8

Isotropic R. vector

A random vector is a random var with values in $\mathbb{R}^n$ or, equiv, a vector with random entries.

$(Z_1, \cdots, Z_N)$

The mean $\mu = EZ$ is given by

$\mu = (EZ_1, \cdots, EZ_n)$

Analog to second moment: ~~sess~~

second moment matrix

$$\mathbb{R}^{n\times n} \ni \Sigma = \Sigma(Z) = E\, Z \otimes Z = E(ZZ^T)$$

outer product

(consists of expectations of products of any two events)

Analog to variance:

Covariance matrix of a random vector Z.

$$\mathrm{Cov}(Z) = E((Z-\mu)\otimes(Z-\mu))$$
$$= E(Z\otimes Z) - \mu\otimes\mu$$

For centered Z, $\mu = 0 \rightarrow \mathrm{Cov}(X) = \Sigma(X)$

Analog to unit variance

Defining 2.23 (isotropic random vector)

A random vector $X \in \mathbb{R}^n$ is called isotropic if $\Sigma(X) = I$.

(another def: $E\langle X, x\rangle^2 = \|x\|^2 \ \forall x \in \mathbb{R}^n$)

Prop 2.24:

X is isotropic iff $E\langle X, x\rangle^2 = \|x\|_2^2$

$\forall\ x \in \mathbb{R}^n$.

Pf: "$\Rightarrow$"

Let x be isotropic. Then

$$E\langle X, x\rangle^2 = E\left(\sum_{j=1}^{n} X_j x_j\right)^2$$

$$= \cdots$$

$$= \sum_j x_j^2$$

$$= \cdots$$

$\Leftarrow$: Let $E\langle X, x\rangle^2 = \|x\|_2^2$ for all $x \in \mathbb{R}^n$

To prove $(E X \otimes X)_{jj} = 1$, consider $x = e_j = (0, \cdots, \underset{\substack{\uparrow \\ j\text{-th}}}{0, 1}, 0, \cdots, 0)$

Then $1 = \|e_j\|_2^2 = E\langle X, e_j\rangle^2 = EX_j^2$

To show that $(Cov(X))_{jk} = 0$ for $j \neq k$

consider $x = e_j + e_k$

Then $2 = \|e_j + e_k\|_2^2 =$

$$= E\langle X, e_j + e_k\rangle^2$$

$$= E(X_j + X_k)^2$$

$$= \cdots$$

$\Rightarrow EX_jX_k = 0$

Remark: If $\Sigma(X)$ is not invertible

hence $\exists\, x \neq 0 \in \mathbb{R}^n$ s.t. $\Sigma(X)x = 0$

Then $E\langle X, x\rangle^2 = Ex\, X^T X x = x(EX^T X)x$

$$= \Sigma(X)x = 0$$

$\Rightarrow \langle X, x\rangle = 0$ a.s.

$\rightsquigarrow$ X is essentially supported on the proper

subspace $\{y : y \perp x\}$

Hence if X does not live in a proper subspace, then $\Sigma(X)$ is invertible

Furthermore, XX^T is positive semidef a.s.

$$\Downarrow$$

$$\Sigma(X) \text{ is } \ldots \text{ (p.s.d.)}$$

$$\Downarrow$$

$\Sigma(X)$ is positive define and we can decompose $(\Sigma(X))^{-\frac{1}{2}}$

Consider $\tilde{X} = \Sigma(X)^{-\frac{1}{2}} X$

Then

$\Sigma(\tilde{X}) = E\tilde{X}\tilde{X}^T$

$$= \Sigma(X)^{-\frac{1}{2}} E XX^T \Sigma(X)^{-\frac{1}{2}}$$

$$= \Sigma(X)^{-\frac{1}{2}} \Sigma(X) \Sigma(X)^{-\frac{1}{2}}$$

$$= I$$

Then every non-degenerate r. vec can be made isotropic by ~~appropriate~~ linear transformation.

However the picture can be destroied when eigenvalues of $\Sigma(X)$ vastly different.

We will mostly study isotropic random vec.

Lemma 2.25

Let X, Y be independent isotropic random vec in $\mathbb{R}^n$. Then $E\|X\|_2^2 = n$ and $E\langle X, y\rangle^2 = n$.

Pf: $E\|X\|_2^2 = E\,tr(X \otimes X)$

$$= tr(E\, X \otimes X)$$

$$= tr\, I$$

$$= n$$

$$E\langle X, y\rangle^2 = E(E\langle X, y\rangle^2 \mid X)$$

Prop 2.24 $= E\|X\|_2^2$

$$= n$$

Two main examples!

1. Product distributions

vector of independent centerend variable of unit var

gaussian vector

rademacher vector

2. (coordinate):

Uniformly distributed is the set $\{e_i\}$ where $\{e_1, \cdots, e_n\}$ is a _____ basis

$$E\, XX^T = \frac{1}{n} \sum_{j=1} n e_j e_j^T = I$$

important because very opposite:

Gaussian: well concentrated, continuous

Coordinate: widely very discrete

(only n choice)

(Rademacher: 2^n choice)

More examples

Fra ? (More vectors form a basis

uniform on certain convex sets)

or the sphere (spherical)

Prime example for Gaussian

What is the analogy of subgaussian?

Look again at Gaussian:

Naive idea: A vector is Gaussian if every entry is Gaussian

Looke $X_1 \sim N(0,1)$, $\varepsilon = \pm 1$

$$X_2 = \varepsilon X_1 \sim N(0,1)$$

then $X = (X_1, X_2)$ has a remarkable property

$$P(X_1 + X_2 = 0) = \frac{1}{2}.$$

This should be avoided.

Instead: we define:

$X \in \mathbb{R}^n$ is a Gaussian vector if all

one-dimensional marginals are Gaussian random variables.

Def 2.26 We say that a <u>random vector</u> X in $\mathbb{R}^n$ is subgaussian if the one-dim marginals $\langle X, x \rangle$ are subgaussian random variables for all $x \in \mathbb{R}^n$.

The subgaussian norm of X is defined as

$$\|X\|_{\psi_2} = \sup_{x \in S^{n-1}} \|\langle X, x \rangle\|_{\psi_2}$$

Remark: In general, it is natural to also use marginals to generalize other contents, eg. sub-exp.

Prime example for Gaussian Vector:

Independent Gaussian entries

Lemma 2.2.7 (Products of subGaussian distribution)

Let $X_1, \cdots, X_n$ be independent centered subgaussian random variables. Then $X = (X_1, \cdots, X_n)$ is a centered subgaussian random vector in $\mathbb{R}^n$ and $\|X\|_{\psi_2} \leq \underbrace{C}_{?} \max_{i \leq n} \|X_i\|_{\psi_2}$

absolute? ✓

Proof: $|\langle X, x \rangle|_{\psi_2}^2 = \| \sum_{i=1}^{n} x_i X_i \|_{\psi_2}^2$

$$\overset{\text{rotation invariance}}{\leq} C \sum_{i=1}^{n} x_i^2 \|X_i\|_{\psi_2}^2$$

$\sum x_i^2 = 1$

$$\leq C \max_i \|X_i\|_{\psi_2}^2$$

Example 1. Gaussian Vector

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2. Rademacher sequences (Lemma 2.27)

3. Coordinate $P(X = \sqrt{n}\, e_i) = \frac{1}{n}$

$\langle X, x \rangle = \sqrt{n} \cdot x_i$ with prob $\frac{1}{n}$

$$\|X\|_{\psi_2} = \sup_{x \in S^{n-1}} \|\langle X, x \rangle\|_{\psi_2}$$

$$= \sup_{\substack{x \in S^{n-1} \\ p \geq 1}} \left(p^{-\frac{1}{2}} E|\langle X, x \rangle|^p\right)^{\frac{1}{p}}$$

$$= \sup_{\substack{x \in S^{n-1} \\ p \geq 1}} \left(p^{-\frac{1}{2}} \frac{1}{n} \sum_{i=1}^{n} (\sqrt{n} \cdot |x_i|)^p \right)^{\frac{1}{p}}$$

$$= \sup_{\substack{p \geq 1 \\ x \in S^{n-1}}} p^{-\frac{1}{2}} \left(n^{\frac{1}{2}p-1} \cdot \sum |x_i|^p \right)^{\frac{1}{p}}$$

$$= \sup_{\substack{p \geq 1 \\ x \in S^{n-1}}} p^{-\frac{1}{2}} n^{\frac{1}{2}-\frac{1}{p}} \|x\|_p$$

$$\overset{p=\log n}{\geq} \sup_{x \in S^{n-1}} (\log n)^{-\frac{1}{2}} (n^{\frac{1}{2}}) n^{-\frac{1}{\log n}} \|x\|_{\log n}$$

$$\overset{x=e_j}{=} \frac{\sqrt{n}}{\sqrt{\log n}} \simeq \sqrt{n}$$

Observation:

subgaussian norms grow Not as well behaved as Gaussian, Rademacher etc.

(课程结尾?)