# Personal Notes on Paper:

## Crack tip fields in soft elastic solids subjected to large quasi-static deformation

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# Contents

# 1 Definitions

$y_\alpha$ where $\alpha = 1, 2$ are deformed coordinates
$(r, \theta)$ are polar coordinates
$m_\alpha, p_\alpha$ are unknown exponents
$v_\alpha(\theta), q_\alpha(\theta)$unknown functions describing the angular variation

$I$ ?
$A, B$ are material constants
$n$ is a material constant where $n = 1$ recovers a neo-Hookean or Mooney-Rivlin material
$\mu$ is the small strain shear modulus

$a, b_0$: unknown positive amplitude
$U(\theta, n)$: dimensionless function
$H(\theta, n)$: angular function
$F$ is a hypergeometric function

# 2 Geometry, Notations, and Basic Equations

## 2.1 Governing Equations: Kinematics and Equilibrium

In-plane displacements because we are considering plane-strain, plane-stress, and anti-plane shear

$$u_\alpha\left(X_1, X_2\right) = x_\alpha\left(X_1, X_2\right) - X_\alpha \quad \alpha = 1, 2 \tag{2.1}$$

In the plane-stress formulation, $(X_1, X_2)$ denote the mid-plane coordinates of a material point in the undeformed reference configuration. The undeformed crack tip is at the origin, $X_1 = X_2 = 0$

Introduction of $y_\alpha$ definition for considering displacement of the crack tip:

$$y_\alpha\left(X_1, X_2\right) \equiv x_\alpha\left(X_1, X_2\right) - x_\alpha\left(X_1 = 0, X_2 = 0\right) \tag{2.2}$$

Therefore, considering we are given displacements from Paraview

$$\begin{aligned} y_\alpha\left(X_1, X_2\right) &= \left(u_\alpha\left(X_1, X_2\right) + X_\alpha\right) - \left(u_\alpha\left(0, 0\right) + X_\alpha(0, 0)\right) && (2.3) \\ &= u_\alpha\left(X_1, X_2\right) - u_\alpha(0, 0) + X_\alpha - X_\alpha(0, 0) && (2.4) \end{aligned}$$

# 3 Crack tip fields under large deformations

## 3.1 Method of asymptotic analysis

The deformed coordinates, $y_1\, y_2$, were assumed to be a series consisting of separable functions of the polar coordinates $(r, \theta)$ in the undeformed material in the vicinity of the crack tip

$$\begin{aligned} &y_a(r, \theta) = r^{m_a} v_a(\theta) + r^{n_0} q_a(\theta) + \cdots \\ &m_a < p_a, \quad \alpha = 1, 2 \end{aligned} \tag{3.1}$$

where $m_\alpha, p_\alpha$ are unknown exponents and $v_\alpha(\theta).q_\alpha(\theta)$ are unknown functions describing the angular variation

## 3.2 Plane strain crack in homogeneous materials

Strain energy density

$$W(I \gg 1) = AI^n + BI^{n-1} + o\left(l^{n-1}\right) \tag{3.2}$$

$n$ is a material constant where $n = 1$ recovers a neo-Hookean or Mooney-Rivlin material. Remove higher order terms.

$$\begin{aligned}
W(I \gg 1) &= AI^1 + BI^{1-1} + o(l^{1-1}) \\
W(I \gg 1) &= AI + B \quad \text{where } A = \frac{\mu}{2}\; B = -\frac{3\mu}{2} \\
&= \frac{\mu}{2}I - \frac{3\mu}{2} \\
W(I \gg 1) &= \frac{\mu}{2}(I - 3)
\end{aligned}$$

### 3.2.1 Crack tip deformation field (Mode I)

The characteristics of the deformation has a transition at n = 3/2. Note that n=1 indicates a Neo-Hookan material.

$$\begin{aligned}
y_1 &= \begin{cases} -b_0 r^{2-\frac{1}{n}}[U(\theta,n)]^2, & 1/2 < n < 3/2 \\ -\frac{1}{a} r^{1+\frac{1}{2\pi}} H(\theta,n), & n > 3/2 \end{cases} \\
y_2 &= ar^{1-\frac{1}{2n}} U(\theta,n)
\end{aligned} \tag{3.3}$$

U is simply a dimensionless function that holds for any n

$$U(\theta,n) = \sin\left(\frac{\theta}{2}\right)\sqrt{1 - \frac{2\kappa^2\cos^2(\theta/2)}{1+\omega(\theta,n)}} \times [\omega(\theta,n) + \kappa\cos\theta]^{\kappa/2}, \quad 0 \le |\theta| \le \pi \tag{3.4}$$

where

$$\kappa = 1 - \frac{1}{n} \tag{3.5}$$

$$\omega(\theta,n) = \sqrt{1 - (\kappa\sin\theta)^2} \tag{3.6}$$

H is an angular function that is only relevant for n > 3/2 Note that we have a typo where it is written that n > 3/2

$$H(\theta,n) = -\frac{n^{5/2}}{m^2}[\omega(\theta,n) + \kappa\cos\theta]^{2-m}\left[\frac{m}{2-m} \times F\left(\frac{1}{2} - \frac{1}{m}, \frac{1}{2}; \frac{3}{2} - \frac{1}{m}; \cos^2\xi_0\right) - \kappa\sin\xi_0\right] \tag{3.7}$$

where F is the hypergeometric function and,

$$\begin{aligned}
m &= 1 - \frac{1}{2n} \\
\cos\xi_0 &= \frac{1}{n\sqrt{2}}\frac{\sqrt{1 + k\sin^2\theta - \omega(\theta,n)\cos\theta}}{\omega(\theta,n) + \kappa\cos\theta} \qquad 0 \le \xi_0 \le \frac{\pi}{2}
\end{aligned}$$

To make sure I have plotted correctly, check the boundaries

$$\begin{aligned}
H(\theta = 0, n) &= -\frac{4n^{9/2}}{(2n-1)^2}\left[2 - \frac{1}{n}\right]^{1+\frac{1}{2m}} \frac{1}{n(2n+1)} \\
H(\theta = \pm\pi, n) &= -\frac{4n^{9/2}}{(2n-1)^2} n^{-1-\frac{1}{n}} \left(\frac{2n-1}{2n+1}\right) \times \frac{\sqrt{\pi}\Gamma\left(\frac{2n-3}{2(2n-1)}\right)}{\Gamma\left(\frac{-1}{2n-1}\right)}
\end{aligned}$$

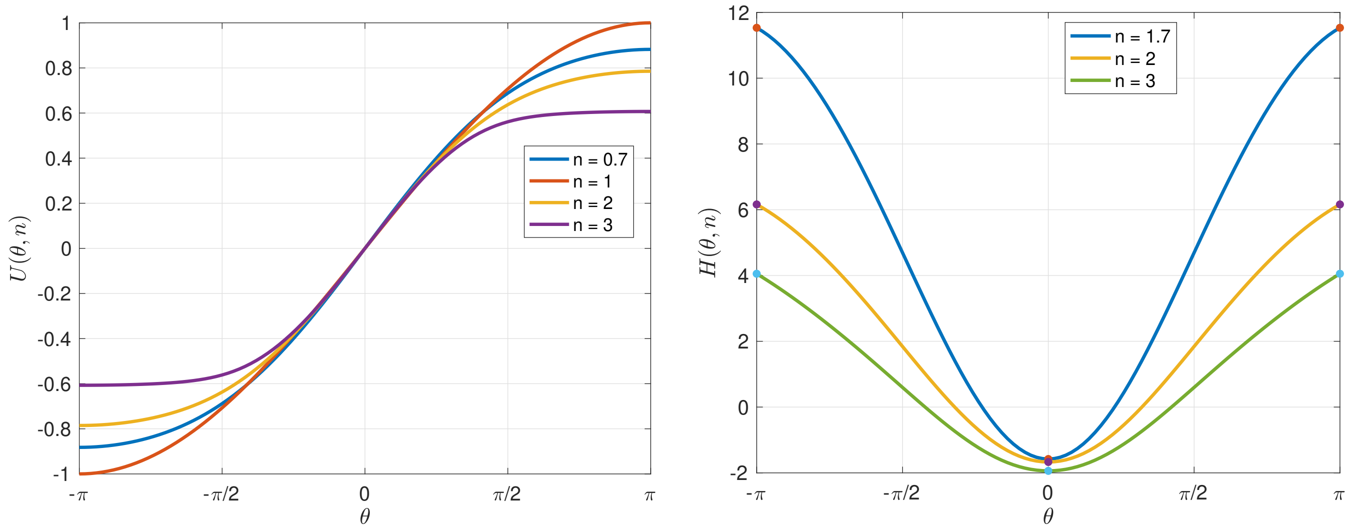


Figure 1: Replication of Figure 3 without n = 10

where the gamma function exists in MATLAB.

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Note that plotting U in Fig 3A is simply a matter of choosing a n value and using Eq. 3.4, 3.5, and 3.6. Plotting H is more complicated and relies on inbuilt functions in MATLAB for hypergeometric and Gamma functions, for example

```
for i = 1:length(x)
    F(i) = hypergeom([1/2 - 1/m, 1/2], 3/2 - 1/m, x(i));
end
```

and to check the boundaries, we list two gamma functions as follows:

```
gamma1 = gamma((2*n(ii)-3)/(2*(2*n(ii)-1)));
gamma2 = gamma(-1/(2*n(ii)-1));
```

---

In the interest of replicating Figure 4A, rewrite the equations for the deformed coordinates near the crack tip, where we are interested in n between 1/2 and 3/2

$$\begin{aligned} y_1 &= -b_0 r^{2-\frac{1}{n}} [U(\theta, n)]^2 \\ y_2 &= a r^{1-\frac{1}{2n}} U(\theta, n) \end{aligned} \tag{3.8}$$

Before normalizing the result, we first rearrange $y_2$

$$\frac{y_2}{a} = r^{1-\frac{1}{2n}} U(\theta, n)$$

and substitute into $y_1$

$$y_1 = -b_0 r^{2-\frac{1}{n}} [U(\theta, n)]^2 \tag{3.9}$$

$$= -b_0 r^{2(1-\frac{1}{2n})} [U(\theta, n)]^2 \tag{3.10}$$

$$y_1 = -b_0 \left(\frac{y_2}{a}\right)^2 \tag{3.11}$$

Therefore, we now have Eq. 40a in the paper

$$y_1 = -b_0\left(\frac{y_2}{a}\right)^2 \tag{3.12}$$

$$\sqrt{\frac{y_1}{-b_0}} = \frac{y_2}{a} \tag{3.13}$$

$$\pm a\sqrt{\frac{y_1}{-b_0}} = y_2 \tag{3.14}$$

Now if we normalize by $a^{2n}$, where normalized quantities are denoted by $\hat{()}$

$$\begin{aligned}
\hat{y}_1 &= -\frac{b_0}{a^{2n}} r^{2-\frac{1}{n}}[U(\theta,n)]^2 \qquad \hat{y}_2 = \frac{a}{a^{2n}} r^{1-\frac{1}{2n}} U(\theta,n) \\
\hat{y}_1 &= -\frac{b_0}{a^{2n}} r^{2-\frac{1}{n}}[U(\theta,n)]^2 \qquad \hat{y}_2 = a^{1-2n} r^{1-\frac{1}{2n}} U(\theta,n) \\
\hat{y}_1 &= -\frac{xa^{2-2n}}{a^{2n}} r^{2-\frac{1}{n}}[U(\theta,n)]^2 \\
\hat{y}_1 &= -xa^{2-4n} r^{2-\frac{1}{n}}[U(\theta,n)]^2
\end{aligned} \tag{3.15}$$

where we can introduce some unknown x according to the figure caption:

$$x = \frac{b_0}{a^{2-2n}} \rightarrow b_0 = xa^{2-2n} \tag{3.16}$$

Finally

$$\hat{y}_1 = -x(\hat{y}_2)^2 = -\frac{b_0}{a^{2-2n}}(\hat{y}_2)^2 \tag{3.17}$$

### 3.2.2 Special case: n = 1 for neo-Hookean or MR solid

First, we want to plot U for n = 1, which makes $\kappa = 0$ and $\omega = 1$

$$\begin{aligned}
U(\theta,n) &= \sin(\frac{\theta}{2})\sqrt{1 - \frac{2\kappa^2\cos^2(\theta/2)}{1+\omega(\theta,n)}} \times [\omega(\theta,n) + \kappa\cos\theta]^{\kappa/2}, \quad 0 \le |\theta| \le \pi \\
&= \sin(\frac{\theta}{2})\sqrt{1 - \frac{2(0)\cos^2(\theta/2)}{2}} \times [1+\cos\theta]^{0/2} \\
U(\theta,1) &= \sin(\frac{\theta}{2})
\end{aligned}$$

Therefore, considering n = 1, starting with Eq. 3.8, we can obtain Eq. 48 in the paper

$$\begin{aligned}
y_1 &= -b_0 r^{2-\frac{1}{n}}[U(\theta,n)]^2 = -b_0 r\left[\sin(\frac{\theta}{2})\right]^2 \\
y_2 &= ar^{1-\frac{1}{2n}}U(\theta,n) = a\sqrt{r}\sin(\frac{\theta}{2})
\end{aligned}$$

In terms of the normalized quantities, we have the following:

$$\begin{aligned}
\hat{y}_1 &= -xa^{2-4n} r^{2-\frac{1}{n}}[U(\theta,n)]^2 \qquad \hat{y}_2 = a^{1-2n} r^{1-\frac{1}{2n}} U(\theta,n) \\
\hat{y}_1 &= -xa^{2-4} r^{2-1}[U(\theta,1)]^2 \qquad \hat{y}_2 = a^{1-2} r^{1-\frac{1}{2}} U(\theta,1) \\
\hat{y}_1 &= -b_0 a^{-2} r\left[\sin(\frac{\theta}{2})\right]^2 \qquad \hat{y}_2 = a^{-1}\sqrt{r}\sin(\frac{\theta}{2}) \\
\hat{y}_1 &= -b_0\hat{y}_2^2
\end{aligned} \tag{3.18}$$

There are several ways to plot Fig. 4, but I plotted the full U function for more generalizability. In the MATLAB code, we simply set the value of x which is a ratio of $b_0$ and $a$.

$$\hat{y}_2 = \frac{1}{a}\sqrt{r}\,U(\theta, 1)$$
$$\hat{y}_1 = -x\hat{y}_2^2$$

Note: neglect the $\frac{\sqrt{r}}{a}$ in the actual code:

```
y2 = real(U);
y1 = -x(ii)*y2.^2;
```

How this is plotted in Figure 4 is by setting n = 1 for the calculation of $\kappa$ for $U$

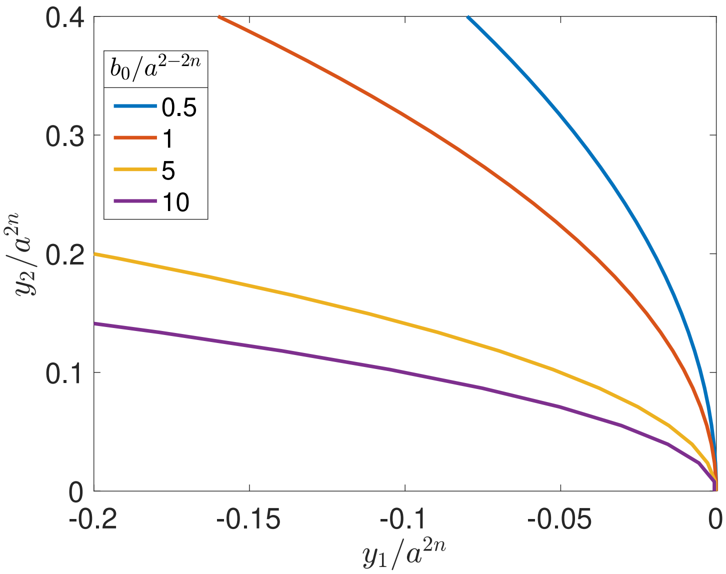


Figure 2: Crack opening profile predicted by the asymptotic solution.

## 3.3 Plane stress crack in homogenous materials

### 3.3.1 Crack tip deformation field for GNH solids

Leading order terms of mode I deformation field $y_1$ and $y_2$:

$$y_1 = cr^d g(\theta, n),\ d < 1 + \frac{1}{4n} \qquad n < n^* \tag{3.19}$$
$$y_2 = ar^{1-\frac{1}{2n}}U(\theta, n) \tag{3.20}$$

For the special case of a neo-Hookean solid with n = 1

$$d = 1 \quad \text{and} \quad g(\theta, n = 1) = \cos\theta \tag{3.21}$$

### 3.3.2 Special case: n = 1 for neo-Hookean solid

Therefore for a neo-Hookean solid we obtain Eq. 65 in the paper

$$y_1 = cr\cos\theta$$
$$y_2 = a\sqrt{r}\sin\left(\frac{\theta}{2}\right)$$

Note 1: There is a distinct difference in the relation between $y_1$ and $y_2$ see Eq. 66 below

$$y_2 = \pm a\sqrt{\frac{-y_1}{c}}$$
$$\left(\frac{y_2}{a}\right)^2 = \frac{-y_1}{c}$$
$$cr\sin^2\left(\frac{\theta}{2}\right) = -y_1$$

but $-\cos(\theta) \neq \sin^2(\theta/2)$

If we want to plot this similarly to Fig. 3 in the paper, we normalize by $a^{2n}$ or $a^2$ for n = 1

$$\hat{y}_1 = \frac{c}{a^{2n}} r\cos\theta$$
$$\hat{y}_1 = \frac{c}{a^2} r\cos\theta \rightarrow r = \frac{\hat{y}_1 a^2}{c\cos\theta}$$

For $\hat{y}_2$ we substitute the above identity for r

$$\begin{aligned}
\hat{y}_2 &= a^{1-2n}\sqrt{r}\sin\left(\frac{\theta}{2}\right) \\
\hat{y}_2^2 &= a^{-2} r\sin^2\left(\frac{\theta}{2}\right) \quad \text{where } \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-\cos\theta) \\
\hat{y}_2^2 &= a^{-2} r\frac{1}{2}(1-\cos\theta) \quad \text{substitute r} \\
\hat{y}_2^2 &= a^{-2}\frac{\hat{y}_1 a^2}{c\cos\theta}\frac{1}{2}(1-\cos\theta) \\
\hat{y}_2^2 &= \frac{1}{2c}\frac{(1-\cos\theta)}{\cos\theta}\hat{y}_1
\end{aligned}$$

Rearrange so we obtain the following properties:

$$\hat{y}_1 = 2c\hat{y}_2^2\frac{\cos\theta}{1-\cos\theta} \quad \hat{y}_2 = \frac{1}{a}\sqrt{r}\sin\left(\frac{\theta}{2}\right)$$

Note 2: we can determine a but not c.

# 4 Energetics: J-integral and energy release rate

The J-integral for plane stress, Mode I opening

$$\begin{aligned}
J &= \frac{\mu\pi}{2}\left(\frac{b}{n}\right)^{n-1}\left(\frac{2n-1}{2n}\right)^{2n-1} n^{1-n}a^{2n} \quad n = 1 \\
J &= \frac{\mu\pi a^2}{4}
\end{aligned}$$

## 4.1 Interpretation of J-integral in experiments

For a pure shear specimen the J-integral is

$$J = 2W(I_1, I_2)h_0 = 2\Psi(\lambda_A)h_0 \tag{4.1}$$

where the stretch in the direction of loading is:

$$\lambda_A = 1 + \frac{\Delta}{h_0}$$

For a neo-hookean solid

$$\Psi = \frac{\mu}{2}(\lambda_A - \lambda_A^{-1})^2$$

where we can substitute this into the J-integral Eq. 4.1 and solve for a

$$\begin{aligned}
J &= 2\Psi(\lambda_A)h_0 \\
J &= \mu(\lambda_A - \lambda_A^{-1})^2 h_0 \\
\frac{\mu\pi a^2}{4} &= \mu(\lambda_A - \lambda_A^{-1})^2 h_0 \\
a^2 &= \frac{4h_0}{\pi}(\lambda_A - \lambda_A^{-1})^2 \\
a &= 2\sqrt{\frac{h_0}{\pi}}(\lambda_A - \lambda_A^{-1})
\end{aligned}$$

The unknown amplitude c decreases monotonically from 1.55 at $\lambda_A = 1.02$ to 1.15 at $\lambda_A = 2$.