0. Measure Theory Foundations

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Reference

1. Theory of Functions of A Real Variable. (Senlin Xu, Chunhua Xue)
2. An Introduction to Measure Theory. (Terence Tao)

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1. Set Theory (Limit Inferior and Limit Superior of Sets)

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Definitions

1. **Upper Limit Set:**
    1. $\overline{\lim\limits_{k\to+\infty}} A_k = \limsup\limits_{k\to+\infty} A_k = \{x \mid \exists \text{ infinite } k, \text{s.t. } x \in A_k\} = \{x \mid \forall n \in \mathbb{N},\ \exists k \geq n,\ \text{s.t. } x \in A_k\}$
2. **Lower Limit Set:**
    1. $\underline{\lim\limits_{k\to+\infty}} A_k = \liminf\limits_{k\to+\infty} A_k = \{x \mid \exists \text{ finite } k, \text{s.t. } x \notin A_k\} = \{x \mid \exists n_0 \in \mathbb{N},\ \forall k \geq n_0,\ \text{s.t. } x \in A_k\}$

Theorems

1. **De Morgan Formula:**
    1. **Statement:**
        1. $X - \bigcup\limits_{\alpha\in\Gamma} A_\alpha = \bigcap\limits_{\alpha\in\Gamma}(X - A_\alpha)$
        2. $X - \bigcap\limits_{\alpha\in\Gamma} A_\alpha = \bigcup\limits_{\alpha\in\Gamma}(X - A_\alpha)$
        3. if $A_\alpha \subset X (\forall \alpha \in \Gamma)$, X is the whole space, then the above formula equals to
            1. $(\bigcup\limits_{\alpha\in\Gamma} A_\alpha)^c = \bigcap\limits_{\alpha\in\Gamma} A_\alpha^c$
            2. $(\bigcap\limits_{\alpha\in\Gamma} A_\alpha)^c = \bigcup\limits_{\alpha\in\Gamma} A_\alpha^c$
2. **Using Countable Intersection and Countable Union to Represent Upper and Lower Limit Sets:**
    1. **Statement:**
        1. $\overline{\lim\limits_{k\to+\infty}} A_k = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} A_k$
        2. $\underline{\lim\limits_{k\to+\infty}} A_k = \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} A_k$
        3. $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_{k+1} \subset \cdots \Rightarrow \lim\limits_{k\to+\infty} A_k = \bigcup\limits_{k=1}^{\infty} A_k$
        4. $A_1 \supset A_2 \supset \cdots \supset A_k \supset A_{k+1} \supset \cdots \Rightarrow \lim\limits_{k\to+\infty} A_k = \bigcap\limits_{k=1}^{\infty} A_k$
    2. **Proof Steps:**
        1. Check the definition
        2. Prove $\overline{\lim\limits_{k\to+\infty}} A_k = \underline{\lim\limits_{k\to+\infty}} A_k = \bigcup\limits_{k=1}^{\infty} A_k$ or $\overline{\lim\limits_{k\to+\infty}} A_k = \underline{\lim\limits_{k\to+\infty}} A_k = \bigcap\limits_{k=1}^{\infty} A_k$

3. **Corollary:**
    1. Since $\left\{\bigcup_{k=n}^{\infty} A_k\right\}_{n=1}^{\infty}$ is monotone decreasing, by Statement 4, $\overline{\lim_{k\to+\infty}} A_k = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \lim_{n\to+\infty}\bigcup_{k=n}^{\infty} A_k$
    2. Since $\left\{\bigcap_{k=n}^{\infty} A_k\right\}_{n=1}^{\infty}$ is monotone increasing, by Statement 3, $\underline{\lim_{k\to+\infty}} A_k = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = \lim_{n\to+\infty}\bigcap_{k=n}^{\infty} A_k$

2. Ring, σ -Ring, Monotone Class

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Basic Assumptions

1. Suppose Ω is a set.
2. $\mathcal{F}$ is a non-empty family of subsets of Ω.

Notations

1. $\mathcal{R}(\mathcal{F})$: $\mathcal{R}(\mathcal{F})$ represents the ring generated by $\mathcal{F}$.
2. $\sigma(\mathcal{F})$: $\sigma(\mathcal{F})$ represents the σ -ring generated by $\mathcal{F}$.
3. $\mathcal{M}(\mathcal{F})$: $\mathcal{F}$ is a collection of subsets, $\mathcal{M}(\mathcal{F})$ represents the monotone class generated by $\mathcal{F}$.
4. $\mathcal{B}$: Borel Algebra.

Definitions

1. **Ring, Algebra/ Field:**
    1. If $\forall E_1, E_2 \in \mathcal{F},\ E_1 \bigcup E_2 \in \mathcal{F}, E_1 - E_2 \in \mathcal{F}$, then we call $\mathcal{F}$ is a **ring** on Ω.
    2. Moreover, if $\Omega \in \mathcal{F}$, then we say $\mathcal{F}$ is an **algebra** or a **field**.
2. **σ -Algebra (σ-Field):**
    1. If $\forall E, F \in \mathcal{F}, E - F \in \mathcal{F}$ and $\forall \{E_i\}_{i=1}^{\infty} \subset \mathcal{F}, \bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$, then we call $\mathcal{F}$ a σ **-ring** or σ **-field** on Ω.
3. **Generated Ring:**
    1. See Theorem 3.
4. **Monotone Class:**
    1. $\mathcal{M}$ is a non-empty family of subsets of Ω, if $\forall$ monotone $\{E_i\}_{i=1}^{\infty}$, $\lim_{i\to+\infty} E_i \in \mathcal{M}$, then we call $\mathcal{M}$ is a **monotone class**.
5. **Monotone Ring:**
    1. If $\mathcal{M}$ is a monotone class and a ring, then we call it a **monotone ring**.
6. **Borel Algebra, Borel Sets:**
    1. All of the σ -algebras generated by the collection of open sets in $\mathbb{R}^n$ are called the **Borel algebra**.
    2. The elements in Borel algebra are called **Borel sets**.
7. **G_δ set:**
    1. A G_δ **set** is a countable intersection of open sets (in $\mathbb{R}^n$).
8. **F_σ set:**
    1. A F_σ **set** is a countable union of closed sets (in $\mathbb{R}^n$).

Theorems

1. **Properties of a σ-ring:**
    1. **Statement:**
        1. Suppose $\mathcal{F}$ is a σ-ring,
            1. $\mathcal{F}$ is closed under infinite intersection
            2. $\mathcal{F}$ is closed under $\overline{\lim_{k\to+\infty}}$, $\underline{\lim_{k\to+\infty}}$, $\lim_{k\to+\infty}$
        2. Suppose $\mathcal{F}_\alpha(\alpha \in \Gamma)$ are σ-rings (or σ-algebras), then $\bigcap_{\alpha\in\Gamma} \mathcal{F}_\alpha$ is still a σ-ring (or σ-algebra).
    2. **Proof Steps:**
        1. $\bigcap_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} E_i - \bigcup_{i=1}^{\infty}\left(\bigcup_{j=1}^{\infty} E_j - E_i\right)$
        2. Using the definition of upper/lower limit sets
        3. Check the definition
2. **Ring on $\mathbb{R}$:**
    1. **Statement:**

1. Suppose $\mathcal{R}_0$ is a collection of sets consisting of the union of a finite number of left-open and right-closed finite intervals ($\forall A = \bigcup_{i=1}^{n}(a_i, b_i] \Rightarrow A \in \mathcal{R}_0$), then $\mathcal{R}_0$ is a ring, not an algebra nor σ-ring.
2. **Idea:**
    1. "left-open and right-closed interval": To construct ring, and to include empty set.
3. **Proof Steps:**
    1. For $A = \bigcup_{i=1}^{n}(a_i, b_i],\ B = \bigcup_{j=1}^{m}(c_j, d_j]$, relabel the interval. Also Theorem 1.2.1 can be helpful.

3. **Generated Ring (Smallest Ring):**
    1. **Statement:**
        1. There exists only one ring (or algebra or σ -ring or σ -algebra) $\mathcal{R}$ satisfy
            1. $\mathcal{F} \subset \mathcal{R}$;
            2. Any ring (or algebra or σ -ring or σ -algebra) $\mathcal{R}'$ contains $\mathcal{F}$ must also contain $\mathcal{R}$.
    2. **Proof Steps:**
        1. Prove that the intersection of all rings which contain $\mathcal{E}$ also contains $\mathcal{R}$, and this intersection is our desired result.
4. **A σ-ring Can Only Be Extended Once:**
    1. **Statement:** $\sigma(\mathcal{F}) = \sigma(\mathcal{R}(\mathcal{F}))$.
5. **The Necessary and Sufficient Condition for a Collection of Sets to Become a σ-algebra:**
    1. **Statement:**
        1. $\mathcal{F}$ is σ-algebra $\iff$
            1. $\varnothing \in \mathcal{F}$;
            2. $E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}$;
            3. $E_i \in \mathcal{F}(i = 1, 2, \cdots) \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$.
6. **Properties of a Monotone Class:**
    1. Suppose $\mathcal{M}_\alpha(\alpha \in \Gamma)$ are monotone classes, then $\bigcap_{\alpha\in\Gamma} \mathcal{M}_\alpha$ is still a monotone class.
7. **Generated Monotone Class (Smallest Monotone Class):**
    1. **Statement:**
        1. There exists only one monotone class $\mathcal{M}$ satisfy
            1. $\mathcal{F} \subset \mathcal{M}$;
            2. Any monotone class $\mathcal{M}'$ contains $\mathcal{F}$ must also contain $\mathcal{M}$.
8. **The Relationship Between Sigma Rings and Monotone Classes:**
    1. **Statement:**
        1. $\mathcal{M}$ is a σ -ring $\iff$ $\mathcal{M}$ is a monotone ring;
        2. Suppose $\mathcal{F}$ is a ring, then $\sigma(\mathcal{F}) = \mathcal{M}(\mathcal{F})$;
        3. Suppose $\mathcal{M}, \mathcal{E}$ are families of subsets of Ω, $\mathcal{M}$ is a monotone class, $\mathcal{E}$ is a ring, and $\mathcal{M} \supset \mathcal{E}$, then $\mathcal{M} \supset \sigma(\mathcal{E})$.

3. Measure Theorem (Measure, Outer Extension)

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Basic Assumptions

1. Ω is a set, $\mathcal{F}$ is a ring of subsets of Ω.
2. μ is a measure defined on $\mathcal{F}$.

Notations

1. $\mathcal{H}(\mathcal{F})$: $\mathcal{H}(\mathcal{F}) = \left\{ E \,\middle|\, E \subset \Omega,\ \exists E_i \in \mathcal{F}(i = 1, 2, \cdots), \text{s.t. } E \subset \bigcup_{i=1}^{\infty} E_i \right\}$.
2. $\mathcal{F}^*$: $\mathcal{F}^*$ is the collection of all μ^* -measurable sets.
3. μ^*: Outer measure introduced by μ.

Definitions

1. **The Set of Extended Real Numbers:**
    1. $\mathbb{R}_* = \mathbb{R} \bigcup \{-\infty, +\infty\}$.
2. **Measure:**
    1. $\mu : \mathcal{F} \to \mathbb{R}_*$, we call μ a **measure** if it satisfy:
        1. $\mu(\varnothing) = 0$;
        2. $\forall\ E \in \mathcal{F},\ \mu(E) \geq 0$;

3. $\forall E_i \in \mathcal{F}(i=1,2,\cdots)$, if $E_i \bigcap E_j = \varnothing (i \neq j)$ and $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$.

3. **H-ring:**
   1. See Notation 1
4. **Outer Measure:**
   1. We define $\mu^* : \mathcal{H}(\mathcal{F}) \to \mathbb{R}_*$ as **outer measure** (introduced by μ): $\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(E_i) \middle| E_i \in \mathcal{F} \text{ and } E \subset \bigcup_{i=1}^{\infty} E_i\right\}$
5. **Caratheodory Condition, μ^* -Measurable set:**
   1. , if $E \in \mathcal{H}(\mathcal{F})$ satisfy the **Caratheodory condition**: $\forall F \in \mathcal{H}(\mathcal{F}),\ \mu^*(F) = \mu^*(F \bigcap E) + \mu^*(F - E)$, then we say E is a μ^* **-measurable set**.
   2. **Idea:** From Theorem 4 and Theorem 5, we could know that if μ^* is a measure on $\mathcal{F}^* \supset \mathcal{F}$, then $\forall E \in \mathcal{R}^*, F \in \mathcal{H}(\mathcal{F}^*) = \mathcal{H}(\mathcal{F})$, we have $\mu^*(F) = \mu^{**}(F) = \mu^{**}(F \bigcap E) + \mu^{**}(F - E) = \mu^*(F \bigcap E) + \mu^*(F - E)$ (Note that in Theorem 5, we have supposed that $\mathcal{F}^* \subset \mathcal{H}(\mathcal{F})$). Therefore, if μ^* and E satisfy the Caratheodory Condition, we will obtain a measurable set from $\mathcal{H}(\mathcal{F})$, and furthermore construct the algebra $\mathcal{F}^*$ (We selected the elements).
6. **Finite Measure Set and Finite Measure:**
   1. If $\forall E \in \mathcal{F},\ \mu(E) < +\infty$, then we call E is **a finite measure set**, and μ is a **finite measure**.
7. **σ -Finite Set and σ -Finite Measure:**
   1. Suppose $E \in \mathcal{F}$, if there exists a sequence of sets $E_i \in \mathcal{F}(i=1,2,\cdots)$, s.t. $\forall E_i$ has finite measure and $E \subset \bigcup_{i=1}^{\infty} E_i$, then we say E is a σ **-finite set**.
   2. *Remark: $\mu(E)$ may not be finite, σ -finite is weaker than finite.*
   3. If $\forall E \in \mathcal{F}$ is σ -finite, then we say μ is a σ **-finite measure**.
8. **Totally Finite Measure:**
   1. If $\mathcal{F}$ is an algebra, and $\mu(\Omega) < +\infty$, then μ is called **totally finite measure**.
9. **Totally σ - Finite Measure:**
   1. If $\mathcal{F}$ is an algebra, and Ω is a σ -finite set, then we say μ is **totally σ -finite measure**.

Theorems

1. **Properties of Measure:**
   1. **Statement:**
      1. **Continuous at the Limit of Monotone Classes:**
         1. $\forall E_1 \subset E_2 \subset E_3 \subset \cdots,\ \bigcup_{i=1}^{\infty} E_i \in \mathcal{F} \Rightarrow \mu(\lim_{i\to+\infty} E_i) = \lim_{i\to+\infty} \mu(E_i)$
         2. $\forall E_1 \supset E_2 \supset E_3 \supset \cdots,\ \bigcap_{i=1}^{\infty} E_i \in \mathcal{F}$, and $\exists E_n$ s.t. $\mu(E_n) < +\infty \Rightarrow \mu(\lim_{i\to+\infty} E_i) = \lim_{i\to+\infty} (E_i)$
      2. **Upper and Lower Limits on a Sigma Ring:** Now suppose $\mathcal{F}$ is a σ-ring, then
         1. $\forall E_i \in \mathcal{F}(i=1,2,\cdots) \Rightarrow \mu(\underline{\lim}_{i\to+\infty} E_i) \leq \underline{\lim}_{i\to+\infty} \mu(E_i)$
         2. $\forall E_i \in \mathcal{F}(i=1,2,\cdots),\ \exists m \in \mathbb{N}$, s.t. $\mu\left(\bigcup_{n=m}^{\infty} E_n\right) < +\infty \Rightarrow \mu(\overline{\lim}_{i\to+\infty} E_i) \geq \overline{\lim}_{i\to+\infty} \mu(E_i)$
         3. $\forall E_i \in \mathcal{F}(i=1,2,\cdots),\ \lim_{i\to+\infty} E_i$ exists, and $\exists m \in \mathbb{N},$ s.t. $\mu\left(\bigcup_{n=m}^{\infty} E_n\right) < +\infty \Rightarrow \mu(\lim_{i\to+\infty} E_i) = \lim_{i\to+\infty} \mu(E_i)$
         4. $\forall E_i \in \mathcal{F}(i=1,2,\cdots),\ \exists m \in \mathbb{N},$ s.t. $\sum_{n=m}^{\infty} \mu(E_n) < +\infty \Rightarrow \mu(\overline{\lim}_{i\to+\infty} E_i) = 0.$
2. **Properties of H-ring:**
   1. **Statement:**
      1. $\mathcal{F} \subset \mathcal{H}(\mathcal{F})$
      2. $\forall E \in \mathcal{H}(\mathcal{F}), F \subset E \Rightarrow F \in \mathcal{H}(\mathcal{F})$
      3. $\mathcal{H}(\mathcal{F})$ is a σ -ring.
3. **Properties of Outer Measure:**
   1. **Statement:**
      1. $\forall E \in \mathcal{F} \Rightarrow \mu^*(E) = \mu(E)$, which means: μ^* is an extension of μ , from $\mathcal{F}$ to $\mathcal{H}(\mathcal{F})$
      2. $\forall \{E_i\}_{i=1}^{\infty} \subset \mathcal{H}(\mathcal{F}) \Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$
      3. $\forall \{E_i\}_{i=1}^{\infty} \subset \mathcal{H}(\mathcal{F}), \mu^*(E_i) = 0 \Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = 0$
   2. **Proof Steps:**
      1. To prove Statement 2, considering the property of infimum, $\forall\ E_i \in \mathcal{H}(\mathcal{F}),\ \epsilon > 0$, there exists a series of sets $\{E_i^j\}_{j=1}^{\infty} \subset \mathcal{F}$ s.t. $E_i \subset \bigcup_{j=1}^{\infty} E_i^j$, and $\sum_{j=1}^{\infty} \mu(E_i^j) < \mu^*(E_i) + \frac{\epsilon}{2^i}$; $\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} E_i^j \Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i,j=1}^{\infty} \mu(E_i^j)$.
      2. Most of time we assume that $\mu < +\infty$, but actually $\mu = +\infty$ is allowed, and in this case the statement holds trivially.

3. To prove Statement 3, using Statement 2.

4. **Decomposition of Outer Measure:**
   1. **Statement:**
      1. $E \in \mathcal{F}, \forall F \in \mathcal{H}(\mathcal{F}) \Rightarrow \mu^*(F) = \mu^*(F \bigcap E) + \mu^*(F - E)$
5. **The Extension of A Measure Can Only Be Done Once on H-ring (with the Rule of Outer Measure):**
   1. **Statement:**
      1. Suppose $\mathcal{F}^*$ is a sub-ring of $\mathcal{H}(\mathcal{F})$. If $\mathcal{F}^* \supset \mathcal{F}$, then $\mathcal{H}(\mathcal{F}^*) = \mathcal{H}(\mathcal{F})$ and $\mu^{**} = \mu^*$.
6. **Properties of $\mathcal{F}^*$ with Its Outer Measure:**
   1. **Statement:** (based on Notation 2)
      1. $\mathcal{F}^*$ is a σ -ring ;
      2. μ^* is a measure on $\mathcal{F}^*$, and it is an extension of μ (from $\mathcal{F}$) ;
7. **The Uniqueness Theorem for Outer Extension:**
   1. **Statement:**
      1. Suppose $\mu_k (k = 1, 2)$ are two measures on $\sigma(\mathcal{F})$. If μ_k are σ -finite on $\mathcal{F}$, and $\forall E \in \mathcal{F}, \mu_1(E) = \mu_2(E)$, then $\forall E \in \sigma(\mathcal{F}), \mu_1(E) = \mu_2(E)$.
   2. **Idea:**
      1. To prove the uniqueness of measures on the ring $\sigma(\mathcal{F})$, we could construct a ring that already satisfy the requirement of uniqueness, then prove the second ring is equivalent to the first ring or is the sup-ring of the first ring.
      2. Furthermore, we construct a ring similar to $\mathcal{H}(\mathcal{R})$ to facilitate the use of the experience of extending a measure to an outer measure (also note that μ_k are finite).
   3. **Proof Steps:**
      1. Construct $\mathcal{M} = \left\{ E \in \sigma(\mathcal{F}) \middle| E \text{ s.t. } 1.\ \exists\ E_n \in \mathcal{R}, \mu_1(E_n) < +\infty, \text{ s.t. } E \subset \bigcup_{i=1}^{\infty} E_i;\ 2.\ \forall A \in \mathcal{F},\ \text{when } \mu_1(A) < +\infty,\ \mu_1(A \bigcap E) = \mu_2(A \bigcap E) \right\}$;
      2. Prove $\mathcal{F} \subset \mathcal{M}$.
      3. Prove $\forall E \in \mathcal{M},\ \mu_1(E) = \mu_2(E)$. We could assume that there exists a disjoint set sequence $\{E_i\}_{i=1}^{\infty}$ s.t. $\mu_1(E_i) < +\infty,\ E \subset \bigcup_{i=1}^{\infty} E_i$, and use $E = \bigcup_{i=1}^{\infty} (E_i \bigcap E)$.
      4. Prove $\mathcal{M}$ is a monotone class, so we can use Section 2 Theorem 8.3, $\mathcal{M} \supset \sigma(\mathcal{F})$ (Hence we finished the Proof).
      5. In order to prove $\mathcal{M}$ is a monotone class, we should check the definition: if we have a monotone sequence $\{F_n \in \mathcal{M} | n = 1, 2, \cdots\}$ in $\mathcal{M}$, is $\lim\limits_{n \to +\infty} F_n$ satisfy $1. \exists E_n \in \mathcal{F}, \mu_1(E_n) < +\infty,\ \lim\limits_{n \to +\infty} F_n \subset \bigcup_{n=1}^{\infty} E_n;\ 2. \forall A \in \mathcal{F},\ \text{when } \mu_1(A) < +\infty, \mu_1(A \bigcap \lim\limits_{n \to +\infty} F_n) = \mu_2(A \bigcap \lim\limits_{n \to +\infty} F_n)$.
      6. By using $\lim\limits_{n \to +\infty} F_n = \bigcup_{n=1}^{\infty} F_n$ or $\bigcap_{n=1}^{\infty} F_n$, and the continuity of measure at the limit of monotone set sequence, we can achieve our goal.
8. **The Extension Must Also be σ -finite:**
   1. **Statement:**
      1. Suppose $\mathcal{F}$ is a ring over the non-empty set Ω, μ is a σ -finite measure over $\mathcal{F}$, then μ^* must also be σ -finite over $\mathcal{F}^*$.
9. **Relationship Between $\sigma(\mathcal{F})$ and $\mathcal{F}^*$:**
   1. **Statement:**
      1. If $E \in \mathcal{F}^*, \mu^*(E) < +\infty$, then $\exists F \in \sigma(\mathcal{F}),\ \text{s.t. } F \supset E$, and $\mu^*(F - E) = 0,\ \mu^*(E) = \mu^*(F)$.

4. Lebesgue Measure, Lebesgue-Stieltjes Measure

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Notations

1. $\mathcal{P} = \{(a, b] | a \leq b\}$
2. $\mathcal{R}_0 = \mathcal{R}(\mathcal{P}) = \left\{ \bigcup_{i=1}^{n} (a_i, b_i] \middle| a_i \leq b_i, i = 1, 2, \cdots, n \right\}$
3. m :
   1. $\mathcal{P} \to \mathbb{R},\ m((a, b]) = b - a$;
   2. $\mathcal{R}_0 \to \mathbb{R},\ m(E) = \sum_{i=1}^{n} m(E_i) = \sum_{i=1}^{n} (b_i - a_i)$, where $E = \bigcup_{i=1}^{n} E_i,\ E_i = (a_i, b_i] \in \mathcal{P}$ are disjoint sets;
   3. Outer measure extension of m, from $\mathcal{R}_0$ to $\mathcal{R}_0^*$ (the collection of sets in $\mathcal{H}(\mathcal{R}_0)$, which satisfy Caratheodory condition).
4. m_g : Suppose $g(x)$ is a nondecreasing right-continuous function,
   1. $\mathcal{P} \to \mathbb{R}$ (here we assume that $\mathcal{P}$ only contains finite intervals), $m_g((a, b]) = g(b) - g(a)$;

2. $\mathcal{R}_0 \to \mathbb{R},\ m_g(E) = \sum\limits_{i=1}^{n} m_g(E_i) = \sum\limits_{i=1}^{n}(g(b_i) - g(a_i))$, where $E = \bigcup\limits_{i=1}^{n} E_i,\ E_i = (a_i, b_i] \in \mathcal{P}$ are disjoint sets;
3. Outer measure extension of m_g, from $\mathcal{R}_0$ to $\mathcal{R}_0^*$ (the collection of sets in $\mathcal{H}(\mathcal{R}_0)$, which satisfy Caratheodory condition).

Definitions

1. **Elementary Decomposition:**
    1. For $E \in \mathcal{R}_0$, its **elementary decomposition** is $\bigcup\limits_{i=1}^{n} F_i,\ F_i \bigcap F_j = \varnothing (i \neq j),\ \forall F_i \in \mathcal{P}$.
    2. *Remark: Any elementary decomposing sets must be finite and disjoint.*
2. **Lebesgue Measure:**
    1. See Notation 3.3.
    2. *Remark: Notation 3.1 and Notation 3.2 are not Lebesgue Measure, though they share the same notation with Notation 3.3. (Actually, in the case described in Notation 3.3, m should be written as $m^$.)**
3. **Lebesgue-Stieltjes Measure (L-S Measure):**
    1. See Notation 4.3.
    2. *Remark 1: Notation 4.1 and Notation 4.2 are not Lebesgue Measure, though they share the same notation with Notation 4.3. (Actually, in the case described in Notation 4.3, m_g should be written as $m_g^$.)**
    3. *Remark 2: Note that when we defined L-S measure, we required the elements in $\mathcal{P},\ \mathcal{R}_0$ to be finite intervals. We also required g to be nondecreasing.*

Theorems

1. **m is Well Defined on $\mathcal{R}_0$ (Notation 3.1):**
    1. **Statement:**
        1. Suppose $E \in \mathcal{R}_0$, $m(E)$ depends only on E and not on its elementary decomposition.
    2. **Proof Steps:**
        1. Prove the statement is true when $E \in \mathcal{P}$.
        2. For $E \in \mathcal{R}_0$, suppose $\bigcup\limits_{i=1}^{n} E_i,\ \bigcup\limits_{j=1}^{l} F_j$ are two different elementary decomposition. We need to prove $\sum\limits_{i=1}^{n} m(E_i) = \sum\limits_{j=1}^{l} m(F_j)$.
        3. Set $G_{ij} = E_i \bigcap F_j,\ E_i = \bigcup\limits_{j=1}^{l} G_{ij},\ F_j = \bigcup\limits_{i=1}^{n} G_{ij}$, we perform elementary decomposition of $E_i,\ F_j$, by doing this we can use Step 1.
2. **Properties of m (Notation 3.2):**
    1. **Statement:**
        1. m has finite additivity;
        2. $E_1, E_2, \cdots, E_n \in \mathcal{R}_0$ are disjoint sets, if $\bigcup\limits_{i=1}^{n} E_i \subset E, E \in \mathcal{R}_0$, then $\sum\limits_{i=1}^{n} m(E_i) \leq m(E)$;
        3. If $E_1, E_2, \cdots, E_n, E \in \mathcal{R}_0,\ E \subset \bigcup\limits_{i=1}^{n} E_i$, then $m(E) \leq \sum\limits_{i=1}^{n} m(E_i)$;
        4. m is a measure on $\mathcal{R}_0$.
    2. **Proof Steps:**
        1. To prove Statement 1, we suppose $E = \bigcup\limits_{i=1}^{n} E_i,\ E_i \in \mathcal{R}_0$ are disjoint sets. From the definition of m, we know that m has finite additivity on $\mathcal{P}$, so we perform an elementary decomposition of E_i.
        2. To prove Statement 2, set $E_{n+1} = E - \bigcup\limits_{i=1}^{n} E_i,\ E = \bigcup\limits_{i=1}^{n+1} E_i$.
        3. Statement 3 is trivial.
        4. For Statement 4,
            1. Here we only prove the countable additivity. Suppose $E = \bigcup\limits_{i=1}^{\infty} E_i \in \mathcal{R}_0\ (E_i \bigcap E_j = \varnothing, i \neq j)$, due to $\bigcup\limits_{i=1}^{n} E_i \subset E$ and Statement 2, $\sum\limits_{i=1}^{n} m(E_i) \leq m(E)$, let $n \to +\infty$, we have $\sum\limits_{i=1}^{\infty} m(E_i) \leq m(E)$.
            2. Due to $E \in \mathcal{R}_0$, we have $E = \bigcup\limits_{j=1}^{l} F_j,\ F_j = (a_j, b_j] \in \mathcal{P},\ F_i \bigcap F_j = \varnothing (i \neq j)$. Similarly, E_i can also be given an elementary decomposition, after relabeling, we have $E = \bigcup\limits_{i=1}^{\infty} (\alpha_i, \beta_i]$, where $(\alpha_i, \beta_i] (i = 1, 2, \cdots,)$ are disjoint sets. Therefore we have $\bigcup\limits_{j=1}^{l} (a_j, b_j] \subseteq \bigcup\limits_{i=1}^{\infty} (\alpha_i, \beta_i]$. We hope the left hand side of the formula is a closed set and the right hand side is an open cover, in order to apply the Heine-Borel theorem. To perform our plan, we need to do approximation.
            3. For $\epsilon > 0$, we require $\forall j,\ a_j + \frac{\epsilon}{l} \leq b_j$, and we have $\bigcup\limits_{j=1}^{l} [a_j + \frac{\epsilon}{l}, b_j] \subset E \subset \bigcup\limits_{i=1}^{\infty} (\alpha_i, \beta_i] \subset \bigcup\limits_{i=1}^{\infty} (\alpha_i, \beta_i + \frac{\epsilon}{2^i})$. By Heine-Borel theorem, we have a finite open cover $\left\{(\alpha_{i_n}, \beta_{i_n} + \frac{\epsilon}{2^{i_n}})\right\}_{n=1,\cdots,k}$ that covers $\left\{[a_j + \frac{\epsilon}{l}, b_j]\right\}_{j=1}^{l}$, and thus $\bigcup\limits_{j=1}^{l} (a_j + \frac{\epsilon}{l}, b_j] \subset \bigcup\limits_{n=1}^{k} (\alpha_{i_n}, \beta_{i_n} + \frac{\epsilon}{2^{i_n}}]$.

4. $\sum\limits_{j=1}^{l}(b_j-a_j-\frac{\epsilon}{l}) \leq \sum\limits_{n=1}^{k}(\beta_{i_n}+\frac{\epsilon}{2^{i_n}}-\alpha_{i_n}) \leq \sum\limits_{i=1}^{\infty}(\beta_i+\frac{\epsilon}{2^i}-\alpha_i)$. Let $\epsilon \to 0^+$, we finished the proof.

3. *Remark:*
    1. *When we need to create a closed set and an open cover in* Step 5*, why does our construction have the form in* Step 6*?*
        1. *Essentially, we utilize the fact that:* $\forall \epsilon > 0$*,* $g(x)$ *is a nondecreasing right-continuous function,* $\exists \eta, \delta_i$ s.t. $g(a_j+\eta) \leq g(a_j)+\epsilon \ (1 \leq j \leq l),\ g(\beta_j+\delta_j) \leq g(\beta_j)+\frac{\epsilon}{2^j}(1 \leq j)$ *, and here* $g(x)$ *happens to be equal to* x*.*
        2. *Our closed sets must be included in* E*, and our open cover must cover* E*, to keep sure, we need it covers* $\bigcup\limits_{i=1}^{\infty} E_i$*.*

3. m_g **is Well Defined on** $\mathcal{R}_0$**:**
    1. **Statement:**
        1. Suppose $E \in \mathcal{R}_0$, $m_g(E)$ depends only on E and not on its elementary decomposition.
4. **Properties of** m **(Notation 4.2):**
    1. **Statement:**
        1. m_g has finite additivity;
        2. $E_1, E_2, \cdots, E_n \in \mathcal{R}_0$ are disjoint sets, if $\bigcup\limits_{i=1}^{n} E_i \subset E, E \in \mathcal{R}_0$, then $\sum\limits_{i=1}^{n} m_g(E_i) \leq m_g(E)$;
        3. If $E_1, E_2, \cdots, E_n, E \in \mathcal{R}_0,\ E \subset \bigcup\limits_{i=1}^{n} E_i$, then $m_g(E) \leq \sum\limits_{i=1}^{n} m_g(E_i)$;
        4. m_g is a measure on $\mathcal{R}_0$.
    2. **Proof Steps:**
        1. We only prove countable additivity that required in Statement 4. Similar to the proof steps 2.2.4 for $E = \bigcup\limits_{i=1}^{\infty} E_i$, we have $\sum\limits_{i=1}^{\infty} m_g(E_i) \leq m_g(E)$.
        2. Similar to proof steps 2.2.5 and proof steps 2.2.6, now suppose $E = \bigcup\limits_{j=1}^{l}(a_j, b_j] = \bigcup\limits_{i=1}^{\infty}(\alpha_i, \beta_i] = \bigcup\limits_{i=1}^{\infty} E_i$. $\forall \epsilon > 0$, $\exists \eta, \delta_i$ s.t. $g(a_j+\eta) \leq g(a_j)+\frac{\epsilon}{l}\ (1 \leq j \leq l),\ g(\beta_j+\delta_j) \leq g(\beta_j)+\frac{\epsilon}{2^j}(1 \leq j)$. We also have $\bigcup\limits_{j=1}^{l}(a_j+\eta, b_j] \subset \bigcup\limits_{n=1}^{k}(\alpha_{i_n}, \beta_{i_n}+\delta_{i_n}]$.
        3. Similar to proof steps 2.2.7, $\sum\limits_{j=1}^{l}(g(b_j)-g(a_j)-\frac{\epsilon}{l}) \leq \sum\limits_{j=1}^{l}(g(b_j)-g(a_j+\eta)) \leq \sum\limits_{n=1}^{k}(g(\beta_{i_n}+\delta_{i_n})-g(\alpha_{i_n})) \leq \sum\limits_{n=1}^{k}(g(\beta_{i_n})+\frac{\epsilon}{2^{i_n}}-g(\alpha_{i_n})) \leq \sum\limits_{i=1}^{\infty}(g(\beta_i)-g(\alpha_i))+\epsilon$.
5. **L-S Measure in Special Cases:**
    1. **Statement:**
        1. $m_g(\{a\}) = g(a)-g(a-0)$;
        2. $m_g((a,b)) = g(b-0)-g(a)$;
        3. $m_g([a,b)) = g(b-0)-g(a-0)$;
        4. $m_g([a,b]) = g(b)-g(a-0)$.

5. Theory of Integration 1 (Measurable Function, Measure space)

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Basic Assumptions

1. $(\Omega, \mathcal{F})$ is a measurable space (see below), $E \subset \Omega$.

Notations

1. $(\Omega, \mathcal{F})$: Measurable space (see below).
2. $\mathcal{R}_0$: See Section 4 Notation 2.
3. $(\mathbb{R}, \mathcal{L}^g)$: Suppose we have set function m_g. After performing the outer extension and selection (using Caratheodory condition), we obtain a family of m_g^* -measurable sets $\mathcal{L}^g \subset \mathcal{H}(\mathcal{R}_0)$.
4. $E(c \leq f)$: f is a function, c $\in \mathbb{R}$, $E(c \leq f) = \{x \in E | c \leq f(x)\}$.
5. $(\Omega, \mathcal{F}, \mu)$: Measure space.
6. $\overline{\lim\limits_{n\to+\infty}} f_n(x)$ / $\limsup\limits_{n\to+\infty} f_n(x)$:
    1. The maximum value of the limit of subsequences of $\{f_n(x)\}_{n=1}^{\infty}$. (lim $=\infty$ is allowed, pointwise definition)
    2. $\forall$ fixed $x \in \mathbb{R}$, $\overline{\lim\limits_{n\to+\infty}} f_n(x) = \lim\limits_{n\to+\infty}\left(\sup\limits_{m\geq n} f_m(x)\right) = \inf\limits_{n\geq 1}\sup\limits_{m\geq n} f_m(x)$. (pointwise definition)
7. $\underline{\lim\limits_{n\to+\infty}} f_n(x)$ / $\liminf\limits_{n\to+\infty} f_n(x)$:
    1. The minimum value of the limit of subsequences of $\{f_n(x)\}_{n=1}^{\infty}$. (lim $=\infty$ is allowed, pointwise definition)

2. $\varliminf_{n\to+\infty} f_n(x) = \lim_{n\to+\infty}\left(\inf_{m\geq n} f_m(x)\right) = \sup_{n\geq 1}\inf_{m\geq n} f_m(x)$. (pointwise definition)

Definitions

1. **Measurable Space, Measurable Set:**
   1. Ω is a basic space, $\mathcal{F}$ is a σ -ring over Ω, if $\Omega = \bigcup_{E\in\mathcal{F}} E$, then $(\Omega, \mathcal{F})$ is a **measurable space**. We also call $E \in \mathcal{F}$ is a **measurable set** (over $(\Omega, \mathcal{F})$).
   2. *Remark1: $\mathcal{F}$ must be σ -ring.*
   3. *Remark2: Ω does not necessarily belong to $\mathcal{F}$.*
2. **Measurable Function:**
   1. $f : E \to \mathbb{R}$ is a bounded real-valued function. If $\forall c \in \mathbb{R}$, the set $E(c \leq f) \in \mathcal{F}$, then we call f is a **measurable function**.
3. **Lebesgue-Stieltjes Measurable Space/ Set/ Function:**
   1. **Lebesgue-Stieltjes Measurable Space:** $(\mathbb{R}, \mathcal{L}^g)$ (See Notation 3).
   2. **Lebesgue-Stieltjes Measurable Set:** $E \in \mathcal{L}^g$.
   3. **Lebesgue-Stieltjes Measurable Function:** The measurable function on the Lebesgue-Stieltjes Measurable Space.
4. **Lebesgue Measurable Space/ Set/ Function:**
   1. **Lebesgue Measurable Space:** $(\mathbb{R}^n, \mathcal{L})$.
   2. **Lebesgue-Stieltjes Measurable Set:** $E \in \mathcal{L}$.
   3. **Lebesgue-Stieltjes Measurable Function:** The measurable function on the Lebesgue Measurable Space.
5. **Borel Measurable Space/ Set/ Function:**
   1. **Borel Measurable Space:** $(\mathbb{R}^n, \mathcal{B}) = (\mathbb{R}^n, \sigma(\mathcal{R}_0))$.
   2. **Borel Measurable Set:** $E \in \mathcal{B}$.
   3. **Borel Measurable Function:** The measurable function on the Borel Measurable Space.
6. **Measure Space:**
   1. μ is a measure on $\mathcal{F}$, then we call $(\Omega, \mathcal{F}, \mu)$ is a **measure space**.
   2. *Remark: $\mathcal{F}$ must be σ -ring.*
7. **Generalized Measurable Function:**
   1. $f : E \to \mathbb{R}_* = \mathbb{R}\bigcup\{-\infty, +\infty\}$ is a **generalized measurable function**, if $\forall c \in \mathbb{R}_*$, $E(c \leq f) \in \mathcal{F}$.

Theorems

1. **The Necessary and Sufficient Condition for a Function to be Measurable:**
   1. **Statement:**
      1. $f:\ E \to \mathbb{R}$ is a bounded real-valued function. The following propositions are equivalent:
         1. f is a measurable function;
         2. $\forall c \in \mathbb{R},\ E(c < f)$ is a measurable set;
         3. $\forall c \in \mathbb{R},\ E(f \leq c)$ is a measurable set;
         4. $\forall c \in \mathbb{R}$, $E(f < c)$ is a measurable set;
         5. $\forall c, d \in \mathbb{R}$, $E(c \leq f < d)$ is a measurable set.
2. **Algebraic Operations on Measurable Functions:**
   1. **Statement:**
      1. $f, g:\ E \to \mathbb{R}$ are two measurable functions. Then
         1. $\forall \alpha \in \mathbb{R},\ \alpha f$ is still a measurable function;
         2. $f + g$ is a measurable function;
         3. fg is a measurable function;
         4. $\frac{f}{g}$ $(g(x) \neq 0, \forall x \in E)$ is a measurable function;
         5. $\min\{f, g\},\ \max\{f, g\}$ are measurable functions;
         6. $|f|$ is a measurable function.
3. **The Limit of A Sequence of Measurable Functions:**
   1. **Statement:**
      1. $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable function on E. When $\sup_{n\in\mathbb{N}} f_n(x),\ \inf_{n\in\mathbb{N}} f_n(x),\ \varlimsup_{n\to+\infty} f_n(x),\ \varliminf_{n\to+\infty} f_n(x)$ are bounded, they are all measurable functions.
      2. *Review 1: The Supremum and Infimum of Sequence of Functions:*
         1. Both $\sup_{n\in\mathbb{N}} f_n(x)$ and $\inf_{n\in\mathbb{N}} f_n(x)$ are pointwise defined. $\forall$ fixed $x \in \mathbb{R}$, they can be regarded as the supremum and infimum of a sequence (In this case, n does not need to go to infinity, but when considering the cases in Notation 6.1 and Notation 7.1, the result must be a limit of the sequence).
      3. *Review 2: An Example of Difference between Supremum and Limit Superior:*

1. $f_n(x) = \frac{1}{nx} + (-x)^n$, when $x = 1,\ \sup\limits_{n\in\mathbb{N}} f_n(1) = \frac{3}{2},\ \overline{\lim\limits_{n\to+\infty}} f_n(1) = 1$.

2. **Proof Steps:**
   1. $E(\sup\limits_{n\in\mathbb{N}} f_n > c) = \bigcup\limits_{n=1}^{\infty} E(f_n > c)$
   2. $E(\inf\limits_{n\in\mathbb{N}} f_n < c) = \bigcup\limits_{n=1}^{\infty} E(f_n < c)$
   3. Using Notation 6.2 and Notation 7.2.

4. **The Relationship between Borel Measurable Functions and Lebesgue Measurable Functions:**
   1. **Statement:**
      1. Suppose $E \subset \mathbb{R}$, $f : E \to \mathbb{R}$ is a bounded real-valued function,
         1. A Borel measurable function must also be Lebesgue measurable.
         2. If f is a Lebesgue measurable function on E, then there must exist a Borel measurable function h defined on whole $\mathbb{R}$, such that $m^*(E(f \neq h)) = 0$.

6. Theory of Integration 2 (Sequence Convergence)

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Basic Assumptions

1. $(\Omega, \mathcal{F}, \mu)$ is a measure space.
2. E is a set in $\mathcal{F}$.
3. Unless specified, all functions are defined on E.

Notations

1. $f =_\mu h\ (a.e.)$: f equals almost everywhere to h.
2. $f_n \to_\mu f\ (a.e.)$: Sequence $\{f_n\}_{n=1}^{\infty}$ converges almost everywhere to f.
3. $\lim\limits_{n\to+\infty} f_n =_\mu f$: Sequence $\{f_n\}_{n=1}^{\infty}$ converges almost everywhere to f.
4. $f_n \Rightarrow_\mu f$: Sequence $\{f_n\}_{n=1}^{\infty}$ converges in measure μ to f.

Definitions

1. **Almost Everywhere (a.e.):**
   1. P is a proposition concerning the points in E. If $\exists E_0 \subset \Omega$, s.t. $\mu(E_0) = 0$, and $\forall x \in E - E_0,\ P$ holds, then we say P holds **almost everywhere** on E.
   2. *Remark:* $\forall \epsilon > 0,\ \mu(E(|\lim\limits_{n\to+\infty} f_n - f| > \epsilon)) = 0$.
2. **Convergence in Measure:**
   1. $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, if $\exists$ bounded real-valued function f, s.t. $\forall \epsilon > 0,\ \lim\limits_{n\to+\infty} \mu(E(|f_n - f| > \epsilon)) = 0$, then we say $\{f_n\}_{n=1}^{\infty}$ **converges in measure to** $f : f_n \Rightarrow_\mu f$.
   2. *Equivalent Description*: $\forall \epsilon > 0,\ \delta > 0, \exists N = N(\epsilon, \delta) \in \mathbb{N}$, s.t. as long as $n \geq N$, $\mu(E(|f_n - f| > \epsilon)) < \delta$.
   3. *Remark: f doesn't have to be a measurable function, but $f_n - f$ should be.*
3. **Cauchy Sequence in Measure Space:**
   1. $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, if $\forall \epsilon > 0,\ \lim\limits_{n\to+\infty,\, m\to+\infty} \mu(E(|f_n - f_m| > \epsilon)) = 0$, then we say $\{f_n\}_{n=1}^{\infty}$ is a **Cauchy Sequence**.
   2. *Equivalent Description*: $\forall \epsilon > 0,\ \delta > 0, \exists N = N(\epsilon, \delta) \in \mathbb{N}$, s.t. $\forall m, n \geq N$, $\mu(E(|f_n - f_m| > \epsilon)) < \delta$.
4. **Complete Measure:**
   1. μ is said to be a **complete measure** if $\forall E \subset F$, where F is a set of measure zero (or "null set"), $\mu(E) = 0$ and $E \in \mathcal{F}$.
   2. *Remark: This concept avoids the case that $\mu(E)$ is not defined, namely requires any subset of null set to be measurable.*

Theorems

1. **Both the Subsequence and the Sequence Converge in Measure:**
   1. **Statement:**
      1.$\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, if there is a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ converges in measure to a measurable function f, then $f_n \Rightarrow_\mu f$.
2. **The Necessary and Sufficient Condition for Convergence in Measure:**
   1. **Statement:**
      1. $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then $\{f_n\}_{n=1}^{\infty}$ converges in measure $\iff$ $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
      2. $\mu(E) < +\infty$, $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, f is a measurable function on E. Then $f_n \Rightarrow_\mu f \iff$ Every subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ has a further subsequence $\{f_{n_{k_i}}\}_{i=1}^{\infty}$ such that $f_{n_{k_i}} \to_\mu f\ (a.e.)$.

2. **Corollary:**
   1. (Based on Statement 1) $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. If there is a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ converges in measure to a bounded real function h (defined on E), then $\exists$ measurable function f, s.t. $f =_\mu h\ (a.e.), f_n \Rightarrow_\mu f$.

3. **The Relationship between Almost Everywhere Convergence and Convergence in Measure:**
   1. **Statement:**
      1. E is a measurable set, and $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions defined on E that converges almost everywhere to a bounded real-valued function h. Then there exists a measurable function f defined on E such that $f_n \to_\mu f\ (a.e.), f =_\mu h\ (a.e.)$.
      2. $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions defined on E. If $\{f_n\}_{n=1}^{\infty}$ converges in measure to f, then it has a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ such that $f_{n_i} \to_\mu f\ (a.e.)$ on E.
      3. $\mu(E) < +\infty,\ \{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. If f is a measurable function, $f_n \to_\mu f\ (a.e.)$, then $f_n \Rightarrow_\mu f$;
   2. **Corollary:**
      1. (Based on Statement 1, Statement 3) $\mu(E) < +\infty,\ \{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions that converges almost everywhere to a bounded real-valued function h. Then there exists a measurable function f defined on E such that $f_n \Rightarrow_\mu h, f =_\mu h\ (a.e.)$.
   3. **Idea:**
      1. To prove Statement 2, we need to construct a sequence that makes both ϵ and δ in Definition 2.2 ("Convergence in Measure") small enough, so f_n converges to f, and $\mu(E(\cdots)) \to 0$.
      2. In most cases, "small enough" means "an item in a power series".
   4. **Proof Steps:**
      1. To prove Statement 2, we use the definition, $\exists n_i \in \mathbb{N}$, s.t. $\forall n \geq n_i$, $\mu(E(|f_n - f| > \frac{1}{2^i})) < \frac{1}{2^i}$. Therefore $\mu(E(|f_{n_i} - f| > \frac{1}{2^i})) < \frac{1}{2^i}$.
      2. We could assume that $n_1 < n_2 < \cdots$, set $F_k = \bigcap\limits_{i=k}^{\infty}\left(E - E(|f_{n_i} - f| > \frac{1}{2^i})\right) = \bigcap\limits_{i=k}^{\infty}\left(E(|f_{n_i} - f| \leq \frac{1}{2^i})\right) = E(|f_{n_i} - f| \leq \frac{1}{2^i}, \forall i \geq k)$. $\{f_{n_i}\}_{i=1}^{\infty}$ converges uniformly to f, and converges almost everywhere to f on $F = \bigcup\limits_{k=1}^{\infty} F_k$.
      3. Now we only need to prove $\mu(E - F) = 0$. $E - F = E - \bigcup\limits_{k=1}^{\infty} F_k = \bigcap\limits_{k=1}^{\infty}(E - F_k) = \bigcap\limits_{k=1}^{\infty}\left(E - \bigcap\limits_{i=k}^{\infty}(E(|f_{n_i} - f| \leq \frac{1}{2^i})\right) = \bigcap\limits_{k=1}^{\infty}\bigcup\limits_{i=k}^{\infty} E(|f_{n_i} - f| > \frac{1}{2^i}) = \overline{\lim\limits_{i\to+\infty}} E(|f_{n_i} - f| > \frac{1}{2^i})$
      4. $\sum\limits_{i=1}^{\infty}\mu(E(|f_{n_i} - f| > \frac{1}{2^i})) < \sum\limits_{i=1}^{\infty}\frac{1}{2^i} = 1,\ \overline{\lim\limits_{i\to+\infty}} E(|f_{n_i} - f| > \frac{1}{2^i}) = 0$

4. **Properties of Convergence in Measure:**
   1. **Statement:**
      1. $\{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty}$ are two sequences of measurable functions, $f_n \Rightarrow_\mu f,\ g_n \Rightarrow_\mu g$. Then
         1. $f_n \Rightarrow_\mu h$ on $E \Rightarrow f =_\mu h$;
         2. $\exists$ A measurable function h s.t. $f =_\mu h$;
         3. f, g are measurable functions, $\alpha, \beta \in \mathbb{R} \Rightarrow\ \alpha f_n + \beta g_n \Rightarrow_\mu \alpha f + \beta g$;
         4. $|f_n| \Rightarrow_\mu |f|$;
         5. $\mu(E) < +\infty$, f and g are measurable functions on $E \Rightarrow\ f_n g_n \Rightarrow_\mu fg$;
         6. $\mu(E) < +\infty, g_n$ and g are almost everywhere nonzero on E, f and g are measurable functions on $E \Rightarrow \frac{f_n}{g_n} \Rightarrow_\mu \frac{f}{g}$ (Here we can define $\frac{f_n}{g_n}, \frac{f}{g}$ take any value on the set of zero measure, where g_n or g equal to zero).
   2. **Idea:**
      1. Why is it required that the function being approximated is measurable?
         1. To facilitate the construction of a sequence of sets related to the function and contained in E, thereby inducing an approximation: $\{E(h_n > c_n)\}_{n=1}^{\infty}$ or $\{E(h_n < c_n)\}_{n=1}^{\infty}$. Usually h_n is a function related to f and f_n, and converges to a constant.
      2. $A \subset B_n, \forall n \in \mathbb{R} \Rightarrow A \subset \bigcap\limits_{n=1}^{\infty} B_n$.
      3. $E(|f + g| > \epsilon) \subset E(|f| > \frac{\epsilon}{2}) \bigcup E(|g| > \frac{\epsilon}{2})$. This comes from triangle inequality.
      4. $E(|f_n h - fh| > \epsilon) \subset E(|h| > K) \bigcup E(|f_n - f| > \frac{\epsilon}{K})$, h is a function
      5. We decompose the target set to the union of two measurable sets, and thus facilitates estimation.

5. **The Relationship between Almost Everywhere Convergence and Uniform Convergence:**
   1. **Statement:**
      1. $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, $\mu(E) < +\infty$. Then $\{f_n\}_{n=1}^{\infty}$ converges almost everywhere to a bounded real-valued function $f \iff \forall\ \delta > 0,\ \exists$ a measurable set E_δ s.t. $\mu(E - E_\delta) < \delta$, and $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f over E_δ.

6. **Lebesgue Measurable Functions are Almost Continuous Functions (Lusin's Theorem):**
   1. **Statement:**
      1. Suppose $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, f is a Lebesgue measurable function on E. $\forall\delta > 0,\ \exists$ closed set $F_\delta \subset E,$ s.t. $m(E - F_\delta) < \delta$, and f is a continuous function on F_δ.
      2. (Equivalent) Suppose $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, f is a Lebesgue measurable function on E. $\forall\delta > 0,\ \exists$ continuous function h on $\mathbb{R}^n$, satisfy $m(E(f \neq h)) < \delta$.

2. **Corollary:**
    1. Furthermore, suppose $E \subset \mathbb{R}^n$ is bounded, then h has compactly supported set, namely $\text{supp}h = \overline{\{x \in \mathbb{R}^n | h(x) \neq 0\}}$ is compact.
    2. Suppose f is a Lebesgue measurable function and bounded almost everywhere on $E \subset \mathbb{R}^n$, then there exists a sequence of continuous functions $\{f_k\}_{k=1}^{\infty}$ s.t. $\lim\limits_{k\to+\infty} f_k(x) =_m f(x)$ on E.

7. Theory of Integration 3 (Definition of Integration)

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Basic Assumptions

1. $(\Omega, \mathcal{F}, \mu)$ is a measure space.
2. E is a set in $\mathcal{F}$.
3. Unless specified, all functions are defined on E.
4. $0 \cdot \infty = 0$.

Notations

1. $\mathcal{X}_A(x)$: Characteristic function, $\mathcal{X}_A(x) = \mathbb{1}(x \in A)$.
2. $f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$
3. $f^-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

Definitions

1. **Simple Function:**
    1. f is a **simple function** if it is constant on each set in its domain.
2. **Integral:**
    1. **The Integral of A Non-negative Measurable Simple Function:**
        1. Suppose h is a non-negative measurable simple function on $(\Omega, \mathcal{F}, \mu)$ and defined on disjoint sets $\{A_i\}_{i=1}^p$, $h(x) = \sum\limits_{i=1}^{p} c_i \mathcal{X}_{A_i}(x)$. The **integral** of h over E is (L) $\int_E h d\mu = \sum\limits_{i=1}^{p} c_i \mu(E \bigcap A_i)$
    2. **The Integral of A Non-negative Measurable Function:**
        1. Suppose h is a non-negative generalized measurable function. The **integral** of h over E is (L) $\int_E f d\mu = \sup\limits_{h \leq f} \{\int_E h d\mu | h$ is a non-negative measurable simple function on $(\Omega, \mathcal{F}, \mu)\}$.
    3. **The Integral of A General Measurable Function:**
        1. Suppose f is a generalized measurable function, if at least one of the integrals of $f^+(x)$ and $f^-(x)$ is bounded, we set $\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$ as the **integral** of f over E.
    4. *Remark:*
        1. *There are three steps to establish the theory of integration, from a non-negative measurable simple function to a general measurable function, the objects of integration are gradually generalized.*
        2. *The difference between "integral" and "integration": Integral is the result of the integration itself, which can be a definite or indefinite integral, while integration is the process or method of finding an integral.*
        3. *"Lebesgue integral" doesn't mean the measure in integral must be Lebesgue measure.*
        4. *A measurable function must have integration (infinity is allowed), but it does not have to be integrable.*
3. **Generalized Integrable Function:**
    1. For General Case:
        1. f is called a **generalized integrable function** if one of $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ is bounded.
4. **Integrable Function:**
    1. For Special Case (When function is non-negative measurable):
        1. f is called a **integrable function** if $\int_E f d\mu$ is bounded.
    2. For General Case:
        1. f is called a **integrable function** if both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are bounded.
    3. *Remark 1: Measurable and integrable are different concepts.*
    4. *Remark 2: From the definition, we know an integrable function must be measurable.*

Theorems

1. **Levi's Monotone Convergence Theorem (Limits and Integrals Can Be Interchanged):**
    1. **Statement:**

1. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of non-negative measurable monotone increasing functions, namely $f_1(x) \leq f_2(x) \leq \cdots f_k(x) \leq \cdots$; $\lim\limits_{k\to+\infty} f_k(x) = f(x),\ \forall x \in E$, then $\lim\limits_{k\to+\infty} \int_E f_k d\mu = \int_E f d\mu = \int_E \lim\limits_{k\to+\infty} f_k d\mu$

2. **Proof Steps:**
   1. Because the limit of measurable function is still measurable, we know f is measurable. $\int_E f d\mu$ exists.
   2. $\int_E f_k d\mu$ is monotone increasing, so it also exists.
   3. $f_k \leq f$, $\lim\limits_{k\to+\infty} \int_E f_k d\mu = \int_E f d\mu$.
   4. We set $\lambda \in (0,1)$, h is a non-negative measurable simple function, $E_k = E(f_k(x) \geq \lambda h(x))$, $\lim\limits_{k\to+\infty} E_k = \bigcup\limits_{k=1}^{\infty} E_k = E$,

      $\lim\limits_{k\to+\infty} \int_{E_k} h d\mu = \int_E h d\mu$
   5. $\lim\limits_{k\to\infty} \int_E f_k d\mu \geq \lim\limits_{k\to+\infty} \lambda \int_{E_k} h d\mu = \lambda \int_E h d\mu$, let $\lambda \to 1^-$.

2. **A Function and Its Absolute Value are Integrable Simultaneously:**
   1. **Statement:**
      1. Suppose f is a generalized measurable function, then f is integrable $\iff$ $|f|$ is integrable.

3. **Properties of Functions with Integral Equal to 0:**
   1. **Statement:**
      1. Suppose f is a non-negative integrable function on E,
         1. $f >_\mu 0\ a.e.$, $\int_E f d\mu = 0 \Rightarrow \mu(E) = 0$;
         2. $\int_E f d\mu = 0 \Rightarrow f =_\mu 0\ a.e.$

8. Theory of Integration 4 (Theorem on Convergence of Integrals)

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Basic Assumptions

1. $(\Omega, \mathcal{F}, \mu)$ is a measure space.
2. E is a set in $\mathcal{F}$.
3. Unless specified, all functions are defined on E.
4. $0 \cdot \infty = 0$.

Definitions

1. **Control Function:**
   1. In dominated convergence theorem (see below), the function F is called "**control function**".

Theorems

1. **Dominated Convergence Theorem:**
   1. **Statement:**
      1. $\{f_n\}_{n=1}^{\infty}$ is a sequence of generalized measurable functions, and there exists a non-negative function F s.t. $|f_n| \leq_\mu F\ (a.e.),\ n = 1, 2, \cdots$. If F is integrable, $\{f_n\}_{n=1}^{\infty}$ converges in measure to a generalized measurable function f, namely $f_n \Rightarrow_\mu f$, then f is also integrable, and $\lim\limits_{n\to+\infty} \int_E f_n d\mu = \int_E f d\mu$.
      2. $\{f_n\}_{n=1}^{\infty}$ is a sequence of generalized measurable functions, and there exists a non-negative function F s.t. $|f_n| \leq_\mu F\ (a.e.),\ n = 1, 2, \cdots$. If F is integrable, $\{f_n\}_{n=1}^{\infty}$ converges almost surely to a generalized measurable function f, namely $f_n \to_\mu f\ (a.e.)$, then f is also integrable, and $\lim\limits_{n\to+\infty} \int_E f_n d\mu = \int_E f d\mu$.
   2. **Corollary:**
      1. (Bounded Convergence Theorem) $\mu(E) < +\infty$, $\{f_n\}_{n=1}^{\infty}$ is a sequence of generalized measurable functions, $|f_n| \leq_\mu K,\ a.e. \quad n = 1, 2, \cdots$, where K is a constant. If $\{f_n\}_{n=1}^{\infty}$ converges almost surely (or in measure) to a generalized measurable function f, then f is integrable, and $\lim\limits_{n\to+\infty} \int_E f_n d\mu = \int_E f d\mu$.

2. **Monotone Convergence Theorem/ Levi's Lemma:**
   1. **Statement:**
      1. $\{f_n\}_{n=1}^{\infty}$ is a sequence of monotone increasing (or decreasing) integrable functions, if $\sup\limits_n\{\int_E f_n d\mu\} < +\infty$ (or $\inf\limits_n\{\int_E f_n d\mu\} > -\infty$), then $\{f_n\}_{n=1}^{\infty}$ converges almost surely to an integrable function f, and $\lim\limits_{n\to+\infty} \int_E f_n d\mu = \int_E f d\mu$.
      2. $\{u_n\}_{n=1}^{\infty}$ is a sequence of non-negative integrable functions, if $\sum\limits_{n=1}^{\infty} \int_E u_n d\mu < +\infty$, then $\left\{\sum\limits_{n=1}^{m} u_n\right\}_{m=1}^{\infty}$ converges almost surely to an integrable function f, and $\int_E f d\mu = \int_E \sum\limits_{n=1}^{\infty} u_n d\mu = \sum\limits_{n=1}^{\infty} \int_E u_n d\mu$

3. **Fatou's Lemma:**
   1. **Statement:**

1. $\{f_n\}_{n=1}^{\infty}$ is a sequence of integrable functions. If there exists an integrable function h, s.t. $f_n \geq_\mu h\ a.e.$ (or $f_n \leq_\mu h\ a.e.$), $n = 1, 2, \cdots$, and $\underline{\lim}_{n\to+\infty} \int_E f_n d\mu < +\infty$ (or $\overline{\lim}_{n\to+\infty} \int_E f_n d\mu > -\infty$), then the function $\underline{\lim}_{n\to+\infty} f_n$ (or $\overline{\lim}_{n\to+\infty} f_n$) is integrable, and $\int_E \underline{\lim}_{n\to+\infty} f_n d\mu \leq \underline{\lim}_{n\to+\infty} \int_E f_n d\mu$ (or $\int_E \overline{\lim}_{n\to+\infty} f_n d\mu \geq \overline{\lim}_{n\to+\infty} \int_E f_n d\mu$).

4. **The Equivalence of Convergence Theorems of Integrals:**
   1. **Statement:**
      1. Theorem 1.1 (Dominated Convergence Theorem) $\iff$ Theorem 2.1 (Monotone Convergence Theorem) $\iff$ Theorem 3.1 (Fatou's Lemma)

9. Theory of Integration 5 (Fubini's Theorem)

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Basic Assumptions

1. If not specified, all integrals are Lebesgue integral.
2. $(L) \int_E f(x)dx$ means $(L) \int_E f(x)dm$, where m is Lebesgue measure.
3. $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$.

Notations

1. $\mathcal{G}$: Suppose $f(\mathbf{x}, \mathbf{y})$ is a non-negative generalized measurable function defined on $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$, $\mathcal{G}$ is the set of all $f(\mathbf{x}, \mathbf{y})$ satisfy
   1. Concerning the Lebesgue measure, for almost all $\mathbf{x} \in \mathbb{R}^p$, $f(\mathbf{x}, \mathbf{y})$ as a function of y is a non-negative generalized measurable function on $\mathbb{R}^q$;
   2. $F_f(\mathbf{x}) = \int_{\mathbb{R}^q} f(\mathbf{x}, \mathbf{y})d\mathbf{y}$ is a non-negative generalized measurable function on $\mathbb{R}^p$;
   3. $\int_{\mathbb{R}^n} f(\mathbf{x}, \mathbf{y})d\mathbf{x}d\mathbf{y} = \int_{\mathbb{R}^p} F_f(\mathbf{x})d\mathbf{x}$.
2. $\mathcal{L}(E)$: The set of all Lebesgue measurable functions defined on E.
3. $\mathcal{P}$: $\{A \times B | A \in \mathcal{F}_X, B \in \mathcal{F}_Y\}$.
4. $\mathcal{F}_X \times \mathcal{F}_Y$: $\mathcal{F}_X$ and $\mathcal{F}_Y$ are σ -rings, $\mathcal{F}_X \times \mathcal{F}_Y = \sigma(\mathcal{P})$.
5. $\mathcal{F}_X \widehat{\times} \mathcal{F}_Y$: $\mathcal{F}_X$ is a ring of subsets of X, $\mathcal{F}_Y$ is a ring of subsets of Y, $\mathcal{F}_X \widehat{\times} \mathcal{F}_Y = \{C | C = \bigcup_{i=1}^{n}(A_i \times B_i), A_i \in \mathcal{F}_X, B_i \in \mathcal{F}_Y, (A_i \times B_i) \bigcap (A_j \times B_j) = \varnothing (i \neq j)\}$.
6. E_x: Cross-section of E determined by x.
7. $\mu \times \upsilon$: Product measure of μ and υ.

Definitions

1. **Product Measurable Space, Measurable Rectangle:**
   1. Suppose $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ are two measurable spaces, we set $\mathcal{P} = \{A \times B | A \in \mathcal{F}_X, B \in \mathcal{F}_Y\}$, $\mathcal{F}_X \times \mathcal{F}_Y = \sigma(\mathcal{P})$, then we call $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)$ a **product measurable space** of $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$, $A \times B \in \mathcal{P}$ a **measurable rectangle**.
2. **Cross-Section, Cross-Section Function:**
   1. Based on Definition 1, suppose $E \subset X \times Y$, $f : E \to \mathbb{R}$ is a function,
      1. $E_x = \{y \in Y | (x, y) \in E\}$ is a **cross-section** of E determined by x;
      2. $E^y = \{x \in X | (x, y) \in E\}$ is a **cross-section** of E determined by y;
      3. For fixed $x \in X$, $f_x : E_x \to \mathbb{R}$, $f_x(y) = f(x, y)$, f_x is called a **cross-section function** determined by x;
      4. For fixed $y \in Y$, $f^y : E^y \to \mathbb{R}$, $f^y(x) = f(x, y)$, f^y is called a **cross-section function** determined by y.
3. **Product Measure:**
   1. Suppose $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \upsilon)$ are two sigma-finite measure spaces, we define a generalized set function λ on $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)$ as follows:
      1. $\forall E \in \mathcal{F}_X \times \mathcal{F}_Y,\ A \times B \in \mathcal{F}_X \times \mathcal{F}_Y$, where $\mu(A) < +\infty, \upsilon(B) < +\infty, E \subset A \times B$, then we set $\lambda(E) = \int_A \upsilon(E_x)d\mu = \int_B \mu(E^y)d\upsilon$;
      2. $\forall E \in \mathcal{F}_X \times \mathcal{F}_Y$, by Theorem 5.3, there exists a sequence of rectangles $\{F_n = A_n \times B_n\}_{n=1}^{\infty}$ satisfy $F_n \in \mathcal{F}_X \times \mathcal{F}_Y, \mu(A_n) < +\infty, \upsilon(B_n) < +\infty, F_1 \subset F_2 \subset \cdots, E \subset \bigcup_{n=1}^{\infty} F_n$. We set $\lambda(E) = \lim_{n\to+\infty} \lambda(E \bigcap F_n)$. Also by Theorem 5.3 we know λ is a sigma-finite measure on $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)$.

      Then we say $\lambda = \mu \times \upsilon$ is the **product measure** of μ and υ.
4. **Multiple Integral, Iterated Integral:**
   1. Suppose $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \upsilon)$ are two sigma-finite measure spaces, $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu \times \upsilon)$ is the product measure space, $E \in \mathcal{F}_X \times \mathcal{F}_Y,\ E = A \times B,\ A \in \mathcal{F}_X,\ B \in \mathcal{F}_Y,\ f : E \to \mathbb{R}$ is a function. If f is $\mu \times \upsilon$ integrable on E,
      1. We call $\int_E f(x, y)d(\mu \times \upsilon)(x, y) = \int_E f(x, y)d(\mu \times \upsilon)$ the **multiple integral** of f on E.
      2. We call $\int_B(\int_A f^y(x)d\mu(x))d\upsilon(y) = \int_B d\upsilon(y) \int_A f d\mu(x)$ and $\int_A(\int_B f_x(y)d\upsilon(y))d\mu(x) = \int_A d\mu(x) \int_B f d\upsilon(y)$ the **iterated integrals** of f on E.

Theorems

1. **Properties of $\mathcal{G}$:**
   1. **Statement:**
      1. If $f \in \mathcal{G}, \alpha > 0$, then $\alpha f \in \mathcal{G}$;
      2. If $f_1, f_2 \in \mathcal{G}$, then $f_1 + f_2 \in \mathcal{G}$;
      3. If $f, g \in \mathcal{G}, f(\mathbf{x}, \mathbf{y}) - g(\mathbf{x}, \mathbf{y}) \geq 0$ and $g \in \mathcal{L}(\mathbb{R}^n)$, then $f - g \in \mathcal{G}$;
      4. $f_k \in \mathcal{G}, f_k(\mathbf{x}, \mathbf{y}) \leq f_{k+1}(\mathbf{x}, \mathbf{y})$, and $\lim\limits_{k \to +\infty} f_k(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}, \mathbf{y})$, then $f \in \mathcal{G}$.
2. **Fubini's Theorem for Lebesgue Integrable Functions:**
   1. **Statement:**
      1. Suppose $f \in \mathcal{L}(\mathbb{R}^n)$, then
         1. Concerning the Lebesgue measure, for almost all $\mathbf{x} \in \mathbb{R}^p$, $f(\mathbf{x}, \mathbf{y})$ is generalized measurable on $\mathbb{R}^q$;
         2. $\int_{\mathbb{R}^q} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$ is integrable on $\mathbb{R}^p$;
         3. $\int_{\mathbb{R}^n} f(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = \int_{\mathbb{R}^p} d\mathbf{x} \int_{\mathbb{R}^q} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^q} d\mathbf{y} \int_{\mathbb{R}^p} f(\mathbf{x}, \mathbf{y}) d\mathbf{x}$
3. **Properties of $\widehat{\mathcal{F}_X \times \mathcal{F}_Y}$:**
   1. **Statement:**
      1. $\widehat{\mathcal{F}_X \times \mathcal{F}_Y}$ is a ring;
      2. $\sigma(\widehat{\mathcal{F}_X \times \mathcal{F}_Y}) = \mathcal{F}_X \times \mathcal{F}_Y$;
4. **Properties of Cross-Section, Cross-Section Function:**
   1. **Statement:**
      1. On the product measurable space $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)$
         1. A cross-section of a measurable set is measurable;
         2. A cross-section function of a measurable function is measurable.
5. **The Foundation of Establishing Product Measures**:
   1. **Statement:**
      1. Suppose $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \upsilon)$ are two totally finite measure spaces, E is a measurable set in $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)$, then $\upsilon(E_x)$ and $\mu(E^y)$ are measurable functions on $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \upsilon)$ respectively, and $\int_X \upsilon(E_x) d\mu = \int_Y \mu(E^y) d\upsilon$.
      2. Suppose $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \upsilon)$ are two measure spaces, $A_0 \in \mathcal{F}_X, B_0 \in \mathcal{F}_Y$, and $\mu(A_0) < +\infty, \upsilon(B_0) < +\infty$. If $E \in \mathcal{F}_X \times \mathcal{F}_Y$ and $E \subset A_0 \times B_0$, then $\upsilon(E_x), \mu(E^y)$ are measurable functions on A_0, B_0 respectively, and $\int_{A_0} \upsilon(E_x) d\mu = \int_{B_0} \mu(E^y) d\upsilon$.
      3. Suppose $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \upsilon)$ are two sigma-finite measure spaces, λ in Definition 3.1.1 is the only sigma-finite measure on $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)$ satisfy $\lambda(A \times B) = \mu(A)\upsilon(B),\ A \in \mathcal{F}_X, B \in \mathcal{F}_Y$.
   2. **Proof Steps:**
      1. To prove Statement 1:
         1. For $E = A \times B \in \mathcal{F}_X \times \mathcal{F}_Y$, since $E_x = \begin{cases} B & \text{if } x \in A \\ \varnothing & \text{if } x \notin A \end{cases}$, we have $\upsilon(E_x) = \upsilon(B)\chi_A(x)$, then $\int_X \upsilon(E_x) d\mu = \mu(A)\upsilon(B)$.
6. **Fubini's Theorem:**
   1. **Statement:**
      1. Suppose E is a sigma-finite measurable rectangle on $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu \times \upsilon)$, $E = A \times B$, $f : E \to \mathbb{R}$,
         1. If f is integrable concerning $\mu \times \upsilon$, then the iterated integrals of f exist and are bounded, and $\int_E f d(\mu \times \upsilon) = \int_A d\mu(x) \int_B f d\upsilon(y) = \int_B d\upsilon(y) \int_A f d\mu(x)$;
         2. If f is measurable, and one of iterated integrals of $|f|$ exists and is bounded ($\int_A d\mu(x) \int_B |f(x, y)| d\upsilon(y)$ or $\int_B d\upsilon(y) \int_A |f(x, y)| d\mu(x)$), then the other also exists and is bounded, and the formula in the previous statement holds.

10. Summary of Proof Techniques

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1. Split up equalities into inequalities:
   1. $f = g \iff f \leq g$ and $g \leq f$
   2. $X = Y \iff X \subseteq Y$ and $Y \subseteq X$
2. Relax the boundaries, give yourself an epsilon of room:
   1. If we have to show $f \leq g$, we could prove $f \leq g + \epsilon$ instead.
   2. If we have to show f and g agree almost everywhere, we could try to show $|f - g| \leq \epsilon$ a.e.
3. . Decompose (or approximate) a rough or general object into (or by) a smoother or simpler one:
   1. Decompose the target to the union or intersection of open, closed, compact, bounded, or elementary term at first.
   2. Construct sequence to approximate.
4. If one needs to flip an upper bound to a lower bound or vice versa, look for a way to take reflections or complements:

    1. Turn f to $F-f$
5. Uncountable unions can sometimes be replaced by countable or finite unions.
6. If it is difficult to work globally, work locally instead:
    1. If we can't prove a proposition for whole space, we could prove it when constrained in a large ball, and utilize properties such as arbitrariness or compactness
7. Abstract away any information that you believe or suspect to be irrelevant.
8. Exploit Zeno's paradox: a single epsilon can be cut up into countably many sub-epsilons:
    1. $\epsilon=\sum_{i=1}^{\infty}\frac{\epsilon}{2^i}$
9. If you expand your way to a double sum, a double integral, a sum of an integral, or an integral of a sum, try interchanging the two operations.
10. Pointwise control, uniform control, and integrated (average) control are all partially convertible to each other.
11. One can often pass to a subsequence to improve the convergence properties.
12. A real limit can be viewed as a meeting of the limit superior and limit inferior.

End

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