FEniCS: 2D Linear Elasticity

Based on Numerical Tours of Continuum Mechanics Using FEniCS

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Contents

1 Problem Definition

This demonstration shows how to compute a small strain solution for a 2D isotropic linear elastic medium in either plane stress or plane strain, in a traditional displacement based finite element formulation.

This document is meant to provide additional information that supplements the 2D linear elasticity demonstration on Numerical Tours of Computational Mechanics with FEniCS , but still assumes the reader has a basic understanding of common solid mechanics notations (ie. indicial notation).

Several modification are made, some of which are purely for convenience and usability. For example, 1) parameters can now be parsed in from command line for running different cases (ie, plane strain vs plane stress). For convergence purposes, 2) the body force can be ramped from a low to high value within a for loop. 3) Post-processing of final values are included to demonstrate code functionality and comparison to an analytical solution

2 Formulation

Equation for strain in terms of displacement

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \tag{1a}$$

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \tag{1b}$$

General expression of the linear elastic isotropic constitutive relationship

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \tag{2}$$

Equation 2 can be inverted,

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij} \tag{3}$$

where the Lamé coefficients are given by:

$$\begin{aligned} \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \\ \mu &= \frac{E}{2(1+\nu)} \end{aligned} \tag{4}$$

The user will be able to specify either a plane strain or stress condition in the code. See the subsections below

2.1 Plane Strain

The strain tensor under plane strain assumptions:

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eq. 2 for plane strain gives the following relationship for σ_{ij}

$$\begin{aligned} \sigma_{ij} &= \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \\ \sigma_{ij} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu \epsilon_{ij} \quad \text{where } \epsilon_{zz} = 0 \end{aligned} \tag{5}$$

Therefore, the constitutive relationship for plane strain is given as follows,

$$\sigma_{\alpha\beta} = \lambda(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{\alpha\beta} \tag{6}$$

where the change to the indices indicates $\alpha, \beta \in 1, 2$ and $i, j \in 1, 2, 3$

2.2 Plane Stress

The stress tensor under plane stress assumptions

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eq. 2 for plane stress gives the following relationship for ϵ_{zz}

$$\begin{aligned}
\sigma_{ij} &= \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \\
0 &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{zz} \\
-(2\mu + \lambda)\epsilon_{zz} &= \lambda(\epsilon_{xx} + \epsilon_{yy}) \\
\epsilon_{zz} &= -\frac{\lambda}{(2\mu + \lambda)}(\epsilon_{xx} + \epsilon_{yy})
\end{aligned}$$

Write a general form of the constitutive relationship by substituting the above into Eq. 2:

$$\begin{aligned}
\sigma_{ij} &= \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \quad \text{where } \epsilon_{kk} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \\
\sigma_{ij} &= \lambda\left[\epsilon_{xx} + \epsilon_{yy} - \frac{\lambda}{2\mu + \lambda}(\epsilon_{xx} + \epsilon_{yy})\right] + 2\mu\epsilon \\
&= \lambda\big[1 - \frac{\lambda}{\lambda + 2\mu}\big](\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon
\end{aligned}$$

Therefore, the constitutive relationship for plane stress is given as follows,

$$\sigma_{\alpha\beta} = \frac{2\lambda\mu}{\lambda + 2\mu}(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon \tag{7}$$

where the change to the indices indicates $\alpha, \beta \in 1, 2$ and $i, j \in 1, 2, 3$

2.3 Strong to Weak Form

Starting from the mechanical equilibrium equation, or strong form,

$$-\mathrm{Div}\boldsymbol{\sigma} = \mathbf{f} \quad \rightarrow -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{f} \tag{8a}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t} \tag{8b}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\mathbf{f}$ is the body force, and $\mathbf{t}$ is the traction force. Eq. 8b for traction is also known as the Neumann or natural boundary condition which is naturally subsumed into the weak form in the following derivation. First, Eq. 8 is converted to indicial notation,

$$
\begin{aligned}
-\frac{\partial}{\partial x_k}\mathbf{e_k} \cdot \sigma_{ij}(\mathbf{e_i} \otimes \mathbf{e_j}) &= f_k \mathbf{e_k} \\
-\frac{\partial \sigma_{ij}}{\partial x_k}\delta_{ki}\mathbf{e_j} &= f_k \mathbf{e_k} \\
-\frac{\partial \sigma_{ij}}{\partial x_i}\mathbf{e_j} &= f_k \mathbf{e_k} \quad \text{Multiply be a test function} \\
-\frac{\partial \sigma_{ij}}{\partial x_i}\mathbf{e_j} \cdot v_p \mathbf{e_p} &= f_k \mathbf{e_k} \cdot v_p \mathbf{e_p} \\
-\frac{\partial \sigma_{ij}}{\partial x_i} v_p \delta_{jp} &= f_k v_p \delta_{kp} \\
-\frac{\partial \sigma_{ij}}{\partial x_i} v_j &= f_k v_k \quad \text{Integrate over domain} \\
-\int_\Omega \frac{\partial \sigma_{ij}}{\partial x_i} v_j dx &= \int_\Omega f_k v_k dx
\end{aligned}
$$

Integration by parts on the left hand side

$$
\begin{aligned}
(fg)' = f'g + fg' \rightarrow f'g &= (fg)' - fg' \\
\frac{\partial \sigma_{ij}}{\partial x_i} v_j &= (\sigma_{ij} v_j)_{,i} - \sigma_{ij}\frac{\partial v_j}{\partial x_i}
\end{aligned}
$$

Substitute the result from integration by parts:

$$
\begin{aligned}
-\int_\Omega (\sigma_{ij} v_j)_{,i} dx + \int_\Omega \sigma_{ij}\frac{\partial v_j}{\partial x_i} dx &= \int_\Omega f_k v_k dx \quad \text{Use the divergence theorem} \\
-\int_{\partial\Omega} \sigma_{ij} v_j n_i ds + \int_\Omega \sigma_{ij}\frac{\partial v_j}{\partial x_i} dx &= \int_\Omega f_k v_k dx \quad \text{Recognize the traction term} \\
-\int_{\partial\Omega} t_i v_j ds + \int_\Omega \sigma_{ij}\frac{\partial v_j}{\partial x_i} dx &= \int_\Omega f_k v_k dx \quad \text{Rearrange} \\
\int_\Omega \sigma_{ij}\frac{\partial v_j}{\partial x_i} dx &= \int_\Omega f_k v_k dx + \int_{\partial\Omega} t_i v_j ds
\end{aligned}
$$

The principle of virtual work states that:

$$
\int_\Omega \delta W dV = \int_\Omega \sigma_{ij}\delta\epsilon_{ij} dV = \int_{\partial\Omega} t_i \delta u_i dS + \int_\Omega b_i \delta u_i dV \tag{9}
$$

Applying this principle to, where $\mathbf{f}$ is the body force:

$$
\begin{aligned}
\int_\Omega \sigma_{ij}\frac{\partial v_j}{\partial x_i} dx &= \int_\Omega f_k v_k dx + \int_{\partial\Omega} t_i v_j ds \\
\int_\Omega \sigma_{ij}\epsilon_{ij} dV &= \int_\Omega f_k v_k dx + \int_{\partial\Omega} t_i v_j ds \quad \text{no traction applied} \\
\int_\Omega \sigma_{ij}\epsilon_{ij} dV &= \int_\Omega f_k v_k dx
\end{aligned}
$$

Writing in direct notation, we have the variational (weak) formulation. Find $\mathbf{u} \in V$ such that:

$$
\int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})\, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega \quad \forall \mathbf{v} \in V \tag{10}
$$

3 FEniCS Implementation

User parameters were added to allow for specification from the command-line prompt

```
parameters.parse()
userpar = Parameters("user")
userpar.add("model", "plane_stress")
userpar.parse()
```

Next, user parameters can be specified by typing in the command line prompt

```
python3 Demo2DLE.py - -model plane_stress
```

(no space between dashes), where any number of user parameters can be added and specified.

```
python3 Demo2DLE.py - -model plane_stress - -MaxStep 20 - -MaxRho 0.2
```

3.1 Validation with Analytical Solution

We can validate our problem according to the analytical solution from Euler-Bernoulli beam theory

$$w_{beam} = \frac{qL^4}{8EI} \tag{11}$$

where w_{beam} is the overall deflection of the beam, q is a distributed load, E is Young's modulus, and I is the inertia defined as

$$I = \frac{1}{12}LH^3$$

We want to define the distributed load, q, in terms of our knowns. Note that the force, f, is defined where f and L are perpendicular

$$q = f \times L \rightarrow q = fL$$

Using the inertia and definition for q in Eq. 11

$$\begin{aligned} w_{beam} &= \frac{qL^4}{8E}\frac{12}{LH^3} \\ &= \frac{3qL^3}{EH^3} \quad \text{Substitute in } q = fL \\ w_{beam} &= \frac{3fL^4}{EH^3} \end{aligned}$$

The percentage error is quantified using the following equation, where S denotes simulation deflection and A denotes the analytical solution.

$$\%\,\text{Error} = 100\,\frac{abs|w_S - w_A|}{w_S}$$

Fig. 1 demonstrates the difference between the simulation result and analytical prediction for an increasing body force. Euler-Bernoulli beam theory has several assumptions, some of which include the assumption of linear elasticity and plane sections remaining plane. As more body force is applied, a larger displacement results in strains larger than the assumptions of linear elasticity (5 % strain).

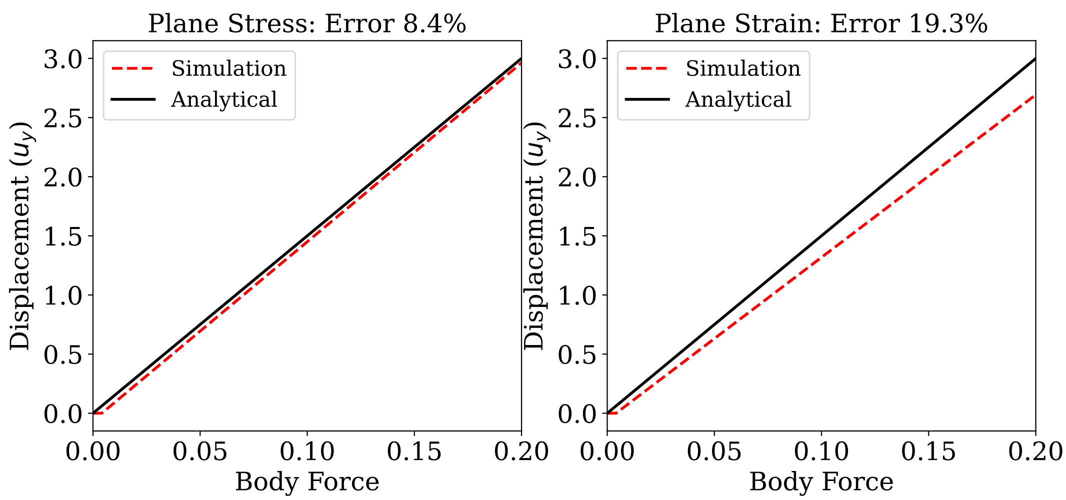

Figure 1: Difference between the simulation and analytical solution for a) Plane Stress and b) Plane Strain for a bar where $E = 1000$, $\nu = 0.3$. Average percentage error for every time step is quantified in the title.