FEniCS: Lagrange formulation for Stokes Equation

Based on DOLFIN 1.4.0 Demo 21: Stokes equations

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1 Problem Definition

The Stokes equations represent a considerable simplification of the full Navier–Stokes equations, especially in the incompressible Newtonian case. There is no longer a time dependency.

1.1 Strong Form

The following equations solve for velocity, u, and pressure, p, with the second equation enforcing incompressibility.

$$\begin{aligned} -\nabla \cdot (\nabla \mathbf{u} + p\mathbf{I}) &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \tag{1}$$

Boundary conditions:

$$\begin{aligned} u &= u_o \quad \text{on } \Gamma_D \\ \nabla u \cdot n + pn &= g \quad \text{on } \Gamma_N \end{aligned} \tag{2}$$

Note that consulting Wikipedia, the classical definition is written in the following form ($\mu =$ dynamic viscosity). Note that the sign of pressure is changed from the classical definition (body forces can be defined as negative or positive).

$$\begin{aligned} \mu\nabla^2 u - \nabla p + f &= 0 \\ \nabla \cdot u &= 0 \end{aligned}$$

The sign difference for pressure is done in order to have a symmetric (but not positive-definite) system of equations rather than a non-symmetric (but positive-definite) system of equations.

1.2 Weak Form

We have two partial differential equations, leading to the introduction of two different test functions, v and q.

1.2.1 Equation 1

$$
\begin{aligned}
-\nabla \cdot \left[\frac{\partial}{\partial x_i}\mathbf{e_i} \otimes u_k \mathbf{e_k} + p\mathbf{I}\right] &= f \\
-\nabla \cdot \left[\frac{\partial u_k}{\partial x_i}\mathbf{e_i} \otimes \mathbf{e_k} + p\right] &= f \\
-\nabla \cdot \frac{\partial u_k}{\partial x_i}\mathbf{e_i} \otimes \mathbf{e_k} - \nabla \cdot p &= f \\
-\frac{\partial}{\partial x_l}\mathbf{e_l} \cdot \frac{\partial u_k}{\partial x_i}\mathbf{e_i} \otimes \mathbf{e_k} - \frac{\partial}{\partial x_l}\mathbf{e_l} \cdot p &= f \\
-\frac{\partial^2 u_k}{\partial x_l x_i}\mathbf{e_l} \cdot \mathbf{e_i} \otimes \mathbf{e_k} - \frac{\partial p}{\partial x_l}\mathbf{e_l} &= f \\
-\frac{\partial^2 u_k}{\partial x_l x_i}\delta_{li}\mathbf{e_k} - \frac{\partial p}{\partial x_l}\mathbf{e_l} &= f_q \mathbf{e_q} \\
-\frac{\partial^2 u_k}{\partial x_l^2}\mathbf{e_k} - \frac{\partial p}{\partial x_l}\mathbf{e_l} &= f_q \mathbf{e_q} \quad \text{Multiply by test function} \\
-\frac{\partial^2 u_k}{\partial x_l^2}\mathbf{e_k} \cdot v_p \mathbf{e_p} - \frac{\partial p}{\partial x_l}\mathbf{e_l} \cdot v_p \mathbf{e_p} &= f_q \mathbf{e_q} \cdot v_p \mathbf{e_p} \\
-\frac{\partial^2 u_k}{\partial x_l^2} v_k - \frac{\partial p}{\partial x_l} v_l &= f_q v_q \quad \text{Integrate over domain} \\
-\int_\Omega \frac{\partial^2 u_k}{\partial x_l^2} v_k dx - \int_\Omega \frac{\partial p}{\partial x_l} v_l dx &= \int_\Omega f_q v_q dx
\end{aligned}
$$

Use integration by parts on the left hand side terms

$$
\begin{aligned}
(fg)' = f'g + fg' \rightarrow f'g &= (fg)' - fg' \\
f''g &= (f'g)' - f'g'
\end{aligned} \tag{3}
$$

Where, we can use integration by parts twice

$$
\begin{aligned}
\frac{\partial^2 u_k}{\partial x_l^2} v_k &= \left(\frac{\partial u_k}{\partial x_l} v_k\right)_{,l} - \frac{\partial u_k}{\partial x_l}\frac{\partial v_k}{\partial x_l} \\
\frac{\partial p}{\partial x_l} v_l &= (p v_l)_{,l} - p\frac{\partial v_l}{\partial x_l}
\end{aligned}
$$

Substitute:

$$
\begin{aligned}
-\int_\Omega \frac{\partial^2 u_k}{\partial x_l^2} v_k dx - \int_\Omega \frac{\partial p}{\partial x_l} v_l dx &= \int_\Omega f_q v_q dx \\
-\int_\Omega \left(\frac{\partial u_k}{\partial x_l} v_k\right)_{,l} dx + \int_\Omega \frac{\partial u_k}{\partial x_l}\frac{\partial v_k}{\partial x_l} dx - \int_\Omega (p v_l)_{,l} dx + \int_\Omega p\frac{\partial v_l}{\partial x_l} dx &= \int_\Omega f_q v_q dx \quad \text{Use divergence theorem} \\
-\int_{\delta\Omega} \frac{\partial u_k}{\partial x_l} v_k n_l ds + \int_\Omega \frac{\partial u_k}{\partial x_l}\frac{\partial v_k}{\partial x_l} dx - \int_{\partial\Omega} p v_l n_l ds + \int_\Omega p\frac{\partial v_l}{\partial x_l} dx &= \int_\Omega f_q v_q dx
\end{aligned}
$$

Rearrange and change from indicial to direct notation:

$$\begin{gathered}
\int_\Omega \frac{\partial u_k}{\partial x_l}\frac{\partial v_k}{\partial x_l}dx + \int_\Omega p\frac{\partial v_l}{\partial x_l}dx - \int_{\partial\Omega} \frac{\partial u_k}{\partial x_l}v_k n_l ds - \int_{\partial\Omega} p v_l n_l ds = \int_\Omega f_q v_q dx \\
\int_\Omega \frac{\partial u_k}{\partial x_l}\frac{\partial v_k}{\partial x_l}dx + \int_\Omega p\frac{\partial v_l}{\partial x_l}dx - \int_{\partial\Omega} \left(\frac{\partial u_k}{\partial x_l}v_k n_l + p v_l n_l\right) ds = \int_\Omega f_q v_q dx \\
\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega p(\nabla \cdot v)dx - \int_{\partial\Omega} \big(\nabla u \cdot n + pn\big) v ds = \int_\Omega f v dx
\end{gathered}$$

1.2.2 Equation 2

Multiply the second equation of Eq. **??** with test function q:

$$\begin{aligned}
\nabla \cdot u = 0 &\quad \text{Multiply by test function q} \\
(\nabla \cdot u)q = 0 &\quad \text{Integrate over domain}
\end{aligned}$$

$$\int_\Omega q(\nabla \cdot u)dx = 0 \tag{4}$$

1.2.3 Boundary Conditions

Relabeling to match with FEniCS implementation, (v $\rightarrow v_u$ and q $\rightarrow v_p$). Utilize the Neumann boundary condition on the surface integral term.

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla v_u dx + \int_\Omega p(\nabla \cdot v_u)dx - \int_{\partial\Omega} g v ds &= f v dx \\
\int_\Omega v_p(\nabla \cdot u)dx &= 0
\end{aligned}$$

2 Lagrange Multiplier

Pressure and lagrange multiplier space are equal order. Displacement is a higher order space.

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla v_u dx + \int_\Omega p(\nabla \cdot v_u)dx + \int_{\partial\Omega} \lambda v_u[1] ds &= 0 \\
\int_\Omega v_p(\nabla \cdot u)dx &= 0 \\
\int_{\partial\Omega} v_l(u[1] - u_i n)ds &= 0
\end{aligned}$$