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On some mixed finite element methods for incompressible and nearly incompressible finite elasticity

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Abstract We compare some mixed methods based on different variational formulations, namely a displacement-pressure formulation employed by de Borst and coworkers, the three-field formulation investigated by Simo and Taylor and a two-field formulation which is directly based on an energy functional. It emerges that all these yield the same discrete results if the stored energy function contains a volumetric contribution $\frac{1}{2}\kappa(J-1)^2$ where J is the volume dilatation, i.e., the Jacobian determinant of the deformation, and κ is the bulk modulus. The equivalence holds for arbitrary 3D and plane strain elements. In the numerical examples the mixed formulations are discretized by the quadrilateral Q1/P0 and Q2/P1 elements and the triangular Crouzeix-Raviart P2+/P1 element. We also compare with standard displacement elements and the enhanced strain Q1/E4 element.

1 Introduction

It is widely accepted that mixed finite elements are an efficient approach to overcome the locking that is often observed in the numerical treatment of almost incompressible materials when a low order pure displacement method is used. In a mixed method, besides the displacement, one or more additional fields are treated as independent variables. Typically, such an extra unknown field is the pressure. However, in the literature comparisons of the various mixed approaches actually used are rare.

For linear elliptic boundary value problems the theory of mixed methods is well established: By the Babuška-Brezzi theory, see Brezzi (1974), certain conditions are imposed on the continuous variational formulation and on the discrete approximation spaces in order to guarantee unique solvability, convergence and robustness in the incompressible limit. These conditions may be rather hard to check, but numerous stable mixed elements are known. When the conditions are violated, spurious modes are frequently observed.

Problems in nonlinear elasticity are usually solved by the Newton or a Newton-like iterative method. This requires the linearization of a weak formulation. In each iteration a linear problem is solved which, at least for sufficiently small deformations, should satisfy the requirements of the Babuška-Brezzi stability theory.

It is possible to derive a two-field formulation for finite elasticity directly from an energy functional. We present a formulation which is a generalization of the approach by Chang, Saleeb and Li (1991). After linearization the method is very similar to a formulation by de Borst and coworkers, see de Borst et al. (1988), van den Bogert et al. (1991, 1994). Another related displacement-pressure method was proposed by Sussman and Bathe (1987). Atluri and Reissner (1989) presented a general framework for the derivation of variational principles involving volume constraints. They proposed formulations with two, three and more independent fields. Some of the formulations are related to other approaches discussed in the present paper.

The main purpose of the present paper is to compare various mixed methods. It emerges that under certain conditions the two-field formulations mentioned above are equivalent to the three-field method investigated by Simo et al. (1985, 1991) which involves the volume dilatation, i.e., the determinant of the deformation gradient, as an additional unknown. Equivalence here means that in each step of the Newton method the same increment in the discrete unknowns is obtained.

In order to apply a mixed method it is a standard procedure to decompose the stored energy function additively into two parts, one of them being responsible for the (almost) incompressible behavior and depending solely on the volume dilatation. The main restriction in the proof of the above equivalence result is that the volumetric part of the stored energy function has a simple form, namely $\frac{1}{2}\kappa(J-1)^2$ with κ being the bulk modulus. However, this function is often used in the modeling of rubber-like materials. We do not require the remaining part of the stored energy function to be independent of the dilatation J.

De Borst et al. (1988) and van den Bogert et al. (1991) used the Q2/Q1 element in the two-dimensional case and for three dimensions the Q1/P0 element as well as an element with quadratic serendipity displacements and four pressure degrees of freedom. We combine their formula-

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tion with some other discrete spaces and exploit a push-forward to the deformed configuration. Specifically, we employ the Q2/P1, the Q1/P0 and the P2+/P1 element; the latter is due to Crouzeix and Raviart (1973) and is triangular or tetrahedral in two or three dimensions, respectively. This geometry is useful especially for adaptive mesh refinement.

For the sake of comparison, numerical results are also given for standard displacement elements and the enhanced strain Q1/E4 element due to Simo and Armero (1992). The latter element, for plane strain, has four extra degrees of freedom besides the displacements, but, as remarked by Reddy and Simo (1995), it does not arise from a saddle point problem in the sense of the Babuška-Brezzi theory. The Q1/E4 element offers the advantage that the material law can be used in the same form as in a standard displacement method.

When comparing numerical results, the number of load steps required in the incremental procedure is an important characteristic besides the accuracy of displacements and stresses. It emerges that properly constructed mixed elements allow higher load increments than displacement elements and the Q1/E4 element.

The main objective of mixed displacement-pressure formulations is to overcome the volume constraint. One may introduce other independent fields, for instance to improve the performance in bending. Seki and Atluri (1994, 1995) employ a four-field variational principle with the deviatoric part of the (unsymmetric) Biot stress, the hydrostatic pressure, the displacement and drilling degrees of freedom as independent variables.

In this paper we do not intend to give a complete review of the huge amount of literature on the treatment of incompressible and nearly incompressible material. Also, we do not discuss the method of selective reduced integration, which is designed to overcome locking and is in many cases equivalent to a mixed finite element method so that the Babuška-Brezzi conditions are a critical point.

This paper is organized as follows: In Section 2 we introduce the boundary value problem of finite elastostatics and three mixed formulations, namely the two-field method by de Borst et al., a new two-field method and the three-field method by Simo and Taylor. Section 3 is devoted to a theoretical investigation of the mixed formulations. In Section 4 first the finite element spaces are detailed, then some constitutive laws for rubber-like materials and their implementation into the mixed elements are discussed, and we report on various numerical computations. Finally, in Section 5 the main conclusions are summarized.

2
Variational formulations

2.1
The boundary value problem of finite elastostatics

In the undeformed state, which is also referred to as the reference configuration, the body occupies the domain $\mathscr{B} \subset R^3$. The deformation is an injective orientation-preserving mapping

$$\left.\begin{aligned} \varphi : \mathscr{B} &\to R^3 \\ \boldsymbol{X} &\mapsto \varphi(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}) \end{aligned}\right\} \tag{2.1}$$

with the displacement field $\boldsymbol{u}$. Further, we introduce the deformation gradient $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{u}) = 1 + \nabla \boldsymbol{u}$ with $\mathbf{1}$ being the second-order unit tensor and ∇ denoting the gradient with respect to the coordinates $\boldsymbol{X} \in \mathscr{B}$. We will make frequent use of the right Cauchy-Green strain tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$.

The boundary value problem of finite elastostatics consists in finding a displacement field $\boldsymbol{u} : \mathscr{B} \to R^3$ such that the equations of equilibrium and the kinematic boundary conditions are satisfied,

$$\left.\begin{aligned} -\mathrm{Div}[\boldsymbol{FS}] &= \boldsymbol{f} \quad \text{in } \mathscr{B} \\ \boldsymbol{S} &= \boldsymbol{S}^T \quad \text{in } \mathscr{B} \\ [\boldsymbol{FS}]\boldsymbol{n} &= \boldsymbol{g} \quad \text{on } \Gamma_N \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_D \end{aligned}\right\} \tag{2.2}$$

Here $\boldsymbol{f}$ denote body forces, and the boundary of $\mathscr{B}$ is partitioned into Γ_D and Γ_N with surface forces prescribed on Γ_N. The exterior unit normal is $\boldsymbol{n}$. The dependence of the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ on the deformation is given by a constitutive law. We assume the material to be hyperelastic, that means there exists a stored energy function W such that

$$\boldsymbol{S} = 2\frac{\partial W(\boldsymbol{X}, \boldsymbol{C})}{\partial \boldsymbol{C}} \ . \tag{2.3}$$

Moreover, if the material is isotropic, the value of the stored energy function W depends on the principal invariants of C only,

$$W(\boldsymbol{X}, \boldsymbol{C}) = \hat{W}(\boldsymbol{X}, I_1, I_2, I_3) \tag{2.4}$$

where $I_1 = \mathrm{tr}\ \boldsymbol{C}, I_2 = \frac{1}{2}((\mathrm{tr}\ \boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2)), I_3 = \det \boldsymbol{C}$. Examples of stored energy functions will be given in Section 4.2.

2.2
The displacement formulation

For completeness and to introduce some concepts, let us begin with the weak formulation of (2.2). We multiply the first equation of (2.2) by a test function $\overset{*}{\boldsymbol{u}}$ lying in the space V of kinematically admissible displacements and integrate by parts,

$$\int_{\mathscr{B}} 2\frac{\partial W(\boldsymbol{C}(\boldsymbol{u}))}{\partial \boldsymbol{C}} : [\boldsymbol{F}(\boldsymbol{u})^T \nabla \overset{*}{\boldsymbol{u}}]\, \mathrm{d}V = L_{\mathrm{ext}}(\overset{*}{\boldsymbol{u}}) \ . \tag{2.5}$$

As usual, the inner product of two second-order tensors is $\boldsymbol{A} : \boldsymbol{B} = \Sigma_{i,j} A^{ij} B_{ij}$. The exact solution $\boldsymbol{u}$ is sought in the space V which, in terms of Sobolev spaces, can be defined as $V = \{\boldsymbol{v} \in W^{1,s}(\mathscr{B}) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}$ where $s \geq 1$ is such that all integrals exist. The exterior loads are

$$L_{\mathrm{ext}}(\overset{*}{\boldsymbol{u}}) = \int_{\mathscr{B}} \overset{*}{\boldsymbol{u}} \cdot \boldsymbol{f}\, \mathrm{d}V + \int_{\Gamma_N} \overset{*}{\boldsymbol{u}} \cdot \boldsymbol{g}\, \mathrm{d}S \ . \tag{2.6}$$

For conservative loads the total potential energy can be written as

$$\Pi(\boldsymbol{u}) = \int_{\mathscr{B}} W(C(\boldsymbol{u}))\,\mathrm{d}V - \Pi_{\mathrm{ext}}(\boldsymbol{u}) \ . \tag{2.7}$$

Solving the weak formulation (2.5) is in a formal sense equivalent to minimizing $\Pi(\boldsymbol{u})$ over the set V. (2.5) can directly be obtained by requiring the Gateaux derivative with respect to $\boldsymbol{u}$ to vanish in all directions $\overset{*}{\boldsymbol{u}} \in V$,

$$D_u\Pi(\boldsymbol{u})\,\overset{*}{\boldsymbol{u}} = 0 \ , \tag{2.8}$$

see, e.g., Stein and Müller-Hoeppe (1987). $D_u\Pi(\boldsymbol{u})\,\overset{*}{\boldsymbol{u}}$ is also referred to as the first variation of $\Pi(\boldsymbol{u})$ and is defined as

$$D_u\Pi(\boldsymbol{u})\,\overset{*}{\boldsymbol{u}} = \lim_{\varepsilon\to 0}\frac{\Pi(\boldsymbol{u}+\varepsilon\,\overset{*}{\boldsymbol{u}}) - \Pi(\boldsymbol{u})}{\varepsilon} \ . \tag{2.9}$$

2.3 Mixed displacement-pressure formulation

Our main objective is the treatment of nearly incompressible materials. They are characterized by the fact that volume-changing deformations require a significantly higher exterior work than volume-preserving deformations. If this is not properly taken into account in numerical methods, a considerable loss of accuracy must be expected and often locking or instabilities occur.

It is standard to employ an additive decomposition of the stored energy function. In this section we assume

$$W = \kappa U(J) + \tilde{W}(\boldsymbol{C}) \ . \tag{2.10}$$

Here κ is the (possibly generalized) bulk modulus which is required to be independent of the deformation. The volumetric part U of the stored energy function depends solely on the Jacobian $J = J(\boldsymbol{u}) = \det \boldsymbol{F} = (\det \boldsymbol{C})^{1/2}$ of the deformation. We assume that

$$U \text{ is strictly convex (at least in a neighborhood of } J = 1) \ , \tag{2.11}$$

$$U \text{ has a minimum at } J = 1 \ . \tag{2.12}$$

One may also impose the condition

$$W \to +\infty \ as \ J \to 0+ \ . \tag{2.13}$$

However, this property may be built into $\tilde{W}$ rather than U. We do not exclude the case that U or $\tilde{W}$ depend on κ, but we require both functions to remain well-behaved as $\kappa \to +\infty$. In particular, for fixed $J > 0$ they must be bounded independently of κ.

The decomposition (2.10) implies the second Piola-Kirchhoff stress to be of the form

$$\mathbf{S} = 2\frac{\partial W}{\partial \boldsymbol{C}} = 2\kappa U'(J)\frac{\partial J}{\partial \boldsymbol{C}} + 2\frac{\partial \tilde{W}}{\partial \boldsymbol{C}} \ . \tag{2.14}$$

Hence the condition

$$\frac{\partial \tilde{W}(\boldsymbol{C})}{\partial \boldsymbol{C}} = \mathbf{0} \quad \text{if} \quad \boldsymbol{C} = \mathbf{1} \tag{2.15}$$

together with (2.12) ensures that the reference configuration is stress-free. As we will see, the above assumptions make it possible to isolate the effect of a large bulk modulus and thus facilitate the development of robust numerical methods.

Remark 2.1 Given a stored energy function W, the additive decomposition (2.10) is in general not unique. Examples will be given in Section 4.2. A unique decomposition can be achieved by stipulating that the contribution $\tilde{W}$ depend on the isochoric part of the deformation only, i.e., $\tilde{W}(C) = \tilde{W}(J^{-2/3}C)$ (note that $\det(J^{-2/3}C) \equiv 1$), but we do not require this assumption. In the developments by Simo and Taylor (1991) this assumption is used, but it means a restriction on the form of the stored energy function.

Remark 2.2 Atluri and Reissner (1989) also introduce the isochoric part $J^{-2/3}C$ of the deformation and then work with the corresponding modified Green-Lagrange strain tensor.

Owing to the decomposition (2.10), the Cauchy stress tensor $\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T$ becomes

$$\boldsymbol{\sigma} = \kappa U'(J)\mathbf{1} + 2J^{-1}\boldsymbol{F}\frac{\partial \tilde{W}}{\partial \boldsymbol{C}}\boldsymbol{F}^T \ , \tag{2.16}$$

where we used the relation $\partial J/\partial \boldsymbol{C} = \frac{1}{2}J\boldsymbol{C}^{-1}$. In this representation of the stress in the deformed configuration, the quantity

$$p = \kappa U'(J(\boldsymbol{u})) \tag{2.17}$$

contributes solely to the hydrostatic pressure. The important point in the subsequent developments is to treat $\boldsymbol{u}$ and p as two independent fields. $\boldsymbol{u}$ is sought in the space V of kinematically admissible displacements as in (2.5), while p is sought in $Q = L^r(\Omega)$ with some $r > 0$. We insert (2.14) into the weak formulation (2.5). Further, (2.17) is enforced in a weak sense. This yields the following *mixed* formulation: Find $(\boldsymbol{u}, p) \in V \times Q$ such that for all $(\overset{*}{\boldsymbol{u}}, \overset{*}{p}) \in V \times Q$

$$\left.\begin{aligned} &\int_{\mathscr{B}} 2\frac{\partial \tilde{W}(C(\boldsymbol{u}))}{\partial \boldsymbol{C}} : [\boldsymbol{F}(\boldsymbol{u})^T\nabla\,\overset{*}{\boldsymbol{u}}]\,\mathrm{d}V \\ &\qquad + \int_{\mathscr{B}} 2p\frac{\partial J(\boldsymbol{u})}{\partial \boldsymbol{C}} : [\boldsymbol{F}(\boldsymbol{u})^T\nabla\,\overset{*}{\boldsymbol{u}}]\,\mathrm{d}V = L_{\mathrm{ext}}(\overset{*}{\boldsymbol{u}}) \\ &\int_{\mathscr{B}} \left\{U'(J(\boldsymbol{u})) - \frac{1}{\kappa}p\right\}\overset{*}{p}\,\mathrm{d}V = 0 \ . \end{aligned}\right\} \tag{2.18}$$

Observe that this weak formulation has not been obtained by taking the first variation of any energy functional. Thus the linearization will not automatically be symmetric.

However, it will emerge that in a very simple way symmetry can be retained without introducing any error.

If we write the first equation of the mixed formulation (2.18) in short-hand notation as

$$G_{(u,p)}(\overset{*}{u}) = L_{\text{ext}}(\overset{*}{u}) \ , \tag{2.19}$$

the linearization leads to

$$D_u[G_{(u,p)}(\overset{*}{u})]\overset{\triangle}{u} + D_p[G_{(u,p)}(\overset{*}{u})]\overset{\triangle}{p} = L_{\text{ext}}(\overset{*}{u}) - G_{(u,p)}(\overset{*}{u}) \tag{2.20}$$

where on the left side the Gateaux derivatives are bilinear forms, denoted by $a_{(u,p)}(.,.)$ and $b_{(u)}(.,.)$,

$$\begin{aligned} a_{(u,p)}(\overset{\triangle}{u},\overset{*}{u}) &:= D_u[G_{(u,p)}(\overset{*}{u})]\overset{\triangle}{u} \\ &= \int_{\mathscr{B}} 4[\boldsymbol{F}^T\nabla\overset{*}{u}]^{\text{sym}} : \left\{\frac{\partial^2\tilde{W}}{\partial C\partial C} + p\frac{\partial^2 J}{\partial C\partial C}\right\}[\boldsymbol{F}^T\nabla\overset{\triangle}{u}]^{\text{sym}}\,\mathrm{d}V \\ &\quad + \int_{\mathscr{B}} 2\left\{\frac{\partial\tilde{W}}{\partial C} + p\frac{\partial J}{\partial C}\right\} : [\nabla\overset{\triangle}{u}^T\nabla\overset{*}{u}]\,\mathrm{d}V \end{aligned} \tag{2.21}$$

$$b_{(u)}(\overset{\triangle}{p},\overset{*}{u}) := D_p[G_{(u,p)}(\overset{*}{u})]\overset{\triangle}{p} = \int_{\mathscr{B}} 2\overset{\triangle}{p}\frac{\partial J}{\partial C} : [\boldsymbol{F}^T\nabla\overset{*}{u}]\,\mathrm{d}V \ , \tag{2.22}$$

using the notation $\boldsymbol{A}^{\text{sym}} = \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T)$. Keep in mind that $\boldsymbol{F}$, $\tilde{W}$ and J depend on $\boldsymbol{u}$ only. The linearization of the second equation in (2.18) is

$$\begin{aligned} &\int_{\mathscr{B}} 2\overset{*}{p}\,U''(J)\frac{\partial J}{\partial C} : [\boldsymbol{F}^T\nabla\overset{\triangle}{u}]\,\mathrm{d}V - \int_{\mathscr{B}}\frac{1}{\kappa}\overset{*}{p}\overset{\triangle}{p}\,\mathrm{d}V \\ &\quad = -\int_{\mathscr{B}}\overset{*}{p}\,U'(J)\,\mathrm{d}V + \int_{\mathscr{B}}\frac{1}{\kappa}\overset{*}{p}p\,\mathrm{d}V \ . \end{aligned} \tag{2.23}$$

To achieve symmetry the essential step is to divide this equation by $U''(J(\boldsymbol{u}))$. Then the method reads: Given the current state $(\boldsymbol{u},p)$ (which is in general not in equilibrium), find increments $(\overset{\triangle}{u},\overset{\triangle}{p}) \in V \times Q$ such that

$$\left.\begin{aligned} a_{(u,p)}(\overset{\triangle}{u},\overset{*}{u}) + b_{(u)}(\overset{\triangle}{p},\overset{*}{u}) &= L_{\text{ext}}(\overset{*}{u}) - G_{(u,p)}(\overset{*}{u}) \quad \forall\,\overset{*}{u} \in V \\ b_{(u)}(\overset{*}{p},\overset{\triangle}{u}) - c_{(u)}(\overset{*}{p},\overset{\triangle}{p}) &= R_{(u)}(\overset{*}{p}) \qquad\qquad \forall\,\overset{*}{p} \in Q \end{aligned}\right\} \tag{2.24}$$

where

$$c_{(u)}(\overset{*}{p},\overset{\triangle}{p}) = \int_{\mathscr{B}}\frac{1}{\kappa U''(J)}\overset{*}{p}\overset{\triangle}{p}\,\mathrm{d}V \tag{2.25}$$

$$R_{(u)}(\overset{*}{p}) = -\int_{\mathscr{B}}\overset{*}{p}\frac{U'(J)}{U''(J)}\,\mathrm{d}V + \int_{\mathscr{B}}\frac{1}{\kappa U''(J)}\overset{*}{p}p\,\mathrm{d}V \ . \tag{2.26}$$

The linearized weak formulation (2.24) was previously used by van den Bogert, de Borst, Luiten and Zeilmaker (1991).

In the implementation it is advantageous to refer certain quantities to the current deformed configuration, which is also called the spatial or Eulerian configuration. In particular, the so-called B-Matrices become sparse. For this purpose let us introduce the notation grad and div for the gradient and the divergence operator with respect to the spatial coordinates $\boldsymbol{x} \in \boldsymbol{\varphi}(\mathscr{B})$. We define a modified Kirchhoff stress tensor $\tilde{\boldsymbol{\tau}}$ by

$$\tilde{\boldsymbol{\tau}} = 2\boldsymbol{F}\frac{\partial\tilde{W}}{\partial C}\boldsymbol{F}^T \tag{2.27}$$

and a modified forth-order elastic tangent tensor $\tilde{\mathbf{e}}$ by

$$\tilde{\mathbf{e}}^{iklm} = 4F_I^iF_K^k\frac{\partial^2\tilde{W}}{\partial C_{IK}\partial C_{LM}}F_L^lF_M^m \ . \tag{2.28}$$

Here the components of the deformation gradient $\boldsymbol{F}$ are $F_I^i = \partial u^i/\partial X^I + \delta_I^i$.

Proposition 2.1 *Push-forward to the deformed configuration yields*

$$G_{(u,p)}(\overset{*}{u}) = \int_{\mathscr{B}}\tilde{\boldsymbol{\tau}} : \operatorname{grad}\overset{*}{u}\,\mathrm{d}V + \int_{\varphi(B)} p\,\operatorname{div}\overset{*}{u}\,\mathrm{d}v \ ,$$

$$\begin{aligned} a_{(u,p)}(\overset{\triangle}{u},\overset{*}{u}) = \int_{\mathscr{B}} \Big\{ &[\operatorname{grad}\overset{*}{u}]^{\text{sym}} : \tilde{\mathbf{e}}[\operatorname{grad}\overset{\triangle}{u}]^{\text{sym}} \\ &+ \tilde{\boldsymbol{\tau}} : [\operatorname{grad}\overset{\triangle}{u}^T\operatorname{grad}\overset{*}{u}]\Big\}\mathrm{d}V \\ &+ \int_{\varphi(\mathscr{B})} p\Big\{\operatorname{div}\overset{\triangle}{u}\operatorname{div}\overset{*}{u} - \operatorname{grad}\overset{\triangle}{u} : \operatorname{grad}\overset{*}{u}^T\Big\}\mathrm{d}v \ , \end{aligned}$$

$$b_{(u)}(\overset{\triangle}{p},\overset{*}{u}) = \int_{\varphi(\mathscr{B})}\overset{\triangle}{p}\ \operatorname{div}\overset{*}{u}\,\mathrm{d}v \ .$$

Proof. Beginning with the terms in $b_{(u)}(\overset{\triangle}{p},\overset{*}{u})$, one observes that $\nabla\overset{*}{u} = \operatorname{grad}\overset{*}{u}\,\boldsymbol{F}(\boldsymbol{u})$ and $\partial J/\partial C = \frac{1}{2}JC^{-1}$. Hence,

$$2\frac{\partial J}{\partial C} : [\boldsymbol{F}(\boldsymbol{u})^T\nabla\overset{*}{u}] = J\mathbb{1} : \operatorname{grad}\overset{*}{u} = J\operatorname{div}\overset{*}{u} \ . \tag{2.29}$$

For an arbitrary second-order tensor $\boldsymbol{A}$,

$$[\partial\boldsymbol{A}^{-1}/\partial\boldsymbol{A}]^{ijkl} = -[\boldsymbol{A}^{-1}]^{ik}[\boldsymbol{A}^{-1}]^{lj} \ , \tag{2.30}$$

$$[\partial\boldsymbol{A}/\partial\boldsymbol{A}]^{ijkl} = \delta^{ik}\delta^{jl} =: [\mathbb{1}]^{ijkl} \ . \tag{2.31}$$

The symmetry of C implies

$$[\partial C^{-1}/\partial C]^{ijkl} = -[C^{-1}]^{ik}[C^{-1}]^{jl} =: -[\mathbb{1}_{C^{-1}}]^{ijkl} \tag{2.32}$$

Now it is elementary to verify

$$\frac{\partial^2 J}{\partial C\partial C} = -\frac{1}{2}J\mathbb{1}_{C^{-1}} + \frac{1}{4}JC^{-1}\otimes C^{-1} \ , \tag{2.33}$$

where $[\boldsymbol{A}\otimes\boldsymbol{B}]^{ijkl} = A^{ij}B^{kl}$. (We remark that, for symmetry reasons, $\mathbb{1}_{C^{-1}}$ can be replaced by $\frac{1}{2}([C^{-1}]^{ik}[C^{-1}]^{jl} + [C^{-1}]^{il}[C^{-1}]^{jk})$.) As in (2.28), a forth-order tensor $\mathbb{C}$ in the reference configuration has the spatial counterpart $\mathbb{C}^\flat$ with

$$[\mathbb{C}^{\natural}]^{iklm} = 4F_I^i F_K^k \mathbb{C}^{IKLM} F_L^l F_M^m \ . \tag{2.34}$$

With this notation,

$$[\mathbb{1}_{C^{-1}}]^{\natural} = \mathbb{1} \tag{2.35}$$

and

$$[C^{-1} \otimes C^{-1}]^{\natural} = 1 \otimes 1 \ . \tag{2.36}$$

Therefore,

$$4[F^T \nabla \overset{*}{u}]^{\mathrm{sym}} : p\frac{\partial^2 J}{\partial C \partial C}[F^T \nabla \overset{\triangle}{u}]^{\mathrm{sym}}$$
$$= pJ\left\{\operatorname{div} \overset{\triangle}{u} \operatorname{div} \overset{*}{u} - 2 \operatorname{grad} \overset{\triangle}{u}^{\mathrm{sym}} : \operatorname{grad} \overset{*}{u}^{\mathrm{sym}}\right\} \ .$$

Further,

$$2p\frac{\partial J}{\partial C} : [\nabla \overset{\triangle}{u}^T \nabla \overset{*}{u}] = pJ \operatorname{grad} \overset{\triangle}{u} : \operatorname{grad} \overset{*}{u} \ . \tag{2.37}$$

Note that

$$\operatorname{grad} \overset{\triangle}{u} : \operatorname{grad} \overset{*}{u} - 2 \operatorname{grad} \overset{\triangle}{u}^{\mathrm{sym}} : \operatorname{grad} \overset{*}{u}^{\mathrm{sym}}$$
$$= -\operatorname{grad} \overset{\triangle}{u} : \operatorname{grad} \overset{*}{u}^T \ .$$

The rest follows by straightforward arguments. □

The choice of suitable finite element spaces for the displacement and the pressure will be motivated in Section 3 by considering linear elasticity.

Remark 2.3 Proposition 2.1 implies that, up to rounding errors in solving the linear systems, discretization of the total Lagrangian formulation with (2.21), (2.22) and of the Eulerian formulation as in Proposition 2.1 lead to the same results.

Remark 2.4 The mixed weak formulation (2.18) does not break down for $\kappa \to +\infty$. If $1/\kappa$ is replaced by zero, (2.18) becomes a weak formulation of the boundary value problem of *fully incompressible* finite elastostatics. The equation

$$\int_{\mathscr{B}} U'(J(u)) \overset{*}{p} \, dV = 0 \tag{2.38}$$

is a weak form of the incompressibility constraint $J(u) = 1$ since we assume that $U'(J) = 0$ holds if and only if $J = 1$. For the case $U'(J) = J - 1$ this incompressible formulation was investigated by Le Tallec (1994). The linearization of the incompressible formulation is given by (2.24), again with $1/\kappa$ replaced by zero.

Remark 2.5 All these developments remain valid for the case of *plane strain*. Then the displacement is independent of the X^3- coordinate and u^3 vanishes. The tensors F and C simplify accordingly.

2.4
A two-field variational principle

The mixed formulation (2.18) was not directly derived from an energy functional. In contrast, the following formulation is based on a two-field energy functional and leads to a method which, to the authors' knowledge, is new. It can be viewed as a specialization of an approach by Atluri and Reissner (1989).

The additive decomposition of the stored energy function is now required to have the form

$$W = \frac{1}{2}\kappa[\hat{U}(J)]^2 + \tilde{W}(C) \tag{2.39}$$

instead of (2.10). We require that

$$\hat{U}(J) = 0 \quad \text{if and only if } J = 1 \ , \tag{2.40}$$

and hence the conditions (2.11) and (2.12) are satisfied provided U is sufficiently smooth.

The minimization problem for the total potential energy $\Pi(u)$ in (2.7) now becomes a saddle point problem:

$$\inf_{u \in V} \sup_{p \in Q} \Pi(u, p) \tag{2.41}$$

with

$$\Pi(u,p) = \int_{\mathscr{B}} \tilde{W}(C(u)) \, dV + \int_{\mathscr{B}} p\hat{U}(J(u)) \, dV$$
$$- \frac{1}{2}\int_{\mathscr{B}} \frac{1}{\kappa} p^2 \, dV - \Pi_{\mathrm{ext}}(u) \ . \tag{2.42}$$

This formulation can be considered as a perturbed Lagrangian obtained by a Legendre transformation.

Proposition 2.2 *Solving the saddle point problem (2.41) formally corresponds to minimizing (2.7) provided that* $\kappa < \infty$.

Proof. The weak formulation of (2.41) is obtained by requiring the Gateaux derivative to vanish,

$$D_u \Pi(u,p) \overset{*}{u} = 0 \quad \text{and} \quad D_p \Pi(u,p) \overset{*}{p} = 0 \tag{2.43}$$

for all directions $(\overset{*}{u}, \overset{*}{p}) \in V \times Q$; more explicitly: Find $(u,p) \in V \times Q$ such that for all $(\overset{*}{u}, \overset{*}{p}) \in V \times Q$

$$\left.\begin{aligned} &\int_{\mathscr{B}} 2\frac{\partial \tilde{W}(C(u))}{\partial C} : [F(u)^T \nabla \overset{*}{u}] \, dV \\ &+ \int_{\mathscr{B}} 2p\hat{U}'(J(u))\frac{\partial J(u)}{\partial C} : [F(u)^T \nabla \overset{*}{u}] \, dV = L_{\mathrm{ext}}(\overset{*}{u}) \\ &\int_{\mathscr{B}} \left\{\hat{U}(J(u)) - \frac{1}{\kappa}p\right\} \overset{*}{p} \, dV = 0 \ . \end{aligned}\right\} \tag{2.44}$$

If (u,p) is a solution of (2.44) then, by the second equation in (2.44),

$$p = \kappa \hat{U}(J(u)) \ . \tag{2.45}$$

Inserting this into the first equation of (2.44) we see that u is a solution of the weak formulation (2.5) of the minimization problem (2.7). Vice versa, if u is a solution of (2.5), then (u,p) with $p := \kappa\hat{U}(J(u))$ is a solution of (2.44). □

Utilizing the special form of the stored energy function and the relation (2.45), the second Piola-Kirchhoff stress can be written as

$$S = 2p\,\hat{U}'(J(u))\frac{\partial J(u)}{\partial C} + 2\frac{\partial \tilde{W}(C(u))}{\partial C} \tag{2.46}$$

and the Cauchy stress is

$$\sigma = p\,\hat{U}'(J(u))1 + 2J(u)^{-1}F(u)\frac{\partial \tilde{W}(C(u))}{\partial C}F(u)^T \ . \tag{2.47}$$

In contrast to the mixed formulation of Section 2.3, now p cannot directly be regarded as a pressure.

The linearization of (2.44) is given by the second variation of (2.42): Given (u , p), find increments $(\overset{\triangle}{u}, \overset{\triangle}{p}) \in V \times Q$ such that

$$\left.\begin{aligned} a_{(u,p)}(\overset{\triangle}{u}, \overset{*}{u}) + b_{(u)}(\overset{\triangle}{p}, \overset{*}{u}) &= -D_u\Pi(u,p)\overset{*}{u} \quad \forall \overset{*}{u} \in V \\ b_{(u)}(\overset{*}{p}, \overset{\triangle}{u}) - c_{(u)}(\overset{*}{p}, \overset{\triangle}{p}) &= -D_p\Pi(u,p)\overset{*}{p} \quad \forall \overset{*}{p} \in Q \end{aligned}\right\} \tag{2.48}$$

where

$$\begin{aligned} a_{(u,p)}(\overset{\triangle}{u}, \overset{*}{u}) &= D_u[D_u\Pi(u,p)\overset{*}{u}]\overset{\triangle}{u} \\ &= \int_{\mathscr{B}} 4[F^T\nabla \overset{*}{u}]^{\mathrm{sym}} : \left\{\frac{\partial^2 \tilde{W}}{\partial C \partial C} + p\hat{U}'(J)\frac{\partial^2 J}{\partial C \partial C} \right. \\ &\qquad \left. + p\hat{U}''(J)\frac{\partial J}{\partial C}\otimes\frac{\partial J}{\partial C}\right\}[F^T\nabla \overset{\triangle}{u}]^{\mathrm{sym}}\mathrm{d}V \\ &\quad + \int_{\mathscr{B}} 2\left\{\frac{\partial \tilde{W}}{\partial C} + p\hat{U}'(J)\frac{\partial J}{\partial C}\right\} : [\nabla \overset{\triangle}{u}^T \nabla \overset{*}{u}]\mathrm{d}V \\ b_{(u)}(\overset{\triangle}{p}, \overset{*}{u}) &= D_p[D_u\Pi(u,p)\overset{*}{u}]\overset{\triangle}{p} \\ &= \int_{\mathscr{B}} 2\overset{\triangle}{p}\,\hat{U}'(J)\frac{\partial J}{\partial C} : [F^T\nabla \overset{*}{u}]\mathrm{d}V \\ c_{(u)}(\overset{*}{p}, \overset{\triangle}{p}) &= -D_p[D_p\Pi(u,p)\overset{*}{p}]\overset{\triangle}{p} = \int_{\mathscr{B}} \frac{1}{\kappa}\overset{*}{p}\overset{\triangle}{p}\ \mathrm{d}V \ . \end{aligned}$$

In analogy to Proposition 2.1, the push-forward results in

$$\begin{aligned} &-D_u\Pi(u,p)\overset{*}{u} \\ &\quad = L_{\mathrm{ext}}(\overset{*}{u}) - \int_{\mathscr{B}} \tilde{\tau} : \mathrm{grad}\,\overset{*}{u}\,\mathrm{d}V - \int_{\varphi(\mathscr{B})} p\hat{U}'(J)\mathrm{div}\,\overset{*}{u}\,\mathrm{d}v \\ &a_{(u,p)}(\overset{\triangle}{u}, \overset{*}{u}) \\ &\quad = \int_{\mathscr{B}} \left\{[\mathrm{grad}\,\overset{*}{u}]^{\mathrm{sym}} : \tilde{\mathrm{e}}\,[\mathrm{grad}\,\overset{\triangle}{u}]^{\mathrm{sym}} \right. \\ &\qquad \left. + \tilde{\tau} : [\mathrm{grad}\,\overset{\triangle}{u}^T\,\mathrm{grad}\,\overset{*}{u}]\right\}\mathrm{d}V \\ &\qquad + \int_{\varphi(\mathscr{B})} p\{\hat{U}'(J) + J\hat{U}''(J)\}\mathrm{div}\,\overset{\triangle}{u}\,\mathrm{div}\,\overset{*}{u}\,\mathrm{d}v \\ &\qquad - \int_{\varphi(\mathscr{B})} p\hat{U}'(J)\mathrm{grad}\,\overset{\triangle}{u} : \mathrm{grad}\,\overset{*}{u}^T\,\mathrm{d}v \\ &b_{(u)}(\overset{\triangle}{p}, \overset{*}{u}) = \int_{\varphi(\mathscr{B})} \overset{\triangle}{p}\,\hat{U}'(J)\mathrm{div}\,\overset{*}{u}\,\mathrm{d}v \ . \end{aligned}$$

Note that $\tilde{\tau}, \tilde{\mathrm{e}}$ (as defined in (2.27), (2.28)) and J depend on u, but not on p. Incompressible material can be treated by replacing $1/\kappa$ by 0.

Remark 2.6 Comparison with Proposition 2.1 reveals that only some scalar factors have changed. If in the additive decomposition (2.10) U has the form $U(J) = \frac{1}{2}(J-1)^2$ then $\hat{U}(J) = J - 1$ so that the present formulation is identical to (2.24) in Section 2.3. In other words, in this special case the formulation by de Borst et al. can be derived from the energy functional (2.42).

Remark 2.7 Also, for $\hat{U}(J) = J - 1$ the energy functional (2.42) is identical to the expression used by Chang, Saleeb and Li (1991).

Remark 2.8 Atluri and Reissner (1989) derived a two-field formulation in a general framework; they start from a five-field variational principle, employ a partial Legendre transformation and then reduce the number of fields. The essential point in our formulation is the decomposition (2.39) of the stored energy function. With (2.39) the derivation of Atluri and Reissner also leads to the functional (2.42).

Remark 2.9 Sussman and Bathe (1987) proposed a mixed formulation which is based on the modified stored energy function

$$W = \check{W}(C(u)) - \frac{1}{2\kappa}(\kappa[J(u)-1]-p)^2 \ , \tag{2.49}$$

cf. Eqs. (2.26) and (3.24) in Sussman and Bathe (1987). The linearization leads to a problem of the form (2.48), but now $a_{(u,p)}(\overset{\triangle}{u}, \overset{*}{u})$ depends on κ. In contrast, in (2.48) and in Section 2.3 the influence of the possibly very large parameter κ is confined to $c_{(u)}(\overset{*}{p}, \overset{\triangle}{p})$. However, for almost incompressible material, one may assume that $\check{W}$ in (2.49) contains a hydrostatic work term, $\check{W} = \tilde{W}(C(u)) + W_H(J(u))$. Sussman and Bathe (1987), in their eq. (3.25), used $W_H(J) = \frac{1}{2}\kappa[J-1]^2$; then W in (2.49) reduces to

$$W = \tilde{W}(C(u)) + p[J(u)-1] - \frac{1}{2\kappa}p^2 \ , \tag{2.50}$$

which exactly leads to the energy functional (2.42) provided that $\hat{U}(J) = J - 1$ in (2.42).

2.5
A three-field variational principle

Simo and Taylor (1985, 1991) investigated a mixed method which involves an approximation of the volume dilatation, i.e., the Jacobian determinant, as a third independent field. They started from the three-field problem

$$\inf_u \inf_\theta \sup_p \Pi(u, \theta, p) \tag{2.51}$$

with

$$\Pi(\boldsymbol{u},\theta,p) = \int_{\mathscr{B}} \{\kappa U(\theta) + \tilde{W}(C(\boldsymbol{u})) + p[J(\boldsymbol{u}) - \theta]\} \mathrm{d}V - \Pi_{\mathrm{ext}}(\boldsymbol{u}) \ . \tag{2.52}$$

Now p is a Lagrange multiplier to enforce the constraint $\theta = J(\boldsymbol{u})$. The scalar function U is a purely volumetric contribution to the stored energy function. The above notation does not exclude the case that $\tilde{W}$ depends solely on $J^{-2/3}C$, i.e., on the isochoric part of the deformation. As usual, the weak formulation is

$$D_{\boldsymbol{u}}\Pi(\boldsymbol{u},\theta,p)\overset{*}{\boldsymbol{u}} = 0, \quad D_\theta \Pi(\boldsymbol{u},\theta,p)\overset{*}{\theta} = 0, \quad D_p\Pi(\boldsymbol{u},\theta,p)\overset{*}{p} = 0 \tag{2.53}$$

for all $\overset{*}{\boldsymbol{u}}$ satisfying $\overset{*}{\boldsymbol{u}} = 0$ on Γ_D and all $\overset{*}{\theta}$, $\overset{*}{p}$, where

$$\begin{aligned} D_{\boldsymbol{u}}\Pi(\boldsymbol{u},\theta,p)\overset{*}{\boldsymbol{u}} &= \int_{\mathscr{B}} 2\left\{\frac{\partial \tilde{W}(C(\boldsymbol{u}))}{\partial C} + p\frac{\partial J(\boldsymbol{u})}{\partial C}\right\} : [\boldsymbol{F}(\boldsymbol{u})^T \nabla \overset{*}{\boldsymbol{u}}] \mathrm{d}V \\ &\quad - L_{\mathrm{ext}}(\overset{*}{\boldsymbol{u}}) \\ D_\theta \Pi(\boldsymbol{u},\theta,p)\overset{*}{\theta} &= \int_{\mathscr{B}} \{\kappa U'(\theta) - p\}\overset{*}{\theta}\mathrm{d}V \\ D_p\Pi(\boldsymbol{u},\theta,p)\overset{*}{p} &= \int_{\mathscr{B}} \{J(\boldsymbol{u}) - \theta\}\overset{*}{p}\mathrm{d}V \ . \end{aligned}$$

For one step of the Newton method the linearized problem can be written in the following form: Given $(\boldsymbol{u},\theta,p)$, find increments $(\overset{\triangle}{\boldsymbol{u}},\overset{\triangle}{\theta},\overset{\triangle}{p})$ such that

$$\left.\begin{aligned} a_{(u,p)}(\overset{\triangle}{\boldsymbol{u}},\overset{*}{\boldsymbol{u}}) + b_{(u)}(\overset{\triangle}{p},\overset{*}{\boldsymbol{u}}) &= -D_{\boldsymbol{u}}\Pi(\boldsymbol{u},\theta,p)\overset{*}{\boldsymbol{u}} \quad \forall \overset{*}{\boldsymbol{u}} \\ -\int_{\mathscr{B}} \overset{\triangle}{p}\overset{*}{\theta}\mathrm{d}V + \int_{\mathscr{B}} \kappa U''(\theta)\overset{\triangle}{\theta}\overset{*}{\theta}\mathrm{d}V &= -D_\theta\Pi(\boldsymbol{u},\theta,p)\overset{*}{\theta} \quad \forall \overset{*}{\theta} \\ b_{(u)}(\overset{*}{p},\overset{\triangle}{\boldsymbol{u}}) - \int_{\mathscr{B}} \overset{*}{p}\overset{\triangle}{\theta}\mathrm{d}V &= -D_p\Pi(\boldsymbol{u},\theta,p)\overset{*}{p} \quad \forall \overset{*}{p} \ . \end{aligned}\right\} \tag{2.54}$$

Here $a_{(u,p)}$ and $b_{(u)}$ are the same bilinear forms as in the two-field formulation in Section 2.3, see (2.21), (2.22).

Remark 2.10 The functional (2.52) resembles the three-field formulation of Atluri and Reissner (1989), which in our notation reads

$$\Pi(\boldsymbol{u},\theta,p) = \int_{\mathscr{B}} \{W^*(\bar{C}(\boldsymbol{u}),\theta) + p[f(J^2(\boldsymbol{u})) - f(\theta)]\}\mathrm{d}V - \Pi_{\mathrm{ext}}(\boldsymbol{u})$$

with $\bar{C}(\boldsymbol{u}) = J^{-2/3}(\boldsymbol{u})C(\boldsymbol{u})$ being the isochoric part of the right Cauchy-Green tensor and f being a suitably chosen scalar function. In contrast to (2.52), here θ is a new variable for $\det C = J^2$.

3
On the analysis of mixed methods

In the development of finite element methods for incompressible or nearly incompressible materials, two key steps are the choice of a suitable variational formulation and the choice of discrete approximation spaces. In the linear case, the Babuška-Brezzi theory of saddle point problems is a powerful tool to assess the quality of some proposed method. Thus, first we consider a mixed formulation of linear elasticity. Then we show that the linearizations of the mixed formulations of Sections 2.3 and 2.4 have the same structure. Finally we prove an equivalence result for the discrete iterative solution procedure arising from the two-field formulations of Sections 2.3 and 2.4 and the three-field formulation.

3.1
Review of the displacement-pressure formulation in linear elasticity

In linear elasticity the stress-strain relationship is

$$\boldsymbol{\sigma} = \lambda \operatorname{tr}\boldsymbol{\varepsilon}\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \tag{3.1}$$

where $\boldsymbol{\varepsilon} = [\operatorname{grad}\boldsymbol{u}]^{\mathrm{sym}}$ and λ and μ denote the Lamé coefficients. We replace the scalar quantity $\lambda \operatorname{tr}\boldsymbol{\varepsilon} = \lambda \operatorname{div}\boldsymbol{u}$ by the new unknown field p. The equilibrium equation $-\operatorname{div}\boldsymbol{\sigma} = f$ in Ω with the boundary conditions $\boldsymbol{\sigma}\boldsymbol{n} = \mathbf{g}$ on Γ_N and $\boldsymbol{u} = 0$ on Γ_D can be cast into the following well-known weak formulation: Find $(\boldsymbol{u},p) \in V \times Q$ such that

$$\left.\begin{aligned} a(\boldsymbol{u},\overset{*}{\boldsymbol{u}}) + b(p,\overset{*}{\boldsymbol{u}}) &= L_{\mathrm{ext}}(\overset{*}{\boldsymbol{u}}) \quad \forall \overset{*}{\boldsymbol{u}} \in V \\ b(\overset{*}{p},\boldsymbol{u}) - c(\overset{*}{p},p) &= 0 \quad \forall \overset{*}{p} \in Q \ . \end{aligned}\right\} \tag{3.2}$$

Here the bilinear forms are

$$a(\boldsymbol{u},\overset{*}{\boldsymbol{u}}) = \int_\Omega 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\overset{*}{\boldsymbol{u}})\mathrm{d}v \ , \tag{3.3}$$

$$b(p,\overset{*}{\boldsymbol{u}}) = \int_\Omega p \operatorname{div}\overset{*}{\boldsymbol{u}}\,\mathrm{d}v \ , \tag{3.4}$$

$$c(\overset{*}{p},p) = \int_\Omega \frac{1}{\lambda}\overset{*}{p}p\,\mathrm{d}v \ , \tag{3.5}$$

the exterior load is

$$L_{\mathrm{ext}}(\overset{*}{\boldsymbol{u}}) = \int_\Omega \overset{*}{\boldsymbol{u}}\cdot\boldsymbol{f}\mathrm{d}v + \int_{\Gamma_N} \overset{*}{\boldsymbol{u}}\cdot\boldsymbol{g}\,\mathrm{d}S \ , \tag{3.6}$$

and the spaces V and Q can be identified as

$$V = \left\{\overset{*}{\boldsymbol{u}} \in H^1(\Omega)^3 : \overset{*}{\boldsymbol{u}} = \mathbf{0} \text{ on } \Gamma_D\right\} \ , \tag{3.7}$$

$$Q = L^2(\Omega) \ . \tag{3.8}$$

A slightly different mixed formulation can be obtained by exploiting

$$\boldsymbol{\sigma} = \kappa \operatorname{tr}\boldsymbol{\varepsilon}\mathbf{1} + 2\mu \operatorname{dev}\boldsymbol{\varepsilon}, \tag{3.9}$$

where dev $\boldsymbol{\varepsilon} := \boldsymbol{\varepsilon} - \frac{1}{3}\operatorname{tr}\boldsymbol{\varepsilon}\mathbf{1}$. The bilinear form a in (3.2) is replaced by

$$a(\boldsymbol{u}, \overset{*}{\boldsymbol{u}}) = \int_\Omega 2\mu \operatorname{dev}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \operatorname{dev}\boldsymbol{\varepsilon}(\overset{*}{\boldsymbol{u}})\mathrm{d}v \tag{3.10}$$

and in c the Lamé coefficient λ is replaced by the bulk modulus κ.

For a comprehensive account of the Babuška-Brezzi theory we refer to Brezzi and Fortin (1991). Here we only state the main results for the above saddle point problem. Provided that Γ_D and Γ_N each have positive surface measure, the bilinear form a in both the cases (3.3) and (3.10) satisfies an ellipticity condition and b satisfies the inf-sup condition on the spaces V and Q in (3.7) and (3.8). Thus (3.2) has a unique solution $(\boldsymbol{u}, p) \in V \times Q$ depending continuously on the data $\boldsymbol{f}$ and $\boldsymbol{g}$. This result remains valid for $\lambda \to \infty$ or if $c(.,.) \equiv 0$.

A discrete version of (3.2) is obtained by replacing V and Q by finite-dimensional conforming subspaces $V_h \subset V$ and $Q_h \subset Q$ with h typically representing the mesh width: Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\left.\begin{aligned} a(\boldsymbol{u}_h, \overset{*}{\boldsymbol{u}}_h) + b(p_h, \overset{*}{\boldsymbol{u}}_h) &= L_{\text{ext}}(\overset{*}{\boldsymbol{u}}_h) \quad \forall \overset{*}{\boldsymbol{u}}_h \in V_h \\ b(\overset{*}{p}_h, \boldsymbol{u}_h) - c(\overset{*}{p}_h, p_h) &= 0 \qquad \forall \overset{*}{p}_h \in Q_h \ . \end{aligned}\right\} \tag{3.11}$$

One of the requirements of the Babuška-Brezzi theory is the existence of an h-independent constant $\beta > 0$ such that

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(q_h, v_h)}{\|q_h\|_Q \|v_h\|_V} \geq \beta \ . \tag{3.12}$$

Under the hypotheses of the Babuška-Brezzi theory the discrete problem is uniquely solvable and $(\boldsymbol{u}_h, p_h)$ converges with optimal order to the exact solution $(\boldsymbol{u}, p)$. The error is independent of λ, and the case $c(.,.) \equiv 0$ is admissible. This facilitates the treatment of incompressible material. To achieve robustness for the nearly incompressible case, fulfillment of the Ladyzenskaya-Babuška-Brezzi condition (3.12) is *necessary*. This is confirmed both by mathematical arguments, see Brezzi and Fortin (1991), and by numerical experiments, e.g. van den Bogert et al. (1991), Chapelle and Bathe (1993), among many other references. The condition (3.12) together with conditions on the bilinear form a imposes severe restrictions on the choice of V_h and Q_h: The spaces for the displacement and the pressure must be balanced properly. Suitable finite element spaces are discussed in Section 4.1.

3.2
Structure of the linearized problems of finite elasticity

We are now going to clarify some analogies between the above mixed formulation of linear elasticity and the linearized mixed problems of Section 2. We begin with the formulation by de Borst et al.

We discretize the linearized formulation (2.24) by conforming subspaces $V_h \subset V, Q_h \subset Q$. In particular, the bilinear form

$$b_{(\boldsymbol{u}_h)}(\overset{\triangle}{p}_h, \overset{*}{\boldsymbol{u}}_h) = \int_{\varphi_h(\mathscr{B})} \overset{\triangle}{p}_h \operatorname{div} \overset{*}{\boldsymbol{u}}_h \, \mathrm{d}v \tag{3.13}$$

very much resembles the linear elasticity counterpart (3.4). When the bilinear forms satisfy the conditions of the Babuška-Brezzi theory, the discrete linearized problem is well posed. Whether or not these conditions are fulfilled depends on the current values of $\boldsymbol{u}_h$ and p_h. Practically, fulfillment cannot be guaranteed in advance. As a reasonable minimum requirement, the conditions should be satisfied in the undeformed state since then the linearized problem is identical to a mixed formulation of linear elasticity (if, e.g., (3.15) holds). This has been pointed out by Le Tallec (1981, 1994), who considered the fully incompressible case. In general the conditions cannot always be satisfied as, for instance, bifurcation points show.

Now consider the linearized problem (2.24) for the undeformed state, i.e., $\boldsymbol{u} = \mathbf{0}$ and $p = 0$. Then we have $J = 1$ and $U'(J) = 0$. If in this case $\tilde{\tau}$ and $\tilde{\mathbf{e}}$ defined in (2.27), (2.28) satisfy

$$\tilde{\tau} = 0 \tag{3.14}$$

and

$$\tilde{\mathbf{e}} = 2\mu\mathbb{1} \tag{3.15}$$

then (2.24) coincides with the linear problem (3.2). This is easily seen from the push-forward in Proposition 2.1. Fulfillment of (3.14) is guaranteed by (2.15). On the other hand, (3.15) is by no means necessary. If in the undeformed state

$$\tilde{\mathbf{e}} = 2\mu\{\mathbb{1} - \tfrac{1}{3}\mathbf{1} \otimes \mathbf{1}\} \ , \tag{3.16}$$

then $\tilde{\mathbf{e}}$ is essentially a deviatoric projection and the linearized problem (2.24) is identical to the linear formulation with (3.10).

Similar conclusions as above can be drawn for the approach in Section 2.4 since it has the same structure. For the three-field formulation of Section 2.5, the linearized problem in the undeformed state is not identical to the linear problem (3.2). The inf-sup condition now is

$$\inf_{\overset{*}{p}_h \in Q_h} \sup_{\overset{*}{\boldsymbol{u}}_h \in V_h} \sup_{\overset{*}{\theta}_h \in \tilde{Q}_h} \frac{b_{(\boldsymbol{u}_h)}(\overset{*}{p}_h, \overset{*}{\boldsymbol{u}}_h) - \int_{\mathscr{B}} \overset{*}{p}_h \overset{*}{\theta}_h \, \mathrm{d}V}{\|\overset{*}{p}_h\|(\|\overset{*}{\boldsymbol{u}}_h\| + \|\overset{*}{\theta}_h\|)} \geq \beta \ , \tag{3.17}$$

where usually the discrete spaces Q_h and $\tilde{Q}_h$ are chosen to be identical. Comparing with the two-field formulations, (3.17) does not pose any additional difficulties. However, in (2.54) the contribution $\int_{\mathscr{B}} \kappa U''(\theta) \overset{\triangle}{\theta}\overset{*}{\theta} \, \mathrm{d}V$ becomes unbounded as $\kappa \to \infty$. In view of the Babuška-Brezzi theory this is undesirable, but the investigations in the next section indicate why in practice no difficulties arise from this point.

3.3
Equivalence between the two-field and the three-field formulations

If in the additive decomposition (2.10) U has the form $U(J) = \frac{1}{2}(J-1)^2$ then, as pointed out in Remark 2.6, the two-field formulations in Sections 2.3 and 2.4 are identical. Now we investigate their relationship to the three-field formulation. The following result holds for arbitrary pairs of discrete displacement and pressure spaces and for both the plane and the three-dimensional case.

Proposition 3.1 *Let $U(J) = \frac{1}{2}(J-1)^2$ and κ be constant on each element. If in the three-field formulation* (2.54) *the pressure p and the dilatation θ are discretized by the same ansatz functions and these functions are discontinuous across interelement sides, then both* (2.24) *and* (2.54) *yield the same result in every step of the Newton method.*

Proof. The degrees of freedom in $\overset{\triangle}{u}, \overset{\triangle}{p}$ and $\overset{\triangle}{\theta}$ for the element Ω_e are denoted by the vectors $\overset{\triangle}{\mathbf{u}}^e, \overset{\triangle}{\mathbf{p}}^e$ and $\overset{\triangle}{\boldsymbol{\theta}}^e$, and we introduce a vector $\mathbf{N}$ of shape functions such that $\overset{\triangle}{p}$ and $\overset{\triangle}{\theta}$ are approximated by $\mathbf{N}^T \overset{\triangle}{\mathbf{p}}^e$ and $\mathbf{N}^T \overset{\triangle}{\boldsymbol{\theta}}^e$, respectively. For the two-field formulation (2.24) the contribution of Ω_e to the global tangential stiffness matrix can be written as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & -\kappa^{-1}\mathbf{M} \end{bmatrix} \begin{bmatrix} \overset{\triangle}{\mathbf{u}}^e \\ \overset{\triangle}{\mathbf{p}}^e \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_u - \mathbf{f}^{\text{other}} \\ -\hat{\mathbf{r}}_p \end{bmatrix} \tag{3.18}$$

where

$$\mathbf{M} = \int_{\Omega_e} \mathbf{N}\mathbf{N}^T \, dV, \quad \hat{\mathbf{r}}_p = \int_{\Omega_e} \mathbf{N}\{J(\boldsymbol{u}_h) - 1 - \kappa^{-1} p_h\} \, dV \tag{3.19}$$

and $\mathbf{f}^{\text{other}}$ contains contributions from other elements when the global system is assembled. The submatrices $\mathbf{A}$ and $\mathbf{B}$ arise from the bilinear forms $a_{(\mathbf{u},p)}(.,.)$ and $b_{(u)}(.,.)$ in (2.24) in a standard fashion. In the three-field case (2.54) one obtains

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{0} & -\mathbf{M} \\ \mathbf{0} & -\mathbf{M} & \kappa\mathbf{M} \end{bmatrix} \begin{bmatrix} \overset{\triangle}{\mathbf{u}}^e \\ \overset{\triangle}{\mathbf{p}}^e \\ \overset{\triangle}{\boldsymbol{\theta}}^e \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_u - \mathbf{f}^{\text{other}} \\ -\mathbf{r}_p \\ -\mathbf{r}_\theta \end{bmatrix} \tag{3.20}$$

with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$ and $\mathbf{r}_u$ as above and

$$\mathbf{r}_p = \int_{\Omega_e} \mathbf{N}\{J(\boldsymbol{u}_h) - \theta_h\} \, dV \ ,$$
$$\mathbf{r}_\theta = \int_{\Omega_e} \mathbf{N}\{\kappa[\theta_h - 1] - p_h\} \, dV \ . \tag{3.21}$$

Condensation yields

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & -\kappa^{-1}\mathbf{M} \end{bmatrix} \begin{bmatrix} \overset{\triangle}{\mathbf{u}}^e \\ \overset{\triangle}{\mathbf{p}}^e \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_u - \mathbf{f}^{\text{other}} \\ -\mathbf{r}_p - \kappa^{-1}\mathbf{r}_\theta \end{bmatrix} \ . \tag{3.22}$$

Now we are going to show by induction that $\mathbf{r}_\theta$ remains zero during all Newton iterations. The reference configuration is in equilibrium which means that $\mathbf{r}_\theta$ vanishes. In each Newton step, after solving the linear system a new residual is calculated as

$$\mathbf{r}_\theta^{\text{new}} = \int_{\Omega_e} \mathbf{N}\{\kappa[\theta_h + \mathbf{N}^T \overset{\triangle}{\boldsymbol{\theta}}^e - 1] - p_h - \mathbf{N}^T \overset{\triangle}{\mathbf{p}}^e\} \, dV \ . \tag{3.23}$$

From the last equation in (3.20) we infer $\overset{\triangle}{\mathbf{p}}^e = \kappa \overset{\triangle}{\boldsymbol{\theta}}^e$, and thus $\mathbf{r}_\theta^{\text{new}} = \mathbf{r}_\theta = \mathbf{0}$. Finally, $\mathbf{r}_\theta = \mathbf{0}$ implies

$$\mathbf{r}_p := \int_{\Omega_e} \mathbf{N}\{J(\boldsymbol{u}_h) - \theta_h\} \, dV$$
$$= \int_{\Omega_e} \mathbf{N}\{J(\boldsymbol{u}_h) - 1 - \kappa^{-1} p_h\} \, dV =: \hat{\mathbf{r}}_p \ .$$

Thus (3.22) is identical to (3.18). □

Usually the integrals appearing above are approximated by some quadrature rule. If the same quadrature rule is used for all integrals, the exact discrete equivalence remains valid.

Remark 3.1 For the above proof it is essential that $U'(J) = J - 1$. This relation is often used in practice. It implies that the hydrostatic pressure contains a contribution $\kappa[J-1]$. For other volumetric functions U usually there holds $U'(J) = J - 1 + O((J-1)^2)$ near $J = 1$, and thus the deviation between the two-field and the three-field formulation can be expected to be very small for nearly incompressible material. As numerical experiments in Section 4.3 show, the three-field formulation makes it possible to use larger load increments in some cases.

Liu, Hofstetter and Mang (1994) compared numerical results of the two formulations, but did not realize the equivalence demonstrated above. The two-field formulation is more economic, since the number of unknowns in each element is smaller.

4
Examples

In this section, first we discuss some finite element spaces which are suitable for all the above mixed formulations. Then we briefly list some constitutive laws. Finally we report on numerical computations.

4.1
Finite element spaces

We suppose that the domain $\mathscr{B}$ is partitioned into either hexahedra or tetrahedra, while in R^2 quadrilaterals or triangles are used. Two different elements are either disjoint or share exactly one common vertex or edge or (in R^3) face, i.e., there are no hanging nodes.

4.1.1
The Q_k/P_{k-1} family

For hexahedra and quadrilaterals, the reference element is $\hat{\Omega} = [-1,1]^N, N = 2,3$. On $\hat{\Omega}$ the usual space of tensorial products of polynomials of degree k is

$$Q_k(\hat{\Omega}) = \text{span}\{(\xi^1)^{i_1} \cdot \ldots \cdot (\xi^N)^{i_N} : 0 \le i_1, \ldots, i_N \le k\} \ . \tag{4.1}$$

Applying the isoparametric concept, the elements in $\mathscr{B}$ are

$$\Omega_e = \psi_e(\hat{\Omega}) \tag{4.2}$$

with mappings $\psi_e \in Q_k(\hat{\Omega})^N$. On each element Ω_e we define the approximation space

$$Q_k(\Omega_e) = \{v \circ \psi_e^{-1} : v \in Q_k(\hat{\Omega})\} \ . \tag{4.3}$$

For the discretization of the linearized mixed problems as well as for the linear mixed problem (3.2), we will now define the family Q_k/P_{k-1} of hexahedral (quadrilateral) elements, also denoted by Q1/P0, Q2/P1 etc. The discrete displacements lie in

$$V_h = \{v_h : v_h \text{ is continuous on } \mathscr{B}, \quad v_h = \mathbf{0} \text{ on } \Gamma_D \ , \quad v_h \in Q_k(\Omega_e)^N \text{ on each element } \Omega_e\} \ . \tag{4.4}$$

The pressure is approximated by complete polynomials of degree $k-1$ in the coordinates of the reference configuration $\mathscr{B}$; on each element,

$$p_h|_{\Omega_e} \in \text{span}\{(X^1)^{i_1} \cdot \ldots \cdot (X^N)^{i_N} : i_1, \ldots, i_N \ge 0 \ , \quad i_1 + \ldots + i_N \le k-1\} \ . \tag{4.5}$$

For the discrete pressure no interelement continuity and no boundary conditions are required. This is admissible since in (3.2) the pressure is sought in $L^2(\Omega)$. Moreover, as long as the bulk modulus is not too large, the pressure degrees of freedom can be eliminated before assembling the global stiffness matrix.

For $k \ge 2$ the Q_k/P_{k-1} elements satisfy the inf-sup condition (3.12), at least under the technical restriction $\psi_e \in Q_1(\hat{\Omega})^N$ (instead of being fully isoparametric). A proof can be found in the monograph by Girault and Raviart (1986). The Q1/P0 element does not satisfy the inf–sup condition. Unless certain restrictions are imposed on the meshes or other measures are taken, an unstable approximation must be expected. The same is true for the Q2/Q1 element (which has a discontinuous Q1 pressure field).

Among the earliest references to the Q2/P1 element for linear problems we mention Fortin (1981) for the theory and Engelman, Sani, Gresho and Bercovier (1982) for the implementation and numerical performance.

4.1.2 The P2+/P1 element

Next, we turn to *triangular* elements and describe the P2+/P1 element introduced and analyzed by Crouzeix and Raviart (1973). For tetrahedra and triangles, the space of complete polynomials of degree k on the reference element $\hat{\Omega} = \{(\xi^1, \ldots, \xi^N) : \xi^1, \ldots, \xi^N \ge 0, \ \xi^1 + \ldots + \xi^N \le 1\}$ is defined by

$$P_k(\hat{\Omega}) = \text{span}\{(\xi^1)^{i_1} \cdot \ldots \cdot (\xi^N)^{i_N} : i_1, \ldots, i_N \ge 0, \quad i_1 + \ldots + i_N \le k\} \ . \tag{4.6}$$

The elements in $\mathscr{B}$ are $\Omega_e = \psi_e(\hat{\Omega})$ with mappings $\psi_e \in P_k(\hat{\Omega})^N$. On the reference element we introduce the barycentric coordinates

$$\hat{\lambda}_1 = \xi^1, \quad \ldots, \quad \hat{\lambda}_N = \xi^N, \quad \hat{\lambda}_{N+1} = 1 - \xi^1 - \ldots - \xi^N \tag{4.7}$$

The product $\hat{\lambda}_1 \cdot \ldots \cdot \hat{\lambda}_{N+1}$ is the bubble function associated with the barycenter of $\hat{\Omega}$. On each triangle Ω_e the displacement is approximated in the 14-dimensional space

$$P_{2+}(\Omega_e) = \{v \circ \psi_e^{-1} : v \in (P_2(\hat{\Omega}) \oplus \{\zeta \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 : \zeta \in R\})^2\} \tag{4.8}$$

Further the discrete displacements are required to be continuous on $\mathscr{B}$ and to vanish on Γ_D. To complete the definition of the P2+/P1 element, for the discrete pressure the space (4.5) with polynomial degree $k-1=1$ is employed. This element satisfies the inf-sup condition (3.12) (if $\psi_e \in P_1(\hat{\Omega})^2$), see Girault and Raviart (1986).

Following Girault and Raviart (1986), the P2+/P1 element can be extended to *tetrahedra*. Discussion is restricted to the case $\psi_e \in P_1(\hat{\Omega})^3$, implying plane faces. The only point to be changed is the definition of $P_{2+}(\Omega_e)$. The barycentric coordinates in Ω_e are $\lambda_i = \hat{\lambda}_i \circ \psi_e^{-1}$. The normal vector on the face on which λ_i vanishes is denoted by $\boldsymbol{n}_i$. Now bubble functions associated with the faces are defined:

$$\boldsymbol{z}_1 = \lambda_2\lambda_3\lambda_4\boldsymbol{n}_1 \ , \quad \boldsymbol{z}_2 = \lambda_1\lambda_3\lambda_4\boldsymbol{n}_2 \ ,$$
$$\boldsymbol{z}_3 = \lambda_1\lambda_2\lambda_4\boldsymbol{n}_3 \ , \quad \boldsymbol{z}_4 = \lambda_1\lambda_2\lambda_3\boldsymbol{n}_4 \ ;$$

each $\boldsymbol{z}_i$ is defined for all $\boldsymbol{X} \in \Omega_e$ (with $\boldsymbol{n}_i$ fixed) and vanishes on all but one face of Ω_e. Now the discrete displacement on Ω_e lies in the 37-dimensional space

$$P_{2+}(\Omega_e) = \{v \circ \psi_e^{-1} : v \in P_2(\hat{\Omega})^3\} \otimes \{\lambda_1\lambda_2\lambda_3\lambda_4\zeta : \zeta \in R^3\} \otimes \text{span}\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_4\} \ .$$

For a proof of the inf–sup condition (3.12) we refer to Girault and Raviart (1986), where also a generalization to higher polynomial degrees can be found.

4.2 Constitutive laws

In view of the mixed formulations of Sections 2.3 and 2.5, we write all stored energy functions in terms of a purely volumetric contribution U and a remainder $\tilde{W}$, bearing in mind that

$$W = \kappa U(J) + \tilde{W}(C) \ . \tag{4.9}$$

As for U, most often (see van den Bogert et al. (1991, 1994), Chang et al. (1991), Chen et al. (1994), Le Tallec (1994), Sussman and Bathe (1987))

$$U(J) = \tfrac{1}{2}(J-1)^2 \ , \tag{4.10}$$

which results in the contribution $\kappa(J-1)\mathbf{1}$ to the Cauchy stress. Alternatives are

$$U(J) = \tfrac{1}{2}(\ln J)^2 \tag{4.11}$$

and (Simo and Armero (1992))

$$U(J) = \tfrac{1}{4}(J^2 - 1) - \tfrac{1}{2}\ln J \ . \tag{4.12}$$

In all these cases, U is strictly convex (for (4.11) we need $J < e = 2.718\ldots$) and has a minimum at $J = 1$, i.e., the conditions (2.11) and (2.12) are satisfied. Further, all but (4.10) enforce (2.13).

Neo-Hooke material may be described by

$$\tilde{W}(C) = -\mu \ln J + \frac{1}{2}\mu[\mathrm{tr}\ C - 3] \ . \tag{4.13}$$

The modified Kirchhoff stress $\tilde{\tau}$ and the modified elastic tangent tensor $\tilde{\mathbf{e}}$ defined by (2.27) and (2.28) are now

$$\tilde{\tau} = \mu[\boldsymbol{b} - 1], \quad \tilde{\mathbf{e}} = 2\mu\mathbb{1} \ . \tag{4.14}$$

where $\boldsymbol{b} = FF^T$ is the left Cauchy-Green tensor. Note that incorporation of the purely volumetric term $-\mu \ln J$ in $\tilde{W}$ is important in order to achieve $\tilde{\tau} = \mathbf{0}$ in the undeformed state. Moreover, (3.15) is satisfied; therefore, the linearized mixed problem in the undeformed state is equivalent to the linear saddle point problem (3.2).

For nearly incompressible material it may be advantageous to make $\tilde{W}$ independent of $J(\boldsymbol{u})$. Then the additive decomposition (4.9) is a volumetric-isochoric split where for neo-Hooke material

$$\tilde{W} = \tfrac{1}{2}\mu[J^{-2/3}\mathrm{tr}C - 3] \ . \tag{4.15}$$

Now

$$\tilde{\tau} = \mu J^{-2/3}\ \mathrm{dev}\ \boldsymbol{b} \tag{4.16}$$

where $\mathrm{dev}\ \boldsymbol{b} = \boldsymbol{b} - \frac{1}{3}[\mathrm{tr}\,\boldsymbol{b}]1$, and

$$\tilde{\mathbf{e}} = \tfrac{2}{3}\mu J^{-2/3}\{[\mathrm{tr}\,C]\mathbb{1} + \tfrac{1}{3}[\mathrm{tr}\,C]\mathbf{1}\otimes 1 - \boldsymbol{b}\otimes 1 - 1\otimes \mathbf{b}\} \ . \tag{4.17}$$

In the undeformed state we have $\tilde{\tau} = \mathbf{0}$ and

$$\tilde{\mathbf{e}} = 2\mu\{\mathbb{1} - \tfrac{1}{3}\mathbf{1}\otimes 1\} \ , \tag{4.18}$$

in correspondence with (3.16).

A generalization is the *Mooney-Rivlin* material

$$\tilde{W} = K_1[J^{-2/3}\mathrm{tr}\,C - 3] + K_2[J^{-4/3}I_2 - 3] \tag{4.19}$$

with the second principal invariant $I_2 = \frac{1}{2}((\mathrm{tr}\,C)^2 - \mathrm{tr}(C^2))$ of C.

Ogden (1984) proposed a stored energy function in terms of the principal stretches $\lambda_j > 0$, which are the square roots of the eigenvalues of C,

$$\tilde{W}(C) = \sum_{i=1}^{m}\frac{\mu_i}{\alpha_i}\{\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\} - \sum_{i=1}^{m}\mu_i \ln J \ . \tag{4.20}$$

The material constants μ_i and α_i are related to the ground-state shear modulus through

$$\mu = \tfrac{1}{2}\sum_{i=1}^{m}\mu_i\alpha_i \ . \tag{4.21}$$

$\tilde{W}$ can be made independent of the volumetric part of the deformation by introducing the deviatoric principal stretches $\tilde{\lambda}_i = J^{-1/3}\lambda_i$,

$$\tilde{W}(C) = \sum_{i=1}^{m}\frac{\mu_i}{\alpha_i}\{\tilde{\lambda}_1^{\alpha_i} + \tilde{\lambda}_2^{\alpha_i} + \tilde{\lambda}_3^{\alpha_i} - 3\} \ . \tag{4.22}$$

4.3 Numerical examples

4.3.1 Cook's plane strain problem

This problem has frequently been used to assess finite elements under combined bending and shear. A tapered panel is clamped on one side while a shearing load acts on the opposite side, see Fig. 1. We choose a neo-Hooke material (4.13) with the volumetric function (4.10). To enable a comparison with the results of Simo and Armero (1992), the material parameters are taken to be $\kappa = 8000\,\mathrm{N/mm^2}$ and $\mu = 0.8\,\mathrm{N/mm^2}$. In Fig. 2 the vertical displacements at the point P are compared for different elements, namely the standard displacement elements Q2 and Q3, the Q1/E4 enhanced strain element by Simo et al. (1992, 1993) and the mixed elements Q1/P0, Q2/P1 and P2+/P1. For the last three elements, the two-field formulations of Sections 2.3 and 2.4 and the three-field formulation of Section 2.5 yield the same result up to machine accuracy.

Static condensation is used for all mixed and enhanced elements. For the Q2 and the Q2/P1 element we employ a 3×3-point Gaussian quadrature rule; for the Q1/P0, Q1/E4, P2+/P1 and Q3 element we use $2 \times 2, 5, 7$ and 4×4 quadrature points, respectively.

In Fig. 2 only the number of degrees of freedom in the global system is taken into account. For a more complete comparison, the number of load increments required and the sparseness or bandwidth of the tangential stiffness matrix have to be considered. In Table 1 the number of load increments is displayed. For simplicity, a plain Newton method is used. If it does not converge within twelve iterations, the load increment is halved. Within one load increment the Newton method in most cases needs

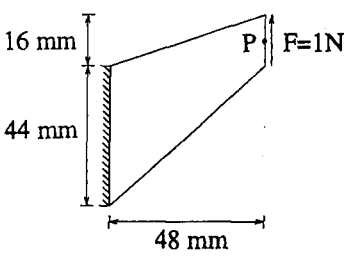

Fig. 1. Cook's problem, plane strain

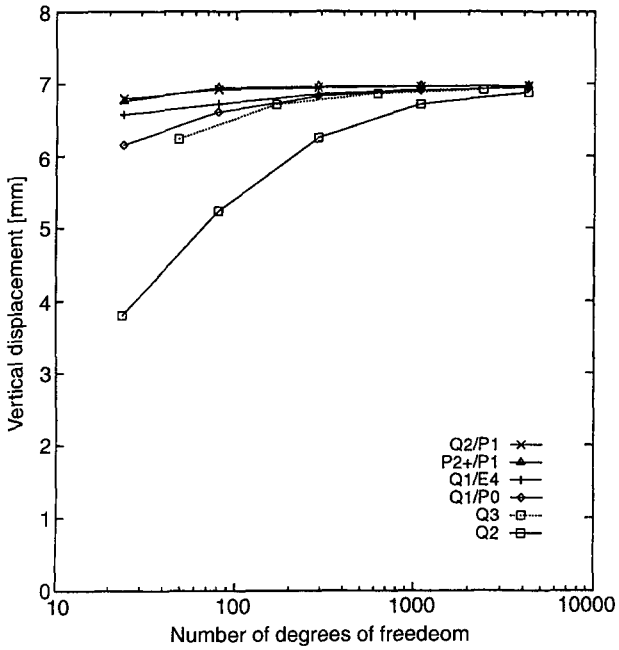

Fig. 2. Cook's problem, vertical displacement at the midpoint of the right-hand side

Table 1. Cook's problem, N_{el} = number of elements, N_{dofs} = number of degrees of freedom, N_{li} = number of load increments required

	coarse grid			fine grid		
	N_{el}	N_{dofs}	N_{li}	N_{el}	N_{dofs}	N_{li}
Q1/P0	32	80	1	512	1088	1
Q1/E4	32	80	4	512	1088	14
P2+/P1	16	80	1	256	1088	1
Q2	8	80	2	128	1088	2
Q2/P1	8	80	1	128	1088	1
Q3	8	168	2	128	2400	7

between three and six iterations. For all mixed elements under consideration, one load step is sufficient.

In Figures 3 to 7, the smallest in-plane principal stress computed from the Cauchy stress tensor is compared. For the graphical output the stresses at the quadrature points have been interpolated. Pure displacement elements exhibit a locking behavior and tend to overestimate the stress, see Fig. 3. The Q1/E4 element does not properly detect the stress singularity at the upper left corner, see Fig. 4. The mixed elements yield accurate results even on coarse meshes, see Figures 5 and 6. Especially the P2+/P1 and the Q2/P1 yield a smooth stress distribution. When refining the mesh, for all elements the results tend to the same distribution; as an example we refer to Fig. 7.

4.3.2
Uniaxially loaded specimen of square cross-section

In this example, we use the material description (4.22) where, following Ogden (1984), p. 498, $m = 3$ and

$$\mu_1 = 0.63\,\mathrm{N/mm^2} \quad \mu_2 = 0.0012\,\mathrm{N/mm^2} \quad \mu_3 = -0.01\,\mathrm{N/mm^2}$$
$$\alpha_1 = 1.3 \qquad \alpha_2 = 5 \qquad \alpha_3 = -2\ . \tag{4.23}$$

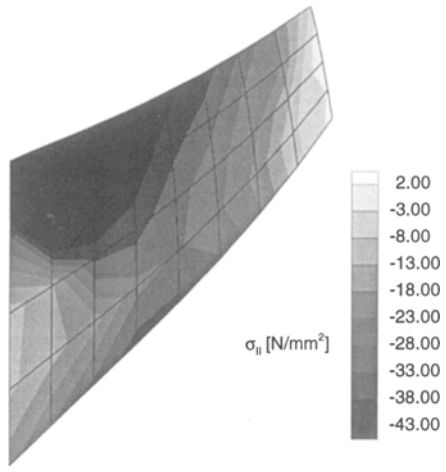

Fig. 3. Cook's problem, smallest in-plane principal stress σ_{II}, 32 elements Q2

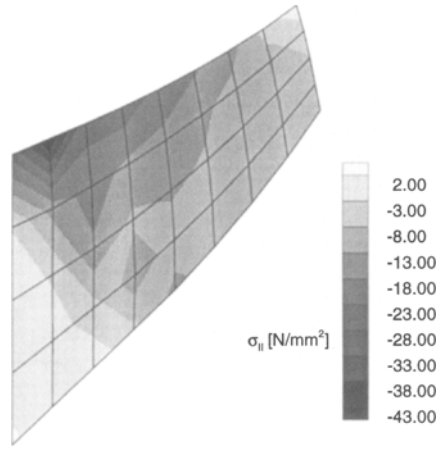

Fig. 4. Cook's problem, σ_{II}, 32 elements Q1/E4

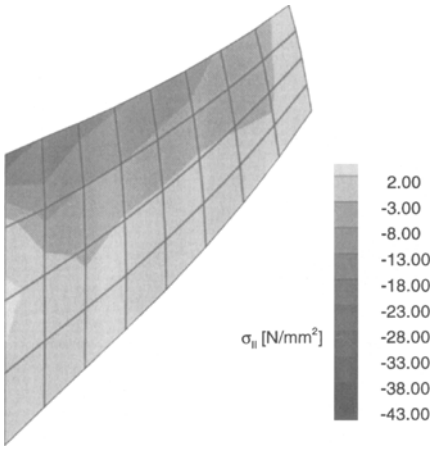

Fig. 5. Cook's problem, σ_{II}, 32 elements Q1/P0

Here the ground-state shear modulus is $\mu = 0.4225$ N/mm^2. Aspects of the implementation were addressed by Simo and Taylor (1991) and Miehe (1993, 1994). As volumetric contribution U to the stored energy function we choose (4.10). The bulk modulus is $\kappa = 1000\,\mathrm{N/mm^2}$. For

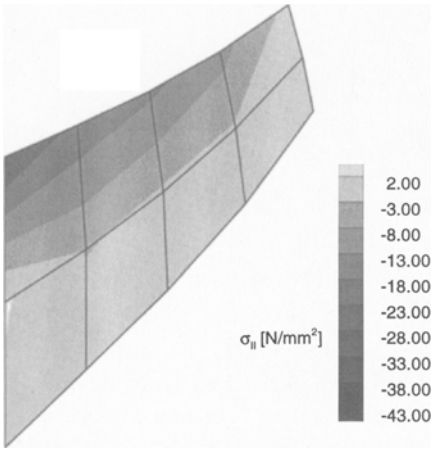

Fig. 6. Cook's problem, σ_{II}, 8 elements Q2/P1

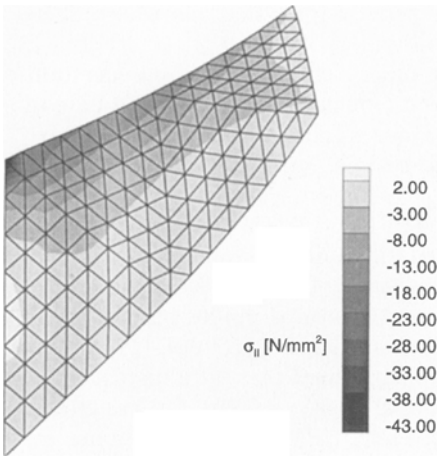

Fig. 7. Cook's problem, σ_{II}, 256 elements P2+/P1

this large strain example, the results deteriorate if $\tilde{W}$ is not purely isochoric as in (4.20).

A specimen of square cross-section of 20 mm × 20 mm is clamped at the left and right-hand side and is subjected to axial elongation or compression. For symmetry reasons, only the upper right quarter of the specimen is discretized, see Fig. 8. We assume plane strain and use uniform meshes unless explicitly stated. This example was also considered by Miehe (1994).

Figures 9 to 11 show the largest in-plane principal stress computed from the Cauchy stress tensor. Here the length of the specimen is three times its original length. The Q2 element yields less accurate stresses than the Q2/P1 element. The Q1/P0 element requires a rather fine mesh if a good stress resolution is desired. For the Q2 element the load increments must be chosen significantly smaller than for the mixed elements. An alternative is to start the incremental process with a low value of κ and to increase κ when the final elongation is reached (which is admissible since the problem is path-independent), or one has to employ an augmented Lagrangian method as described by Simo and Taylor (1991). We remark that for the Q2 and the Q2/P1 element the displayed deformation is inaccurate since the edges of the elements are curved.

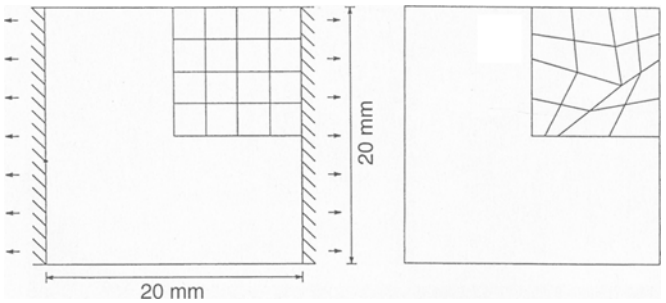

Fig. 8. Square, plane strain, one quarter discretized by uniform and distorted mesh

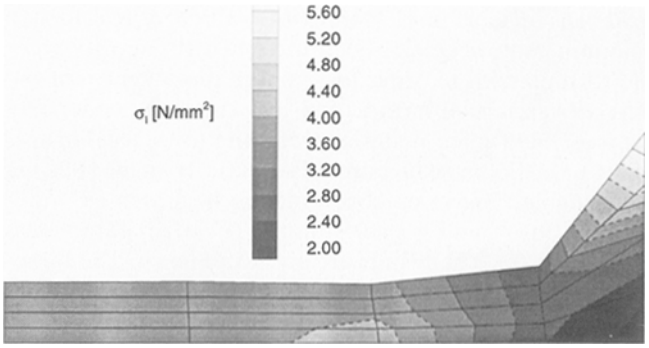

Fig. 9. Tension test, nearly incompressible Ogden material, largest in-plane principal stress σ_I, 16 elements Q2

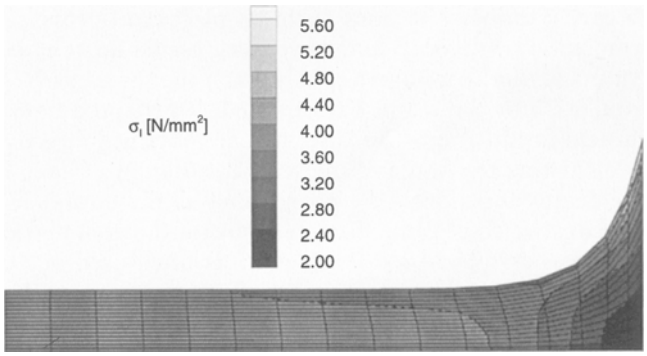

Fig. 10. Tension test, σ_I, 16 elements Q2/P1

In Fig. 12 the load is plotted versus the elongation of one quarter of the specimen. An elongation of 20 mm corresponds to triple length as depicted in Figures 9 to 11. Under compression the Q2 element yields a too stiff response while under tension the Q1/P0 is slightly too stiff. On the distorted mesh shown in Fig. 8 the results are almost identical.

Mixed elements of polynomial degree two for the displacement yield accurate results even on coarse meshes. For the P2+/P1 element this is confirmed by Fig. 13. If the same compression test is performed with the Q1/E4 element, hourglass modes occur, especially for lower values of the bulk modulus κ.

For nearly incompressible material the choice of the volumetric function U has no significant influence on the final results since J is forced to be close to 1 anyway.

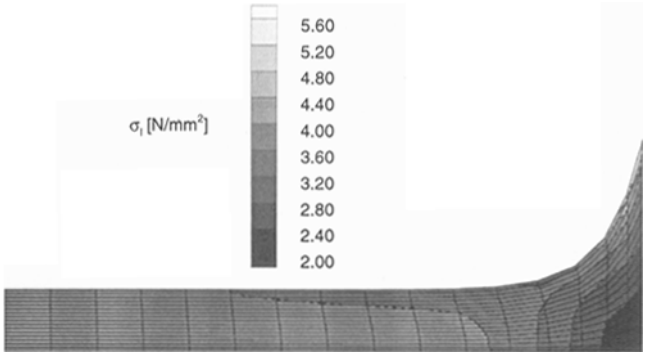

Fig. 11. Tension test, σ_1, 256 elements Q1/P0

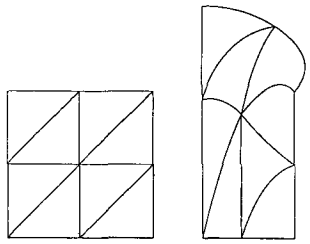

Fig. 13. Compression test, initial mesh and deformed mesh at a load of -1.2 N/mm^2, 8 elements P2+/P1

However, for U as in (4.11) or (4.12) the two-field formulations require smaller load increments than the three-field formulation. In order to compare the accuracy of the different variational formulations, we choose a moderately compressible Ogden material according to (4.20). For low values of κ the use of a purely isochoric $\tilde{W}$ as in (4.22) is questionable. The volumetric function is chosen as $U(J) = \frac{1}{2}(\ln J)^2$, and $\kappa = 10\,\mathrm{N/mm^2}$. The other parameters are as in (4.23). The results are given in Fig. 14. The three-field formulation needed at most two load increments, the new two-field formulation (Section 2.4) and the formulation by de Borst et al. (Section 2.3) required up to three and four load steps, respectively.

5 Conclusions

We gave a unifying account of the displacement-pressure formulation by de Borst and coworkers, see de Borst et al. (1988) and van den Bogert et al. (1991), the three-field formulation by Simo and Taylor (1985, 1991) and a new two-field formulation. The latter two are obtained directly by calculating the first and the second variation of a suitable energy functional. We proved that all these formulations yield the same discrete results if the volumetric function is $\kappa U(J) = \frac{1}{2}\kappa(J-1)^2$, which is often used in practice. Thus in this case it is not efficient to employ the three-field formulation. For other functions $U(J)$ the three-field formulation allows larger load increments than the other formulations and is therefore preferable.

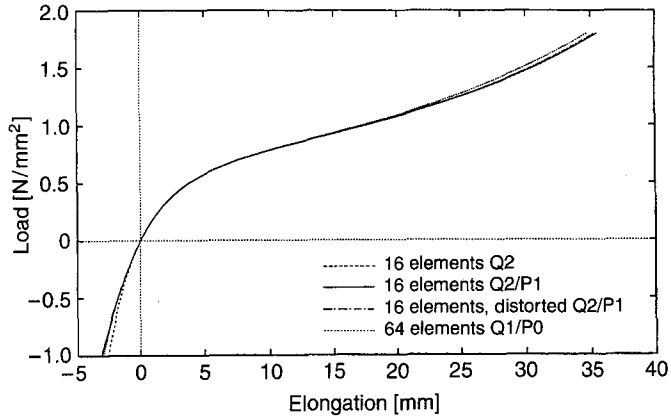

Fig. 12. Load-displacement curve for 10 mm × 10 mm square, nearly incompressible Ogden material

For nearly incompressible material and large strains, it was advantageous to employ a true volumetric-isochoric split involving, e.g., for neo-Hooke material, (4.15) rather than (4.13).

In the new two-field formulation of Section 2.4 the volumetric contribution to the stored energy function is required to have a less general form than elsewhere; this restricts the applicability.

When constructing finite element spaces one has to find a compromise between efficiency, reliability and ease of implementation. Thus for nonlinear problems the polynomial degree should be low. Further, as discussed in Section 3.2, fulfillment of the inf–sup condition is desirable. Therefore, the Q2/P1 and P2+/P1 elements are preferable; in our computations no spurious modes in the displacements and pressures occurred. If the inf-sup condition is not satisfied, spurious modes may occur as, for example, results by van den Bogert et al. (1991) show.

Our numerical tests confirmed that pure displacement methods with polynomial degree ≤ 3 are far less efficient than mixed methods. The Q1/E4 element sometimes exhibits spurious modes. Moreover, it yields less accurate stresses, involves more non-displacement degrees of free-

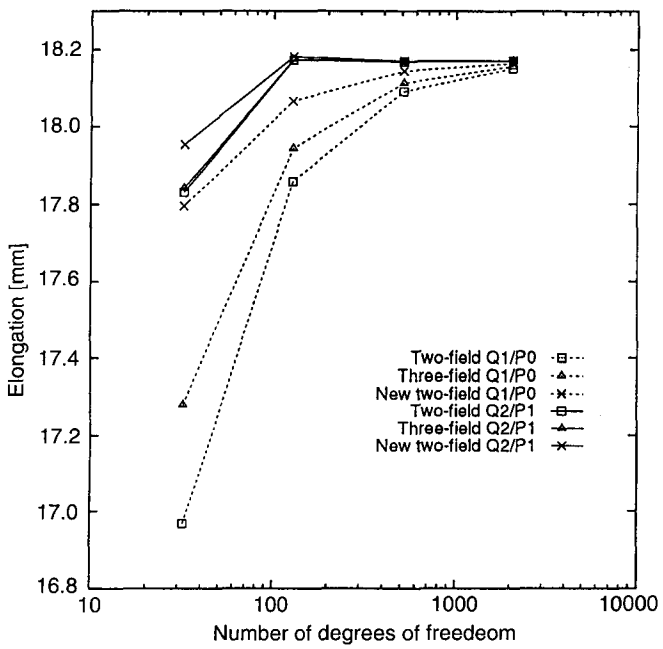

Fig. 14. Tension test, Ogden material (4.20) with volumetric function (4.11), $\kappa = 10$ N/mm^2, load 1.0 N/mm^2

dom to be eliminated and requires smaller load increments than other comparable mixed methods.

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