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A generalized inf–sup test for multi-field mixed-variational methods

A. Krischok, C. Linder*

Department of Civil and Environmental Engineering, Stanford University, Stanford, CA 94305, USA

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Abstract

We present a variational approach towards identifying conditions for the stability of Galerkin methods for multi-field saddle point problems in continuum mechanics and continuum thermodynamics with free energy functions that have positive second variations at critical points. The framework aims to generalize the discrete inf–sup theory and its numerical verification in the context of general problems in an arbitrary number of fields for both linear and nonlinear problems. In utilizing a linearized second derivative test the proposed scheme is purely based on uniqueness properties of a mixed Lagrangian around the solution, thus providing a purely variational requirement for the well-posedness of Galerkin approximations. In maintaining the dependence on parameters and in incorporating the notion of multifold inf–sup conditions the scheme provides a generalization of the classical LBB theory that allows the study of multi-field saddle point problems in the limit of vanishing parameters and their connection to generalized numerical eigenvalue tests. The proposed generalization is trusted to provide a helpful tool for the development of mixed methods that arise in many novel engineering problems due to the coupling of multiple irreversible phenomena or the incorporation of Lagrange multipliers in advanced methods. An emphasis is given with regard to time-dependent problems in irreversible thermodynamics that arise from Biot-type variational formulations if conjugate variables are employed by means of a Legendre–Fenchel transformation. Examples are given for a two-field variational principle in finite deformation poroelasticity in the undrained limit, a three-field variational principle for elasticity in the incompressible limit, and a recent four-field variational principle in gradient-extended plasticity.

Keywords: Inf–sup condition; Multi-field saddle point principles; Uniqueness of mixed problems

1. Introduction

Since the pioneering works of Ladyzhenskaya, Babuška, and Brezzi [1–3] it has been known that the identically named *LBB condition* is a requirement for the well-posedness of Galerkin methods that aim to approximate solutions of saddle point problems. In its most typical form the LBB condition, often referred to as *inf–sup condition*, appears in the form of an incompressibility constraint with relevance in the study of two-field problems such as Stokes flow or Hellinger–Reissner-type elasticity. In addition to the requirement that the problem is coercive upon application of appropriate boundary conditions, the satisfaction of the condition ensures the boundedness of *a priori estimates* in both fields and thus uniqueness of a solution associated with a Galerkin discretization. In terms of providing

* Correspondence to: Department of Civil and Environmental Engineering, Stanford University, Stanford, CA 94305, USA.
E-mail address: linder@stanford.edu (C. Linder).

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strategies for proving the latter [4] provides an extensive overview for a variety of problems. Over the decades a lot of element types have been proposed to satisfy this constraint for incompressible problems [5]. Expectedly, similar problems appeared for plate problems [6], electromagnetic problems [7], and problems in topology optimization [8], to name a few. More precisely, the satisfaction of an inf–sup condition will provide the continuity of an operator which couples the weak equations of a given problem in a monolithic scheme in the presence of operator kernels. Proving that a certain condition holds for a chosen element interpolation is usually not trivial and analytical proofs are quite often missing or only provided for undistorted meshes, even for a number of element types that are successfully employed in practice. An alternative to proving element formulations is the so-called *inf–sup test* [9,10] which tests in a general eigenvalue problem whether stability constants decrease towards zero for a chosen mesh with varying size. It should be emphasized that such a test is by no means able to prove stability in the strict mathematical sense as it would need to be conducted for an infinite number of meshes and boundary value problems to provide a strict proof. However, it provides a valuable tool that is widely used in the engineering community.

Various applications in engineering motivate researchers to develop highly coupled continuum models with multi-field saddle point structures. Recent applications include for example thermo-hydro-mechanical simulations [11], multi-field saddle point principles for gradient-extended plasticity [12,13], and improved finite element schemes for polyconvex material models [14] but the notion of multi-field saddle point principles goes back to a variety of well-known problems in finite element analysis such as Hu–Washizu principles [15] or enhanced strain methods in the context of pressure degrees of freedom [16–18]. In particular for these multi-field problems the identification is vital and a little more circuitous than in the standard two-field case. While the associated structure can be seen as a single saddle point problem on a product space, the development of corresponding conditions can be notoriously difficult to trace and establishing proper conditions on single operators is advised to ensure the proper treatment of compound operators [19,20]. These models often also bear the question how stability problems are to be treated in the context of nonlinear functional analysis, most prominently in hyperelasticity [21], as nonlinear functionals may only provide the positivity of an operator in a neighborhood around the solution.

The goal of this paper is to provide an *energy approach* for identifying crucial conditions for the stability of multi-field saddle point problems upon discretization which, to the best knowledge of the authors, has not yet been pointed out in the literature to the extent at which it is presented here. The expression energy approach is owed to the fact that we employ an *infinitesimal stability criterion* for a given saddle point functional of interest, noting that the supplement *infinitesimal* will be redundant in the context of linear analysis. The notion of infinitesimal stability that builds on an infinitesimal perturbation of a functional around a solution and the corresponding investigation of a Taylor expansion within a neighborhood of this solution is a general result in nonlinear functional analysis [22] that has been early established in the context of hyperelasticity (see Truesdell and Noll [23] chapters 68 and 89) and represents a suitable criterion that is widely used in the computational context to test the stability of hyperelastic materials with microstructural effects [24]. These ideas are generalized here in the context of typical saddle point problems in continuum mechanics and directly linked to numerical eigenvalue tests.

The proposed scheme is hoped to provide a clearer path between energy methods in continuum mechanics and continuum thermodynamics and the mathematical theory of mixed Galerkin methods. Clearly, variational formulations can serve not only to unify diverse fields but also to suggest new theories, and they provide a powerful means for studying the existence of solutions to partial differential equations [25]. Not least because the connection between energy principles and variational methods has been the one of the key reasons for the success of the finite element method, artificial energy principles have been constructed to exploit this advantage in the context of problems where the typical ansatz is the method of mean weighted residuals for problems with no underlying potential function, for example in the case of irreversible problems. Nevertheless, variational extremum principles have been successfully applied even to such problems since Onsager's observation that in the case of transport processes such as the conduction of heat, electricity, and diffusion, the rate of entropy, expressed by a dissipation function, follows an extremum principle [26,27] and it has been pointed out in an overwhelming number of cases that similar principles do exist, particularly in the context of thermomechanics. Most notably, a lot of variational formulations with applications to viscoelasticity, thermoelasticity, and poroelasticity [28–31] were established by Biot. Other works in the context of plasticity were made by Ziegler [32] and Halphen and Nguyen [33]. In the stricter context of numerical analysis notable contributions include variational formulations in elasto-viscoplasticity [34,35] and finite-strain plasticity [36]. More recently, principles were established for gradient-extended plasticity [37], poroelastic problems [38], Cahn–Hilliard-type diffusion [39] as well as combinations of principles such as Cahn–Hilliard-type diffusion in elastic bodies [40] or diffusion-induced fracture in silicon electrodes [41]. While not all variational principles depict saddle point problems, or even require a Galerkin discretization of all the fields, it is important to note that new problems may very well be expressed in this context, if only by changing arguments of thermodynamic potentials by Legendre transformations and, consequently, the question arises whether a variational principle is unique upon discretization.

As quite often researchers restrict themselves to counting the degrees of freedom, which only provides a necessary criterion for non-singularity but not a sufficient condition for stability [42,43], the presented framework is hoped to provide a roadmap, particularly for the engineering community, towards the identification of crucial stability conditions and corresponding numerical inf–sup tests that can verify the robustness of a chosen discretization scheme for a newly developed highly coupled multi-field model. For a non-standard problem or a multi-field problem it may not be trivial how conditions are to be evaluated [19] and to be linked to numerical tests which easily results in an ambiguity of the results [44,45].

We point out a few highlights of the proposed framework: The proposed principle results in a stability condition that maintains material parameters and hence provides the means to identify critical limits in which LBB-type conditions are to be satisfied to maintain the well-posedness of a Lagrangian and further allows to avoid ambiguous results in the context of numerical tests. Due to its general interpretation it creates a basis for understanding multi-field inf–sup conditions and their additive and multiplicative structure in the presence of Helmholtz free energy functions which, depending of course on the problem, can be cumbersome to identify by an investigation of a linear system which is compounded on product spaces and complicated in nonlinear analysis. As a result of linking LBB-type conditions with the help of an infinitesimal stability criterion, stability conditions appear naturally in quadratic form due to their incorporation of Hessian matrices such that, in contrast to previous inf–sup tests, the result of the test is identical to the condition that is to be tested.

The paper is organized as follows: Section 2 discusses the proposed framework and identifies a generalized stability condition for problems in continuum mechanics and continuum thermodynamics. In Section 3, a numerical inf–sup test for general problems is set in the context of the proposed framework. Finally, Section 4 discusses examples including a classical two-field variational formulation for finite deformation poroelasticity, a three-field variational formulation for elasticity in the incompressible limit, and a four-field variational formulation in gradient-extended plasticity with an embedded Hellinger–Reissner-type structure. The numerical test in the context of gradient-extended plasticity tests some elements investigated in [12] for which a mathematically more rigorous investigation has been lacking so far.

2. A generalized framework for multi-field saddle point functionals

Consider a generalized functional with saddle point structure Π in an arbitrary number of fields that describes the internal and external energetic response of a reference body $\mathcal{B}$ and that is associated with the search for the (possibly local) critical point

$$\{\mathfrak{u}^*, \mathfrak{v}^*\} = \operatorname{Arg}\left\{ \inf_{\mathfrak{u}\in\mathfrak{U}} \sup_{\mathfrak{v}\in\mathfrak{V}} \Pi(\mathfrak{u}, \mathfrak{v}) \right\} \ . \tag{1}$$

The lists of separate fields $\mathfrak{u} = (\mathfrak{u}^1, \mathfrak{u}^2, \ldots, \mathfrak{u}^M) \in \mathfrak{U}$ and fields $\mathfrak{v} = (\mathfrak{v}^1, \mathfrak{v}^2, \ldots, \mathfrak{v}^N) \in \mathfrak{V}$ are defined such that the solution $\{\mathfrak{u}^*, \mathfrak{v}^*\} \in \mathfrak{U} \oplus \mathfrak{V}$ is obtained through minimizing $\Pi(\mathfrak{u}, \mathfrak{v}) : \mathfrak{U} \oplus \mathfrak{V} \to \mathbb{R}$ among all possible $\mathfrak{u} \in \mathfrak{U}$ and through maximizing Π among all possible $\mathfrak{v} \in \mathfrak{V}$. We demand that $\mathfrak{u} = \bar{\mathfrak{u}}$ on $\partial\mathcal{B}_u$ and $\mathfrak{v} = \bar{\mathfrak{v}}$ on $\partial\mathcal{B}_v$ on parts of the boundary $\partial\mathcal{B}_u \cup \partial\mathcal{B}_v$ where Dirichlet-type boundary conditions are applied. The indexed set of Sobolev spaces $\mathfrak{U}^m(\mathcal{B})$ is each equipped with a derivative of order k_m and compactly supported in the bounded Lipschitz domain $\mathcal{B} \in \mathbb{R}^{n_{\dim}}$ with reference coordinates $\boldsymbol{X}$ such that the product space $\mathfrak{U}$ reads

$$\mathfrak{U} = \prod_{m=1}^{M} \mathfrak{U}^m(\mathcal{B}) \qquad \text{with} \qquad \mathfrak{U}^m(\mathcal{B}) = \left\{ \mathfrak{u}^m : \ D^\alpha \mathfrak{u}^m \in L_2\,(\mathcal{B})\ \forall\ \alpha \text{ such that } |\alpha| \ \leq k_m \right\} , \tag{2}$$

for the set of $n_{\dim}$ ordered tuples of non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots$ with associated norms

$$\|\mathfrak{u}\|_U^2 = \sum_{m=1}^{M} \left\|\mathfrak{u}^m\right\|_{U^m}^2 \qquad \text{with} \qquad \left\|\mathfrak{u}^m\right\|_{U^m}^2 = \sum_{|\alpha|\leq k_m} \int_{\mathcal{B}} |D^\alpha \mathfrak{u}^m|^2 dV < \infty \ . \tag{3}$$

The list of Sobolev spaces $\mathfrak{V}^n(\mathcal{B})$ of fields $\mathfrak{v} = (\mathfrak{v}^1, \mathfrak{v}^2, \ldots, \mathfrak{v}^N)$ with derivatives up to order k_n is defined in an analogous way and forms the product space $\mathfrak{V}$. If $\{\mathfrak{u}^*, \mathfrak{v}^*\}$ is the solution to problem (1) then the necessary conditions for the first directional derivatives are

$$D_u \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot \boldsymbol{u} = \frac{d}{d\epsilon}\left\{\Pi(\mathfrak{u}^* + \epsilon \boldsymbol{u}, \mathfrak{v}^*)\right\}\bigg|_{\epsilon=0} = 0 \qquad \forall\, \boldsymbol{u} \in U \ , \tag{4}$$

and

$$D_v \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot \boldsymbol{v} = \frac{d}{d\eta}\left\{\Pi(\mathfrak{u}^*, \mathfrak{v}^* + \eta \boldsymbol{v})\right\}\bigg|_{\eta=0} = 0 \qquad \forall\, \boldsymbol{v} \in V \ , \tag{5}$$

where the lists $\boldsymbol{u} = (u^1, u^2, \ldots, u^M) \in U$ and $\boldsymbol{v} = (v^1, v^2, \ldots, v^N) \in V$ contain elements of indexed sets of Sobolev spaces U^m and V^n such that their corresponding product spaces U and V are analogous to $\mathfrak{U}$ and $\mathfrak{V}$, respectively, but satisfy the homogeneous boundary conditions $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\mathcal{B}_u$ and $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\mathcal{B}_v$ on parts of the boundary $\partial\mathcal{B}_u \cup \partial\mathcal{B}_v$ where Dirichlet-type boundary conditions are applied. Note that $D_u\Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot \boldsymbol{u}$ and $D_v\Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot \boldsymbol{v}$ mean $D_u\Pi(\mathfrak{u}^*, \mathfrak{v}^*)$ and $D_v\Pi(\mathfrak{u}^*, \mathfrak{v}^*)$ applied to the directions $\boldsymbol{u} \in U$ and $\boldsymbol{v} \in V$, respectively. As Π is a function of multiple field variables, we can write $D_u\Pi(\mathfrak{u}^*, \mathfrak{v}^*) = \sum_{m=1}^{M} D_{u^m}\Pi(\mathfrak{u}^*, \mathfrak{v}^*)$ and $D_v\Pi(\mathfrak{u}^*, \mathfrak{v}^*) = \sum_{n=1}^{N} D_{v^n}\Pi(\mathfrak{u}^*, \mathfrak{v}^*)$ [46].

We focus here on two typical appearances of saddle point principles in continuum mechanics and combinations of the two. The first sort falls into the category of principles with Lagrange-multiplier structure with

$$\Pi(\mathfrak{u}, \mathfrak{v}) = \int_{\mathcal{B}} \psi(\mathfrak{u})\, dV - \int_{\mathcal{B}} \mathfrak{v} \cdot f(\mathfrak{u}, \mathfrak{v})\, dV - \Pi^{\text{ext}}(\mathfrak{u}, \mathfrak{v}) \ , \tag{6}$$

for a list of functions f to which the problem is subjected. The second sort are rate-type principles with dissipation functions that account for some sort of entropy production and may be expressed at a finite time increment $[t_n, t]$ with the current time t and the previous time t_n as

$$\Pi(\mathfrak{u}, \mathfrak{v}) = \int_{\mathcal{B}} \int_{t_n}^{t} \pi(\dot{\mathfrak{u}}, \mathfrak{v})\, dt\, dV - \int_{t_n}^{t} \Pi^{\text{ext}}(\dot{\mathfrak{u}}, \mathfrak{v})\, dt \ , \tag{7}$$

where $\Pi^{\text{ext}}(\dot{\mathfrak{u}}, \mathfrak{v})$ accounts for external contributions and $\pi(\dot{\mathfrak{u}}, \mathfrak{v})$ is an internal incremental potential per unit volume as a function of a free energy function ψ and a dissipation function ϕ such that

$$\pi(\dot{\mathfrak{u}}, \mathfrak{v}) = \partial_{\mathfrak{u}}\psi \cdot \dot{\mathfrak{u}} + \dot{\mathfrak{u}} \cdot \mathfrak{v} - \phi(\mathfrak{v}; \mathfrak{u}) \ , \tag{8}$$

where ϕ depends implicitly on the field $\mathfrak{u}$. In both cases, physically, the entries $\mathfrak{u}$ have the character of kinematic quantities while the entries $\mathfrak{v}$ are driving forces which may be understood as a list of thermodynamically conjugate fields that are dual to entries of the list $\mathfrak{u}$. These principles are generally obtained from a Legendre–Fenchel transformation of a Biot-type equation [29] where the local rate of entropy production, expressed through a dissipation function χ, is minimized. The two expressions for the dissipation functions can be linked via the Legendre–Fenchel transformation

$$\chi(\dot{\mathfrak{u}}; \mathfrak{u}) = \sup_{\mathfrak{v}}[\mathfrak{v} \cdot \dot{\mathfrak{u}} - \phi(\mathfrak{v}; \mathfrak{u})] \ , \tag{9}$$

to obtain a dual representation. The entries of the lists $\dot{\mathfrak{u}}$ and $\mathfrak{v}$ couple with corresponding fields to which they are dual. Typical examples include yield- and threshold functions in plasticity and viscoelasticity as well as diffusion laws of Darcy- and Fourier-type. Often, principles of this type are governed by a non-smooth representation of $\chi(\dot{\mathfrak{u}}; \mathfrak{u})$ which is difficult to handle and hard to implement. Further, depending on the structure of Π^{ext}, the variational principle corresponding to a certain field will be either local or global and (9) may have to be formulated as a *generalized Legendre transformation* [40].

The cases (6) and (7) depict scenarios where the stored free energy density only possesses second variations with respect to $\boldsymbol{u}$. This implies that

$$\Pi(\mathfrak{u}^* + \boldsymbol{u}, \mathfrak{v}^*) - \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \geq 0 \qquad \forall\, \boldsymbol{u} \in U \ . \tag{10}$$

The equality sign of this global condition only holds for some $\boldsymbol{u}$ [47] and plays a role in situations where material instabilities and structural instabilities are of concern which is only of interest in the computational tracking of post-critical states [48]. Especially from the computational point of view it is more suitable to restrict the analysis to $\delta \boldsymbol{u} = \epsilon \boldsymbol{u}$ for a positive factor ϵ that is small enough to act in the vicinity of the solution. We consider here, and henceforth, thermodynamic scenarios for which the stored free energy gives a stable response around the local solution, that is, $\Pi(\mathfrak{u}^* + \epsilon \boldsymbol{u}, \mathfrak{v}^*) - \Pi(\mathfrak{u}^*, \mathfrak{v}^*) > 0$ and the second variation in $\boldsymbol{u}$ satisfies

$$C_1 \, \|\boldsymbol{u}\|_U^2 \leq D^2_{uu} \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot (\boldsymbol{u}, \boldsymbol{u}) \leq C_2 \, \|\boldsymbol{u}\|_U^2 \ , \tag{11}$$

for all $\boldsymbol{u} \in U$ and some constants $C_1 > 0$ and $C_2 > 0$. This treatment can be seen as a generalized expression of the notion of infinitesimal static stability of the configuration of an elastic body under dead loading [23] for which (4) is a necessary condition. Hence, in all relevant cases, provided that Dirichlet boundary conditions are applied properly, we consider the properties

$$A \quad > 0 \quad \forall \ \boldsymbol{u} \in U \ , \tag{12}$$

$$B \begin{cases} = 0 & \forall \ \boldsymbol{u} \in \operatorname{Ker} B \ , \\ = 0 & \forall \ \boldsymbol{v} \in \operatorname{Ker} B^T \ , \\ \lesseqgtr 0 & \text{else} \ , \end{cases} \tag{13}$$

$$C \begin{cases} = 0 & \forall \ \boldsymbol{v} \in \operatorname{Ker} C \ , \\ < 0 & \forall \ \boldsymbol{v} \in \operatorname{Ker} C^{\perp} \ , \end{cases} \tag{14}$$

for second directional derivatives

$$A = D^2_{uu} \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot (\boldsymbol{u}, \boldsymbol{u}) = \frac{d^2}{d\epsilon^2} \{ \Pi(\mathfrak{u}^* + \epsilon \boldsymbol{u}, \mathfrak{v}^*) \} \Big|_{\epsilon=0} \ , \tag{15}$$

$$B = D^2_{uv} \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot (\boldsymbol{u}, \boldsymbol{v}) = \frac{d^2}{d\epsilon \ d\eta} \{ \Pi(\mathfrak{u}^* + \epsilon \boldsymbol{u}, \mathfrak{v}^* + \eta \boldsymbol{v}) \} \Big|_{\epsilon,\eta=0} \ , \tag{16}$$

$$C = D^2_{vv} \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot (\boldsymbol{v}, \boldsymbol{v}) = \frac{d^2}{d\eta^2} \{ \Pi(\mathfrak{u}^*, \mathfrak{v}^* + \eta \boldsymbol{v}) \} \Big|_{\eta=0} \ , \tag{17}$$

with their respective abbreviations A, B, and C which we introduce for ease of notation. Note that a second variation is only evaluated at the solution $\{\mathfrak{u}^*, \mathfrak{v}^*\}$ in the case of nonlinear analysis.

The second variation in $\boldsymbol{v}$ may disappear on parts of V in the limit of vanishing time-steps or parameters for the problem types (6) and (7), even if a unique solution $\{\mathfrak{u}^*, \mathfrak{v}^*\}$ exists. We therefore introduce

$$\operatorname{Ker} B = \left\{ \ \boldsymbol{u} \in U \mid D^2 \Pi_{uv}(\mathfrak{u}^*, \mathfrak{v}^*) \cdot (\boldsymbol{u}, \boldsymbol{v}) = 0 \quad \forall \ \boldsymbol{v} \in V \ \right\} \ , \tag{18}$$

$$\operatorname{Ker} B^T = \left\{ \ \boldsymbol{v} \in V \mid D^2 \Pi_{vu}(\mathfrak{u}^*, \mathfrak{v}^*) \cdot (\boldsymbol{v}, \boldsymbol{u}) = 0 \quad \forall \ \boldsymbol{u} \in U \ \right\} \ , \tag{19}$$

$$\operatorname{Ker} C = \left\{ \ \boldsymbol{v} \in V \mid D^2 \Pi_{vv}(\mathfrak{u}^*, \mathfrak{v}^*) \cdot (\boldsymbol{v}, \boldsymbol{v}) = 0 \quad \forall \ \boldsymbol{v} \in V \ \right\} \ , \tag{20}$$

and $\operatorname{Ker} C^{\perp}$ as the quotient space $V \setminus \operatorname{Ker} C$. While the definition of a kernel for variations $D^2_{vv} \Pi$ is intuitive, the definition of a kernel for mixed variations $D^2_{uv} \Pi$ is somewhat unsatisfactory and provides more meaningful results in the finite dimensional case. Evidently, if multiple fields $n = 1, \ldots, N$ are considered, the expression for C should be further decomposed into

$$C^n = D^2_{v^n v^n} \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot (v^n, v^n) \begin{cases} < 0 & \forall \ v^n \in \operatorname{Ker} C^{n\perp} \ , \\ = 0 & \forall \ v^n \in \operatorname{Ker} C^n \ , \end{cases} \tag{21}$$

for

$$\operatorname{Ker} C^n = \left\{ \ v^n \in V^n \mid C^n = 0 \quad \forall \ v^n \in V^n \ \right\} \ . \tag{22}$$

Note that for $o = 1, \ldots, N$ and $n \neq o$ we consider problems for which

$$D^2_{v^n v^o} \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot (v^n, v^o) = 0 \qquad \forall \ v^n, v^o \in V \ , \tag{23}$$

in the context of thermodynamically consistent dual principles that are considered here, as further discussed in Appendix C.

Consider the perturbation of the potential

$$\Delta \Pi = \Pi(\mathfrak{u}^* + \delta \boldsymbol{u}, \mathfrak{v}^* + \delta \boldsymbol{v}) - \Pi(\mathfrak{u}^*, \mathfrak{v}^*) \ , \tag{24}$$

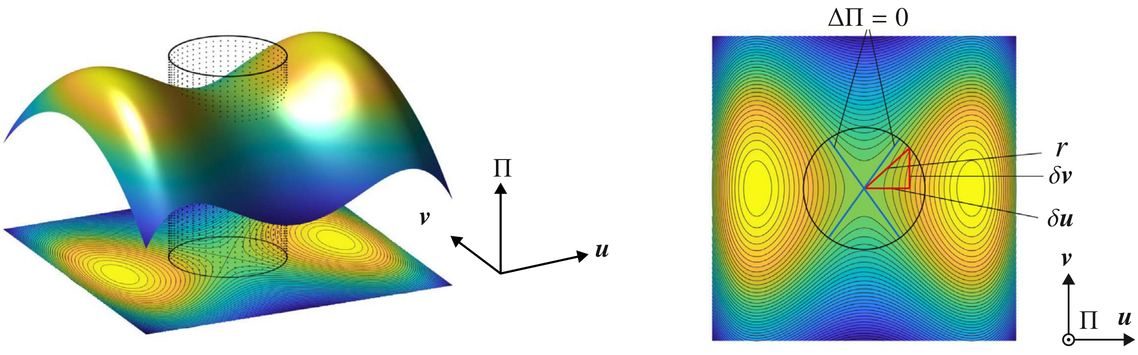

Fig. 1. Abstract depiction of a saddle point functional on $\mathfrak{U} \oplus \mathfrak{V}$ and the ball $B_r(\mathfrak{u}^*, \mathfrak{v}^*)$ around the solution.

which describes infinitesimal changes of the Lagrangian Π which can be approximated by a Taylor expansion. The solution of typical systems relies on search methods such as the Newton–Raphson scheme with Taylor formulas of order two. Clearly, $\{\delta \boldsymbol{u}, \delta \boldsymbol{v}\}$ has to be chosen sufficiently small to justify an appropriate second-order Taylor approximation to perturb the Lagrangian Π only in a neighborhood around the solution in nonlinear cases where a solution might only be locally unique, i.e. $\Delta\Pi$ should be infinitesimally small in accordance with considerations discussed above. To achieve this, we define the open ball $B_r(\mathfrak{u}^*, \mathfrak{v}^*)$ about the solution points $\mathfrak{u}^*$ and $\mathfrak{v}^*$ such that a perturbation $\Delta\Pi = \Pi(\mathfrak{u}^* + \delta \boldsymbol{u}, \mathfrak{v}^* + \delta \boldsymbol{v}) - \Pi(\mathfrak{u}^*, \mathfrak{v}^*)$ is infinitesimally small as the non-zero deflection $(\delta \boldsymbol{u}, \delta \boldsymbol{v}) = (\epsilon \boldsymbol{u}, \eta \boldsymbol{v})$ is constrained by small enough factors ϵ, η to lie inside an open ball $B_r(\mathfrak{u}^*, \mathfrak{v}^*)$ around $(\mathfrak{u}^*, \mathfrak{v}^*)$ defined by

$$B_r(\mathfrak{u}^*, \mathfrak{v}^*) = \left\{ (\delta \boldsymbol{u}, \delta \boldsymbol{v}) \in U \oplus V \;\middle|\; \sqrt{\epsilon^2 \, \|\boldsymbol{u}\|_U^2 + \eta^2 \, \|\boldsymbol{v}\|_V^2} < r \right\} , \tag{25}$$

with arbitrarily small radius $r \in \mathbb{R}^+$, assuming that Π is twice Fréchet-differentiable within the ball about $\{\mathfrak{u}^*, \mathfrak{v}^*\}$ and continuous at the solution. Note that we have so far not specified the properties of the derivatives with regard to their differentiability in the Fréchet-sense (F-derivative) or Gâteaux-sense (G-derivative). This is owed to the fact that we may assume that upon introduction of a small enough open neighborhood within the ball the maps A, B, and C will be linear with respect to the directions $\boldsymbol{u}$ and $\boldsymbol{v}$ in which they act and a G-derivative will thus be equivalent to an F-derivative [22,49]. Employing a Taylor expansion around the solution and exploiting the linearity in $\boldsymbol{u}$ and $\boldsymbol{v}$ gives the approximation

$$\Delta\Pi \approx \frac{\epsilon^2}{2} A + \frac{\eta^2}{2} C + \epsilon\eta \; B \; . \tag{26}$$

Based on what we have so far, an appropriate requirement for stability would be that for all admissible configurations of the form $(\mathfrak{u}^* + \delta \boldsymbol{u}, \mathfrak{v}^* + \delta \boldsymbol{v})$ the response will be positive for some perturbations and negative for other perturbations and for a chosen combination of directions $\boldsymbol{u}$ and $\boldsymbol{v}$, as depicted in Fig. 1, there are exactly two combinations of ϵ and η such that $\Delta\Pi = 0$ within the infinitesimal range defined by $B_r(\mathfrak{u}^*, \mathfrak{v}^*)$, that is, for an arbitrary direction $(\boldsymbol{u}, \boldsymbol{v})$

$$\Delta\Pi = 0 \qquad \exists^{!2} \epsilon, \forall \eta \;\; \text{with} \;\; (\delta \boldsymbol{u}, \delta \boldsymbol{v}) \in B_r(\mathfrak{u}^*, \mathfrak{v}^*) \; . \tag{27}$$

It should be remarked that the notion of infinitesimal stability in the context of pure minimizers includes neutral stability by allowing zero responses which does not suffice as premise for a uniqueness theorem [50]. The exclusion of neutral infinitesimal perturbations is sometimes referred to as infinitesimal superstability [23] and implies the desired uniqueness of a configuration. We expect this to be the case for the part that relates to the minimizers, as implied by Eq. (11), but will henceforth maintain the notion of infinitesimal stability as some neutral perturbations are clearly allowed and enforced in Eq. (27) as the means to make a statement about the uniqueness by demanding that an arbitrary deflection from the solution is associated with two zero perturbations. We will, for brevity, assume that the latter is satisfied in the continuous case and only compromised upon introduction of a corresponding discretization.

Let now $\mathfrak{U}_h$ and $\mathfrak{V}_h$ be closed subspaces of $\mathfrak{U}$ and $\mathfrak{V}$. Problem (1) can now be replaced by the *approximated problem*

$$\{\mathfrak{u}_h^*, \mathfrak{v}_h^*\} = \text{Arg}\left\{ \inf_{\mathfrak{u}_h \in \mathfrak{U}_h} \sup_{\mathfrak{v}_h \in \mathfrak{V}_h} \Pi(\mathfrak{u}_h, \mathfrak{v}_h) \right\} \ , \tag{28}$$

and the corresponding necessary conditions for stationarity

$$D_u \Pi(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \cdot \boldsymbol{u}_h = 0 \qquad \forall \ \boldsymbol{u}_h \in U_h \ , \tag{29}$$

and

$$D_v \Pi(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \cdot \boldsymbol{v}_h = 0 \qquad \forall \ \boldsymbol{v}_h \in V_h \ , \tag{30}$$

where U_h and V_h are the closed subspaces of U and V corresponding to the discretization of permissible variations. The perturbation (26) in discrete form now reads

$$\Delta\Pi_h \approx \frac{\epsilon^2}{2} A_h + \frac{\eta^2}{2} C_h + \epsilon\eta \ B_h \ . \tag{31}$$

The properties of the operator $A_h = D^2_{uu}\Pi(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \cdot (\boldsymbol{u}_h, \boldsymbol{u}_h)$ are exemplified here analogous to (11) as

$$C_1 \, \|\boldsymbol{u}_h\|_U^2 \le A_h \le C_2 \, \|\boldsymbol{u}_h\|_U^2 \ , \tag{32}$$

for all $\boldsymbol{u}_h \in U_h$ and some constants $C_1 > 0$ and $C_2 > 0$. Accordingly, the operators A_h, B_h, and C_h are defined as in (12)–(17) but in the context of the chosen discretization. Analogous to (27) we require that

$$\Delta\Pi_h = 0 \qquad \exists^{!2}\epsilon, \forall\eta \ \text{ with } \ (\delta\boldsymbol{u}_h, \delta\boldsymbol{v}_h) \in B_r(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \ . \tag{33}$$

So far, this description lacks a clear investigation of individual kernels that may exist on V_h^n and a distinction should be made. The statement can be decomposed using (23) to be of the form

$$\Delta\Pi_h \approx \frac{\epsilon^2}{2} A_h + \sum_{n=1}^{N} \left(\frac{\eta^2}{2} C_h^n + \epsilon\eta \ B_h^n \right) \ , \tag{34}$$

for operators $B_h^n = D^2_{uv^n}\Pi(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \cdot (\boldsymbol{u}_h, v_h^n)$ and $C_h^n = D^2_{v^n v^n}\Pi(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \cdot (v_h^n, v_h^n)$. Eq. (33) can alternatively be tested by the individual expression

$$\Delta\Pi_h^n = \frac{\epsilon^2}{2} A_h + \frac{\eta^2}{2} C_h^n + \epsilon\eta \ B_h^n \ , \tag{35}$$

if we identify $\delta v_h^n = \epsilon v_h^n$ with $(0, \dots, \delta v_h^n, \dots, 0)$ for which we want to satisfy

$$\Delta\Pi_h^n = 0 \qquad \exists^{!2}\epsilon, \forall\eta \ \text{ with } \ (\delta\boldsymbol{u}_h, \delta v_h^n) \in B_r(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \ , \tag{36}$$

which we want to be true in the sense that exactly two zero perturbations associated with the specific choice δv_h^n exist. To test this, we introduce a Hessian matrix $\boldsymbol{H}_h^n$ with

$$\boldsymbol{H}_h^n = \begin{bmatrix} A_h & B_h^n \\ B_h^n & C_h^n \end{bmatrix} \ , \tag{37}$$

that is to be tested to guarantee that (36) holds. This is the case if the formula

$$\beta^n = \inf_{v_h^n \in V_h^n} \sup_{\boldsymbol{u}_h \in U_h} \left\{ \frac{-\det \boldsymbol{H}_h^n}{\|\boldsymbol{u}_h\|_U^2 \ \|v_h^n\|_{V^n}^2} \right\} \ , \tag{38}$$

is satisfied which can be recast into

$$\beta^n = \inf_{v_h^n \in V_h^n} \sup_{\boldsymbol{u}_h \in U_h} \frac{(B_h^n)^2 - A_h C_h^n}{\|\boldsymbol{u}_h\|_U^2 \ \|v_h^n\|_{V^n}^2} = \underbrace{\inf_{v_h^n \in V_h^n} \sup_{\boldsymbol{u}_h \in U_h} \frac{(B_h^n)^2}{\|\boldsymbol{u}_h\|_U^2 \ \|v_h^n\|_{V^n}^2}}_{\alpha^n} \ + \ \mathbb{R}_{\ge 0} \ . \tag{39}$$

Proposition 2.1. *Let A_h satisfy the properties in (32). If*

$$\prod_{n=1}^{N} \beta^n = \beta > 0 \ , \tag{40}$$

holds with β^n given in (38), then we can find exactly two zero perturbations for the discrete perturbation $\Delta\Pi_h$ in (31) for arbitrary non-zero deflections $(\delta\boldsymbol{u}_h, \delta\boldsymbol{v}_h)$, that is, (33) holds.

Proof. The perturbation (35) is a second order polynomial in both ϵ and η. The quadratic formula yields for fixed $\eta > 0$ the solutions with

$$\epsilon_{1/2} = -\eta \frac{B_h^n}{A_h} \pm \eta \frac{\sqrt{(B_h^n)^2 - A_h C_h^n}}{A_h} \ . \tag{41}$$

To obtain exactly two solutions for arbitrary choices of $\boldsymbol{u}_h \in U_h$ and $v_h^n \in V_h^n$ we demand $(B_h^n)^2 - A_h C_h^n > 0$ in all cases and thus

$$(v_h^n \in \operatorname{Ker} C_h^n \wedge \boldsymbol{u}_h \in \operatorname{Ker} B_h^{n\perp}) \vee (v_h^n \in \operatorname{Ker} C_h^{n\perp}) \ , \tag{42}$$

has to be satisfied such that $\Delta\Pi_h^n = 0$ holds for exactly two ϵ. The first case demands that there is a positive constant C_b independent of the mesh such that

$$\left\| v_h^n \right\|_{V^n}^2 C_b \leq \inf_{\boldsymbol{u}_h \in \operatorname{Ker} B_h^{n\perp}} \frac{(B_h^n)^2}{\|\boldsymbol{u}_h\|_U^2} = \sup_{\boldsymbol{u}_h \in U_h} \frac{(B_h^n)^2}{\|\boldsymbol{u}_h\|_U^2} \qquad \forall\ v_h^n \in \operatorname{Ker} C_h^n \ . \tag{43}$$

The equality is admittedly not intuitive and will be shown in the next section for an example in matrix form. The second case demands that there is a positive constant C_{ac} independent of the mesh such that

$$\left\| v_h^n \right\|_{V^n}^2 C_{ac} \leq \inf_{\boldsymbol{u}_h \in U_h} \frac{-A_h C_h^n}{\|\boldsymbol{u}_h\|_U^2} \leq \sup_{\boldsymbol{u}_h \in U_h} \frac{-A_h C_h^n}{\|\boldsymbol{u}_h\|_U^2} \qquad \forall\ v_h^n \in \operatorname{Ker} C_h^{n\perp} \ , \tag{44}$$

which exploits (32), that is, the response of an infinitesimal perturbation in $\boldsymbol{u}_h$ is assumed to be bounded from above and below. Combination of both cases into a single statement demands that there is a real constant C_{abc} such that

$$\left\| v_h^n \right\|_{V^n}^2 C_{abc} \leq \sup_{\boldsymbol{u}_h \in U_h} \frac{(B_h^n)^2 - A_h C_h^n}{\|\boldsymbol{u}_h\|_U^2} \qquad \forall\ v_h^n \in V_h^n \ , \tag{45}$$

or alternatively

$$0 < \beta^n = \inf_{v_h^n \in V_h^n} \sup_{\boldsymbol{u}_h \in U_h} \frac{(B_h^n)^2 - A_h C_h^n}{\|\boldsymbol{u}_h\|_U^2 \left\| v_h^n \right\|_{V^n}^2} = \inf_{v_h^n \in V_h^n} \sup_{\boldsymbol{u}_h \in U_h} \frac{-\det \boldsymbol{H}_h^n}{\|\boldsymbol{u}_h\|_U^2 \left\| v_h^n \right\|_{V^n}^2} \ . \tag{46}$$

To guarantee that $\Delta\Pi_h$ given in (31) has two zero perturbations for general choices $\delta\boldsymbol{v}_h$, instead of individual δv_h^n, it is sufficient to show that $\beta^n > 0$ for all $n = 1, \ldots, N$ or

$$\prod_{n=1}^{N} \beta^n = \beta > 0 \ . \quad \square \tag{47}$$

The main result of this section is summarized in Theorem 2.2.

Theorem 2.2. *Let $(\mathfrak{u}_h^*, \mathfrak{v}_h^*)$ be the solution to problem (28) and let A_h satisfy the properties in (32). If there exists a constant β with*

$$\beta = \prod_{n=1}^{N} \beta^n > 0, \qquad \textit{(I)}$$

independent of the mesh and β^n given in (38) then, for arbitrary deflections $(\delta\boldsymbol{u}_h, \delta\boldsymbol{v}_h)$ inside a small enough neighborhood around the solution $(\mathfrak{u}_h^*, \mathfrak{v}_h^*)$, we can find exactly two zero perturbations

$$\Delta\Pi_h = 0 \qquad \exists^{!2}\epsilon, \forall\eta \ \ \textit{with} \ \ (\delta\boldsymbol{u}_h, \delta\boldsymbol{v}_h) \in B_r(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \ , \qquad \textit{(II)}$$

and, in addition, the a priori estimate

$$\left\| \mathfrak{u}^* - \mathfrak{u}_h^* \right\|_U + \left\| \mathfrak{v}^* - \mathfrak{v}_h^* \right\|_V \leq C \left(\inf_{\boldsymbol{u}_h \in U_h} \left\| \mathfrak{u}^* - \boldsymbol{u}_h \right\|_U + \inf_{\boldsymbol{v}_h \in V_h} \left\| \mathfrak{v}^* - \boldsymbol{v}_h \right\|_V \right) , \qquad (III)$$

will be satisfied with

$$\inf_{\boldsymbol{u}_h \in U_h} \left\| \mathfrak{u}^* - \boldsymbol{u}_h \right\|_U \to 0 \quad \textit{and} \quad \inf_{\boldsymbol{v}_h \in V_h} \left\| \mathfrak{v}^* - \boldsymbol{v}_h \right\|_V \to 0 \quad \textit{as} \;\; h \to 0 \, .$$

(*II*) follows from Proposition 2.1 and the proof of (*III*) is attached in Appendix A and uses standard arguments as employed in [3,4], however in the context of nonlinear problems with operators inside an open ball around the solution.

Theorem 2.2 constitutes the central theorem of this chapter and demands through (I) and the evaluation of corresponding Hessian matrices $\boldsymbol{H}_h^n$ that the problem is always well-posed in the sense that an infinitesimal perturbation is unique such that (II) is satisfied, and of course (III) which guarantees that the error is bounded and the solution is unique within its neighborhood.

In comparison to the classical treatment of mixed methods, some differences should be outlined. First of all, the proposed framework is purely defined by a perturbation of a multi-field saddle point functional. This has the following implications and benefits.

The system of equations is symmetric and incorporates limit cases where no LBB-type conditions have to be satisfied. To illustrate this, we reevaluate the definition of β_h^n in (39) with

$$\beta^n = \inf_{v_h^n \in V_h^n} \sup_{\boldsymbol{u}_h \in U_h} \frac{(B_h^n)^2 - A_h C_h^n}{\|\boldsymbol{u}_h\|_U^2 \; \|v_h^n\|_{V^n}^2} = \underbrace{\inf_{v_h^n \in V_h^n} \sup_{\boldsymbol{u}_h \in U_h} \frac{(B_h^n)^2}{\|\boldsymbol{u}_h\|_U^2 \; \|v_h^n\|_{V^n}^2}}_{\alpha^n} + \mathbb{R}_{\geq 0} \, , \tag{48}$$

from which it is clear that α^n is a condition purely on the discretization itself and only crucial if a nontrivial $\operatorname{Ker} C_h^n$ exists. The latter has implications for the successful evaluation of corresponding tests which may not be trivial [51].

The definition of α^n is in its meaning similar to problems that satisfy conditions under the hypothesis of the usual Babuška–Brezzi theory [2,3]. However, the operator acts in the presence of individual fields $n = 1, \ldots, N$. The notion of this treatment is akin to the recent investigation of two-fold saddle point principles in [19] for which individual conditions are to be considered if a treatment as a standard saddle point problem is cumbersome.

Note in this context that $B_h^n = D^2_{uv^n} \Pi(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \cdot (\boldsymbol{u}_h, v_h^n)$ is applied to the list $\boldsymbol{u}_h = (u_h^1, \ldots, u_h^N)$ of variations related to the minimization problem and can be decomposed such that

$$B_h^n = \sum_{m=1}^{M} D^2_{u^m v^n} \Pi(\mathfrak{u}_h^*, \mathfrak{v}_h^*) \cdot (u_h^m, v_h^n). \tag{49}$$

The structure of those fields that are part of the free energy function is hence additive in this context. This means that the operators B_h^n couples with all mixed variations for the entire list $\boldsymbol{u}$.

As a final remark it should be noted that the statement appears naturally in quadratic form due to its motivation by means of a Taylor expansion that is linked to the infinitesimal stability criterion and the corresponding notion of Hessian matrices. This has the implication that α^n is a quadratic counterpart to classical LBB-type conditions. In the context of the generalized inf–sup test, which is discussed in the next section, this involves the simplification that the result of the corresponding general eigenvalue problem is identical to the value itself.

3. Generalized inf–sup test

To evaluate whether α^n is bounded for a certain discretization, we employ a generalized eigenvalue test that follows several key ideas and key steps proposed in [9,10,52]. While the testing procedure is similar, we want to emphasize that the test computes the previously introduced α^n. In [9,10] the square-root of the smallest eigenvalue is taken to give a lower bound for the inf–sup value. In contrast, α^n appears naturally in quadratic form due to its energetic nature. Consequently, the smallest eigenvalue of interest is identical to the desired stability condition and taking the square root will be redundant.

Proposition 3.1. *Let $\mathbb{M}_h \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_u}$ and $\mathbb{N}_h^n \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_v}$ be symmetric positive definite matrices that correspond to the norms of elements in U_h and V_h^n, i.e.*

$$\|\boldsymbol{u}_h\|_U^2 = \sum_{m\in M} \|u_h^m\|_{U^m}^2 = \sum_{m\in M}\sum_{\alpha\le k_m} \left\{ \int_{\mathcal{B}} |D^\alpha u_h^m|^2 dV \right\} = \mathbb{U}_h^T \mathbb{M}_h \mathbb{U}_h \ , \tag{50}$$

and

$$\left\| v_h^n \right\|_{V^n}^2 = \mathbb{V}_h^T \mathbb{N}_h^n \mathbb{V}_h \ , \tag{51}$$

where $\mathbb{U}_h$ and $\mathbb{V}_h$ correspond to vectors of nodal degrees of freedom with dimensions n_u and n_v, respectively. Further, let $\mathbb{K}_h^n \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_v}$ be a matrix such that

$$\mathbb{U}_h^T \mathbb{K}_h^n \mathbb{V}_h = D^2_{uv^n} \Pi_h(\mathbf{u}_h^*, \mathfrak{v}_h^*) \bullet (\boldsymbol{u}_h, v_h^n) \ . \tag{52}$$

The discrete problem corresponding to the generalized condition

$$\inf_{\mathbb{V}_h} \sup_{\mathbb{U}_h} \left\{ \frac{(\mathbb{U}_h^T \mathbb{K}_h^n \mathbb{V}_h)^2}{\mathbb{U}_h^T \mathbb{M}_h \mathbb{U}_h \ \mathbb{V}_h^T \mathbb{N}_h^n \mathbb{V}_h} \right\} = \alpha^n \ , \tag{53}$$

is identical to the smallest eigenvalue $(\lambda_p)_{p=1,n_v}$ of the general eigenvalue problem

$$\mathbb{K}_h^{nT} \mathbb{M}_h^{-1} \mathbb{K}_h^n \Phi_h = \lambda_p \mathbb{N}_h^n \Phi_h \ . \tag{54}$$

Proof. Let $\mathbb{C}_h$ and $\mathbb{L}_h$ be the Cholesky decomposition of $\mathbb{M}_h$ and $\mathbb{N}_h^n$, respectively, that is, $\mathbb{M}_h = \mathbb{C}_h \mathbb{C}_h^T$ and $\mathbb{N}_h^n = \mathbb{L}_h \mathbb{L}_h{}^T$. Rewriting the statement using $\mathbb{C}_h^T \mathbb{U}_h = \mathbb{P}_h$ and $\mathbb{L}_h^T \mathbb{V}_h = \mathbb{Q}_h$ yields

$$\inf_{\mathbb{Q}_h} \sup_{\mathbb{P}_h} \left\{ \frac{(\mathbb{P}_h^T \mathbb{C}_h^{-1} \mathbb{K}_h^n \mathbb{L}_h^{-T} \mathbb{Q}_h)^2}{\mathbb{P}_h^T \mathbb{P}_h \ \mathbb{Q}_h^T \mathbb{Q}_h} \right\} = \alpha^n \ . \tag{55}$$

The singular value decomposition $\mathbb{R}_h \Sigma_h \mathbb{S}_h^T = \mathbb{C}_h^{-1} \mathbb{K}_h^n \mathbb{L}_h^{-T}$ with orthogonal matrices $\mathbb{R}_h$ and $\mathbb{S}_h$ motivates the expression

$$\inf_{\mathbb{Y}_h} \sup_{\mathbb{X}_h} \left\{ \frac{(\mathbb{X}_h^T \Sigma_h \mathbb{Y}_h)^2}{\mathbb{X}_h^T \mathbb{X}_h \ \mathbb{Y}_h^T \mathbb{Y}_h} \right\} = \alpha^n \ , \tag{56}$$

which is further simplified using $\mathbb{P}_h^T \mathbb{R}_h = \mathbb{X}_h^T$ and $\mathbb{S}_h^T \mathbb{Q}_h = \mathbb{Y}_h$ and exploiting the properties $\mathbb{R}_h^T \mathbb{R}_h = \mathbb{1}$ and $\mathbb{S}_h^T \mathbb{S}_h = \mathbb{1}$. If we fix a random $\mathbb{Y}_h$ it is not difficult to see that the sup is obtained when $\mathbb{X}_h = \Sigma_h \mathbb{Y}_h$ such that

$$\sup_{\mathbb{X}_h} \left\{ \frac{(\mathbb{X}_h^T \Sigma_h \mathbb{Y}_h)^2}{\mathbb{X}_h^T \mathbb{X}_h \ \mathbb{Y}_h^T \mathbb{Y}_h} \right\} = \frac{(\mathbb{Y}_h^T \Sigma_h^T \Sigma_h \ \mathbb{Y}_h)^2}{\mathbb{Y}_h^T \Sigma_h^T \Sigma_h \mathbb{Y}_h \ \mathbb{Y}_h^T \mathbb{Y}_h} = \frac{\mathbb{Y}_h^T \Sigma_h^T \Sigma_h \mathbb{Y}_h}{\mathbb{Y}_h^T \mathbb{Y}_h} \ . \tag{57}$$

Application of the inf yields

$$\inf_{\mathbb{Y}_h} \left\{ \frac{\mathbb{Y}_h^T \Sigma_h^T \Sigma_h \ \mathbb{Y}_h}{\mathbb{Y}_h^T \mathbb{Y}_h} \right\} = \inf_{\mathbb{Y}_h} \left\{ \frac{\sum_{i=1}^{n_v} \sigma_i^2 y_i^2}{\sum_{i=1}^{n_v} y_i^2} \right\} = \sigma_{\min}^2 \ , \tag{58}$$

where σ_i are the singular values of Σ_h and y_i is the ith entry of the vector $\mathbb{Y}_h$. $\sigma_{\min}$ denotes the smallest singular value of Σ_h. α^n will hence be equivalent to the smallest eigenvalue of $\Sigma_h^T \Sigma_h = \mathbb{S}_h^T \mathbb{L}_h^{-1} \mathbb{K}_h^{nT} \mathbb{M}_h^{-1} \mathbb{K}_h^n \mathbb{L}_h^{-T} \mathbb{S}_h$ or $\mathbb{L}_h^{-1} \mathbb{K}_h^{nT} \mathbb{M}_h^{-1} \mathbb{K}_h^n \mathbb{L}_h^{-T}$ according to the spectral theorem. Thus, the smallest eigenvalue of the general eigenvalue problem

$$\mathbb{K}_h^{nT} \mathbb{M}_h^{-1} \mathbb{K}_h^n \ \Phi_h = \lambda_p \mathbb{N}_h^n \ \Phi_h \ , \tag{59}$$

is identical to α^n. Note that, so far, we did not account for elements in $\operatorname{Ker} \mathbb{K}^{nT}$ which, depending in the problem, may exist if certain spurious modes are present for which no interaction exists between the fields. While spurious modes should be avoided, they affect the question of solvability and may appear for certain meshes and boundary conditions and do not relate to the question of stability in the context of the convergence behavior of an element. The test should be conducted for an undistorted and a distorted mesh [9,10]. □

Remark 1. Here we would like to pick up on the step in (43). One might be inclined to apply the more intuitive condition

$$\inf_{\mathbb{V}_h} \inf_{\mathbb{U}_h \setminus \operatorname{Ker} \mathbb{K}_h^n} \left\{ \frac{(\mathbb{U}_h^T \mathbb{K}_h^n \mathbb{V}_h)^2}{\mathbb{U}_h^T \mathbb{M}_h \mathbb{U}_h \ \mathbb{V}_h^T \mathbb{N}_h^n \mathbb{V}_h} \right\} = \alpha^n \ , \tag{60}$$

or after singular value decomposition

$$\inf_{\mathbb{Y}_h} \inf_{\mathbb{X}_h \setminus \operatorname{Ker} \Sigma_h} \left\{ \frac{(\mathbb{X}_h^T \Sigma_h \mathbb{Y}_h)^2}{\mathbb{X}_h^T \mathbb{X}_h \ \mathbb{Y}_h^T \mathbb{Y}_h} \right\} = \alpha^n \ , \tag{61}$$

where $\operatorname{Ker} \mathbb{K}_h^n$ and $\operatorname{Ker} \Sigma_h$ are the kernels of $\mathbb{K}_h^n$ and Σ_h in $\mathbb{U}_h$ and $\mathbb{X}_h$, respectively. They correspond to the previously introduced $\operatorname{Ker} B_h^n$. Knowing that $n_u \geq n_v$ arises as a necessary condition for solvability [42,43], we consider the matrix $\Sigma_h \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_v}$

$$\Sigma_h = \begin{bmatrix} \sigma_1 & 0 & \cdot & 0 \\ 0 & \sigma_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \sigma_{n_v} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{62}$$

Clearly, both $\operatorname{Ker} \mathbb{K}_h^n$ and $\operatorname{Ker} \Sigma_h$ span a $(n_u - n_v)$-dimensional vector-space. Then

$$\begin{aligned} \alpha^n &= \inf_{\mathbb{V}_h} \inf_{\mathbb{U}_h \setminus \operatorname{Ker} \mathbb{K}_h^n} \left\{ \frac{(\mathbb{U}_h^T \mathbb{K}_h^n \mathbb{V}_h)^2}{\mathbb{U}_h^T \mathbb{M}_h \mathbb{U}_h \ \mathbb{V}_h^T \mathbb{N}_h^n \mathbb{V}_h} \right\} \\ &= \inf_{\mathbb{Y}_h} \inf_{\mathbb{X}_h \setminus \operatorname{Ker} \Sigma_h} \left\{ \frac{(\mathbb{X}_h^T \Sigma_h \mathbb{Y}_h)^2}{\mathbb{X}_h^T \mathbb{X}_h \ \mathbb{Y}_h^T \mathbb{Y}_h} \right\} \\ &= \inf_{\mathbb{Y}_h} \inf_{\mathbb{X}_h \setminus \operatorname{Ker} \Sigma_h} \left\{ \frac{(\sum_i^{n_v} x_i \sigma_i y_i)^2}{\sum_i^{n_u} x_i^2 \sum_j^{n_v} y_j^2} \right\} = \sigma_{\min}^2 \ , \end{aligned} \tag{63}$$

which implies the equivalence

$$\inf_{\mathbb{V}_h} \inf_{\mathbb{U}_h \setminus \operatorname{Ker}(\mathbb{K}_h)} \left\{ \frac{(\mathbb{U}_h^T \mathbb{K}_h^n \mathbb{V}_h)^2}{\mathbb{U}_h^T \mathbb{M}_h \mathbb{U}_h \ \mathbb{V}_h^T \mathbb{N}_h^n \mathbb{V}_h} \right\} \equiv \inf_{\mathbb{V}_h} \sup_{\mathbb{U}_h} \left\{ \frac{(\mathbb{U}_h^T \mathbb{K}_h^n \mathbb{V}_h)^2}{\mathbb{U}_h^T \mathbb{M}_h \mathbb{U}_h \ \mathbb{V}_h^T \mathbb{N}_h^n \mathbb{V}_h} \right\} \ , \tag{64}$$

that we used in (43).

4. Examples

4.1. Two-field Biot-type poroelasticity

The response of a porous solid matrix, fully saturated with a solvent that gives rise to a pore pressure, may be expressed by a time-discrete potential function at time $[t_n, t]$

$$\Pi^{\Delta}(\boldsymbol{\varphi}, \mathfrak{p}, \theta) = \int_{\mathcal{B}} \pi^{\Delta}(\nabla \boldsymbol{\varphi}, \theta, p, \mathbb{P}) \, dV - \Pi^{\text{ext}}(\boldsymbol{\varphi}, p) \ , \tag{65}$$

as a function of the deformation map $\boldsymbol{\varphi} = \boldsymbol{X} + \mathfrak{u}$ with a displacement field $\mathfrak{u}$, the fluid volume ratio θ, and the pore pressure $\mathfrak{p}$ where $\boldsymbol{F} = \nabla \boldsymbol{\varphi}$ is the deformation gradient with $J = \det \boldsymbol{F}$ and $\mathbb{P} = -\nabla \mathfrak{p}$ the negative material gradient of the pressure such that the incremental potential density π^{Δ} reads

$$\pi^{\Delta}(\nabla \boldsymbol{\varphi}, \theta, p, \mathbb{P}) = \psi(\boldsymbol{F}, \theta) - \mathfrak{p}(\theta - \theta_{t_n}) - \Delta t \ \phi(\mathbb{P}; \boldsymbol{F}_{t_n}, \theta_{t_n}) \ , \tag{66}$$

where we neglect the evaluation of the free energy ψ at the last time-step as it will vanish in the variational context. The expression can be seen as a time-discrete counterpart to a rate-type expression $\pi(\nabla \dot{\boldsymbol{\varphi}}, \dot{\theta}, \mathfrak{p}, \mathbb{P})$ in the interval $[t_n, t]$. Mixed-variational principles have been applied to problems in poroelasticity in [53] or more recently in [38]. The external contribution $\Pi^{\text{ext}}(\boldsymbol{\varphi}, \mathfrak{p})$ accounts for the external mechanical work on parts of the boundary $\partial \mathcal{B}_T$ due to prescribed tractions $\bar{\boldsymbol{T}}$ as well as solvent out-flux $\bar{H}$ normal to the boundary $\partial \mathcal{B}_H$ such that

$$\Pi^{\text{ext}}(\boldsymbol{\varphi}, p) = \int_{\partial \mathcal{B}_T} \bar{\boldsymbol{T}} \cdot \boldsymbol{\varphi} \, dV - \int_{\partial \mathcal{B}_H} \bar{H} \mathfrak{p} \, dV \ . \tag{67}$$

The stored free energy is decomposed into an elastic part and a fluid part such that $\psi(\boldsymbol{F},\theta)=\psi_{\mathrm{el}}(\boldsymbol{F})+\psi_{\mathrm{fl}}(\boldsymbol{F},\theta)$. While the elastic energy may be chosen for a given problem the fluid part takes the form

$$\psi_{\mathrm{fl}}(\boldsymbol{F},\theta)=\frac{M}{2}\left\{B^2(J-1)^2-2B\theta(J-1)+\theta^2\right\} \ , \tag{68}$$

where B is Biot's coefficient and M is Biot's modulus. The dissipation function

$$\phi(\mathbb{P};\boldsymbol{F}_n)=\frac{J_n\ K}{2\ \rho^f\ g}\ \|\boldsymbol{F}_n^{-T}\mathbb{P}\|^2 \ , \tag{69}$$

accounts for the redistribution of solvent due to gradients of the pressure. Here K is the (isotropic) hydraulic conductivity and ρ^f is the fluid density. g refers to the gravitational acceleration but gravitational effects are neglected here for the redistribution of solvent. To maintain variational consistency, the push-forward is evaluated at the previous time step, as implied by the subscript $(\cdot)_n$.

The solution of the overall problem is governed by the principle

$$\{\mathfrak{u}^*,\mathfrak{p}^*,\theta^*\}=\operatorname{Arg}\left\{\inf_{\substack{\mathfrak{u}\in\mathfrak{U}\\ \theta\in T}}\sup_{\mathfrak{p}\in\mathfrak{P}}\Pi^{\Delta}(\mathfrak{u},\mathfrak{p},\theta)\right\} \ , \tag{70}$$

with $\theta\in L^2(\mathcal{B})$ and the associated spaces for displacement and pressure field

$$\mathfrak{U}=\left\{\mathfrak{u}\in[H^1(\mathcal{B})]^{n_{\dim}}\ |\ \mathfrak{u}=\bar{\mathfrak{u}}\text{ on }\partial\mathcal{B}_u\right\} \ , \tag{71}$$

$$\mathfrak{P}=\left\{\mathfrak{p}\in H^1(\mathcal{B})\ |\ \mathfrak{p}=\bar{\mathfrak{p}}\text{ on }\partial\mathcal{B}_p\right\} \ , \tag{72}$$

considering the boundaries $\partial\mathcal{B}_u$ and $\partial\mathcal{B}_p$ where displacement and pressure boundary conditions are imposed, respectively. Due to the local structure of θ one may condense the overall functional, that is,

$$\pi^{\Delta}_{\mathrm{red}}(\nabla\boldsymbol{\varphi},\mathfrak{p},\mathbb{P})=\inf_{\theta}\pi^{\Delta}(\nabla\boldsymbol{\varphi},\theta,\mathfrak{p},\mathbb{P}) \quad \text{and} \quad \Pi^{\Delta}_{\mathrm{red}}(\boldsymbol{\varphi},\mathfrak{p})=\int_{\mathcal{B}}\pi^{\Delta}_{\mathrm{red}}(\nabla\boldsymbol{\varphi},\mathfrak{p},\mathbb{P})dV-\Pi^{\mathrm{ext}}(\boldsymbol{\varphi},p) \ . \tag{73}$$

The local principle

$$\{\theta^*\}=\operatorname{Arg}\left\{\inf_{\theta}\pi^{\Delta}(\nabla\boldsymbol{\varphi},\theta,\mathfrak{p},\mathbb{P})\right\} \ , \tag{74}$$

yielding

$$\partial_\theta\psi-\mathfrak{p}=0 \ , \tag{75}$$

is equivalent to the well-known Biot equation

$$\theta=\frac{\mathfrak{p}}{M}+B(J-1) \ . \tag{76}$$

The solution of the reduced problem follows as

$$\{\mathfrak{u}^*,\mathfrak{p}^*\}=\operatorname{Arg}\left\{\inf_{\mathfrak{u}\in\mathfrak{U}}\sup_{\mathfrak{p}\in\mathfrak{P}}\Pi^{\Delta}_{\mathrm{red}}(\mathfrak{u},\mathfrak{p})\right\} \ . \tag{77}$$

For the stability condition of interest consider the discrete counterpart

$$\{\mathfrak{u}_h^*,\mathfrak{p}_h^*\}=\operatorname{Arg}\left\{\inf_{\mathfrak{u}_h\in\mathfrak{U}_h}\sup_{\mathfrak{p}_h\in\mathfrak{P}_h}\Pi_h^{\Delta}(\mathfrak{u}_h,\mathfrak{p}_h)\right\} \ , \tag{78}$$

for which, based on the proposed framework, the corresponding Hessian matrix of the discrete problem follows as

$$\boldsymbol{H}_h^p=\begin{bmatrix} D^2_{uu}\Pi_h^{\Delta}(\mathfrak{u}_h^*)\cdot(\boldsymbol{u}_h,\boldsymbol{u}_h) & D^2_{up}\Pi_h^{\Delta}(\mathfrak{u}_h^*)\cdot(\boldsymbol{u}_h,p_h)\\ D^2_{up}\Pi_h^{\Delta}(\mathfrak{u}_h^*)\cdot(\boldsymbol{u}_h,p_h) & D^2_{pp}\Pi_h^{\Delta}(p_h,p_h)\end{bmatrix} \ , \tag{79}$$

for $\boldsymbol{u}_h\in U_h\subset U$ and $p_h\in P_h\subset P$ with

$$\mathfrak{U}_h=\left\{\mathfrak{u}_h\in[H^1(\mathcal{B})]^{n_{\dim}}\ |\ \mathfrak{u}_h=\bar{\mathfrak{u}}_h\text{ on }\partial\mathcal{B}_u\right\} \ , \tag{80}$$

$$\mathfrak{P}_h=\left\{\mathfrak{p}_h\in H^1(\mathcal{B})\ |\ \mathfrak{p}_h=\bar{\mathfrak{p}}_h\text{ on }\partial\mathcal{B}_p\right\} \ , \tag{81}$$

$$U_h=\left\{\boldsymbol{u}_h\in[H^1(\mathcal{B})]^{n_{\dim}}\ |\ \boldsymbol{u}_h=0\text{ on }\partial\mathcal{B}_u\right\} \ , \tag{82}$$

$$P_h = \left\{ p_h \in H^1(\mathcal{B}) \mid p_h = 0 \text{ on } \partial\mathcal{B}_p \right\} \ , \tag{83}$$

and the properties

$$C_{u1} \, \|\boldsymbol{u}_h\|^2_{H^1} \leq D^2_{uu} \Pi^{\Delta}_h(\mathbf{u}^*_h) \cdot (\boldsymbol{u}_h, \boldsymbol{u}_h) \leq C_{u2} \, \|\boldsymbol{u}_h\|^2_{H^1} \qquad \forall \ \boldsymbol{u}_h \in U_h \ , \tag{84}$$

$$D^2_{up} \Pi^{\Delta}_h(\mathbf{u}^*_h) \cdot (\boldsymbol{u}_h, p_h) = -B \int_{\mathcal{B}} p_h \ D_u J(\mathbf{u}^*_h) \cdot \boldsymbol{u}_h \ dV \ , \tag{85}$$

$$D^2_{pp} \Pi^{\Delta}_h(p_h, p_h) = -\frac{1}{M} \, \|p_h\|^2_{L^2} - \Delta t C_K \, \|p_h\|^2_1 \ , \tag{86}$$

for positive real constants $C_{u1} \leq C_u \leq C_{u2}$ and a real constant C_K which depends on the hydraulic conductivity K and, as the latter is evaluated in the deformed configuration, the bounded deformation gradient such that C_K is real and bounded and only vanishes if $K = 0$. The evaluation of the Hessian matrix leads to the general formula

$$\beta^p = \inf_{p_h \in P_h} \sup_{\boldsymbol{u}_h \in U_h} \frac{(\int_{\mathcal{B}} B p_h D_u J(\mathbf{u}^*_h) \cdot \boldsymbol{u}_h dV)^2 + (\frac{1}{M} \, \|p_h\|^2_{L^2} + \Delta t C_K \, \|p_h\|^2_1) C_u \, \|\boldsymbol{u}_h\|^2_{H^1}}{\|\boldsymbol{u}_h\|^2_{H^1} \, \|p_h\|^2_{H^1}} \ . \tag{87}$$

This problem is bounded for $\Delta t C_K$ sufficiently large. For $\Delta t C_K \to 0$ the problem is set in L^2, that is,

$$\begin{aligned} \beta^p &= \inf_{p_h \in P_h} \sup_{\boldsymbol{u}_h \in \boldsymbol{U}_h} \frac{(\int_{\mathcal{B}} B p D_u J(\mathbf{u}^*_h) \cdot \boldsymbol{u}_h dV)^2 + (\frac{1}{M} \, \|p_h\|^2_{L^2}) C_u \, \|\boldsymbol{u}_h\|^2_{H^1}}{\|\boldsymbol{u}_h\|^2_{H^1} \, \|p_h\|^2_{L^2}} \\ &= \underbrace{\inf_{p_h \in P_h} \sup_{\boldsymbol{u}_h \in \boldsymbol{U}_h} \frac{(\int_{\mathcal{B}} B p D_u J(\boldsymbol{u}^*_h) \cdot \boldsymbol{u}_h dV)^2}{\|\boldsymbol{u}_h\|^2_{H^1} \, \|p_h\|^2_{L^2}}}_{\alpha^p} + \frac{C_u}{M} \ , \end{aligned} \tag{88}$$

and α^p will need to be bounded in the limit $M \to \infty$. The evaluation of the test follows from

$$\mathbb{U}^T_h \mathbb{M}_h \mathbb{U}_h = \int_{\mathcal{B}} (\boldsymbol{u}_h \cdot \boldsymbol{u}_h + \nabla \boldsymbol{u}_h \cdot \nabla \boldsymbol{u}_h) \ dV = \|\boldsymbol{u}_h\|^2_{H^1} \ , \tag{89}$$

$$\mathbb{V}^T_h \mathbb{N}^p_h \mathbb{V}_h = \int_{\mathcal{B}} p^2_h \ dV = \|p_h\|^2_{L^2} \ , \tag{90}$$

$$\mathbb{U}^T_h \mathbb{K}^p_h \mathbb{V}_h = -B \int_{\mathcal{B}} p_h D_u J(\mathbf{u}^*_h) \cdot \boldsymbol{u}_h \ dV \ . \tag{91}$$

Fig. 2 shows the contour plots of a soil deposit loaded by a rigid foundation for Q2/Q1 and Q1/Q1 elements in the limit of $\Delta t K \to 0$ and $M \to \infty$. α^p decreases towards zero upon mesh refinement if linear interpolations are chosen for both fields while α^p is bounded from below if quadratic interpolations are chosen for the displacement field and linear interpolations for the pressure, as expected [54].

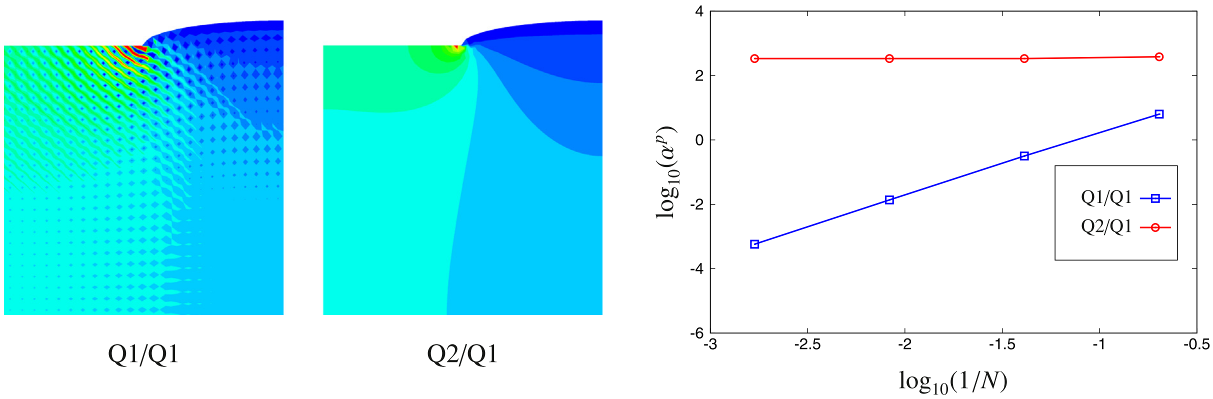

Fig. 2. Contour plots of the pore pressure in a soil deposit loaded by a rigid foundation for Q2/Q1 and Q1/Q1 elements (left) and inf–sup test for different formulations using $N \times N$ finite elements (right). Only the Q2/Q1 element passes the test as α^p does not decrease towards zero upon mesh refinement.

4.2. Three-field variational principle for elasticity

The purpose of this section is to apply the introduced framework to an example that falls into the category of the problems that we discussed above. A three-field principle of this sort, involving a volume dilation θ in addition to displacements $\mathfrak{u}$ and the pressure p, was proposed by [15,55] with the Lagrangian

$$\Pi(\mathfrak{u},\theta,p)=\int_{\mathfrak{B}}\left\{\kappa\ U(\theta)+\bar{W}(\mathfrak{u})+p[J(\mathfrak{u})-\theta]\right\}\ dV-\Pi_{\text{ext}}(\mathfrak{u})\ . \tag{92}$$

Here, U only accounts for volumetric contributions to the stored free energy and $\bar{W}$ accounts only for deviatoric deformations depending on $J^{-2/3}\boldsymbol{C}$ where $J=\det\boldsymbol{F}$ is the determinant of the deformation gradient $\boldsymbol{F}=\boldsymbol{I}+\nabla\mathfrak{u}$ and $\boldsymbol{C}=\boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy–Green tensor. p is the hydrostatic pressure which enforces the constraint $J(\mathfrak{u})=\theta$ and κ is the bulk modulus. The problem is governed by the approximated principle

$$\{\mathfrak{u}_h^*,\theta_h^*,p_h^*\}=\text{Arg}\left\{\inf_{\boldsymbol{u}_h,\theta_h}\sup_{p_h}\Pi_h(\boldsymbol{u}_h,\theta_h,p_h)\right\}\ , \tag{93}$$

corresponding to the weak equations

$$D_u\Pi_h(\mathfrak{u}_h^*)\cdot\boldsymbol{u}_h=\int_{\mathcal{B}}\left\{D_u\bar{W}(\mathfrak{u}_h^*)\cdot\boldsymbol{u}_h+p_hD_uJ(\mathfrak{u}_h^*)\cdot\boldsymbol{u}_h\right\}dV=0\ , \tag{94}$$

$$D_\theta\Pi_h(\theta_h^*)\cdot\theta_h=\int_{\mathcal{B}}\left\{\kappa D_\theta U(\theta_h^*)\cdot\theta_h-p_h\theta_h\right\}dV=0\ , \tag{95}$$

$$D_p\Pi_h(\mathfrak{u}_h^*)\cdot p_h=\int_{\mathcal{B}}\{p_h(J(\boldsymbol{u}_h)-\theta_h)\}\,dV=0\ , \tag{96}$$

for all $p_h\in L^2(\mathcal{B})$, $\theta_h\in L^2(\mathcal{B})$, and $\boldsymbol{u}_h\in U_h\subset U$ where the spaces for trial and test function for the displacements are defined such that

$$\mathfrak{U}=\left\{\mathfrak{u}\in[H^1(\mathcal{B})]^{n_{\dim}}\mid\mathfrak{u}=\bar{\mathfrak{u}}\text{ on }\partial\mathcal{B}_u\right\}\ , \tag{97}$$

$$U=\left\{\boldsymbol{u}\in[H^1(\mathcal{B})]^{n_{\dim}}\mid\boldsymbol{u}=0\text{ on }\partial\mathcal{B}_u\right\}\ , \tag{98}$$

with a subpart of the boundary $\partial\mathcal{B}_u$ where Dirichlet boundary conditions are imposed. The evaluation follows via the Hessian matrix

$$\boldsymbol{H}_h^p=\begin{bmatrix}D^2_{uu}\Pi_h+D^2_{\theta\theta}\Pi_h & D^2_{up}\Pi_h+D^2_{\theta p}\Pi_h\\ D^2_{up}\Pi_h+D^2_{\theta p}\Pi_h & 0\end{bmatrix}\ , \tag{99}$$

with

$$C_{u1}\,\|\boldsymbol{u}_h\|^2_{H^1}\le D^2_{uu}\Pi_h(\mathfrak{u}_h^*)\cdot(\boldsymbol{u}_h,\boldsymbol{u}_h)\le C_{u2}\,\|\boldsymbol{u}_h\|^2_{H^1}\qquad\forall\ \boldsymbol{u}_h\in U_h\ , \tag{100}$$

$$C_{t1}\,\|\theta_h\|^2_{L^2}\le D^2_{\theta\theta}\Pi_h(\theta_h^*)\cdot(\theta_h,\theta_h)\le C_{t2}\,\|\theta_h\|^2_{L^2}\qquad\forall\ \theta_h\in L^2(\mathcal{B})\ , \tag{101}$$

$$D_{up}\Pi_h(\mathfrak{u}_h^*)\cdot(\boldsymbol{u}_h,p_h)=\int_{\mathcal{B}}p_hD_uJ(\mathfrak{u}_h^*)\cdot\boldsymbol{u}_h\ dV\ , \tag{102}$$

$$D_{\theta p}\Pi_h(\theta_h,p_h)=-\int_{\mathcal{B}}p_h\theta_h\ dV\ , \tag{103}$$

$$D^2_{pp}\Pi_h(p_h,p_h)=0\ , \tag{104}$$

from which the condition, for example derived in [56], follows based on the framework in quadratic form as

$$\alpha^p=\beta^p=\inf_{p_h\in L^2}\sup_{\boldsymbol{u}_h\in U_h}\sup_{\theta_h\in L^2}\frac{\left(\int_{\mathcal{B}}(p_hD_uJ(\mathfrak{u}_h^*)\cdot\boldsymbol{u}_h-\theta_hp_h)\ dV\right)^2}{(\|\theta_h\|^2_{L^2}+\|\boldsymbol{u}_h\|^2_{H^1})\,\|p_h\|^2_{L^2}}\ . \tag{105}$$

This formulation is typically employed for the widely used extended Q1P0 element with incompatible and constant pressure and dilation modes for which it is well known that the condition above is violated.

4.3. Four-field gradient-extended plasticity

We investigate a three-field mixed-variational principle for von Mises plasticity at small deformations with gradient-extended hardening, proposed in [12], and the corresponding mixed finite element design for which a more rigorous mathematical analysis has been lacking so far. To avoid locking one may extend the principle by an additional pressure degree of freedom to a four-field principle.

The starting point is the discrete potential function

$$\Pi^{\Delta}(\mathfrak{u}, \alpha, f, \boldsymbol{\varepsilon}^p, \bar{\boldsymbol{s}}, p) = \int_{\mathcal{B}} \left\{ \pi^{\Delta}(\bar{\boldsymbol{\varepsilon}}, \alpha, f, \boldsymbol{\varepsilon}^p, \bar{\boldsymbol{s}}) + pe - \frac{1}{2\kappa} p^2 \right\} dV - \Pi^{\text{ext}}(\mathfrak{u}) \ , \tag{106}$$

as a function of the displacement field $\mathfrak{u}$, the hardening/softening variable α, the plastic driving force f, the plastic strain field $\boldsymbol{\varepsilon}^p$, which is dual to the isochoric stress $\bar{\boldsymbol{s}}$, and the hydrostatic pressure p. $\boldsymbol{\varepsilon} = \nabla^s \mathfrak{u} = \text{sym}[\nabla \mathfrak{u}]$ denotes the strain tensor with $e = \text{tr}[\nabla^s \mathfrak{u}]$ and $\bar{\boldsymbol{\varepsilon}} = \text{dev}[\nabla^s \mathfrak{u}]$. κ is the bulk modulus which controls to which extent a volumetric deformation is possible in the elastic range. Above we used the internal incremental potential density

$$\pi^{\Delta}(\bar{\boldsymbol{\varepsilon}}, \alpha, f, \boldsymbol{\varepsilon}^p, \bar{\boldsymbol{s}}) = \bar{\psi}(\bar{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}^p, \alpha, \nabla\alpha) + \bar{\boldsymbol{s}} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}^p_n) - f(\alpha - \alpha_n) - \Delta t \phi(\bar{\boldsymbol{s}}, f) \ , \tag{107}$$

which is a time-discrete counterpart to a rate-type potential density $\pi(\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}, f, \dot{\boldsymbol{\varepsilon}}^p, \bar{\boldsymbol{s}})$. $\bar{\psi}$ denotes the free energy density for which the contribution due to the elastic deformation accounts only for deviatoric contributions with

$$\bar{\psi}(\bar{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}^p, \alpha, \nabla\alpha) = \mu|\bar{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}^p|^2 + \frac{h}{2}\alpha^2 + \frac{\mu L_e^2}{2}|\nabla\alpha|^2 \ , \tag{108}$$

as a function of the shear modulus μ, the hardening constant h, and the energetic length scale L_e. $\bar{\psi}$ neglects the evaluation at the last time step as this becomes redundant in the variational context. The volumetric contribution to the elastic energy density is enforced by the hydrostatic pressure p using a perturbed Lagrange multiplier method. Further, ϕ denotes the dual dissipation function

$$\phi(\bar{\boldsymbol{s}}, f) = \frac{1}{2\eta} \left\langle |\bar{\boldsymbol{s}}| - \sqrt{2/3}[y_0 + f] \right\rangle^2 \ , \tag{109}$$

where $\langle x \rangle = (x + |x|)/2$ is the Macaulay bracket, y_0 the critical yield stress, and η the viscosity of the plastic over-stress. Due to the local structure of $\boldsymbol{\varepsilon}^p$ and $\bar{\boldsymbol{s}}$ one may condense the overall functional, that is,

$$\pi^{\Delta}_{\text{red}}(\bar{\boldsymbol{\varepsilon}}, \alpha, f) = \inf_{\boldsymbol{\varepsilon}^p} \sup_{\bar{\boldsymbol{s}}} \ \pi^{\Delta}(\bar{\boldsymbol{\varepsilon}}, \alpha, f, \boldsymbol{\varepsilon}^p, \bar{\boldsymbol{s}}) \ , \tag{110}$$

such that

$$\Pi^{\Delta}_{\text{red}}(\bar{\boldsymbol{\varepsilon}}, \alpha, f, p) = \int_{\mathcal{B}} \left\{ \pi^{\Delta}_{\text{red}}(\bar{\boldsymbol{\varepsilon}}, \alpha, f) + pe - \frac{1}{2\kappa} p^2 \right\} dV - \Pi^{\text{ext}}(\mathfrak{u}) \ . \tag{111}$$

The equilibrium at the unique point $\{\mathfrak{u}^*, \alpha^*, f^*, p^*\}$ is obtained among all displacements $\mathfrak{u} \in [H^1(\mathcal{B})]^{\text{dim}}$ such that $\mathfrak{u} = \bar{\mathfrak{u}}$ on $\partial\mathcal{B}_u$, hardening/softening variables $\alpha \in H^1(\mathcal{B})$, energetic driving forces $f \in L^2(\mathcal{B})$, and hydrostatic pressures $p \in L^2(\mathcal{B})$. After introducing the subspaces of the corresponding solution spaces $\mathfrak{U}_h \subset \mathfrak{U}$, $A_h \subset A$, $F_h \subset F$, and $P_h \subset P$ the solution follows from the principle

$$\{\mathfrak{u}^*_h, \alpha^*_h, f^*_h, p^*_h\} = \text{Arg}\left\{ \inf_{\substack{\mathfrak{u}_h \in \mathfrak{U}_h \\ \alpha_h \in A_h}} \ \sup_{\substack{p_h \in P_h \\ f_h \in F_h}} \ \Pi^{\Delta}_h(\boldsymbol{u}_h, \alpha_h, f_h, p_h) \right\} \ , \tag{112}$$

corresponding to the equations

$$D_u \Pi^{\Delta}_h(\mathfrak{u}^*_h, p^*_h, f^*_h) \cdot \boldsymbol{u}_h = \int_{\mathcal{B}} (\bar{\boldsymbol{s}}(\mathfrak{u}^*_h, f^*_h) + p^*_h \boldsymbol{I}) : \boldsymbol{\varepsilon}(\boldsymbol{u}_h) \, dV - D_u \Pi^{\text{ext}}_h(\boldsymbol{u}_h) = 0 \ , \tag{113}$$

$$D_\alpha \Pi^{\Delta}_h(\alpha^*_h, f^*_h) \cdot \alpha_h = \int_{\mathcal{B}} \alpha_h(h\alpha^*_h - f^*_h) \, dV + \int_{\mathcal{B}} \nabla\alpha_h \cdot (\mu L_e^2 \nabla\alpha^*_h) \, dV = 0 \ , \tag{114}$$

$$D_f \Pi^{\Delta}_h(\mathfrak{u}^*_h, \alpha^*_h, f^*_h) \cdot f_h = -\int_{\mathcal{B}} f_h \Big(-\Delta\alpha^*_h + \sqrt{2/3}\gamma(\mathfrak{u}^*_h, f^*_h) \, dV \Big) = 0 \ , \tag{115}$$

$$D_p \Pi^{\Delta}_h(\mathfrak{u}^*_h, p^*_h) \cdot p_h = \int_{\mathcal{B}} \Big(e(\mathfrak{u}^*_h) - \frac{p^*_h}{\kappa} \Big) p_h \, dV = 0 \ , \tag{116}$$

for all $\boldsymbol{u}_h \in [H^1(\mathcal{B})]^{\dim}$ such that $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\mathcal{B}_u$, $\alpha_h \in H^1(\mathcal{B})$, $f_h \in L^2(\mathcal{B})$, and $p_h \in L^2(\mathcal{B})$. For a time-step $\Delta t = t - t_n$ the update of the hardening/softening parameter $\Delta\alpha_h^* = \alpha_h^* - \alpha_{h,n}^*$ is governed by the incremental plastic parameter γ which is obtained from

$$\gamma = \begin{cases} \gamma^{\text{trial}} = \dfrac{1}{2\mu + \frac{\eta}{\Delta t}}(|\bar{\boldsymbol{s}}^{\text{trial}}| - \sqrt{\dfrac{2}{3}}(y_0 - f)) & \chi > 0\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{117}$$

where $\bar{\boldsymbol{s}}^{\text{trial}} = 2\mu(\bar{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_n^p)$ and χ is the yield function with $\chi = |\bar{\boldsymbol{s}}^{\text{trial}}| - 2\mu\gamma - \sqrt{\frac{2}{3}}[y_0 + f]$.

The second derivative test can be evaluated using the matrices

$$\boldsymbol{H}^p = \begin{bmatrix} D^2_{uu}\Pi_h^\Delta + D^2_{\alpha\alpha}\Pi_h^\Delta & D^2_{up}\Pi_h^\Delta \\ D^2_{up}\Pi_h^\Delta & D^2_{pp}\Pi_h^\Delta \end{bmatrix}\ , \tag{118}$$

$$\boldsymbol{H}^f = \begin{bmatrix} D^2_{uu}\Pi_h^\Delta + D^2_{\alpha\alpha}\Pi_h^\Delta & D^2_{uf}\Pi_h^\Delta + D^2_{\alpha f}\Pi_h^\Delta \\ D^2_{uf}\Pi_h^\Delta + D^2_{\alpha f}\Pi_h^\Delta & D^2_{ff}\Pi_h^\Delta \end{bmatrix}\ , \tag{119}$$

with second variations

$$C_{u1}\,\|\boldsymbol{u}_h\|^2_{H^1} \le D^2_{uu}\Pi_h^\Delta(\mathbf{u}_h^*)\cdot(\boldsymbol{u}_h, \boldsymbol{u}_h) \le C_{u2}\,\|\boldsymbol{u}_h\|^2_{H^1}\ , \tag{120}$$

$$D^2_{\alpha\alpha}\Pi_h^\Delta(\alpha_h, \alpha_h) = h\,\|\alpha_h\|^2_{L^2} + \mu L_e^2\,\|\alpha_h\|^2_1\ , \tag{121}$$

$$D^2_{ff}\Pi_h^\Delta(f_h, f_h) = -\frac{2/3}{2\mu + \eta/\Delta t}\int_{\mathcal{B}} s\ f_h^2\ dV = -C_f\,\|f_h\|^2_{L^2}\ , \tag{122}$$

$$D^2_{pp}\Pi_h^\Delta(p_h, p_h) = -\frac{1}{\kappa}\,\|p_h\|^2_{L^2}\ , \tag{123}$$

where $D^2_{uu}\Pi_h^\Delta$ is bounded by positive constants C_{u1} and C_{u2} for all $\boldsymbol{u}_h \in U_h$. s is one for $\chi > 0$ and zero otherwise and $C_f = 0$ if $s = 0$ at all Gauss points. Note that we consider positive hardening here to allow for the positivity of the stored energy density with respect to all its arguments. The mixed variations are

$$D^2_{up}\Pi_h^\Delta(\mathbf{u}_h^*)\cdot(\boldsymbol{u}_h, p_h) = \int_{\mathcal{B}} p_h D_u e(\mathbf{u}_h^*)\cdot\boldsymbol{u}_h\ dV\ , \tag{124}$$

$$D^2_{\alpha f}\Pi_h^\Delta(\alpha_h, f_h) = -\int_{\mathcal{B}} \alpha_h f_h\ dV\ , \tag{125}$$

$$D^2_{fu}\Pi_h^\Delta(\mathbf{u}_h^*, f_h)\cdot(f_h, \boldsymbol{u}_h) = -\int_{\mathcal{B}} f_h s\sqrt{2/3}D_u\gamma(\mathbf{u}_h^*, f_h)\cdot\boldsymbol{u}_h\ dV\ . \tag{126}$$

The evaluation of the test follows in the limit of a purely elastic update at all Gauss points with $s = 0$ as well as the incompressible limit with $\kappa \to \infty$ in which we have to satisfy

$$\begin{aligned} \alpha^f &= \left(\inf_{f_h\in F_h}\ \sup_{\alpha_h\in A_h}\ \sup_{\boldsymbol{u}_h\in U_h} \frac{(-\int_{\mathcal{B}}\alpha_h f_h\ dV)^2}{(\|\boldsymbol{u}_h\|^2_{H^1} + \|\alpha_h\|^2_{H^1})\,\|f_h\|^2_{L^2}}\right) \\ &= \left(\inf_{f_h\in F_h}\ \sup_{\alpha_h\in A_h} \frac{(-\int_{\mathcal{B}}\alpha_h f_h\ dV)^2}{\|\alpha_h\|^2_{H^1}\,\|f_h\|^2_{L^2}}\right)\ , \end{aligned} \tag{127}$$

and

$$\begin{aligned} \alpha^p &= \left(\inf_{p_h\in P_h}\ \sup_{\alpha_h\in A_h}\ \sup_{\boldsymbol{u}_h\in U_h} \frac{(\int_{\mathcal{B}} p_h D_u e(\mathbf{u}_h^*)\cdot\boldsymbol{u}_h\ dV)^2}{(\|\boldsymbol{u}_h\|^2_{H^1} + \|\alpha_h\|^2_{H^1})\,\|p_h\|^2_{L^2}}\right) \\ &= \left(\inf_{p_h\in P_h}\ \sup_{\boldsymbol{u}_h\in U_h} \frac{(\int_{\mathcal{B}} p_h D_u e(\mathbf{u}_h^*)\cdot\boldsymbol{u}_h\ dV)^2}{\|\boldsymbol{u}_h\|^2_{H^1}\,\|p_h\|^2_{L^2}}\right)\ . \end{aligned} \tag{128}$$

The matrices for the numerical test follow from

$$\mathbb{U}_h^T\mathbb{M}_h^u\mathbb{U}_h = \int_{\mathcal{B}}(\boldsymbol{u}_h\cdot\boldsymbol{u}_h + \nabla\boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h)\ dV = \|\boldsymbol{u}_h\|^2_{H^1}\ , \tag{129}$$

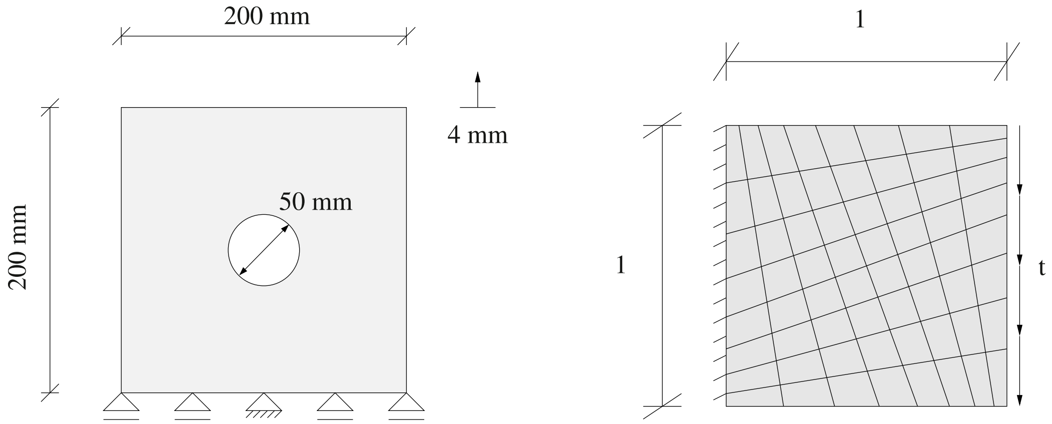

Fig. 3. Geometry and boundary conditions of the model problems: A plate with a hole (left) and a quadrilateral with a non-regular mesh for the inf–sup test shown for $N = 8$ elements per side (right).

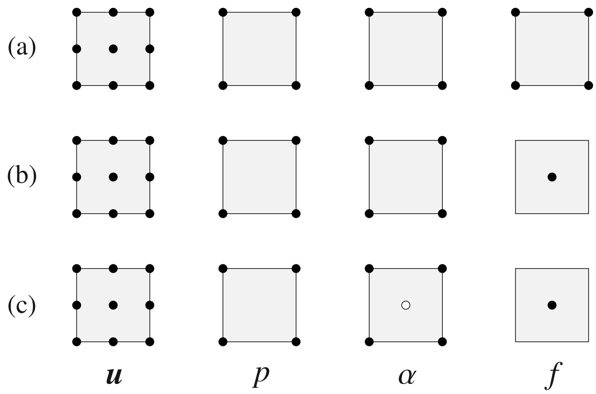

Fig. 4. Mixed finite element design for the four-field problem in gradient-extended plasticity in the fields $\{\boldsymbol{u}, p, \alpha, f\}$. Interpolations are investigated for the following elements: (a) Q2Q1–Q1Q1 with bilinear α and f; (b) Q2Q1–Q1Q0 with bilinear α and constant f; and (c) Q2Q1–Q1$_B$Q0 with a bubble mode that can be condensed out at the element level.

$$\mathbb{U}_h^T \mathbb{M}_h^a \mathbb{U}_h = \int_{\mathcal{B}} (\alpha_h \cdot \alpha_h + \nabla\alpha_h \cdot \nabla\alpha_h)\, dV = \|\alpha_h\|^2_{H^1} \ , \tag{130}$$

$$\mathbb{V}_h^T \mathbb{N}_h^p \mathbb{V}_h = \int_{\mathcal{B}} p_h^2 \, dV = \|p_h\|^2_{L^2} \ , \tag{131}$$

$$\mathbb{V}_h^T \mathbb{N}_h^f \mathbb{V}_h = \int_{\mathcal{B}} f_h^2 \, dV = \|f_h\|^2_{L^2} \ , \tag{132}$$

$$\mathbb{U}_h^T \mathbb{K}_h^p \mathbb{V}_h = \int_{\mathcal{B}} p_h D_u e(\mathbf{u}_h^*) \bullet \boldsymbol{u}_h \, dV \ , \tag{133}$$

$$\mathbb{U}_h^T \mathbb{K}_h^f \mathbb{V}_h = -\int_{\mathcal{B}} \alpha_h f_h \, dV \ . \tag{134}$$

As a model problem we study the plastic deformation of a plate with a circular hole which is pulled vertically within 0.5 s, as shown in Fig. 3, for the different elements Q2Q1–Q1$_B$Q0, Q2Q1–Q1Q0, and Q2Q1–Q1Q1 depicted in Fig. 4 where the subscript B refers to a MINI-type finite element design where the hardening variable α is interpolated with an additional bubble which can be condensed out at the element level. The study is extended to linear choices for the interpolation of the displacements, that is, including the three elements Q1Q1–Q1$_B$Q0, Q1Q1–Q1Q0, and Q1Q1–Q1Q1. For the inf–sup test a quadrilateral with unit length is chosen which is clamped at

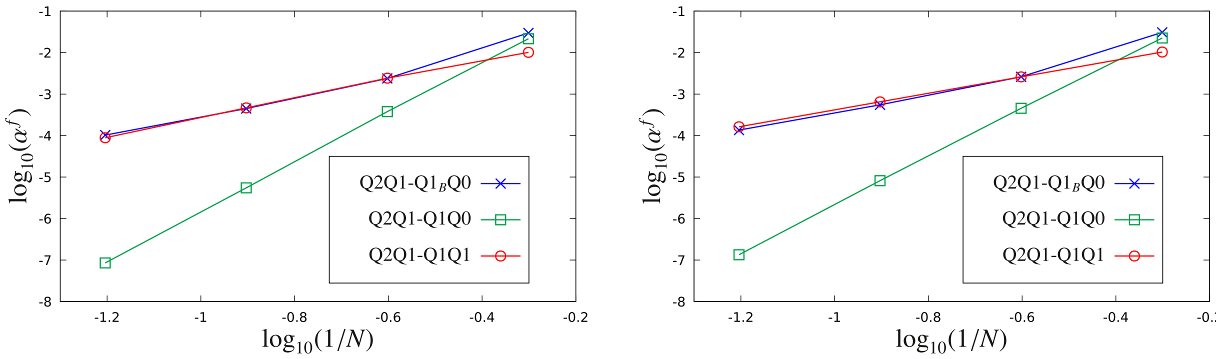

Fig. 5. Inf–sup test of the plastic driving force f for a distorted mesh (left) and an undistorted mesh (right). α^f decreases towards zero for all the elements.

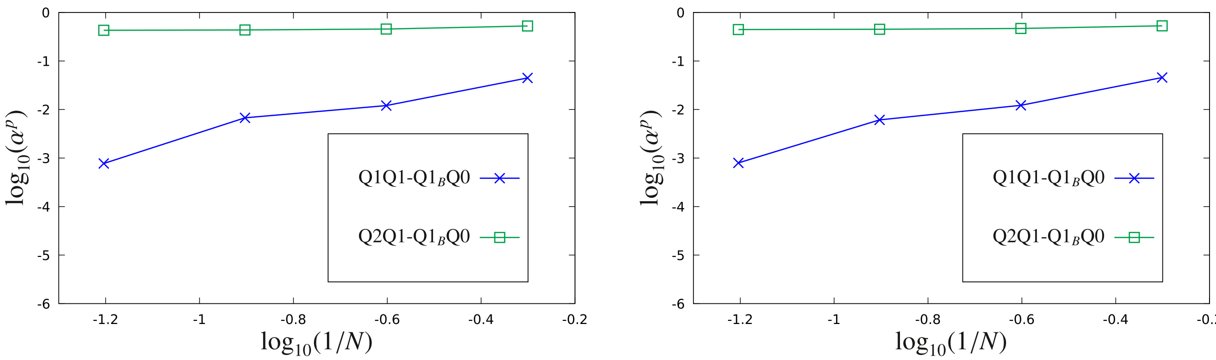

Fig. 6. Inf–sup test of the hydrostatic pressure p for a distorted mesh (left) and an undistorted mesh (right). α^p decreases towards zero for all the elements for linear choices for the displacements and is bounded for Taylor–Hood-type interpolations.

its left edge. On the right edge a traction is applied such that the problem remains purely elastic which, as pointed out above, relates to the critical limit in which the stability criterion is to be evaluated.

The parameters are chosen as for the numerical examples in [12], that is, 164.21 kN/mm^2 for the bulk modulus κ, 80.19 kN/mm^2 for the shear modulus μ, 0.129 kN/mm^2 for the hardening parameter h, 0.1 mm for the length-scale parameter L_e, 0.45 kN/mm^2 for the yield stress y_0, and 10^{-7} kN/mm^2 for the viscosity η.

Fig. 5 shows the stability parameter α^f for the quadrilateral with unit length. α^f decreases towards zero for all the three elements Q2Q1–Q1$_B$Q0, Q2Q1–Q1Q0, and Q2Q1–Q1Q1 for a distorted and an undistorted mesh if the number of elements per side N is increased from 2 to 16 which means that all the investigated elements are unstable in the sense that they cannot provide a unique and stable plastic driving force f. In this context it should be noted that mixed elements may exhibit local oscillations in regions of sharp gradients, even if a formulation is stable [54]. Since we are dealing with a plasticity model with sharp gradients, we would expect oscillations at the sharp interface between the elastic and the plastic zone, even if we had an element that we knew is stable. Making clear statements about the stability of an element is hence rather difficult without the results of this test.

By inspecting the distribution of the plastic zone of the plate with a circular hole, it can be observed that this result is not immediately apparent by investigating only the contour plots. Fig. 7 shows the contour of the hardening field α for the three elements Q2Q1–Q1$_B$Q0, Q2Q1–Q1Q0, and Q2Q1–Q1Q1. A plastic zone forms at the hole upon pulling the plate vertically which propagates towards the corners. The contour of the hardening field is almost identical for the three different elements. For the same three elements the evolution of the plastic driving force f is shown in Fig. 8. Strong oscillations of the plastic driving force can be observed for the Q2Q1–Q1Q0 element, even outside the plastic domain, while the oscillations of the Q2Q1–Q1$_B$Q0 element and the Q2Q1–Q1Q1 element are difficult to trace.

Figs. 9 and 10 show the distribution of the hydrostatic pressure p for all the elements with quadratic and linear shape functions for the displacement field, respectively. As shown in Fig. 6, α^p has a lower bound only for quadratic

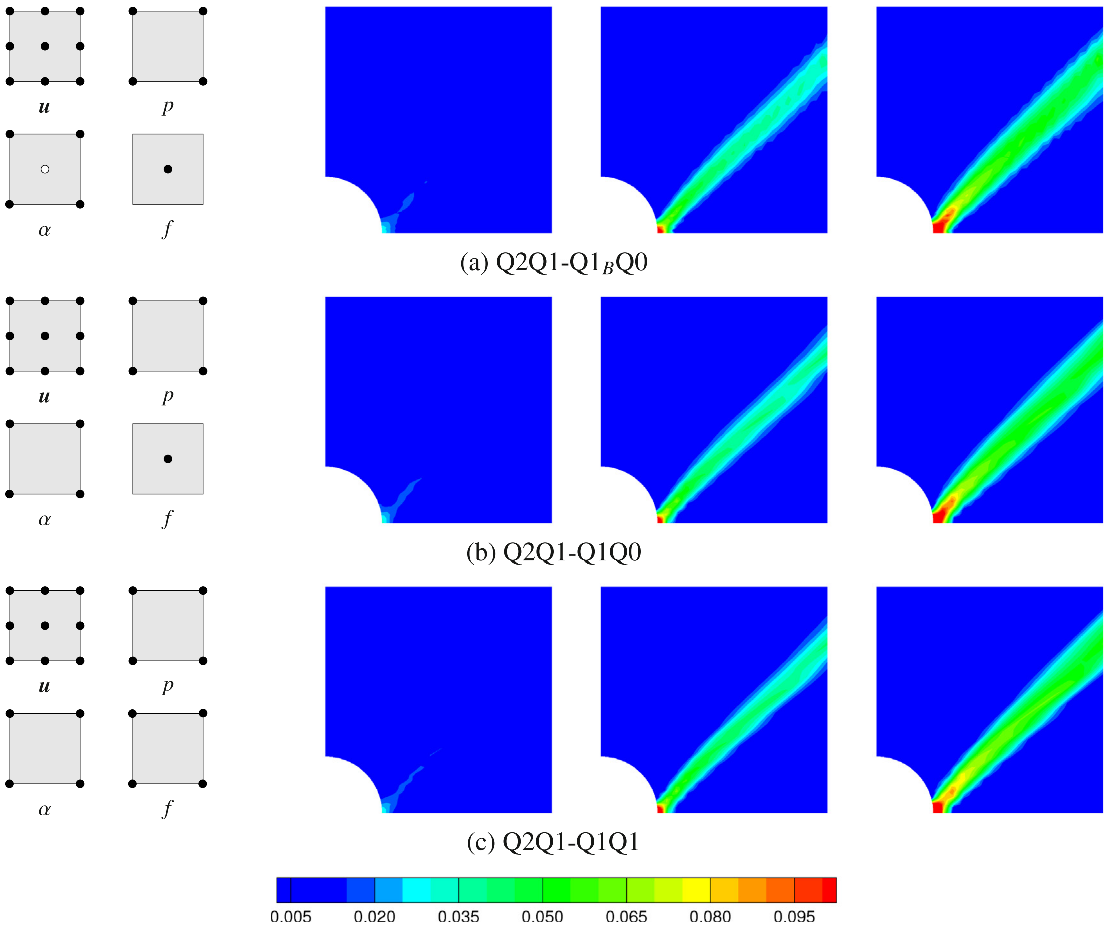

Fig. 7. Distribution of the hardening variable after 0.125 s (left), 0.3125 s (center), and 0.5 s (right) for the Q2Q1–Q1$_B$Q0 element (a), the Q2Q1–Q1Q0 element (b), and the Q2Q1–Q1Q1 element (c).

interpolations, a result that is to be expected for Taylor–Hood-type interpolations. While the compression modulus is chosen as too low to cause pressure oscillations in the elastic regime with linear interpolations for the displacement field, it is important to note that an incompressible case is reached for von Mises-type gradient plasticity once the plastic update takes place in which case linear interpolations will not be able to suppress pressure oscillations in the plastic zone, as can be seen in Fig. 10.

5. Conclusion

We have presented a variational approach towards identifying conditions for stability and uniqueness for multi-field Galerkin methods in continuum mechanics and continuum thermodynamics that arise from discretizing weak equations corresponding to the first variations of a saddle point functional. In this context, we discussed typical multi-field problems with stored free energy functions for which the identification of LBB-type conditions may be cumbersome and pointed out a generalized condition that ensures stability. The proposed framework incorporates material parameters which aims to prevent erroneous results in the context of the numerical test to which the framework is directly linked.

The framework links these non-trivial LBB-type conditions to the notion of Hessian matrices to unify the well-known inf–sup theory in the discrete setting with the well-known infinitesimal stability criterion in a generalized saddle point setting. In doing so, we identify parameters β^n that appear for a list of $n = 1, \ldots, N$ fields of maximizers which are able to make statements about the necessity of satisfying stability conditions related to the

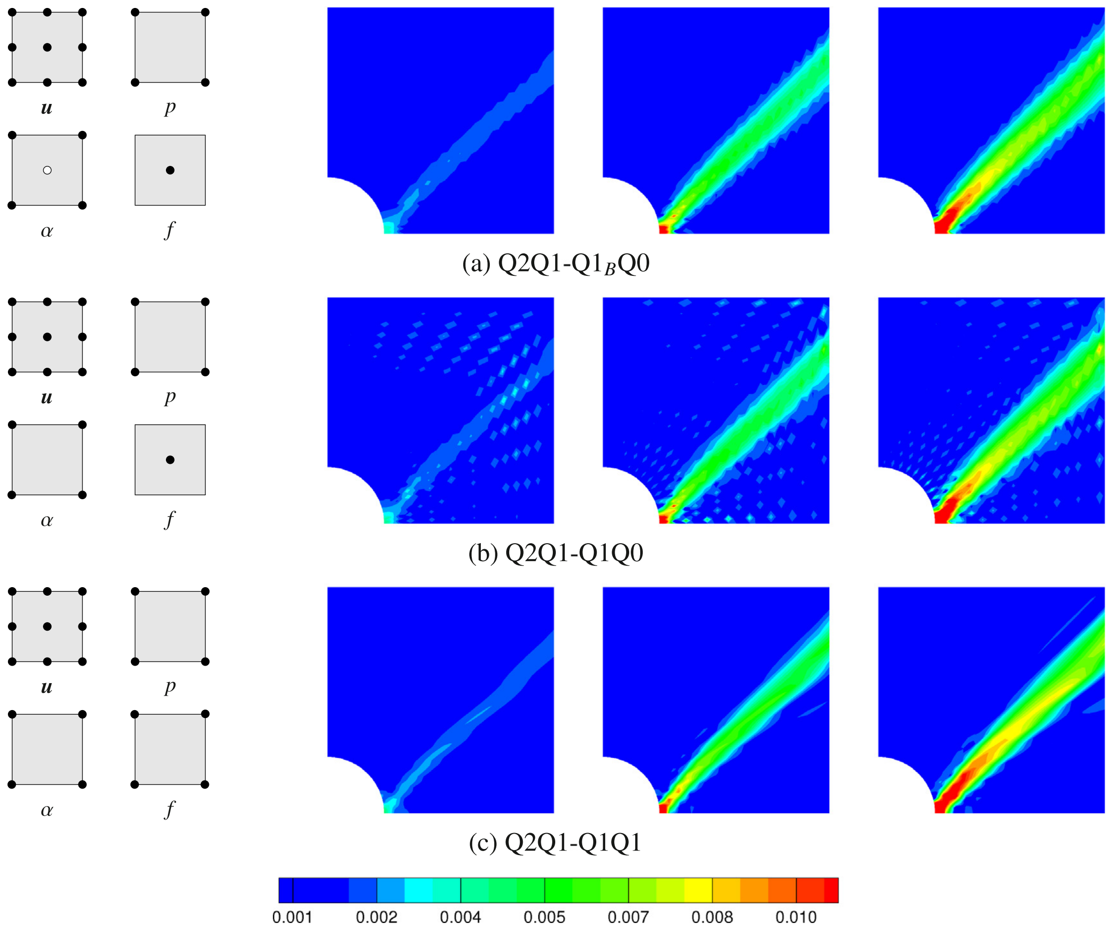

Fig. 8. Distribution of the driving force variable after 0.125 s (left), 0.3125 s (center), and 0.5 s (right) for the Q2Q1–Q1$_B$Q0 element (a), the Q2Q1–Q1Q0 element (b), and the Q2Q1–Q1Q1 element (c).

discretization in unfavorable limits of vanishing parameters. Further, as a result of this treatment, critical parameters α^n appear naturally in quadratic form and simplify the evaluation of eigenvalue tests that investigate the robustness of a discretization for a given problem.

The framework was applied to a two-field functional in finite deformation poroelasticity, a three-field formulation for elasticity in the incompressible limit, and a four-field formulation in gradient-extended plasticity for which all types of elements were shown to be deficient.

Acknowledgments

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Appendix A. Proof of Theorem 2.2 part *(III)*

Proof. Satisfaction of *(III)* is shown first for the homogeneous and conforming case $\{\boldsymbol{u}^*, \boldsymbol{v}^*\} \in U \oplus V$. For convenience we decompose the energy potential into internal and external parts, i.e. $\Pi = \Pi^{\mathrm{i}} - \Pi^{\mathrm{e}}$ and assume all external loads to be *dead* in the sense that they do not change their directions if $\partial\mathcal{B}$ should change its location

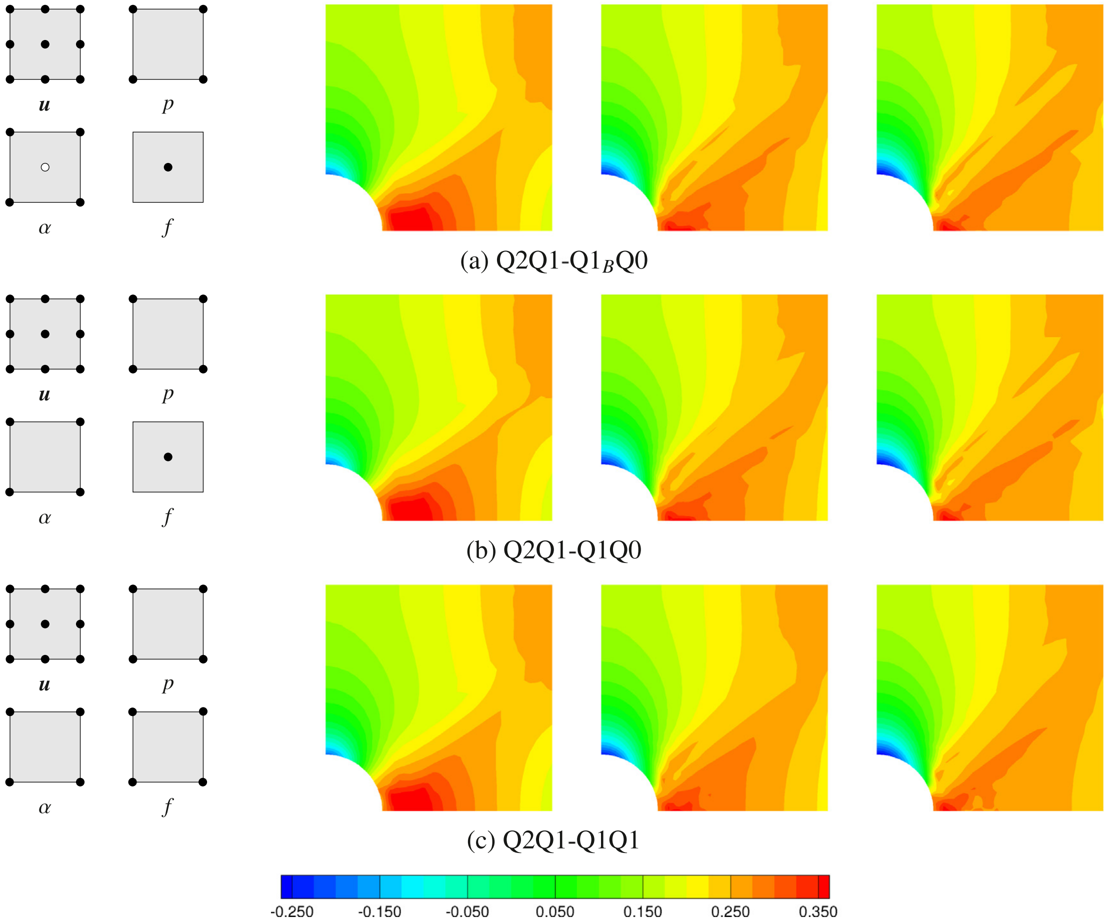

Fig. 9. Distribution of the hydrostatic pressure in kN/mm^2 after 0.125 s (left), 0.3125 s (center), and 0.5 s (right) for the Q2Q1–Q1$_B$Q0 element (a), the Q2Q1–Q1Q0 element (b), and the Q2Q1–Q1Q1 element (c).

which will assure that second variations of Π^{e} will vanish, i.e. Π^{e} is linear. Consequently, we may assume the equality of the first variation to satisfy

$$D_u \Pi^{\mathrm{i}}(\boldsymbol{u}^*, \boldsymbol{v}^*) \cdot \boldsymbol{u} = D_u \Pi^{\mathrm{e}}(\cdot) \cdot \boldsymbol{u} \ , \tag{A.1}$$

$$D_u \Pi^{\mathrm{i}}(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*) \cdot \boldsymbol{u}_h = D_u \Pi^{\mathrm{e}}(\cdot) \cdot \boldsymbol{u}_h \ , \tag{A.2}$$

$$D_v \Pi^{\mathrm{i}}(\boldsymbol{u}^*, \boldsymbol{v}^*) \cdot \boldsymbol{v} = D_v \Pi^{\mathrm{e}}(\cdot) \cdot \boldsymbol{v} \ , \tag{A.3}$$

$$D_v \Pi^{\mathrm{i}}(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*) \cdot \boldsymbol{v}_h = D_v \Pi^{\mathrm{e}}(\cdot) \cdot \boldsymbol{v}_h . \tag{A.4}$$

Combining these stationarity demands and exploiting $U_h \subset U$ and $V_h \subset V$ for conforming methods as well as subtracting variations evaluated at the arbitrary point $\{\boldsymbol{u}_I, \boldsymbol{v}_I\} \in U_h \oplus V_h$ yields

$$D_u \Pi^{\mathrm{i}}(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*) \cdot \boldsymbol{u}_h - D_u \Pi^{\mathrm{i}}(\boldsymbol{u}_I, \boldsymbol{v}_I) \cdot \boldsymbol{u}_h = D_u \Pi^{\mathrm{i}}(\boldsymbol{u}^*, \boldsymbol{v}^*) \cdot \boldsymbol{u}_h - D_u \Pi^{\mathrm{i}}(\boldsymbol{u}_I, \boldsymbol{v}_I) \cdot \boldsymbol{u}_h \ , \tag{A.5}$$

$$D_v \Pi^{\mathrm{i}}(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*) \cdot \boldsymbol{v}_h - D_v \Pi^{\mathrm{i}}(\boldsymbol{u}_I, \boldsymbol{v}_I) \cdot \boldsymbol{v}_h = D_v \Pi^{\mathrm{i}}(\boldsymbol{u}^*, \boldsymbol{v}^*) \cdot \boldsymbol{v}_h - D_v \Pi^{\mathrm{i}}(\boldsymbol{u}_I, \boldsymbol{v}_I) \cdot \boldsymbol{v}_h \ , \tag{A.6}$$

for all $\boldsymbol{u}_h \in U_h$ and for all $\boldsymbol{v}_h \in V_h$. For choices $\{\boldsymbol{u}_h^* - \boldsymbol{u}_I, \boldsymbol{v}_h^* - \boldsymbol{v}_I\} \in B_r(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*)$, linearity inside the open ball (see Appendix B) allows to rewrite the system in the form

$$\underbrace{D_{uu}^2 \Pi^{\mathrm{i}}(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*) \cdot (\boldsymbol{u}_h^* - \boldsymbol{u}_I, \boldsymbol{u}_h)}_{A_h^{\boldsymbol{u}_I}} + \underbrace{D_{uv}^2 \Pi^{\mathrm{i}}(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*) \cdot (\boldsymbol{v}_h^* - \boldsymbol{v}_I, \boldsymbol{u}_h)}_{B_h^{\boldsymbol{u}_I}} = \langle \mathcal{F}, \boldsymbol{u}_h \rangle_{U_h' \times U_h} \ , \tag{A.7}$$

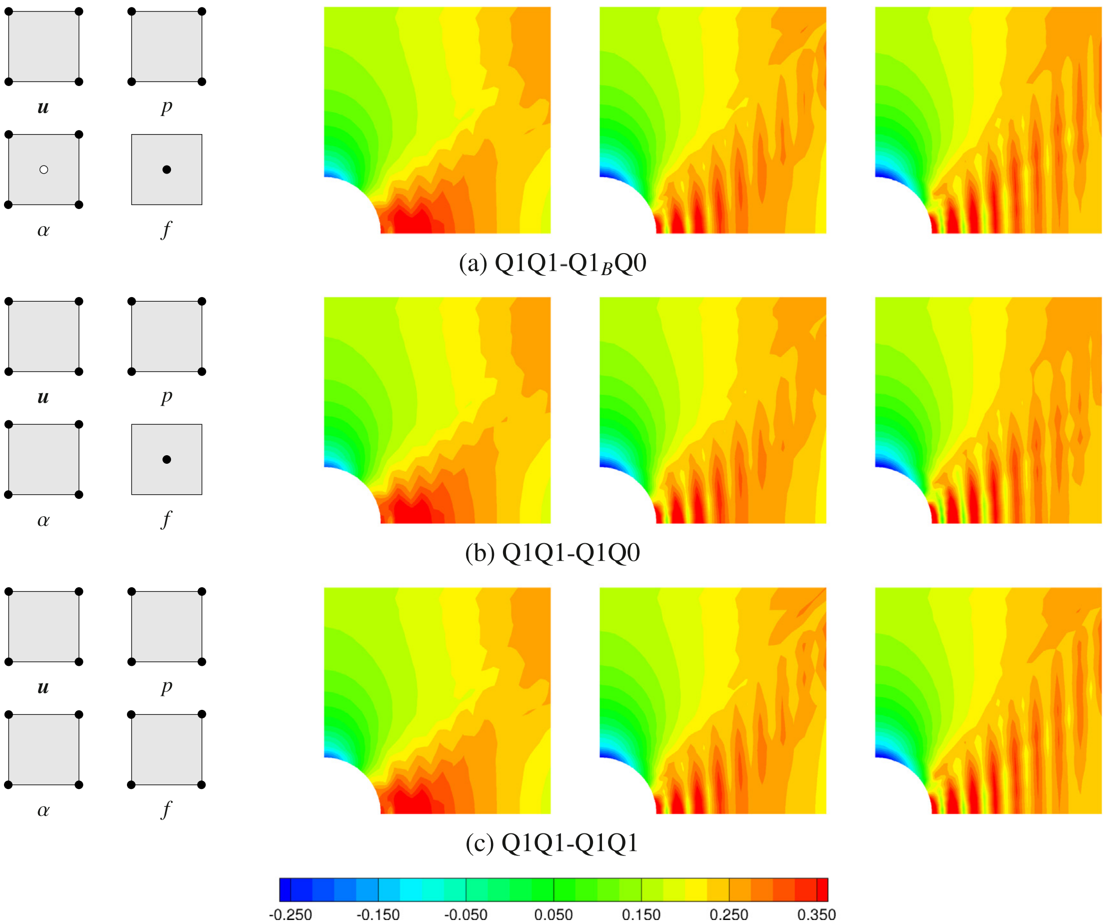

Fig. 10. Distribution of the hydrostatic pressure in kN/mm^2 after 0.125 s (left), 0.3125 s (center), and 0.5 s (right) for the Q1Q1–Q1$_B$Q0 element (a), the Q1Q1–Q1Q0 element (b), and the Q1Q1–Q1Q1 element (c).

$$\underbrace{D^2_{uv}\Pi^{\mathrm{i}}(\boldsymbol{u}^*_h, \boldsymbol{v}^*_h) \cdot (\boldsymbol{u}^*_h - \boldsymbol{u}_I, \boldsymbol{v}_h)}_{B^{\boldsymbol{v}_I}_h} + \underbrace{D^2_{vv}\Pi^{\mathrm{i}}(\boldsymbol{u}^*_h, \boldsymbol{v}^*_h) \cdot (\boldsymbol{v}^*_h - \boldsymbol{v}_I, \boldsymbol{v}_h)}_{C^{\boldsymbol{v}_I}_h} = \langle \mathcal{G}, \boldsymbol{v}_h \rangle_{V'_h \times V_h} \ , \tag{A.8}$$

where, by way of exception, we refrain from using simply the abbreviations A, B, C to emphasize on their specific arguments. We further introduced

$$\begin{aligned} \langle \mathcal{F}, \boldsymbol{u}_h \rangle_{U'_h \times U_h} &:= D^2_{uu}\Pi^{\mathrm{i}}(\boldsymbol{u}^*, \boldsymbol{v}^*) \cdot (\boldsymbol{u}^* - \boldsymbol{u}_I, \boldsymbol{u}_h) + D^2_{uv}\Pi^{\mathrm{i}}(\boldsymbol{u}^*, \boldsymbol{v}^*) \cdot (\boldsymbol{v}^* - \boldsymbol{v}_I, \boldsymbol{u}_h) \\ &\leq \|\mathcal{F}\|_{U_h{}'} \, \|\boldsymbol{u}_h\|_U \ , \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} \langle \mathcal{G}, \boldsymbol{v}_h \rangle_{V'_h \times V_h} &:= D^2_{vv}\Pi^{\mathrm{i}}(\boldsymbol{u}^*, \boldsymbol{v}^*) \cdot (\boldsymbol{v}^* - \boldsymbol{v}_I, \boldsymbol{v}_h) + D^2_{uv}\Pi^{\mathrm{i}}(\boldsymbol{u}^*, \boldsymbol{v}^*) \cdot (\boldsymbol{u}^* - \boldsymbol{u}_I, \boldsymbol{v}_h) \\ &\leq \|\mathcal{G}\|_{V_h{}'} \, \|\boldsymbol{v}_h\|_V \ , \end{aligned} \tag{A.10}$$

with the dual norms

$$\|\mathcal{F}\|_{U_h{}'} := \sup_{\boldsymbol{u}_h \in U_h} \frac{\langle \mathcal{F}, \boldsymbol{u}_h \rangle_{U'_h \times U_h}}{\|\boldsymbol{u}_h\|_U} \leq C_{uu} \left\| \boldsymbol{u}^* - \boldsymbol{u}_I \right\|_U + C_{uv} \left\| \boldsymbol{v}^* - \boldsymbol{v}_I \right\|_V \ , \tag{A.11}$$

$$\|\mathcal{G}\|_{V_h{}'} := \sup_{\boldsymbol{v}_h \in V_h} \frac{\langle \mathcal{G}, \boldsymbol{v}_h \rangle_{V'_h \times V_h}}{\|\boldsymbol{v}_h\|_V} \leq C_{vv} \left\| \boldsymbol{v}^* - \boldsymbol{v}_I \right\|_V + C_{uv} \left\| \boldsymbol{u}^* - \boldsymbol{u}_I \right\|_U \ , \tag{A.12}$$

where we introduced

$$C_{uv} := \sup_{\boldsymbol{u}_h \in U_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{D^2_{uv} \Pi^{\mathrm{i}}(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*) \cdot (\boldsymbol{u}_h, \boldsymbol{v}_h)}{\|\boldsymbol{u}_h\|_U \, \|\boldsymbol{v}_h\|_V} \ , \tag{A.13}$$

$$C_{uu} := \sup_{\boldsymbol{u}_h^1 \in U_h} \sup_{\boldsymbol{u}_h^2 \in U_h} \frac{D^2_{uu} \Pi^{\mathrm{i}}(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*) \cdot (\boldsymbol{u}_h^1, \boldsymbol{u}_h^2)}{\|\boldsymbol{u}_h^1\|_U \, \|\boldsymbol{u}_h^2\|_U} \ , \tag{A.14}$$

$$C_{vv} := \sup_{\boldsymbol{v}_h^1 \in V_h} \sup_{\boldsymbol{v}_h^2 \in V_h} \frac{D^2_{vv} \Pi^{\mathrm{i}}(\boldsymbol{u}_h^*, \boldsymbol{v}_h^*) \cdot (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2)}{\|\boldsymbol{v}_h^1\|_V \, \|\boldsymbol{v}_h^2\|_V} \ . \tag{A.15}$$

Setting $\boldsymbol{u}_h = \boldsymbol{u}_h^* - \boldsymbol{u}_I$ and $\boldsymbol{v}_h = \boldsymbol{v}_h^* - \boldsymbol{v}_I$ yields the estimate

$$\begin{aligned} & C_{u1} \left\|\boldsymbol{u}_h^* - \boldsymbol{u}_I\right\|_U^2 + C_{v1} \left\|\boldsymbol{v}_h^* - \boldsymbol{v}_I\right\|_{\mathrm{Ker}\, C_h^\perp}^2 \leq \\ & \leq \|\mathcal{F}\|_{U_h'} \left\|\boldsymbol{u}_h^* - \boldsymbol{u}_I\right\|_U + \|\mathcal{G}\|_{V_h'} \left\|\boldsymbol{v}_h^* - \boldsymbol{v}_I\right\|_V \ . \end{aligned} \tag{A.16}$$

Clearly, this estimate is lacking an appropriate stability estimate for elements in $\mathrm{Ker}\, C_h$. However, the existence of α^n guarantees that

$$\alpha^n = \inf_{v_h^n \in V_h^n} \sup_{\boldsymbol{u}_h \in U_h} \frac{(B_h^n)^2}{\|\boldsymbol{u}_h\|_U^2 \ \|v_h^n\|_V^2} > 0 \ , \tag{A.17}$$

which for choosing $v_h^n = v_h^{*n} - v_I^n \in V_h^n$ and $\boldsymbol{u}_h = \boldsymbol{u}_h^* - \boldsymbol{u}_I \in U_h$ at fixed $\{\boldsymbol{u}_h^*, v_h^{*n}\}$ and variable $\{\boldsymbol{u}_I, v_I^n\}$ allows the combination with the first stationarity condition yielding

$$\begin{aligned} \left\|v_h^{n*} - v_I^n\right\|_{V^n}^2 & \leq \frac{1}{\alpha^n} \sup_{\boldsymbol{u}_I \in U_h} \left\{ \frac{(B_h^n)^2}{\|\boldsymbol{u}_h\|_U^2} \right\} \\ & \leq \frac{1}{\alpha^n} \sup_{\boldsymbol{u}_I \in U_h} \left\{ \frac{\left(\langle \mathcal{F}, \boldsymbol{u}_h \rangle_{U_h' \times U_h} - A_h^{\boldsymbol{u}_I} \right)^2}{\|\boldsymbol{u}_h\|_U^2} \right\} \\ & \leq \frac{1}{\alpha^n} \left(\|\mathcal{F}\|_{U_h'} + C_{uu} \left\|\boldsymbol{u}_h^* - \boldsymbol{u}_I\right\|_U \right)^2 \ , \end{aligned} \tag{A.18}$$

such that we can find a constant C_α which only explodes if some α^n goes to zero for elements V_h^n associated with $\mathrm{Ker}\, C_h^n$ and hence

$$\left\|\boldsymbol{v}_h^* - \boldsymbol{v}_I\right\|_{\mathrm{Ker}\, C}^2 \leq C_\alpha (\left\|\boldsymbol{u}^* - \boldsymbol{u}_I\right\|_U^2 + \left\|\boldsymbol{v}^* - \boldsymbol{v}_I\right\|_V^2) \ , \tag{A.19}$$

and we can find a real positive C such that

$$\left\|\boldsymbol{u}_h^* - \boldsymbol{u}_I\right\|_U + \left\|\boldsymbol{v}_h^* - \boldsymbol{v}_I\right\|_V \leq C \left(\left\|\boldsymbol{u}^* - \boldsymbol{u}_I\right\|_U + \left\|\boldsymbol{v}^* - \boldsymbol{v}_I\right\|_V \right) \ , \tag{A.20}$$

or after adding $\|\boldsymbol{u}_I - \boldsymbol{u}^*\|_U$ and $\|\boldsymbol{v}_I - \boldsymbol{v}^*\|_V$ on both sides and using the triangle inequality

$$\left\|\boldsymbol{u}^* - \boldsymbol{u}_h^*\right\|_U + \left\|\boldsymbol{v}^* - \boldsymbol{v}_h^*\right\|_V \leq (C+1) \left(\left\|\boldsymbol{u}^* - \boldsymbol{u}_I\right\|_U + \left\|\boldsymbol{v}^* - \boldsymbol{v}_I\right\|_V \right) \ , \tag{A.21}$$

for all $\boldsymbol{u}_I \in U_h$ and $\boldsymbol{v}_I \in V_h$ or alternatively

$$\left\|\boldsymbol{u}^* - \boldsymbol{u}_h^*\right\|_U + \left\|\boldsymbol{v}^* - \boldsymbol{v}_h^*\right\|_V \leq (C+1) \left(\inf_{\boldsymbol{u}_h \in U_h} \left\|\boldsymbol{u}^* - \boldsymbol{u}_h\right\|_U + \inf_{\boldsymbol{v}_h \in V_h} \left\|\boldsymbol{v}^* - \boldsymbol{v}_h\right\|_V \right) \ , \tag{A.22}$$

where we replaced the for all statement by the extremal cases for $\boldsymbol{u}_h$ and $\boldsymbol{v}_h$ that minimize the distance between $\boldsymbol{u}^*$ and $\boldsymbol{u}_I$ as well as between $\boldsymbol{v}^*$ and $\boldsymbol{v}_I$ which is expected to vanish as $h \to \infty$. The typical strategy to show the latter in the non-homogeneous case $\{\mathfrak{u}^*, \mathfrak{v}^*\} \in \mathfrak{U} \oplus \mathfrak{V}$ is to introduce $\{\tilde{\mathfrak{u}}, \tilde{\mathfrak{v}}\} \in \mathfrak{U} \oplus \mathfrak{V}$ with $\tilde{\mathfrak{u}} = \bar{\mathfrak{u}}$ on $\partial\mathcal{B}_{\boldsymbol{u}}$ and $\tilde{\mathfrak{v}} = \bar{\mathfrak{u}}$ on $\partial\mathcal{B}_{\boldsymbol{v}}$ such that $\mathfrak{u}^* = \boldsymbol{u}^* + \tilde{\boldsymbol{u}}$ and $\mathfrak{v}^* = \boldsymbol{v}^* + \tilde{\boldsymbol{v}}$ and to treat the additional effects as external contributions, i.e. non-zero Dirichlet conditions are condensed and treated as part of Π^{e} such that the resulting problem can be reduced to the homogeneous problem that we just investigated. We will, for the sake of brevity, restrict ourselves to the result thus far and assume that proving boundedness of the error estimate in the homogeneous case will imply boundedness of the non-homogeneous problem in our problems of interest. Note that in the case of non-conforming methods $U_h \subset U$ and $V_h \subset V$ do not hold and additional spaces need to be introduced that contain U_h and V_h. □

Appendix B. Properties of the first and second F-derivatives inside an open ball

The derivatives are twice differentiable inside the open ball $B_r(\mathfrak{u}^*, \mathfrak{v}^*)$ about $\{\mathfrak{u}^*, \mathfrak{v}^*\} \in \mathfrak{U} \oplus \mathfrak{V}$ and uniformly continuous with respect to $\{\boldsymbol{u}, \boldsymbol{v}\} \in U \oplus V$. In particular:

$$|\Pi(\mathfrak{u}^* + \epsilon\boldsymbol{u}, \mathfrak{v}^*) - \Pi(\mathfrak{u}^*, \mathfrak{v}^*) - D_u\Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot \boldsymbol{u}| \leq C_u(\epsilon)\, \|\boldsymbol{u}\|_U \ , \tag{B.1}$$

$$|\Pi(\mathfrak{u}^*, \mathfrak{v}^* + \eta\boldsymbol{v}) - \Pi(\mathfrak{u}^*, \mathfrak{v}^*) - D_v\Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot \boldsymbol{v}| \leq C_v(\eta)\, \|\boldsymbol{v}\|_V \ , \tag{B.2}$$

and

$$|D_u\Pi(\mathfrak{u}^* + \epsilon\boldsymbol{u}, \mathfrak{v}^* + \eta\boldsymbol{v}) \cdot \boldsymbol{u} - D_u\Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot \boldsymbol{u} - A - B| \leq C_{uv}(\epsilon, \eta)\, \|\boldsymbol{u}\|_U\, \|\boldsymbol{v}\|_V \ , \tag{B.3}$$

$$|D_v\Pi(\mathfrak{u}^* + \epsilon\boldsymbol{u}, \mathfrak{v}^* + \eta\boldsymbol{v}) \cdot \boldsymbol{v} - D_v\Pi(\mathfrak{u}^*, \mathfrak{v}^*) \cdot \boldsymbol{v} - B - C| \leq C_{vu}(\epsilon, \eta)\, \|\boldsymbol{u}\|_U\, \|\boldsymbol{v}\|_V \ , \tag{B.4}$$

for all $\boldsymbol{u} \in U$, $\boldsymbol{v} \in V$, and positive constants $C_v(\epsilon)$, $C_v(\eta)$, $C_{uv}(\epsilon, \eta)$, and $C_{vu}(\eta, \epsilon)$.

Appendix C. A note on dual principles

It should be noted that we focus here on dual principles where the list of arguments $\mathfrak{v}$ is dual to entries of the list $\mathfrak{u}$. This is the case for typical principles based on Lagrange multipliers, i.e.

$$\Pi(\mathfrak{u}, \mathfrak{v}) = \int_{\mathcal{B}} \psi(\mathfrak{u})\, dV - \sum_{n=1}^{N} \int_{\mathcal{B}} \mathfrak{v}^n f(\mathfrak{u}, \mathfrak{v}^n)\, dV - \Pi^{\mathrm{ext}}(\mathfrak{u}, \mathfrak{v}) \ . \tag{C.1}$$

Note here that in this case we impose a constraint $f(\mathfrak{u})$ [14,15] using elements of the list $\mathfrak{u}$ while, in the case of perturbed Lagrange multiplier methods, $\mathfrak{v}_n$ may appear as well and we consider a constraint of the type $f(\boldsymbol{u}, \mathfrak{v}^n)$ [57]. Clearly, these are cases where the assumption (23) always holds.

In the case of numerous irreversible problems the overall dissipation $\mathcal{D}$ can be expressed by a representation of dual variables [37,58,59], which may be linked to a dissipation function χ or its dual representation ϕ, in the form

$$\mathcal{D} = \mathfrak{v} \cdot \dot{\mathfrak{u}} = \partial_{\dot{u}}\, \chi \cdot \dot{\boldsymbol{u}} = \partial_v\phi \cdot \boldsymbol{v} \geq 0 \ , \tag{C.2}$$

with the relations

$$\phi(\mathfrak{v}; \mathfrak{u}) = \sup_{\dot{\mathfrak{u}}}[\mathfrak{v} \cdot \dot{\mathfrak{u}} - \chi(\dot{\mathfrak{u}}; \mathfrak{u})] \quad \text{and} \quad \chi(\dot{\mathfrak{u}}; \mathfrak{u}) = \sup_{\mathfrak{v}}[\mathfrak{v} \cdot \dot{\mathfrak{u}} - \phi(\mathfrak{v}; \mathfrak{u})] \ , \tag{C.3}$$

introduced in (9) and the necessary conditions $\dot{\mathfrak{u}} = \partial_{\mathfrak{v}}\, \phi(\boldsymbol{v})$ and $\mathfrak{v} = \partial_{\dot{\mathfrak{u}}}\, \chi$. For ϕ to be thermodynamically consistent we require that

1. $\phi \geq 0$ (positive) ,
2. $\phi(\mathfrak{v} = \boldsymbol{0}) = 0$ (zero at origin) ,
3. $\partial^2_{vv}\phi \geq 0$ (convex) .

If $\mathfrak{v}$ is an array of driving forces and dual to the array $\dot{\mathfrak{u}}$, a decomposed expression of (C.2) based on the introduced notation yields (not necessarily all the entries need to be filled)

$$\mathcal{D} = \sum_{n=1}^{N} \mathfrak{v}^n \cdot \dot{\mathfrak{u}}^n \quad \text{with} \quad \dot{\mathfrak{u}}^n = \partial_{v^n}\phi \quad \text{and} \quad \mathfrak{v}^n = \partial_{\dot{\mathfrak{u}}^n}\chi \ . \tag{C.4}$$

An overall saddle point problem is then of the form (7). In a typical situation with different physical mechanisms one may assume that specific dissipation functions are independent, that is, $\phi = \sum_{n=1}^{N} \phi(\mathfrak{v}^n)$ and mixed derivatives vanish such that $\partial^2_{v^n v^o}\phi = 0$ and (23) holds.

References

[1] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Vol. 12, (3) Gordon & Breach, 1969.
[2] I. Babuška, The finite element method with Lagrangian multipliers, Numer. Math. 20 (3) (1973) 179–192.
[3] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, Rev. Fr. Autom. Inform. Rech. Oper. Anal. Numer. 8 (2) (1974) 129–151.
[4] D. Boffi, F. Brezzi, M. Fortin, Mixed Finite Element Methods and Applications, Vol. 44, Springer, Heiderlberg, 2013.
[5] K.J. Bathe, Finite Element Procedures, 2006.
[6] A. Iosilevich, K.J. Bathe, F. Brezzi, On evaluating the inf-sup condition for plate bending eelement, Internat. J. Numer. Methods Engrg. 40 (19) (1997) 3639–3663.
[7] R. Hiptmair, Finite elements in computational electromagnetism, Acta Numer. 11 (2002) 237–339.
[8] C. Jog, R. Haber, Stability of finite element models for distributed-parameter optimization and topology design, Comput. Methods Appl. Mech. Engrg. (1996).
[9] D. Chapelle, K.J. Bathe, The inf-sup test, Comput. Struct. (1993).
[10] K.J. Bathe, The inf-sup condition and its evaluation for mixed finite element methods, Comput. Struct. 79 (2) (2001) 243–252.
[11] W. Sun, A stabilized finite element formulation for monolithic thermo-hydro-mechanical simulations at finite strain, Internat. J. Numer. Methods Engrg. 103 (11) (2015) 789–839.
[12] C. Miehe, F. Aldakheel, S. Mauthe, Mixed variational principles and robust finite element implementations of gradient plasticity at small strains, Internat. J. Numer. Methods Engrg. 94 (11) (2013) 1037–1074.
[13] C. Miehe, F. Welschinger, F. Aldakheel, Variational gradient plasticity at finite strains. part II: Local-global updates and mixed finite elements for additive plasticity in the logarithmic strain space, Comput. Methods Appl. Mech. Engrg. 268 (2014) 704–734.
[14] J. Schröder, P. Wriggers, D. Balzani, A new mixed finite element based on different approximations of the minors of deformation tensors, Comput. Methods Appl. Mech. Engrg. 200 (49) (2011) 3583–3600.
[15] J. Simo, R. Taylor, K. Pister, Variational and projection methods for the volume constraint in finite deformation elasto-plasticity, Comput. Methods Appl. Mech. Engrg. 51 (1–3) (1985) 177–208.
[16] D. Pantuso, K.J. Bathe, A four-node quadrilateral mixed-interpolated element for solids and fluids, Math. Models Methods Appl. Sci 5 (8) (1995) 1113–1128.
[17] D. Pantuso, K.J. Bathe, On the stability of mixed finite elements in large strain analysis of incompressible solids, Finite Elem. Anal. Des. 28 (2) (1997) 83–104.
[18] A. Krischok, C. Linder, On the enhancement of low-order mixed finite element methods for the large deformation analysis of diffusion in solids, Internat. J. Numer. Methods Engrg. 106 (4) (2016) 278–297.
[19] J. Howell, N. Walkington, Inf-sup conditions for twofold saddle point problems, Numer. Math. 118 (4) (2011).
[20] G. Gatica, N. Heuer, Conjugated gradient method for dual-dual mixed formulations, Math. Comp. 71 (240) (2002) 1455–1472.
[21] P. Le Tallec, Existence and approximation results for nonlinear mixed problems; application to incompressible finite elasticity, Numer. Math. 38 (2) (1982) 365–382.
[22] E. Zeidler, Nonlinear Functional Analysis and its Applications III: Variational Methods and Optimization, Springer Science & Business media, 2013.
[23] C. Truesdell, W. Noll, The Non-Linear Field Theories of Mechanics, Springer Berlin Heidelberg, 2004.
[24] J. Schröder, K. Hackl (Eds.), Plasticity and Beyond: Microstructures, Crystal-Plasticity and Phase Transitions, Springer Science & Business Media, 2013.
[25] J. Oden, J. Reddy, Variational Methods in Theoretical Mechanics, Springer –Verlag, 1976.
[26] L. Onsager, Reciprocal relations in irreversible processes. I, Phys. Rev. 31 (4) (1931) 405–426.
[27] L. Onsager, Reciprocal relations in irreversible processes. II, Phys. Rev. 38 (12) (1931) 2265–2279.
[28] M. Biot, Variational principles in irreversible thermodynamics with application to viscoelasticity, Phys. Rev. 97 (6) (1955) 1463–1469.
[29] M. Biot, Thermoelasticity and irreversible thermodynamics, J. Appl. Phys. 27 (3) (1956) 240–253.
[30] M. Biot, G. Temple, Theory of finite deformations of porous solids, Indiana Univ. Math. J. 21 (7) (1972) 597–620.
[31] M. Biot, Mechanics of Incremental Deformation, John Wiley & Sons Inc., New York, 1965.
[32] H. Ziegler, An Introduction to Thermomechanics, Vol. 21, 2012.
[33] B. Halphen, Q.S. Nguyen, Sur les matériaux standard généralisés, J. Mec. 14 (1975) 39–63.
[34] J. Simo, T. Honein, Variational formulation, discrete conservation laws, and path-domain independent integrals for elasto-viscoplasticity, J. Appl. Mech. 57 (3) (1990) 488–497.
[35] M. Ortiz, L. Stainier, The variational formulation of viscoplastic constitutive updates, Comput. Methods Appl. Mech. Engrg. 171 (3–4) (1999) 419–444.
[36] C. Carstensen, K. Hackl, A. Mielke, Non–convex potentials and microstructures in finite-strain plasticity, Proc. R. Soc. A 458 (2002) 299–317.
[37] C. Miehe, A multi-field incremental variational framework for gradient-extended standard dissipative solids, J. Mech. Phys. Solids 59 (4) (2011) 898–923.
[38] C. Miehe, S. Mauthe, S. Teichtmeister, Minimization principles for the coupled problem of Darcy-Biot-type fluid transport in porous media linked to phase field modeling of fracture, J. Mech. Phys. Solids 82 (2015) 186–217.
[39] C. Miehe, F. Hildebrand, L. Böger, Mixed variational potentials and inherent symmetries of the Cahn-Hilliard theory of diffusive phase separation, Proc. R. Soc. A 470 (2164) (2014) 20130641.
[40] C. Miehe, S. Mauthe, H. Ulmer, Formulation and numerical exploitation of mixed variational principles for coupled problems of Cahn-Hilliard-type and standard diffusion in elastic solids, Internat. J. Numer. Methods Engrg. 99 (10) (2014) 737–762.

[41] X. Zhang, A. Krischok, C. Linder, A variational framework to model diffusion induced large plastic deformation and phase field fracture during initial two-phase lithiation of silicon electrodes, Comput. Methods Appl. Mech. Engrg. 312 (2016) 51–77.
[42] O. Zienkiewicz, S. Qu, R. Taylor, S. Nakazawa, The patch test for mixed formulations, Internat. J. Numer. Methods Engrg. 23 (10) (1986) 1873–1883.
[43] T. Hughes, The Finite Element Method: Linear Static and Dynamic Finite Element Analysis, Courier Corporation, 2012.
[44] P. Mira, M. Pastor, T. Li, X. Liu, A new stabilized enhanced strain element with equal order of interpolation for soil consolidation problems, Comput. Methods Appl. Mech. Engrg. 192 (37–38) (2003) 4257–4277.
[45] S. Na, W. Sun, Computational thermo-hydro-mechanics for multiphase freezing and thawing porous media in the finite deformation range, Comput. Methods Appl. Mech. Engrg. 318 (2017) 667–700.
[46] J.E. Marsden, T.J. Hughes, Mathematical Foundations of Elasticity, Courier Corporation, 1994.
[47] R.W. Ogden, Non-linear Elastic Deformations, Dover Publications, 1984.
[48] C. Miehe, D. Vallicotti, S. Teichtmeister, Homogenization and multiscale stability analysis in finite magneto-electro-elasticity. Application to soft matter EE, ME and MEE composites, Comput. Methods Appl. Mech. Engrg. 300 (2016) 294–346.
[49] M.M. Vainberg, Variational Methods for the Study of Nonlinear Equations, Holden–Day, San Francisco, 1964.
[50] J. Ericksen, R. Toupin, Implications of Hadamard's conditions for elastic stability with respect to uniqeness theorems, Canad. J. Math. 8 (1956) 432–436.
[51] P. Mira, M. Pastor, T. Li, X. Liu, A new stabilized enhanced strain element with equal order of interplation for soil condiolidation problems, Comput. Methods Appl. Mech. Engrg. 192 (37) (2003).
[52] D. Malkus, Eigenproblems associated with the discrete LBB condition for incompressible finite elements, Internat. J. Engrg. Sci. (1981).
[53] M. Biot, Variational Lagrangian thermodynamics of nonisothermal finite strain mechanics of porous solids and thermomolecular diffusion, Int. J. Solids Struct. 13 (6) (1977) 579–597.
[54] J.A. White, R. Borja, Stabilized low-order finite elements for coupled solid-deformation/fluid-diffusion and their application to fault zone transients, Comput. Methods Appl. Mech. Engrg. 197 (49) (2008) 4353–4366.
[55] J. Simo, R. Taylor, Quasi-incompressible finite elasticity in principal stretches. continuum basis and numerical algorithms, Comput. Methods Appl. Mech. Engrg. 85 (3) (1991) 273–310.
[56] U. Brink, E. Stein, On some mixed finite element methods for incompressible and nearly incompressible finite elasticity, Comput. Mech. 19 (1) (1996) 105–119.
[57] S.N. Atluri, E. Reissner, On the formulation of variational theorems involving volume constraints, Comput. Mech. 5 (5) (1989) 337–344.
[58] C. Miehe, Variational gradient plasticity at finite strains. Part I: Mixed potentials for the evolution and update problems of gradient-extended dissipative solids, Comput. Methods Appl. Mech. Engrg. 268 (2014) 677–703.
[59] C. Miehe, S. Mauthe, Phase field modeling of fracture in multi-physics problems. Part III. Crack driving forces in hydro-poro-elasticity and hydraulic fracturing of fluid saturated porous media, Comput. Methods Appl. Mech. Engrg. 304 (2016) 619–655.