

Personal Notes on Paper:

Crack tip fields in soft elastic solids subjected to large quasi-static deformation

Rong Long, Chung-Yuen Hui

Contents

1	Definitions	2
2	Geometry, Notations, and Basic Equations	2
2.1	Governing Equations: Kinematics and Equilibrium	2
3	Crack tip fields under large deformations	2
3.1	Method of asymptotic analysis	2
3.2	Plane strain crack in homogeneous materials	3
3.2.1	Crack tip deformation field (Mode I)	3
3.2.2	Special case: $n = 1$ for neo-Hookean or MR solid	5
3.3	Plane stress crack in homogenous materials	6
3.3.1	Crack tip deformation field for GNH solids	6
3.3.2	Special case: $n = 1$ for neo-Hookean solid	6
4	Energetics: J-integral and energy release rate	7
4.1	Interpretation of J-integral in experiments	8

1 Definitions

y_α where $\alpha = 1, 2$ are deformed coordinates

(r, θ) are polar coordinates

m_α, p_α are unknown exponents

$v_\alpha(\theta), q_\alpha(\theta)$ unknown functions describing the angular variation

I ?

A, B are material constants

n is a material constant where $n = 1$ recovers a neo-Hookean or Mooney-Rivlin material

μ is the small strain shear modulus

a, b_0 : unknown positive amplitude

$U(\theta, n)$: dimensionless function

$H(\theta, n)$: angular function

F is a hypergeometric function

2 Geometry, Notations, and Basic Equations

2.1 Governing Equations: Kinematics and Equilibrium

In-plane displacements because we are considering plane-strain, plane-stress, and anti-plane shear

$$u_\alpha(X_1, X_2) = x_\alpha(X_1, X_2) - X_\alpha \quad \alpha = 1, 2 \quad (2.1)$$

In the plane-stress formulation, (X_1, X_2) denote the mid-plane coordinates of a material point in the undeformed reference configuration. The undeformed crack tip is at the origin, $X_1 = X_2 = 0$

Introduction of y_α definition for considering displacement of the crack tip:

$$y_\alpha(X_1, X_2) \equiv x_\alpha(X_1, X_2) - x_\alpha(X_1 = 0, X_2 = 0) \quad (2.2)$$

Therefore, considering we are given displacements from Paraview

$$y_\alpha(X_1, X_2) = (u_\alpha(X_1, X_2) + X_\alpha) - (u_\alpha(0, 0) + X_\alpha(0, 0)) \quad (2.3)$$

$$= u_\alpha(X_1, X_2) - u_\alpha(0, 0) + X_\alpha - X_\alpha(0, 0) \quad (2.4)$$

3 Crack tip fields under large deformations

3.1 Method of asymptotic analysis

The deformed coordinates, $y_1 y_2$, were assumed to be a series consisting of separable functions of the polar coordinates (r, θ) in the undeformed material in the vicinity of the crack tip

$$y_a(r, \theta) = r^{m_a} v_a(\theta) + r^{n_0} q_a(\theta) + \dots \quad (3.1)$$

$$m_a < p_a, \quad \alpha = 1, 2$$

where m_α, p_α are unknown exponents and $v_\alpha(\theta), q_\alpha(\theta)$ are unknown functions describing the angular variation

3.2 Plane strain crack in homogeneous materials

Strain energy density

$$W(I \gg 1) = AI^n + BI^{n-1} + o(l^{n-1}) \quad (3.2)$$

n is a material constant where $n = 1$ recovers a neo-Hookean or Mooney-Rivlin material. Remove higher order terms.

$$\begin{aligned} W(I \gg 1) &= AI^1 + BI^{1-1} + o(l^{1-1}) \\ W(I \gg 1) &= AI + B \quad \text{where } A = \frac{\mu}{2} B = -\frac{3\mu}{2} \\ &= \frac{\mu}{2} I - \frac{3\mu}{2} \\ W(I \gg 1) &= \frac{\mu}{2} (I - 3) \end{aligned}$$

3.2.1 Crack tip deformation field (Mode I)

The characteristics of the deformation has a transition at $n = 3/2$. Note that $n=1$ indicates a Neo-Hookean material.

$$\begin{aligned} y_1 &= \begin{cases} -b_0 r^{2-\frac{1}{n}} [U(\theta, n)]^2, & 1/2 < n < 3/2 \\ -\frac{1}{a} r^{1+\frac{1}{2n}} H(\theta, n), & n > 3/2 \end{cases} \\ y_2 &= ar^{1-\frac{1}{2n}} U(\theta, n) \end{aligned} \quad (3.3)$$

U is simply a dimensionless function that holds for any n

$$U(\theta, n) = \sin\left(\frac{\theta}{2}\right) \sqrt{1 - \frac{2\kappa^2 \cos^2(\theta/2)}{1 + \omega(\theta, n)}} \times [\omega(\theta, n) + \kappa \cos \theta]^{\kappa/2}, \quad 0 \leq |\theta| \leq \pi \quad (3.4)$$

where

$$\kappa = 1 - \frac{1}{n} \quad (3.5)$$

$$\omega(\theta, n) = \sqrt{1 - (\kappa \sin \theta)^2} \quad (3.6)$$

H is an angular function that is only relevant for $n > 3/2$ **Note that we have a typo where it is written that $n > 3/2$**

$$H(\theta, n) = -\frac{n^{5/2}}{m^2} [\omega(\theta, n) + \kappa \cos \theta]^{2-m} \left[\frac{m}{2-m} \times F\left(\frac{1}{2} - \frac{1}{m}, \frac{1}{2}; \frac{3}{2} - \frac{1}{m}; \cos^2 \xi_0\right) - \kappa \sin \xi_0 \right] \quad (3.7)$$

where F is the hypergeometric function and,

$$\begin{aligned} m &= 1 - \frac{1}{2n} \\ \cos \xi_0 &= \frac{1}{n\sqrt{2}} \frac{\sqrt{1 + k \sin^2 \theta - \omega(\theta, n) \cos \theta}}{\omega(\theta, n) + \kappa \cos \theta} \quad 0 \leq \xi_0 \leq \frac{\pi}{2} \end{aligned}$$

To make sure I have plotted correctly, check the boundaries

$$\begin{aligned} H(\theta = 0, n) &= -\frac{4n^{9/2}}{(2n-1)^2} \left[2 - \frac{1}{n} \right]^{1+\frac{1}{2m}} \frac{1}{n(2n+1)} \\ H(\theta = \pm\pi, n) &= -\frac{4n^{9/2}}{(2n-1)^2} n^{-1-\frac{1}{n}} \left(\frac{2n-1}{2n+1} \right) \times \frac{\sqrt{\pi} \Gamma\left(\frac{2n-3}{2(2n-1)}\right)}{\Gamma\left(\frac{-1}{2n-1}\right)} \end{aligned}$$

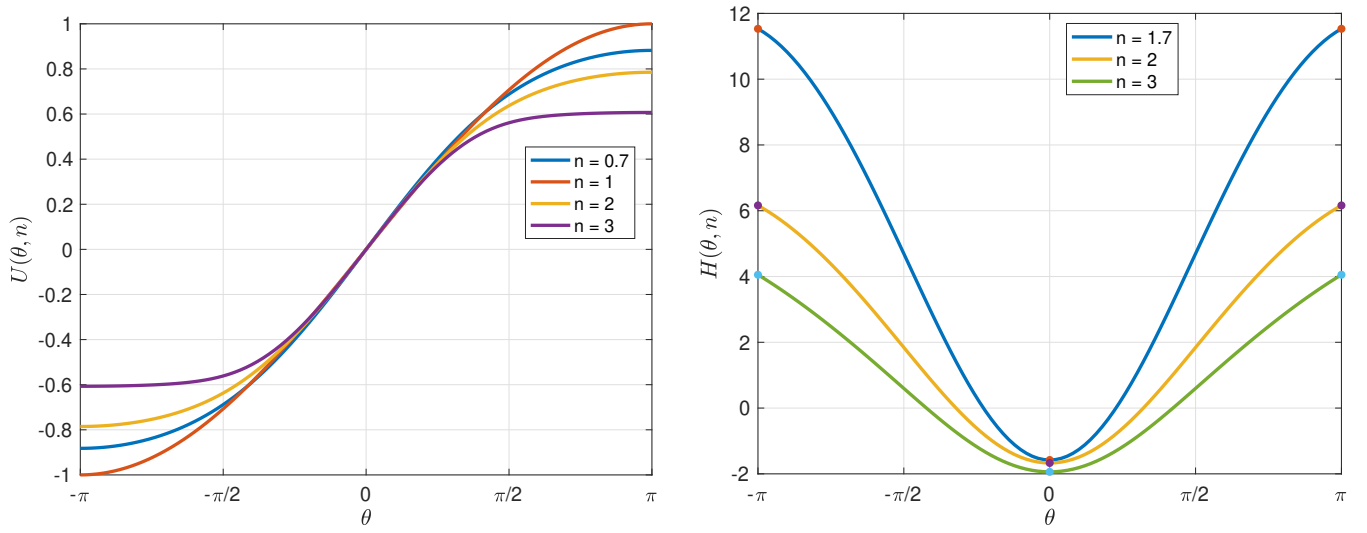


Figure 1: Replication of Figure 3 without $n = 10$

where the gamma function exists in MATLAB.

Note that plotting U in Fig 3A is simply a matter of choosing a n value and using Eq. 3.4, 3.5, and 3.6. Plotting H is more complicated and relies on inbuilt functions in MATLAB for hypergeometric and Gamma functions, for example

```
for i = 1:length(x)
    F(i) = hypergeom([1/2 - 1/m, 1/2], 3/2 - 1/m, x(i));
end
```

and to check the boundaries, we list two gamma functions as follows:

```
gamma1 = gamma((2*n(ii) - 3)/(2*(2*n(ii) - 1)));
gamma2 = gamma(-1/(2*n(ii) - 1));
```

In the interest of replicating Figure 4A, rewrite the equations for the deformed coordinates near the crack tip, where we are interested in n between $1/2$ and $3/2$

$$\begin{aligned} y_1 &= -b_0 r^{2-\frac{1}{n}} [U(\theta, n)]^2 \\ y_2 &= a r^{1-\frac{1}{2n}} U(\theta, n) \end{aligned} \quad (3.8)$$

Before normalizing the result, we first rearrange y_2

$$\frac{y_2}{a} = r^{1-\frac{1}{2n}} U(\theta, n)$$

and substitute into y_1

$$y_1 = -b_0 r^{2-\frac{1}{n}} [U(\theta, n)]^2 \quad (3.9)$$

$$= -b_0 r^{2(1-\frac{1}{2n})} [U(\theta, n)]^2 \quad (3.10)$$

$$y_1 = -b_0 \left(\frac{y_2}{a} \right)^2 \quad (3.11)$$

Therefore, we now have Eq. 40a in the paper

$$y_1 = -b_0 \left(\frac{y_2}{a} \right)^2 \quad (3.12)$$

$$\sqrt{\frac{y_1}{-b_0}} = \frac{y_2}{a} \quad (3.13)$$

$$\pm a \sqrt{\frac{y_1}{-b_0}} = y_2 \quad (3.14)$$

Now if we normalize by a^{2n} , where normalized quantities are denoted by $\hat{()}$

$$\begin{aligned} \hat{y}_1 &= -\frac{b_0}{a^{2n}} r^{2-\frac{1}{n}} [U(\theta, n)]^2 & \hat{y}_2 &= \frac{a}{a^{2n}} r^{1-\frac{1}{2n}} U(\theta, n) \\ \hat{y}_1 &= -\frac{b_0}{a^{2n}} r^{2-\frac{1}{n}} [U(\theta, n)]^2 & \hat{y}_2 &= a^{1-2n} r^{1-\frac{1}{2n}} U(\theta, n) \\ \hat{y}_1 &= -\frac{x a^{2-2n}}{a^{2n}} r^{2-\frac{1}{n}} [U(\theta, n)]^2 \\ \hat{y}_1 &= -x a^{2-4n} r^{2-\frac{1}{n}} [U(\theta, n)]^2 \end{aligned} \quad (3.15)$$

where we can introduce some unknown x according to the figure caption:

$$x = \frac{b_0}{a^{2-2n}} \rightarrow b_0 = x a^{2-2n} \quad (3.16)$$

Finally

$$\hat{y}_1 = -x (\hat{y}_2)^2 = -\frac{b_0}{a^{2-2n}} (\hat{y}_2)^2 \quad (3.17)$$

3.2.2 Special case: $n = 1$ for neo-Hookean or MR solid

First, we want to plot U for $n = 1$, which makes $\kappa = 0$ and $\omega = 1$

$$\begin{aligned} U(\theta, n) &= \sin\left(\frac{\theta}{2}\right) \sqrt{1 - \frac{2\kappa^2 \cos^2(\theta/2)}{1 + \omega(\theta, n)}} \times [\omega(\theta, n) + \kappa \cos \theta]^{\kappa/2}, \quad 0 \leq |\theta| \leq \pi \\ &= \sin\left(\frac{\theta}{2}\right) \sqrt{1 - \frac{2(0) \cos^2(\theta/2)}{2}} \times [1 + \cos \theta]^{0/2} \\ U(\theta, 1) &= \sin\left(\frac{\theta}{2}\right) \end{aligned}$$

Therefore, considering $n = 1$, starting with Eq. 3.8, we can obtain Eq. 48 in the paper

$$\begin{aligned} y_1 &= -b_0 r^{2-\frac{1}{n}} [U(\theta, n)]^2 = -b_0 r \left[\sin\left(\frac{\theta}{2}\right) \right]^2 \\ y_2 &= a r^{1-\frac{1}{2n}} U(\theta, n) = a \sqrt{r} \sin\left(\frac{\theta}{2}\right) \end{aligned}$$

In terms of the normalized quantities, we have the following:

$$\begin{aligned} \hat{y}_1 &= -x a^{2-4n} r^{2-\frac{1}{n}} [U(\theta, n)]^2 & \hat{y}_2 &= a^{1-2n} r^{1-\frac{1}{2n}} U(\theta, n) \\ \hat{y}_1 &= -x a^{2-4} r^{2-1} [U(\theta, 1)]^2 & \hat{y}_2 &= a^{1-2} r^{1-\frac{1}{2}} U(\theta, 1) \\ \hat{y}_1 &= -b_0 a^{-2} r \left[\sin\left(\frac{\theta}{2}\right) \right]^2 & \hat{y}_2 &= a^{-1} \sqrt{r} \sin\left(\frac{\theta}{2}\right) \\ \hat{y}_1 &= -b_0 \hat{y}_2^2 \end{aligned} \quad (3.18)$$

There are several ways to plot Fig. 4, but I plotted the full U function for more generalizability. In the MATLAB code, we simply set the value of x which is a ratio of b_0 and a .

$$\hat{y}_2 = \frac{1}{a}\sqrt{r}U(\theta, 1)$$

$$\hat{y}_1 = -x\hat{y}_2^2$$

Note: neglect the $\frac{\sqrt{r}}{a}$ in the actual code:

```
y2 = real(U);
y1 = -x(ii)*y2.^2;
```

How this is plotted in Figure 4 is by setting $n = 1$ for the calculation of κ for U

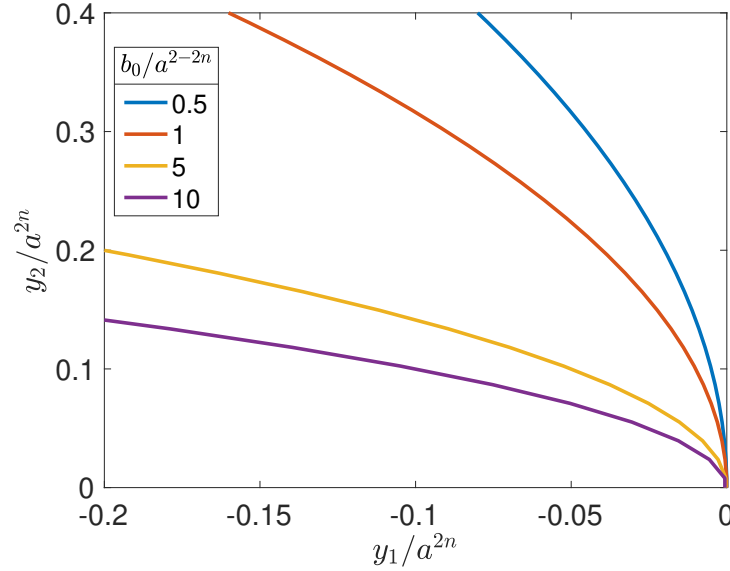


Figure 2: Crack opening profile predicted by the asymptotic solution.

3.3 Plane stress crack in homogenous materials

3.3.1 Crack tip deformation field for GNH solids

Leading order terms of mode I deformation field y_1 and y_2 :

$$y_1 = cr^d g(\theta, n), \quad d < 1 + \frac{1}{4n} \quad n < n^* \quad (3.19)$$

$$y_2 = ar^{1-\frac{1}{2n}} U(\theta, n) \quad (3.20)$$

For the special case of a neo-Hookean solid with $n = 1$

$$d = 1 \quad \text{and} \quad g(\theta, n = 1) = \cos \theta \quad (3.21)$$

3.3.2 Special case: $n = 1$ for neo-Hookean solid

Therefore for a neo-Hookean solid we obtain Eq. 65 in the paper

$$y_1 = cr \cos \theta$$

$$y_2 = a\sqrt{r} \sin \left(\frac{\theta}{2} \right)$$

Note 1: There is a distinct difference in the relation between y_1 and y_2 see Eq. 66 below

$$y_2 = \pm a \sqrt{\frac{-y_1}{c}}$$

$$\left(\frac{y_2}{a}\right)^2 = \frac{-y_1}{c}$$

$$cr \sin^2\left(\frac{\theta}{2}\right) = -y_1$$

but $-\cos(\theta) \neq \sin^2(\theta/2)$

If we want to plot this similarly to Fig. 3 in the paper, we normalize by a^{2n} or a^2 for $n = 1$

$$\hat{y}_1 = \frac{c}{a^{2n}} r \cos \theta$$

$$\hat{y}_1 = \frac{c}{a^2} r \cos \theta \rightarrow r = \frac{\hat{y}_1 a^2}{c \cos \theta}$$

For \hat{y}_2 we substitute the above identity for r

$$\hat{y}_2 = a^{1-2n} \sqrt{r} \sin\left(\frac{\theta}{2}\right)$$

$$\hat{y}_2^2 = a^{-2} r \sin^2\left(\frac{\theta}{2}\right) \quad \text{where } \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos \theta)$$

$$\hat{y}_2^2 = a^{-2} r \frac{1}{2}(1 - \cos \theta) \quad \text{substitute } r$$

$$\hat{y}_2^2 = a^{-2} \frac{\hat{y}_1 a^2}{c \cos \theta} \frac{1}{2}(1 - \cos \theta)$$

$$\hat{y}_2^2 = \frac{1}{2c} \frac{(1 - \cos \theta)}{\cos \theta} \hat{y}_1$$

Rearrange so we obtain the following properties:

$$\hat{y}_1 = 2c \hat{y}_2^2 \frac{\cos \theta}{1 - \cos \theta} \quad \hat{y}_2 = \frac{1}{a} \sqrt{r} \sin\left(\frac{\theta}{2}\right)$$

Note 2: we can determine a but not c .

4 Energetics: J-integral and energy release rate

The J-integral for plane stress, Mode I opening

$$J = \frac{\mu\pi}{2} \left(\frac{b}{n}\right)^{n-1} \left(\frac{2n-1}{2n}\right)^{2n-1} n^{1-n} a^{2n} \quad n = 1$$

$$J = \frac{\mu\pi a^2}{4}$$

4.1 Interpretation of J-integral in experiments

For a pure shear specimen the J-integral is

$$J = 2W(I_1, I_2)h_0 = 2\Psi(\lambda_A)h_0 \quad (4.1)$$

where the stretch in the direction of loading is:

$$\lambda_A = 1 + \frac{\Delta}{h_0}$$

For a neo-hookean solid

$$\Psi = \frac{\mu}{2}(\lambda_A - \lambda_A^{-1})^2$$

where we can substitute this into the J-integral Eq. 4.1 and solve for a

$$\begin{aligned} J &= 2\Psi(\lambda_A)h_0 \\ J &= \mu(\lambda_A - \lambda_A^{-1})^2h_0 \\ \frac{\mu\pi a^2}{4} &= \mu(\lambda_A - \lambda_A^{-1})^2h_0 \\ a^2 &= \frac{4h_0}{\pi}(\lambda_A - \lambda_A^{-1})^2 \\ a &= 2\sqrt{\frac{h_0}{\pi}}(\lambda_A - \lambda_A^{-1}) \end{aligned}$$

The unknown amplitude c decreases monotonically from 1.55 at $\lambda_A = 1.02$ to 1.15 at $\lambda_A = 2$.