E} = \{yz \in X_H \mid \partial_H(y, z) = 2\}. \qquad (7.14)$$

Observe

$$X_J \rightarrow \tilde{X} \qquad (7.15)$$

$$S \mapsto \hat{S}, \qquad (7.16)$$

where

$$\hat{S} = a_1 \cdots a_N, \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$$

induces an isomorphism of graphs $\Gamma_J \rightarrow \tilde{\Gamma}$.

Pf.

$$ST \in E_J \Leftrightarrow |S \cap T| = D - 1 \quad (7.17)$$

$$\Leftrightarrow \partial_H(\hat{S}, \hat{T}) = 2 \quad (7.18)$$

$$\Leftrightarrow (\hat{S}, \hat{T}) \in \tilde{E}. \quad (7.19)$$

Identify, Γ_J with $\tilde{\Gamma}$. Then the standard module V_J of Γ_J becomes $\tilde{V} = E_D^* V_H$, where V_H is the standard module of Γ_H , and $E_D^* \equiv E_D^*(x)$.

Let R be the raising matrix with respect to x in Γ_H , and

let L be the lowering matrix with respect to x in Γ_H .

Recall

$$(RL - DE_D^*)|_{\tilde{V}}$$

is the adjacency map in $\tilde{\Gamma}$.

To find eigenvalues of \tilde{A} , pick any irreducible $T(x)$ -module W with the endpoint $r \leq D$. Then by Theorem 5.1

$$\text{diam}(W) = N - 2r + 1.$$

Let $w_0, w_1, \dots, w_{N-2r}$ denote a basis for W as in Theorem 5.1. Then,

$$w_{D-r} \in E_D^* W \subseteq \tilde{V}.$$

Observe:

$$\tilde{A}w_{D-r} = RLw_{D-r} - DE_D^*w_{D-r} \quad (7.20)$$

$$= R(N - 2r - D + r + 1)w_{D-r-1} - Dw_{D-r} \quad (7.21)$$

$$= ((N - D - r + 1)(D - r) - D)w_{D-r}. \quad (7.22)$$

Note that this is valid for $D = r$ as well.

Hence,

$$\tilde{A}w_{D-r} = ((N - D - r)(D - r) - r)w_{D-r}.$$

Let

$$V_H = \sum W \quad (\text{direct sum of irreducible } T(x) \text{ - modules.})$$

Then,

$$V_J = E_D^* V_H \quad (7.23)$$

$$= \sum_{W: r(W) \leq D} E_D^* W \quad (7.24)$$

$$= \text{a direct sum of 1 dimensional eigenspaces for } \tilde{A}. \quad (7.25)$$

The eigenspace for eigenvalue

$$(N - D - r)(D - r) - r \quad (\text{monotonously decreasing with respect to } r)$$

appears with multiplicity

$$\binom{N}{r} - \binom{N}{r-1}$$

in this sum by Theorem 5.1 (iv). \square

Theorem 7.1. *Let $\Gamma = (X, E)$ be any graph. For a fixed vertex $x \in X$, let*

$$E_i^* \equiv E_i^*(x), \quad T \equiv T(x), \quad D \equiv D(x), \quad \text{and } K = \mathbb{C}.$$

Then we have the following implications of conditions:

$$TH \Leftrightarrow C \Leftarrow S \Leftarrow G.$$

where

(TH) Γ is thinn with respect to x .

*(C) $E_i^*TE_i^*$ is commutative for every i , $(0 \leq i \leq D)$.*

*(S) $E_i^*TE_i^*$ is symmetric for every i , $(0 \leq i \leq D)$.*

(G) For every $y, z \in X$ with $\partial(x, y) = \partial(x, z)$, there exists $g \in \text{Aut}(\Gamma)$ such that

$$gx = x, \quad gy = z, \quad gz = y.$$

Proof.

$(TH) \Rightarrow (C)$

Fix i with $0 \leq i \leq D$. Let

$V = \sum W$. The standard module written as a direct sum of irreducible T -modules.

The,

$E_i^*V = \sum E_i^*W$. The direct sum of 1-dimensional $E_i^*TE_i^*$ -modules.

Since $\dim E_i^*W = 1$, for $a, b \in E_i^*TE_i^*$, $ab - ba|_{E_i^*W} = 0$. Hence $ab - ba = 0$.

$(C) \Rightarrow (TH)$

Suppose $\dim E_i^*W \geq 2$ for some irreducible T -module W with some i with $1 \leq i \leq D$.

Claim: E_i^*W is an irreducible $E_i^*TE_i^*$ -module.

Pf. Suppose

$$0 \subsetneq U \subsetneq E_i^*W,$$

where U is a $E_i^*TE_i^*$ -module. Then by the irreducibility,

$$TU = W.$$

So

$$U \supseteq E_i^*TE_i^*U = E_i^*TU = E_i^*W.$$

This is a contradiction.

Claim 2: Each irreducible $S = E_i^*TE_i^*$ -module U has dimension 1. In particular, Γ is thin with respect to x .

Pf. Pick

$$0 \neq a \in E_i^*TE_i^*.$$

Since \mathbb{C} is algebraically closed, a has an eigenvector $w \in U$ with eigenvalue θ . Then,

$$(a - \theta I)U = (a - \theta I)Sw \tag{7.26}$$

$$= S(a - \theta I)w \tag{7.27}$$

$$= 0. \tag{7.28}$$

Hence,

$$a|_U = \theta I|_U \quad \text{for all } a \in S.$$

Thus each 1 dimensional subspace of U is an S -module. We have

$$\dim U = 1.$$

By Claim 1 and Claim 2, we have (TH).

□

Chapter 8

Thin Graphs

Friday, February 5, 1993

Proof of Theorem 7.1 continued.

(S) \Rightarrow (C)

Fix i and pick $a, b \in E_i^* T E_i^*$.

Since a , b and ab are symmetric,

$$ab = (ab)^\top = b^\top a^\top = ba.$$

Hence $E_i^* T E_i^*$ is commutative.

(G) \Rightarrow (S)

Fix i and pick $a \in E_i^* T E_i^*$. Pick vertices $y, z \in X$.

We want to show that

$$a_{yz} = a_{zy}.$$

We may assume that

$$\partial(x, y) = \partial(x, z) = i,$$

otherwise

$$a_{yz} = a_{zy} = 0.$$

By our assumption, there exists $g \in G$ such that

$$g(y) = z, \quad g(z) = y, \quad g(x) = x.$$

Let \hat{g} denote the permutation matrix representing g , i.e.,

$$\hat{g}\hat{y} = \widehat{g(y)} \quad \text{for all } y \in X, \quad \hat{g} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow y.$$

If $g \in \text{Aut}(\Gamma)$, then

$$\hat{g}A = A\hat{g} \quad \text{Exercise.}$$

Also we have

$$\hat{g}E_j^* = E_j^*\hat{g} \quad (0 \leq j \leq D),$$

since

$$\partial(x, y) = \partial(g(x), g(y)) = \partial(x, g(y)).$$

Hence \hat{g} commutes with each element of T . We have

$$a_{yz} = (\hat{g}^{-1}a\hat{g})_{yz}, \quad (\hat{g})_{yz} = \begin{cases} 1 & g(z) = y \\ 0 & \text{else.} \end{cases} \quad (8.1)$$

$$= \sum_{y', z'} (\hat{g}^{-1})_{yy'} a_{y'z'} \hat{g}_{z'z} \quad (8.2)$$

$$(\text{zero except for } g^{-1}(y') = y, g(z) = z'.) \quad (8.3)$$

$$= a_{g(y)g(z)} \quad (8.4)$$

$$a_{zy}. \quad (8.5)$$

This proves Theorem 7.1. □

Open Problem: Find all the graphs that satisfy the condition (G) for every vertex x .

$H(N, 2)$ is one example, because

$$\text{Aut}\Gamma_{1\dots 1} \simeq S_\Omega, \quad x = (1 \dots 1), \Gamma_i(x) = \{\hat{S} \mid |S| = i\}.$$

Property (G) is clearly related to the distance-transitive property.

Definition 8.1. Let $\Gamma = (X, E)$ be any graph. Γ with $G \subseteq \text{Aut}(\Gamma)$ is said to be distance-transitive (or two-point homogeneous), whenever

$$\text{for all } x, x', y, y' \in X \text{ with } \partial(x, y) = \partial(x', y'),$$

there exists $g \in G$ such that

$$g(x) = y, \quad g(x') = y'.$$

(This means G is as close to being doubly transitive as possible.)

Lemma 8.1. Suppose a graph $\Gamma = (X, E)$ satisfies the property $(G) = (G(x))$ for every $x \in X$. Then,

- (i) either
- (ia) Γ is vertex transitive; or
- (iia) Γ is bipartite ($X = X^+ \cup X^-$) with X^+, X^- each an orbit of $\text{Aut}(\Gamma)$.
- (ii) if (ia) holds, then Γ is distance-transitive.

Proof. (i) Claim. Suppose $y, z \in X$ are connected by a path of even length. Then y, z are in the same orbit of $\text{Aut}(\Gamma)$.

Pf. It suffices to assume that the path has length 2, $y \sim w \sim z$.

Now $\partial(y, w) = \partial(w, z) = 1$. So there exists $g \in \text{Aut}(\Gamma)$ such that $gw = w$, $gy = z$, $gz = y$. This proves Claim.

Fix $x \in X$. Now suppose that Γ is not vertex transitive, and we shall show (ib).

Observe that $X = X^+ \cup X^-$, where

$$X^+ = \{y \in X \mid \text{there exists a path of even length connecting } x \text{ and } y\} \quad (8.6)$$

$$X^- = \{y \in X \mid \text{there exists a path of odd length connecting } x \text{ and } y\} \quad (8.7)$$

As X^+ is contained in an orbit O^+ of $\text{Aut}(\Gamma)$, and X^- is contained in an orbit O^- of $\text{Aut}(\Gamma)$.

Now $O^+ \cap O^- = \emptyset$ (else $O^+ = O^- = X$ and vertex transitive). So, $X = O^+$, and $X^- = O^-$.

Also $X^+ \cup X^- = X$ is a bipartition by construction.

(ii) Fix x, y, x', y' with $\partial(x, y) = \partial(x', y')$.

By vertex transitivity, there exists an element

$$g_1 \in G \text{ such that } g_1x = x'.$$

Observe that

$$\partial(x', y') = \partial(x, y) = \partial(g_1x, g_1y) = \partial(x', g_1y).$$

Hence, there exists an element

$$g_2 \in G \text{ such that } g_1x' = x', g_2y' = g_1y', g_2g_1y = y'$$

by $(G(x'))$ property.

Set $g = g_2g_1$. Then

$$gx = x', gy = y'$$

by construction. □

The following graphs $\Gamma = (X, E)$ are vertex transitive, and satisfy the property $(G(x))$ for all $x \in X$.

$$J(D, N), \quad H(D, r), \quad J_q(D, N),$$

where

$$H(D, r):$$

$$X = \{a_1 \cdots a_D \mid a_i \in F, 1 \leq i \leq D\} \quad (8.8)$$

$$F : \text{ any set of cardinality } r \quad (8.9)$$

$$E = \{xy \mid y, x \in X, x \text{ and } y \text{ differ in exactly one coordinate}\}. \quad (8.10)$$

$J_q(D, N)$:

X = the set of all D -dimensional subspaces of N -dimensional vector space over $GF(q)$.
(8.11)

$$F : \text{ any set of cardinality } r \quad (8.12)$$

$$E = \{xy \mid y, x \in X, \dim(x \cap y) = D - 1\}. \quad (8.13)$$

The following graph is distance-transitive but does not satisfy $(G(x))$ for any $x \in G$.

$H_q(D, N)$:

$$X = \text{the set of all } D \times N \text{ matrices with entries in } GF(q). \quad (8.14)$$

$$E = \{xy \mid y, x \in X, \text{rank}(x - y) = 1\}. \quad (8.15)$$

Remark.

$$H(D, r): G = S_r \text{wr} S_D, G_x = S_{r-1} \text{wr} S_D,$$

For $x, y \in X$ with $\partial(x, y) = \partial(x, z) = i$,

$$Y = \{j \in \Omega \mid x_j \neq y_j\} \leftrightarrow Z = \{j \in \Omega \mid x_j \neq z_j\} \quad (8.16)$$

$$(y_{j_1}, \dots, y_{j_i}) \leftrightarrow (z_{\ell_1}, \dots, z_{\ell_i}) \quad (8.17)$$

$$J(D, N): G = S_N, G_x = S_D \times S_{N-D}.$$

$$X \cap Y \leftrightarrow X \cap Z \quad (8.18)$$

$$(\Omega - X) \cap Y \leftrightarrow (\Omega - X) \cap Z. \quad (8.19)$$

The following graph is distance-transitive but does not satisfy $(G(x))$ for any $x \in G$.

$J_q(D, N)$:

$$X \cap Y \leftrightarrow X \cap Z.$$

The theory of single thin irreducible T -module.

Let $\Gamma = (X, E)$ be any graph.

$$M = \text{Bose-Mesner algebra over } K/\mathbb{C} \text{ generated by the adjacency matrix } A. \quad (8.20)$$

$$= \text{Span}(E_0, \dots, E_R). \quad (8.21)$$

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix $x \in X$, let $D \equiv D(x)$ be the x -diameter, and $k = k(x)$ be the valency of x .

Chapter 9

Thin T -Module, I

Monday, February 8, 1993

Let $\Gamma = (X, E)$ be any graph.

M : Bose-Mesner algebra over K/\mathbb{C} generated by the adjacency matrix A .

$$M = \text{Span}(E_0, \dots, E_R).$$

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix $x \in X$, let $D \equiv D(x)$ be the x -diameter, and $k = k(x)$ be the valency of x .

Definition 9.1. Pick $x \in X$ and write $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

Let W be an irreducible thin T -module with endpoint r , diameter d .

Let $a_i = a_i(W) \in \mathbb{C}$ satisfying

$$E_{r+i}^* A E_{r+i}^* |_{E_{r+i}^* W} = a_i 1 |_{E_{r+i}^* W} \quad (0 \leq i \leq d).$$

Let $x_i = x_i(W) \in \mathbb{C}$ satisfying

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* |_{E_{r+i}^* W} = x_i 1 |_{E_{r+i}^* W} \quad (0 \leq i \leq d).$$

Lemma 9.1. *With above notation, the following hold.*

(i) $a_i \in \mathbb{R} \quad (0 \leq i \leq d)$.

(ii) $x_i \in \mathbb{R}^{>0} \quad (0 \leq i \leq d)$.

(iii) Pick $0 \neq w_0 \in E_r^* W$. Set $w_i = E_{r+i}^* A^i w_0$ for all i . Then

(iiia) w_0, w_1, \dots, w_d is a basis for W , $w_{-1} = w_{d+1} = 0$.

(iiib) $A w_i = w_{i+1} + a_i w_i + x_i w_{i-1} \quad (0 \leq i \leq d)$.

(iv) Define $p_0, p_1, \dots, p_{d+1} \in \mathbb{R}[\lambda]$ by

$$p_0 = 1, \quad \lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1} \quad (0 \leq i \leq d), \quad p_{-1} = 0.$$

(iva) $p_i(A)w_0 = w_i$, $(0 \leq i \leq d+1)$.

(ivb) p_{d+1} is the minimal polynomial of $A|_W$.

Proof. (i) a_i is an eigenvalue of a real symmetric matrix $E_{r+i}^* A E_{r+i}^*$.

(ii) x_i is an eigenvalue of a real symmetric matrix $B^\top B$, where

$$B = E_{r+i}^* A E_{r+i-1}^*.$$

Hence, $x_i \in \mathbb{R}$.

Since $B^\top B$ is positive semidefinite,

$$x_i \geq 0.$$

Pf. If $B^\top Bv = \sigma v$ for some $\sigma \in \mathbb{R}$, $v \in \mathbb{R}^m \setminus \{0\}$, then

$$0 \leq \|Bv\|^2 = v^\top B^\top Bv = \sigma v^\top v = \sigma \|v\|^2, \quad \|v\|^2 > 0.$$

Hence, $\sigma \geq 0$.

Moreover, $x_i \neq 0$ by Lemma 4.1 (iv).

(iiia) Observe

$$w_i = E_{r+i}^* A E_{r+i-1}^* w_{i-1} \quad (1 \leq i \leq d).$$

So $w_i \neq 0$ $(1 \leq i \leq d)$ by Lemma 4.1 (iv).

Hence,

$$W = \text{Span}(w_0, \dots, w_d)$$

by Lemma 4.1. (iii).

(iiib) We have that

$$Aw_i = E_{r+i+1}^* Aw_i + E_{r+i}^* Aw_i + E_{r+i-1}^* Aw_i \quad (9.1)$$

$$= w_{i+1} + E_{r+i}^* A E_{r+i}^* w_i + E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* w_{i-1} \quad (9.2)$$

$$= w_{i+1} + a_i w_i + x_i w_{i-1} \quad (9.3)$$

(iva) Clear for $i = 0$. Assume it is valid for $0, \dots, i$.

$$p_{i+1}(A)w_0 = (A - a_i I)w_i - x_i w_{i-1} = w_{i+1}.$$

(ivb) By definition,

$$p_{d+1}(A)w_0 = 0.$$

Moreover, $p_{d+1}(A)W = 0$. For every $w \in W$, write

$$w = \sum_{i=0}^d \alpha_i w_i \quad (9.4)$$

$$= \sum_{i=0}^d \alpha_i p_i(A)w_0 \quad \text{for some } \alpha_i \in \mathbb{C} \quad (9.5)$$

$$= p(A)w_0 \quad \text{for some } p \in \mathbb{C}[\lambda] \quad (9.6)$$

Hence,

$$p_{d+1}(A)w = p_{d+1}(A)p(A)w_0 \quad (9.7)$$

$$= p(A)p_{d+1}(A)w_0 \quad (9.8)$$

$$= 0. \quad (9.9)$$

Note that p_{d+1} is the minimal polynomial.

Pf. Suppose $q(A)W = 0$ for some $0 \neq q \in \mathbb{C}[\lambda]$ with $\deg q < \deg p_{d+1} = d + 1$. Then,

$$q = \sum_{i=0}^d \beta_i p_i \quad \text{for some } \beta_i \in \mathbb{C}.$$

We have,

$$0 = q(A)w_0 = \sum_{i=0}^d \beta_i w_i.$$

Hence $\beta_0 = \dots = \beta_d = 0$ by (iiia). Thus $q = 0$ and a contradiction. \square

Corollary 9.1. *Let Γ , W , r , d be as above. Then*

(i) *W is dual thin, that is,*

$$\dim E_i W \leq 1 \quad (1 \leq i \leq d).$$

(ii) $d = |\{i \mid E_i W \neq 0\}| - 1$.

Proof. (i) Set as in Lemma 9.1,

$$w_i = p_i(A)w_0 \in E_{r+i}^* W.$$

Then w_0, w_1, \dots, w_d is a basis for W . We have

$$W = Mw_0.$$

So,

$$E_i W = E_i M w_0 = \text{Span}(E_i w_0).$$

Thus,

$$\dim E_i W = \begin{cases} 1 & \text{if } E_i w_0 \neq 0 \\ 0 & \text{if } E_i w_0 = 0. \end{cases}$$

In particular,

$$\dim E_i^* W \leq 1.$$

(ii) Immediate as

$$\dim W = d + 1.$$

This proves the lemma. \square

Lemma 9.2. *Given an irreducible $T(x)$ -module W with endpoint $r = r(W)$, diameter $d = d(W)$. Write*

$$x_i = x_i(W) \ (0 \leq i \leq d), \quad w_i = p_i(A)w_0 \in E_{r+i}^* W \ (0 \leq i \leq d), \quad 0 \neq w_0 \in E_r^* W.$$

Then,

$$\frac{\|w_i\|^2}{\|w_0\|^2} = x_1 x_2 \cdots x_i \quad (1 \leq i \leq d).$$

Proof. It suffices to show that

$$\|w_i\|^2 = x_i \|w_i\|^2 \quad (1 \leq i \leq d).$$

Recall by Lemma 9.1 (iiib) that

$$Aw_j = w_{j+1} + a_j w_j + x_j w_{j-1} \quad (0 \leq j \leq d), \quad w_{-1} = w_{d+1} = 0.$$

Now observe,

$$\langle w_{i-1}, Aw_i \rangle = \langle w_{i-1}, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle \quad (9.10)$$

$$= \overline{x_i} \|w_{i-1}\|^2 \quad (9.11)$$

$$= x_i \|w_{i-1}\|^2. \quad (9.12)$$

by Lemma 9.1 (ii). Also,

$$\langle w_{i-1}, Aw_i \rangle = \langle Aw_{i-1}, w_i \rangle \quad (\text{since } \bar{A}^\top = A) \quad (9.13)$$

$$= \langle x_i + a_{i-1} w_{i-1} + x_{i-1} w_{i-2}, w_i \rangle \quad (9.14)$$

$$= \|w_i\|^2. \quad (9.15)$$

This proves the lemma. \square

Definition 9.2. Let W be an irreducible thin $T(x)$ module with endpoint r , $E_i^* \equiv E_i^*(x)$.

The measure $m = m_W$ is the function

$$m : \mathbb{R} \rightarrow \mathbb{R}$$

such that

$$m(\theta) = \begin{cases} \frac{\|E_i w\|^2}{\|w\|^2} & \text{where } 0 \neq w \in E_r^* W \\ & \text{if } \theta = \theta_i \text{ is an eigenvalue for } \Gamma \\ 0 & \text{if } \theta \text{ is not an eigenvalue for } \Gamma. \end{cases}$$

Chapter 10

Thin T -Module, II

Wednesday, February 10, 1993

Let $\Gamma = (X, E)$ be any graph.

Fix a vertex $x \in X$. Let $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$, the subconstituent algebra over \mathbb{C} , and $V = \mathbb{C}^{|X|}$ the standard module.

Lemma 10.1. *With above notation, let W denote a thin irreducible $T(x)$ -module with endpoint r and diameter d . Let*

$$a_i = a_i(W) \quad (0 \leq i \leq d) \quad (10.1)$$

$$x_i = x_i(W) \quad (1 \leq i \leq d) \quad (10.2)$$

$$p_i = p_i(W) \quad (0 \leq i \leq d+1) \quad (10.3)$$

be from Lemma 9.1, and measure $m = m_W$. Then,

(i) p_0, \dots, p_{d+1} are orthogonal with respect to m , i.e.,

$$\sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \leq i, j \leq d+1) \text{ with } x_{d+1} = 0.$$

$$(ia) \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 m(\theta) = x_1 \cdots x_i \quad (0 \leq i \leq d).$$

$$(iia) \sum_{\theta \in \mathbb{R}} m(\theta) = 1.$$

$$(iiia) \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) = x_1 \cdots x_i a_i \quad (0 \leq i \leq d).$$

Proof. Pick $0 \neq w_0 \in E_r^* W$. Set

$$w_i = p_i(A) w_0 \in E_{r+i}^* W.$$

Since E_i^*W and E_j^*W are orthogonal if $i \neq j$,

$$\delta_{ij}\|w_i\|^2 = \langle w_i, w_j \rangle \quad (10.4)$$

$$= \langle p_i(A)w_0, p_j(A)w_0 \rangle \quad (10.5)$$

$$= \left\langle p_i(A) \left(\sum_{\ell=0}^R E_\ell \right) w_0, p_j(A) \left(\sum_{\ell=0}^R E_\ell \right) w_0 \right\rangle \quad (10.6)$$

$$= \left\langle \sum_{\ell=0}^R p_i(\theta_\ell) E_\ell w_0, \sum_{\ell=0}^R p_j(\theta_\ell) E_\ell w_0 \right\rangle \quad (\text{as } AE_j = \theta_j E_j) \quad (10.7)$$

$$= \sum_{\ell=0}^R p_i(\theta_\ell) \overline{p_j(\theta_\ell)} \|E_\ell w_0\|^2 \quad (10.8)$$

$$(\text{as } p_j \in \mathbb{R}[\lambda], \quad \theta_\ell \in \mathbb{R}, \quad m(\theta_i)\|w_0\|^2 = \|E_i w_0\|^2) \quad (10.9)$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) \|w_0\|^2. \quad (10.10)$$

Now we are done by Lemma 9.2 as

$$\|w_i\|^2 = \|w_0\|^2 x_1 x_2 \dots x_i.$$

For (ia), set $i = j$, and for (ib), set $i = j = 0$.

(ii) We have

$$\langle w_i, Aw_i \rangle = \langle w_i, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle \quad (10.11)$$

$$= \overline{a_i} \|w_i\|^2 \quad (10.12)$$

$$= a_i x_1 \dots x_i \|w_0\|^2, \quad (10.13)$$

as $a_i \in \mathbb{R}$ by Lemma 9.1.

Also,

$$\langle w_i, Aw_i \rangle = \langle p_i(A)w_0, Ap_i(A)w_0 \rangle \quad (10.14)$$

$$= \left\langle p_i(A) \left(\sum_{\ell=0}^R E_\ell \right) w_0, Ap_i(A) \left(\sum_{\ell=0}^R E_\ell \right) w_0 \right\rangle \quad (\text{as in (i)}) \quad (10.15)$$

$$= \sum_{\ell=0}^D p_i(\theta_\ell)^2 \theta_\ell \|E_\ell w_0\|^2 \quad (10.16)$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) \|w_0\|^2. \quad (10.17)$$

Thus, we have (ii). \square

Lemma 10.2. *With above notation, let W be a thin irreducible $T(x)$ -module with measure m . Then m determines diameter $d(W)$,*

$$a_i = a_i(W) \quad (0 \leq i \leq d) \quad (10.18)$$

$$x_i = x_i(W) \quad (1 \leq i \leq d) \quad (10.19)$$

$$p_i = p_i(W) \quad (0 \leq i \leq d+1). \quad (10.20)$$

Proof. Note that $d+1$ is the number of $\theta \in \mathbb{R}$ such that $m(\theta) \neq 0$. Hence m determines d .

Apply (ia), (ii) of Lemma 10.1.

$$\sum_{\theta \in \mathbb{R}} m(\theta) = 1 \quad p_0 = 1. \quad (10.21)$$

$$\sum_{\theta \in \mathbb{R}} \theta m(\theta) = a_0 \quad p_1 = \lambda - a_0 \quad (10.22)$$

$$\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 m(\theta) = x_1 \quad (10.23)$$

$$\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 \theta m(\theta) = x_1 a \quad \rightarrow a_1 \quad (10.24)$$

$$p_2 = (\lambda - a_1)p_1 - x_1 p_0 \quad (10.25)$$

$$\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 m(\theta) = x_1 x_2 \quad \rightarrow x_2 \quad (10.26)$$

$$\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 \theta m(\theta) = x_1 x_2 a_2 \quad \rightarrow a_2 \quad (10.27)$$

$$p_3 = (\lambda - a_2)p_2 - x_2 p_1 \quad (10.28)$$

$$\vdots \quad (10.29)$$

$$\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 m(\theta) = x_1 x_2 \cdots x_d \quad \rightarrow x_d \quad (10.30)$$

$$\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 \theta m(\theta) = x_1 x_2 \cdots x_d a_d \quad \rightarrow a_d \quad (10.31)$$

$$p_{d+1} = (\lambda - a_d)p_d - x_d p_{d-1}. \quad (10.32)$$

$$(10.33)$$

This proves the assertions. \square

Corollary 10.1. *With above notation, let W, W' denote thin irreducible $T(x)$ -modules. The following are equivalent.*

(i) W, W' are isomorphic as T -modules.

(ii) $r(W) = r(W')$ and $m_W = m_{W'}$.

(iii) $r(W) = r(W')$, $d(W) = d(W')$, $a_i(W) = a_i(W')$ and $x_i(W) = x_i(W')$ ($0 \leq i \leq d$).

Proof. (i) \Rightarrow (iii) Write $r \equiv r(W)$, $r' \equiv r(W')$, $d = d(W)$, $d' = d(W')$, $a_i = a_i(W)$, $a'_i = a_i(W')$, $x_i = x_i(W)$ and $x'_i = x_i(W')$.

Let $\sigma : W \rightarrow W'$ denote an isomorphism of T -modules. (See Definition 5.1.)

For every i ,

$$\sigma E_i^* W = E_i^* \sigma W = E_i^* W'.$$

So, $r = r'$ and $d = d'$.

To show $a_i = a'_i$, pick $w \in E_{r+i}^* W \setminus \{0\}$. Then,

$$E_{r+i}^* A E_{r+i}^* \sigma(W) = \sigma(E_{r+i}^* A E_{r+i}^* w) = \sigma(a_i w) = a_i \sigma(w),$$

and $\sigma w \neq 0$. So,

$$a_i = \text{eigenvalue of } E_{r+i}^* A E_{r+i}^* \text{ on } E_{r+i}^* W \quad (10.34)$$

$$= a'_i \quad (10.35)$$

It is similar to show $x = x'$.

Remark. Pick $w \in E_{r+i-1}^* W \setminus \{0\}$

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* \sigma(W) = \sigma(E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* w) = x_i \sigma(w).$$

Hence, x_i is the eigenvalue of $E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^*$ on $E_{r+i-1}^* W = x'_i$.

(iii) \Rightarrow (i)

Pick $0 \neq w_0 \in E_r^* W$, $0 \neq w'_0 \in E_r^* W'$. Let p_i be in Lemma 9.1, and set

$$w_i = p_i(A)w_0 \in E_{r+i}^* W \quad (0 \leq i \leq d) \quad (10.36)$$

$$w'_i = p'_i(A)w'_0 \in E_{r+i}^* W' \quad (0 \leq i \leq d) \quad (10.37)$$

Define a linear transformation,

$$\sigma : W \rightarrow W' \quad (w_i \mapsto w'_i).$$

Since $\{w_i\}$ and $\{w'_i\}$ are bases with $d = d'$, σ is an isomorphism of vector spaces.

We need to show

$$a\sigma = \sigma a \quad (\text{for all } a \in T).$$

Take $a = E_j^*$ for some j ($0 \leq j \leq d(x)$). Then for all i , we have

$$E_j^* \sigma w_i = E_j^* w'_i = \delta_{ij} w'_i,$$

$$\sigma E_j^* w_i = \delta_{ij} \sigma(w_i) = \delta_{ij} w'_i.$$

$$E_j^* \sigma w_i = \sigma E_j^* w_i?$$

Take an adjacency matrix A of a . Then,

$$A \sigma w_i = A w'_i = w'_{i+1} + a'_i w'_i + x'_i w'_{i-1} = \sigma(w_{i+1} + a_i w_i + x_i w_{i-1}) = \sigma A w_i.$$

(ii) \Rightarrow (iii) Lemma 10.2.

(iii) \Rightarrow (ii) Given d, a_i, x_i , we can compute the polynomial sequence

$$p_0, p_1, \dots, p_{d+1}$$

for W .

Show p_0, p_1, \dots, p_{d+1} determines $m = m_W$. Set

$$\Delta = \{\theta \in \mathbb{R} \mid p_{d+1}(\theta) = 0\}.$$

Observe: $|\Delta| = d + 1$. See ‘An Introcuction to Interlacing’.

$m(\theta) = 0$ if $\theta \notin \Delta$ ($\theta \in \mathbb{R}$). So it suffices to find $m(\theta)$, $\theta \in \Delta$.

By Lemma 10.1 (i),

$$\begin{cases} \sum_{\theta \in \Delta} m(\theta) p_0(\theta) &= 1 \\ \sum_{\theta \in \Delta} m(\theta) p_1(\theta) &= 0 \\ \vdots & \\ \sum_{\theta \in \Delta} m(\theta) p_d(\theta) &= 0 \end{cases}$$

$d + 1$ linear equation with $d + 1$ unknowns $m(\theta)$ ($\theta \in \Delta$).

But the coefficient matrix is essentially Vander Monde (since $\deg p_i = i$). Hence the system is nonsingular and there are unique values for $m(\theta)$ ($\theta \in \Delta$). \square

Remark.

$$\begin{pmatrix} \theta - a_0 & -1 & \cdots & 0 & 0 \\ -x_1 & \theta - a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta - a_{d-1} & -1 \\ 0 & 0 & \cdots & -x_d & \theta - a_d \end{pmatrix} \begin{pmatrix} p_0(\theta) \\ \vdots \\ \vdots \\ \vdots \\ p_d(\theta) \end{pmatrix} = 0,$$

where θ is an eigenvalue of a diagonalizable matrix

$$L = \begin{pmatrix} a_0 & 1 & \cdots & 0 & 0 \\ x_1 & a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{d-1} & 1 \\ 0 & 0 & \cdots & x_d & \theta a_d \end{pmatrix}$$

with multiplicity $\dim(\text{Ker}(\theta I - L)) = 1$.

Chapter 11

Examples of T -Module

Friday, February 12, 1993

Let $\Gamma = (X, E)$ be a connected graph.

Let θ_0 be the maximal eigenvalue of Γ , and δ its corresponding eigenvector.

$$\delta = \sum_{y \in X} \delta_y \hat{y}.$$

Without loss of generality, we may assume that $\delta_y \in \mathbb{R}^*$ for all $y \in X$.

Lemma 11.1. *Fix a vertex $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$.*

- (i) $T\delta = T\hat{x}$ is an irreducible T -module.
- (ii) *Given any irreducible T -module W , the following are equivalent:*
 - (iia) $W = T\delta$.
 - (iib) *The diameter $d(W) = d(x)$.*
 - (iic) *The endpoint $r(W) = 0$.*

Proof. (i) Observe: there exists an irreducible T -module W that contains δ .

Let $V = \sum_i W_i$ be a direct sum decomposition of the standard module. Then

$$\text{Span}(\delta) = E_0 V = \sum_i E_0 W_i.$$

So, $E_0 W_i \neq 0$ for some i . Then,

$$\delta \in E_0 W_i \subseteq W_i.$$

Observe: $T\delta$ is an irreducible T -module.

Since $\delta \in W$, where W is a T -module. As $T\delta \subseteq W$ and W is irreducible, $T\delta = W$.

Observe: $T\delta = T\hat{x}$.

Since $\hat{x} = \delta_x^{-1} E_0^* \delta \in T\delta$, $T\hat{x} \subseteq T\delta$. Since $T\delta$ is irreducible, $T\hat{x} = T\delta$.

(ii) (a) \rightarrow (b):

$$E_i^* \delta = \sum_{y \in X, \partial(x,y)=i} \delta_y \hat{y} \neq 0, \quad (0 \leq i \leq d(x)),$$

because $\delta_y > 0$ for every $y \in X$.

Hence,

$$E_i^* T\delta \neq 0, \quad (0 \leq i \leq d(x)).$$

Thus, $d(x) = d(W)$.

(b) \rightarrow (c): Immediate.

(c) \rightarrow (a): Since $r(W) = 0$, $E_0^* W \neq 0$. Hence, $\hat{x} \in W$ and $T\hat{x} \subseteq W$.

By the irreducibility, we have $T\hat{x} = W$. □

Lemma 11.2. Assume Γ is bipartite ($X = X^+ \cup X^-$) (X^+ and X^- are nonempty). Then the following are equivalent.

(i) There exist α^+ and $\alpha^- \in \mathbb{R}$ such that

$$\delta_x = \begin{cases} \alpha^+ & \text{if } x \in X^+ \\ \alpha^- & \text{if } x \in X^-. \end{cases}$$

/(ii) There exist k^+ and $k^- \in \mathbb{Z}^{>0}$ such that

$$k(x) = \begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^-. \end{cases}$$

In this case, $k^+ k^- = \theta_0^2$, and Γ is called bi-regular.

Proof. (i) \rightarrow (ii)



$$A\delta = A \left(\alpha^+ \sum_{x \in X^+} \hat{x} + \alpha^- \sum_{y \in X^-} \hat{y} \right) \quad (11.1)$$

$$= \alpha^+ \sum_{y \in X^-} k(y) \hat{y} + \alpha^- \sum_{x \in X^+} k(x) \hat{x} \quad (11.2)$$

$$= \theta_0 \delta. \quad (11.3)$$

So,

$$k(x)\alpha^- = \theta_0\alpha^+, \quad k(y)\alpha^+ = \theta_0\alpha^-.$$

As $\alpha^+ \neq 0$ and $\alpha^- \neq 0$,

$$k^+ := k(x) \text{ is independent of the choice of } x \in X^+, \text{ and} \quad (11.4)$$

$$k^- := k(y) \text{ is independent of the choice of } y \in X^-. \quad (11.5)$$

Moreover, $k^+k^- = \theta_0^2$.

(ii) \rightarrow (i) Set

$$\delta' = \sum_{y \in X} \alpha_y \hat{y} \quad \text{where } \alpha = \begin{cases} 1/\sqrt{k^-} & \text{if } y \in X^+ \\ 1/\sqrt{k^+} & \text{if } y \in X^-. \end{cases}$$

Then one checks

$$A\delta' = A \left(\frac{1}{\sqrt{k^-}} \sum_{y \in X^+} \hat{y} + \frac{1}{\sqrt{k^+}} \sum_{y \in X^-} \hat{y} \right) \quad (11.6)$$

$$= \frac{k^-}{\sqrt{k^-}} \sum_{y \in X^-} \hat{y} + \frac{k^+}{\sqrt{k^+}} \sum_{y \in X^+} \hat{y} \quad (11.7)$$

$$= \sqrt{k^+k^-} \delta' \quad (11.8)$$

Since $\delta' > 0$, $\delta' \in \text{Span}(\delta)$, and $\theta_0 = \sqrt{k^+k^-}$. \square

Definition 11.1. For any graph $\Gamma = (X, E)$, fix a vertex $x \in X$. Set $d = d(x)$.

Γ is distance-regular with respect to x , if for all $i : (0 \leq i \leq d)$, and all $y \in X$ such that $\partial(x, y) = i$:

$$c_i(x) := |\{z \in X \mid \partial(x, z) = i-1, \partial(y, z) = 1\}| \quad (11.9)$$

$$a_i(x) := |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = 1\}| \quad (11.10)$$

$$b_i(x) := |\{z \in X \mid \partial(x, z) = i+1, \partial(y, z) = 1\}| \quad (11.11)$$

depends only on i , x , and not on y .

(In this case, $c_0(x) = a_0(x) = b_d(x) = 0$, $c_1(x) = 1$, $b_0(x) = k(x)$ is the valency of x .)

We call $c_i(x)$, $a_i(x)$ and $b_i(x)$ the intersection numbers with respect to x .

Example 11.1.



$$c_0 = 1 \qquad c_1 = 1 \qquad c_2 = 1 \qquad (11.12)$$

$$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = 1 \qquad (11.13)$$

$$b_0 = 2 \qquad b_1 = 1 \qquad b_2 = 0 \qquad (11.14)$$

Chapter 12

Distance-Regular

Monday, February 15, 1993

Lemma 12.1. *For any connected graph $\Gamma = (X, E)$, the following are equivalent.*

(i) *The trivial $T(x)$ -module is thin for all $x \in X$.*

(ii) $\left\{ \sum_{y \in X, d(x,y)=i} \hat{y} \mid 0 \leq i \leq d(x) \right\}$ *is a basis for the trivial $T(x)$ -module for every $x \in X$.*

(iii) Γ *is distance-regular with respect to x for all $x \in X$.*

Note. Let $\Gamma = (X, E)$ be a graph, with $X = \{x, y_1, y_2, y_3, z_1, z_2, z_3\}$, $E = \{xy_1, xy_2, xy_3, y_1z_1, y_1z_2, y_2z_3, y_3z_3\}$.



Then (i), (ii) are not equivalent for a single vertex x .

$$E_0^* T \hat{x} = \langle \hat{x} \rangle, \quad (12.1)$$

$$E_1^* T \hat{x} = \langle y_1 + y_2 + y_3 \rangle, \quad (12.2)$$

$$E_2^* T \hat{x} = \langle z_1 + z_2 + 2z_3 \rangle. \quad (12.3)$$

Proof of Lemma 12.1. (i) \rightarrow (ii) Let $\delta = \sum_{y \in X} \delta_y \hat{y}$ be an eigenvector for the maximal eigenvalue θ_0 . Then,

$$\sum_{y \in X, \partial(x,y)=1} \hat{y} = A\hat{x} \in T(x)\hat{x} = T(x)\delta \ni E_1^* \delta \quad (12.4)$$

$$= \sum_{y \in X, \partial(x,y)=1} \delta_y \hat{y} \quad (12.5)$$

If the trivial $T(x)$ -module is thin,

$$\delta_y = \delta_z \text{ for } y, z \in X, \partial(x, y) = \partial(x, z) = 1.$$

Hence, $\delta_y = \delta_z$ if y and z in X are connected by a path of even length.

So, Γ is regular or bipartite biregular by Lemma 11.2.

In particular, $\delta_y = \delta_z$ if $\partial(x, y) = \partial(x, z)$, as there is a path of length $2 \cdot \partial(x, y)$;

$$y \sim \dots \sim x \sim \dots \sim z.$$

Hence,

$$E_i^* \delta \in \text{Span} \left(\sum_{y \in X, \partial(x,y)=i} \hat{y} \right).$$

Since $E_0^* \delta, E_1^* \delta, \dots, E_d^* \delta$ forms a basis for $T(x)\delta$, we have (ii).

(ii) \rightarrow (iii) Fix $x \in X$, and let $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, and $d \equiv d(x)$.

$$A \sum_{y \in X, \partial(x,y)=i} \hat{y} = \sum_{z \in X} |\{y \in X \mid \partial(y, z) = 1, \partial(x, y) = i\}| \hat{z} \quad (12.6)$$

$$= \sum_{z \in X, \partial(x,y)=i-1} b_{i-1}(x, z) \hat{z} \quad (12.7)$$

$$+ \sum_{z \in X, \partial(x,y)=i} a_i(x, z) \hat{z} \quad (12.8)$$

$$+ \sum_{z \in X, \partial(x,y)=i+1} c_{i+1}(x, z) \hat{z} \quad (12.9)$$

$$\in \text{Span} \left\{ \sum_{z \in X, \partial(x,z)=j} \hat{z} \mid j = 0, 1, \dots, d \right\}. \quad (12.10)$$

Hence, $b_{i-1}(x, z)$, $a_i(x, z)$ and $c_{i+1}(x, z)$ depend only on i and x , and not on z . Therefore, Γ is distance-regular with respect to x .

(iii) \rightarrow (i) Fix $x \in X$, and let $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, and $d \equiv d(x)$. By definition of distance-regularity, for every i ($0 \leq i \leq d$),

$$A \left(\sum_{y \in X, \partial(x,y)=i} \hat{y} \right) = b_{i-1}(x) \sum_{y \in X, \partial(x,y)=i-1} \hat{y} \quad (12.11)$$

$$+ a_i(x) \sum_{y \in X, \partial(x,y)=i} \hat{y} \quad (12.12)$$

$$+ c_{i+1}(x) \sum_{y \in X, \partial(x,y)=i+1} \hat{y}. \quad (12.13)$$

Hence,

$$W = \left\{ \sum_{y \in X, \partial(x,y)=i} \hat{y} \mid 0 \leq i \leq d \right\}$$

is A -invariant and so T -invariant. Since $\hat{x} \in W$, $T\hat{x} = W$ is the trivial module and $T\hat{x}$ is thin. \square

Next, we show more is true if (i) – (iii) hold in Lemma 12.1.

In fact, $d(x)$, $a_i(x)$, $c_i(x)$, and $b_i(x)$ are

$$\begin{cases} \text{independent of } X & \text{if } \Gamma \text{ is regular; or} \\ \text{constant over } X^+ \text{ and } X^- & \text{if } \Gamma \text{ is biregular.} \end{cases}$$

Let $\Gamma = (X, E)$ be any (connected) graph. Pick vertices $x, y \in X$.

Let W be a thin, irreducible $T(x)$ -module, and measure $m : \mathbb{R} \rightarrow \mathbb{R}$ determined by W .

Let W' be a thin, irreducible $T(y)$ -module, and measure $m' : \mathbb{R} \rightarrow \mathbb{R}$ determined by W' .

Recall W, W' are orthogonal if

$$\langle w, w' \rangle = 0 \quad \text{for all } w \in W, w' \in W'.$$

We shall show if W and W' are not orthogonal, then m and m' are related:

$$m \cdot \text{poly}_1 = m' \cdot \text{poly}_2$$

for some polynomials with

$$\deg \text{poly}_1 + \deg \text{poly}_2 \leq 2 \cdot \partial(x, y).$$

Notation. V : standard module of Γ .

H : any subspace of V .

$$V = H + H^\perp \quad \text{orthogonal direct sum,}$$

and for $v = v_1 + v_2$ $\text{proj}_H : V \rightarrow H$ ($v \mapsto v_1$): linear transformation.

Observe: For every $v \in V$,

$$v - \text{proj}_H v \in H^\perp.$$

So,

$$\langle v - \text{proj}_H v, h \rangle = 0 \quad \text{for all } h \in H \text{ or,}$$

$$\langle v, h \rangle = \langle \text{proj}_H v, h \rangle \quad \text{for all } v \in V, \text{ and for all } h \in H.$$

Theorem 12.1. *Let $\Gamma = (X, E)$ be any graph. Pick vertices $x, y \in X$ and set $\Delta = \partial(x, y)$. Assume*

W : thin irreducible $T(x)$ -module with endpoint r , diameter d , and measure m .

W' : thin irreducible $T(y)$ -module with endpoint r' , diameter d' , and measure m' .

W and W' are not orthogonal.

Now pick

$$0 \neq w \in E_r^*(x)W, \quad 0 \neq w' \in E_{r'}^*(x)W'.$$

Then,

$$(i) \quad \text{proj}_{W'} w = p(A) \frac{\|w\|}{\|w'\|} w'$$

for some $0 \neq p \in \mathbb{C}[\lambda]$ with $\deg p \leq \Delta - r' + r, d'$,

$$\text{proj}_W w' = p'(A) \frac{\|w'\|}{\|w\|} w$$

for some $0 \neq p' \in \mathbb{C}[\lambda]$ with $\deg p' \leq \Delta - r + r', d$.

(ii) For all eigenvalues θ_i of Γ ,

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = m(\theta_i) \overline{p'(\theta_i)}.$$

(iii) For all eigenvalues θ_i of Γ ,

$$p(\theta_i) p'(\theta_i)$$

is in a real number in interval $[0, 1]$.

Proof. (i) Since W, W' are not orthogonal, there exist

$$v \in W, v' \in W' \text{ such that } \langle v, v' \rangle \neq 0.$$

Then there exists $a \in M$ such that

$$v' = aw'.$$

(This is because $w'_i = p'_i(A)w'_0$ and hence for every $v' \in W'$, there is a polynomial $q \in \mathbb{C}[\lambda]$, $q(A)w'_0 = v$.)

We have

$$0 \neq \langle v', v \rangle = \langle aw', v \rangle = \langle w', a^*v \rangle$$

and $a^*v \in W$.

Hence, $\text{proj}_W w' \neq 0$.

Let $p_0, \dots, p_d \in \mathbb{C}[\lambda]$ be from Lemma 9.1.

Then, $w_i = p_i(A)w$ is a basis for $E_{r+i}^*(x)W$ ($0 \leq i \leq d$).

Hence,

$$\text{proj}_W w' = \alpha_0 w_0 + \dots + \alpha_d w_d \text{ for some } \alpha_j \in \mathbb{C}.$$

Set

$$p' := \frac{\|w\|}{\|w'\|} \sum_{i=0}^d \alpha_i p_i.$$

Then $0 \neq p' \in \mathbb{C}[\lambda]$ and $\deg p' \leq d$.

Claim: $\alpha_i = 0$ ($\Delta - r + r' < i \leq d$).

In particular, $\deg p' \leq \Delta - r + r'$.

Pf. Observe:

$$w' \in E_{r'}^*(y)V, \quad w \in E_r^*(x)V,$$

for $\partial(x, y) = \Delta$.

$$E_{r'}^*(y)V \cap E_{r+i}^*(x)V = 0$$

by triangle inequality.

($\Delta = \partial(x, y) < r + i - r'$ or $\Delta + r' < r + i$ by our choice of i .)



Hence,

$$E_{r'}^*(y)V \perp E_{r+i}^*(x)V,$$

or

$$0 = \langle w', w_i \rangle \quad (12.14)$$

$$= \langle \text{proj}_W w', w_i \rangle \quad (12.15)$$

$$= \sum_{j=0}^d \alpha_j \langle w_j, w_i \rangle \quad (12.16)$$

$$= \alpha_i \|w_i\|^2. \quad (12.17)$$

Hence, $\alpha_i = 0$. Thus,

$$\text{proj}_W w' = \sum_{i=0}^{\Delta+r'-r} \alpha_i w_i \quad (12.18)$$

$$= \sum_{i=0}^{\Delta+r'-r} \alpha_i p_i(A) w_0 \quad (12.19)$$

$$= p'(A) \frac{\|w'\|}{\|w\|} w. \quad (12.20)$$

(ii) We have

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = \frac{\langle E_i w, w' \rangle}{\|w\| \|w'\|} \quad (12.21)$$

$$= \frac{\langle E_i w, \text{proj}_W w' \rangle}{\|w\| \|w'\|} \quad \text{as } \text{proj}_W w' = p'(A) \frac{\|w\|}{\|w'\|} w \quad (12.22)$$

$$= \frac{\langle E_i w, p'(A) w \rangle}{\|w\|^2} \quad (12.23)$$

$$= \frac{\langle E_i w, E_i p'(A) w \rangle}{\|w\|^2} \quad (12.24)$$

$$= \overline{p'(\theta_i)} \frac{\|E_i W\|^2}{\|w\|^2} \quad (12.25)$$

$$= \overline{p'(\theta_i)} m(\theta_i). \quad (12.26)$$

Moreover, as $m(\theta_i), m'(\theta_i) \in \mathbb{R}$,

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = \frac{\overline{\langle E_i w, E_i w' \rangle}}{\|w'\| \|w\|} = \overline{p(\theta_i) m'(\theta_i)} = p(\theta_i) m'(\theta_i).$$

(iii) Since,

$$\frac{|\langle E_i w, E_i w' \rangle|^2}{\|w\|^2 \|w'\|^2} = p(\theta_i) p'(\theta_i) m(\theta_i) m'(\theta_i),$$

$$p(\theta_i)p'(\theta_i) = \frac{|\langle E_i w, E_i w' \rangle|^2}{m(\theta_i)m'(\theta_i)\|w\|^2\|w'\|^2} \in \mathbb{R} \quad (12.27)$$

$$= \frac{|\langle E_i w, E_i w' \rangle|^2}{\frac{\|E_i w\|^2}{\|w\|^2} \frac{\|E_i w'\|^2}{\|w'\|^2} \|w\|^2 \|w'\|^2}. \quad (12.28)$$

By Cauchy-Schwartz inequality,

$$(|\langle a, b \rangle| \leq \|a\| \|b\|,)$$

$$\frac{|\langle E_i w, E_i w' \rangle|^2}{\|E_i w\|^2 \|E_i w'\|^2} \leq 1.$$

Hence, we have the assertion. \square

Chapter 13

Modules of a DRG

Wednesday, February 17, 1993

Lemma 13.1. *Let $\Gamma = (X, E)$ be any graph. Pick an edge $xy \in E$.*

Assume the trivial $T(x)$ -module $T(x)\delta$ is thin with measure m_x ,

and the trivial $T(y)$ -module $T(y)\delta$ is thin with measure m_y .

Then,

$$(ia) \quad \frac{m_x(\theta)}{k_x} = \frac{m_y(\theta)}{k_y} \text{ for all } \theta \in \mathbb{R} \setminus \{0\}.$$

$$(ib) \quad \frac{m_x(0) - 1}{k_x} = \frac{m_y(0) - 1}{k_y} \text{ for all } \theta \in \mathbb{R} \setminus \{0\}.$$

$$(\delta = \sum_{y \in X} \delta_y \hat{y} \quad \text{eigenvector corresponding to the maximal eigenvalue})$$

Proof. Apply Theorem 12.1,

$$W = T(x)\delta \quad r = 0, \quad d = d(x) \tag{13.1}$$

$$W' = T(y)\delta \quad r' = 0, \quad d' = d(y). \tag{13.2}$$

Take $w = \hat{x}$, $w' = \hat{y}$.

Claim. $\text{proj}_{T(y)\delta} \hat{x} = k_y^{-1} A \hat{y}$.

Pf. Since

$$\hat{y} \in T(y)\delta, \quad A\hat{y} \in T(y)\delta.$$

Show

$$(\hat{x} - k_y^{-1} A \hat{y}) \perp (T(y)\delta).$$

Recall

$$A\hat{y} = \sum_{z \in X, yz \in E} \hat{z}.$$

$$\hat{x} - k_y^{-1}Ay \in E_1^*(y)V.$$

So,

$$\hat{x} - \frac{1}{k_y}A\hat{y} \perp E_j^*(y)T(y)\delta \quad \text{if } j \neq 1 \quad (0 \leq j \leq k(y)).$$

And we have,

$$\left\langle \hat{x} - \frac{1}{k_y}A\hat{y}, A\hat{y} \right\rangle = \left\langle \hat{x}, \sum_{z \in X, yz \in E} \hat{z} \right\rangle - \frac{1}{k_y} \left\| \sum_{z \in X, yz \in E} \hat{z} \right\|^2 \quad (13.3)$$

$$= 1 - 1 \quad (13.4)$$

$$= 0 \quad (13.5)$$

This proves Claim.

Similarly,

$$\text{prof}_{T(x)\delta} \hat{y} = k_x^{-1}A\hat{x}.$$

Hence, the polynomials $p, p' \in \mathbb{C}[\lambda]$ from Theorem 12.1 equal

$$\frac{\lambda}{k_y} \quad \text{and} \quad \frac{\lambda}{k_x}$$

respectively.

By Theorem 12.1,

$$\frac{m_x(\theta)\theta}{k_x} = m_x(\theta)\overline{p'(\theta)} = m_y(\theta)\overline{p(\theta)} = \frac{m_y(\theta)\theta}{k_y}.$$

If $\theta \neq 0$, we have (ia).

Also,

$$\frac{1 - m_x(0)}{k_x} = \left(\sum_{\theta \in \mathbb{R} \setminus \{0\}} m_x(0) \right) \frac{1}{k_x} \quad \text{by (ia)} \quad (13.6)$$

$$= \left(\sum_{\theta \in \mathbb{R} \setminus \{0\}} m_y(0) \right) \frac{1}{k_y} \quad (13.7)$$

$$= \frac{1 - m_y(0)}{k_y} \quad (13.8)$$

Hence, we have (ib). □

Theorem 13.1. *Suppose any graph $\Gamma = (X, E)$ is distance-regular with respect to every vertex $x \in X$. (So Γ is regular or biregular by Lemma 12.1.)*

Then,

Case Γ is regular: the diameter $d(x)$ and the intersection numbers $a_i(x)$, $b_i(x)$, $c_i(x)$ ($0 \leq i \leq d(x)$) are independent of $x \in X$.

(And Γ is called distance-regular.)

Case Γ is biregular: ($X = X^+ \cup X^-$)

$d(x)$ and $a_i(x)$, $b_i(x)$, $c_i(x)$ ($0 \leq i \leq d(x)$) are constant over X^+ and X^- . (And Γ is called distance-biregular.)

Proof. We apply Lemma 13.1.

Case Γ : regular.

Then $m_x = m_y$ for all $xy \in E$. Hence, the measure of the trivial $T(x)$ -module is independent of $x \in X$.

Case Γ is biregular.

Then $m_x = m_{x'}$ for all $x, x' \in X$ with $\partial(x, x') = 2$.

Hence, the measure of the trivial $T(x)$ -module is constant over $x \in X^+$, X^- .

Fix $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, $W = T\delta$ with measure m , diameter $d = d(x)$.

We know by Corollary 10.1 that m determines

$$d, \quad a_i(W) \ (0 \leq i \leq d), \quad x_i(W) \ (1 \leq i \leq d)$$

(as $d = D(x) = d(W)$ by Lemma 11.1.)

We shall show that m determines

$$a_i(x), \ c_i(x), \ b_i(x) \quad (0 \leq i \leq d).$$

Observe:

$$a_i(W) = a_i(x) \quad (0 \leq i \leq d) \tag{13.9}$$

$$x_i(W) = b_{i-1}c_i(x) \quad (1 \leq i \leq d) \tag{13.10}$$

Remark. $a_i = a_i(W)$ is an eigenvalue of

$$E_i^* A E_i^* \text{ on } E_i^* W = \langle \sum_{y \in \Gamma_i(x)} \hat{y} \rangle.$$

(See Lemma 12.1.)

$x_i = x_i(W)$ is an eigenvalue of

$$E_{i-1}^* A E_i^* A E_{i-1}^* \text{ on } E_{i-1}^* W,$$

and

$$A \sum_{y \in X, \partial(x,y)} \hat{y} = b_{i-1}(x) \sum_{y \in X, \partial(x,y)=i-1} \hat{y} \quad (13.11)$$

$$+ a_i(x) \sum_{y \in X, \partial(x,y)=i} \hat{y} \quad (13.12)$$

$$+ c_{i+1} \sum_{y \in X, \partial(x,y)=i+1} \hat{y} \quad (13.13)$$

So $x_i = b_{i-1}(x)c_i(x)$.

Set $k^+ = k_x$. Define

$$k^- = \frac{\theta_0^2}{k^+},$$

where θ_0 is the maximal eigenvalue. (See Lemma 11.1.)

(So, $k^+ = k^-$ is the valency, if Γ is regular.)

For every i ($0 \leq i \leq d$) and for every $z \in X$ with $\partial(x, z) = i$,

$$k_z = c_i(x) + a_i(x) + b_i(x) \quad (13.14)$$

$$= \begin{cases} k^+ & \text{if } i \text{ is even,} \\ k^- & \text{if } i \text{ is odd.} \end{cases} \quad (13.15)$$

Now m determines

$$c_0(x) = a_0(x) = 0, \quad c_1(x) = 1,$$

$$b_0(x) = b_0(x)c_1(x) = x_1(W).$$

$$k^+ = b_0(x) \quad (13.16)$$

$$k^- = \theta_0^2 / k^+ \quad (13.17)$$

$$c_i(x) = x_i(W) / b_{i-1}(x) \quad (1 \leq i \leq d) \quad (13.18)$$

$$b_i(x) = \begin{cases} k^+ - a_i(x) - c_i(x) & i: \text{ even,} \\ k^- - a_i(x) - c_i(x) & i: \text{ odd.} \end{cases} \quad (13.19)$$

This proves the assertions. \square

Proposition 13.1. *Under the assumption of Theorem 13.1, the following hold.*

Case Γ : regular.

- (i) $\dim E_i V = |X| m(\theta_i)$.
- (ii) Γ has exactly $d + 1$ distinct eigenvalues
- ($d = \text{diam} \Gamma = d(x)$, for all $x \in X$).

Case Γ : biregular.

- (i) $\dim E_V = |X^+| m^+(\theta_i) + |X^-| m^-(\theta_i)$.
- (ii) Γ has exactly $d^+ + 1$ distinct eigenvalues ($d^+ \geq d^-$).
- (iii) If d^+ is odd, the Γ is regular.
- (iv) $d^+ = d^-$, or $d^+ = d^- + 1$ is even.
- (v) $a_i(x) = 0$ for all i and for all x .

Proof. (i) Suppose Γ is regular.

Let m_x be the measure of the trivial $T(x)$ -module,

$$m_x(\theta_i) = \|E_i \hat{x}\|^2, \quad \text{as } \|\hat{x}\| = 1.$$

Now,

$$|X| m_x(\theta_i) = \sum_{x \in X} m_x(\theta_i) \tag{13.20}$$

$$= \sum_{x \in X} \|E_i \hat{x}\|^2 \tag{13.21}$$

$$= \sum_{y, z \in X} |(E_i)_{yz}|^2 \tag{13.22}$$

$$= \text{trace} E_i \overline{E_i}^\top. \tag{13.23}$$

Since A is real symmetric and

$$E_i \overline{E_i}^\top = E_i^2 = E_i$$

with E_i symmetric

$$E_i \sim \begin{pmatrix} I & O \\ O & I \end{pmatrix}.$$

$$\text{trace} E_i = \text{rank} E_i = \dim E_i V.$$

Thus, we have the assertion in this case.

Suppose Γ is biregular.

Then, same except,

$$\sum_{x \in X} m_x(\theta_i) = |X^+| m^+(\theta_i) + |X^-| m^-(\theta_i).$$

- (ii) Γ : regular. Immediately, if θ is an eigenvalue of Γ , then $m(\theta) \neq 0$.

Γ : biregular. For each $\theta = \theta_i \in \mathbb{R} \setminus \{0\}$,

$$m^-(\theta) \neq 0 \Leftrightarrow m^+(\theta) \neq 0 \quad (13.24)$$

$$\Leftrightarrow \theta \text{ is an eigenvalue of } \Gamma \quad (13.25)$$

$$\left(\frac{m^+(\theta)}{k^+} = \frac{m^-(\theta)}{k^-} \right) \quad (13.26)$$

(iv) and (v) are clear.

Remark. (iii) If d^+ is odd, $d^+ = d^-$ and Γ has even number of eigenvalues, i.e., 0 is not an eigenvalue. So A is nonsingular, and Γ is regular.

□

Chapter 14

Parameters of Thin Modules, I

Friday, February 19, 1993

Summary.

Definition 14.1. Assume $\Gamma = (X, E)$ is distance-regular with respect to every vertex $x \in X$.

Notation: Let $x \in X$. The data of the trivial $T(x)$ -module.

	Case DR	Case DBR
valency k_x	k	$\begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^- \end{cases}$
x -diameter D_x	D	$\begin{cases} D^+ & \text{if } x \in X^+ \\ D^- & \text{if } x \in X^- \end{cases}$
measure m_x	m	$\begin{cases} m^+ & \text{if } x \in X^+ \\ m^- & \text{if } x \in X^- \end{cases}$
int. number $c_i(x)$	c_i	$\begin{cases} c_i^+ & \text{if } x \in X^+ \\ c_i^- & \text{if } x \in X^- \end{cases}$
int. number $b_i(x)$	b_i	$\begin{cases} b_i^+ & \text{if } x \in X^+ \\ b_i^- & \text{if } x \in X^- \end{cases}$
int. number $a_i(x)$	a_i	$\begin{cases} a_i^+ & \text{if } x \in X^+ \\ a_i^- & \text{if } x \in X^- \end{cases}$

Call $m, m^{\pm 1}$ the measure of Γ .

Assume $\Gamma = (X, E)$ is distance-regular.

To what extent do a_i 's, b_i 's and c_i 's determine the structure of irreducible $T(x)$ -modules? In general the following hold.

Lemma 14.1. *Assume $\Gamma = (X, E)$ is distance-regular. Pick $x \in X$. Let X be a thin irreducible $T(x)$ -module with endpoint r , diameter d and measure m_W .*

(i) *There is a unique polynomial $f_W \in \mathbb{C}[\lambda]$ with the following properties.*

(ia) $\deg f_W \leq D$ (diameter of Γ).

(ib) $m_W(\theta) = m(\theta)f_W(\theta)$ for every $\theta \in \mathbb{R}$, where m is the measure of Γ .

Moreover, $f_W \in \mathbb{R}[\lambda]$, and

(ii) $\deg f_W \leq 2r$.

(iii) *For all eigenvalues θ_i of Γ , $\lambda - \theta_i$ is a factor of f_W whenever, $E_i W = 0$.*

In particular, $2r - D + d \geq 0$.

Proof. Let $\theta_0, \dots, \theta_D$ denote distinct eigenvalues of Γ . Then $m(\theta_i) \neq 0$ ($0 \leq i \leq D$) by Proposition 13.1.

There exists a unique $f_W \in \mathbb{C}[\lambda]$ with $\deg f_W \leq D$ such that

$$f_W(\theta_i) = \frac{m_W(\theta_i)}{m(\theta_i)} \quad (0 \leq i \leq D)$$

by polynomial interpolation.

$f_W \in \mathbb{R}[\lambda]$ since

$$\theta_0, \dots, \theta_D \in \mathbb{R} \quad \text{and} \quad f_W(\theta_0), \dots, f_W(\theta_D) \in \mathbb{R}.$$

(ii) Without loss of generality, we may assume $r < D/2$, else trivial.

Pick $0 \neq w \in E_r^*(x)W$.

$$w = \sum_{y \in W, \partial(x,y)=r} \alpha_y \hat{y} \quad \text{some } \alpha_y \in \mathbb{C}.$$

Pick $y \in X$ such that $\alpha_y \neq 0$.

Set W' be the trivial $T(y)$ -module. ($\langle w, \hat{y} \rangle \neq 0$, as $W \not\perp W'$.)

$$r' = 0, \quad m' = m, \quad \Delta = r.$$

Apply Theorem 12.1, we have

$$\deg p \leq \Delta - r' + r = 2r, \quad p \neq 0 \tag{14.1}$$

$$\deg p' \leq \Delta - r + r' = 0, \quad p' \neq 0. \tag{14.2}$$

$$m_W(\theta)\overline{p'(\theta)} = m(\theta)p(\theta) \quad (\text{for all } \theta \in \mathbb{R}).$$

So,

$$\deg p/\bar{p}' \leq 2r,$$

and p/\bar{p}' satisfies the conditions of f_W .

$$\left(\frac{p(\theta)}{\bar{p}'(\theta)} = \frac{m_W(\theta)}{m(\theta)} \right)$$

(iii)

$$E_i W = 0 \rightarrow m_W(\theta_i) = 0 \rightarrow f_W(\theta_i) = 0.$$

that is, $E_i W = 0$. Hence θ_i is a root of $f_W(\lambda) = 0$. So,

$$2r \geq \deg f_W \geq |\{\theta_i \mid E_i W = 0\}| = D - d.$$

Hence,

$$2r - D + d \geq 0.$$

This proves the assertions. \square

Lemma 14.2. *Let $\Gamma = (X, E)$ be any distance-regular graph with valency k , diameter D ($d \geq 2$), measure m , and eigenvalues*

$$k = \theta_0 > \theta_1 > \dots > \theta_D.$$

Pick $x \in X$. Let W be a thin irreducible $T(x)$ -module with endpoint $r = 1$, diameter D and measure $m_W = mf_W$. Then one fo the following cases (i)–(iv) occurs.

Case	d	$f_W(\lambda)$	$a_0(W)$
(i)	$D - 2$	$\frac{(\lambda-k)(\lambda-\theta_1)}{k(\theta_1+1)}$	$-\frac{b_1}{\theta_1+1} - 1$
(ii)	$D - 2$	$\frac{(\lambda-k)(\lambda-\theta_D)}{k(\theta_D+1)}$	$-\frac{b_1}{\theta_1+1} - 1$
(iii)	$D - 1$	$\frac{k-\lambda}{k}$	-1
(iv)	$D - 1$	$\frac{(\lambda-k)(\lambda-\beta)}{k(\beta+1)}$	$-\frac{b_1}{\beta+1} - 1$

for some $\beta \in \mathbb{R}$ with $\beta \in (-\infty, \theta_D) \cup (\theta_1, \infty)$. Moreover, the isomorphism class of W is determined by $a_0(W)$.

Note. By (iii), the possible “shapes” of a thin irreducible $T(x)$ -modules are:

$$r = 0 \quad d = D \tag{14.3}$$

$$r = 1 \quad d = D - 1 \tag{14.4}$$

$$r = 1 \quad d = D - 2 \tag{14.5}$$

Chapter 15

Parameters of Thin Modules, II

Monday, February 22, 1993

Proof of Lemma 14.2 Continued.

We have $\deg f_W \leq 2$ by Lemma 14.1 (ii).

Also by Lemma 11.1, $E_0 W = 0$.

(As otherwise $\langle \delta \rangle = E_0 V \subseteq W$ and $r(W) = 0$.)

Hence, $\lambda - \theta_0 = \lambda - k$ is a factor of f_W by Lemma 14.1 (iii).

Let p_0, p_1, \dots, p_D denote the polynomials for the trivial $T(x)$ -module from Lemma 9.1.

Recall,

$$\sum_{\theta \in \mathbb{R}} m(\theta) p_i(\theta) p_j(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \leq i, j \leq D) \quad (15.1)$$

$$= \delta_{ij} b_0 b_1 \cdots b_{i-1} c_1 c_2 \cdots c_i. \quad (15.2)$$

Note that $x_i = b_{i-1} c_i$ is in the proof of Theorem 7.1.

By construction,

$$p_0(\lambda) = 1.p_1(\lambda) \quad \quad \quad = \lambda.p_2(\lambda)\lambda^2 - a_1\lambda - k. \quad (15.3)$$

Apparently,

$$f_W = \sigma_0 p_0 + \sigma_1 p_1 + \sigma_2 p_2$$

for some $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{C}$.

Claim:

$$\sigma_0 = 1, \quad (15.4)$$

$$\sigma_1 = \frac{a_0(W)}{k}, \quad (15.5)$$

$$\sigma_2 = \frac{1 + a_0(W)}{kb_1}. \quad (15.6)$$

Pf of Claim.

$$1 = \sum_{\theta \in \mathbb{R}} m_W(\theta) \quad (15.7)$$

$$= \sum_{\theta \in \mathbb{R}} m(\theta) f_W(\theta) \quad (15.8)$$

$$= \sum_{j=0}^2 \sigma_j \left(\sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) \right) \quad (15.9)$$

$$= \sigma_0. \quad (15.10)$$

We applied Lemma 10.1 (ib), Lemma 14.1 (ib), and Lemma 10.1 (i) in this order.

Next by Lemma 10.1 (ii), and $p_1(\theta) = \theta$,

$$a_0(W) = \sum_{\theta \in \mathbb{R}} m_W(\theta) \theta \quad (15.11)$$

$$= \sum_{\theta \in \mathbb{R}} f_W(\theta) \theta \quad (15.12)$$

$$= \sum_{j=0}^2 \sigma_j \sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) p_1(\theta) \quad (15.13)$$

$$= \sigma_1 x_1(T\delta) \quad (15.14)$$

$$= \sigma_1 b_0 c_1 \quad (15.15)$$

$$= \sigma_1 k. \quad (15.16)$$

So for,

$$f_W(\lambda) = 1 + \frac{a_0(W)}{k} \lambda + \sigma_2 (\lambda^2 - a_1 \lambda - k).$$

But,

$$0 = f_W(k) \quad (15.17)$$

$$= 1 + a_0(W) + \sigma_2 k(k - a_1 - 1) \quad (15.18)$$

$$1 + a_0(W) + \sigma_2 k b_1. \quad (15.19)$$

Thus,

$$\sigma_2 = -\frac{1 + a_0(W)}{k b_1}.$$

This proves Claim.

Case: $a_0(W) = -1$.

Here, $\sigma_2 = 0$ and

$$f_W(\lambda) = 1 + \frac{a_0(W)\lambda}{k} = 1 - \frac{\lambda}{k}.$$

Also,

$$d + 1 = |\{\theta \mid \theta \text{ is an eigenvalue of } \Gamma, f_W(\theta) \neq 0\}| = D.$$

Case: $a_0(W) \neq -1$.

Here, $\sigma_2 \neq 0$, and $\deg f_W = 2$. So,

$$f_W(\lambda) = (\lambda - k)(\lambda - \beta)\alpha$$

for some $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$.

Comparing the coefficients in

$$(\lambda - k)(\lambda - \beta)\alpha = 1 + \frac{a_0(W)}{k}\lambda - \frac{a_0(W) + 1}{kb_1}(\lambda^2 - a_1\lambda - k),$$

we find

$$\alpha = -\frac{a_0(W) + 1}{kb_1}, \quad (15.20)$$

$$-(k + \beta)\alpha = \frac{a_0(W)}{k} + \frac{a_0(W) + 1}{kb_1}a_1, \quad (15.21)$$

$$k\beta\alpha = 1 + \frac{a_0(W) + 1}{b_1}. \quad (15.22)$$

Hence,

$$-\beta(a_0(W) + 1) = b_1 + (a_0(W) + 1).$$

Thus, we have

$$(1 + a_0(W))(1 + \beta) = -b_1. \quad (15.23)$$

In particular, $\beta \neq -1$, and

$$\alpha = -\frac{1 + a_0(W)}{kb_1} = \frac{1}{k(\beta + 1)}.$$

Also, by Definition 9.2,

$$0 \leq m_W(\theta) \quad (15.24)$$

$$= m(\theta)f_W(\theta) \quad (\text{for all } \theta \in \mathbb{R}). \quad (15.25)$$

But if θ is an eigenvalue of Γ ,

$$0 < m(\theta).$$

So,

$$0 \leq f_W(\theta) \quad (15.26)$$

$$= \frac{(\theta - k)(\theta - \beta)}{k(\beta + 1)}. \quad (15.27)$$

Either

$$\beta + 1 > 0 \rightarrow \theta - \beta \leq 0 \text{ or } \beta \geq \theta_1,$$

or

$$\beta + 1 < 0 \rightarrow \theta - \beta \geq 0 \text{ or } \beta \leq \theta_D.$$

If $\beta = \theta_1$,

$$a_0(W) = -\frac{b_1}{\beta + 1} - 1 = -\frac{b_1}{\theta_1 + 1} - 1 \quad (15.28)$$

$$f_W(\lambda) = \frac{(\lambda - k)(\lambda - \theta_1)}{k(\theta_1 + 1)}, \quad (15.29)$$

and we have (i).

If $\beta = \theta_D$,

$$a_0(W) = -\frac{b_1}{\theta_D + 1} - 1 \quad (15.30)$$

$$f_W(\lambda) = \frac{(\lambda - k)(\lambda - \theta_D)}{k(\theta_D + 1)}, \quad (15.31)$$

and we have (ii).

If $\beta \notin \{\theta_1, \theta_2\}$,

$$\theta \in (-\infty, \theta_D) \cup (\theta_1, \infty),$$

we have (iv).

Note using (15.23), we have (iv).

□

Note. Using (15.23),

$$a_0(W) \rightarrow \beta \rightarrow f_W \rightarrow m_W \rightarrow \text{isomorphism class of } W.$$

Note on Lemma 14.2. In fact, $\theta_1 > -1$, $\theta_D < -1$ if $D \geq 2$.

Definition 15.1. The complete graph K_n has n vertices and diameter $D = 1$, i.e., $xy \in E$ for all vertices x, t .

K_n is distance-regular with valency $k = n - 1$ and $a_1 = n - 2$, $D = 1$. Moreover, it has two distance eigenvalues θ_0, θ_1 .

Recall, $\theta_0, \dots, \theta_D$ are roots of p_{D+1} , i.e., $D + 1$ st polynomial for the trivial module/

$$p_0 = 1 \quad (15.32)$$

$$p_1 = \lambda \quad (15.33)$$

$$p_2 = \lambda^2 - a_1\lambda - k \quad (15.34)$$

$$= \lambda^2 - (n - 2)\lambda - (n - 1) \quad (15.35)$$

$$= (\lambda - (n - 1))(\lambda + 1). \quad (15.36)$$

The roots are $\theta_0 = n - 1 = k$ and $\theta_1 = -1$.

Lemma 15.1. *Let $\Gamma = (X, E)$ be distance-regular of diameter $D \geq 1$ with distinct eigenvalues*

$$k = \theta_0 > \theta_1 > \dots > \theta_D.$$

(i) $\theta_D \leq -1$ with equality if and only if $D = 1$.

(ii) $\theta_1 \geq -1$ with equality if and only if $D = 1$.

Proof. (i) Suppose $\theta_D \geq -1$.

Then $I + A$ is positive semi-definite.

By Lemma 2.1, there exists vectors $\{v_x \mid x \in X\}$ in a Euclidean space such that

$$\langle v_x, v_y \rangle = (I + A)_{xy} \quad (15.37)$$

$$= \begin{cases} 1 & \text{if } x = y \text{ or } xy \in E, \\ 0 & \text{otherwise.} \end{cases} \quad (15.38)$$

For every $xy \in E$,

$$\langle v_x, v_y \rangle = \|v_x\| \|v_y\| = 1.$$

Hence, $v_x = v_y$, and v_x is independent of $x \in X$.

Shus $\langle v_x, v_y \rangle = 1$ for all $x, y \in X$.

We have $I + A = J$, (all 1's matrix), and $D = 1$.

(ii) Let m be the trivial measure. Then,

$$1 = \sum_{\theta \in \mathbb{R}} m(\theta) + \sum_{\theta \in \mathbb{R}} m(\theta)\theta \quad (15.39)$$

$$= \sum_{\theta \in \mathbb{R}} m(\theta)(\theta + 1) \quad (15.40)$$

$$= m(k)(k + 1) + \sum_{\theta \neq k} m(\theta)(\theta + 1) \quad (15.41)$$

$$\leq (k + 1)|X|^{-1}. \quad (15.42)$$

Note that $m(k) = |X|^{-1} \dim d_0 V = |X|^{-1}$.

So $k+1 \geq |X|$ or $k = |X| - 1$. Thus, $xy \in E$ for every $x, y \in X$, and $D = 1$. \square

Note. Lemma 15.1 does not require distance-regular assumption.

Chapter 16

Thin Modoles of a DRG

Wednesday, February 24, 1993

Let $\Gamma = (X, E)$ denote any graph of diameter D .

Definition 16.1. For all integer i , the i -th incidence matrix $A_i \in \text{Mat}_X(\mathbb{C})$ satisfies

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \quad (x, y \in X).$$

Observe,

$$A_0 = I \quad (\text{identity}) \quad (16.1)$$

$$A_1 = A \quad (\text{adjacency matrix}) \quad (16.2)$$

$$A_0 + A_1 + \cdots + A_D = J \quad (\text{all 1's matrix}). \quad (16.3)$$

In general, A_i may not belong to Bose-Mesner algebra.

Lemma 16.1. Assume $\Gamma = (X, E)$ is distance-regular with diameter $D \geq 1$ and intersection numbers c_i, a_i, b_i .

(i)

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1}, \quad (0 \leq i \leq D, A_{-1} = A_{D+1} = O).$$

(ii) $A_i = \frac{p_i(A)}{c_1 c_2 \cdots c_i}$, $(0 \leq i \leq D)$, where p_0, p_1, \dots, p_D are polynomials for the trivial module from Lemma 9.1.

(iii) A_0, A_1, \dots, A_D form a basis for Bose-Mesner algebra M .

(iv) For all distances h, i, j $(0 \leq i, j, h \leq D)$, and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the constant

$$p_{i,j}^h = |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|$$

depends only on h, i, j and not on x, y .

$$(v) \ E_0 = \frac{1}{|X|} J.$$

Proof.

(i) Pick $x \in X$. Apply each side to \hat{x} , we want to show that

$$AA_i\hat{x} = c_{i+1}A_{i+1}\hat{x} + a_iA_i\hat{x} + b_{i-1}A_{i-1}\hat{x}.$$

$$\text{LHS} = A \left(\sum_{y \in X, \partial(x, y) = i} \hat{y} \right) \tag{16.4}$$

$$= c_{i+1} \left(\sum_{z \in X, \partial(x, z) = i+1} \hat{z} \right) + a_i \left(\sum_{z \in X, \partial(x, z) = i} \hat{z} \right) + b_{i-1} \left(\sum_{z \in X, \partial(x, z) = i-1} \hat{z} \right) \tag{16.5}$$

$$= \text{RHS}. \tag{16.6}$$

(ii) Recall (Lemma 9.1)

$$Ap_i(A) = p_{i+1}(A) + a_i p_i(A) + b_{i-1} c_i p_{i-1}(A) \quad (0 \leq i \leq D).$$

Dividing by $c_1 c_2 \cdots c_i$, we have

$$A \frac{p_i(A)}{c_1 c_2 \cdots c_i} = c_{i+1} \frac{p_{i+1}(A)}{c_1 c_2 \cdots c_{i+1}} + a_i \frac{p_i(A)}{c_1 c_2 \cdots c_i} + b_{i-1} \frac{p_{i-1}(A)}{c_1 c_2 \cdots c_i}.$$

So, $A_i, p_i(A)/(c_1 c_2 \cdots c_i)$ satisfy the same recurrence.

Also boundary condition,

$$A_0 = p_0(A) = I.$$

Hence,

$$A_i = \frac{p_i(A)}{c_1 c_2 \cdots c_i} \quad (0 \leq i \leq D).$$

(iii) Since E_0, E_1, \dots, E_D form a basis for M , $\dim M = D + 1$.

Observe $A_0, A_1, \dots, A_D \in M$ by (ii), A_0, A_1, \dots, A_D are linearly independent, since p_0, p_1, \dots, p_D are linearly independent.

Thus, A_0, A_1, \dots, A_D form a basis for M .

(iv) A_0, A_1, \dots, A_D form a basis for an algebra M ,

$$A_i A_j = \sum_{\ell=0}^D p_{ij}^{\ell} A_{\ell} \quad \text{for some } p_{ij}^{\ell} \in \mathbb{C}. \quad (16.7)$$

Fix h ($0 \leq h \leq D$). Pick $x, y \in X$ with $\partial(x, y) = h$.

Compute x, y entry in (16.7),

$$(A_i A_j)_{xy} = \sum_{z \in X} (A_i)_{xz} (A_j)_{zy} \quad (16.8)$$

$$= \sum_{z \in X, \partial(x, z)=i, \partial(y, z)=j} 1 \cdot 1 \quad (16.9)$$

$$= |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|. \quad (16.10)$$

On the other hand,

$$\left(\sum_{\ell=0}^D p_{ij}^{\ell} A_{\ell} \right)_{xy} = p_{ij}^h (A_h)_{xy} = p_{ij}^h.$$

(v) $\frac{1}{|X|}J$ is the orthogonal projection onto $\text{Span}(\delta) = E_0 V$. Hence,

$$\frac{1}{|X|} = E_0.$$

This proves the assertions. □

Theorem 16.1. *Let $\Gamma = (X, E)$ be distance-regular with diameter $D \geq 2$ and intersection numbers c_i, a_i, b_i . Pick a vertex $x \in X$. Let W be a thin irreducible $T(x)$ -module with endpoint $r = 1$ and diameter d ($d = D - 2$ or $D - 1$). Set $\gamma_0 = a_0(W) + 1$.*

(i) *The scalars*

$$\gamma_i := \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_1(W) x_2(W) \cdots x_i(W)} \quad (0 \leq i \leq d) \quad (16.11)$$

$a_i(W), x_i(W)$ are algebraic integers in $\mathbb{Q}[\gamma_0]$. In particular, if $\gamma_0 \in \mathbb{Q}$, then $\gamma_i, a_i(W)$ and $x_i(W)$ are integers for all i .

(ii) *The numbers, $\gamma_i, a_i(W), x_i(W)$ can all be determined from γ_0 and the intersection numbers of Γ in order*

$$x_1(W), \gamma_1, a_1(W), x_2(W), \gamma_2, a_2(W), \dots$$

using (i),

$$x_i(W) = c_i b_i + \gamma_{i-1} (a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad (1 \leq i \leq D - 1), \quad (16.12)$$

and

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad (1 \leq i \leq D). \quad (16.13)$$

Note.

$$p_i = p_1^W + \gamma_{i-1}p_{i-1}^W - c_i(p_{i-1}^W + \gamma_{i-2}^W), \quad (\gamma_{-1} = -\gamma_{-2} = 0, \quad 0 \leq i \leq d+1).$$

Proof. Set

$$\tilde{A}_i = A_0 + A_1 + \cdots + A_i \quad (0 \leq i \leq D).$$

$$\text{Claim 1. } A\tilde{A}_i = c_{i+1}\tilde{A}_{i+1} + (a_i - c_{i+1} + c_i)\tilde{A}_i + b_i\tilde{A}_{i-1} \quad (0 \leq i \leq D-1).$$

Proof of Claim 1.

$$\text{LHS} = \sum_{j=0}^i AA_j \quad (16.14)$$

$$= \sum_{j=0}^i (c_{j+1}A_{j+1} + a_jA_j + b_{j-1}A_{j-1}) \quad (16.15)$$

$$= \sum_{j=0}^{i-1} A_j(c_j + a_j + b_j) + A_i(c_i + a_i) + A_{i+1}c_{i+1} \quad (16.16)$$

$$= k(A_0 + \cdots + A_{i-1}) + (a_i + c_i)A_i + c_{i+1}A_{i+1}. \quad (16.17)$$

$$\text{RHS} = c_{i+1}(A_0 + A_1 + \cdots + A_{i-1} + A_i + A_{i+1}) \quad (16.18)$$

$$+ (a_i - c_{i+1} + c_i)(A_0 + A_1 + \cdots + A_{i-1} + A_i) \quad (16.19)$$

$$+ b_i(A_0 + A_1 + \cdots + A_{i-1}) \quad (16.20)$$

$$= k(A_0 + \cdots + A_{i-1}) + A_i(a_i + c_i) + A_{i+1}c_{i+1}. \quad (16.21)$$

This proves Claim 1.

Now pick $0 \neq w \in E_1^*(x)W$ and let

$$w = \sum_{z \in X, \partial(x,z)=1} \alpha_z \hat{z}.$$

Pick y , where $\alpha_y \neq 0$.

For i ($0 \leq i \leq D$), define

$$B_i = \tilde{A}_i(\hat{x} - \hat{y}) \quad (16.22)$$

$$= \sum_{z \in X, \partial(x,z) \leq i} \hat{z} - \sum_{z \in X, \partial(y,z) \leq i} \hat{z} \quad (16.23)$$

$$= \sum_{z \in X, \partial(x,z)=i, \partial(y,z)=i+1} \hat{z} - \sum_{z \in X, \partial(y,z)=i+1, \partial(y,z)=i} \hat{z}. \quad (16.24)$$

Note that $B_D = O$, $B_0 = \hat{x} - \hat{y}$, and

$$\langle B_0, w_0 \rangle = -\alpha_y \neq 0.$$

From Claim 1,

$$AB_i = c_{i+1}B_{i+1} + (a_i - c_{i+1} + c_i)B_i + b_iB_{i-1} \quad (0 \leq i \leq D), \quad B_{-1} = O.$$

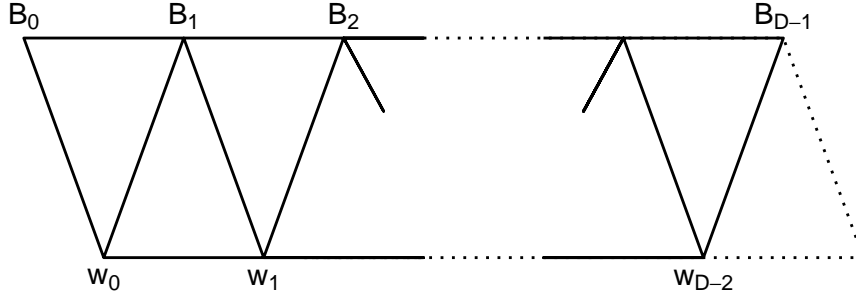
Let p_0^W, \dots, p_d^W denote polynomials for W from Lemma 9.1. So,

$$w_i = p_i^W(A)w \in E_{1+i}^*(x)W, \quad (0 \leq i \leq d).$$

Claim 2. $\langle w_i, B_j \rangle = 0$ if $j \notin \{i, i+1\}$, $(0 \leq i \leq d, 0 \leq j \leq D)$.

Proof of Claim 2.

$$w_i \in E_{1+i}^*W, \quad B_j \in E_j^*(x)W + E_{j+1}^*(x)W.$$



Vertical lines indicate possible non-orthogonality.

Compute

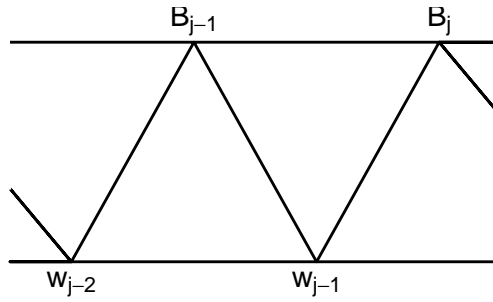
$$\langle Aw_i, B_j \rangle = \langle w_i, AB_j \rangle, \quad \text{quad}(0 \leq i \leq D, 0 \leq j \leq D-1). \quad (16.25)$$

$$\text{LHS} = \langle w_{i+1}, B_j \rangle + a_i(W)\langle w_i, B_j \rangle + x_i(W)\langle w_{i-1}, B_j \rangle \quad (16.26)$$

$$\text{RHD} = b_j\langle w_i, B_{j-1} \rangle + (a_j - c_{j+1} + c_j)\langle w_i, B_j \rangle + c_{j+1}\langle w_i, B_{j+1} \rangle. \quad (16.27)$$

Evaluate for $i = j-2, j-1, j, j+1$.

Set $i = j-2$.



Then (16.25) becomes

$$\langle w_{j-1}, B_j \rangle = b_j \langle w_{j-2}, B_{j-1} \rangle \quad (2 \leq j \leq D-1).$$

By induction,

$$\langle w_{j-1}, B_j \rangle = b_2 b_3 \cdots b_j \langle w_0, B_1 \rangle \quad (1 \leq j \leq D-1).$$

Define

$$\gamma_0 = \frac{\langle w_0, B_1 \rangle}{\langle w_0, B_0 \rangle}.$$

(We will show $\gamma_0 = 1 + a_0(W)$.)

Then,

$$\langle w_{j-1}, B_j \rangle = b_2 b_3 \cdots b_j \gamma_0 \langle w_0, B_0 \rangle. \quad (16.28)$$

Set $i = j + 1$. Then (16.25) becomes

$$x_{j+1}(W) \langle w_j, B_j \rangle = c_{j+1} \langle w_0, B_{j+1} \rangle \quad (0 \leq j \leq d).$$

Hence,

$$\langle w_j, B_j \rangle = \frac{x_1(W) \cdots x_j(W)}{c_1 c_2 \cdots c_j} \langle w_0, B_0 \rangle \quad (0 \leq j \leq d). \quad (16.29)$$

Set $i = j - 1$. Then (16.25) becomes

$$\langle w_j, B_j \rangle + a_{j-1}(W) \langle w_{j-1}, B_j \rangle = (a_j - c_{j+1} + c_j) \langle w_{j-1}, B_j \rangle + b_j \langle w_{j-1}, B_{j-1} \rangle.$$

Evaluate this using (16.28) and (16.29). ($\langle w_0, B_0 \rangle \neq 0$). Then we have

$$\frac{w_1(W) \cdots x_j(W)}{c_1 \cdots c_j} + (a_{j-1}(W) - a_j + c_{j+1} - c_j) b_2 \cdots b_j \gamma_0 = b_j \frac{x_1(W) \cdots x_{j-1}(W)}{c_1 \cdots c_{j-1}},$$

$$\left(\gamma_i := \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_0(W) x_2(W) \cdots x_i(W)} \right).$$

$$\frac{x_j(W)}{c_j} = b_j + \frac{c_1 c_3 \cdots c_{j-1} b_2 b_3 \cdots b_j \gamma_0}{x_0(W) x_2(W) \cdots x_{j-1}(W)} (a_j + c_j - c_{j+1} - a_{j-1}).$$

So,

$$x_j(W) = c_j b_j + \gamma_{j-1} (a_j + c_j - c_{j+1} - a_{j-1}(W)).$$

This proves (16.12).

Set $i = j$. Then (16.25) becomes

$$a_j(W) \langle w_j, B_j \rangle + x_j(W) \langle w_{j-1}, B_j \rangle = (a_j - c_{j+1} + c_j) \langle w_j, B_j \rangle + c_{j+1} \langle w_j, B_{j+1} \rangle.$$

$$(a_j(W) - (a_j - c_{j+1} + c_j)) \frac{x_1(W) \cdots x_j(W)}{c_1 \cdots c_j} x_j(W) b_2 \cdots b_j \gamma_0 - c_{j+1} b_2 \cdots b_{j+1} \gamma_0 = 0.$$

Thus,

$$a_j(W) - (a_j - c_{j+1} + c_j) + \frac{c_1 \cdots c_j b_2 \cdots b_j \gamma_0}{x_1(W) \cdots x_{j-1}(W)} - \frac{c_1 \cdots c_j c_{j+1} b_2 \cdots b_{j+1} \gamma_0}{x_1(W) \cdots x_j(W)} = 0,$$

or

$$a_j(W) = a_j + c_j - c_{j+1} - \gamma_{j-1} + \gamma_j.$$

This proves (16.13).

Also by setting $i = j = 0$, we have

$$a_0(W) \langle w_0, B_0 \rangle = (a_0 - c_1 + c_0) \langle w_0, B_0 \rangle + c_1 \langle w_0, B_1 \rangle \quad (16.30)$$

$$= -\langle w_0, B_0 \rangle + \gamma_0 \langle w_0, B_0 \rangle. \quad (16.31)$$

Hence,

$$\gamma_0 = 1 + a_0(W).$$

Both $a_i(W)$ and $x_i(W)$ are algebraic integers, since they are eigenvalues of matrices with integer entries, namely,

$$E_{i+1}^*(x) A E_{i+1}^*(x) \quad \text{and} \quad E_i^*(x) A E_{i+1}^*(x) A E_i^*(x).$$

Also $\gamma_0 = 1 + a_0(W)$ is an algebraic integer, and $\gamma_i - \gamma_{i-1}$ is an algebraic integer by (16.12).

Hence, γ_i is an algebraic integer by induction.

This completes the proof of Theorem 16.1. \square

Example 16.1 (D=2).

$$D = 2 \Leftrightarrow \text{strongly regular.}$$

Free parameters are k, a_1, c_2 . Let W be an irreducible module of endpoint 1. The matrix representation of $A|_W$ is

$$\begin{pmatrix} a_0(W) & x_1(W) \\ 1 & a_1(W) \end{pmatrix}.$$

$a_0(W)$: free.

$$x_1(W) = c_1 b_1 + (a_0(W) + 1)(a_1 + c_1 - c_2 - a_0(W)) \quad (16.32)$$

$$= k - a_1 - 1 + a_1 a_0(W) + a_0(W) - c_2 a_0(W) - a_0(W)^2 + a_1 + a - c_2 - a_0(W) \quad (16.33)$$

$$= a_1 a_0(W) - c_2 a_0(W) + k - c_2 - a_0(W)^2, \quad (16.34)$$

$$\gamma_1 = 0, \quad (16.35)$$

$$a_1(W) = -(a_0(W) + 1) + a_1 + c_1 - c_2 \quad (16.36)$$

$$= -a_0(W) + a_1 - c_2. \quad (16.37)$$

Then the matrix has eigenvalues θ, θ_1 . There is one feasible condition: $a_0(W)$ is an algebraic integer.

Example 16.2 (D=3). Free parameters c_2, c_3, k, a_1, a_2 . The matrix representation becomes

$$A|_W = \begin{pmatrix} a_0(W) & x_1(W) & 0 \\ 1 & a_1(W) & x_2(W) \\ 0 & 1 & a_2(W) \end{pmatrix}.$$

Here, $a_0(W)$ is free ($= \gamma - 1$)

$$x_1(W) = k - 1 - a_1 + \gamma_0(a_1 + 1 - c_2 - a_0(W)) \quad (16.38)$$

$$= \gamma_0(a_1 - c_2 - a_0(W)) + k - a_1 + a_0(W). \quad (16.39)$$

Set

$$\gamma_1(W) = \frac{c_2 b_2 \gamma_0}{x_1(W)}.$$

$$a_1(W) = \gamma_1 - \gamma_0 + a_1 + 1 - c_2 \quad (16.40)$$

$$x_2(W) = \gamma_1(a_2 - c_3 - a_1(W)) + c_2(\gamma_0 + b_1 - a_2 + a_1(W)) \quad (16.41)$$

$$a_2(W) = -\gamma_1 + a_2 + c_2 - c_3. \quad (16.42)$$

The matrix has eigenvalues, $\theta, \theta_2, \theta_3$.

There are two feasibility conditions; γ_0, γ_1 are algebraic integers.

For arbitrary D , there are $D - 1$ feasibility conditions; $\gamma_0, \gamma_1, \dots, \gamma_{D-1}$ are algebraic integers.

Lemma 16.2. *With the notation of Theorem 16.1, suppose*

$$f_W = \frac{k - \lambda}{k} \quad (\text{so, } a_0(W) = -1).$$

Then,

$$a_i(W) = a_i + c_i - c_{i+1} \quad (0 \leq i \leq D - 1) \quad (16.43)$$

$$x_i(W) = b_i c_i \quad (1 \leq i \leq D - 1) \quad (16.44)$$

$$\gamma_i(W) = 0. \quad (16.45)$$

Proof. Since $\gamma_0 = a_0(W) = 1$, $\gamma_i = 0$. □

Chapter 17

Association Schemes

Monday, March 1, 1993

Review

Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 2$. Pick a vertex $x \in X$.

Let W be a thin irreducible $T(x)$ -module with endpoint $r = 1$, diameter $d = D - 1$ or $D - 2$, and $r_0 = a(W) + 1$.

Show

$$\gamma_i = \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_1(W) \cdots x_i(W)},$$

$a_i(W)$ and $x_i(W)$ are all algebraic integers in $\mathbb{Q}[\gamma_0]$, where

$$x_i(W) = c_i b_i + \gamma_{i-1}(a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad (1 \leq i \leq d) \quad (17.1)$$

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad (1 \leq i \leq d) \quad (17.2)$$

Certainly, $x_i(W)$, γ_i , and $a_i(W)$ are in $\mathbb{Q}[\gamma_0]$ by the above lines and so on.

$$\gamma_0 \rightarrow a_0(W) \rightarrow x_1(W) \rightarrow \gamma_1 \rightarrow a_1(W) \rightarrow x_1(W) \rightarrow \cdots$$

Recall some $B \in \text{Mat}_n(\mathbb{C})$ is integral whenever

$$B \in \text{Mat}_n(\mathbb{Z}).$$

In this case, the characteristic polynomial

$$\det(\lambda I - B) = \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_0, \quad \text{some } \alpha_0, \dots, \alpha_{n-1} \in \mathbb{Z}.$$

Hence, eigenvalues of B are algebraic integers. But $a_i(W)$ is an eigenvalue of an integral matrices,

$$B = E_{i+1}^*(x) A E_{i+1}^*(x).$$

Hence, $a_i(W)$ is an algebraic integer.

Also, $x_i(W)$ is an eigenvalue of an integral matrix

$$B = E_i^*(x)AE_{i+1}^*(x)AE_i^*(x).$$

So $x_i(W)$ is an algebraic integer.

$$\gamma_i - \gamma_{i-1} = a_i(W) - a_i - c_i + c_{i+1}$$

is an algebraic integer.

Since $\gamma_0 = a_0(W) + 1$ is an algebraic integer, we find γ is an algebraic integer for all i .

Definition 17.1. A (commutative) association scheme is a configuration $Y = (X, \{R_i\}_{0 \leq i \leq D})$, where X is a finite nonempty set (of vertices), R_0, R_1, \dots, R_D are nonempty subsets of $X \times X$ such that

- (i) $R_0 = \{(x, x) \mid x \in X\}$,
- (ii) $R_0 \cup \dots \cup R_D = X \times X$ (disjoint union),
- (iii) for every i , $R_i^\top = \{(y, x) \mid xy \in R\} = R_{i'}$ some $i' \in \{0, 1, \dots, D\}$,
- (iv) for every h, i, j ($0 \leq h, i, j \leq D$), and every $x, y \in X$ such that $(x, y) \in R_h$,

$$p_{ij}^h = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|$$

depends only on h, i, j and not on x, y ; and

- (v) $p_{ij}^h = p_{ji}^h$ for all h, i, j .

If $i' = i$ for all i , we say Y is symmetric. We call D the class of scheme and R_i , the i th relation of Y . We say vertices $x, y \in X$ are i -related, or ‘at distance i ’, whenever $(x, y) \in R_i$.

We always assume that a ‘scheme’ is a commutative association scheme.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be an association scheme.

Definition 17.2. The i -the association matrix $A_i \in \text{Mat}_X(\mathbb{C})$

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i, \end{cases} \quad (x, y \in X, 0 \leq i \leq D) \quad (17.3)$$

Then,

$$(i') \quad A_0 = I.$$

$$(ii') \quad A_0 + A_1 + \dots + A_D = J \text{ (= all 1's matrix).}$$

$$(iii') \quad A_i^\top = A_i \quad (0 \leq i \leq D).$$

$$(iv') \quad A_i A_j = \sum_{h=0}^D p_{ij}^h A_h \quad (0 \leq i, j \leq D).$$

$$(v') \quad A_i A_j = A_j A_i.$$

$M := \text{Span}_{\mathbb{C}}(A_0, \dots, A_D)$ (Bose-Mesner algebra of Y) is a commutative \mathbb{C} -algebra of dimension $D + 1$.

Observe:

$$Y \text{ is symmetric} \leftrightarrow A_i^\top = A_i \text{ for all } i \leftrightarrow M \text{ is symmetric.}$$

Example 17.1. Let $\Gamma = (X, E)$ be distance-regular of diameter D . Set

$$R_i = \{(x, y) \mid \partial(x, y) = i\} \quad (0 \leq i \leq D). \quad (17.4)$$

Then,

$$Y = (X, \{R_i\}_{0 \leq i \leq D})$$

is a symmetric scheme.

$$i\text{-th association matrix} = i\text{-th distance matrix} \quad \text{for all } i.$$

Example 17.2. Suppose a group G acts transitively on a set X . Assume G is generously transitive, i.e.,

$$\text{for all } x, y \in X, \text{ there exists } g \in G \text{ such that } gx = y, gy = x.$$

Then G acts on $X \times X$ by rule;

$$g(x, y) = (gx, gy), \quad \text{for all } g \in G, \text{ and for all } x, y \in X.$$

Let R_0, \dots, R_D denote orbits of G on $X \times X$.

Observe that $R_i^\top = R_i$ for all i by generous transitivity, and

$$Y = (X, \{R_i\}_{0 \leq i \leq D})$$

is a symmetric scheme.

Exercise 17.1. In Example Example 17.2, Bose-Mesner algebra

$$M = \{B \in \text{Mat}_X(\mathbb{C}) \mid Bg = gB, \text{ for all } g \in G\} \quad (17.5)$$

$$= \text{the commuting algebra of } G \text{ on } X. \quad (17.6)$$

Here, we view each $g \in G$ as a permutation matrix in $\text{Mat}_X(\mathbb{C})$ satisfying

$$g\hat{x} = \widehat{gx}, \quad \text{for all } x \in G.$$

Example 17.3. Let G be any finite group. G acts on $X = G$ by conjugation.

$$G \times X \rightarrow X, \quad (g, x) \mapsto gxg^{-1}.$$

Let C_0, C_1, \dots, C_D denote orbits (i.e., conjugacy classes), and let $C_0 = \{1_G\}$. Claim that $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is a commutative scheme (not symmetric in general).

(i) $R_0 = \{xx \mid x \in X\}$ as $C_0 = \{1_G\}$.

(ii) R_0, \dots, R_D is a partition of $X \times X$ since C_0, \dots, C_D is a partition of $X = G$.

(iii) $R_i^\top = R_{i'}$, where $C_{i'} = \{g^{-1} \mid g \in C_i\}$.

(iv) Set $H = G \oplus G$, the direct sum. Then H acts on $X = G$:

$$\text{for all } h = (g, gz), \text{ for all } x \in X, \quad h(x) = gx(gx)^{-1} = gxz^{-1}g^{-1}.$$

$$R_i = \{(x, y) \mid x^{-1}y \in C_i\}, \quad h_i \in C_i, \quad x^{-1}y = gh_i g^{-1}.$$

$$(x, y) = (x, xgh_i g^{-1}) \tag{17.7}$$

$$= (xgg^{-1}, xgh_i g^{-1}) \tag{17.8}$$

$$= (xg, g)(1, h_i). \tag{17.9}$$

So, R_0, \dots, R_D are the orbits of H on $X \times X$.

(v) $p_{ij}^h = p_{ji}^h$?

Fix i, j, h and $x, y \in X$ with $(x, y) \in R_h$. Set

$$S = \{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\} \tag{17.10}$$

$$T = \{z \in X \mid (x, z) \in R_j, (z, y) \in R_i\}. \tag{17.11}$$

Show $|S| = |T|$.

For all $z \in S$, set $\hat{z} = xz^{-1}y$.

Observe, $\hat{z} \in T$.

$$x^{-1}z \in C_i x^{-1}\hat{z} = x^{-1}xz^{-1}y \in C_j \tag{17.12}$$

$$z^{-1}y \in C_j \hat{z}^{-1}y = y^{-1}zx^{-1}x^{-1}y = y^{-1}x(x^{-1}z)x^{-1}y \in C_i. \tag{17.13}$$

Observe

$$S \rightarrow T \quad (z \mapsto z^{-1}) \quad \text{is one-to-one and onto.}$$

Chapter 18

Polynomial Schemes

Wednesday, March 3, 1993

Lemma 18.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote the symmetric scheme with associated matrices A_0, A_1, \dots, A_D . Then the following are equivalent.*

(i) *The graph $\Gamma = (X, R_1)$ is distance-regular, and R_0, \dots, R_D are labelled so that*

$$R_i = \{xy \mid \partial(x, y) = i\}.$$

(ii) *There exists $f_i \in \mathbb{C}[\lambda]$, $\deg f_i = i$ such that $f_i(A_1) = A_i$ for all i with $0 \leq i \leq D$.*

(iii) *The parameter p_{ij}^h*

$$\begin{cases} = 0 & \text{if one of } h, i, j \text{ is larger than the sum of the other two} \\ \neq 0 & \text{if one of } h, i, j \text{ is equal to the sum of the other two.} \end{cases}$$

Proof.

(i) \Rightarrow (ii): Lemma 16.1.

(ii) \Rightarrow (iii): Define

$$k_i \equiv p_{ii}^0 = |\{z \mid z \text{ in } X, \partial(x, z) = i \text{ } ((x, z) \in R_i)\}|$$

for any $x \in X$. Then $k_i \neq 0$ ($0 \leq i \leq D$), $k_0 = 1$.

(By symmetricity, $(x, y) \in R_i$ if and only if $(y, x) \in R_i$.)

Claim.

$$k_h p_{ij}^h = k_i p_{hj}^i = k_j p_{ih}^j \quad (18.1)$$

$$= |X|^{-1} |\{xyz \in X^3 \mid \partial(x, y) = h, \partial(x, z) = i, \partial(y, z) = j\}|. \quad (18.2)$$

Pf. The number of $xyz \in X^3$, $\partial(x, y) = h, \partial(x, z) = i, \partial(y, z) = j$ is equal to

$$|X| k_h p_{ij}^h = |X| k_i p_{hj}^i = k_j p_{ih}^j.$$

In particular,

$$p_{ij}^h = 0 \leftrightarrow p_{hj}^i = 0 \leftrightarrow p_{ih}^j = 0.$$

Hence, it suffices to show

$$\begin{cases} p_{ij}^h = 0 & \text{if } h > i + j \\ p_{ij}^h \neq 0 & \text{if } h = i + j. \end{cases}$$

Fix i, j . Without loss of generality, we may assume that $i + j \leq D$ as trivial otherwise.

$$f_i(A) f_j(A) = A_i A_j = \sum_{\ell=0}^D p_{ij}^\ell A_\ell = \sum_{\ell=0}^D p_{ij}^\ell f_\ell(A).$$

$$i + j = \deg \text{LHS} \quad (18.3)$$

$$= \deg \text{RHS} \quad (18.4)$$

$$= \max\{\ell \mid p_{ij}^\ell \neq 0\}. \quad (18.5)$$

(iii) \Rightarrow (i)

Let $A = A_1$, and consider a graph Γ with adjacency matrix A .

$$A A_j = \sum_h p_{1j}^h A_h \quad (18.6)$$

$$= p_{1j}^{j+1} A_{j+1} + p_{1j}^j A_j + p_{1j}^{j-1} A_{j-1}. \quad (18.7)$$

Then, $p_{1j}^{j+1} \neq 0 \neq p_{1j}^{j-1}$.

Fix a vertex $x \in X$, and set $R_i(x) = \{y \mid (x, y) \in R_i\}$.

Then each $y \in R_i(x)$ is adjacent in Γ to exactly

$$p_{1,i+1}^i \neq 0 \quad \text{vertices in } R_i(x), \quad (18.8)$$

$$p_{1i}^i \quad \text{vertices in } R_{i+1}(x), \quad (18.9)$$

$$p_{1,i-1}^i \neq 0 \quad \text{vertices in } R_{i-1}(x). \quad (18.10)$$

Hence, by induction,

$$R_i(x) = \{y \mid \partial(x, y) = i \text{ in } \Gamma\} \quad (0 \leq i \leq D), \quad (18.11)$$

and Γ is distance regular.

□

Chapter 19

Commutative Association Schemes

Friday, March 5, 1993

Lemma 19.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme with Bose-Mesner algebra M .*

Then there exists a basis E_0, E_1, \dots, E_D for M such that

- (i) $E_0 = |X|^{-1}J$.
- (ii) $E_i E_j = E_j E_i = \delta_{ij} E_i \quad (0 \leq i, j \leq D)$.
- (iii) $E_0 + E_1 + \dots + E_D = I$.
- (iv) $E_i^\top = \overline{E_i} = E_{\hat{i}}$ for some $\hat{i} \in \{0, 1, \dots, D\}$.

Proof. M acts on Hermitean space $V = \mathbb{C}^n$ ($n = |X|$).

If W is an M -module, so is W^\perp .

Each irreducible M -module is 1 dimensional by commutativity of M . So V is orthognal direct sum of 1-dimensional M -modules.

Let v_1, \dots, v_n be an orthonormal basis for V consisiting of eigenvectors for all $m \in M$.

Set $P \in \text{Mat}_X(\mathbb{C})$ so that the i -th column of P is equal to v_i . So,

$$\bar{P}^\top P = I = P \bar{P}^\top = \bar{P} P^\top,$$

and P is unitary.

Also, for all $m \in M$,

$$P^{-1}mP = \text{diagonal} \quad (19.1)$$

$$= \text{diag}(\theta_1(m), \dots, \theta_n(m)). \quad (19.2)$$

for some functions

$$\theta_i : M \longrightarrow \mathbb{C}.$$

Observe: each $\theta = \theta_i$ is a character of M , i.e.,

$$\theta : M \longrightarrow \mathbb{C}$$

is a \mathbb{C} -algebra homomorphism.

Observe: the $\theta_1, \dots, \theta_n$ are not all distinct.

Let $\sigma_0, \dots, \sigma_r$ denote distinct elements of

$$\theta_1, \dots, \theta_n.$$

Say σ_i appears m_i times. Without loss of generality, we may assume that

$$P^{-1}mP = \begin{pmatrix} \sigma_0(m)I_{m_0} & O & O & O \\ O & \sigma_1(m)I_{m_1} & O & O \\ O & O & \ddots & O \\ O & O & O & \sigma_r(m)I_{m_r} \end{pmatrix}.$$

Set

$$E_i = P \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix} P^{-1},$$

where I_{m_i} is in the i -th block.

Then,

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq r),$$

$$E_0 + E_1 + \dots + E_r = I.$$

Hence for all $m \in M$,

$$m = \sum_{i=0}^r \sigma_i(m) E_i \in \text{Span}(E_0, \dots, E_r).$$

So,

$$M \subseteq \text{Span}(E_0, \dots, E_r).$$

Since E_0, \dots, E_r are linearly independent, $r \geq D$.

Show $E_i \in M$.

Claim 1. For all distinct i, j ($0 \leq i, j \leq D$), there exists $m \in M$ such that $\sigma_i(m) \neq 0$, $\sigma_j(m) = 0$.

Pf of Claim 1. $\sigma_i \neq \sigma_j$ implies that there exists $m' \in M$ such that $\sigma_i(m') \neq \sigma_j(m')$.

Set $m = m' - \sigma_j(m')I$. Then,

$$\sigma_j(m)\sigma_j(m') - \sigma_j(m') = 0, \quad (19.3)$$

$$\sigma_i(m)\sigma_i(m') - \sigma_j(m') \neq 0. \quad (19.4)$$

Claim 2. $E_i \in M$ ($0 \leq i \leq D$).

Pf of Claim 2. Fix a vertex $x \in X$. For all $j \neq i$, there exists $m_j \in M$ such that $\sigma_i(m_j) \neq 0$, $\sigma_j(m_j) = 0$, $i \neq j$. Observe

$$s = \sigma_i \left(\prod_{\ell \neq i} m_\ell \right) \neq 0.$$

Set

$$m^* = \sigma_i \left(\prod_{\ell \neq i} m_\ell \right) s^{-1}.$$

Observe

$$\sigma_i(m^*) = 1, \quad \sigma_j(m^*) = 0, \quad \text{for all } j \neq i \quad (0 \leq j \leq D).$$

So

$$P^{-1}m^*P = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}.$$

We have

$$E_i = m^* \in M.$$

Now $r = D$, $M = \text{Span}(E_0, \dots, E_D)$ and E_0, \dots, E_D is a basis for M .

Observe

$$P^{-1}E_iP = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}$$

implies

$$P^{-1}\overline{E_i}^\top P = \overline{P}^\top \overline{E_i}^\top \overline{P^{-1}}^\top = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}^\top = P^{-1}E_iP.$$

Hence,

$$\overline{E_i}^\top = E_i.$$

$E_0^\top, \dots, E_D^\top$ are nonzero matrices satisfying

$$E_i^\top E_j^\top = \delta_{ij} E_i^\top,$$

$$E_0^\top + E_1^\top + \cdots + E_D^\top = I.$$

Each E_i^\top is a linear combination of E_0, \dots, E_D with coefficientss that are 0 or 1, and for no two E_i 's are coefficients of any E_j both 1's.

So, $E_0^\top, \dots, E_D^\top$ is a permutation of E_0, \dots, E_D .

Observe $J = A_0 + \cdots + A_D \in M$.

The matrix $|X|^{-1}J$ is an idempotent of rank 1.

So, without loss of generality we may assume that

$$E_0 = \frac{1}{|X|}J.$$

We have the assertions. □

Define entry-wise product \circ on $\text{Mat}_X(\mathbb{C})$.

$$A_i \circ A_j = \delta_{ij}A_i.$$

So, M is closed under \circ .

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h.$$

The numbers q_{ij}^h is called Krein parameters of Y .

Claim. $q_{ij}^h \in \mathbb{R}$.

Pf.

$$\frac{1}{|X|} \sum_{h=0}^D \overline{q_{ij}^h} E_h = \frac{1}{|X|} \sum_{h=0}^D \overline{q_{ij}^h} \overline{E_h}^\top \quad (19.5)$$

$$= (\overline{E_i} \circ \overline{E_j})^\top \quad (19.6)$$

$$= E_i \circ E_j \quad (19.7)$$

$$= \frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h. \quad (19.8)$$

Hence, $q_{ij}^h = \overline{q_{ij}^h}$.

Observe $A_0, \dots, A_D, E_0, \dots, E_D$ are bases for M . Hence, there exist $p_i(j), q_i(j) \in \mathbb{C}$ such that

$$A_i = \sum_{j=0}^D p_i(j) E_j \quad (19.9)$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j. \quad (19.10)$$

Taking transpose and conjugate we find,

$$\overline{p_i(j)} = p_i(j) = p_{i'}(\hat{j}) \quad (0 \leq i, j \leq D) \quad (19.11)$$

$$\overline{q_i(j)} = q_i(j) = q_{\hat{i}}(j') \quad (0 \leq i, j \leq D). \quad (19.12)$$

Fix a vertex $x \in X$. Define

$$E_i^* \equiv E_i^*(x) \in \text{Mat}_X(\mathbb{C})$$

to be a diagonal matrix such that

$$(E_i^*)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases} \quad (0 \leq i \leq D, y \in X.)$$

Then,

$$\begin{aligned} E_i^* E_j^* &= \delta_{ij} E_i^*, \\ E_0^* + \cdots + E_D^* &= I, \\ (E_i^*)^\top &= \overline{E_i^*} = E_i^*. \end{aligned}$$

Definition 19.1. Dual Bose-Mesner algebra $M^* \equiv M^*(x)$ with respect to x is

$$\text{Span}(E_0^*, \dots, E_D^*).$$

Define dual associate matrices A_0^*, \dots, A_D^* . Indeed $A_i^* \equiv A_i^*(x) \in \text{Mat}_X(\mathbb{C})$ is a diagonal matrix with

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad (y \in X).$$

A_i^* is a diagonal matrix having the row x of E_i^* on the diagonal.

Observe

$$A_i^* = \sum_{j=0}^D q_i(j) E_j^* \quad \left(E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j \right) \quad (19.13)$$

$$E_i^* = \frac{1}{|X|} \sum_{j=0}^D p_i(j) A_j^* \quad \left(A_i = \sum_{j=0}^D p_i(j) E_j \right). \quad (19.14)$$

So, A_0^*, \dots, A_D^* form a basis for M^* .

Also,

$$A_i^* E_j^* = q_i(j) E_j^*.$$

$$\left(A_i^* E_j^* = \sum_{h=0}^D q_i(h) E_h^* E_j^* = q_i(j) E_j^* \right)$$

So, $q_i(j)$ are dual eigenvalues of A_i^* .

Observe,

$$A_0^* = I, \quad A_0^* + \cdots + A_D^* = |X|E_0^*, \quad \overline{A_i^*} = A_i^*,$$

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* \quad (0 \leq i, j \leq D).$$

Remark. Proof.

$$(A_0^*)_{yy} = |X|(E_0)_{xy} = (J)_{xy} = 1.$$

$$A_0^* + \cdots + A_D^* = \sum_{i=0}^D \sum_{j=0}^D q_i(j) E_j^* = |X|E_0^*.$$

Note that

$$I = E_0 + \cdots + E_D = \frac{1}{|X|} \sum_{i=0}^D \sum_{j=0}^D q_i(j) A_j.$$

$$\sum_{i=0}^D q_i(j) = \delta_{j0}|X|.$$

$$\overline{A_i^*} = \sum_{j=0}^D \overline{q_i(j) E_j^*} = \sum_{j=0}^D q_i(j) E_j^* = A_i^*.$$

$$(A_i^* A_j^*)_{yy} = |X|^2 (E_i)_{xy} (E_j)_{xy} \tag{19.15}$$

$$= |X|^2 (E_i \circ E_j)_{xy} \tag{19.16}$$

$$= |X| \sum_{h=0}^D q_{ij}^h (E_h)_{xy} \tag{19.17}$$

$$= \sum_{h=0}^D q_{ij}^h (A_h^*)_{yy}. \tag{19.18}$$

The following statements will be proved after a couple of lemmas in the next lecture.

Lemma. Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme. Fix a vertex $x \in X$, and set $E^* \equiv E_i^*(x)$ and $A_i^* \equiv A^*(x)$. Then the following hold.

(i) $E_i^* A_j E_k^* = O$ if and only if $p_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

(ii) $E_i A_j^* E_k = O$ if and only if $q_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

Chapter 20

Vanishing Conditions

Monday, March 15, 1993 (Monday after Spring break)

Lemma 20.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme.*

(i) $p_0(i) = 1$.

(ii) $p_i(0) = k_i$, where

$$k_i = p_{ii'}^0 = |\{y \in X \mid (x, y) \in R_i\}|.$$

(iii) $q_0(i) = 1$.

(iv) $q_i(0) = m_i$, where

$$m_i = \text{rank} E_i.$$

Proof.

(i) Since $A_0 = I$ and

$$A_0 = p_0(0)E_0 + p_0(1)E_1 + \cdots + p_0(D)E_D \quad (20.1)$$

$$I = E_0 + E_1 + \cdots + E_D, \quad (20.2)$$

$p_0(i) = 1$ for all i .

(ii) Since

$$A_i = p_i(0)E_0 + p_i(1)E_1 + \cdots + p_i(D)E_D,$$

$A_i E_0 = p_i(0) E_0$, and

$$k_i J = A_i J = p_i(0) J$$

as there are k_i 1's in each row of A_i , we have $k_i = p_i(0)$.

(iii) Since $E_0 = |X|^{-1} J$ and

$$E_0 = |X|^{-1} (q_0(0) A_0 + q_0(1) A_1 + \cdots + q_0(D) A_D) \quad (20.3)$$

$$|X|^{-1} J = |X|^{-1} (A_0 + A_1 + \cdots + A_D), \quad (20.4)$$

$q_0(i) = 1$ for all i .

(iv) $E_i = |X|^{-1} (q_i(0) A_0 + q_i(1) A_1 + \cdots + q_i(D) A_D)$, $E_i^2 = E_i$, and E_i is similar to a matrix

$$\begin{pmatrix} I_{m_i} & O \\ O & O \end{pmatrix}.$$

So,

$$m_i = \text{rank} E_i = \text{trace} E_i = \sum_{x \in X} (E_i)_{xx} = |X| |X|^{-1} q_i(0) = q_i(0).$$

Note that as

$$E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j \rightarrow (E_i)_{xx} = \frac{1}{|X|} q_i(0) (A_0)_{xx}.$$

Hence, we have all formulas.

□

Lemma 20.2. *With the above notation*

$$(i) \ p_{ij}^h = p_{j'i'}^{h'}.$$

$$(ii) \ k_h p_{ij}^h = k_j p_{i'h}^j = k_{hj'}^i.$$

$$(iii) \ q_{ij}^h = q_{ji}^{\hat{h}}.$$

$$(iv) \ m_h q_{ij}^h = m_j q_{ih}^j = m_i q_{hj}^i.$$

Proof.

(i) We have

$$\sum_{h=0}^D p_{ij}^h A_{h'} \left(\sum_{h=0}^D p_{ij}^h A_h \right)^\top \quad (20.5)$$

$$= (A_i A_j)^\top \quad (20.6)$$

$$= A_j^\top A_i^\top \quad (20.7)$$

$$= A_{j'} A_{i'} \quad (20.8)$$

$$= \sum_{h=0}^D p_{j'i'}^{h'} A_h'. \quad (20.9)$$

(ii) Count the following number,

$$|\{xyz \in X^3 \mid (x, y) \in R_h, (x, z) \in R_i, (z, y) \in R_j\}| \quad (20.10)$$

$$= |X| k_h p_{ij}^h = |X| k_j p_{i'h}^j = |X| k_{hj'}^i. \quad (20.11)$$

(iii)

$$\frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_{\hat{h}} = \left(\frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h \right)^\top \quad (20.12)$$

$$= (E_i \circ E_j)^\top \quad (20.13)$$

$$= E_j^\top \circ E_i^\top \quad (20.14)$$

$$= E_{\hat{j}} E_{\hat{i}} \quad (20.15)$$

$$= \frac{1}{|X|} \sum_{h=0}^D q_{\hat{j}\hat{i}}^{\hat{h}} E_{\hat{h}}. \quad (20.16)$$

(iv) Let $\tau(B)$ denote the sum of the entries in the matrix B .

Observe: $\tau(B \circ C) = \text{trace}(BC^\top)$.

Observe

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \tau((E_i \circ E_j \circ E_{\hat{k}})^\top) = \tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}).$$

Compute each one.

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \text{trace}((E_i \circ E_j)E_k) = \text{trace} \left(\left(\frac{1}{|X|} \sum_h q_{ij}^h E_h \right) E_k \right) \quad (20.17)$$

$$= \text{trace} \left(\frac{1}{|X|} q_{ij}^k E_k \right) = \frac{1}{|X|} m_k q_{ij}^k, \quad (20.18)$$

$$\tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \text{trace}((E_{\hat{i}} \circ E_k)E_{\hat{j}}) = \text{trace} \left(\left(\frac{1}{|X|} \sum_h q_{ik}^h E_h \right) E_{\hat{j}} \right) \quad (20.19)$$

$$= \text{trace} \left(\frac{1}{|X|} q_{ik}^j E_k \right) = \frac{1}{|X|} m_j q_{ik}^j, \quad (20.20)$$

$$\tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}) = \text{trace}((E_k \circ E_{\hat{j}})E_i) = \text{trace} \left(\left(\frac{1}{|X|} \sum_h q_{kj}^h E_h \right) E_i \right) \quad (20.21)$$

$$= \text{trace} \left(\frac{1}{|X|} q_{kj}^i E_i \right) = \frac{1}{|X|} m_i q_{kj}^i. \quad (20.22)$$

Hence, we have (iv).

□

Lemma 20.3. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme. Fix a vertex $x \in X$, and set $E^* \equiv E_i^*(x)$ and $A_i^* \equiv A^*(x)$. Then the following hold.*

(i) $E_i^* A_j E_k^* = O$ if and only if $p_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

(ii) $E_i A_j^* E_k = O$ if and only if $q_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

Proof.

(i) Partition rows and columns by $R_0(x), R_1(x), \dots, R_D(x)$. Then,

$$E_i^*(x) A_j E_h^*(x)$$

is the (i, h) block of A_j .

Hence this submatrix is zero if and only if there exists no $y, z \in X$ such that $(x, y) \in R_i$, $(x, z) \in R_h$ and $(y, z) \in R_j$. This is exactly when $p_{ij}^h = 0$.

(ii) The sum of the squares of norms of entries in $E_i A_j^* E_k$

$$= \tau((E_i A_j^* E_k) \circ (\overline{E_j A_j^* E_k})) \quad (20.23)$$

$$= \text{trace}(E_i A_j^* E_k (\overline{E_j A_j^* E_k})^\top) \quad (20.24)$$

$$= \text{trace}(E_i A_j^* E_k A_j^* E_i) \quad (20.25)$$

$$= \text{trace}(E_i A_j^* E_k A_j^*) \quad \text{as } \text{trace}(XY) = \text{trace}(YX) \quad (20.26)$$

$$= \sum_{y \in X} (E_i A_j^* E_k A_j^*)_{yy} \quad (20.27)$$

$$= \sum_{y \in X} \left(\sum_{z \in X} (E_i)_{yz} (A_j^*)_{zz} (E_k)_{zy} (A_j^*)_{yy} \right) \quad (20.28)$$

$$= \sum_{y \in X} \left(\sum_{z \in X} (E_i)_{zy} (|X| (E_j)_{xz}) (E_k)_{zy} (|X| (E_j)_{yx}) \right) \quad (20.29)$$

$$= |X|^2 (E_j (E_i \circ E_k) E_j)_{xx} \quad (20.30)$$

$$= |X| q_{ik}^j (E_j)_{xx} \quad (20.31)$$

$$= q_{ik}^j m_j \quad (20.32)$$

$$= m_k q_{ij}^k. \quad (20.33)$$

Note that since $|X|E_j = q_j(0)A_0 + q_j(1)A_1 + \cdots q_j(D)A_D$,

$$(E_j)_{xx} = \frac{1}{|X|} q_j(0) = \frac{m_j}{|X|}.$$

Thus, we have (ii). □

Corollary 20.1 (Krein Condition). *For any commutative scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$, q_{ij}^h is a non-negative real number for $0 \leq h, i, j \leq D$.*

Proof. Since $q_{ij}^h m_h$ is a non-negative real by the proof of Lemma 20.3 (ii).

Note that m_h is a positive integer. □

An interpretation of the Krein parameters.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme with standard module V .

Pick a vector $v \in V$ with

$$v = \sum_{x \in X} \alpha_x \hat{x}.$$

View v as a function

$$v : X \longrightarrow \mathbb{C} \quad (x \mapsto \alpha_x).$$

View V as the set of all functions $V \rightarrow \mathbb{C}$. Then the vector space V together with product of functions is a \mathbb{C} -algebra.

For

$$v = \sum_{x \in X} \alpha_x \hat{x}, \quad w = \sum_{x \in X} \beta_x \hat{x} \in V,$$

write

$$v \circ w = \sum_{x \in X} \alpha_x \beta_x \hat{x}$$

to represent the product of v and w viewed as functions.

Lemma 20.4. *With the above notation,*

- (i) $A_j^*(x)v = |X|(E_j \hat{x} \circ v)$ for all $v \in V$ and for all $x \in X$.
- (ii) $E_i V \circ E_j V \subseteq \sum_{h: q_{ij}^h \neq 0} E_h V$ for all $0 \leq i, j \leq D$.
- (iii) $E_h(E_i \circ E_j V) = E_h V$ if $q_{ij}^h \neq 0$ for all $0 \leq h, i, j \leq D$.

Chapter 21

Norton Algebras

Wednesday, March 17, 1993

Proof of Lemma 20.4.

(i) Suppose

$$v = \sum_{x \in X} \alpha_x \hat{x}.$$

Pick a vertex $z \in X$ and compare z -coordinate of each side in (i).

$$(A_j^*(x)v)_z = (A_j^*(x))_{zz}v_z = |X|(E_j)_{xz}\alpha_z. \quad (21.1)$$

$$|X|(E_{\hat{j}}\hat{x} \circ v)_z = |X|(E_{\hat{j}}\hat{x})_z \cdot \alpha_z = |X|(E_j)_{xz}\alpha_z. \quad (21.2)$$

Note that $E_{\hat{j}}\hat{x}$ is the column x of $E_{\hat{j}}$ is the row x of E_j .

(ii) Fix i, j, h such that $q_{ij}^h = 0$.

Claim. $E_h(E_i V \circ E_j V) = 0$.

$$E_h(E_i V \circ E_j V) = E_h(\text{Span}(v \circ w \mid v \in E_i V, w \in E_j V)) \quad (21.3)$$

$$= E_h(\text{Span}(E_i \hat{y} \circ E_j \hat{z} \mid y, z \in X)) \quad (21.4)$$

$$= \text{Span}(E_h(E_j \hat{z} \circ E_i \hat{y}) \mid y, z \in X) \quad (21.5)$$

$$= \text{Span}((E_h A_j^*(z) E_i) \hat{y} \mid y, z \in X) \quad \text{by (i)} \quad (21.6)$$

But $q_{ij}^h = 0$ implies $q_{ji}^{\hat{h}} = 0$.

So, by Lemma 20.3 (ii),

$$0 = (E_{\hat{i}} A_{\hat{j}}^* E_{\hat{h}})^\top = E_h A_j^* E_i.$$

Hence, $E_h(E_i V \circ E_j V) = 0$.

(iii) Fix i, j, h such that $q_{ij}^h \neq 0$. Then,

$$E_h(E_i V \circ E_j V) \subseteq E_h V$$

is clear. We show the other inclusion. Since

$$E_i \hat{y} \circ E_j \hat{y} = (\text{column } y \text{ of } E_i \circ \text{column } y \text{ of } E_j) \quad (21.7)$$

$$= \text{column } y \text{ of } E_i \circ E_j \quad (21.8)$$

$$= (E_i \circ E_j) \hat{y} \quad (21.9)$$

$$= \left(\frac{1}{|X|} \sum_{h=0}^D q_{ij}^h E_h \right) \hat{y}, \quad (21.10)$$

we have,

$$E_h(E_i V \circ E_j V) = E_h \text{Span}(E_i \hat{y} \circ E_j \hat{z} \mid y, z \in X) \quad (21.11)$$

$$\supseteq E_h \text{Span}(E_i \hat{y} \circ E_j \hat{y} \mid y \in X) \quad (21.12)$$

$$= \text{Span}(q_{ij}^h E_h \hat{y} \mid y \in X) \quad (21.13)$$

$$= \text{Span}(E_h \hat{y} \mid y \in X) \quad \text{since } q_{ij}^h \neq 0 \quad (21.14)$$

$$= E_h V. \quad (21.15)$$

This proves the assertion. □

Lemma 21.1. *Given a commutative scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$, fix j ($0 \leq j \leq D$). Define binary multiplication:*

$$E_j V \times E_j V \longrightarrow E_j V \quad ((v, w) \mapsto v * w = E_j(v \circ w)).$$

Then,

(i) $v * w = w * v$, for all $v, w \in E_j V$,

(ii) $v * (w + w') = v * w + v * w'$ for all $v, w, w' \in E_j V$, and

(iii) $(\alpha v) * w = \alpha(v * w)$ for all $\alpha \in \mathbb{C}$.

In particular, the vector space $E_j V$ together with $*$ is a commutative \mathbb{C} -algebra, (not associative in general).

$(N_j : (E_j V, *))$ is called the Norton algebra on $E_j V$.

(iv) $v * w = 0$ for all $v, w \in E_j V$ if and only if $q_{jj}^j = 0$.

Proof.

(i) – (iii) Immediate.

(iv) Immediate from Lemma 20.4 (ii), (iii).

□

Let Y, j, N_j be as in Lemma 21.1, and M Bose-Mesner algebra of Y . Let

$$\text{Aut}Y = \{\sigma \in \text{Mat}_X(\mathbb{C}) \mid \sigma : \text{permutation matrix}, \sigma \cdot m = m \cdot \sigma \text{ for all } m \in M\} \quad (21.16)$$

$$= \{\sigma \in \text{Mat}_X(\mathbb{C}) \mid \sigma : \text{permutation matrix}, \quad (21.17)$$

$$(x, y) \in R_i \rightarrow (\sigma x, \sigma y) \in R_i, \text{ for all } i, \text{ and for all } x, y \in X\} \quad (21.18)$$

$$\text{Aut}(N_j) = \{\sigma : E_j V \rightarrow E_j V \mid \sigma \text{ is } \mathbb{C}\text{-algebra isomorphisms, i.e.,} \quad (21.19)$$

$$\sigma(v * w) = \sigma(v) * \sigma(w) \text{ for all } v, w \in E_j V\}. \quad (21.20)$$

Lemma 21.2. *Let $Y, j, *$ be as in Lemma 21.1.*

(i) $E_j V$ is a module for $\text{Aut}(Y)$.

(ii) $\sigma|_{E_j V} \in \text{Aut}(N_j)$ for all $\sigma \in \text{Aut}(Y)$.

(iii) $\text{Aut}Y \longrightarrow \text{Aut}(N_j)$, $(\sigma \mapsto \sigma|_{E_j})$ is a homomorphism of groups,

(i.e., a representation of $\text{Aut}(Y)$).

(iv) Suppose R_0, \dots, R_D are orbits of $\text{Aut}(Y)$ acting on $X \times X$, (so, we are in Example 17.2) then above representation is irreducible.

Proof.

(i) Pick $\sigma \in \text{Aut}Y$ and $v \in V$. Then,

$$\sigma E_j v = E_j \sigma v,$$

since σ commutes with each element of M .

(ii) $\sigma|_{E_j V} : E_j V \rightarrow E_j V$ is an isomorphism of a vector space. Since σ is invertible, for all $v, w \in E_j V$,

$$\sigma(v * w) = \sigma(E_j(E_j v \circ E_j w)) = E_j \sigma(E_j v \circ E_j w) = E_j(E_j \sigma v \circ E_j \sigma w) = \sigma(v) * \sigma(w).$$

(iii) Immediate from (i) and (ii).

(iv) Here Bose-Mesner algebra M is the full commuting algebra, i.e.,

$$M = \{m \in \text{Mat}_X(\mathbb{C}) \mid \sigma \cdot m = m \cdot \sigma, \text{ for all } \sigma \in \text{Aut}(Y)\}.$$

Suppose there is a nonzero proper subspace $0 \neq W \subsetneq E_j V$ that is $\text{Aut}(Y)$ -invariant.

Set

$$W^\perp = \{v \in E_j V \mid \langle w, v \rangle = 0, \text{ for all } w \in W\}.$$

Then, W^\perp is a module for $\text{Aut}(Y)$, since $\text{Aut}(Y)$ is closed under transpose conjugate.

Let $e : V \rightarrow W$ and $f : V \rightarrow W^\perp$ be orthogonal projection such that $e + f = E_j$,

$$e^e = e, f^e = f, ef = fe = 0, eE_h = 0, \text{ if } h \neq j.$$

Since e commutes with all $\sigma \in \text{Aut}(Y)$, $e \in M$ and

$$e = \sum_{i=0}^D \alpha_i E_i.$$

If $h \neq j$, then $0 = eE_h$ and $\alpha_h = 0$. Thus, $e = \alpha_j E_j$, i.e., $e = 0$ or $f = 0$.

A contradiction.

□

Norton algebras were used in original construction of Monster, a finite simple group G .

Compute character table of G ,

- p_{ij}^h, q_{ij}^h of group scheme on G ,
- find j where $m_j = \dim E_j V$ is small and $q_{jj}^j \neq 0$,
- guess abstract structure of N_j using the knowledge of p_{ij}^h 's and q_{ij}^h 's,
- compute $\text{Aut}(N_j)$,
- G .

Chapter 22

Q-Polynomial Schemes

Friday, March 19, 1993

Lemma 22.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme.*

$$(i) \ p_{0j}^h = p_{j0}^h = \delta_{jh}.$$

$$(ii) \ p_{ij}^0 = \delta_{ij} k_i.$$

$$(iii) \ q_{0j}^h = q_{j0}^h = \delta_{jh}.$$

$$(iv) \ q_{ij}^0 = \delta_{ij} m_i.$$

$$(v) \ \sum_{j=0}^D p_{ij}^h = k_i.$$

$$(vi) \ \sum_{j=0}^D q_{ij}^h = m_i.$$

Proof.

(i), (ii) These are trivial.

(iii) We have

$$|X|^{-1} \sum_{\ell=0}^D q_{0j}^{\ell} E_{\ell} = E_0 \circ E_j = |X|^{-1} J \circ E_j = |X|^{-1} E_j.$$

(iv) Recall from Lemma 20.2

$$|X|^{-1} m_h q_{ij}^h = \tau(E_i \circ E_j \circ E_{\hat{h}}),$$

(where $\tau(B)$ is the sum of entries in matrix B .)

$$|X|^{-1}m_0q_{ij}^0 = \tau(E_i \circ E_j \circ E_0) \quad (22.1)$$

$$= |X|^{-1}\tau(E_i \circ E_j) \quad (E_0 = |X|^{-1}J) \quad (22.2)$$

$$= |X|^{-1}\text{trace}(E_i E_j) \quad (22.3)$$

$$= |X|^{-1}\delta_{ij}\text{trace}E_i \quad (22.4)$$

$$= |X|^{-1}\delta_{ij}m_i. \quad (22.5)$$

(v) Pick $x, y \in X$ with $(x, y) \in R_h$. Then,

$$\{j=0\} \wedge D \quad p \wedge h\{ij\} \quad \& \quad = \quad |\{z \in X \mid (x,z) \in R_{\perp i}, \quad ; \quad (z,y) \in R_{\perp j} \quad ; \quad \text{for some } j\}| \quad \& \quad = \\ |\{z \in X \mid (x,z) \in R_{\perp i}\}| \quad \& \quad k_{\perp i}. \quad \backslash \text{end}\{\text{align}\}$$

(vi)

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h.$$

So,

$$\sum_{j=0}^D E_i \circ E_j = |X|^{-1} \sum_{h=0}^D \left(\sum_{j=0}^D q_{ij}^h \right) E_h \quad (22.6)$$

$$= E_i \circ \sum_{j=0}^D E_j \quad (22.7)$$

$$= E_i \circ I \quad (22.8)$$

$$= |X|^{-1}(q_i(0)A_0 + q_i(1)A_1 + \cdots + q_i(D)A_D) \circ I \quad (22.9)$$

$$= |X|^{-1}q_i(0)I \quad (22.10)$$

$$= |X|^{-1}m_i(E_0 + E_1 + \cdots + E_D). \quad (22.11)$$

This proves the assertions. □

Definition 22.1. Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme.

Y is Q -polynomial with respect to ordering E_0, E_1, \dots, E_D of primitive idempotents, if

$$q_{ij}^h \begin{cases} = 0 & \text{if one of } h, i, j \text{ is greater than the sum of the other two} \\ \neq 0 & \text{if one of } h, i, j \text{ is equal to the sum of the other two.} \end{cases}$$

In this case, set

$$c_i^* = q_{1,i-1}^i, \quad a_i^* = q_{1,i}^i, \quad b_i^* = q_{1,i+1}^i \quad (0 \leq i \leq D), \quad (c_0^* = b_D^* = 0).$$

Observe: Q -polynomial $\rightarrow Y$ is symmetric.

Suppose $i \neq \hat{i}$ for some i . Then, by the condition in Definition 22.1,

$$0 = q_{i\hat{i}}^0 = m_i \ (\neq 0)$$

by Lemma 22.1 (iv). This is a contradiction.

Hence, $E_i^\top = E_{\hat{i}} = E_i$ for all i .

Therefore M is symmetric and Y is symmetric.

Observe: If Y is Q -polynomial,

$$c_i^* + a_i^* + b_i^* = m_1 \quad (0 \leq i \leq D)$$

(just as $c_i + a_i + b_i = k$ for P -polynomial.)

By Lemma 22.1 (iv),

$$m_1 = q_{10}^i + q_{11}^i + \cdots + q_{1,i-1}^i + q_{1i}^i + q_{1,i+1}^i + \cdots$$

and $q_{10}^i = q_{11}^i = 0$, $q_{1,i-1}^i = c_i^*$, $q_{1i}^i = a_i^*$, and $q_{1,i+1}^i = b_i^*$.

Lemma 22.2. Assume $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is a symmetric scheme. Pick $x \in X$, and set $E_i^* \equiv E_i^*(x)$, $A^* \equiv A^*(x)$. Then the following are equivalent.

- (i) Γ is Q -polynomial with respect to E_0, \dots, E_D .
- (ii) $\begin{aligned} & \text{\$} q_{h-j} \text{\$} \begin{cases} \text{\$} \text{\$} q_{h-j} \text{\$} \end{cases} \begin{cases} \text{\$} \text{\$} q_{h-j} \text{\$} \end{cases} = 0 \text{ \& if } |h-j| > 0 \text{ \& } \\ & 0 \text{ \& if } |h-j| = 1. \end{cases} \quad (0 \leq h, j \leq D).$
- (iii) There exists $f_i^* \in \mathbb{C}[\lambda]$, $\deg f_i^* = i$, and

$$A_i^* = f_i^*(A_1^*) \quad (0 \leq i \leq D).$$

- (iv) E_0^*V, \dots, E_D^*V are maximal eigenspaces of A_1^* , and

$$E_i A_1^* E_j = 0 \quad \text{if } |i-j| > 0, \quad (0 \leq i, j \leq D).$$

(Compare (iv) with the definition of Q -polynomial in Definition 6.2.)

Proof.

- (i) \rightarrow (ii) Clear.

- (ii) \rightarrow (iii) $A_0^* = I$,

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* \quad (22.12)$$

$$A_1^* A_j^* = q_{1j}^{j-1} A_{j-1}^* + q_{1j}^j A_j^* + q_{1j}^{j+1} A_{j+1}^* \quad (q_{1j}^{j+1} \neq 0, 1 \leq j \leq D-1). \quad (22.13)$$

Hence A_j^* is a polynomial of degree exactly j in A_1^* by induction on j .

$$\lambda f_j^*(\lambda) = b_{j-1}^* f_{j-1}^*(\lambda) + a_j^* f_j^*(\lambda) + c_{j+1}^* f_{j+1}^*(\lambda) \quad \text{with } c_{j+1}^* \neq 0,$$

and $f_{-1}^* = 0$, $f_0^*(\lambda) = 1$.

(iii) \rightarrow (i) Pick i, j, h with $0 \leq i, j, h \leq D$ and $h \geq i + j$. Since

$$m_h q_{ij}^h = m_j q_{ih}^j = m_i q_{hj}^i$$

by Lemma 20.2, it suffices to show that

$$q_{ij}^h \begin{cases} = 0 & \text{if } h > i + j \\ \neq 0 & \text{if } h = i + j. \end{cases}$$

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* \quad (22.14)$$

$$f_i^*(A_1) f_j^*(A_1) = \sum_{h=0}^D q_{ij}^h f_h^*(A_1). \quad (22.15)$$

Hence,

$$f_i^*(\lambda) f_j^*(\lambda) = \sum_{h=0}^D q_{ij}^h f_h^*(\lambda).$$

Note that since $A_0^*, A_1^*, \dots, A_D^*$ are linearly independent, $f(A_1^*) = 0$ implies $\deg f > D$.

$$\deg \text{LHS} = i + j \rightarrow q_{ij}^{i+j} \neq 0, \quad q_{ij}^h = 0, \quad \text{if } h > i + j.$$

(iii) \rightarrow (iv) Recall

$$A_1^* = q_1(0)E_0^* + q_1(1)E_1^* + \dots.$$

Each A_i^* is a polynomial in A_1^* . Then A_1^* generates the dual Bose-Mesner algebra. So, $q_1(0), q_1(1), \dots, q_1(D)$ are distinct.

So, $E_0^* V, \dots, E_D^* V$ are maximal eigenspaces.

Also, $|i - j| > 1$ implies $q_{11}^j = 0$.

Thus, $E_i A_1^* E_j = 0$ by Lemma 20.3 (ii).

(iv) \rightarrow (ii) $q_{1j}^i = 0$ if $|i - j| > 1$. since in this case,

$E_i A_1^* E_j = 0$ implies $q_{1j}^i = 0$ by Lemma 20.3 (ii).

Suppose $q_{1j}^{j+1} = 0$ for some j ($0 \leq j \leq D - 1$).

Without loss of generalith, choose j minimum. Then A_h^* is a polynomial of degree h in A_1^* ($0 \leq h \leq j$), and

$$A_1^* A_j^* - q_{1j}^{j-1} A_{j-1}^* - q_{1j}^j A_j^* = 0.$$

the left hand side is a polynomial in A_1^* of degree $j + 1$.

Hence, the minimal polynomial of A_1^* has degree less than or equal to $j + 1 \leq D$. But A_1^* has $D + 1$ distinct eigenvalues.

This is a contradiction.

□

Chapter 23

Representation of a Scheme

Monday, March 22, 1993

Theorem 23.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a symmetric scheme. (View the standard module V as an algebra of functions from X to \mathbb{C} .) Then the following are equivalent.*

- (i) Y is Q -polynomial with respect to ordering E_0, E_1, \dots, E_D of primitive idempotents.
- (ii) For all i ($0 \leq i \leq D$),

$$E_0V + E_1V + (E_1V)^2 + \dots + (E_1V)^i = E_0V + E_1V + \dots + E_iV.$$

Proof.

By Lemma 20.4 (ii), (iii).

$$E_h(E_iV \circ E_jV) = 0 \text{ if and only if } q_{ij}^h = 0 \quad (0 \leq i, j, h \leq D).$$

(i) \rightarrow (ii) By our assumption,

$$q_{1j}^h = 0 \text{ if } |h - j| > 1, \text{ and } q_{1j}^{j+1} \neq 0.$$

So,

$$E_1V \circ E_jV \subseteq E_{j-1}V + E_jV + E_{j+1}V \quad (0 \leq j \leq D), \quad (23.1)$$

$$E_{j+1}(E_1V \circ E_jV) = E_{j+1}V \quad (0 \leq j \leq D-1), \quad (23.2)$$

by Lemma 20.4.

Also $E_0V \subseteq \text{Span}(\delta)$, where δ is all 1's vector, i.e., 1 as a function $X \rightarrow \mathbb{C}$. So,

$$E_0 \circ E_j V = E_j V \quad (0 \leq j \leq D). \quad (23.3)$$

Show (ii) by induction on i .

The cases $i = 0, 1$ are trivial.

$i > 1$: \subseteq .

$$E_0V + E_1V + (E_1V)^2 + \cdots + (E_1V)^i \quad (23.4)$$

$$= E_0V + E_1V \circ (E_0V + E_1V + \cdots + (E_1V)^{i-1}) \quad (23.5)$$

$$= E_0V + E_1V \circ (E_0V + E_1V + \cdots + E_{i-1}V) \quad (23.6)$$

$$\subseteq E_0V + E_1V + \cdots + E_iV \quad (23.7)$$

by (23.1).

\supseteq .

Claim. $E_i \subseteq E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V \quad (2 \leq i \leq D)$.

Proof of Claim. By (23.2),

$$E_i(E_1V \circ E_{i-1}V) = E_iV.$$

For all $v \in E_iV$, there exists $u \in E_1V \circ E_{i-1}V$ such that $E_iu = v$.

On the other hand, by (23.1),

$$E_1V \circ E_{i-1}V \subseteq E_{i-2}V + E_{i-1}V + E_{i-2}V.$$

So, $u = w + v$, where $w \in E_{i-2}V + E_{i-1}V$. We have,

$$w = u - v \in E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V$$

as desired.

Remark.

$$E_iV \circ E_jV = \text{Span}(u \circ v \mid u \in E_iV, v \in E_jV).$$

By claim,

$$E_0V + E_1V + \cdots + E_iV \quad (23.8)$$

$$\subseteq E_0V + E_1V + \cdots + E_iV + E_1V \circ E_{i-1}V \quad (23.9)$$

$$\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + E_1V(E_0V + E_1V + \cdots + (E_1V)^{i-1}) \quad (23.10)$$

$$\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + (E_1V)^i. \quad (23.11)$$

(ii) \rightarrow (i)

Claim 1. Pick i, j ($0 \leq i, j \leq D$) with $j > i + 1$. Then $q_{1i}^j = 0$.

Proof of Claim 1.

$$E_j(E_1 \circ E_j V) \subseteq E_j(E_1 V \circ (E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^i)) \quad (23.12)$$

$$\subseteq E_j(E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^{i+1}) \quad (23.13)$$

$$= E_j(E_0 V + E_1 V + \cdots + E_{i+1} V) \quad (23.14)$$

$$= 0. \quad (23.15)$$

So $q_{1i}^j = 0$ by Lemma 20.4.

Claim 2. $q_{1i}^{i+1} \neq 0$ ($0 \leq i < D$).

Proof of Claim 2.

$$E_0 V + E_1 V + \cdots + E_{i+1} V \quad (23.16)$$

$$= E_0 V + E_1 V + \cdots + (E_1 V)^{i+1} \quad (23.17)$$

$$= E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + (E_1 V)^i) \quad (23.18)$$

$$= E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + E_i V) \quad (23.19)$$

$$= E_0 V + E_1 V \circ (E_0 V + \cdots + E_i V). \quad (23.20)$$

So,

$$E_{i+1} V = E_{i+1} (E_1 V \circ (E_0 V + \cdots + E_i V)) \quad (23.21)$$

$$= E_{i+1} (E_1 V \circ E_i V) \quad (23.22)$$

by Claim 1 and Lemma 20.4.

Hence, $q_{1i}^{i+1} \neq 0$ by Lemma 20.4.

□

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme with standard module V .

Definition 23.1. A representation of Y is a pair (ρ, H) , where H is a non-zero Hermitean space (with inner product $\langle \cdot, \cdot \rangle$) and $\rho : X \rightarrow H$ is a map satisfying the following.

R1. $H = \text{Span}(\rho(x) \mid x \in X)$.

R2. $\langle \rho(x), \rho(y) \rangle$ depends only on i for which $(x, y) \in R_i$ ($x, y \in X$).

R3. For every $x \in X$ and for all i ($0 \leq i \leq D$),

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) \in \text{Span}(\rho(x)).$$

Above representation is nondegenerate if $\{\rho(x) \mid x \in X\}$ are distinct.

Example 23.1. $Y = H(D, 2)$, $X = \{a_1 \cdots a_D \mid a_i \in \{1, -1\}, 1 \leq i \leq D\}$. Let $H = \mathbb{C}^D$ and $\langle \cdot, \cdot \rangle$ usual Hermitean dot product.

For a vertex $x = a_1 \cdots a_D \in X$, define

$$\rho(x) = a_1 \cdots a_D \in H.$$

Then, R1 – R3 hold.

Remark. R1, R2 are obvious. For R3, we may assume that $x = 1 \cdots 1$. Restrict

$$\sum_{y \in X, (y, x) \in R_i} \rho(y)$$

on the first coordinate. Then,

$$-1 \text{ appers } \binom{D-1}{i-1} \text{ times} \quad (23.23)$$

$$1 \text{ appers } \binom{D-1}{i} \text{ times.} \quad (23.24)$$

Hence,

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) = \left(\binom{D-1}{i} - \binom{D-1}{i-1} \right) \rho(x).$$

Let (ρ, H) be a representation of arbitrary commutative scheme Y . Set

$$E = (\langle \rho(x), \rho(y) \rangle)_{x, y \in X}$$

Gram matrix of the representation.

Definition 23.2. Representations (ρ, H) , (ρ', H') of Y are equivalent, whenever, Gram matrices are related by

$$E' \in \text{Span} E.$$

We do not distinguish between equivalent representations.

Note. Suppose (ρ, H) is a representation of a symmetric scheme Y . Pick $x, y \in X$ with $(x, y) \in R_j$.

Then $(y, x) \in R_j$. So, by R2,

$$\langle \rho(x), \rho(y) \rangle = \langle \rho(y), \rho(x) \rangle = \overline{\langle \rho(x), \rho(y) \rangle},$$

since $\langle \cdot, \cdot \rangle$ is Hermitean.

Hence, the Gram matrix E of ρ is real symmetirc. Without loss of generality, we can view H as a real Euclidean space in this case.

Lemma 23.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme and V a standard module.*

Let E_j be any primitive idempotent of Y .

(i) (ρ, H) is a representation of Y , where $H = E_j V$ (with inner product inherited from Y).

$$\rho : X \rightarrow H \quad (x \mapsto E_j \hat{x})$$

(i.e., $\rho(x)$ is the x -th column of E_j .)

(ii) $\langle \rho(x), \rho(y) \rangle = |X|^{-1} q_j(i)$, if $(x, y) \in R_i$, $(x, y \in X)$.

(iii) For $0 \leq i \leq D$ and $x, y \in X$,

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) = p_i(j) \rho(x).$$

(iv) (ρ, H) is nondegenerate if and only if $q_j(i) \neq q_j(0)$ for all i , $(0 \leq i \leq D)$.

(v) Every representation of Y is equivalent to a representation of the above type for some j ($0 \leq j \leq D$), and j is unique.

Proof.

(i) – (iii).

R1: $\text{Span}(\rho X)$ is the column space of E_j which is equal to H .

R2:

$$\langle \rho(x), \rho(y) \rangle = \langle E_j \hat{x}, E_j \hat{y} \rangle \quad (23.25)$$

$$= (\overline{E_j \hat{x}})^\top E_j \hat{y} \quad (23.26)$$

$$= \hat{x}^\top \overline{E_j}^\top E_j \hat{y} \quad (23.27)$$

$$= \hat{x}^\top E_j \hat{y} \quad (23.28)$$

$$(E_j)_{xy}. \quad (23.29)$$

Note that $\overline{E_j}^\top = E_j$ by Lemma 19.1.

Recall

$$E_j = |X|^{-1} (q_j(0)A_0 + \cdots + q_j(D)A_D).$$

So,

$$(E_j)_{xy} = |X|^{-1} q_j(i), \quad \text{where } (x, y) \in R_i.$$

R2: Recall

$$A_i = p_i(0)E_0 + \cdots + p_i(D)E_D.$$

So, $E_j A_i = p_i(j) E_j$, and

$$p_i(j) \rho(x) = p_i(j) E_j \hat{x} = E_j A_i \hat{x} = E_j \sum_{y \in X, (y, x) \in R_i} \hat{y} = \sum_{y \in X, (y, x) \in R_i} \rho(y).$$

Note.

$$A_i \hat{x} = \sum_{y \in X, (x, y) \in R_{i'}} \hat{y}.$$

Pf.

$$z \text{ entry of LHS} = (A_i \hat{x})_z \quad (23.30)$$

$$= \sum_{w \in X} (A_i)_{zw} \hat{x}_w \quad (23.31)$$

$$= (A_i)_{zx} \quad (23.32)$$

$$= \begin{cases} 1 & \text{if } (x, z) \in R_{i'} \\ 0 & \text{else.} \end{cases} \quad (23.33)$$

$$z \text{ entry of RHS} = \sum_{y \in X, (x, y) \in R_{i'}, z=y} 1 \quad (23.34)$$

$$= \begin{cases} 1 & \text{if } (x, z) \in R_{i'} \\ 0 & \text{else.} \end{cases} \quad (23.35)$$

(iv) By (ii),

$$\|\rho(x)\|^2 = \langle \rho(x), \rho(y) \rangle \quad (23.36)$$

$$|X|^{-1} q_j(0) \quad (23.37)$$

$$|X|^{-1} m_j, \quad (23.38)$$

as $m_j = \dim E_j V$, and is independent of $x \in X$.

Pick distinct $x, y \in X$ such that $(x, y) \in R_i$ with $i \neq 0$.

Then,

$$\rho(x) = \rho(y) \Leftrightarrow \langle \rho(x), \rho(y) \rangle = \|\rho(x)\|^2 = |X|^{-1} q_j(0) \quad (23.39)$$

$$\Leftrightarrow |X|^{-1} q_j(i) = |X|^{-1} q_j(0) \quad (23.40)$$

$$\Leftrightarrow q_j(i) = q_j(0). \quad (23.41)$$

Hence, we have (iv). To be continued.

□

Chapter 24

Balanced Conditions, I

Wednesday, March 23, 1993

No Class on Friday (another conference).

Proof of Lemma 23.1 continued. Let E_j be a primitive idempotent, $H = E_j V$ and

$$\rho : X \rightarrow H \quad (x \mapsto E_j \hat{x}).$$

(v) Every representation (ρ, H) of Y is equivalent to a representation of above type, for some j ($0 \leq j \leq D$) and j is unique.

Let $E := (\langle \rho(x), \rho(y) \rangle)_{x, y \in X}$.

By R2,

$$E = \sum_{i=0}^D \sigma_i A_i, \quad \text{some } \sigma_0, \sigma_1, \dots, \sigma_D \in \mathbb{C}.$$

Hence, E belongs to the Bose-Mesner algebra \mathcal{M} of Y .

We want to show that E is a scalar multiple of a primitive idempotent.

Fix $x \in X$ and fix i ($0 \leq i \leq D$).

By R3,

$$\sum_{y \in X, (y, x) \in R_i} \rho(y) = \alpha \rho(x), \quad \text{some } \alpha \in \mathbb{C}. \quad (24.1)$$

So,

$$k_i \bar{\sigma}_i = \left\langle \sum_{y \in X, (y, x) \in R_i} \rho(y), \rho(x) \right\rangle = \bar{\alpha} \langle \rho(x), \rho(x) \rangle = \bar{\alpha} \sigma_0.$$

Hence, α is independent of x . In maatrix form (24.1) becomes

$$EA_i \hat{x} = \alpha E \hat{x}.$$

Remark.

$$Eu = Ev \Leftrightarrow \langle z, Eu \rangle = \langle z, Ev \rangle \text{ for all } z \in X \Leftrightarrow (Eu)_z = (Ev)_z \text{ for all } z \in X.$$

$$(EA_i \hat{x})_z = \left\langle \rho(z), \sum_{y \in X, (y, x) \in R_i} \rho(y) \right\rangle \quad (24.2)$$

$$= \alpha \langle \rho(z), \rho(x) \rangle \quad (24.3)$$

$$= (\alpha E \hat{x})_z. \quad (24.4)$$

Hence,

$$EA_i \hat{x} = \alpha E \hat{x}.$$

Since x is arbitrary,

$$EA_i = \alpha E.$$

So,

$$EA_i \in \text{Span} E \text{ and } EM = \text{Span} E.$$

We have $E \in E_j$ for unique j ($0 \leq j \leq D$). □

Remark.

$$E = \tau_0 E_0 + \cdots + \tau_D E_D, \quad \tau_j \in \mathbb{C} \quad (0 \leq j \leq D).$$

And, at least one of τ_j is nonzero, and

$$\tau_j E_j = EE_j \in \text{Span} E.$$

So,

$$\tau_j E_j = E$$

as E_0, \dots, E_D are linearly independent.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a symmetric scheme, and let E be a primitive idempotent.

Definition 24.1. Y is Q -polynomial with respect to E , if and only if Y is Q -polynomial with respect to some ordering E_0, E_1, \dots, E_D of primitive idempotents, where $E_0 = |X|^{-1}J$, and $E_1 = E$.

Theorem 24.1. Assume $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is P -polynomial (i.e., (X, R_1) is distance-regular). Let E be any primitive idempotent of Y . Let (ρ, H) be the corresponding representation.

(i) The following are equivalent.

(ia) Y is Q -polynomial with respect to E .

(ib) (ρ, H) is nondegenerate and for all $x, y \in X$, and for all i, j ($0 \leq i, j \leq D$),

$$\sum_{z \in X, (x, z) \in R_i, (y, z) \in R_j} \rho(z) - \sum_{z' \in X, (x, z') \in R_j, (y, z') \in R_i} \rho(z') \in \text{Span}(\rho(x) - \rho(y)).$$

(ic) (ρ, H) is nondegenerate and for all $x, y \in X$,

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') \in \text{Span}(\rho(x) - \rho(y)).$$

(ii) Wirte

$$E = |X|^{-1} \sum_{j=0}^D \theta_j^* A_j,$$

and suppose (ia) – (ic) hold. Then the coefficient in (ib) is

$$p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \quad (1 \leq h \leq D, 0 \leq i, j \leq D).$$

Proof.

(ia) \rightarrow (ib) Without loss of generality, assume $E \equiv E_1$, and Y is Q -polynomial with respect to E .

Then by Lemma 22.2, $\theta_0^*, \dots, \theta_D^*$ are distinct. So $\theta_h^* \neq \theta_0^*$ for all $h \in \{1, 2, \dots, D\}$, and (ρ, H) is nondegenerate.

Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $A_i^* \equiv A_i^*(x)$, $A^* \equiv A_1^*$.

Let M be the Bose-Mesner algebra. Set

$$L = \{mA^*n - nA^*m \mid m, n \in M\}.$$

Claim 1. $\dim L \leq D$.

Proof of Claim 1.

$$L = \text{Span}(E_i A^* E_j - E_j A^* E_i \mid 0 \leq i < j \leq D) \quad (24.5)$$

$$= \text{Span}(E_i A^* E_{i+1} - E_{i+1} A^* E_i \mid 0 \leq i \leq D-1). \quad (24.6)$$

Since $E_i A^* E_j = O$ if $q_{ij}^1 = 0$ by Lemma 20.2 and Lemma 20.3, and this occurs if $|i - j| > 1$ by Q -polynomial property.

Hence, $\dim L \leq D$.

Claim 2. (i) $\{A^* A_h - A_h A^* \mid 1 \leq h \leq D\}$ is a basis for L . In particular,

(ii) there exist $r_{ij}^h \in \mathbb{C}$ ($1 \leq h \leq D, 0 \leq i, j \leq D$) such that

$$A_i A^* A_j - A_j A^* A_i = \sum_{h=1}^D r_{ij}^h (A^* A_h - A_h A^*).$$

Proof of Claim 2.

(i) The column x of $A^* A_h - A_h A^*$ is a nonzero scalar $\theta_h^* - \theta_0^*$ times the column x of A_h .

Remark.

$$((A^* A_h - A_h A^*) \hat{x})_y = E_{xy}(A_h)_{yx} - (A_h)_{yx} E_{xx} = (\theta_h^* - \theta_0^*)(A_h)_{yz}.$$

Also the column x of A_0, A_1, \dots, A_D are linearly independent.

Hence, the matrices given are linearly independent.

They are in L by construction, so they form a basis for L by Claim 1.

(ii) This is immediate since

$$A_i A^* A_j - A_j A^* A_i \in L, \quad \text{for all } i, j.$$

Claim 3.

$$r_{ij}^\ell = p_{ij}^\ell \left(\frac{\theta^* - \theta_j^*}{\theta_0^* - \theta_\ell^*} \right) \quad (1 \leq \ell \leq D, 0 \leq i, j \leq D).$$

Proof of Claim 3. Fix i, j ,

$$A_i A^* A_j - A_j A^* A_i - \sum_{h=1}^D r_{ij}^h (A^* A_h - A_h A^*) = 0.$$

Pick ℓ ($1 \leq \ell \leq D$). Pick $y \in X$ such that $(x, y) \in R_\ell$.

$$(A_i A^* A_j)_{xy} = \sum_{z \in X} (A_i)_{xz} (A^*)_{zz} (A_j)_{zy} \quad (24.7)$$

$$= \sum_{z \in X, (x, z) \in R_i, (y, z) \in R_j} (A^*)_{zz} \quad (24.8)$$

$$= |X|^{-1} p_{ij}^\ell \theta_i^*. \quad (24.9)$$

Similarly,

$$(A_j A^* A_i)_{xy} = |X|^{-1} p_{ij}^\ell \theta_j^*.$$

$$(A^*A_h - A_hA^*)_{xy} = (A_0A^*A_h - A_hA^*A_0)_{xy} \quad (24.10)$$

$$= |X|^{-1}p_{0h}^\ell(\theta_0^* - \theta_h^*) \quad (24.11)$$

$$= \begin{cases} 0 & \text{if } \ell \neq h \\ |X|^{-1}(\theta_0^* - \theta_h^*) & \text{if } \ell = h. \end{cases} \quad (24.12)$$

Hence,

$$\sum_{h=1}^D r_{ij}^h (A^*A_h - A_hA^*)_{xy} = |X|^{-1}r_{ij}^\ell(\theta_0^* - \theta_\ell^*).$$

Comparing terms, we have

$$p_{ij}^\ell(\theta_i^* - \theta_j^*) - r_{ij}^\ell(\theta_0^* - \theta_\ell^*) = 0.$$

Claim 4. For all h ($1 \leq h \leq D$), for all i, j ($0 \leq i, j \leq D$), for all $w, y \in X$, $(w, y) \in R_h$,

$$\sum_{z \in X, (w, z) \in R_i, (y, z) \in R_j} \rho(z) - \sum_{z' \in X, (w, z') \in R_j, (y, z') \in R_i} \rho(z') - r_{ij}^h(\rho(w) - \rho(y)) = 0. \quad (24.13)$$

Proof of Claim 4. Set $L = \langle \text{LHS of (24.13)}, \rho(x) \rangle$. It suffices to show that $L = 0$.

Note that since x is arbitrary, if LHS of (24.13) is zero.

$$L = \sum_{z \in X, (w, z) \in R_i, (y, z) \in R_j} \langle \rho(z), \rho(x) \rangle - \sum_{z' \in X, (w, z') \in R_j, (y, z') \in R_i} \langle \rho(z'), \rho(x) \rangle \quad (24.14)$$

$$- r_{ij}^h \langle \rho(w) - \rho(y), \rho(x) \rangle \quad (24.15)$$

$$= |X|^{-1}(A_iA^*A_j)_{wy} - |X|^{-1}(A_jA^*A_i)_{wy} - |X|^{-1} \sum_{\ell=1}^D r_{ij}^\ell (A^*A_\ell - A_\ell A^*)_{wy} \quad (24.16)$$

$$= |X|^{-1} \text{times } wy \text{ entry of a matrix known to be zero by Claim 2} \quad (24.17)$$

$$= 0. \quad (24.18)$$

□

Remark.

$$|X|^{-1} \sum_{\ell=1}^D r_{ij}^\ell (A^*A_\ell - A_\ell A^*)_{wy} = |X|^{-1} r_{ij}^h (A^*A_h - A_h A^*)_{wy} \quad (24.19)$$

$$= r_{ij}^h (\langle \rho(x), \rho(w) \rangle - \langle \rho(x), \rho(y) \rangle) \quad (24.20)$$

Chapter 25

Balanced Conditions, II

Monday, March 29, 1993

Proof of Theorem 24.1 continued.

(ib) \rightarrow (ic) Obvious.

(ic) \rightarrow (ia) Without loss of generality, we may assume $D \geq 3$, else trivial.

Remark. The case $D = 2$ should be treated somewhere, but the assumption $D \geq 3$ is not used.

Fix $w \in X$, and write $E_i^* \equiv E_i^*(w)$, $A_i^* \equiv A_i^*(w)$, $A^* \equiv A_1^*$, and A_i , i -th distance matrix. Set

$$E \equiv E_1 = |X|^{-1} \sum_{i=0}^D \theta_i^* A_i.$$

Since (ρ, H) is nondegenerate,

$$\theta_0^* \neq \theta_h^*, \text{ for all } h \in \{1, 2, \dots, D\}$$

See Lemma 23.1 (iv).

Claim 1. Pick h ($1 \leq h \leq D$), and x, y with $(x, y) \in R_h$. Then

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') = r_{12}^h (\rho(x) - \rho(y)),$$

where

$$r_{12}^h = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}.$$

Proof of Claim 1. By our assumption,

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') = \alpha (\rho(x) - \rho(y)).$$

Hence,

$$|X|^{-1}p_{12}^h(\theta_1^* - \theta_2^*) = \left\langle \sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z'), \rho(x) \right\rangle \quad (25.1)$$

$$= \alpha \langle \rho(x) - \rho(y), \rho(x) \rangle \quad (25.2)$$

$$= \alpha |X|^{-1}(\theta_0^* - \theta_h^*). \quad (25.3)$$

We have

$$\alpha = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}.$$

$$\text{Claim 2. } A_1 A^* A_2 - A_2 A^* A_1 = \sum_{h=1}^D r_{12}^h (A^* A_h - A_h A^*).$$

Proof of Claim 2. The xy entry of the LHS – RHS is

$$|X| \left\langle \sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z') - r_{12}^h (\rho(x) - \rho(y)), \rho(w) \right\rangle,$$

where $(x, y) \in R_h$, $h = 1, 2, \dots, D$, and the xy entry of the LHS – RHS is 0 if $x = y$.

But the vector on the left in the above inner product is 0 by Claim 1, so the inner product is 0.

Thus, the xy entry of the LHS – RHS is always 0, and we have Claim 2.

$$\text{Claim 3. } A^* A_3 - A_3 A^* \in \text{Span}(AA^* A_2 - A_2 A^* A, A^* A_2 - A_2 A^*, A^* A - AA^*).$$

Proof of Claim 3. Since $p_{12}^h = 0$, if $h > 3$, and $p_{12}^h \neq 0$, if $h = 3$, we have $r_{12}^h = 0$ if $h > 0$, and $r_{12}^h \neq 0$, if $h = 3$. Note that $\theta_1^* \neq \theta_2^*$.

Now we are done by Claim 2.

Claim 4. There exist $\beta, \gamma, \delta \in \mathbb{R}$ such that

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(AA^* + A^* A) - \delta A^*] \quad (25.4)$$

$$= A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A). \quad (25.5)$$

Proof of Claim 4. There exists $f_i \in \mathbb{R}[\lambda]$, $\deg f_i = i$ such that $A_i = f_i(A_1)$.

Writing A_2, A_3 as polynomials in A in Claim 3 and simplifying, we find

$$A^3 A^* - A^* A^3 \in \text{Span}(A^2 A^* A - AA^* A^2, A^2 A^* - A^* A^2, AA^* - A^* A).$$

Remark. Let $A_3 = \beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I$ with $\beta_3 \neq 0$, and $A_2 = \gamma_2 A^2 + \gamma_1 A + \gamma_0 I$, with $\gamma_2 \neq 0$. Then

$$A^* A_3 - A_3 A^* = A^*(\beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I) - (\beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I)A^*. \quad (25.6)$$

$$A^3 A^* - A^* A^3 \in \text{Span}(A^* A_3 - A_3 A^*, A^2 A^* - A^* A^2, AA^* - A^* A) \quad (25.7)$$

$$\subseteq \text{Span}(AA^* A_2 - A_2 A^* A, A^* A_2 - A_2 A^*, A^2 A^* - A^* A^2, AA^* - A^* A) \quad (25.8)$$

$$A^* A_2 - A_2 A^* = A^*(\gamma_2 A^2 + \gamma_1 A + \gamma_0 I) - (\gamma_2 A^2 + \gamma_1 A + \gamma_0 I)A^* \quad (25.9)$$

$$AA^* A_2 - A_2 A^* A = AA^*(\gamma_2 A^2 + \gamma_1 A + \gamma_0 I) - (\gamma_2 A^2 + \gamma_1 A + \gamma_0 I)A^* A \quad (25.10)$$

$$A^* A_2 - A_2 A^* \in \text{Span}(A^2 A^* - A^* A^2, AA^* - AA^*) \quad (25.11)$$

$$AA^* A_2 - A_2 A^* A \in \text{Span}(A^2 A^* A - AA^* A^2, AA^* - AA^*) \quad (25.12)$$

$$A^3 A^* - A^* A^3 \in \text{Span}(A^2 A^* A - AA^* A^2, A^2 A^* - A^* A^2, AA^* - A^* A). \quad (25.13)$$

Hence, we can find δ, γ, δ satisfying

$$0 = A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A).$$

On the other hand,

$$[A, A^2 A^* - \beta AA^* A + A^* A^2 - \gamma(AA^* + A^* A) - \delta A^*] \quad (25.14)$$

$$= A^3 A^* - A^2 A^* A - \beta A^2 A^* A + \beta AA^* A^2 + AA^* A^2 - A^* A^3 \quad (25.15)$$

$$- \gamma A^2 A^* - \gamma AA^* A + \gamma AA^* A + \gamma A^* A^2 - \delta AA^* + \delta A^* A \quad (25.16)$$

$$= A^3 A^* - A^* A^3 - (\beta + 1)(A^2 A^* A - AA^* A^2) - \gamma(A^2 A^* - A^* A^2) - \delta(AA^* - A^* A). \quad (25.17)$$

Thus we have (i) and (ii).

Define a diagram D_E on nodes $0, 1, \dots, D$.

Connect distinct nodes, by undirected arc if $q_{ij}^1 \neq 0$. (Note $q_{ij}^1 = q_{ji}^1$).

Since $q_{0j}^1 = \delta_{1j}$, the 0-node is adjacent to the 1-node and no other node.

Y is Q -polynomial with respect to E if and only if E_E is a path.

Claim 5. D_E is connected.

Proof of Claim 5. Suppose there exists $\Delta \subseteq \{0, 1, \dots, D\}$ such that i, j not connected for every $i \in \Delta$ and $j \in \{0, 1, \dots, D\} \setminus \Delta$.

Set

$$f = \sum_{i \in \Delta} E_i.$$

Observe

$$fA^* = \sum_{i \in \Delta} E_i A^* \left(\sum_{j=0}^D E_j \right) \quad (25.18)$$

$$= \sum_{i \in \Delta, j \in \Delta} E_i A^* E_j \quad (\text{since } E_i A^* E_j = O \text{ if } q_{ij}^1 = 0) \quad (25.19)$$

$$= fA^* f. \quad (25.20)$$

Also, $A^* f = fA^* f$.

Hence, f commutes with A^* .

But f is an element of the Bose-Mesner algebra

$$f = \sum_{i=0}^D \alpha_i A_i \quad \text{for some } \alpha_0, \dots, \alpha_D \in \mathbb{C}.$$

We have

$$0 = fA^* - A^* f = \sum_{i=1}^D \alpha_i (A_i A^* - A^* A_i).$$

But $\{A_h A^* - A^* A_h \mid 1 \leq h \leq D\}$ are linearly independent. (The column w of $A_h A^* - A^* A_h$ is $\theta_h^* - \theta_0^*$ times the column w of A_h .)

Hence, $\alpha_1 = \dots = \alpha_D = 0$, and $f = \alpha_0 I$. Since $f^2 = f$, α_0 is 0 or 1.

If $\alpha_0 = 0$, $f = O$ and $\Delta = \emptyset$.

If $\alpha_0 = 1$, $f = I$ and $\Delta = \{0, 1, \dots, D\}$.

This proves Claim 5. □

Remark. Claim 5 proves the following in general.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a symmetric association scheme. Fix a vertex $x \in X$, and let

$$E = \frac{1}{|X|} \sum_{j=0}^D \theta_j^* A_j \quad (\theta_j^* = q_1(j) \text{ if } E = E_1)$$

be a primitive idempotent and $E_j^* \equiv E_j^*(x)$.

$$A^* = \sum_{j=0}^D \theta_j^* E_j^*.$$

If $\theta_0 = \theta_h^*$, $h = 1, \dots, D$, then the following hold.

(i) $\{A_h A^* - A^* A_h \mid 1 \leq h \leq D\}$ are linearly independent.

(ii) The diagram D_E on nodes $0, 1, \dots, D$ defined by

$$i \sim j \Leftrightarrow E(E_i \circ E_j) \neq O$$

is connected.

$$(iii) C_M(A^*) = \{L \in M \mid LA^* = A^*L\} = \text{Span}(I).$$

Proof. | (i) The column x of $A_h A^* - A^*(A_h)$ is $\theta_0^* - \theta_h^*$ times the column x of A_h .

$$(iii) 0 = [\sum_{h=0}^D \alpha_h A_h, A^*] = \sum_{h=1}^D \alpha_h (A_h A^* - A^* A_h). \text{ Hence, } \alpha_0 = \dots = \alpha_D = 0.$$

(ii) Δ is a connected component. Let $f = \sum_{i \in \Delta} E_i$, then $f \in C_M(A^*)$.

Let $Y = (X, \{R_i\}_{0 \leq i \leq 2})$ be a symmetric association scheme with $D = 2$. Let

$$E = \frac{1}{|X|} \sum_{j=0}^2 \theta_j^* A_j$$

be a primitive idempotent. If $\theta_0^*, \theta_1^*, \theta_2^*$.

Then Y is Q -polynomial with respect to E .

Proof. By the previous lemma, D_E is connected.

Note. It seems $\theta_1^* \neq \theta_2^*$ is necessary. Clarify the condition $\theta_1^* = \theta_2^*$.

Terwilliger claims that $\theta_1^* = \theta_2^*$ does not occur under the assumption (ic).
(March 7, 1995)

Chapter 26

Representation Diagrams

Wednesday, March 31, 1993

Proof of Theorem 24.1 continued. Assume $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is P -polynomial. Let E be a primitive idempotent of Y such that the corresponding representation (ρ, H) is nondegenerate.

Show for all $x, y \in X$,

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') \in \text{Span}(\rho(x) - \rho(y))$$

implies that Y is Q -polynomial with respect to E .

Define a diagram D_E on nodes $0, 1, \dots, D$, for $i \neq j$,

$$i \frown j \leftrightarrow q_{ij}^1 \neq 0$$

by setting $E = E_1$.

We showed that $0 \frown j \leftrightarrow j = 1$ ($1 \leq j \leq D$) and D_E is connected.

Now it is sufficient to show the following.

Claim 6. Let i be a node in D_E . Then i is adjacent to at most 2 arcs.

Proof of Claim 6. Suppose the node j is adjacent to i in D_E . By claim 4,

$$0 = E_i(A^3A^* - A^*A^3 - (\beta + 1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A) - \delta(AA^* - A^*A))E_j \quad (26.1)$$

$$= E_iA^*E_j(\theta_i^3 - \theta_j^3 - (\beta + 1)(\theta_i^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j)) \quad (26.2)$$

$$= E_iA^*A_j(\theta_i - \theta_j)p(\theta_i, \theta_j), \quad (26.3)$$

where

$$p(s, t) = s^2 - \beta st + t^2 - \gamma(s + t) - \delta.$$

Remark.

$$(\theta_i - \theta_j)(\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \delta) \quad (26.4)$$

$$= \theta_i^3 - \theta_j^3 - (\beta + 1)(\theta^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j) \quad (26.5)$$

Since i is adjacent to j , $q_{ij}^1 \neq 0$ and

$$E_i A^* E_j \neq O$$

by Lemma 20.3 (ii). Since Y is P -polynomial,

$$\theta_i \neq \theta_j \quad \text{if } i \neq j.$$

Hence $p(\theta_i, \theta_j) = 0$. But p is quadratic in t . So $p(\theta_i, t) = 0$ has at most two solutions for θ_j .

Now D_E is a pth, and Γ is Q -polynomial with respect to E .

This proves Theorem 24.1. □

Corollary 26.1. *Assume $Y = (X, \{R_i\}_{0 \leq i \leq D})$ is P -polynomial, and Q -polynomial with respect to a primitive idempotent*

$$E = \frac{1}{|X|} \sum_{i=0}^D \theta_i^* A_i.$$

Then,

$$\beta = \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*}$$

is independent of i ($0 \leq i \leq D-3$).

Proof. Fix i . Without loss of generality, $D \geq 3$, else vacuous.

Pick $x, y \in X$ with $(x, y) \in R_3$.

Let (ρ, H) be the representation for E .

$$\sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z') = \frac{p_{12}^3(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_3^*} (\rho(x) - \rho(y)), \quad (26.6)$$

and $p_{12}^3 = c_3$.

Since $p_{i, i+3}^3 \neq 0$, there exists $w \in X$ such that $(x, w) \in R_{i+3}$, $(y, w) \in R_i$.

Take inner product of (26.6) with $\rho(w)$. We have

$$P_{12}^3(x, y) \subseteq P_{1, i+2}^{i+3}(x, w) \cap P_{2, i+2}^i(y, w) \quad (26.7)$$

$$P_{21}^3(x, y) \subseteq P_{2, i+1}^{i+3}(x, w) \cap P_{2, i+1}^i(y, w). \quad (26.8)$$

Hence,

$$\left\langle \sum_{z \in X, (x, z) \in R_1, (y, z) \in R_2} \rho(z) - \sum_{z' \in X, (x, z') \in R_2, (y, z') \in R_1} \rho(z'), \rho(w) \right\rangle = |X|^{-1} c_3 (\theta_{i+2}^* - \theta_{i+1}^*),$$

$$\left\langle \frac{c_3(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_3^*} (\rho(x) - \rho(y)), \rho(w) \right\rangle = \frac{c_3(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_3^*} |X|^{-1} (\theta_{i+3}^* - \theta_{i+1}^*).$$

We have,

$$\sigma = \frac{\theta_{i+1}^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+3}^*} = \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_3^*}.$$

Remark. Note that since Y is P and Q with respect to A_1 and E_1 , $\theta_0^*, \theta_1^*, \dots, \theta_D^*$, $\theta_0, \theta_1, \dots, \theta_D$ are all distinct.

So

$$\beta = \frac{1}{\sigma} - 1 = \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} = \frac{\theta_0^* - \theta_1^* + \theta_2^* - \theta_3^*}{\theta_1^* - \theta_2^*}.$$

We have the assertion. \square

Given the intersection number of a distance-regular graph Γ . The following 2 lemmas give an efficient method to determine if Γ is Q -polynomial with respect to some primitive idempotent.

Lemma 26.1. *Let Γ be a distance-regular graph of diameter $D \geq 1$. Pick $\theta, \theta_0^*, \theta_1^*, \dots, \theta_D^* \in \mathbb{R}$ such that $\theta_0^* \neq 0$, and set*

$$E = \frac{1}{|X|} \sum_{i=0}^D \theta_i^* A_i.$$

(i) *The following are equivalent.*

(ia) *θ is an eigenvalue of Γ , and E is a corresponding primitive idempotent.*

(ib)

$$\begin{pmatrix} a_0 & b_0 & 0 & \cdots & \cdots & 0 \\ c_1 & a_1 & b_1 & 0 & \cdots & 0 \\ 0 & c_2 & a_2 & b_2 & \ddots & \vdots \\ \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_{D-1} & a_{D-1} & b_{D-1} \\ 0 & \cdots & \cdots & 0 & c_D & a_D \end{pmatrix} \begin{pmatrix} \theta_0^* \\ \theta_1^* \\ \vdots \\ \vdots \\ \vdots \\ \theta_D^* \end{pmatrix} = \theta \cdots \begin{pmatrix} \theta_0^* \\ \theta_1^* \\ \vdots \\ \vdots \\ \vdots \\ \theta_D^* \end{pmatrix},$$

and $\theta_0^* = \text{rank } E$.

(ii) *Suppose (ia), (ib) hold. Then,*

$$\frac{\theta_1^*}{\theta_0^*}, \dots, \frac{\theta_D^*}{\theta_0^*}$$

can be computed from θ using

$$\frac{\theta_i^*}{\theta_0^*} = \frac{p_i(\theta)}{kb_1 \cdots b_{i-1}}, \quad (1 \leq i \leq D),$$

where $p_0 = 1$, $p_1(\lambda) = \lambda$, and

$$\lambda p_i(\lambda) = p_{i+1}(\lambda) + a_i p_i(\lambda) + b_{i-1} c_i p_{i-1}(\lambda) \quad (0 \leq i \leq D).$$

Proof.

(i) We have

$$(ia) \leftrightarrow (A - \theta I)E = O \text{ and } E^2 = E \quad (26.9)$$

$$\leftrightarrow 0 = \sum_{i=0}^D (A - \theta I) \theta_i^* A_i \text{ and } \text{rank} E = \text{trace} E = \theta_0^* \quad (26.10)$$

$$= \sum_{i=0}^D \theta_i^* (c_{i+1} A_{i+1} + a_i A_i + b_{i-1} A_{i-1} - \theta A_i) \quad (26.11)$$

$$= \sum_{j=0}^D A_j (c_j \theta_{j-1}^* + a_j \theta_j^* + b_j \theta_{j+1}^* - \theta \theta_j^*) \quad (26.12)$$

$$\leftrightarrow c_j \theta_{j-1}^* + a_j \theta_j^* + b_j \theta_{j+1}^* = \theta \theta_j^* \quad (0 \leq j \leq D) \text{ and } \text{rank} E = \theta_0^* \quad (26.13)$$

$$\leftrightarrow (ib). \quad (26.14)$$

Remark. The first \leftrightarrow . \rightarrow is clear.

\leftarrow : By the first condition, $AE = \theta E$. So E is a scalar multiple of the primitive idempotent corresponding to θ . Hence, $\text{rank} E = \text{trace} E$ implies E is the primitive idempotent.

(ii) We prove by induction on i .

$i = 0$ is trivial.

$i = 1$: Set $j = 0$ above $c_0 = 0, a_0 = 0, b_0 = k$. We have

$$k\theta_1^* = \theta\theta_0^*.$$

So

$$\frac{\theta_1^*}{\theta_0^*} = \frac{\theta}{k} = \frac{p_1(\theta)}{k}.$$

$i \geq 2$: Set $j = i - 1$ above. We have

$$c_{i-2}\theta_{i-2}^* + a_{i-1}\theta_{i-1}^* + b_{i-1}\theta_i^* = \theta\theta_{i-1}^*.$$

So,

$$\frac{\theta_i^*}{\theta_0^*} = \frac{\theta\theta_{i-1}^* - a_{i-1}\theta_{i-1}^* - c_{i-1}\theta_{i-2}^*}{b_{i-1}\theta_0^*} \quad (26.15)$$

$$= \left((\theta - a_{i-1}) \frac{\theta_{i-1}^*}{\theta_0^*} - c_{i-1} \frac{\theta_{i-2}^*}{\theta_0^*} \right) \frac{1}{b_{i-1}} \quad (26.16)$$

$$= \left((\theta - a_{i-1}) \frac{p_{i-1}(\theta)}{kb_1 \cdots b_{i-2}} - c_{i-1} \frac{p_{i-2}(\theta)}{kb_1 \cdots b_{i-3}} \right) \frac{1}{b_{i-1}} \quad (26.17)$$

$$= \frac{p_i(\theta)}{kb_1 \cdots b_{i-2}b_{i-1}}, \quad (26.18)$$

as desired.

□

Chapter 27

P -and Q -Polynomial Schemes

Friday, April 2, 1993

Theorem 27.1. *Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$.*

Let θ denote an eigenvalue of Γ with associated primitive idempotent

$$E = \frac{1}{|X|} \sum_{i=0}^D \theta_i^* A_i.$$

Then the following are equivalent.

- (i) Γ is Q -polynomial with respect to E .
- (ii) $\theta_0^* \neq \theta_h^*$ for all $h \in \{1, 2, \dots, D\}$ and for $i \in \{3, \dots, D\}$,

$$c_i \left(\theta_2^* - \theta_i^* - \frac{(\theta_1^* - \theta_{i-1}^*)^2}{\theta_0^* - \theta_i^*} \right) + b_{i-1} \left(\theta_2^* - \theta_{i-1}^* - \frac{(\theta_1^* - \theta_i^*)^2}{\theta_0^* - \theta_{i-1}^*} \right) \quad (27.1)$$

$$= (k - \theta)(\theta_1^* + \theta_2^* - \theta_{i-1}^* - \theta_i^*) - (\theta + 1)(\theta_0^* - \theta_2^*) \quad (27.2)$$

- (iii) $\theta_0^* \neq \theta_h^*$ for all $h \in \{1, 2, \dots, D\}$ and (27.2) holds for $i = 3$.

Remark. Note (27.2) is trivial for $i = 1, 2$.

$i = 1$:

$$\text{LHS} = \left(\theta_2^* - \theta_1^* - \frac{\theta_1^* - \theta_0^*}{\theta^* - \theta_1^*} \right) + k(\theta^* - \theta_0^*) \quad (27.3)$$

$$= \theta_2^* - \theta_1^* - \theta_0^* + \theta_1^* + k(\theta_2^* - \theta_0^*) \quad (27.4)$$

$$= (k+1)(\theta_2^* - \theta_0^*) \quad (27.5)$$

$$\text{RHS} = (k-\theta)(\theta_1^* + \theta_2^* - \theta_0^* - \theta_1^*) - (\theta+1)(\theta_0^* - \theta_2^*) \quad (27.6)$$

$$= (k+1)(\theta_2^* - \theta_0^*). \quad (27.7)$$

$i = 2$:

$$\text{LHS} = b_1 \left(\theta_2^* - \theta_1^* - \frac{\theta_1^* - \theta_0^*}{\theta^* - \theta_1^*} \right) \quad (27.8)$$

$$= b_1 \frac{(\theta_2^* - \theta_1^*)(\theta_0^* - \theta_1^* - \theta_2^* + \theta_1^*)}{\theta_0^* - \theta_1^*} \quad (27.9)$$

$$= b_1 \frac{(\theta_2^* - \theta_1^*)(\theta_0^* - \theta_2^*)}{\theta_0^* - \theta_1^*} \quad (27.10)$$

$$\text{RHS} = (\theta+1)(\theta_0^* - \theta_2^*). \quad (27.11)$$

Hence,

$$\text{LHS} = \text{RHS} \leftrightarrow b_1 \frac{\theta_2^* - \theta_1^*}{\theta_0^* - \theta_1^*} + (\theta+1) = 0 \quad (27.12)$$

$$= b_1(\theta_2^* - \theta_1^*) + (\theta+1)(\theta_0^* - \theta_1^*) = 0. \quad (27.13)$$

On the other hand,

$$b_1\theta_2^* + a_1\theta_1^* + c_1\theta_0^* = \theta\theta_1^* \quad (27.14)$$

$$b_1\theta_1^* + a_1\theta_1^* + c_1\theta_1^* = k\theta_1^*, \quad (27.15)$$

as $\theta\theta_0^* = k\theta_1^*$ We have

$$b_1(\theta_2^* - \theta_1^*) + (\theta_0^* - \theta_1^*) = \theta(\theta_1^* - \theta_0^*).$$

Proof. Immediate from the proof of Theorem 2.1 in ‘A new inequality for distance-regular graphs’ (Terwilliger, 1995) and Theorem 24.1. \square

Note. Suppose (i) – (iii) hold. In particular, $\theta_0^*, \theta_1^*, \dots, \theta_D^*$ are distinct. Then,

$$c_i + a_i + b_i = k \quad (0 \leq i \leq D).$$

$$c_i\theta_{i-1}^* + a_i\theta_i^* + b_i\theta_{i+1}^* = \theta\theta_j^* \quad (0 \leq i \leq D).$$

$$\frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i-3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} \quad \text{is independent of } i \quad (0 \leq i \leq D-3).$$

$$c_i \left(\theta_2^* - \theta_i^* - \frac{(\theta_1^* - \theta_{i-1}^*)^2}{\theta_0^* - \theta_i^*} \right) + b_{i-1} \left(\theta_2^* - \theta_{i-1}^* - \frac{(\theta_1^* - \theta_i^*)^2}{\theta_0^* - \theta_{i-1}^*} \right) \quad (27.16)$$

$$= (k - \theta)(\theta_1^* + \theta_2^* - \theta_{i-1}^* - \theta_i^*) - (\theta + 1)(\theta_0^* - \theta_2^*). \quad (27.17)$$

Furthermore, we can show for $c_1, \dots, c_D, a_1, \dots, a_D, b_0, b_1, \dots, b_{D-1}$ in terms of 5 parameters.

In general, we can take the 5 parameters to be

$$D, q, s^*, r_1, r_2$$

and get

$$b_i = \frac{h(1 - q^{i-D})(1 - s^*q^{i+1})(1 - r_1q^{i+1})(1 - r_2q^{i+1})}{(1 - s^*q^{2i+1})(1 - s^*q^{2i+2})} \quad (0 \leq i \leq D), \quad (27.18)$$

$$c_i = \frac{h(1 - q^i)(1 - s^*q^{D+i+1})(r_1 - s^*q^i)(r_2 - s^*q^i)}{s^*q^D(1 - s^*q^{2i})(1 - s^*q^{2i+1})} \quad (0 \leq i \leq D), \quad (27.19)$$

$$a_i = b_0 - c_i - b_i \quad (0 \leq i \leq D), \quad (27.20)$$

where h variable is chosen so that $c_1 = 1$.

(We must also consider limiting cases $h \rightarrow 0, s^* \rightarrow 0, q^* \rightarrow \pm 1$.)

See Theorem 2.1 in “The subconstituent algebra of an association scheme, I, II, III, (Terwilliger, 1992), (Terwilliger, 1993a), (Terwilliger, 1993b).

Definition 27.1. Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$. Choose $q \in \mathbb{R} \setminus \{0, -1\}$, set

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = 1 + q + \dots + q^{i-1} = \begin{cases} \frac{q^i - 1}{q - 1} & q \neq 1 \\ i & q = 1. \end{cases}$$

Definition 27.2. Γ has classical parameters if

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \quad (27.21)$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (27.22)$$

for some $\sigma, \alpha \in \mathbb{R}$.

(This happens for essentially all known families of distance-regular graphs with unbounded diameter, and is essentially equivalent to $s^* = 0$.)

Lemma 27.1. *With above notation, suppose (27.21), (27.22) hold. Then,*

(i) $\theta = \frac{b_1}{q} - 1$ is an eigenvalue of Γ with $\theta \neq k$.

(ii) Let $E = |X|^{-1} \sum_{i=0}^D \theta_i^* A_i$ be associated primitive idempotent. Then

$$\frac{\theta_i^*}{\theta_0^*} = 1 + \left(\frac{\theta}{k} - 1 \right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \quad (0 \leq i \leq D).$$

In particular, $\theta_i^* \neq \theta_0^*$ for all $i \in \{1, 2, \dots, D\}$.

(iii) Γ is Q -polynomial with respect to E .

Proof.

(i), (ii). Need to check

$$c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta \theta_i^* \quad (0 \leq i \leq D),$$

where $a_i = k - c_i - b_i \quad (0 \leq i \leq D)$.

(equivalently: check

$$c_i(\theta_{i-1}^* - \theta_i^*) + b_i(\theta_i^* - \theta_{i+1}^*) = (\theta - k)\theta_i^* \quad (0 \leq i \leq D), \quad (27.23)$$

where $c_i, b_i, \theta_i^*, \theta$ are as given.)

Remark.

$$\theta = \frac{b_1}{q} - 1, \quad \frac{\theta_i^*}{\theta_0^*} = 1 + \left(\frac{\theta}{k} - 1 \right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i}, \quad b_0 = \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma.$$

$i = 0$.

$$\frac{\theta_i^*}{\theta_0^*} = \frac{\theta}{k}, \quad -k \left(1 - \frac{\theta_1^*}{\theta_0^*} \right) = -k \left(1 - \frac{\theta}{k} \right) = \theta - k.$$

$$\frac{\theta_{i-1}^* - \theta_i^*}{\theta_0^*} = \left(\frac{\theta}{k} - 1 \right) \left(\begin{bmatrix} i-1 \\ 1 \end{bmatrix} q^{2-i} - \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) = - \left(\frac{\theta}{k} - 1 \right) q^{1-i}.$$

$$\theta - k = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - 1 \right) (\sigma - \alpha)/q - 1 \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma = \begin{bmatrix} D-1 \\ 1 \end{bmatrix} (\sigma - \alpha) - 1 - \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma.$$

$$(c_i(\theta_{i-1}^* - \theta_i^*) + b_i(\theta_i^* - \theta_{i+1}^*) - (\theta - k)\theta_i^*)/\theta_0^* \quad (27.24)$$

$$= - \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \left(\frac{\theta}{k} - 1 \right) q^{1-i} + \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\frac{\theta}{k} - 1 \right) q^{-i} \quad (27.25)$$

$$- (\theta - k) \left(1 + \left(\frac{\theta}{k} - 1 \right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) \quad (27.26)$$

$$= \left(\frac{\theta}{k} - 1 \right) \left(- \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) q^{1-i} + \begin{bmatrix} D-i \\ 1 \end{bmatrix} \left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \right) \quad (27.27)$$

$$- \left(\begin{bmatrix} D \\ 1 \end{bmatrix} \sigma + \left(\begin{bmatrix} D-1 \\ 1 \end{bmatrix} (\sigma - \alpha) - 1 - \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma \right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) \quad (27.28)$$

$$= \left(\frac{\theta}{k} - 1 \right) \left(- \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} - \alpha \left(\begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} q^{1-i} + \begin{bmatrix} D-i \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) \right) \quad (27.29)$$

$$+ \sigma \left(\begin{bmatrix} D-i \\ 1 \end{bmatrix} - \begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} + \begin{bmatrix} D \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \right) + \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \quad (27.30)$$

Check $\theta \neq k$. Suppose $\theta = k$. Then

$$\frac{b_1}{q} - 1 = k, \quad \text{and } q > 0.$$

By (27.21), (27.22),

$$qc_i - b_i - q(qc_{i-1} - b_{i-1}) = (k - \theta)q \quad (1 \leq i \leq D) \quad (27.31)$$

$$= 0. \quad (27.32)$$

Remark. With the notation of Lemma 27.1, we have the above equality in

general.

$$qc_i - b_i - q(qc_{i-1} - b_{i-1}) \quad (27.33)$$

$$= q \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) - \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (27.34)$$

$$- q \left(q \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-2 \\ 1 \end{bmatrix} \right) - \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \right) \quad (27.35)$$

$$= \left(q \begin{bmatrix} i \\ 1 \end{bmatrix} - q^2 \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \quad (27.36)$$

$$+ \alpha \left(q \begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} + \begin{bmatrix} D \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (27.37)$$

$$- q^2 \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \begin{bmatrix} i-2 \\ 1 \end{bmatrix} - q \begin{bmatrix} D \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} + q \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \quad (27.38)$$

$$+ \sigma \left(- \begin{bmatrix} D \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ 1 \end{bmatrix} + q \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \quad (27.39)$$

$$= q + \alpha \left(- \begin{bmatrix} i \\ 1 \end{bmatrix} + \begin{bmatrix} D \\ 1 \end{bmatrix} + q \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) + \sigma(q^D - 1 + 1) \quad (27.40)$$

$$= q \left(1 + \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \alpha + q^{D-1} \sigma \right) \quad (27.41)$$

$$= q \left(\begin{bmatrix} D \\ 1 \end{bmatrix} \sigma - \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \sigma + \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \alpha + 1 \right) \quad (27.42)$$

$$= q \left(k - \frac{\begin{bmatrix} D \\ 1 \end{bmatrix} - 1}{q} (\sigma - \alpha) + 1 \right) \quad (27.43)$$

$$= q(k - \theta). \quad (27.44)$$

Hence,

$$qc_i - b_i = q(qc_{i-1} - b_{i-1}) \quad (1 \leq i \leq D) \quad (27.45)$$

$$= q^i(qc_0 - b_0) \quad (27.46)$$

$$= -q^i k. \quad (27.47)$$

If $i = D$, $qc_D = -q^D k$, $c_D = -q^{D-1} k < 0$, a contradiction.

(iii) Check the equation (ii) of Theorem 27.1 holds for $i = 3$.

Remark. $\theta_0^* \neq \theta_h^*$ for all $h \in \{1, 2, \dots, D\}$ and

$$c_3 \left(\theta_2^* - \theta_3^* - \frac{(\theta_1^* - \theta_2^*)^2}{\theta_0^* - \theta_3^*} \right) - b_2 \frac{(\theta_1^* - \theta_3^*)^2}{\theta_0^* - \theta_2^*} = (k - \theta)(\theta_1^* - \theta_3^*) - (\theta + 1)(\theta_0^* - \theta_2^*).$$

□

Chapter 28

Title of the Chapter

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Index

- Q -polynomial, 126, 138
- association matrix, 104
- association scheme, 104
- automorphism, 19
- bipartite, 13, 17
- Cayley graph, 19
- character, 22
- complete graph, 92
- connected, 10
- diameter, 29
- distance, 10
- distance-regular, 69
- distance-transitive, 50
- dual associate matrix, 113
- dual Bose-Mesner algebra, 113
- dual thin, 42
- endpoint, 29
- graph, 7
- isomorphic, 32
- measure, 58
- module, 10
- multiplicity, 32
- Norton algebra, 122
- path, 9
- reducible, 13
- regular, 9
- restricted, 42
- subconstituent algebra, 10
- symmetrix, 104
- thin, 42
- valency, 9
- vertex transitive, 19