5

Where do we stand?

Rademacher chaos $\sum_{j,k=1}^{n} \xi_j \xi_k A_{jk}$

(A has zero diagonal)

We aim to prove:

$$P(|X| > t) \leq \begin{cases} 2\exp\left(-\frac{3\varepsilon^2}{128\|A\|_F^2}\right) & t \leq \frac{4\|A\|_F^2}{3\|A\|} \\ 2\exp\left(-\frac{t}{32\|A\|}\right) & \text{else} \end{cases}$$

so far: For $0 < \theta < \frac{1}{16\|A\|}$: $P(X \geq t) \leq \exp\left(-\theta t + \frac{32}{3}\theta^2\|A\|_F^2\right)$

Optimal choose of θ $\quad \theta_{opt} = \frac{3t}{64\|A\|_F^2}$

Recall: We need $0 < \theta \leq \frac{1}{16\|A\|}$

$$\Rightarrow \frac{3t}{64\|A\|_F^2} \leq \frac{1}{16\|A\|}$$

For $t > \dfrac{4\|A\|_F^2}{3\|A\|}$, set $\theta = \dfrac{1}{16\|A\|}$ (as large as possible)

$$P(X \geq t) \leq \exp\left(-\theta t + 3\theta^2 \cdot \frac{\|A\|_F^2}{3}\right)$$

$$\leq \exp(-\theta t + 8\theta^2 \|A\| t)$$

$\frac{\|A\|_F^2}{3} < \frac{\|A\| t}{4}$

$$\leq \exp\left(-\theta t + \frac{\theta}{2} t\right)$$

$$= \exp\left(-\frac{\theta t}{2}\right) = \exp\left(-\frac{t}{32\|A\|}\right)$$

By symmetry of A ← Don't flip ε, otherwise we get the same thing

$$\tilde{X} = \sum_{j,k=1}^{m} \varepsilon_j \varepsilon_k (-A_{jk}) \quad \text{yields}$$

$$P(|X| > t) = P(X > t) + P(-X > t)$$

$$= P(X > t) + P(\tilde{X} > t)$$

$$\leq \begin{cases} 2 \cdots \\ 2 \cdots \end{cases}$$

$\|A\| = \|-A\|$

$\|A\|_F = \|-A\|_F$

2.4 Sub-gaussian r.v.

Motivation: Gaussian distribution are well behaved. What exactly do we mean by well behaved?

Recall: standard normed of X

$\rightarrow$ dis of X has density $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

(denoted by $N(0,1)$)

$\rightarrow$ Normed distribution with mean μ and variance σ^2

$\rightarrow$ distribution has density $\frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

(denoted by $N(\mu, \sigma^2)$)

for standard only ↓

$$\begin{cases} \text{Well-behavior tails: } P(|X|>t) \le 2e^{-\frac{t^2}{2}}, t\ge 1 \\ \text{Well-behaviored moment (finite)} \\ \qquad (E|X|^p)^{\frac{1}{p}} = O(\sqrt{p}) \cdot p \ge 1 \\ \text{Well-behaviored moment generate function } E\exp(tX) = e^{\frac{t^2}{2}} \end{cases}$$

These three properties shall serve our idea of well-behaviored distribution

A r.v. (or more property associated with distri) satisfy above three then will be called subgaussian.

Actually these three are equiv

Lemma: $p^p \ge p! \ge \left(\frac{p}{e}\right)^p$ p-factorial,

Pf: $p! \le p^p$ follows from the def of $p!$

second inequ : <u>induction</u>

$$p! = p \cdot (p-1)! \geq p \cdot \left(\frac{p-1}{e}\right)^{p-1}$$

$$= \frac{p^p}{e^p} \frac{e}{\left(1+\frac{1}{p-1}\right)^{p-1}}$$

$$\geq \frac{p^p}{e^p}$$

Lemma 2.12 (equivalance of subgaussian properties)

Let X be r.v.. Then the following properties are equiv with parameter $K_i > 0$ diff from each other by at most an <u>absolute constant</u>

C does not depend on X

factor.

(More precisely, there exists an absolute constant C s.t. $K_j \le C K_i$ for any two parameters $i, j = 1, 2, 3, 4$)

1. Tail: $P\{|X| > t\} \le \exp\left(1 - \frac{t^2}{K_1^2}\right) \quad \forall t \ge 0$

2. Moment: $(E|X|^p)^{\frac{1}{p}} \le K_2 \sqrt{p}$ for all $p \ge 1$

3. Super exponential moment: $E\exp\left(\frac{X^2}{K_3^2}\right) \le e$

Moreover, if $EX = 0$, then properties 1-3 are also equiv to

4. Moment generating function

$$E\exp(tX) \le \exp(t^2 K_4^2) \quad \forall t \in \mathbb{R}$$

Pf: All properties are non-generating,

that is X satisfies the property with K_i

$\Leftrightarrow$ $\tilde{X} = \frac{X}{K_i}$ satisfy the property with $\tilde{K}_i = 1$

For example, for property 4

$$E\exp(t\tilde{X}) = E\exp\left(t\frac{X}{K_4}\right)$$
$$\leq \exp\left(\frac{t}{K_4}\right)^2$$

So we can always assume $K_i = 1$ and

show $K_j \leq C$

$1 \Rightarrow 2$: $E|X|^p = E\int_0^\infty 1_{\{y \leq |X|^p\}}\, dy$

$$\overset{\text{Fubini}}{=} \int_0^\infty \left(E\, 1_{\{|X|^p \geq y\}}\right) dy$$

$$= \int_0^\infty P(|X|^p \geq y)\, dy$$

$$= \int_0^\infty P(|X| \geq t)\, p t^{p-1}\, dt$$

property 1

$$\leq \int_0^\infty e^{1-t^2}\, p t^{p-1}\, dt$$

$$= \frac{ep}{2}\, \Gamma\left(\frac{p}{2}\right)$$

$$\leq \left(\frac{ep}{2}\right)\left(\frac{p}{2}\right)^{\frac{p}{2}}$$

$$(E|X|^p)^{\frac{1}{p}} \leq \left(\frac{ep}{2}\right)^{\frac{1}{p}}\left(\frac{p}{2}\right)^{\frac{1}{2}}$$

$p^{\frac{1}{p}}$ is bounded

$2 \Rightarrow 3$: Let $c > 0$.

Writing the Taylor series of the exp function, we obtain

$$E\exp(cX^2) = 1 + \sum_{i=1}^{\infty} c^i \frac{EX^{2p}}{p}$$ as all terms ≥ 0

$$\overset{\text{property 2}}{\leq} 1 + \sum_{p=1}^{\infty} \frac{c^p (2p)^p}{p!}$$

$$\overset{p! \geq (\frac{p}{e})^p}{\leq} 1 + \sum_{p=1}^{\infty} (2ce)^p$$

$$\leq \frac{1}{1-2ce} = \frac{e}{e-2ce^2} < e$$

$2ce < 1$ for $c < \frac{e-1}{2e^2}$

$$E\exp\left(\frac{e-1}{2e^2} X^2\right) < e$$

3.31:

$$P\{|X|>t\} = P\{e^{X^2} \geq e^{t^2}\}$$

$$\leq e^{-t^2} E e^{X^2}$$

$$\underset{\substack{\uparrow \\ \text{property 3}}}{\leq} e^{1-t^2}$$