Lecture Note on Terwilliger Algebra

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Setting

sudo This note is created by `bookdown` package on RStudio.

For `bookdown` See (Xie, 2015), (Xie, 2017), (Yihui Xie, 2018).

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Chapter 1

Subconstituent Algebra of a Graph

Wednesday, January 20, 1993

A graph (undirected, without loops or multiple edges) is a pair $\Gamma = (X, E)$, where

$$X = \text{finite set (of vertices)} \tag{1.1}$$
$$E = \text{set of (distinct) 2-element subsets of } X \ (= \text{edges of }) \ \Gamma. \tag{1.2}$$

vertices x and $y \in X$ are adjacent if and only if $xy \in E$.

Example 1.1. Let Γ be a graph. $X = \{a, b, c, d\}$, $E = \{ab, ac, bc, bd\}$.

Set $n = |X|$, the order of Γ.

Pick a field K ($= \mathbb{R}$ or $\mathbb{C}$). Then $\mathrm{Mat}_X(K)$ denotes the K algebra of all $n \times n$ matrices with entries in K. (rows and columns are indexed by X)

Adjacency matrix $A \in \mathrm{Mat}_X(K)$ is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E \\ 0 & \text{else .} \end{cases} \tag{1.3}$$

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Example 1.2. Let a, b, c, d be labels of rows and columns. Then

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{cccc} a & b & c & d \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The subalgebra M of $\mathrm{Mat}_X(K)$ generated by A is called the *Bose-Mesner algebra* of Γ.

Set $V = K^n$, the set of n-dimensional column vectors, the coorinates are indexed by X.

Let $\langle\,,\,\rangle$ denote the Hermitean inner product:

$$\langle u, v \rangle = u^\top \cdot v \quad (u, v \in V)$$

V with $\langle\,,\,\rangle$ is the *standard module* of Γ.

M acts on V: For every $x \in X$, write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where 1 is at the x position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since A is a real symmetrix matrix,

$$V = V_0 + V_1 + \cdots + V_r \quad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal A-eigenspaces.

Let $E_i \in \mathrm{Mat}_X(K)$ denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then $E_0, \ldots, E_r$ are the primitive idempotents of M.

$$M = \mathrm{Span}_K(E_0, \ldots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all } i, j, \quad E_0 + \cdots + E_r = I.$$

Let θ_i denote the eigenvalue of A for V_i in $\mathbb{R}$. Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \cdots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \theta_i = \dim V_i = \text{rank} E_i.$$

Set

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \cdots, & \theta_r \\ m_0, & m_1, & \cdots, & m_r \end{pmatrix}.$$

Problem. What can we say about Γ when $\text{Spec}(\Gamma)$ is given?

The following Lemma1.1, is an example of Problem.

For every $x \in X$,

$$k(x) \equiv \text{ valency of } x \equiv \text{ degree of } x \equiv |\{y \mid y \in X, \ xy \in E\}|.$$

Definition 1.1. The graph Γ is regular of valency k if $k = k(x)$ for every $x \in X$.

Lemma 1.1. *With the above notation,*

(i) $\theta_0 \leq \max\{k(x) \mid x \in X\} = k^{\max}$.
(ii) If Γ is regular of valency k, then $\theta_0 = k$.

Proof.

(i) Without loss of generality we may assume that $\theta_0 > 0$, else done. Let $v := \sum_{x \in X} \alpha_x \hat{x}$ denote the eivenvector for θ_0.

Pick $x \in X$ with $|\alpha_x|$ maximal. Then $|\alpha_x| \neq 0$.

Since $Av = \theta_0 v$,

$$\theta_0 \alpha_x = \sum_{y \in X, xy \in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \leq \sum_{y \in X, xy \in E} |\alpha_y| \leq k(x) |\alpha_x| \leq k^{\max} |\alpha_x|.$$

(ii) All 1's vector $v = \sum_{x \in X} \hat{x}$ satisfies $Av = kv$.

$\square$

Subconstituent Algebra

Let $x, y \in X$ and $\ell \in \mathbb{Z}^{\geq 0}$.

Definition 1.2. A path of length ℓ connecting x, y is a sequence

$$x = x_0, x_1, \ldots, x_\ell = y, \quad x_i \in X, \ 0 \leq i \leq \ell$$

such that $x_i x_{i+1} \in E$ for $0 \leq i \leq \ell - 1$.

Definition 1.3. The distance $\partial(x, y)$ is the length of a shortest path connecting x and y.

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

Definition 1.4. The graph Γ is connected if and only if $\partial(x, y) < \infty$ for all $x, y \in X$.

From now on, assume that Γ is connected with $|X| \geq 2$.

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \text{the diameter of } \Gamma.$$

Fix a 'base' vertex $x \in X$.

Definition 1.5.

$$d(x) = \text{the diameter with respect to } x = \max\{\partial(x, y) \mid y \in X\} \leq d.$$

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad \text{(orthogonal direct sum)},$$

where

$$V_i^* = \operatorname{Span}_K(\hat{y} \mid \partial(x, y) = i) \equiv V_i * (x)$$

and $V_i^* = V_i^*(x)$ is called the i-the subconstituent with respect to x.

Let $E_i^* = E_i^*(x)$ denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View $E_i^*(x) \in \operatorname{Mat}_X(K)$. So, $E_i^*(x)$ is diagonal with yy entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{else,} \end{cases} \quad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \operatorname{Span}_K(E_0^*(x), \ldots, E_{d(x)}^*(x)).$$

Then $M^*(x)$ is a commutative subalgebra of $\operatorname{Mat}_X(K)$ and is calle the *dual Bose-Mesner algbara with respect to x*.

Definition 1.6 (Subconstituent Algebra)**.** Let $\Gamma = (X, E)$, x, M, $M^*(x)$ be as above. Let $T = T(x)$ denote the subalgebra of $\operatorname{Mat}_X(K)$ generated by M and $M^*(x)$. T is the *subconstituent algebra* of Γ with respect to x.

Definition 1.7. A T-module is any subspace $W \subset V$ such that $aw \in W$ for all $a \in T$ and $w \in W$.

T-module W is *irreducible* if and only if $W \neq 0$ and W does not properly contain a nonzero T-module.

For any $a \in \mathrm{Mat}_X(K)$, let a^* denbote the conjugate transpose of a.

Observe that

$$\langle au, v \rangle = \langle u, a^*v \rangle \quad \text{for all} \ \ a \in \mathrm{Mat}_X(K), \text{ and for all} \ \ u, v \in V.$$

Lemma 1.2. *Let* $\Gamma = (X, E)$, $x \in X$ *and* $T \equiv T(x)$ *be as above.*

(i) If $a \in T$, *then* $a^* \in T$.

(ii) For any T*-module* $W \subset V$,

$$W^{\perp} := \{v \in V \mid \langle w, v \rangle = 0, \ \text{for all } w \in W\}$$

is a T*-module.*

(iii) V *decomposes as an orthogonal direct sum of irreducible* T*-modules.*

Proof.

(i) It is becase T is generated by symmetric real matrices

$$A, E_0^*(x), E_1^*(x), ..., E_{d(x)(x)}^*.$$

(ii) Pick $v \in W^{\perp}$ and $a \in T$, it suffices to show that $av \in W^{\perp}$. For all $w \in W$,

$$\langle w, av \rangle = \langle a^*w, v \rangle = 0$$

as $a^* \in T$.

(iii) This is proved by the induction on the dimension of T-modules. If W is an irreducible T-module of V, then

$$V = W + W^{\perp} \quad \text{(orthogonal direct sum).}$$

$\square$

Problem. What does the structure of the $T(x)$-module tell us about Γ?

Study those Γ whose modules take 'simple' form. The Γ's involved are highly regular.

Remark.

1. The subconstituent algebra T is semisimple as the left regular representation of T is completely reducible. See Curtis-Reiner 25.2 (Charles W. Curtis, 2006).
2. The inner product $\langle a, b \rangle_T = \mathrm{tr}(a^{\top}\bar{b})$ is nondegenerate on T.

3. In general,

$$T\text{: Semisimple and Artinian} \Leftrightarrow T\text{: Artinian with } J(T) = 0$$
$$\Leftarrow T\text{: Artinian with nonzero nilpotent element}$$
$$\Leftarrow T \subset \mathrm{Mat}_X(K) \text{ such that for all } a \in T \text{ is normal.}$$

Chapter 2

Perron-Frobenius Theorem

Friday, January 22, 1993

In this lecture we use the Perron Frobenius theory of nonnegative matrices to obtain informaiton on eigenvalues of a graph.

Let $K = \mathbb{R}$. For $n \in \mathbb{Z}^{>0}$, pick a symmetrix matrix $C \in \text{Mat}_n(\mathbb{R})$.

Definition 2.1. The matrix C is *reducible* if and only if there is a bipartition $\{1, 2, ..., n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i \in X^+$, and for all $j \in X^-$, and for all $i \in X^-$, and for all $j \in X^+$,i.e.,

$$C \sim \begin{pmatrix} * & O \\ O & * \end{pmatrix}.$$

Definition 2.2. The matrix C is *bipartite* if and only if there is a bipartition $\{1, 2, ..., n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i, j \in X^+$, and for all $i, j \in X^-$, i.e.,

$$C \sim \begin{pmatrix} O & * \\ * & O \end{pmatrix}.$$

Note.

1. If C is bipatite, for every eigenvalue θ of C, $-\theta$ is an eigenvalue of C such that $\text{mult}(\theta) = \text{mult}(-\theta)$.

Indeed, let $C = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$,

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix},$$

where $Ay = \theta x$ and $Bx = \theta y$.

2. If C is bipartite, C^2 is reducible.

3. The matrix C is irreducible and C^2 is reducible, if $C_{ij} \geq 0$ for all i, j and C is reducible. (Exercise)

Remark. Note 1. Even if C is not symmetric

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

holds. So the geometrix multiplicities coincide. How about the algebraic multiplicities?

Note 3. Set $x \sim y$ if and only if $C_{xy} > 0$. So the graph may have loops. Then

$$(C^2)_{xy} > 0 \Leftrightarrow \text{ if there exists } z \in X \text{ such that } x \sim z \sim y.$$

Note that C is irreducible if and only if $\Gamma(C)$ is connected. Let

$$X^+ = \{y \mid \text{there is a path of even length from } x \text{ to } y\} \tag{2.1}$$
$$X^- = \{y \mid \text{there is no path of even length from } x \text{ to } y\} \neq \emptyset. \tag{2.2}$$

If there is an edge $y \sim z$ in X^+ and $w \in X^-$. Then there would be a path from x to y of even length. So $e(X^+, X^+) = e(X^-, X^-) = 0..$

Theorem 2.1 (Perron-Frobenius). *Given a matrix C in $\mathrm{Mat}_n(\mathbb{R})$ such that*

(a) C is symmetric.

(b) C is irreducible.

(c) $C_{ij} \geq 0$ for all i, j.

Let θ_0 be the maximal eigenvalue of C with eigenspace $V_0 \subseteq \mathbb{R}^n$, and let θ_r be the maximal eigenvalue of C with eigenspace $V_r \subseteq \mathbb{R}^n$. Then the following hold.

(i) Suppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$. Then $\alpha_0 > 0$ for all i, or $\alpha_i < 0$ for all i.

(ii) $\dim V_0 = 1$.

(iii) $\theta_r \geq -\theta_0$.

(iv) $\theta_r = \theta_0$ if and only if C is bipartite.

First, we prove the following lemma.

Lemma 2.1. *Let $\langle \, , \, \rangle$ be the dot product in $V = \mathbb{R}^n$. Pick a symmetric matrix $B \in \mathrm{Mat}_n(\mathbb{R})$. Suppose all eigenvalues of B are nonnegative. (i.e., B is positive semidefinite.) Then there exist vectors $v_1, v_2, \ldots, v_n \in V$ such that $B_{ij} = \langle v_i, v_j \rangle$ for $(1 \leq i, j \leq n)$.*

Proof. By elementary linear algebra, there exists an orthonormal basis $w_1, w_2, \ldots, w_n$ of V consisting of eigenvectors of B. Set the i-th column of P is w_i and $D = \mathrm{diag}(\theta_1, \ldots, \theta_n)$. Then $P^\top P = I$ and $BP = PD$.

Hence,

$$B = PDP^{-1} = PDP^\top = QQ^\top,$$

where

$$Q = P \cdot \mathrm{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \ldots, \sqrt{\theta_n}) \in \mathrm{Mat}_n(\mathbb{R}).$$

Now, let v_i be the i-th column of $Q^\top$. Then

$$B_{ij} = v_i^\top \cdot v_j^- = \langle v_i, v_j \rangle.$$

$\square$

Now we start the proof of Theorem 2.1.

Proof of Theorem 2.1(i)

Let $\langle , \rangle$ denote the dot product on $V = \mathbb{R}^n$. Set

$$B = \theta I - C \tag{2.3}$$
$$= \text{symmetric matrix with eigenvalues } \theta_0 - \theta_i \geq 0 \tag{2.4}$$
$$= (\langle v_i, v_j \rangle)_{1 \leq i, j \leq n} \tag{2.5}$$

with the same $v_1, \ldots, v_n \in V$ by Lemma 2.1.

Observe: $\sum_{i=1}^n \alpha_i v_i = 0$.

Pf.

$$\left\| \sum_{i=1}^n \alpha_i v_i \right\|^2 = \left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \right\rangle \tag{2.6}$$

$$= (\alpha_1 \quad \ldots \quad \alpha_n) B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{2.7}$$

$$= v^\top B v \tag{2.8}$$

$$= 0, \tag{2.9}$$

since $Bv = (\theta_0 I - C)v = 0$.

Now set

$$s = \text{the number of indices } i, \text{ where } \alpha_i > 0.$$

Replacing v by $-v$ if necessary, without loss of generality we may assume that $s \geq 1$. We want to show $s = n$.

Assume $s < n$. Without loss of generality, we may assume that $\alpha_i > 0$ for $1 \leq s \leq s$ and $\alpha_i = 0$ for $s + 1 \leq i \leq n$. Set

$$\rho = \alpha_1 v_1 + \cdots + \alpha_s v_s = -\alpha_{s+1} v_{s+1} - \cdots - \alpha_n v_n.$$

Then, for $i = 1, \dots, s$,

$$\langle v_i, \rho \rangle = \sum_{j=s+1}^{n} -\alpha_j \langle v_i, v_j \rangle \quad (\langle v_i, v_j \rangle = B_{ij}, B = \theta_0 I - C) \qquad (2.10)$$

$$= \sum_{j=s+1}^{n} (-\alpha_{ij})(-C_{ij}) \qquad (2.11)$$

$$\leq 0. \qquad (2.12)$$

Hence

$$0 \leq \langle \rho, \rho \rangle = \sum_{i=1}^{s} \alpha_i \langle v_i, \rho \rangle \leq 0,$$

as $\alpha > 0$ and $\langle v_i, \rho \rangle \leq 0$. Thus, we have $\langle, \rho, \rho \rangle = 0$ and $\rho = 0$. For $j = s+1, \dots, n$,

$$0 = \langle \rho, v_j \rangle = \sum_{i=1}^{s} \alpha_i \langle v_i, v_j \rangle \leq 0,$$

as $\langle v_i, v_j \rangle = -C_{ij}$.

Therefore,

$$0 = \langle v_i, v_j \rangle = -C_{ij} \text{ for } 1 \leq i, \leq s, \ s+1 \leq j \leq n.$$

Since C is symmetric,

$$C = \begin{pmatrix} * & O \\ O & * \end{pmatrix}$$

Thus C is reducible, which is not the case. Hence $s = n$.

Proof of Theorem 2.1 (ii).

Suppose $\dim V_0 \geq 2$. Then,

$$\dim \left(V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{\perp} \right) \geq 1.$$

So, there is a vector

$$0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with $\alpha_1 = 0$. This contradicts 1.

Now pick

$$0 \neq w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r.$$

Proof of Theorem 2.1 (iii).

Suppose $\theta_r < -\theta_0$. Since the eigenvalues of C^2 are the squares of those of C, θ_r^2 is the maximal eigenvalue of C^2.

Also we have $C^2 w = \theta_r^2 w$.

Observe that C^2 is irreducible. (As otherwise, C is bipartite by Note 3, and we must have $\theta_r = -\theta_0$.) Therefore, $\beta_i > 0$ for all i or $\beta_i < 0$ for all i. We have

$$\langle v, w \rangle = \sum_{i=1}^{n} \alpha_i \beta_j \neq 0.$$

This is a contradiction, as $V_0 \perp V_r$.

Proof of Theorem 2.1 (iv)

$\Rightarrow$: Let $\theta_r = -\theta_0$. Then $\theta = \theta_1^2 = \theta_0^2$ is the maximal eigenvalue of C^2, and v and w are linearly independent eigenvalues for θ for C^2. Hence, for C^2, $\text{mult}(\theta) \geq 2$.

Thus by 2, C^2 must be reducible. Therefore, C is bipartite by Note 3.

$\Leftarrow$: This is Note 1. $\square$

Let $\Gamma = (X, E)$ be any graph.

Definition 2.3. Γ is said to be *bipartite* if the adjacency matrix A is bipartite. That is, X can be written as a disjoint union of X^+ and X^- such that X^+, X^- contain no edges of Γ.

Corollary 2.1. *For any (connected) graph Γ with*

$$\text{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_1 & m_1 & \cdots & m_r \end{pmatrix} \quad \text{with } \theta_0 > \theta_1 > \cdots > \theta_r.$$

Let V_i be the eigenspace of θ_i. Then the following holds.

1. *Suppppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \in \mathbb{R}^n$. Then $\alpha_i > 0$ for all i or $\alpha_i < 0$ for all i.*

2. *$m_0 = 1$.*

3. *$\theta_r \geq -\theta_0$ if and only if Γ is bipartite. In this case,*

$$-\theta_i = \theta_{r-i} \text{ and } m_i = m_{r-i} \quad (0 \leq i \leq r)$$

Proof. This is a direct consequences of Theorem 2.1 and Note 3. $\square$

Chapter 3

Cayley Graphs

Monday, January 25, 1993

Given graphs $\Gamma = (X, E)$ and $\Gamma' = (X', E')$.

Definition 3.1. A map $\sigma : X \to X'$ is an *isomorphism*\index{isomorophism of graphs whenever;

    i. σ is one-to-one and onto,
    ii. $xy \in E$ if and only if $\sigma x \sigma y \in E'$ for all $x, y \in X$.

We do not distinguish between isomorphic graphs.

Definition 3.2. Suppose $\Gamma = \Gamma'$. Above isomorphism σ is called an *automorphism* of Γ. Then set $\mathrm{Aut}(\Gamma)$ of all automorphisms of Γ becomes a finite group under composition.

Definition 3.3. If $\mathrm{Aut}(\Gamma)$ acts transitive on X, Γ is called *vertex transitive*.

Example 3.1. A Cayley graphs:

Definition 3.4 (Cayley Graphs)**.** Let G be any finite group, and Δ any generating set for G such that $1_G \notin \Delta$ and $g \in \Delta \to g^{-1} \in \Delta$. Then Cayley graph $\Gamma = \Gamma(G, \Delta)$ is defined on the vetex set $X = G$ with the edge set E define by the following.

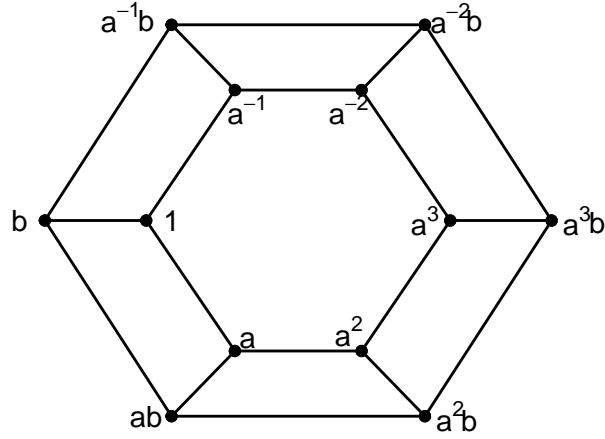$$E = \{(h_1, h_2) \mid h_1, h_2 \in G, h_1^{-1} h_2 \in \Delta\} = \{(h, hg) \mid h \in G, g \in \Delta\}$$

Example 3.2. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}\}$.

Example 3.3. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}, a^2, a^{-2}\}$.

Example 3.4. $G = \langle a, b \mid a^6 = 1, b^2, ab = ba \rangle$, $\Delta = \{a, a^{-1}, b\}$.

Remark. $\mathrm{Aut}(\Gamma) \simeq D_6 \times \mathbb{Z}_2$ contains two regular subgroups isomorphic to D_6 and $\mathbb{Z}_5 \times \mathbb{Z}_2$ and Γ is obtained as Cayley graphs in two ways.

Cayley graphs are vertex transitive, indeed.

Theorem 3.1. *The following hold.*

(i) For any Cayley graph $\Gamma = \Gamma(G, \Delta)$, the map

$$G \to \mathrm{Aut}(\Gamma) \ (g \mapsto \hat{g})$$

is an injective homomorphism of groups, where

$$\hat{g}(x) = gx \quad \text{for all } g \in G \text{ and for all } x \in X(= G).$$

Also, the image $\hat{G}$ is regular on X. i.e., the image $\hat{G}$ acts transitively on X with trivial vertex stabilizers.

(ii) For any graph $\Gamma = (X, E)$, suppose there exists a subgroup $G \subseteq \mathrm{Aut}(\Gamma)$ that is regular on X. Pick $x \in X$, and let

$$\Delta = \{g \in G \mid \langle x, g(x) \rangle \in E\}.$$

Then $1 \notin \Delta$, $g \in \Delta \to g^{-1} \in \Delta$, and Δ generates G. Moreover, $\Gamma \simeq \Gamma(G, \Delta)$.

Proof. (i) Let $g \in G$. We want to show that $\hat{g} \in \mathrm{Aut}(\Gamma)$. Let $h_1, h_2 \in X = G$. Then,

$$(h_1, h_2) \in E \to h_1^{-1}h_2 \in \Delta \tag{3.1}$$
$$\to (gh_1)^{-1}(gh_2) \in \Delta \tag{3.2}$$
$$\to (gh_1, gh_2) \in E \tag{3.3}$$
$$\to (\hat{g}(h_1), \hat{g}(h_2)) \in E. \tag{3.4}$$

Hence, $\hat{g} \in \mathrm{Aut}(\Gamma)$.

Observe: $g \mapsto \hat{g}$ is a homomorphism of groups:

$$\hat{1}_G = 1, \ \widehat{g_1 g_2} = \widehat{g_1}\widehat{g_2}.$$

Observe: $g \mapsto \hat{g}$ is one-to-one:

$$\widehat{g_1} = \widehat{g_2} \to g_1 = \widehat{g_1}(1_G) = \widehat{g_2}(1_G) = g_2.$$

Observe: $\hat{G}$ is regular on X: Clear by construction.

(ii) $1_G \notin \Delta$: Since Γ has not loops, $(x, 1_G x) \notin E$.

$g \in \Delta \to g^{-1} \in \Delta$:

$$g \in \Delta \to (x, g(x)) \in E \to E \ni (g^{-1}(x), g^{-1}(g(x))) = (g^{-1}(x), x).$$

Δ generates G: Suppose $\langle \Delta \subsetneq G$. Let $\hat{X} = \{g(x) \mid g \in \langle \Delta \rangle\} \subsetneq X$. ($\hat{X} \subsetneq X$ as G acts regularly on X.)

Since Γ is connected, there exists $y \in \hat{X}$ and $z \in X \ \hat{X}$ with $yz \in E$.

Let $y = g(x)$, $g \in \langle \Delta \rangle$, $z \in h(x)$, $h \in G \ \langle \Delta \rangle$. Then

$$(y, z) = (g(x), h(x)) \in E \to (x, g^{-1}h(x)) \in E \to g^{-1}h \in \langle \Delta \rangle \to h \in \langle \Delta \rangle.$$

This is a contradition. Therefore, Δ generates G.

Let $\Gamma' = (X', E')$ denote $\Gamma(G, \Delta)$. We shall show that

$$\theta : X' \to X \ (g \mapsto g(x))$$

is an isomorphism of graphs.

θ is one-to-one: For $h_1, h_2 \in X' = G$,

$$\theta(h_1) = \theta(h_2) \to h_1(x) = h_2(x) \to h_2^{-1}h_1(x) = x \to h_2^{-1}h_1 \in \text{Stab}_G(x) = \{1_G\} \to h_1 = h_2.$$

$(\text{Stab}_G = \{g \in G \mid g(x) = x\}.)$

θ is onto: Since G is transitive,

$$X = \{g(x) \mid g \in G\} = \theta(X') = \theta(G).$$

θ respects adjacency: For $h_1, h_2 \in X' = G$,

$$(h_1, h_2) \in E' \leftrightarrow h_1^{-1}h_2 \in \Delta \leftrightarrow (x, h_1^{-1}h_2(x)) \in E \leftrightarrow (h_1(x), h_2(x)) \in E \leftrightarrow (\theta(h_1), \theta(h_2)) \in E.$$

Therefore θ is an isomorphism between graphs $\Gamma(G, \Delta)$ and $\Gamma(X, E)$. $\square$

How to compute the eigenvalues of the Cayley graph of and abelian group.

Let G be any finite abelian group. Let $\mathbb{C}^*$ be the multiplicative group on $\mathbb{C} \ \{0\}$.

Definition 3.5. A (linear) G-character is any group homomorphism $\theta : G \to \mathbb{C}^*$.

Example 3.5. $G = \langle a \mid a^3 = 1 \rangle$ has three characters, $\theta_0, \theta_1, \theta_2$.

$$
\begin{array}{c|ccc}
\theta_i(a^j) & 1 & a & a^2 \\
\hline
\theta_0 & 1 & 1 & 1 \\
\theta_1 & 1 & \omega & \omega^2 \\
\theta_2 & 1 & \omega^2 & \omega
\end{array}
\quad , \quad \text{with } \omega = \frac{-1 + \sqrt{-3}}{2}.
$$

Here ω is a primitive cube root pf q in $\mathbb{C}^*$, i.e., $1 + \omega + \omega^2 = 0$.

For arbitraty group G, let $X(G)$ be the set of all characters of G.

Observe: For $\theta_1, \theta_2 \in X(G)$, one can define product $\$ \_1 \_2$:

$$\theta_1\theta_2(g) = \theta_1(g)\theta_2(g) \quad \text{for all } g \in G.$$

Then $\theta_1\theta_2 \in X(G)$.

Observe: $X(G)$ with this product is an (abelian) group.

Lemma 3.1. *The groups G and $X(G)$ are isomorphic for all finite abelian groups G.*

Proof. G is a direct sum of cyclic groups;

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m, \quad \text{where } G_i = \langle a_i \mid a_i^{d_i} = 1 \rangle \quad (1 \le i \le m).$$

Pick any alement ω_i of order d_i in $\mathbb{C}^*$, i.e., a primitive d_i-the root of 1. Define

$$\theta_i : G \to \mathbb{C}^* \quad (a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \mapsto \omega_i^{\varepsilon_i} \quad \text{where } 0 \le \varepsilon_i < d_i, 1 \le i \le m).$$

Then $\theta_i \in X(G)$. (Exercise)

Claim: There exists an isomorphism of groups $G \to X(G)$ that sends a_i to θ_i.

Observe: $\theta_i^{d_i} = 1$. For every $g = a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \in G$,

$$\theta_i^{d_i}(g) = (\theta_i(g))^{d_i} = (\omega_i^{\varepsilon_i})^{d_i} = (\omega_i^{d_i})^{\varepsilon_i} = 1.$$

Observe: If $\theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m} = 1$ for some $0 \le \varepsilon_i < d_i, 1 \le i \le m$. Then $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = 0$.

Pf. $1 = \theta_1^{\varepsilon_1} \theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m}(a_i) = \omega_i^{\varepsilon_i}$, Since ω_i is a primitive d_i-th root of 1, $\varepsilon_i = 0$ for $1 \le i \le m$.

Observe: $\theta_1, \ldots, \theta_m$ generate $X(G)$. Pick $\theta \in X(G)$. Since $a_i^{d_i} = 1$, $1 = \theta(a_i^{d_i}) = \theta(a_i)^{d_i}$.

Hence $\theta(a_i) = \omega^{\varepsilon_i}$ for some ε_i with $0 \le \varepsilon_i < d_i$.

Now $\theta = \theta_1^{\varepsilon_1} \cdots \theta_m^{\varepsilon_m}$, since these are both equal to $\omega_i^{\varepsilon_i}$ at a_i for $1 \le i \le m$.

Therefore,

$$G \to X(G) \quad (a_i \mapsto \theta_i)$$

is an isomorphism of groups. $\square$

Note. The correspondence above is clearly a group homomorphism.

Chapter 4

Examples

Wednesday, January 27, 1993

Theorem 4.1. *Given a Cayley graph* $\Gamma = \Gamma(G, \Delta)$. *View the standard module* $V \equiv \mathbb{C}G$ *(the group algebra), so*

$$\left\langle \sum_{g \in G} \alpha_g g, \ \sum_{g \in G} \beta_g g \right\rangle = \sum_{g \in G} \alpha_g \overline{\beta_g}, \quad \text{with } \alpha_g, \beta_g \in \mathbb{C}.$$

For any $\theta \in X(G)$, *write*

$$\hat{\theta} = \sum_{g \in G} \theta(g^{-1}) g.$$

Then the following hold.

(i) $\langle \hat{\theta}_1, \hat{\theta}_2 \rangle = |G|$ *if* $\theta_1 = \theta_2$ *and* 0 *othewise for* $\theta_1, \theta_2 \in X(G)$. *In particular,* $\{\hat{\theta} \mid \theta \in X(G)\}$ *forms a basis for* V.

(ii) $A\hat{\theta} = \Delta_\theta \hat{\theta}$ *for* $\theta \in X(G)$, *where* A *is the adjacency matrix and*

$$\Delta_\theta = \sum_{g \in \Delta} \theta(g).$$

In particular, the eigenvalues of Γ *are precisely*

$$\Delta_\theta \mid \theta \in X(G)\}.$$

Proof.

(i) Claim: For every $\theta \in X(G)$, let

$$s := \sum_{g \in G} \theta(g^{-1}) = \begin{cases} |G| & \text{if } \theta = 1 \\ 0 & \text{if } \theta \neq 1. \end{cases}$$

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Pf. Clear if $\theta = 1$.

Let $\theta \neq 1$. Then $\theta(h) \neq 1$ for some $h \in G$.

$$s \cdot \theta(h) = \left(\sum_{g \in G} \theta(g^{-1}) \right) \theta(h) = \sum_{g \in G} \theta(g^{-1}h) = \sum_{g' \in G} \theta(g'^{-1}) = s.$$

Since $\theta(h) \neq 1$, $s = 0$.

Claim. $\theta(g^{-1}) = \overline{\theta(g)}$ for every $\theta \in X(G)$ and every $g \in G$.

Since $\theta(g) \in \mathbb{C}$ is a root of 1,

$$|\theta(g)|^2 = \theta(g)\overline{\theta(g)} = 1.$$

On the other hand, since θ is a homomorphism,

$$\theta(g)\theta(g^{-1}) = \theta(1) = 1.$$

Hence $\theta(g^1) = \overline{\theta(g)}$.

Now

$$\langle \widehat{\theta_1}, \widehat{\theta_2} \rangle = \sum_{g \in G} \theta_1(g^{-1})\overline{\theta_2(g^{-1})} \tag{4.1}$$

$$= \sum_{g \in G} \theta_1(g^{-1})\theta_2(g) \tag{4.2}$$

$$= \sum_{g \in G} \theta_1\theta_2^{-1}(g^{-1}) \tag{4.3}$$

$$= \begin{cases} |G| & \text{if} \quad \theta_1\theta_2^{-1} = 1 \\ 0 & \text{if} \quad \theta_1\theta_2^{-1} \neq 1. \end{cases} \tag{4.4}$$

Since $|G| = |X(G)|$ by Lemma 3.1, and $\widehat{\theta_i}$'s are orthogonal nonzero elements in V, thet form a basis of V.

(ii) Let $\Delta = \{g_1, \dots, g_r\}$. Then

$$A\hat{\theta} = A\left(\sum_{g \in G} \theta(g^{-1}g)\right) \tag{4.5}$$

$$= \sum_{g \in G} \theta(g^{-1})(gg_1 + \cdots + gg_r) \quad (\Gamma(g) = \{gg_1, \dots, gg_r\}) \tag{4.6}$$

$$= \sum_{i=1}^{r}\left(\sum_{g \in G} \theta(g^{-1})(gg_i)\right) \tag{4.7}$$

$$= \sum_{i=1}^{r}\left(\sum_{g \in G} \theta(g_i g_i^{-1} g^{-1})(gg_i)\right) \tag{4.8}$$

$$= \sum_{i=1}^{r}\left(\sum_{g \in G} \theta(g_i)\theta((gg_i)^{-1})gg_i\right) \tag{4.9}$$

$$= \sum_{i=1}^{r} \theta(g_i) \sum_{h \in G} \theta(h^{-1})h \tag{4.10}$$

$$= \Delta_\theta \cdot \hat{\theta}. \tag{4.11}$$

Since $\{\hat{\theta} \mid \theta \in X(G)\}$ forms a basis, the eigenvalues of Γ are precisely,
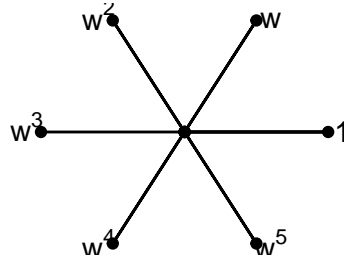
$$\{\Delta_\theta \mid \theta \in X(G)\}.$$

This completes the proof.

$\square$

Example 4.1. Let $G = \langle a \mid a^6 = 1 \rangle$, and $\Delta = \{a, a^{-1}\}$. Pick a primitive 6-th root of 1, ω. Then

$$X(G) = \{\theta^i \mid 0 \le i \le 5\} \quad \text{such that} \quad \theta(a) = \omega, \ \omega + \omega^{-1} = 1.$$

$\varphi \in X(G)$	$\varphi(a)$	$\Delta_\varphi = \theta(a) + \theta(a)^{-1}$
1	1	2
θ	ω	$\omega + \omega^{-1} = 1$
θ^2	ω^2	-1
θ^3	$\omega^3 = -1$	-2
θ^4	ω^4	-1
θ^5	ω^5	1

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

Example 4.2. D-cube, $H(D, 2)$. Let

$$X = \{(a_1, \dots, a_D) \mid a_i \in \{1, -1\}, \ 1 \le i \le D\},$$

$$E = \{xy \mid x, y \in X, \ x, y: \text{ different in exactly one coordinate}\}.$$

Also $H(D, 2)$ is a Cayley graph $\Gamma(G, \Delta)$, where

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_D,$$

$$G_i = \langle a_i \mid a_i^2 = 1 \rangle, \quad \Delta = \{a_1, \dots, a_D\}.$$

Homework: The spectrum of $H(D, 2)$ is

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_D \\ m_0 & m_1 & \cdots & m_D \end{pmatrix},$$

where

$$\theta_i = D - 2i \quad (0 \le i \le D), \quad m_i = \binom{D}{i}.$$

Remark. Let $\theta \in X(G)$. Then $\theta : X \to \{\pm 1\}$. If

$$\nu(\theta) = |\{i \mid \theta(a_i) = -1\}|,$$

then $\Delta_\theta = D - 2i$. Since there are $\binom{D}{i}$ such θ, we have te assertion.

We want to compute the subconstituent algebra for $H(D, 2)$. First, we make a few observations about arbitrary graphs.

Let $\Gamma = (X, E)$ be any graph, A, the adjacemcy matrix of Γ, and V, the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$. Write $E_i^* = E_i^*(x)$, and

$$T \equiv T(x) = \text{the algebra generated by } A, E_0^*, E_1^*, \dots.$$

Definition 4.1. Let W be any orreducible T-module $(\subseteq V)$. Then the endpoint $r \equiv r(W)$ satisfied

$$r = \min\{i \mid E_i^* W \neq 0\}.$$

The diameter $d = d(W)$ satisfied

$$d = |\{i \mid E_i^* W \neq 0\}| - 1.$$

Lemma 4.1. *With the above notation, let W be an irreducible T-module. Then*

(i) $E_i^* A E_j^* = 0$ if $|i - j| = 1$, $\neq 0$ if $|i - j| = 1$, $0 \leq i, j \leq d(x)$.
(ii) $A E_j^* W \subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$, $0 \leq j \leq d(x)$. ($E_i^* W = 0$ if $i < j$ or $i > d(x)$.)
(iii) $E_j^* W \neq 0$ if $r \leq j \leq r + d$, $= 0$ if $0 \leq j \leq r$ or $r + d < j \leq d(x)$.
(iv) $E_i^* A E_j^* W \neq 0$, if $|i - j| = 1$ $(r \leq i, j \leq r + d)$.

Proof.

(*i*) Pick $y \in X$ with $\partial(x, y) = j$. We want to find $E_i^* A E_j^* \hat{y}$. Note,

$$E_j^* \hat{y} = \begin{cases} 0 & \text{if } \partial(x.y) \neq j \\ \hat{y} & \text{if } \partial(x, y) = j. \end{cases}.$$

$$E_i^* A E_j^* \hat{y} = E_i^* A \hat{y} \tag{4.12}$$

$$= E_i^* \sum_{z \in X, yz \in E} \hat{z} \tag{4.13}$$

$$= \sum_{z \in X, yz \in E, \partial(x,z)=i} \hat{z} \quad (*) \tag{4.14}$$

$$= 0 \quad \text{if } |i - j| > 1 \text{ by triangle inequality.} \tag{4.15}$$

If $|i - j| = 1$, there exist $y, y' \in X$ such that $\partial(x, y) = j$, $\partial(x, y') = i$, $yy' \in E$ by connectivity of Γ. Hence (*) contains $\widehat{y'}$ and $* \neq 0$

(*ii*) We have

$$A E_j^* W = \left(\sum_{i=0}^{d(x)} E_i^* \right) A E_j^* W \tag{4.16}$$

$$= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W \tag{4.17}$$

$$\subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W. \tag{4.18}$$

(*iii*) Suppose $E_j^* W = 0$ for some j $(r \leq j \leq r + d)$. Then $r < j$ by the definition of r. Set

$$\tilde{W} = E_r^* W + E_{r+1}^* W + \cdots + E_{j-1}^* W.$$

Observe $0 \subsetneq \tilde{W} \subsetneq W$. Also $A\tilde{W} \subseteq \tilde{W}$ by (ii) and $E_i^* \tilde{W} \subseteq \tilde{W}$ for every i by construction.

Thus $T\tilde{W} \subseteq \tilde{W}$, contradicting W beging irreducible.

$\square$

Chapter 5

T-Modules of $H(D, 2)$, I

Friday, January 29, 1993

Let $\Gamma = (X, E)$ be a graph, A the adjacency matrix, and V the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$ and write $E_I^* \equiv E_i^*(x)$, and $T \equiv T(x)$.

Let W be an irreducible T-module with endpoint $r := \min\{i \mid E_i^* W \neq 0\}$ and diameter $d := |\{i \mid E_i^* W \neq 0\}| - 1$.

We have

$$E_i^* W \neq 0 \qquad\qquad\qquad r \leq i \leq r + d \qquad\qquad (5.1)$$
$$= 0 \qquad\qquad 0 \leq i < r \ \text{ or } \ r + d < i \leq d(x). \qquad (5.2)$$

Claim: $E_i^* A E_j^* W \neq 0$ if $|i - j| = 1$ for $r \leq i, j \leq r + d$. (See Lemma 4.1.)

Suppose $E_{j+1}^* A E_j^* W = 0$ for some j with $r \leq j < r + d$. Observe that

$$\tilde{W} = E_r^* W + \cdot E_j^* W$$

is T-invariant with

$$0 \subsetneq \tilde{W} \subsetneq W.$$

Becase $A\tilde{W} \subseteq \tilde{W}$ since $A E_j^* W \subseteq E_{j-1}^* W + E_j^* W$,

$$E_k^* \tilde{W} \subseteq \tilde{W} \quad \text{ for all } \ k,$$

we have $T\tilde{W} \subseteq W$.

Suppose $E_{i-1}^* A E_i^* W = 0$ for some i with $r \leq i < r + d$.

Similarly,

$$\tilde{W} = E_i^* W + \cdot E_{r+d}^* . W$$

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is a T-module with $0 \subsetneq \tilde{W} \subsetneq W$.

Definition 5.1. Let Γ, E_i^*, and T be as above. Irreducible T-modules W and W' are isomorphic whenever there is an isomorphism $\sigma : W \to W'$ of vector spaces such that $a\sigma = \sigma a$ for all $a \in T$.

Recall that the standard module V is an orthogonal direct sum of irreducible T-modules $W_1 \oplus W_2 \oplus \cdots$. Given W in this list, the multiplicity of W in V is

$$|\{j \mid W_j \simeq W\}|.$$

Remark. It is known that the multiplicity does not depend on the decomposition.

Now assume that Γ is the D-cube, $H(D, 2)$ with $D \geq 1$. Vew

$$X = \{a_1 \cdots a_D \mid a_i \in \{1, -1\}, 1 \leq i \leq D\}, \tag{5.3}$$
$$E = \{xy \mid x, y \in X, \ x, y \ \text{differ in exactly 1 coordinate.}\}. \tag{5.4}$$

Find T-modules.

Claim: $H(D, 2)$ is bipartite with a partition $X = X^+ \cup X^-$, where

$$X^+ = \{a_1 \cdots a_D \in X \mid \prod a_i > 0\} \tag{5.5}$$
$$X^- = \{a_1 \cdots a_D \in X \mid \prod a_i < 0\} \tag{5.6}$$

Observe: for all $y, z \in X$,

$$\partial(y, z) = i \Leftrightarrow y, z \ \text{differ in exactly in} \ i \ \text{coorinates with} \ 0 \leq i \leq D.$$

Here, the diameter of $H(D, 2) = D = d$ for all $x \in X$.

Theorem 5.1. *Let $\Gamma = H(D, 2)$ be as above. Fix $x \in X$, and write $E_i^* = E_i^*(x)$, and $T = T(x)$.*

Let W be an irreducible T-module with endpoint r, and diameter d with $0 \leq r \leq r + d \leq D$.

(i) W has a basis $w_0, w_1, \ldots, w_d$ with $w_i \in E_{i+r}^* W$ for $0 \leq i \leq d$. With respect to which the matrix representing A is

$$\begin{pmatrix}
0 & d & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & d-1 & \cdots & 0 & 0 & 0 \\
0 & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 3 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 2 & 0 \\
0 & 0 & 0 & \cdots & d-1 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & d & 0
\end{pmatrix}$$

(*ii*) $d = D - 2r$. *In particular,* $0 \leq r \leq D/2$.

(*iii*) *Let* W' *denote an irreducible* T*-module with endpoint* r'. *Then* W *and* W' *are isormorphic as* T*-modules if and only if* $r = r'$.

(*iv*) *The multiplicity of the irreducible* T*-module with endpoint* r *is*

$$\binom{D}{r} - \binom{D}{r-1} \quad \text{if } 1 \leq r \leq R/2,$$

and 1 *if* $r = 0$.

Proof. Recall that Γ is vertex transitive. It is a Cayley graph.

Hence without loss of generality, we may assume that $x = \overbrace{11\cdots 1}^{D}$.

Notation: Set $\Omega = \{1, 2, \ldots, D\}$. For every subset $S \subseteq \Omega$, let

$$\hat{S} = a_1 \cdot a_d \in X \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S. \end{cases}$$

In particular, $\widehat{emptyset} = x$ and

$$|S| = i \Leftrightarrow \partial(x, \hat{S}) = i \Leftrightarrow \hat{S} \in E_i^* V.$$

For all $S, T \subseteq \Omega$, we say S covers T if and only if $S \supseteq T$ and $|S| = |T| + 1$.

Observe that $\hat{S}, \hat{T}$ are adjacent in Γ if and only if either T coverse S or S coverr T.

Define the 'raising matrix'

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*.$$

Observe that

$$R E_i^* V \subseteq E_{i+1}^* V \quad \text{for } 0 \leq i \leq D, \quad \text{and } E_{D+1}^* V = 0.$$

Indeed for any $S \subseteq \Omega$ with $|S| = i$,

$$R\hat{S} = R E_i^* \hat{S} \tag{5.7}$$

$$= E_{i+1}^* A \hat{S} \tag{5.8}$$

$$= \sum_{T_1 \subseteq \Omega, S \text{ covers } T_1} E_{i+1}^* \widehat{T_1} + \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T} \tag{5.9}$$

$$= \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T}. \tag{5.10}$$

Define the 'lowering matrix'

$$L = \sum_{i=0}^{D} E_{i-1}^* A E_i^*.$$

Observe that

$$LE_i^* V \subseteq E_{i-1V}^* \quad \text{for } 0 \le i \le D, \text{ and } E_{-1}^* V = 0.$$

Indeed for any $S \subseteq \Omega$,

$$L\hat{S} = \sum_{T \subseteq \Omega, S \text{ covers } T} \hat{T}.$$

Observe that $A = L + R$.

For convenience, set

$$A^* = \sum_{i=0}^{D} (D - 2i) E_i^*.$$

Claim: The following hold.

(a) $LR - RL = A^*$.
(b) $A * L - LA^* = 2L$.
(c) $A^* R - RA^* = -2R$.

In particular $\text{Span}(R, L, A^*)$ is a 'representation of Lie algebra $\text{sl}_2(\mathbb{C})$.

Remark (Lie Algebra sl2(C)).

$$\text{sl}_2(\mathbb{C}) = \{X \mid \text{Mat}(\mathbb{C} \mid \text{tr}(X) = 0\}.$$

For $X, Y \in \text{sl}_2(\mathbb{C})$, define a binary operation $[X, Y] = XY - YX$.

$$A^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then these satisfy the relations (a) - (c) above.

Proof of Claim. Apply both sides to $\hat{S}$ ($S \subseteq \Omega$). Say $|S| = i$.

Proof of (a):

$$(LR - RL)\hat{S} = L \left(\sum_{\substack{T \subseteq \Omega, T \text{ covers } S \\ (D-i \text{ of them})}} \hat{T} \right) - R \left(\sum_{\substack{U \subseteq \Omega, S \text{ covers } U \\ (i \text{ of them})}} \hat{T} \right) \tag{5.11}$$

$$= (D - i)\hat{S} + \sum_{\substack{V \subseteq \Omega, |V| = i, |S \cap V| = i-1}} \hat{V} - \left(i\hat{S} + \sum_{\substack{V \subseteq \Omega, |V| = i, |S \cap V| = i-1}} \hat{V} \right) \tag{5.12}$$

$$= (D - 2i)\hat{S} \tag{5.13}$$

$$= A^* \hat{S}. \tag{5.14}$$

Proof of (b):

$$(A^*L - LA^*)\hat{S} = (D - 2(i-1))L\hat{S} - (D - 2i)L\hat{S} \quad \text{(since } L\hat{S} \in E_{i-1}^* V\text{)} \quad (5.15)$$
$$= 2L\hat{S}. \quad (5.16)$$

Proof of (c):

$$(A^*R - RA^*)\hat{S} = (D - 2(i+1))R\hat{S} - (D - 2i)R\hat{S} \quad \text{(since } R\hat{S} \in E_{i+1}^* V\text{)} \quad (5.17)$$
$$= 2R\hat{S}. \quad (5.18)$$

Let W be an irreducible T-module with endpoint r and diameter d ($0 \leq r \leq r + d \leq D$).

Proof of (i) and (ii):

Pick $0 \neq w \in E_r^* W$.

Claim: $LRw = (D - 2r)w$.

Pf.

$$LRw = (A^* + RL)w \quad \text{(by Claim } (a)) \quad (5.19)$$
$$= A^*w \quad (Lw \in E_{r-1}^* W = 0) \quad (5.20)$$
$$(D - 2r)w. \quad (5.21)$$

Define

$$w_i = \frac{1}{i!} R^i w \in E_{r+i}^* W \quad (0 \leq i \leq d).$$

Then,

$$Rw_i = (i+1)w_{i+1} \quad (0 \leq i \leq d) \quad (5.22)$$
$$Rw_d = 0 \quad \text{(by definition of } d) \quad (5.23)$$

Claim: $Lw_0 = 0$ and

$$Lw_i = (D - 2r - i + 1)w_{i-1} \quad (1 \leq i \leq d).$$

Pf. We prove by induction on i. The case $i = 0$ is trivial, and the case $i = 1$ follows from above claim. Let $i \geq 2$,

$$Lw_i = \frac{1}{i} LRw_{i-1} = \frac{1}{i}(A^* + RL)w_{i-1} \quad \text{(by Claim (a))} \quad (5.24)$$
$$\text{(by induction hypothesis)} \quad (5.25)$$
$$= \frac{1}{i}((D - 2(r + i - 1))w_{i-1} + (D - 2r - (i-1) + 1)Rw_{i-2} \quad (Rw_{i-2} = (i-1)w_{i-1}) \quad (5.26)$$
$$= \frac{1}{i}i(D - 2r - i + 1)w_{i-1} \quad (5.27)$$
$$= (D - 2r - i + 1)w_{i-1}. \quad (5.28)$$

Claim: $w_0, \dots, w_d$ is a basis for W.

Pf. Let $W' = \mathrm{Span}\{w_0, \dots, w_d\}$. Then W' is R and L invariant. So it is $A = R + L$ invariant.

Also it is E_i^*-invariant for every i.

Hence W' is a T-module.

Since W is irreducible, $W' = W$.

As w_i's are orthogonal, theyy are linearly independent. Note that $w_i \neq 0$ by the definition of d and Lemma 4.1 (iv).

Claim: $d = D - 2r$.

Pf. By (a),

$$
\begin{align}
0 &= (LR - RL - A^*)w_d \tag{5.29} \\
&= 0 - (D - 2r - d + 1)Rw_{d-1} - (D - 2(r+d))w_d \tag{5.30} \\
&= -d(D - 2r - d + 1)w_d - (D - 2(r+d))w_d \tag{5.31} \\
&= (-dD + 2rd + d^2 - d - D + 2r + 2d)w_d \tag{5.32} \\
&= (d^2 + (2r - D + 1)d + 2r - D)w_d \tag{5.33} \\
&= (d + 2r - D)(d + 1)w_d. \tag{5.34}
\end{align}
$$

Hence $d = D - 2r$.

Therefore, with respect to a bais $w_0, w_1, \dots, w_d$, $A = L + R$, $w_{-1} = w_{d+1} = 0$,

$$
Lw_i = (d - i + 1)w_{i-1}, \quad Rw_i = (i + 1)w_{i+1}.
$$

$$
L = \begin{pmatrix}
0 & d & 0 & \cdots & 0 & 0 \\
0 & 0 & d-1 & \cdots & 0 & 0 \\
 & & \cdots & \cdots & & \\
 & & & & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad
R = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 2 & 0 & \cdots & & \\
 & & & & 0 & 1 \\
0 & 0 & 0 & \cdots & d & 0
\end{pmatrix}.
$$

This completes the proof of (i) and (ii). $\square$

Chapter 6

T-Modules of $H(D,2)$, II

Proof of Theorem 5.1 Continued.

(iii) Let $r = r'$,

$w_0, \dots, w_d$: a basis for W with $w_i \in E_i^* W$, and

$w_0', \dots, w_d'$: a basis for W' with $w_i' \in E_i^* W'$.

Then $d = D - 2r = D - 2r' = d'$, and

$$\sigma : W \to W' \quad (w_i \mapsto w_i')$$

is an isomorsphism of T-modules by (i).

If $r \neq r'$, then

$$d = D - 2r \neq D - 2r' = d',$$

hence, $\dim W \neq \dim W'$.

(iv) Let W_i be the irreducible T-module with endpoint i. Then

$$\dim E_r^* V = \binom{D}{r} = \sum_{i=0}^{r} \text{mult}(W_i).$$

Hence, we have that

$$\text{mult}(W_r) = \binom{D}{r} - \binom{D}{r-1}$$

by induction on r.

$\square$

Theorem 6.1. *Let* $\Gamma = H(D,2)$ *with* $D \geq 1$. *Fix a vertex* $x \in X$ *and write*

$$E_i^* \equiv E_i^*(x), \quad T = T(x), and A^* \equiv \sum_{i=0}^{D}(D-2i)E_i^*.$$

Let W *be an irreducible* T-*module with endpoint* r *with* $0 \leq r \leq D/2$. *Then,*

(i) W *has a basis*

$$w_0^*, w_1^*, ..., w_d^* \quad (d = D - 2r), \quad such \ that \ \ w_i^* \in E_{i+r}W \ (0 \leq i \leq d)$$

with respect to which the matrix corresponding to A^* *is*

$$\begin{pmatrix} 0 & d & 0 & & & & \\ 1 & 0 & d-1 & & & & \\ 0 & 2 & 0 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & 2 & 0 \\ & & & d-1 & 0 & 1 \\ & & & 0 & d & 0 \end{pmatrix}.$$

In particular, / *(ii)* $E_i A^* E_j = 0$ *if* $|i - j| \neq 1$ *for* $0 \leq i, j \leq D$.

Proof. We use the notation,

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \ (= -[\beta, \alpha]).$$

Recall that

(a) $[L, R] = A^*$,
(b) $[A^*, L] = wL$,
(c) $[A^*, R] = -2R$,

and $A = L + R$.

Write $(a) - (c)$ in terms of A and A^*, we have,

$$[A, A^*] = [L, A^*] + [R, A^*] = 2(R - L).$$

$$\begin{cases} R + L & = A \\ R - L & = [A, A^*]/2. \end{cases}$$

Hence,

$$R = \frac{1}{4}(2A + [A, A^*]) \quad \text{and} \tag{6.1}$$

$$L = \frac{1}{4}(2A - [A, A^*]). \tag{6.2}$$

Now (a), (b) become

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0 \qquad (6.3)$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0 \qquad (6.4)$$

Pf.   By (b),

$$2A - AA^* + A^*A = 4L \qquad (6.5)$$

$$= 2[A^*, L] \qquad (6.6)$$

$$= A^*\frac{2A - [A, A^*]}{2} - \frac{2A - [A, A^*]}{2}A^* \qquad (6.7)$$

So we have (\2)

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0.$$

By (a),

$$-16A^* = [2A + [A, A^*], 2A - [A, A^*]] \qquad (6.8)$$

$$(2A + [A, A^*])(2A - [A, A^*]) - (2A - [A, A^*])(2A + [A, A^*]) \qquad (6.9)$$

$$= [4A^2 - 2A[A, A^*] + [A, A^*](2A) - [A, A^*]^2 \qquad (6.10)$$

$$- 4A^2 - 2A[A, A^*] + [A, A^*](2A) + [A, A^*]^2 \qquad (6.11)$$

$$= -4A^2A^* + 4AA^*A + 4AA^*A - 4A^*A^2. \qquad (6.12)$$

So,

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0.$$

Claim: $E_i^*A^*E_j = 0$ if $|i - j| \neq 1$ for $0 \leq i, j \leq D$.

Pf. We have,

$$0 = E_i(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_j \qquad (6.13)$$

$$= E_iA^*E_j(\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4) \qquad (6.14)$$

$$(AE_j = \theta_jE_j, \ E_iA = (AE_j)^\top = (\theta_iE_i)^\top = \theta_iE_i) \qquad (6.15)$$

$$= E_iA^*E_j(\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) \qquad (6.16)$$

$$= E_iA^*E_j(D - 2i - (D - 2j) - 2)(D - 2i - (D - 2j) + 2) \qquad (6.17)$$

$$(\theta_k = D - 2k) \qquad (6.18)$$

$$= E_iA^*E_j \cdot 4(i - j + 1)(i - j - 1) \qquad (6.19)$$

and $i - j + 1 \neq 0$, $i - j - 1 \neq 0$. Hence, $E_i^*A^*E_j = 0$.

Now define "dual raising matrix",

$$R^* = \sum_{i=0}^{D} E_{i+1}A^*E_i.$$

So,
$$R^* E_i V \subseteq E_{i+1} V, \quad (0 \le i \le D, \ E_{D+1} V = 0).$$

Define "dual lowering matrix"

$$L^* = \sum_{i=0}^{D} E_{i-1} A^* E_i.$$

Then
$$L^* E_i V \subseteq E_{i-1} V \quad (0 \le i \le D, \ E_{-1} V = 0).$$

Observe that
$$A^* = \left(\sum_{i=0}^{D} E_i \right) A^* \left(\sum_{j=0}^{D} E_j \right) = L^* + R^*$$

by Claim 1.

Claim 2. We have | (a) $[L^*, R^*] = A$, | (b) $[A, L^*] = 2L^*$, | (c) $[A, R^*] = -2R^*$.

Pf. (b)

$$AL^* - L^* A = \sum_{i=0}^{D} (A E_{i-1} A^* E_i - E_{i-1} A^* E_i A) \tag{6.20}$$

$$= \sum_{i=0}^{D} E_{i-1} A^* E_i (\theta_{i-1} - \theta_i) \tag{6.21}$$

$$(\theta_k = D - 2k, \quad \theta_{i-1} - \theta_i = 2I - 2(i-1) = 2 \tag{6.22}$$
$$= 2L^*. \tag{6.23}$$

(c) Similar.

Remark.

$$AR^* - R^* A = \sum_{i=0}^{D} (A E_{i+1} A^* E_i - E_{i+1} A^* E_i A) \tag{6.24}$$

$$= \sum_{i=0}^{D} E_{i+1} A^* E_i (\theta_{i+1} - \theta_i) \tag{6.25}$$

$$= 2R^*. \tag{6.26}$$

(a) We have, by (b), (c)
$$[A, A^*] = [A, L^*] + [A, R^*] = 2(L^* - R^*). \tag{6.27}$$

Since $A^* = L^* + R^*$,

$$R^* = \frac{2A^* + [A^*, A]}{4}, \quad L^* = \frac{2A^* - [A^* - A]}{4}.$$

Now (a) is seen to be equivalent to (@) upon evaluation. This proves Claim 2.

Remark.

$$[L^*, R^*] = \frac{1}{16}((2A^* - [A^*, A])(2A^* + [A^*, A]) - (2A^* + [A^*, A])(2A^* - [A, A^*])) \tag{6.28}$$

$$= \frac{1}{16}(4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* - [A^*, A]^2 - 4A^{*2} + 2A^*[A^*, A] - [A^*, A]2A^* + [A^*, A]^2) \tag{6.29}$$

$$= \frac{1}{4}(A^{*2}A - 2A^*AA^* + AA^{*2}) \tag{6.30}$$

$$= A, \tag{6.31}$$

by (@).

Now apply same argument as for (@), (@) of Theorem @(thm:hd2-module1) and observe A^* has $D + 1$ distinct eigenvalues. So,

$$A^* = \sum_{i=0}^{D}(D - 2i)E_i^*$$

generates

$$M^* = \text{Span}(E_0^*, \dots, E_D^*).$$

Hence, $E_0, \dots, E_D$, A^* generates T.

Take an irreducible T-module W with endpoint r with $0 \le r \le D/2$. Set $t = \min\{i \mid E_iW\}$.

Pick $0 \ne w_0^* \in E_tW$. Set

$$w_i^* = \frac{1}{i!}R^{*i}w_0^* \in E_{t+i}W \quad \text{for all } i.$$

Then,

$$R^*w_i^* = (i + 1)w_{i+1}^* \quad \text{for all } i.$$

By (a), we get by induction, $L^*w_i^* = (D - 2t - i + 1)w_{i-1}^*$,

$$L^*w_i^* = \frac{1}{i}L^*R^*w_{i-1}^* \tag{6.32}$$

$$= \frac{1}{i}(A + R^*L^*)w_{i-1}^* \tag{6.33}$$

$$= \frac{1}{i}((D - 2(t + i - 1))w_{i-1}^* + (i - 1)(D - 2t - i + 2)w_{i-1}^*) \tag{6.34}$$

$$= (D - 2t - i + 1)w_{i-1}^*. \tag{6.35}$$

So $\text{Span}(w_0^*, w_1^*, \dots)$ is L^*, R^*, A^*-invariant. Hence, $W = (Span)(w_0^*, w_1^*, \dots, w_d^*)$, $w_0^*, w_1^*, \dots, w_d^* \ne 0$, $w_i^* = 0$ for every $i > d$ by dimension.

Thus $d = D - 2t$.

Pf.

$$(D - 2(t + d))w_d^* = Aw_d^* \tag{6.36}$$
$$= (L^*R^* - R^*L^*)w_d^* \tag{6.37}$$
$$= -(D - 2t - d + 1)R^*w_{d-1}^* \tag{6.38}$$
$$= -(D - 2t - d + 1)dw_d^*. \tag{6.39}$$

Hence,

$$0 = d^2 + (2t - D - 1 + 2)d - (D - 2t) = (d - D + 2t)(d + 1)$$

So $d = D - 2t$. $\square$

Definition 6.1. For any graph $\Gamma = (X, E)$, pick a vertex $x \in X$ and set $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

(i) an irreducible T-module W is thin if $\dim E_i^* W \le 1$ for every i,

(ii) Γ is thin with respet to x, if every irreducible $T(x)$-module is thin,

(iii) an irreducible T-module W is dual thin if $\dim E_i W \le 1$ for every i,

(iv) Γ is dual thin with respect to x, if every irreducible $T(x)$-module is dual thin.

Observe: $H(D, 2)$ is thin, dual thin with respect to each $x \in X$.

With above notation, write $D \equiv D(x)$.

(i) an ordering $E_0, E_1, \dots, E_R$ of primitive idempotents of Γ is restricted if E_0 corresponds to the maximal eigenvalue.

Fix a restricted ordering,

(ii) Γ is Q-polynomial with respect to x, above ordering if there exists $A^* \equiv A^*(x)$ such that

    (a) $E_0^* V, \dots, E_D^* V$ are the maximal eigenspaces for A^*.

    (b) $E_i A^* E_j = 0$ if $|i - j| > 1$ for $0 \le i, j \le R$.

Observe $H(D, 2)$ is Q-polynomial with respect to the natural ordering of the idempotents and every vetex.

Program. Study graphs that are thin and Q-polynomial with respect to each vertex.

(In fact, thin with respect to x implies dual thin with respect to x.)

GEt a situation like $H(D, 2)$, where T is generated by A, A^*. Except $\mathrm{sl}_s(\mathbb{C})$ is repalaced by a quantum Lie algebra.

Chapter 7

The Johnson Graph $J(D, n)$

Wednesday, February 3, 1993

Definition 7.1. The Johnson graph, $\Gamma = J(D, N)$ $(1 \leq D \leq N - 1)$ satisfies

$$X = \{S \mid S \subset \Omega, \; |S| = D\} \quad \text{where} \quad \Omega = \{1, 2, \dots, N\} \qquad (7.1)$$

$$E = \{ST \mid S, T \in X, \quad |S \cap T| = D - 1\}. \qquad (7.2)$$

Chapter 8

Thin Graphs

Friday, February 5, 1993

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