

Neo-Hookean Strain Energy Density Functions

Updated on -

Ida Ang

Contents

1	Definitions	2
2	Compressible Cases	2
3	Incompressible Cases	3
4	Summary	4

1 Definitions

Young's Modulus: E

Poisson's Ratio: ν

Shear Modulus or first Lamé Parameter: μ

Second Lamé Parameter: λ

$$\mu = \frac{E}{2(1+\nu)} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad (1.1)$$

Bulk Modulus: κ

$$\kappa = \lambda + \frac{2}{3}\mu \quad (1.2)$$

Deformation Gradient: \mathbf{F}

Right Cauchy Green Tensor: $\mathbf{C} = \mathbf{F}^T \mathbf{F}$

1st Piola Kirchhoff Stress: \mathbf{P}

The third invariant of the deformation gradient

$$I_3 = J = \det \mathbf{F} \quad (1.3)$$

The first invariant of the right Cauchy Green Tensor:

$$I_1^{\mathbf{C}} = \text{tr } \mathbf{C} \quad (1.4)$$

Lastly, we note a relationship between pressure and the bulk modulus

$$p = -\kappa I_3(I_3 - 1) \quad (1.5)$$

Proof of the derivative $\frac{\partial \text{tr}(\mathbf{F}^T \mathbf{F})}{\partial \mathbf{F}}$ in indicial:

$$\begin{aligned} \frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} &= \frac{\partial \text{tr}(\mathbf{F}^T \mathbf{F})}{\partial \mathbf{F}} \\ &= \frac{\partial F_{kI} F_{kI}}{\partial F_{pQ}} \quad \text{Product Rule} \\ &= \frac{\partial F_{kI}}{\partial F_{pQ}} F_{kI} + F_{kI} \frac{\partial F_{kI}}{\partial F_{pQ}} \\ &= \delta_{kp} \delta_{IQ} F_{kI} + F_{kI} \delta_{kp} \delta_{IQ} \\ &= F_{pQ} + F_{pQ} = 2F_{pQ} \end{aligned}$$

Therefore, we have the following relationships:

$$\begin{aligned} \frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} &= 2\mathbf{F} \\ \frac{\partial I_3}{\partial \mathbf{F}} &= I_3 \mathbf{F}^{-T} \end{aligned} \quad (1.6)$$

2 Compressible Cases

There are several variations for the strain energy density function of a compressible neo-Hookean material

$$\begin{aligned} W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2 \ln I_3) + \frac{\kappa}{2}(I_3 - 1)^2 \\ W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2 \ln I_3) + \frac{\lambda}{2}(\ln I_3)^2 \end{aligned} \quad (2.1)$$

We can calculate the first PK for the first expression using Eq. 1.6

$$\begin{aligned}
\mathbf{P} &= \frac{\partial W}{\partial \mathbf{F}} \quad \text{using chain rule} \\
&= \frac{\partial W}{\partial I_1^C} \frac{\partial I_1^C}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3} \frac{\partial I_3}{\partial \mathbf{F}} \\
&= \frac{\mu}{2}(2\mathbf{F}) + \left[\frac{\mu}{2} \left(-2 \frac{\partial \ln I_3}{\partial I_3} \right) + \frac{\kappa}{2} \frac{\partial (I_3 - 1)^2}{\partial I_3} \right] I_3 \mathbf{F}^{-T} \\
&= \mu \mathbf{F} + \left[-\frac{\mu}{I_3} + \kappa(I_3 - 1) \right] I_3 \mathbf{F}^{-T} \\
&= \mu \mathbf{F} - \mu \mathbf{F}^{-T} + \kappa I_3 (I_3 - 1) \mathbf{F}^{-T} \\
\mathbf{P} &= \mu(\mathbf{F} - \mathbf{F}^{-T}) + \kappa I_3 (I_3 - 1) \mathbf{F}^{-T}
\end{aligned} \tag{2.2}$$

For the second expression

$$\begin{aligned}
\mathbf{P} &= \frac{\partial W}{\partial I_1^C} \frac{\partial I_1^C}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3} \frac{\partial I_3}{\partial \mathbf{F}} \\
&= \frac{\mu}{2}(2\mathbf{F}) + \left[\frac{\mu}{2} \left(-2 \frac{\partial \ln I_3}{\partial I_3} \right) + \frac{\lambda}{2} \frac{\partial (\ln I_3)^2}{\partial I_3} \right] I_3 \mathbf{F}^{-T} \\
&= \mu \mathbf{F} + \left[-\frac{\mu}{I_3} + \lambda \ln I_3 \right] I_3 \mathbf{F}^{-T} \\
&= \mu \mathbf{F} - \mu \mathbf{F}^{-T} + \lambda I_3 (\ln I_3) \mathbf{F}^{-T} \\
\mathbf{P} &= \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda I_3 (\ln I_3) \mathbf{F}^{-T}
\end{aligned} \tag{2.3}$$

3 Incompressible Cases

For an incompressible material, $I_1 = \det(\mathbf{F}) = 1$; therefore, Eq. 3.1 can be modified,

$$\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^C - 3 - 2 \ln(1)) + \frac{\kappa}{2}(1 - 1)^2 \\
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^C - 3 - 2 \ln(1)) + \frac{\lambda}{2}(\ln(1))^2
\end{aligned}$$

Therefore, we can write

$$W(\mathbf{F}) = \frac{\mu}{2}(I_1^C - 3) \tag{3.1}$$

To enforce incompressibility, we can utilize the Lagrange multiplier method where the lagrange multiplier, p , enforces the constraint that $J=1$:

$$\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^C - 3) + p(I_3 - 1) \\
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^C - 3 - 2 \ln I_3) + p(I_3 - 1)
\end{aligned} \tag{3.2}$$

The 1st PK of Eq. 3.2 the first expression

$$\begin{aligned}
\mathbf{P} &= \frac{\partial W}{\partial I_1^C} \frac{\partial I_1^C}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3} \frac{\partial I_3}{\partial \mathbf{F}} \\
&= \frac{\mu}{2}(2\mathbf{F}) + p I_3 \mathbf{F}^{-T} \\
\mathbf{P} &= \mu \mathbf{F} + p I_3 \mathbf{F}^{-T}
\end{aligned}$$

The 1st PK of the second expression

$$\begin{aligned}
\mathbf{P} &= \frac{\partial W}{\partial I_1^{\mathbf{C}}} \frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3} \frac{\partial I_3}{\partial \mathbf{F}} \\
&= \frac{\mu}{2} (2\mathbf{F}) + \left[\frac{\mu}{2} \left(-2 \frac{\partial \ln I_3}{\partial I_3} \right) + p \right] I_3 \mathbf{F}^{-T} \\
&= \mu \mathbf{F} + \left[-\frac{\mu}{I_3} + p \right] I_3 \mathbf{F}^{-T} \\
&= \mu \mathbf{F} - \mu \mathbf{F}^{-T} + p I_3 \mathbf{F}^{-T} \\
\mathbf{P} &= \mu (\mathbf{F} - \mathbf{F}^{-T}) + p I_3 \mathbf{F}^{-T}
\end{aligned} \tag{3.3}$$

4 Summary

Compressible neo-Hookean

$$\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2} (I_1^{\mathbf{C}} - 3 - 2 \ln I_3) + \frac{\kappa}{2} (I_3 - 1)^2 \\
\mathbf{P} &= \mu (\mathbf{F} - \mathbf{F}^{-T}) + \kappa I_3 (I_3 - 1) \mathbf{F}^{-T}
\end{aligned} \tag{4.1}$$

Compressible neo-Hookean

$$\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2} (I_1^{\mathbf{C}} - 3 - 2 \ln I_3) + \frac{\lambda}{2} (\ln I_3)^2 \\
\mathbf{P} &= \mu (\mathbf{F} - \mathbf{F}^{-T}) + \lambda I_3 (\ln I_3) \mathbf{F}^{-T}
\end{aligned} \tag{4.2}$$

Incompressible neo-Hookean

$$\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2} (I_1^{\mathbf{C}} - 3) + p (I_3 - 1) \\
\mathbf{P} &= \mu \mathbf{F} + p I_3 \mathbf{F}^{-T}
\end{aligned} \tag{4.3}$$

Incompressible neo-Hookean

$$\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2} (I_1^{\mathbf{C}} - 3 - 2 \ln I_3) + p (I_3 - 1) \\
\mathbf{P} &= \mu (\mathbf{F} - \mathbf{F}^{-T}) + p I_3 \mathbf{F}^{-T}
\end{aligned} \tag{4.4}$$