3

Where do we stand?

ε-net, covering number (cardinality of smallest ε-net)

covering number of the sphere in $(\mathbb{R}^n, \|\ \|) \leq (1+\frac{2}{\varepsilon})^n$

Lemma: for $A \in \mathbb{R}^{n\times n}$ and W_ε ε-net of S^{n-1}, $\varepsilon \in (0,1)$

$$\sup_{x\in N_\varepsilon} \|Ax\|_2 \leq \|A\| \leq (1-\varepsilon)^{-1} \sup_{x\in W_\varepsilon} \|Ax\|_2$$

Lemma 2.7: Let A be a symmetric $n\times n$ matrix and let N_ε be a ~~finite~~ X (don't need) (we take sup) ε-net of S^{n-1} for some $\varepsilon \in (0,1)$ ← w.r.t. $\|\cdot\|_2$

Then

$$\|A\| = \overset{\overset{\text{max}}{\parallel}}{\sup_{x\in S^{n-1}}} |\langle Ax, x\rangle| \leq (1-2\varepsilon)^{-1} \sup_{x\in N_\varepsilon} |\langle Ax, x\rangle|$$

$\subseteq S$

Proof. Let us choose $x \in S^{n-1}$ for which $\|A\| = |\langle Ax, x\rangle|$, and choose $y \in \mathcal{N}_\varepsilon$ which approximates x as $\|x - y\|_2 \le \varepsilon$. By the triangle inequality we have

$$|\langle Ax, x\rangle - \langle Ay, y\rangle| \ominus |\langle Ax, x - y\rangle + \langle A(x - y), y\rangle|$$
$$\le \|A\|\|x\|_2\|x - y\|_2 + \|A\|\|x - y\|_2\|y\|_2 \le 2\varepsilon\|A\|.$$

← we can't use tri-ineq for both components, but only seperately

It follows that $|\langle Ay, y\rangle| \ge |\langle Ax, x\rangle| - 2\varepsilon\|A\| = (1 - 2\varepsilon)\|A\|$. Taking the maximum over all $y \in \mathcal{N}_\varepsilon$ in this inequality completes the proof. □

2.3 Non-asy results in 1-D

↳ number of r.v. $\not\to$ infty

Last time: Asymptotic → Non-Asy theory

Most results discussed in Prob Theory are asy

— — — — —

Here are some non-asy results:

Prop 2.8 (Hoeffding's ineq)

Let $X_1, \cdots, X_n$ be a sequence of independent real-valued r.v. s.t. $E(X_i) = 0$ and $|X_i| \le B_i$ a.s. for all $i = 1, \cdots, n$ for some $B_i > 0$. Then

$$P(\sum X_i > t) \le e^{-\frac{t^2}{2\sum B_i^2}}$$

(we want to see how fast the convergence in CLT is)

and consequently

$$P(|\sum X_i| > t) \le 2e^{-\frac{t^2}{2\sum B_i^2}}$$

(preview: bound $\rightarrow$ sub-Gaussian)

proposition 2.9 (Berstein ineq)

Let $X_1, \cdots, X_n$ are inde mean-zero r.v.

s.t. $E(|X_i|^m) \le m!\, K^{m-2} \frac{\sigma_i}{2}$ for all

$i = 1, \cdots, n$, $m \in \mathbb{N}$, $n \ge 2$

for some constant $K > 0$ and $\sigma_i > 0$, $i = 1, \cdots, n$

Then $\forall t > 0$

$$P(|\sum_{i=1}^{n} X_i| > t) \le 2e^{-\frac{\frac{t^2}{2}}{\sigma^2 + kt}}$$

who dominates? $\begin{cases} t > \sqrt{n} & kt \\ t \text{ small} & \sigma^2 \end{cases}$

where $\sigma^2 := \sum \sigma_i^2$

individual	VS.	Gauss
n small		n large

high moments describe
indivi

both _____ : sum of indep r.v.
?

Now: One look for sum of depe r.v.

simplest possible dependence:

products of two independent r.v.

$$X = \sum_{i,j} a_{ij} X_i X_j \quad \text{where } X_i \text{ i.i.d.} \quad 1 \le i \le n$$

Even most specific

ε_i : Rademacher variable (symmetric Bernoulli)

$$P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$$

Rade class:

$$\sum_{i \neq j} a_{ij} \varepsilon_i \varepsilon_j$$

$\uparrow$

diagonal terms are deterministic

prob: ε_i and ε_j are the "same" variable

cannot be treated seperately

Idea: Decoupling

$\rightarrow$ compare to the case where ε_i and ε_j' are inde sequences

Lemma 2.10: let $\xi = \{\xi_j\}_{j=1}^n$ be a sequence of inde r.v. with $E\xi_j = 0 \ \forall j$,

let A_{jk} $(1 \le j, k \le M)$

let a doubling-indexed sequence of elements in a vector space X.

Let $F: X \to \mathbb{R}$ be a convex function

Then $EF(\sum_{j,k} q_j q_k A_{jk}) \le EF(4 \sum_{j,k} q_j q_k' A_{jk})$

ξ_j' is an independent copy of ξ_j

Jensen's inequality:

let $(\Omega, \mathcal{B}, P)$ be a prob space,

$X \in$ vector space V

$\phi: V \to \mathbb{R}$ convex

$E|X| < \infty$, $E|\phi(X)| < \infty$

Then $E\phi(X) \geq \phi(EX)$

Markov's inequa

Let X be a r.v. as

$$P(|X| \geq t) \leq \frac{E|X|}{t} \quad \forall t > 0$$