

Stabilized Mixed Finite Element Formulation

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October 28, 2020

Definitions

\mathbf{F} : Deformation gradient

\mathbf{I} : second-order unit tensor

\mathbf{u} : Displacement

J : determinant of the deformation gradient

\mathbf{C} : Right Cauchy-Green Strain Tensor

$\mathcal{W}(\mathbf{F})$: strain energy function

\mathbf{P} : first Piola-Kirchhoff stress tensor

\mathbf{S} : second PK stress tensor

α : cracks are represented by a scalar phase-field variable

p : Lagrange multiplier, hydrostatic pressure field

κ : bulk modulus

$$\kappa = \frac{E}{3(1 - 2\nu)} \quad (0.1)$$

μ : shear modulus

$$\mu = \frac{E}{2(1 + \nu)} \quad (0.2)$$

λ : Lamé modulus

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad (0.3)$$

\mathcal{E}_ℓ : potential energy functional $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

c_w is a normalization constant

1 Hyperelastic Phase-Field Fracture Models

Deformation Gradient

$$\mathbf{F} = \mathbf{I} + \nabla \otimes \mathbf{u} \quad (1.1)$$

where $J = \det \mathbf{F}$ and $\mathbf{C} = \mathbf{F}^T \mathbf{F}$.

The strain energy function $\mathcal{W}(\mathbf{F})$ is defined per unit reference volume such that the first PK and second PK

$$\mathbf{P} = \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \quad (1.2a)$$

$$\mathbf{S} = 2 \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{C}} \quad (1.2b)$$

where $\mathbf{P} = \mathbf{F}\mathbf{S}$.

1.1 Phase-Field Fracture Model

Non-modified strain energy function is the compressible Neo-Hookean:

$$\mathcal{W}(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2 \ln J) \quad (1.3)$$

For incompressible hyperelastic materials, the strain energy function is defined as

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p(J - 1), \quad (1.4)$$

We instead consider a damage dependent relaxation of the incompressibility constraint and the phase-field couples to the strain energy through the modified function

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J - 1)^2 \quad (1.5)$$

where the decreasing stiffness modulation function and $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

$$a(\alpha) = (1 - \alpha)^2 \quad w(\alpha) = \alpha \quad (1.6)$$

In the code we have the following definition

$$b(\alpha) = (1 - \alpha)^3 = \sqrt{a^3(\alpha)}$$

To circumvent numerical difficulties, we resort to the classical mixed formulation and introduce the pressure-like field

$$p = -\sqrt{a^3(\alpha)}\kappa(J - 1), \quad (1.7)$$

as an independent variable along with the displacement field.

Lastly, the normalization constant is defined as:

$$c_w = \int_0^1 \sqrt{w(\alpha)} d\alpha \quad (1.8)$$

The first Piola-Kirchhoff stress tensor is given:

$$\begin{aligned} \mathbf{P} &= \frac{\partial \widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{\partial \mathbf{F}} \\ &= \frac{\partial}{\partial \mathbf{F}} \left[a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J - 1)^2 \right] \\ &= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\frac{1}{2}\kappa\frac{\partial (J - 1)^2}{\partial \mathbf{F}} \\ \mathbf{P} &= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa(J - 1)\frac{\partial J}{\partial \mathbf{F}} \quad \text{where } \frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T} \\ \mathbf{P} &= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa(J - 1)J\mathbf{F}^{-T} \end{aligned} \quad (1.9)$$

Note that we DO NOT consider this form of the modified functional; therefore, we do not consider this PK stress.

1.1.1 According to the rough draft

Therefore our modified function:

$$\begin{aligned}\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J-1)^2 \\ &= a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\frac{-p}{\sqrt{a^3(\alpha)}}(J-1) \\ &= a(\alpha)\mathcal{W}(\mathbf{F}) - \sqrt{a^3(\alpha)}\frac{p}{2}(J-1) \\ \widetilde{\mathcal{W}}(\mathbf{F}, \alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)\frac{p}{2}(J-1)\end{aligned}$$

In the rough draft of the paper we have Eq. 5: energy functional of a possibly fractured elastic body with isotropic surface energy

$$\begin{aligned}\mathcal{E}_\ell(\mathbf{u}, \alpha) &= \int_{\Omega} \widetilde{\mathcal{W}}(\mathbf{F}, \alpha) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \mathbf{u} dA \\ \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \int_{\Omega} \left[a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)\frac{p}{2}(J-1) \right] d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega \\ \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha)\mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} b(\alpha)\frac{p}{2}(J-1) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega\end{aligned}$$

Note that we are missing one term.

1.1.2 According to the code

In the code we have the following for the energy functional of the energy problem

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}$$

Energy functional, where we ignore the surface term

$$\begin{aligned}\mathcal{E}_\ell(\mathbf{u}, \alpha) &= \int_{\Omega} \widetilde{\mathcal{W}}(\mathbf{F}, \alpha) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \mathbf{u} dA \\ &= \int_{\Omega} a(\alpha)\mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} b(\alpha)p(J-1) d\Omega - \int_{\Omega} \frac{p^2}{2\lambda} d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega\end{aligned}$$

1.1.3 According to a derivation from Bin2020 Paper

Versus Eq. 21 where we drop λ_b which is not a consideration in this formulation

$$\mathcal{E}_\ell(\mathbf{u}, \alpha) = \int_{\Omega} a(\alpha)\mathcal{W}(\mathbf{F}, \alpha) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega \quad (1.10)$$

Starting from Eq. 25 in the 2020 Li and Bouklas paper where κ is the bulk modulus

$$\begin{aligned}
\mathcal{E}_\ell(\mathbf{u}, p, \Lambda, \alpha) &= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega \Lambda(p + \sqrt{a^3(\alpha)}\kappa(J-1)) d\Omega \quad \Lambda = -p/\kappa \\
&= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega -\frac{p}{\kappa}(p + \sqrt{a^3(\alpha)}\kappa(J-1)) d\Omega \\
&= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \frac{p^2}{\kappa} d\Omega - \int_\Omega \frac{p}{\kappa} \sqrt{a^3(\alpha)}\kappa(J-1) d\Omega \\
\mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \mathcal{E}_\ell(\mathbf{u}, \alpha) - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1) d\Omega
\end{aligned} \tag{1.11}$$

Substitute in $\mathcal{E}_\ell(\mathbf{u}, \alpha)$ and substitute Eq. 1.3

$$\mathcal{E}_\ell(\mathbf{u}, p, \alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w} \int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2 \right) d\Omega - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1) d\Omega$$

After taking the directional derivative of the prior equation, we can introduce the stabilization term

$$-\frac{\varpi h^2}{2\mu} \sqrt{a^3(\alpha)} \sum_{e=1}^{n_{el}} \int_{\Omega^e} J\mathbf{C}^{-1} : (\nabla p \cdot \nabla q) dV = 0$$

1.2 Changes for 2D Plane-Stress Models

Following the code, we have the following energy functional of the energy problem

$$\begin{aligned}
\widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda} \\
\widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}
\end{aligned}$$

Where we know k_ℓ is a modeling parameter, so we can list the energy functional as:

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda} \tag{1.12}$$

Derive the 1st PK stress

$$\begin{aligned}
\mathbf{P} &= \frac{\partial \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}} \\
&= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T}) - b(\alpha)p \frac{\partial \det \mathbf{F}}{\partial \mathbf{F}} \\
\mathbf{P} &= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T}) - b(\alpha)pJ\mathbf{F}^{-T}
\end{aligned}$$

Taking the third component to be zero, in the plane stress case

$$\begin{aligned}
\mathbf{P}_{33} &= a(\alpha)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha)pJ\mathbf{F}_{33}^{-1} = 0 \\
(1-\alpha)^2\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1-\alpha)^3pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1-\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mathbf{F}_{33} - \mathbf{F}_{33}^{-1} - \frac{(1-\alpha)pJ}{\mu}\mathbf{F}_{33}^{-1} &= 0 \\
\mathbf{F}_{33}\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}\mathbf{F}_{33} - \frac{(1-\alpha)pJ}{\mu}\mathbf{F}_{33}^{-1}\mathbf{F}_{33} &= 0 \\
\mathbf{F}_{33}^2 - 1 - \frac{(1-\alpha)pJ}{\mu} &= 0
\end{aligned}$$

This can be multiplied by its associated test function to obtain the weak form

$$\int_{\Omega} \left(\mathbf{F}_{33}^2 - 1 - \frac{(1 - \alpha)pJ}{\mu} \right) v_{F_{33}} dV = 0$$

1.2.1 Changes for 2D Discrete Crack Model

If we are considering a discrete fracture method

$$\begin{aligned} \mathcal{E}_{\ell}(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega \\ \mathcal{E}_{\ell}(\mathbf{u}, p, \alpha) &= \int_{\Omega} \mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} p (J - 1) d\Omega \end{aligned}$$

with the stabilization term and plane stress in the weak form

$$\begin{aligned} -\frac{\varpi h^2}{2\mu} \int_{\Omega} J \mathbf{C}^{-1} : (\nabla p \cdot \nabla q) dV &= 0 \\ \int_{\Omega} \left(\mathbf{F}_{33}^2 - 1 - \frac{pJ}{\mu} \right) v_{F_{33}} dV &= 0 \end{aligned}$$

where we have assumed for the energy functional

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = \frac{\mu}{2} (I_c - 3 - 2 \ln J) - p(J - 1) - \frac{p^2}{2\lambda}$$

Therefore, we can calculate the 1st Piola Kirchoff Stress as:

$$\begin{aligned} \mathbf{P} &= \frac{\mu}{2} (2\mathbf{F} - \frac{2}{J} J \mathbf{F}^{-T}) - p J \mathbf{F}^{-T} \\ &= \mu (\mathbf{F} - \mathbf{F}^{-T}) - p J \mathbf{F}^{-T} \end{aligned}$$

Taking the third component to be zero

$$\begin{aligned} P_{33} &= \mu (F_{33} - F_{33}^{-1}) - p J F_{33}^{-1} = 0 \\ &= F_{33} - F_{33}^{-1} - \frac{pJ}{\mu} F_{33}^{-1} = 0 \\ P_{33} &= F_{33}^2 - 1 - \frac{pJ}{\mu} = 0 \end{aligned}$$

1.2.2 Changes for 2D displacement formulation

Removing pressure terms

$$\begin{aligned} \mathcal{E}_{\ell}(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega \\ \mathcal{E}_{\ell}(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega \end{aligned}$$

with plane stress in the weak form (no need for stabilization terms)

$$\int_{\Omega} (a(\alpha) \mu (F_{33} - F_{33}^{-1})) v_{F_{33}} dV = 0$$

We have assumed the modified energy functional

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha) \frac{\mu}{2} (I_c - 3 - 2 \ln J) \quad (1.13)$$

Therefore, we can calculate the 1st Piola Kirchhoff Stress as:

$$\begin{aligned} \mathbf{P} &= a(\alpha) \frac{\mu}{2} (2\mathbf{F} - \frac{2}{J} J\mathbf{F}^{-T}) \\ &= a(\alpha) \mu (\mathbf{F} - \mathbf{F}^{-T}) \end{aligned}$$

Taking the third component to be zero

$$P_{33} = a(\alpha) \mu (F_{33} - F_{33}^{-1}) = 0$$

2 Following Borden: Derivations of Analytical Phase Field

Note the full potential energy functional, which can also be called the lagrangian

$$\mathcal{E}_\ell(\mathbf{u}, p, \alpha) = \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega$$

We can use the Euler-Lagrange equations to arrive at the equations of motion by taking the derivative with respect to displacement, pressure, and the scalar damage field. Starting with displacement:

$$\frac{\partial \mathcal{E}_\ell}{\partial \mathbf{u}} = \int_{\Omega} a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{u}} d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial p} = - \int_{\Omega} \frac{p}{\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} (J - 1) d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial \alpha} = - \int_{\Omega} 2(1 - \alpha) \mathcal{W}(\mathbf{F}) d\Omega + \int_{\Omega} 3p(1 - \alpha)^2 (J - 1) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] d\Omega$$

Therefore we have three equations:

First should be mechanical eq, second is a an equation for pressure,

$$\begin{aligned} -\frac{p}{\kappa} - \sqrt{a^3(\alpha)} (J - 1) &= 0 \\ -\frac{p}{\kappa} - (1 - \alpha)^3 (J - 1) &= 0 \\ -\kappa (J - 1) (1 - \alpha)^3 &= p \end{aligned}$$

Lastly,

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) + 3p(1 - \alpha)^2 (J - 1) + \frac{G_c}{c_w} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] = 0$$

Substitute second equation into third

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) - 3\kappa(1 - \alpha)^5 (J - 1)^2 + \frac{G_c}{c_w} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] = 0$$

2.1 Homogeneous Solution

Therefore, we can study the homogeneous solution by ignoring spatial derivatives of α

$$-2(1 - \alpha_h) \mathcal{W}(\mathbf{F}) - 3\kappa(1 - \alpha_h)^5(J - 1)^2 + \frac{G_c}{c_w \ell} = 0$$

We can expand and arrange

$$\begin{aligned} -2\mathcal{W}(\mathbf{F}) + 2\alpha_h \mathcal{W}(\mathbf{F}) - 3\kappa(J - 1)^2(1 - 5\alpha_h + 10\alpha_h^2 - 10\alpha_h^3 + 5\alpha_h^4 - \alpha_h^5) + \frac{G_c}{c_w \ell} &= 0 \\ -2\mathcal{W}(\mathbf{F}) + 2\alpha_h \mathcal{W}(\mathbf{F}) + \frac{G_c}{c_w \ell} & \\ -3\kappa(J - 1)^2 + 15\kappa(J - 1)^2\alpha_h - 30\kappa(J - 1)^2\alpha_h^2 + 30\kappa(J - 1)^2\alpha_h^3 - 15\kappa(J - 1)^2\alpha_h^4 + \kappa(J - 1)^2\alpha_h^5 &= 0 \\ -2\mathcal{W}(\mathbf{F}) - 3\kappa(J - 1)^2 + \frac{G_c}{c_w \ell} + [2\mathcal{W}(\mathbf{F}) + 15\kappa(J - 1)^2]\alpha_h & \\ -30\kappa(J - 1)^2\alpha_h^2 + 30\kappa(J - 1)^2\alpha_h^3 - 15\kappa(J - 1)^2\alpha_h^4 + \kappa(J - 1)^2\alpha_h^5 &= 0 \end{aligned}$$

2.2 Non-Homogeneous Solution

Now for the Non-homogenous solution, we have the following

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) - 3\kappa(1 - \alpha)^5(J - 1)^2 + \frac{G_c}{c_w} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] = 0$$

(2.1)