

Stabilized Mixed Finite Element Formulation

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Definitions

F: Deformation gradient

I: second-order unit tensor

u: Displacement

J: determinant of the deformation gradient

C: Right Cauchy-Green Strain Tensor

$\mathcal{W}(\mathbf{F})$: strain energy function

P: first Piola-Kirchhoff stress tensor

S: second PK stress tensor

α : cracks are represented by a scalar phase-field variable

p: Lagrange multiplier, hydrostatic pressure field

κ : bulk modulus

$$\kappa = \frac{E}{3(1 - 2\nu)} \quad (0.1)$$

μ : shear modulus

$$\mu = \frac{E}{2(1 + \nu)} \quad (0.2)$$

λ : Lamé modulus

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad (0.3)$$

For Plane Stress

$$\kappa = \frac{3 - \nu}{1 + \nu}, \quad \lambda = \frac{E\nu}{(1 - \nu)^2} \quad (0.4)$$

\mathcal{E}_ℓ : potential energy functional

$a(\alpha)$ is the decreasing stiffness modulation function

$w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

c_w is a normalization constant

1 Hyperelastic Phase-Field Fracture Models

Deformation Gradient

$$\mathbf{F} = \mathbf{I} + \nabla \otimes \mathbf{u} \quad (1.1)$$

where $J = \det \mathbf{F}$ and $\mathbf{C} = \mathbf{F}^T \mathbf{F}$.

The strain energy function $\mathcal{W}(\mathbf{F})$ is defined per unit reference volume such that the first PK and second PK

$$\mathbf{P} = \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \quad (1.2a)$$

$$\mathbf{S} = 2 \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{C}} \quad (1.2b)$$

where $\mathbf{P} = \mathbf{F}\mathbf{S}$.

1.1 Phase-Field Fracture Model

Non-modified strain energy function is the compressible Neo-Hookean:

$$\mathcal{W}(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2 \ln J) \quad (1.3)$$

For incompressible hyperelastic materials, the strain energy function is defined as

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p(J - 1), \quad (1.4)$$

We instead consider a damage dependent relaxation of the incompressibility constraint and the phase-field couples to the strain energy through the modified function

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J - 1)^2 \quad (1.5)$$

where the decreasing stiffness modulation function is $a(\alpha)$ and $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

$$a(\alpha) = (1 - \alpha)^2 \quad w(\alpha) = \alpha \quad (1.6)$$

In the code, we have the following definition

$$b(\alpha) = (1 - \alpha)^3 = \sqrt{a^3(\alpha)}$$

To circumvent numerical difficulties, we resort to the classical mixed formulation and introduce the pressure-like field

$$\begin{aligned} p &= -\sqrt{a^3(\alpha)}\kappa(J - 1) \\ p &= -b(\alpha)\kappa(J - 1) \end{aligned} \quad (1.7)$$

as an independent variable along with the displacement field.

Following the code, we have the following energy functional of the energy problem

$$\begin{aligned} \widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J - 1) - \frac{p^2}{2\kappa} \\ \widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\frac{\mu}{2}(I_c - 3 - 2 \ln J) - b(\alpha)p(J - 1) - \frac{p^2}{2\kappa} \end{aligned}$$

The first Piola-Kirchhoff stress tensor is given:

$$\begin{aligned}
\mathbf{P} &= \frac{\partial \widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{\partial \mathbf{F}} \\
&= \frac{\partial}{\partial \mathbf{F}} \left[a(\alpha) \mathcal{W}(\mathbf{F}) + a^3(\alpha) \frac{1}{2} \kappa (J - 1)^2 \right] \\
&= a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha) \frac{1}{2} \kappa \frac{\partial (J - 1)^2}{\partial \mathbf{F}} \\
&= a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha) \kappa (J - 1) \frac{\partial J}{\partial \mathbf{F}} \quad \text{where } \frac{\partial J}{\partial \mathbf{F}} = J \mathbf{F}^{-T} \\
&= a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha) \kappa (J - 1) J \mathbf{F}^{-T} \quad \text{substituting in pressure equation} \\
\mathbf{P} &= a(\alpha) \mu (\mathbf{F} - \mathbf{F}^{-T}) - b(\alpha) p J \mathbf{F}^{-T}
\end{aligned} \tag{1.8}$$

1.1.1 Derivation from Bin2020 Paper

Here, unlike Eq. 21 from Bin2020, we drop λ_b which is not a consideration in this formulation

$$\mathcal{E}_\ell(\mathbf{u}, \alpha) = \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}, \alpha) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega \tag{1.9}$$

The normalization constant is defined as:

$$c_w = \int_0^1 \sqrt{w(\alpha)} d\alpha \tag{1.10}$$

Starting from Eq. 25 in the 2020 Li and Bouklas paper where κ is the bulk modulus

$$\mathcal{E}_\ell(\mathbf{u}, p, \Lambda, \alpha) = \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_{\Omega} \frac{p^2}{2\kappa} d\Omega + \int_{\Omega} \Lambda (p + \sqrt{a^3(\alpha)} \kappa (J - 1)) d\Omega \tag{1.11}$$

Identify the stationary point of the energy functional with respect to **pressure** (not Λ)

$$\begin{aligned}
\frac{\partial \mathcal{E}_\ell}{\partial p} &= \int_{\Omega} \frac{p}{\kappa} d\Omega + \int_{\Omega} \Lambda d\Omega \\
0 &= \frac{p}{\kappa} + \Lambda \rightarrow \Lambda = -p/\kappa
\end{aligned}$$

Substituting this relationship into the energy functional yields:

$$\begin{aligned}
\mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_{\Omega} \frac{p^2}{2\kappa} d\Omega + \int_{\Omega} -\frac{p}{\kappa} (p + \sqrt{a^3(\alpha)} \kappa (J - 1)) d\Omega \\
&= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \frac{p^2}{\kappa} d\Omega - \int_{\Omega} \frac{p}{\kappa} \sqrt{a^3(\alpha)} \kappa (J - 1) d\Omega \\
\mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \mathcal{E}_\ell(\mathbf{u}, \alpha) - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega
\end{aligned} \tag{1.12}$$

Substitute in $\mathcal{E}_\ell(\mathbf{u}, \alpha)$ and substitute Eq. 1.3

$$\mathcal{E}_\ell(\mathbf{u}, p, \alpha) = \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega$$

The prior equation includes the full weak form, unless we want to consider linear interpolation of all fields. In that case, we can introduce the stabilization term

$$-\frac{\varpi h^2}{2\mu} \sqrt{a^3(\alpha)} \sum_{e=1}^{n_{el}} \int_{\Omega^e} J \mathbf{C}^{-1} : (\nabla p \cdot \nabla q) dV = 0$$

1.2 Changes for 2D Plane-Stress Models

Recalling the 1st PK stress in Eq. 1.8.

$$\mathbf{P} = a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T}) - b(\alpha)pJ\mathbf{F}^{-T}$$

In a plane-stress case, the \mathbf{P}_{33} component is zero:

$$\mathbf{P}_{33} = a(\alpha)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha)pJ\mathbf{F}_{33}^{-1} = 0$$

This can be multiplied by its associated test function to obtain the weak form

$$\int_{\Omega} \left(a(\alpha)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha)pJ\mathbf{F}_{33}^{-1} \right) v_{F_{33}} dV = 0$$

In the FEniCS code, we expand the solution space to include displacement, pressure, and a component of the deformation gradient \mathbf{F}_{33} . Therefore, we include a change to the invariants of the deformation tensors:

$$\begin{aligned} J &= \det(\mathbf{F}) * F_{33} \\ I_c &= \text{tr}(\mathbf{C}) + F_{33} ** 2 \end{aligned}$$

Together with the weak form from above:

$$\begin{aligned} \text{F_u} &= \text{derivative}(\text{elastic_potential}, \text{w_p}, \text{v_q}) \setminus \\ &+ (a(\alpha) * \mu * (F_{33} - 1/F_{33}) - b(\alpha) * p * J/F_{33}) * v_F_{33} * dx \end{aligned}$$

1.2.1 Changes for 2D Discrete Crack Model

If we are considering a discrete fracture method

$$\begin{aligned} \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega \\ \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \int_{\Omega} \mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} p (J - 1) d\Omega \end{aligned}$$

where we have assumed for the energy functional

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = \frac{\mu}{2} (I_c - 3 - 2 \ln J) - p(J - 1) - \frac{p^2}{2\lambda}$$

Therefore, we can calculate the 1st Piola Kirchhoff Stress as:

$$\begin{aligned} \mathbf{P} &= \frac{\mu}{2} (2\mathbf{F} - \frac{2}{J} J\mathbf{F}^{-T}) - pJ\mathbf{F}^{-T} \\ &= \mu(\mathbf{F} - \mathbf{F}^{-T}) - pJ\mathbf{F}^{-T} \end{aligned}$$

Taking the third component to be zero

$$\begin{aligned}
P_{33} &= \mu(F_{33} - F_{33}^{-1}) - pJF_{33}^{-1} = 0 \\
&= F_{33} - F_{33}^{-1} - \frac{pJ}{\mu}F_{33}^{-1} = 0 \\
P_{33} &= F_{33}^2 - 1 - \frac{pJ}{\mu} = 0
\end{aligned}$$

with the stabilization term and plane stress in the weak form

$$\begin{aligned}
-\frac{\varpi h^2}{2\mu} \int_{\Omega} J\mathbf{C}^{-1} : (\nabla p \cdot \nabla q) dV &= 0 \\
\int_{\Omega} \left(\mathbf{F}_{33}^2 - 1 - \frac{pJ}{\mu} \right) v_{F_{33}} dV &= 0
\end{aligned}$$

1.2.2 Changes for 2D displacement formulation

Removing pressure terms

$$\begin{aligned}
\mathcal{E}_{\ell}(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega \\
\mathcal{E}_{\ell}(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega
\end{aligned}$$

with plane stress in the weak form (no need for stabilization terms)

$$\int_{\Omega} (a(\alpha) \mu (F_{33} - F_{33}^{-1})) v_{F_{33}} dV = 0$$

We have assumed the modified energy functional

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha) \frac{\mu}{2} (I_c - 3 - 2 \ln J) \tag{1.13}$$

Therefore, we can calculate the 1st Piola Kirchhoff Stress as:

$$\begin{aligned}
\mathbf{P} &= a(\alpha) \frac{\mu}{2} (2\mathbf{F} - \frac{2}{J} J \mathbf{F}^{-T}) \\
&= a(\alpha) \mu (\mathbf{F} - \mathbf{F}^{-T})
\end{aligned}$$

Taking the third component to be zero

$$P_{33} = a(\alpha) \mu (F_{33} - F_{33}^{-1}) = 0$$

2 Obtaining the Critical Stretch

In order to obtain the critical stretch of a particular specimen in Krishnan 2008, we can equate our dissipated energy or critical fracture energy density

$$\int_{\Gamma} G_c d\mathbf{S} = \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega$$

to Eq. 15 for the energy release rate of the strip geometry.

$$J = h\mu \left(\lambda_a - \frac{1}{\lambda_a} \right)^2$$

where the total height of the strip is $2h$ and μ is the shear modulus

3 Strain energy decomposition

The Heaviside function is defined as

$$H(x) = \frac{x + |x|}{2x} = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

The Macaulay bracket is defined

$$M(x) = \frac{x + |x|}{2} = \begin{cases} x, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

3.1 Following Ye 2020 and Tang 2019

In the Ye 2020 paper, the internal energy is expressed as:

$$W_{int}(\mathbf{F}, \alpha, \nabla \alpha) = [a(\alpha) + k_{\ell}] W_{act} + W_{pas} + G_c \left(\frac{\alpha^2}{2\ell} + \frac{\ell}{2} |\nabla \alpha|^2 \right)$$

Now in section 3.2.3 of Ye 2020, is stated the decomposition for a Mooney Rivlin constitutive law:

$$\begin{aligned} W_{MR}(I_1, I_2) &= C_1(I_1 - 3) + C_2(I_2 - 3) \\ W_{MR}(\lambda_1, \lambda_1, \lambda_3) &= C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3) \\ &= C_1 \sum_{i=1}^3 (\lambda_i^2 - 1) + C_2 \sum_{i=1}^3 (\lambda_i^{-2} - 1) \end{aligned}$$

Which can be decomposed to active and passive internal energy terms. First we can rewrite:

$$\begin{aligned} W_{MR}(\lambda_1, \lambda_1, \lambda_3) &= C_1 \sum_{i=1}^3 H(\lambda_i - 1)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^3 H(\lambda_i - 1)(\lambda_i^{-2} - 1) \\ &\quad + C_1 \sum_{i=1}^3 H(1 - \lambda_i)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^3 H(1 - \lambda_i)(\lambda_i^{-2} - 1) \end{aligned}$$

The active and passive terms can be stated as follows where the active part represents the crack-driven energy.

$$W_{act} = C_1 \sum_{i=1}^3 H(\lambda_i - 1)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^3 H(\lambda_i - 1)(\lambda_i^{-2} - 1)$$

$$W_{pas} = C_1 \sum_{i=1}^3 H(1 - \lambda_i)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^3 H(1 - \lambda_i)(\lambda_i^{-2} - 1)$$

One way to better understand these is to consider some cases 1) triaxial tension $\lambda_i > 1$ 2) other stress states where $\lambda_i < 1$:

For $\lambda_i > 1$:

$$W_{act} = C_1 \sum_{i=1}^3 (\lambda_i^2 - 1) + C_2 \sum_{i=1}^3 (\lambda_i^{-2} - 1)$$

$$W_{pas} = 0$$

For $\lambda_i < 1$:

$$W_{act} = 0$$

$$W_{pas} = C_1 \sum_{i=1}^3 (\lambda_i^2 - 1) + C_2 \sum_{i=1}^3 (\lambda_i^{-2} - 1)$$

For this second case, we end up with a negative energy component (first term of the passive energy).

We can also note the definitions within Tang 2019 for Model M_I . In this model, we consider the free energy density of a neo-Hookean constitutive law:

$$W(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2 \ln J) + \frac{\kappa}{2}(\ln J)^2$$

which can also be rephrased in terms of stretches

$$W(\lambda_1, \lambda_2, \lambda_3) = W_1 + W_2$$

$$= \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i) + \frac{\kappa}{2}(\ln J)^2$$

where we note that W_1 is a linear function of $\ln \lambda_i$ and W_2 is a nonlinear function of $\ln J$. The free energy is stated as

$$G_{rub} = [(1 - K)\alpha^2 + K]W^+ + W^-$$

Note that K is not κ . No definition is provided in the paper. This is another way of coupling the damage to the free energy density, and I believe we can rewrite our own version where:

$$G_{rub} = [a(\alpha) + k_\ell]W^+ + W^-$$

Now we turn to the definition of W^+ and W^- which refers to the energy with tensile stretching

$$W^+ = W(\lambda_i^+, J^+) = \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^+)^2 - 1 - 2 \ln \lambda_i^+) + \frac{\kappa}{2}(\ln J^+)^2$$

and the energy with compression respectively.

$$W^- = W(\lambda_i^-, J^-) = \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^-)^2 - 1 - 2 \ln \lambda_i^-) + \frac{\kappa}{2} (\ln J^-)^2$$

The definitions for these superscript + and - terms gives us

$$\lambda_i^+ = \begin{cases} \lambda_i, & \lambda_i > 1, \\ 1, & \lambda_i \leq 1 \end{cases} \quad J^+ = \begin{cases} J, & J > 1, \\ 1, & J \leq 1 \end{cases}$$

$$\lambda_i^- = \begin{cases} \lambda_i, & \lambda_i < 1, \\ 1, & \lambda_i \geq 1 \end{cases} \quad J^- = \begin{cases} J, & J < 1, \\ 1, & J \geq 1 \end{cases}$$

This isn't the definition for the heaviside function, but it could be a shifted Macaulay bracket

$$M_s(x) = \frac{x - 1 + |x - 1|}{2} + 1 = \begin{cases} x, & x > 1, \\ 1, & x \leq 1 \end{cases}$$

Now we can consider some examples.

For $J > 1$:

$$W^+ = \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^+)^2 - 1 - 2 \ln \lambda_i^+) + \frac{\kappa}{2} (\ln J)^2$$

$$W^- = \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^-)^2 - 1 - 2 \ln \lambda_i^-)$$

For $J < 1$:

$$W^+ = \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^+)^2 - 1 - 2 \ln \lambda_i^+)$$

$$W^- = \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^-)^2 - 1 - 2 \ln \lambda_i^-) + \frac{\kappa}{2} (\ln J)^2$$

If we consider the same stress states as in Ye2020, 1) triaxial tension $\lambda_i > 1$ 2) all other stress states $\lambda_i < 1$:

For $J > 1, \lambda_i > 1$:

$$W^+ = \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i) + \frac{\kappa}{2} (\ln J)^2$$

$$W^- = 0$$

For $J < 1, \lambda_i > 1$:

$$W^+ = \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i)$$

$$W^- = \frac{\kappa}{2} (\ln J)^2$$

Now for all other stress states:

For $J > 1$, $\lambda_i < 1$:

$$W^+ = \frac{\kappa}{2}(\ln J)^2$$

$$W^- = \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i)$$

For $J < 1$, $\lambda_i < 1$:

$$W^+ = 0$$

$$W^- = \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i) + \frac{\kappa}{2}(\ln J)^2$$

This should be roughly equivalent to the considerations in the Ye2020 paper.

3.2 Our strain energy decomposition

Following the section above, we can consider our modified strain energy

$$\begin{aligned} \widetilde{W}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} (I_c - 3 - 2 \ln J) - b(\alpha) p (J - 1) - \frac{p^2}{2\kappa} \quad \text{where } p = -b(\alpha) \kappa (J - 1) \\ \widetilde{W}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} (I_c - 3 - 2 \ln J) + \frac{\kappa}{2} a(\alpha)^3 (J - 1)^2 \end{aligned}$$

We can also rewrite the first term with regards to stretches

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i) + \frac{\kappa}{2} a(\alpha)^3 (J - 1)^2$$

Following Tang 2019 we rewrite the strain energy as

$$\widetilde{W}(\mathbf{F}, \alpha) = \widetilde{W}_{\text{act}}(\mathbf{F}, \alpha) + \widetilde{W}_{\text{pas}}(\mathbf{F}, \alpha) \quad (3.1)$$

where the active and passive parts of the strain energy can be written as:

$$\begin{aligned} \widetilde{W}_{\text{act}}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^+)^2 - 1 - 2 \ln \lambda_i^+) + \frac{\kappa}{2} a(\alpha)^3 (J^+ - 1)^2 \\ \widetilde{W}_{\text{pas}}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^-)^2 - 1 - 2 \ln \lambda_i^-) + \frac{\kappa}{2} a(\alpha)^3 (J^- - 1)^2 \end{aligned}$$

where the definitions of the superscript + and - terms remain the same as in Tang2020:

For $J > 1$:

$$\begin{aligned}\widetilde{W}_{act}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^+)^2 - 1 - 2 \ln \lambda_i^+) + \frac{\kappa}{2} a(\alpha)^3 (J - 1)^2 \\ \widetilde{W}_{pas}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^-)^2 - 1 - 2 \ln \lambda_i^-)\end{aligned}\tag{3.2}$$

For $J \leq 1$:

$$\begin{aligned}\widetilde{W}_{act}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^+)^2 - 1 - 2 \ln \lambda_i^+) \\ \widetilde{W}_{pas}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 ((\lambda_i^-)^2 - 1 - 2 \ln \lambda_i^-) + \frac{\kappa}{2} a(\alpha)^3 (J - 1)^2\end{aligned}$$

Now considering the same two cases of, triaxial tension and

For $J > 1, \lambda_i > 1$:

$$\begin{aligned}\widetilde{W}_{act}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i) + \frac{\kappa}{2} a(\alpha)^3 (J - 1)^2 \\ \widetilde{W}_{pas}(\mathbf{F}, \alpha) &= 0\end{aligned}$$

For $J \leq 1, \lambda_i > 1$:

$$\begin{aligned}\widetilde{W}_{act}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i) \\ \widetilde{W}_{pas}(\mathbf{F}, \alpha) &= \frac{\kappa}{2} a(\alpha)^3 (J - 1)^2\end{aligned}$$

all other cases:

For $J > 1, \lambda_i < 1$:

$$\begin{aligned}\widetilde{W}_{act}(\mathbf{F}, \alpha) &= \frac{\kappa}{2} a(\alpha)^3 (J - 1)^2 \\ \widetilde{W}_{pas}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i)\end{aligned}$$

For $J \leq 1, \lambda_i < 1$:

$$\begin{aligned}\widetilde{W}_{act}(\mathbf{F}, \alpha) &= 0 \\ \widetilde{W}_{pas}(\mathbf{F}, \alpha) &= a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 (\lambda_i^2 - 1 - 2 \ln \lambda_i) + \frac{\kappa}{2} a(\alpha)^3 (J - 1)^2\end{aligned}$$

These can be concisely summarized with the following expressions, where the active part of the strain energy is

$$\widetilde{W}_{act}(\mathbf{F}, \alpha) = a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 H(\lambda_i - 1) (\lambda_i^2 - 1 - 2 \ln \lambda_i) + a^3(\alpha) H(J - 1) \frac{1}{2} \kappa (J - 1)^2, \tag{3.3}$$

and the passive part of the strain energy is

$$\widetilde{W}_{\text{pas}}(\mathbf{F}, \alpha) = a(\alpha) \frac{\mu}{2} \sum_{i=1}^3 H(1 - \lambda_i) (\lambda_i^2 - 1 - 2 \ln \lambda_i) + a^3(\alpha) H(1 - J) \frac{1}{2} \kappa (J - 1)^2, \quad (3.4)$$

3.2.1 Compute the principal stretches λ_i

The eigenvalues of Cauchy-Green strain tensor \mathbf{C} are λ_i^2 , $i = 1, 2, 3$. With following definitions

$$d = \frac{\text{Tr} \mathbf{C}}{3}, \quad e = \sqrt{\frac{\text{Tr}(\mathbf{C} - d\mathbf{I})^2}{6}}, \quad f = \frac{1}{e} (\mathbf{C} - d\mathbf{I}), \quad g = \frac{\det f}{2}, \quad (3.5)$$

and assuming the eigenvalues satisfying $\lambda_3^2 \leq \lambda_2 \leq \lambda_1$, we could obtain (?)

$$\lambda_1^2 = d + 2e \cos\left(\frac{\arccos g}{3}\right), \quad \lambda_3^2 = d + 2e \cos\left(\frac{\arccos g}{3} + \frac{2\pi}{3}\right), \quad \lambda_2^2 = 3d - \lambda_1^2 - \lambda_3^2. \quad (3.6)$$

4 Following Borden: Derivations of Analytical Phase Field

Note the full potential energy functional, which can also be called the lagrangian

$$\mathcal{E}_\ell(\mathbf{u}, p, \alpha) = \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega$$

We can use the Euler-Lagrange equations to arrive at the equations of motion by taking the derivative with respect to displacement, pressure, and the scalar damage field. Starting with displacement:

$$\frac{\partial \mathcal{E}_\ell}{\partial \mathbf{u}} = \int_{\Omega} a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{u}} d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial p} = - \int_{\Omega} \frac{p}{\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} (J - 1) d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial \alpha} = - \int_{\Omega} 2(1 - \alpha) \mathcal{W}(\mathbf{F}) d\Omega + \int_{\Omega} 3p(1 - \alpha)^2 (J - 1) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] d\Omega$$

Therefore we have three equations:

First should be mechanical eq, second is a an equation for pressure,

$$\begin{aligned} -\frac{p}{\kappa} - \sqrt{a^3(\alpha)} (J - 1) &= 0 \\ -\frac{p}{\kappa} - (1 - \alpha)^3 (J - 1) &= 0 \\ -\kappa (J - 1) (1 - \alpha)^3 &= p \end{aligned}$$

Lastly,

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) + 3p(1 - \alpha)^2 (J - 1) + \frac{G_c}{c_w} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] = 0$$

Substitute second equation into third

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) - 3\kappa (1 - \alpha)^5 (J - 1)^2 + \frac{G_c}{c_w} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] = 0$$

4.1 Homogeneous Solution

Therefore, we can study the homogeneous solution by ignoring spatial derivatives of α

$$-2(1 - \alpha_h) \mathcal{W}(\mathbf{F}) - 3\kappa(1 - \alpha_h)^5(J - 1)^2 + \frac{G_c}{c_w \ell} = 0$$

We can expand and arrange

$$\begin{aligned} -2\mathcal{W}(\mathbf{F}) + 2\alpha_h \mathcal{W}(\mathbf{F}) - 3\kappa(J - 1)^2(1 - 5\alpha_h + 10\alpha_h^2 - 10\alpha_h^3 + 5\alpha_h^4 - \alpha_h^5) + \frac{G_c}{c_w \ell} &= 0 \\ -2\mathcal{W}(\mathbf{F}) + 2\alpha_h \mathcal{W}(\mathbf{F}) + \frac{G_c}{c_w \ell} & \\ -3\kappa(J - 1)^2 + 15\kappa(J - 1)^2\alpha_h - 30\kappa(J - 1)^2\alpha_h^2 + 30\kappa(J - 1)^2\alpha_h^3 - 15\kappa(J - 1)^2\alpha_h^4 + \kappa(J - 1)^2\alpha_h^5 &= 0 \\ -2\mathcal{W}(\mathbf{F}) - 3\kappa(J - 1)^2 + \frac{G_c}{c_w \ell} + [2\mathcal{W}(\mathbf{F}) + 15\kappa(J - 1)^2]\alpha_h & \\ -30\kappa(J - 1)^2\alpha_h^2 + 30\kappa(J - 1)^2\alpha_h^3 - 15\kappa(J - 1)^2\alpha_h^4 + \kappa(J - 1)^2\alpha_h^5 &= 0 \end{aligned}$$

4.2 Non-Homogeneous Solution

Now for the Non-homogenous solution, we have the following

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) - 3\kappa(1 - \alpha)^5(J - 1)^2 + \frac{G_c}{c_w} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] = 0$$

(4.1)