Stabilized Mixed Finite Element Formulation

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Definitions

$\mathbf{F}$: Deformation gradient
$\mathbf{I}$: second-order unit tensor
$\boldsymbol{u}$: Displacement
J: determinant of the deformation gradient
$\mathbf{C}$: Right Cauchy-Green Strain Tensor
$\mathcal{W}(\mathbf{F})$: strain energy function
$\mathbf{P}$: first Piola-Kirchhoff stress tensor
$\mathbf{S}$: second PK stress tensor

α: cracks are represented by a scalar phase-field variable
p: Lagrange multiplier, hydrostatic pressure field
κ: bulk modulus

$$\kappa = \frac{E}{3(1-2\nu)} \tag{0.1}$$

μ: shear modulus

$$\mu = \frac{E}{2(1+\nu)} \tag{0.2}$$

λ: Lamé modulus

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{0.3}$$

$\mathcal{E}_\ell$: potential energy functional $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume
c_w is a normalization constant

1 Hyperelastic Phase-Field Fracture Models

Deformation Gradient

$$\mathbf{F} = \mathbf{I} + \nabla \otimes \boldsymbol{u} \tag{1.1}$$

where $J = \det \mathbf{F}$ and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

The strain energy function $\mathcal{W}(\mathbf{F})$ is defined per unit reference volume such that the first PK and second PK

$$\mathbf{P} = \mathcal{W}(\mathbf{F})\mathbf{F} \tag{1.2a}$$

$$\mathbf{S} = 2\mathcal{W}(\mathbf{F})\mathbf{C} \tag{1.2b}$$

where $\mathbf{P} = \mathbf{F}\mathbf{S}$.

1.1 Phase-Field Fracture Model

Non-modified strain energy function is the compressible Neo-Hookean:

$$\mathcal{W}(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2\ln J) \tag{1.3}$$

For incompressible hyperelastic materials, the strain energy function is defined as

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p\,(J-1)\,, \tag{1.4}$$

We instead consider a damage dependent relaxation of the incompressibility constraint and the phase-field couples to the strain energy through the modified function

$$\widetilde{\mathcal{W}}(\mathbf{F},\alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa\,(J-1)^2 \tag{1.5}$$

where the decreasing stiffness modulation function and $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

$$a(\alpha) = (1-\alpha)^2 \quad w(\alpha) = \alpha \tag{1.6}$$

In the code we have the following definition

$$b(\alpha) = (1-\alpha)^3 = \sqrt{a^3(\alpha)}$$

To circumvent numerical difficulties, we resort to the classical mixed formulation and introduce the pressure-like field

$$p = -\sqrt{a^3(\alpha)}\kappa\,(J-1)\,, \tag{1.7}$$

as an independent variable along with the displacement field.

Lastly, the normalization constant is defined as:

$$c_w = \int_0^1 \sqrt{w(\alpha)}d\alpha \tag{1.8}$$

The first Piola-Kirchhoff stress tensor is given:

$$\begin{aligned}
\mathbf{P} &= \tilde{\mathcal{W}}(\mathbf{F},\alpha)\mathbf{F} \\
&= \mathbf{F}\left[a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa\,(J-1)^2\right] \\
&= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial\mathbf{F}} + a^3(\alpha)\frac{1}{2}\kappa(J-1)^2\mathbf{F} \\
\mathbf{P} &= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial\mathbf{F}} + a^3(\alpha)\kappa\,(J-1)\,\frac{\partial J}{\partial\mathbf{F}} \quad \text{where } J\mathbf{F} = J\mathbf{F}^{-T} \\
\mathbf{P} &= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial\mathbf{F}} + a^3(\alpha)\kappa(J-1)J\mathbf{F}^{-T}
\end{aligned} \tag{1.9}$$

Note that we DO NOT consider this form of the modified functional; therefore, we do not consider this PK stress.

1.1.1 According to the rough draft

Therefore our modified function:

$$\begin{aligned}
\widetilde{\mathcal{W}}(\mathbf{F},\alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J-1)^2 \\
&= a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\frac{-p}{\sqrt{a^3(\alpha)}}(J-1) \\
&= a(\alpha)\mathcal{W}(\mathbf{F}) - \sqrt{a^3(\alpha)}\frac{p}{2}(J-1) \\
\widetilde{\mathcal{W}}(\mathbf{F},\alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)\frac{p}{2}(J-1)
\end{aligned}$$

In the rough draft of the paper we have Eq. 5: energy functional of a possibly fractured elastic body with isotropic surface energy

$$\begin{aligned}
\mathcal{E}_\ell(\boldsymbol{u},\alpha) &= \int_\Omega \widetilde{\mathcal{W}}(\mathbf{F},\alpha)d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_{\partial_N\Omega} \tilde{\mathbf{g}}_\mathbf{0}\cdot\mathbf{u}dA \\
\mathcal{E}_\ell(\boldsymbol{u},p,\alpha) &= \int_\Omega \left[a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)\frac{p}{2}(J-1)\right]d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega \\
\mathcal{E}_\ell(\boldsymbol{u},p,\alpha) &= \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega - \int_\Omega b(\alpha)\frac{p}{2}(J-1)d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega
\end{aligned}$$

Note that we are missing one term.

1.1.2 According to the code

In the code we have the following for the energy functional of the energy problem

$$\widetilde{W}(\mathbf{F},\alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}$$

Energy functional, where we ignore the surface term

$$\begin{aligned}
\mathcal{E}_\ell(\boldsymbol{u},\alpha) &= \int_\Omega \widetilde{\mathcal{W}}(\mathbf{F},\alpha)d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_{\partial_N\Omega} \tilde{\mathbf{g}}_\mathbf{0}\cdot\mathbf{u}dA \\
&= \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega - \int_\Omega b(\alpha)p(J-1)d\Omega - \int_\Omega \frac{p^2}{2\lambda}d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega
\end{aligned}$$

1.1.3 According to a derivation from Bin2020 Paper

Versus Eq. 21 where we drop λ_b which is not a consideration in this formulation

$$\mathcal{E}_\ell(\boldsymbol{u},\alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F},\alpha)d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega \tag{1.10}$$

Starting from Eq. 25 in the 2020 Li and Bouklas paper where κ is the bulk modulus

$$
\begin{aligned}
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \Lambda, \alpha\right) &= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega \Lambda(p + \sqrt{a^3(\alpha)}\kappa(J-1)) d\Omega \quad \Lambda = -p/\kappa \\
&= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega -\frac{p}{\kappa}(p + \sqrt{a^3(\alpha)}\kappa(J-1)) d\Omega \\
&= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \frac{p^2}{\kappa} d\Omega - \int_\Omega \frac{p}{\kappa}\sqrt{a^3(\alpha)}\kappa(J-1)) d\Omega \\
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) &= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)) d\Omega
\end{aligned}
\tag{1.11}
$$

Substitute in $\mathcal{E}_\ell(\boldsymbol{u}, \alpha)$ and substitute Eq. 1.3

$$
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right) d\Omega - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1) d\Omega
$$

1.2 Changes for 2D Code Version

Following the code, we have the following energy functional of the energy problem

$$
\begin{aligned}
\widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda} \\
\widetilde{W}(\mathbf{F}, \alpha) &= \big(a(\alpha) + k_\ell\big)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}
\end{aligned}
$$

Where we know k_ℓ is a modeling parameter, so we can list the energy functional as:

$$
\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}
\tag{1.12}
$$

Derive the 1st PK stress

$$
\begin{aligned}
\mathbf{P} &= \widetilde{W}(\mathbf{F})\mathbf{F} \\
&= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-1}) - b(\alpha)p\det\mathbf{F}\mathbf{F} \\
\mathbf{P} &= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-1}) - b(\alpha)pJ\mathbf{F}^{-1}
\end{aligned}
$$

Taking the third component to be zero, in the plane stress case

$$
\begin{aligned}
\mathbf{P}_{33} = a(\alpha)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
(1-\alpha)^2\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1-\alpha)^3 pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1-\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu\mathbf{F}_{33} - \mu\mathbf{F}_{33}^{-1} - (1-\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu\mathbf{F}_{33} &= \mu\mathbf{F}_{33}^{-1} + (1-\alpha)pJ\mathbf{F}_{33}^{-1} \\
\mu\mathbf{F}_{33} &= \big[\mu + (1-\alpha)pJ\big]\mathbf{F}_{33}^{-1} \\
\mathbf{F}_{33}^2 &= \frac{\mu + (1-\alpha)pJ}{\mu} \\
\mathbf{C}_{33} &= 1 + \frac{(1-\alpha)pJ}{\mu}
\end{aligned}
$$

Treating F_{33} as an independent unknown, we can state the governing equation

$$\begin{aligned}
\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1-\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mathbf{F}_{33} - \mathbf{F}_{33}^{-1} - \frac{(1-\alpha)pJ}{\mu}\mathbf{F}_{33}^{-1} &= 0 \\
\mathbf{F}_{33}\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}\mathbf{F}_{33} - \frac{(1-\alpha)pJ}{\mu}\mathbf{F}_{33}^{-1}\mathbf{F}_{33} &= 0 \\
\mathbf{F}_{33}^{2} - 1 - \frac{(1-\alpha)pJ}{\mu} &= 0
\end{aligned}$$

This can be multiplied by its associated test function to obtain the weak form

2 Following Borden: Derivations of Analytical Phase Field

Note the full potential energy functional, which can also be called the lagrangian

$$\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega$$

We can use the Euler-Lagrange equations to arrive at the equations of motion by taking the derivative with respect to displacement, pressure, and the scalar damage field. Starting with displacement:

$$\frac{\partial \mathcal{E}_\ell}{\partial \boldsymbol{u}} = \int_\Omega a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{u}}d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial p} = -\int_\Omega \frac{p}{\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}(J-1)d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial \alpha} = -\int_\Omega 2(1-\alpha)\,\mathcal{W}(\mathbf{F})\,d\Omega + \int_\Omega 3p(1-\alpha)^2(J-1)d\Omega + \frac{G_c}{c_w}\int_\Omega \left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right]d\Omega$$

Therefore we have three equations:
First should be mechanical eq

$$-\frac{p}{\kappa} - \sqrt{a^3(\alpha)}(J-1) = 0$$

$$\begin{aligned}
-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) + 3p(1-\alpha)^2(J-1) + \frac{G_c}{c_w}\left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right] &= 0 \\
-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) + 3p\,a(\alpha)(J-1) + \frac{G_c}{c_w\ell}\left[1 + 2\ell^2\nabla^2\alpha\right] &= 0
\end{aligned}$$

Obtain for the final equation

$$\begin{aligned}
&\frac{G_c}{c_2\ell} - 2\mathcal{W}(\mathbf{F}) + 3p(J-1) + \\
&\quad 2\alpha\mathcal{W}(\mathbf{F}) - 6p\alpha(J-1) + \\
&\quad 3p(J-1)\alpha^2 + \frac{G_c}{c_w}2\ell\nabla^2\alpha = 0
\end{aligned}$$

3 Stabilized Finite Element Method

From this point, the notes haven't been revised in a long time

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3.1 Gateaux Derivative

The Gateaux derivative with respect to $(\boldsymbol{u}, \alpha)$ in direction $(\boldsymbol{v}, \beta)$ under the irreversibility condition $\dot{\alpha} \geq 0$.

$$d\mathcal{E}_\ell\left(\boldsymbol{u}, \alpha; \boldsymbol{v}, \beta\right) \geq 0. \tag{3.1}$$

Calculation of the Gateaux derivative

$$\begin{aligned}
d\mathcal{E}_\ell(\boldsymbol{u}, \boldsymbol{v})(\alpha, \beta) &= \frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u} + \delta\boldsymbol{v}, \alpha + \delta\beta)\big|_{\delta=0} \\
&= \frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u} + \delta\boldsymbol{v}, \alpha)\big|_{\delta=0} + \frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}, \alpha + \delta\beta)\big|_{\delta=0}
\end{aligned}$$

Starting with the first term:

$$\begin{aligned}
\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u} + \delta\boldsymbol{v}, \alpha)\big|_{\delta=0} &= \frac{d}{d\delta}\left[\int_\Omega \widetilde{\mathcal{W}}(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v}), \alpha)\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot (\boldsymbol{u} + \delta\boldsymbol{v})\, dA\right]\Bigg|_{\delta=0} \\
&= \left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v}), \alpha)}{d\delta}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \frac{d(\boldsymbol{u} + \delta\boldsymbol{v})}{d\delta}\, dA\right]\Bigg|_{\delta=0} \quad \text{chain rule} \\
&= \left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v}), \alpha)}{d(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v})}\frac{d(\mathbf{I} + \nabla(\boldsymbol{u} + \delta\boldsymbol{v}))}{d\delta}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \frac{d(\boldsymbol{u} + \delta\boldsymbol{v})}{d\delta}\, dA\right]\Bigg|_{\delta=0} \\
&= \left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v}, \alpha)}{d(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v})}\frac{d(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v})}{d\delta}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \boldsymbol{v}\, dA\right]\Bigg|_{\delta=0} \\
&= \left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v}, \alpha)}{d(\mathbf{I} + \nabla\boldsymbol{u} + \delta\nabla\boldsymbol{v})}\nabla\boldsymbol{v}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \boldsymbol{v}\, dA\right]\Bigg|_{\delta=0} \\
&= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla\boldsymbol{u}, \alpha)}{d(\mathbf{I} + \nabla\boldsymbol{u})}\nabla\boldsymbol{v}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \boldsymbol{v}\, dA \\
\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u} + \delta\boldsymbol{v}, \alpha)\big|_{\delta=0} &= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\mathbf{F}}\nabla\boldsymbol{v}\, d\Omega - \int_{\partial_N\Omega} \tilde{\boldsymbol{g}}_0 \cdot \boldsymbol{v}\, dA
\end{aligned}$$

Second term:

$$
\begin{aligned}
&\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u},\alpha+\delta\beta)\big|_{\delta=0}\\
&=\frac{d}{d\delta}\left[\int_\Omega \widetilde{\mathcal{W}}(\mathbf{F},\alpha+\delta\beta)\,d\Omega+\frac{\mathcal{G}_c}{c_w}\int_\Omega\left(\frac{w(\alpha+\delta\beta)}{\ell}+\ell\|\nabla(\alpha+\delta\beta)\|^2\right)dV\right]\Bigg|_{\delta=0}\\
&=\left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha+\delta\beta)}{d\delta}\,d\Omega+\frac{\mathcal{G}_c}{c_w}\frac{d}{d\delta}\int_\Omega\left(\frac{w(\alpha+\delta\beta)}{\ell}+\ell\|\nabla(\alpha+\delta\beta)\|^2\right)dV\right]\Bigg|_{\delta=0}\\
&=\left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha+\delta\beta)}{d(\alpha+\delta\beta)}\frac{d(\alpha+\delta\beta)}{d\delta}\,d\Omega+\frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega\frac{dw(\alpha+\delta\beta)}{d\delta}\,dV+\frac{\mathcal{G}_c}{c_w}\ell\int_\Omega\frac{\|\nabla(\alpha+\delta\beta)\|^2}{d\delta}\,dV\right]\Bigg|_{\delta=0}\\
&=\left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha+\delta\beta)}{d(\alpha+\delta\beta)}\beta\,d\Omega+\frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega\frac{dw(\alpha+\delta\beta)}{d(\alpha+\delta\beta)}\frac{d(\alpha+\delta\beta)}{d\delta}\,dV+\frac{\mathcal{G}_c}{c_w}\ell\int_\Omega 2\nabla(\alpha+\delta\beta)\frac{\nabla(\alpha+\delta\beta)}{d\delta}\,dV\right]\Bigg|_{\delta=0}\\
&=\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\,d\Omega+\left[\frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega\frac{dw(\alpha+\delta\beta)}{d(\alpha+\delta\beta)}\beta\,dV+\frac{\mathcal{G}_c}{c_w}\ell\int_\Omega 2\nabla(\alpha+\delta\beta)\nabla\beta\,dV\right]\Bigg|_{\delta=0}\\
&=\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\,d\Omega+\frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega\frac{dw(\alpha)}{d\alpha}\beta\,dV+2\ell\frac{\mathcal{G}_c}{c_w}\int_\Omega\nabla\alpha\cdot\nabla\beta\,dV\\
&=\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\,d\Omega+\frac{\mathcal{G}_c}{c_w\ell}\int_\Omega\left[\frac{dw(\alpha)}{d\alpha}\beta+2\ell^2(\nabla\alpha\cdot\nabla\beta)\right]dV
\end{aligned}
$$

First, consider Eq. 1.7

$$
\begin{aligned}
p&=-\sqrt{a^3(\alpha)}\kappa(J-1)\\
\frac{p}{\kappa}&=-\sqrt{a^3(\alpha)}(J-1)\\
0&=-\sqrt{a^3(\alpha)}(J-1)-\frac{p}{\kappa}
\end{aligned}
$$

Multiplying this by test function q and integrating over volume, we obtain an equation that can be combined with the equations from the Gateaux Derivative.

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\mathbf{F}}\nabla\boldsymbol{v}\,d\Omega-\int_{\partial_N\Omega}\tilde{\boldsymbol{g}}_0\cdot\boldsymbol{v}\,dA=0 \tag{3.2a}$$

$$\int_\Omega\left(-\sqrt{a^3(\alpha)}(J-1)-\frac{p}{\kappa}\right)q dV=0 \tag{3.2b}$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\,d\Omega+\frac{\mathcal{G}_c}{c_w\ell}\int_\Omega\left[\frac{dw(\alpha)}{d\alpha}\beta+2\ell^2(\nabla\alpha\cdot\nabla\beta)\right]dV\geq 0 \tag{3.2c}$$

The strong form

$$\operatorname{Div}\mathbf{P}=0 \quad \text{in} \quad \Omega \tag{3.3a}$$

$$\mathbf{u}=\widetilde{\mathbf{u}}_0 \quad \text{in} \quad \partial_D\Omega \tag{3.3b}$$

$$[\mathbf{FS}]\,\boldsymbol{n}=\tilde{\boldsymbol{g}}_0 \quad \text{on} \quad \partial_N\Omega, \tag{3.3c}$$

where from Eq. 1.9 we can substitute Eq. 1.7

$$
\begin{aligned}
\mathbf{P}&=a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial\mathbf{F}}+a^3(\alpha)\kappa\,(J-1)\,\frac{\partial J}{\partial\mathbf{F}} \quad \text{where}\, p=-\sqrt{a^3(\alpha)}\kappa\,(J-1)\\
\mathbf{P}&=a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial\mathbf{F}}-p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial\mathbf{F}}
\end{aligned}
$$

and write the mechanical equilibrium equation in Eq. 3.3:

$$\operatorname{Div}\left[a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial\mathbf{F}} - p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial\mathbf{F}}\right] = 0 \tag{3.4}$$

Derivation of the KKT condition equations where $\nabla\beta\cdot\nabla\alpha = \nabla(\beta\nabla\alpha) - \beta\Delta\alpha$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\,d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega\left[\frac{dw(\alpha)}{d\alpha}\beta + 2\ell^2(\nabla\alpha\cdot\nabla\beta)\right]dV \geq 0$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\,d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta dV + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega 2\ell^2(\nabla\alpha\cdot\nabla\beta)\,dV \geq 0$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\,d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta dV + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega 2\ell^2(\nabla(\beta\nabla\alpha))\,dV - \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega 2\ell^2(\beta\Delta\alpha)\,dV \geq 0$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\,d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta dV - \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega 2\ell^2(\beta\Delta\alpha)\,dV \geq 0$$

$$\left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\,d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega\left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right)dV\right]\beta \geq 0$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\,d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega\left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right)dV \geq 0$$

Grouping terms, we obtain

$$\frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha} + \frac{\mathcal{G}_c}{c_w\ell}\left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right) \geq 0 \quad \text{in} \quad \Omega \tag{3.5a}$$

$$\dot{\alpha} \geq 0 \quad \text{in} \quad \Omega \tag{3.5b}$$

$$\dot{\alpha}\left[\frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha} + \frac{\mathcal{G}_c}{c_w\ell}\left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right)\right] \geq 0 \quad \text{in} \quad \Omega \tag{3.5c}$$

$$\tag{3.5d}$$

Lastly, we have the following, (Neumann?)

$$\alpha\mathbf{n} \geq 0 \quad \text{and} \quad \dot{\alpha}\alpha\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \tag{3.6}$$

Multiply Eq. 3.4 with weighting function $\boldsymbol{v} + (\varpi h^2)/(2\mu)\mathbf{F}^{-T}\nabla q$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \left[\boldsymbol{v} + \frac{\varpi h^2}{2\mu}\mathbf{F}^{-T}\nabla q\right] dV = 0$$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV + \int_\Omega \mathrm{Div}\mathbf{P} \cdot \left[\frac{\varpi h^2}{2\mu}\mathbf{F}^{-T}\nabla q\right] dV = 0$$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV + \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \mathrm{Div}\mathbf{P} \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV = 0$$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV$$

$$+\frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \mathrm{Div}\left[a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}}\right] \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \mathrm{Div}\left[p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial \mathbf{F}}\right] \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV = 0$$

$$"" + "" - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \nabla\left(p\sqrt{a^3(\alpha)}\right) J\mathbf{F}^{-T} \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV = 0$$

$$"" + "" - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \nabla\left(p\sqrt{a^3(\alpha)}\right) J\mathbf{F}^{-1}\mathbf{F}^{-T} \cdot \left(\mathbf{F}^{-1}\mathbf{F}^{-T}\nabla q\right)\, dV = 0$$

$$"" + "" - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \nabla\left(p\sqrt{a^3(\alpha)}\right) J \cdot \left(\mathbf{C}^{-1}\nabla q\right)\, dV = 0$$

$$"" + "" - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} J\mathbf{C}^{-1} : \left[\nabla\left(p\sqrt{a^3(\alpha)}\right) \cdot \nabla q\right] dV = 0$$

where $\mathbf{P} = a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} - p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial \mathbf{F}}$

We also want to deal with the first term where $(fg)' = f'g + fg'$

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV = \int_\Omega (\mathbf{F} \cdot \boldsymbol{v})_{,X}\, dV - \int_\Omega \mathbf{P} \cdot \boldsymbol{v}X\, dV$$

Leaving

$$\int_\Omega \mathrm{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV + \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \mathrm{Div}\left[a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}}\right] \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} J\mathbf{C}^{-1} : \left[\nabla\left(p\sqrt{a^3(\alpha)}\right) \cdot \nabla q\right] dV \tag{3.7}$$

$$\tag{3.8}$$