# Stabilized Mixed Finite Element Formulation

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August 14, 2020

## Definitions

$\mathbf{F}$: Deformation gradient
$\mathbf{I}$: second-order unit tensor
$\boldsymbol{u}$: Displacement
$J$: determinant of the deformation gradient
$\mathbf{C}$: Right Cauchy-Green Strain Tensor
$\mathcal{W}(\mathbf{F})$: strain energy function
$\mathbf{P}$: first Piola-Kirchhoff stress tensor
$\mathbf{S}$: second PK stress tensor

$\alpha$: cracks are represented by a scalar phase-field variable
$p$: Lagrange multiplier, hydrostatic pressure field
$\kappa$: bulk modulus

$$\kappa = \frac{E}{3(1-2\nu)} \tag{0.1}$$

$\mu$: shear modulus

$$\mu = \frac{E}{2(1+\nu)} \tag{0.2}$$

$\lambda$: Lamé modulus

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{0.3}$$

$\mathcal{E}_\ell$: potential energy functional $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume
$c_w$ is a normalization constant

## 1 Hyperelastic Phase-Field Fracture Models

Deformation Gradient

$$\mathbf{F} = \mathbf{I} + \nabla \otimes \boldsymbol{u} \tag{1.1}$$

where $J = \det \mathbf{F}$ and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

The strain energy function $\mathcal{W}(\mathbf{F})$ is defined per unit reference volume such that the first PK and second PK

$$\mathbf{P} = \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \tag{1.2a}$$

$$\mathbf{S} = 2\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{C}} \tag{1.2b}$$

where $\mathbf{P} = \mathbf{F}\mathbf{S}$.

## 1.1 Phase-Field Fracture Model

For incompressible hyperelastic materials, the strain energy function is defined as (**?** )

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p\left(J-1\right), \tag{1.3}$$

We instead consider a damage dependent relaxation of the incompressibility constraint and the phase-field couples to the strain energy through the modified function

$$\widetilde{\mathcal{W}}(\mathbf{F},\alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa\left(J-1\right)^2 \tag{1.4}$$

where the decreasing stiffness modulation function.

$$a(\alpha) = (1-\alpha)^2 \tag{1.5}$$

Use Eq. 1.2 to solve for the first Piola-Kirchhoff stress tensor

$$\begin{aligned}
\mathbf{P} &= \frac{\partial \tilde{\mathcal{W}}(\mathbf{F},\alpha)}{\partial \mathbf{F}} \\
&= \frac{\partial}{\partial \mathbf{F}}\left[a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa\left(J-1\right)^2\right] \\
&= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\frac{1}{2}\kappa\frac{\partial (J-1)^2}{\partial \mathbf{F}} \\
\mathbf{P} &= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa\left(J-1\right)\frac{\partial J}{\partial \mathbf{F}}
\end{aligned} \tag{1.6}$$

where $\partial J/\partial \mathbf{F} = J\mathbf{F}^{-T}$. To circumvent this numerical difficulties, we resort to the classical mixed formulation and introduce the pressure-like field

$$p = -\sqrt{a^3(\alpha)}\kappa\left(J-1\right), \tag{1.7}$$

as an independent variable along with the displacement field.

We first consider the energy functional of a possibly fractured elastic body with isotropic surface energy, this equation is found in Bin2020 Eq. 21. (We drop $\lambda_b$ which is not a consideration in this formulation)

$$\mathcal{E}_\ell(\boldsymbol{u},\alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F},\alpha)d\Omega + \frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega \tag{1.8}$$

where $a(\alpha)$ is a (decreasing) stiffness modulation function and $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume.

$$a(\alpha) = (1-\alpha)^2 \quad w(\alpha) = \alpha \tag{1.9}$$

The normalization constant is defined as:

$$c_w = \int_0^1 \sqrt{w(\alpha)} d\alpha \tag{1.10}$$

In the code we have the following definition

$$b(\alpha) = (1-\alpha)^3$$

Non-modified strain energy function is the compressible Neo-Hookean:

$$\mathcal{W}(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2\ln J) \tag{1.11}$$

Starting from Eq. 25 in the 2020 Li and Bouklas paper where $\kappa$ is the bulk modulus

$$\begin{aligned}
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \Lambda, \alpha\right) &= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega \Lambda(p + \sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \quad \lambda = -p/\kappa \\
&= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega -\frac{p}{\kappa}(p + \sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \\
&= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \frac{p^2}{K} d\Omega - \int_\Omega \frac{p}{\kappa}\sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \\
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) &= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1))d\Omega
\end{aligned} \tag{1.12}$$

Substitute in $\mathcal{E}_\ell(\boldsymbol{u}, \alpha)$ and substitute Eq. 1.11

$$\begin{aligned}
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) &= \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1))d\Omega \\
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) &= \int_\Omega a(\alpha)\frac{\mu}{2}(I_1 - 3 - 2\ln J)d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1))d\Omega \\
&\quad + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega
\end{aligned}$$

Note that the last term is the dissipative energy term, defined separately in the code.

## 1.2 Changes for 2D Code Version

In the code we have the following for the energy functional of the energy problem

$$\widetilde{W}(\mathbf{F}, \alpha) = \big(a(\alpha) + k_\ell\big)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}$$

where $\mu$ is the shear modulus and $\lambda$ is the 1st Lamé parameter.

In my derivation

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$$
\begin{aligned}
\mathbf{P} &= \frac{\partial \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}} \\
&= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-1}) + b(\alpha)p\frac{\partial \det \mathbf{F}}{\partial \mathbf{F}} \\
\mathbf{P} &= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-1}) + b(\alpha)pJ\mathbf{F}^{-1}
\end{aligned}
$$

Taking the third component to be zero

$$
\begin{aligned}
\mathbf{P}_{33} = a(\alpha)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) + b(\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
(1-\alpha)^2\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) + (1-\alpha)^3 pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) + (1-\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu\mathbf{F}_{33} - \mu\mathbf{F}_{33}^{-1} + (1-\alpha)pJ\mathbf{F}_{33}^{-1} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\mu\mathbf{F}_{33} &= \mu\mathbf{F}_{33}^{-1} - (1-\alpha)pJ\mathbf{F}_{33}^{-1} \\
\mu\mathbf{F}_{33} &= \big[\mu - (1-\alpha)pJ\big]\mathbf{F}_{33}^{-1} \\
\mathbf{F}_{33}^{2} &= \frac{\mu - (1-\alpha)pJ}{\mu} \\
\mathbf{C}_{33} &= 1 - \frac{(1-\alpha)pJ}{\mu}
\end{aligned}
$$

Treating $F_{33}$ as an independent unknown, we can state the governing equation

$$
\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) + (1-\alpha)pJ\mathbf{F}_{33}^{-1} = 0
$$

This can be multiplied by its associated test function

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Jason's

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$$
\begin{aligned}
\mathbf{P} &= \frac{\partial \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}} \\
&= \big(a(\alpha) + k_\ell\big)\mu(\mathbf{F} - \mathbf{F}^{-T}) + b(\alpha)p\frac{\partial \det \mathbf{F}}{\partial \mathbf{F}} \\
\mathbf{P} &= \big(a(\alpha) + k_\ell\big)\mu(\mathbf{F} - \mathbf{F}^{-T}) + b(\alpha)pJ\mathbf{F}^{-T}
\end{aligned}
$$

Taking the third component to be zero

$$
\begin{aligned}
\mathbf{P}_{33} = \big(a(\alpha) + k_\ell\big)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-T}) + b(\alpha)pJ\mathbf{F}_{33}^{-T} &= 0 \\
\big(a(\alpha) + k_\ell\big)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-T}) + b(\alpha)pJ\mathbf{F}_{33}^{-T} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\big(a(\alpha) + k_\ell\big)\mu\mathbf{F}_{33}^{-1}\mathbf{F}_{33} &= \Big[\big(a(\alpha) + k_\ell\big)\mu - b(\alpha)pJ\Big]\mathbf{F}_{33}^{-1}\mathbf{F}_{33}^{-T} \\
\big(a(\alpha) + k_\ell\big)\mu &= \Big[\big(a(\alpha) + k_\ell\big)\mu - b(\alpha)pJ\Big](\mathbf{F}_{33}^{T}\mathbf{F}_{33})^{-1} \\
\big(a(\alpha) + k_\ell\big)\mu &= \Big[\big(a(\alpha) + k_\ell\big)\mu - b(\alpha)pJ\Big]\mathbf{C_{33}}^{-1} \\
\mathbf{C_{33}} &= \frac{\big(a(\alpha) + k_\ell\big)\mu - b(\alpha)pJ}{\mu\big(a(\alpha) + k_\ell\big)} \\
\mathbf{C_{33}} &= 1 - \frac{b(\alpha)pJ}{\mu\big(a(\alpha) + k_\ell\big)}
\end{aligned}
$$

## 2 Stabilized Finite Element Method

### 2.1 Gateaux Derivative

The Gateaux derivative with respect to $(\boldsymbol{u}, \alpha)$ in direction $(\boldsymbol{v}, \beta)$ under the irreversibility condition $\dot{\alpha} \geq 0$.

$$d\mathcal{E}_\ell\left(\boldsymbol{u}, \alpha; \boldsymbol{v}, \beta\right) \geq 0. \tag{2.1}$$

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Calculation of the Gateaux derivative

$$\begin{aligned} d\mathcal{E}_\ell(\boldsymbol{u}, \boldsymbol{v})(\alpha, \beta) &= \frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u} + \delta\boldsymbol{v}, \alpha + \delta\beta)\big|_{\delta=0} \\ &= \frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u} + \delta\boldsymbol{v}, \alpha)\big|_{\delta=0} + \frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}, \alpha + \delta\beta)\big|_{\delta=0} \end{aligned}$$

Starting with the first term:

$$\begin{aligned} \frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u} + \delta\boldsymbol{v}, \alpha)\big|_{\delta=0} &= \frac{d}{d\delta}\left[\int_\Omega \widetilde{\mathcal{W}}(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v}), \alpha)\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot (\boldsymbol{u} + \delta\boldsymbol{v})\, dA\right]\Bigg|_{\delta=0} \\ &= \left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v}), \alpha)}{d\delta}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \frac{d(\boldsymbol{u} + \delta\boldsymbol{v})}{d\delta}\, dA\right]\Bigg|_{\delta=0} \quad \text{chain rule} \\ &= \left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v}), \alpha)}{d(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v})}\frac{d(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v}))}{d\delta}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \frac{d(\boldsymbol{u} + \delta\boldsymbol{v})}{d\delta}\, dA\right]\Bigg|_{\delta=0} \\ &= \left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v}, \alpha)}{d(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v})}\frac{d(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v})}{d\delta}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \boldsymbol{v}\, dA\right]\Bigg|_{\delta=0} \\ &= \left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v}, \alpha)}{d(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v})}\nabla\boldsymbol{v}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \boldsymbol{v}\, dA\right]\Bigg|_{\delta=0} \\ &= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla\boldsymbol{u}, \alpha)}{d(\mathbf{I} + \nabla\boldsymbol{u})}\nabla\boldsymbol{v}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \boldsymbol{v}\, dA \\ \frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u} + \delta\boldsymbol{v}, \alpha)\big|_{\delta=0} &= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\mathbf{F}}\nabla\boldsymbol{v}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \boldsymbol{v}\, dA \end{aligned}$$

Second term:

$$
\begin{aligned}
&\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}, \alpha+\delta\beta)\big|_{\delta=0}\\
&= \frac{d}{d\delta}\bigg[\int_\Omega \widetilde{\mathcal{W}}(\mathbf{F}, \alpha+\delta\beta)\, d\Omega + \frac{\mathcal{G}_c}{c_w}\int_\Omega \bigg(\frac{w(\alpha+\delta\beta)}{\ell} + \ell\|\nabla(\alpha+\delta\beta)\|^2\bigg)\, dV\bigg]\bigg|_{\delta=0}\\
&= \bigg[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha+\delta\beta)}{d\delta}\, d\Omega + \frac{\mathcal{G}_c}{c_w}\frac{d}{d\delta}\int_\Omega \bigg(\frac{w(\alpha+\delta\beta)}{\ell} + \ell\|\nabla(\alpha+\delta\beta)\|^2\bigg)\, dV\bigg]\bigg|_{\delta=0}\\
&= \bigg[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha+\delta\beta)}{d(\alpha+\delta\beta)}\frac{d(\alpha+\delta\beta)}{d\delta}\, d\Omega + \frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega \frac{dw(\alpha+\delta\beta)}{d\delta}\, dV + \frac{\mathcal{G}_c}{c_w}\ell\int_\Omega \frac{\|\nabla(\alpha+\delta\beta)\|^2}{d\delta}\, dV\bigg]\bigg|_{\delta=0}\\
&= \bigg[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha+\delta\beta)}{d(\alpha+\delta\beta)}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega \frac{dw(\alpha+\delta\beta)}{d(\alpha+\delta\beta)}\frac{d(\alpha+\delta\beta)}{d\delta}\, dV + \frac{\mathcal{G}_c}{c_w}\ell\int_\Omega 2\nabla(\alpha+\delta\beta)\frac{\nabla(\alpha+\delta\beta)}{d\delta}\, dV\bigg]\bigg|_{\delta=0}\\
&= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha}\beta\, d\Omega + \bigg[\frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega \frac{dw(\alpha+\delta\beta)}{d(\alpha+\delta\beta)}\beta\, dV + \frac{\mathcal{G}_c}{c_w}\ell\int_\Omega 2\nabla(\alpha+\delta\beta)\nabla\beta\, dV\bigg]\bigg|_{\delta=0}\\
&= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta\, dV + 2\ell\frac{\mathcal{G}_c}{c_w}\int_\Omega \nabla\alpha\cdot\nabla\beta\, dV\\
&= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega \bigg[\frac{dw(\alpha)}{d\alpha}\beta + 2\ell^2(\nabla\alpha\cdot\nabla\beta)\bigg]\, dV
\end{aligned}
$$

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First, consider Eq. 1.7

$$
\begin{aligned}
p &= -\sqrt{a^3(\alpha)}\kappa(J-1)\\
\frac{p}{\kappa} &= -\sqrt{a^3(\alpha)}(J-1)\\
0 &= -\sqrt{a^3(\alpha)}(J-1) - \frac{p}{\kappa}
\end{aligned}
$$

Multiplying this by test function $q$ and integrating over volume, we obtain an equation that can be combined with the equations from the Gateaux Derivative.

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\mathbf{F}}\nabla\boldsymbol{v}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0\cdot\boldsymbol{v}\, dA = 0 \tag{2.2a}$$

$$\int_\Omega \bigg(-\sqrt{a^3(\alpha)}(J-1) - \frac{p}{\kappa}\bigg)q dV = 0 \tag{2.2b}$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega \bigg[\frac{dw(\alpha)}{d\alpha}\beta + 2\ell^2(\nabla\alpha\cdot\nabla\beta)\bigg]\, dV \geq 0 \tag{2.2c}$$

The strong form

$$\mathrm{Div}\,\mathbf{P} = 0 \quad \text{in} \quad \Omega \tag{2.3a}$$

$$\mathbf{u} = \widetilde{\mathbf{u}}_0 \quad \text{in} \quad \partial_D\Omega \tag{2.3b}$$

$$[\mathbf{FS}]\,\boldsymbol{n} = \tilde{\boldsymbol{g}}_0 \quad \text{on} \quad \partial_N\Omega, \tag{2.3c}$$

where from Eq. 1.6 we can substitute Eq. 1.7

$$
\begin{aligned}
\mathbf{P} &= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial\mathbf{F}} + a^3(\alpha)\kappa\,(J-1)\,\frac{\partial J}{\partial\mathbf{F}} \quad \text{where}\, p = -\sqrt{a^3(\alpha)}\kappa\,(J-1)\\
\mathbf{P} &= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial\mathbf{F}} - p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial\mathbf{F}}
\end{aligned}
$$

and write the mechanical equilibrium equation in Eq. 2.3:

$$\mathrm{Div}\left[a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} - p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial \mathbf{F}}\right] = 0 \tag{2.4}$$

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Derivation of the KKT condition equations where $\nabla\beta \cdot \nabla\alpha = \nabla(\beta\nabla\alpha) - \beta\Delta\alpha$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \left[\frac{dw(\alpha)}{d\alpha}\beta + 2\ell^2(\nabla\alpha \cdot \nabla\beta)\right] dV \geq 0$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta dV + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega 2\ell^2(\nabla\alpha \cdot \nabla\beta)\, dV \geq 0$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta dV + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega 2\ell^2(\nabla(\beta\nabla\alpha))\, dV - \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega 2\ell^2(\beta\Delta\alpha)\, dV \geq 0$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta dV - \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega 2\ell^2(\beta\Delta\alpha)\, dV \geq 0$$

$$\left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right) dV\right]\beta \geq 0$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right) dV \geq 0$$

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Grouping terms, we obtain

$$\frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha} + \frac{\mathcal{G}_c}{c_w \ell}\left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right) \geq 0 \quad \text{in} \quad \Omega \tag{2.5a}$$

$$\dot{\alpha} \geq 0 \quad \text{in} \quad \Omega \tag{2.5b}$$

$$\dot{\alpha}\left[\frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha} + \frac{\mathcal{G}_c}{c_w \ell}\left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right)\right] \geq 0 \quad \text{in} \quad \Omega \tag{2.5c}$$

$$\tag{2.5d}$$

Lastly, we have the following, (Neumann?)

$$\frac{\partial \alpha}{\partial \mathbf{n}} \geq 0 \quad \text{and} \quad \dot{\alpha}\frac{\partial \alpha}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega \tag{2.6}$$

Multiply Eq. 2.4 with weighting function $\boldsymbol{v} + (\varpi h^2)/(2\mu)\mathbf{F}^{-T}\nabla q$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \left[\boldsymbol{v} + \frac{\varpi h^2}{2\mu}\mathbf{F}^{-T}\nabla q\right] dV = 0$$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV + \int_\Omega \mathrm{Div}\mathbf{P} \cdot \left[\frac{\varpi h^2}{2\mu}\mathbf{F}^{-T}\nabla q\right] dV = 0$$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV + \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \mathrm{Div}\mathbf{P} \cdot \left(\mathbf{F}^{-T}\nabla q\right) dV = 0$$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV$$

$$+\frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \mathrm{Div}\left[a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}}\right] \cdot \left(\mathbf{F}^{-T}\nabla q\right) dV - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \mathrm{Div}\left[p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial \mathbf{F}}\right] \cdot \left(\mathbf{F}^{-T}\nabla q\right) dV = 0$$

$$\text{''''} + \text{''''} - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \nabla\big(p\sqrt{a^3(\alpha)}\big) J\mathbf{F}^{-T} \cdot \left(\mathbf{F}^{-T}\nabla q\right) dV = 0$$

$$\text{''''} + \text{''''} - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \nabla\big(p\sqrt{a^3(\alpha)}\big) J\mathbf{F}^{-1}\mathbf{F}^{-T} \cdot \left(\mathbf{F}^{-1}\mathbf{F}^{-T}\nabla q\right) dV = 0$$

$$\text{''''} + \text{''''} - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \nabla\big(p\sqrt{a^3(\alpha)}\big) J \cdot \left(\mathbf{C}^{-1}\nabla q\right) dV = 0$$

$$\text{''''} + \text{''''} - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} J\mathbf{C}^{-1} : \left[\nabla\big(p\sqrt{a^3(\alpha)}\big) \cdot \nabla q\right] dV = 0$$

where $\mathbf{P} = a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} - p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial \mathbf{F}}$

We also want to deal with the first term where $(fg)' = f'g + fg'$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV = \int_\Omega (\mathbf{F} \cdot \boldsymbol{v})_{,X}\, dV - \int_\Omega \mathbf{P} \cdot \frac{\partial \boldsymbol{v}}{\partial X}\, dV$$

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Leaving

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV + \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \mathrm{Div}\left[a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}}\right] \cdot \left(\mathbf{F}^{-T}\nabla q\right) dV - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} J\mathbf{C}^{-1} : \left[\nabla\big(p\sqrt{a^3(\alpha)}\big) \cdot \nabla q\right] dV \tag{2.7}$$

# 3 Numerical Examples

## 3.1 Benchmark: Uniaxial Tension of a Hyperelastic Bar

## 3.2 Revisiting Crack Nucleation in an Elastomer

$$(3.1)$$