

The Thermodynamics of Elastic Materials with Heat Conduction and Viscosity

BERNARD D. COLEMAN & WALTER NOLL

1. Introduction

The basic physical concepts of classical continuum mechanics are *body*, *configuration* of a body, and *force system* acting on a body. In a formal rational development of the subject, one first tries to state precisely what mathematical entities represent these physical concepts: a body is regarded to be a smooth manifold whose elements are the material points; a configuration is defined as a mapping of the body into a three-dimensional Euclidean space, and a force system is defined to be a vector-valued function defined for pairs of bodies¹. Once these concepts are made precise one can proceed to the statement of *general principles*, such as the principle of objectivity or the law of balance of linear momentum, and to the statement of specific *constitutive assumptions*, such as the assertion that a force system can be resolved into body forces with a mass density and contact forces with a surface density, or the assertion that the contact forces at a material point depend on certain local properties of the configuration at the point. While the general principles are the same for all work in classical continuum mechanics, the constitutive assumptions vary with the application in mind and serve to define the *material* under consideration. When one has stated the mathematical nature of bodies, configurations and forces, and has laid down the ways in which these concepts occur in the general principles and the constitutive assumptions, then the properties of these concepts are fixed, and one can present rigorous arguments without recourse to "operational definitions" and other metaphysical paraphernalia, which may be of some use in deciding on the applicability of a theory to a specific physical situation but seem to have no place in its mathematical development.

To discuss the *thermodynamics* of continua, it appears that to the concepts of continuum mechanics one must add five new basic concepts: these are *temperature*, specific *internal energy*², specific *entropy*³, *heat flux*, and *heat supply*⁴ (due to radiation). Once mechanics is axiomatized, it is easy to give the mathematical entities representing the thermodynamic concepts: temperature, specific internal energy, specific entropy and heat supply are scalar fields defined over

¹ For more extensive discussions of the foundations of continuum mechanics see references [1]—[4].

² Sometimes called "internal energy density".

³ Sometimes called "entropy density".

⁴ Sometimes called "density of absorbed radiation".

the body, while heat flux is a vector field over the body. We believe that in presenting thermodynamics one should retain all the general principles of mechanics but add to them two new principles: the *first law* of thermodynamics, *i.e.* the law of balance of energy¹, and the *second law*, which for continua takes the form of the Clausius-Duhem inequality². The constitutive assumptions of our present work are the following:

(1) That there exists a caloric equation of state relating the specific internal energy to the "strain"³ and the specific entropy.

(2) That there exists a temperature equation relating the temperature to the strain and specific entropy.

(3) That the stress tensor of mechanics is the sum of two terms, one of which, the "elastic" term, depends on only the strain and the specific entropy, and the other, the "viscous" term, depends on both these variables and, in addition, has a linear dependence on the "rate of strain"⁴.

(4) That the heat flux depends on only the strain, the specific entropy and the spatial gradient of the temperature. (We assume smoothness but not linearity for this dependence.)

We allow the heat supply to be assignable in any way compatible with the general principles, just as body forces are often left assignable in mechanics.

The constitutive assumptions considered here are not the most general imaginable⁵, but they are sufficiently general to cover many applications; in particular, they include the starting assumptions of thermoelasticity theory and the classical theory of viscous fluids with heat conduction.

We feel that a statement of the role of the thermodynamic fields in the general laws of thermodynamics and in specific constitutive assumptions should serve to fix the mathematics of thermodynamics and permit a rational development of the subject without requiring the introduction of non-mathematical concepts, such as Heat Reservoirs, Perpetual Motion Machines, and Reversibly Added Heat⁶.

Here we proceed as follows. We first give a precise meaning to the term *thermodynamic process*; such a process is defined to be a time-dependent set of configurations, force systems, and temperature, internal energy, entropy, heat supply and heat flux fields, compatible with the principles of mechanics and the law of conservation of energy. A thermodynamic process is said to be *admissible* if it is compatible with the constitutive assumptions under consideration. We then demand that the Clausius-Duhem inequality hold for *all*

¹ Cf. §§ 241 and 242 of [4].

² Cf. § 257 of [4].

³ More precisely, the deformation gradient.

⁴ More precisely, the velocity gradient.

⁵ For example, they do not allow for all the long range memory effects covered in the purely mechanical theory of simple materials; cf. [5], [2].

⁶ We do not believe it impossible to assign a mathematical meaning to Heat Reservoirs, Perpetual Motion Machines, and Reversibly Added Heat, but we feel that this has not yet been done, and we doubt its necessity. In several of the extant "axiomatizations" of thermodynamics, reference to these vague concepts appears to disguise the absence of, rather than to furnish, a mathematical justification for the conclusions drawn.

admissible processes. This requirement places restrictions on the constitutive assumptions.

Our main interest here is in the rigorous derivation of a set of necessary and sufficient restrictions on the assumptions (1)–(4) for the Clausius-Duhem inequality to hold for all admissible processes. Each of the restrictions we find is, by itself, a familiar formula or inequality. What is new here is the logical connection.

In a future article we shall apply the general principles of the present framework to more general constitutive assumptions and in so doing obtain a thermodynamic theory compatible with long range memory and non-linear viscoelastic effects, such as stress relaxation.

2. Thermodynamic processes

Consider a body consisting of material points X . A thermodynamic process for this body is described by eight functions of X and the time t , with physical interpretations as follows:

- (1) The spatial *position* $\mathbf{x} = \boldsymbol{\chi}(X, t)$; here the function $\boldsymbol{\chi}$, called the *deformation function*, describes a *motion* of the body.
- (2) The symmetric *stress* tensor $\mathbf{T} = \mathbf{T}(X, t)$.
- (3) The *body force* $\mathbf{b} = \mathbf{b}(X, t)$ per unit mass (exerted on the body by the external world).
- (4) The specific *internal energy* $\varepsilon = \varepsilon(X, t)$.
- (5) The *heat flux* vector $\mathbf{q} = \mathbf{q}(X, t)$.
- (6) The *heat supply* $r = r(X, t)$ per unit mass and unit time (absorbed by the material and furnished by radiation from the external world).
- (7) The specific *entropy* $\eta = \eta(X, t)$.
- (8) The local *temperature* $\vartheta = \vartheta(X, t)$, which is assumed to be always positive, $\vartheta > 0$.

We say that such a set of eight functions is a *thermodynamic process* if the following two conservation laws¹ are satisfied not only for the body but also for each of its parts \mathcal{B} :

(A) *The law of balance of linear momentum:*

$$\int_{\mathcal{B}} \dot{\mathbf{x}} \, dm = \int_{\mathcal{B}} \mathbf{b} \, dm + \int_{\partial\mathcal{B}} \mathbf{T}\mathbf{n} \, ds. \quad (2.1)$$

(B) *The law of balance of energy*

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{B}} \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} \, dm + \int_{\mathcal{B}} \dot{\varepsilon} \, dm = \int_{\mathcal{B}} (\dot{\mathbf{x}} \cdot \mathbf{b} + r) \, dm + \int_{\partial\mathcal{B}} (\dot{\mathbf{x}} \cdot \mathbf{T}\mathbf{n} - \mathbf{q} \cdot \mathbf{n}) \, ds. \quad (2.2)$$

In (2.1) and (2.2), dm denotes the element of mass in the body, $\partial\mathcal{B}$ the surface of \mathcal{B} , ds the element of surface area in the configuration at time t , and \mathbf{n} the exterior unit normal vector to $\partial\mathcal{B}$ in the configuration at time t ; superimposed dots denote time-derivatives.

¹ A thorough discussion of these conservation laws is given in [4], §§ 196–205, 240, 241.

The assumed symmetry of the stress tensor \mathbf{T} insures that the moment of momentum is automatically balanced. Couple stresses, body couples and other mechanical interactions not included in \mathbf{T} or \mathbf{b} are assumed to be absent.

Under suitable smoothness assumptions the balance equations (2.1) and (2.2) in integral form are equivalent to the following two balance equations in differential form¹:

$$\operatorname{div} \mathbf{T} - \rho \ddot{\mathbf{x}} = -\rho \mathbf{b}, \quad (2.3)$$

$$\operatorname{tr}\{\mathbf{T}\mathbf{L}\} - \operatorname{div} \mathbf{q} - \rho \dot{\varepsilon} = -\rho r; \quad (2.4)$$

here, ρ denotes the mass density and \mathbf{L} the velocity gradient.

In order to define a thermodynamic process it is sufficient to prescribe the six functions χ , \mathbf{T} , ε , \mathbf{q} , η , and ϑ . The two remaining functions \mathbf{b} and r are then uniquely determined by (2.3) and (2.4).

3. The constitutive equations

A *material* is defined by a *constitutive assumption*, which is a restriction on the processes that are admissible in a body consisting of the material. An *elastic material with heat conduction and viscosity* is defined by five *response functions* $\hat{\varepsilon}$, $\hat{\vartheta}$, $\hat{\mathbf{T}}$, \mathbf{I} and $\hat{\mathbf{q}}$. A process is said to be *admissible* in a homogeneous body consisting of such a material if the following constitutive equations hold at each material point X and at all times t :

$$\varepsilon = \hat{\varepsilon}(\mathbf{F}, \eta), \quad (3.1)$$

$$\vartheta = \hat{\vartheta}(\mathbf{F}, \eta), \quad (3.2)$$

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}, \eta) + \mathbf{I}(\mathbf{F}, \eta)[\mathbf{L}], \quad (3.3)$$

$$\mathbf{q} = \hat{\mathbf{q}}(\mathbf{F}, \eta, \operatorname{grad} \vartheta). \quad (3.4)$$

Here \mathbf{F} denotes the deformation of gradient, at X and t , computed relative to a fixed homogeneous reference configuration, and \mathbf{L} denotes, as before, the velocity gradient; *i.e.*, if we identify the material point X with its position \mathbf{X} in the reference configuration, we have

$$\mathbf{F} = \nabla_{\mathbf{X}} \chi(\mathbf{X}, t) \quad (3.5a)$$

and

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \quad (3.5b)$$

The value $\mathbf{I}(\mathbf{F}, \eta)$ of the response function \mathbf{I} is a linear transformation over the nine-dimensional space of tensors, and the square brackets in (3.3) indicate that this transformation operates on the tensor \mathbf{L} .² The response functions $\hat{\varepsilon}$, $\hat{\vartheta}$, $\hat{\mathbf{T}}$, \mathbf{I} and $\hat{\mathbf{q}}$ depend on the choice of the reference configuration.

¹ See the sections of [4] cited above.

² The assumed linear dependence of the "viscous stress" $\mathbf{I}(\mathbf{F}, \eta)[\mathbf{L}]$ on \mathbf{L} is not essential to our present arguments; for example, for our derivation of (5.5) it suffices that $\mathbf{I}(\mathbf{F}, \eta)[\mathbf{L}] \rightarrow 0$ as $\mathbf{L} \rightarrow 0$. Our work on materials with fading memory [6] suggests, however, that if the extra-stress depends non-linearly on the velocity gradient then it should depend also on acceleration gradients. Since we here do not allow for an effect of acceleration gradients on the viscous stress, we prefer to let $\mathbf{I}(\mathbf{F}, \eta)[\mathbf{L}]$ be linear in \mathbf{L} and thus stay within the theory of "linearly viscous materials", a theory which gives, in the sense of [6], a complete first-order approximation to the non-linear theory of materials with fading memory.

To specify an admissible process, it is sufficient to prescribe the deformation function χ and the entropy distribution η as functions of time. The deformation gradient F and the velocity gradient L can be computed from χ , and then ε , ϑ , and T can be determined from (3.1)–(3.3). With the temperature distribution ϑ thus obtained, the heat flux q can be calculated from (3.4). Finally b and r can be chosen so that (2.3) and (2.4) hold¹. In Sections 4 and 5 we shall make frequent use of this observation that χ and η can be chosen *independently* and that to every such choice corresponds a unique admissible thermodynamic process.

The constitutive equations (3.1)–(3.4) must obey the *principle of material objectivity*², which states that an admissible process must remain admissible after a change of frame or observer. Such a change of frame is defined by a time-dependent orthogonal tensor Q . The scalars ε , η , and ϑ remain unaffected by a change of frame, but F , T , L , and $\text{grad } \vartheta$ transform as follows:

$$\begin{aligned} F &\rightarrow QF, \\ T &\rightarrow QTQ^T, \\ L &\rightarrow QLQ^T + \dot{Q}Q^T, \\ q &\rightarrow Qq, \\ \text{grad } \vartheta &\rightarrow Q \text{grad } \vartheta. \end{aligned} \tag{3.6}$$

Using the methods of reference [7], it is easy to show that the principle of material objectivity is satisfied if and only if the response functions obey the following identities:

$$\begin{aligned} \hat{\varepsilon}(F, \eta) &= \hat{\varepsilon}(QF, \eta), \\ \hat{\vartheta}(F, \eta) &= \hat{\vartheta}(QF, \eta), \\ Q\hat{T}(F, \eta)Q^T &= \hat{T}(QF, \eta), \\ Q\hat{I}(F, \eta)[L]Q^T &= \hat{I}(QF, \eta)[QLQ^T + W], \\ Q\hat{q}(F, \eta, v) &= \hat{q}(QF, \eta, Qv) \end{aligned} \tag{3.7}$$

for all scalars η , all vectors v , all tensors F and L , all orthogonal tensors Q , and all skew tensors W .

¹ In many treatments of continuum thermo-mechanics b and r are regarded as assigned *a priori*. We do not follow this practice here. Nor are we disturbed by the fact that it might be difficult to control b and r experimentally. Physical considerations suggest, however, that for each set of assigned values of F , L , and η the heat supply r should have a negative lower bound $r_0 < 0$ whose magnitude $|r_0|$ is the radiation the body would emit into an environment at zero absolute temperature.

To account explicitly for this lower bound it would suffice to restrict the admissible fields χ and η to those which give rise to values of T , L , $\text{div } q$, ϱ , and $\dot{\varepsilon}$ such that the left side of (2.4) is greater than ϱr_0 . Our arguments would not be affected much by this restriction.

² See [1]–[3].

The identities (3.7) can be used to derive reduced forms for the constitutive equations (3.1)–(3.4). One set of such reduced constitutive equations is

$$\begin{aligned}\varepsilon &= \hat{\varepsilon}(\mathbf{U}, \eta), \\ \vartheta &= \hat{\vartheta}(\mathbf{U}, \eta), \\ \mathbf{R}^T \mathbf{T} \mathbf{R} &= \hat{\mathbf{T}}(\mathbf{U}, \eta) + \mathbf{I}(\mathbf{U}, \eta) [\mathbf{R}^T \mathbf{D} \mathbf{R}], \\ \mathbf{q} &= \mathbf{R} \hat{\mathbf{q}}(\mathbf{U}, \eta, \mathbf{R}^T \text{grad } \vartheta).\end{aligned}\tag{3.8}$$

Here \mathbf{U} is the (symmetric and positive-definite) right stretch tensor and \mathbf{R} the (orthogonal) rotation tensor, determined by the polar decomposition $\mathbf{F} = \mathbf{R} \mathbf{U}$, and $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$ is the stretching tensor.

An alternative set of reduced constitutive equations is

$$\begin{aligned}\varepsilon &= \hat{\varepsilon}(\mathbf{C}, \eta), \\ \vartheta &= \hat{\vartheta}(\mathbf{C}, \eta), \\ \mathbf{F}^T \mathbf{T} \mathbf{F} &= \tilde{\mathbf{T}}(\mathbf{C}, \eta) + \tilde{\mathbf{I}}(\mathbf{C}, \eta) [\mathbf{F}^T \mathbf{D} \mathbf{F}], \\ \mathbf{q} &= \mathbf{F} \hat{\mathbf{q}}(\mathbf{C}, \eta, \mathbf{F}^T \text{grad } \vartheta),\end{aligned}\tag{3.9}$$

where $\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T \mathbf{F}$ is the right Cauchy-Green tensor. Noting that

$$\mathbf{F}^T \text{grad } \vartheta = V_{\mathbf{X}} \vartheta,\tag{3.10}$$

where $V_{\mathbf{X}} \vartheta$ is the gradient of the temperature with respect to the position \mathbf{X} in the reference configuration, we find that the last equation of (3.9) has the form¹

$$\mathbf{q} = \mathbf{F} \hat{\mathbf{q}}(\mathbf{C}, \eta, V_{\mathbf{X}} \vartheta).\tag{3.11}$$

In rough terms, the *isotropy group* of a material is the set of density-preserving changes of reference configuration which leave the response of the material unaltered. A more formal definition is the following². The isotropy group \mathcal{J} of an elastic material with heat conduction and viscosity is the set of all unimodular tensors \mathbf{H} for which the following identities hold:

$$\begin{aligned}\hat{\varepsilon}(\mathbf{F}, \eta) &= \hat{\varepsilon}(\mathbf{F} \mathbf{H}, \eta), \\ \hat{\vartheta}(\mathbf{F}, \eta) &= \hat{\vartheta}(\mathbf{F} \mathbf{H}, \eta), \\ \tilde{\mathbf{T}}(\mathbf{F}, \eta) &= \tilde{\mathbf{T}}(\mathbf{F} \mathbf{H}, \eta), \\ \mathbf{I}(\mathbf{F}, \eta) [\mathbf{L}] &= \mathbf{I}(\mathbf{F} \mathbf{H}, \eta) [\mathbf{L}], \\ \hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v}) &= \hat{\mathbf{q}}(\mathbf{F} \mathbf{H}, \eta, \mathbf{v}),\end{aligned}\tag{3.12}$$

for all scalars η , all vectors \mathbf{v} and all tensors \mathbf{F} and \mathbf{L} . It is easy to show that the tensors \mathbf{H} in \mathcal{J} form a group. Of course, the group \mathcal{J} depends not only on the material under consideration but also on the choice of reference configuration. It can be shown, however, that the isotropy groups corresponding

¹ This is the form derived by PIPKIN & RIVLIN ([8], equation (17.2)), and by GREEN & ADKINS ([9], equation (8.5.17)).

² This definition is analogous to that given in [2] and discussed in detail for elastic materials (without heat conduction or viscosity) in [10].

to two different reference configurations of the same material are conjugate and hence isomorphic.

The unit tensor \mathbf{I} always belongs to \mathcal{J} . Suppose now that the inversion $-\mathbf{I}$ is in \mathcal{J} . (We note that if $-\mathbf{I}$ is in the isotropy group corresponding to one reference configuration of a material, then $-\mathbf{I}$ is in the isotropy groups corresponding to all other reference configurations of the same material.) In this case (3.12)₅ for $\mathbf{H} = -\mathbf{I}$ yields the identity

$$\hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v}) = \hat{\mathbf{q}}(-\mathbf{F}, \eta, \mathbf{v}). \quad (3.13)$$

On the other hand, (3.7)₅ with $\mathbf{Q} = -\mathbf{I}$ and \mathbf{v} replaced by $-\mathbf{v}$, gives

$$-\hat{\mathbf{q}}(\mathbf{F}, \eta, -\mathbf{v}) = \hat{\mathbf{q}}(-\mathbf{F}, \eta, \mathbf{v}). \quad (3.14)$$

Combining (3.13) and (3.14), we obtain the identity

$$\hat{\mathbf{q}}(\mathbf{F}, \eta, -\mathbf{v}) = -\hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v}). \quad (3.15)$$

In words: if $-\mathbf{I}$ is in \mathcal{J} , then, the heat flux must be given by an *odd* function of the temperature gradient. If we put $\mathbf{v} = \mathbf{0}$ in (3.15), it follows that

$$\hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{0}) = \mathbf{0}. \quad (3.16)$$

Of course, (3.16) holds whenever $\hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v})$ is linear in \mathbf{v} , *i.e.*, when Fourier's law is assumed. We have shown here that even if $\hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v})$ is not linear in \mathbf{v} , the presence of the inversion $-\mathbf{I}$ in the isotropy group implies that \mathbf{q} vanishes when $\text{grad } \vartheta$ vanishes¹.

PIPKIN & RIVLIN [8], § 18, have referred to the equation (3.16) as expressing the "non-existence of a piezo-caloric effect".

4. The Clausius-Duhem inequality

We regard \mathbf{q}/ϑ to be a vectorial *flux of entropy* and r/ϑ to be a scalar *supply of entropy*. In other words, for each process we define the *production of entropy* in the body \mathcal{B} to be

$$\Gamma = \frac{d}{dt} \int_{\mathcal{B}} \eta \, dm - \int_{\mathcal{B}} \frac{1}{\vartheta} r \, dm + \int_{\partial \mathcal{B}} \frac{1}{\vartheta} \mathbf{q} \cdot \mathbf{n} \, ds. \quad (4.1)$$

Under suitable smoothness assumptions one can write

$$\Gamma = \int_{\mathcal{B}} \gamma \, dm, \quad (4.2)$$

where

$$\begin{aligned} \gamma &= \dot{\eta} - \frac{r}{\vartheta} + \varrho^{-1} \text{div}(\mathbf{q}/\vartheta) \\ &= \dot{\eta} - \frac{r}{\vartheta} + \frac{1}{\varrho \vartheta} \text{div } \mathbf{q} - \frac{1}{\varrho \vartheta^2} \mathbf{q} \cdot \text{grad } \vartheta \end{aligned} \quad (4.3)$$

is the *specific production of entropy*.

¹ There are real materials, such as crystals in the pedial class, whose material symmetry does not imply that $-\mathbf{I}$ be in \mathcal{J} .

We now lay down the following

Postulate. *For every process admissible in a body consisting of a given material and for every part \mathcal{B} of this body the inequality*

$$\Gamma \geq 0 \quad (4.4)$$

is valid.

The inequality (4.4) is called *the Clausius-Duhem inequality*, and the postulate above is a statement of the second law of thermodynamics within the framework presented here. The main purpose of the present paper is to derive logical consequences from the postulate.

In order that the inequality (4.4) hold for *all* parts \mathcal{B} of the body, it is necessary and sufficient that the specific entropy production (4.3) be non-negative at all material points X and at all times t . Using (2.4) and $\vartheta > 0$, we see that this condition is equivalent to

$$\varrho \vartheta \gamma = \varrho (\vartheta \dot{\eta} - \dot{\varepsilon}) + \text{tr} \{ \mathbf{T} \mathbf{L} \} - \frac{1}{\vartheta} \mathbf{q} \cdot \text{grad } \vartheta \geq 0. \quad (4.5)$$

It follows from (3.1) and the chain rule for the differentiation of composite functions that

$$\dot{\varepsilon} = \text{tr} \{ \hat{\varepsilon}_{\mathbf{F}}(\mathbf{F}, \eta) \dot{\mathbf{F}} \} + \hat{\varepsilon}_{\eta}(\mathbf{F}, \eta) \dot{\eta}, \quad (4.6)$$

where $\hat{\varepsilon}_{\mathbf{F}}$ is the gradient¹ of $\hat{\varepsilon}$ with respect to the tensor \mathbf{F} , and $\hat{\varepsilon}_{\eta}$ the derivative of $\hat{\varepsilon}$ with respect to the scalar η . Substituting (4.6) into (4.5) and using the relation $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$, we obtain

$$\varrho [\vartheta - \hat{\varepsilon}_{\eta}(\mathbf{F}, \eta)] \dot{\eta} + \text{tr} \{ [\mathbf{T} - \varrho \mathbf{F} \hat{\varepsilon}_{\mathbf{F}}(\mathbf{F}, \eta)] \mathbf{L} \} - \frac{1}{\vartheta} \mathbf{q} \cdot \text{grad } \vartheta \geq 0. \quad (4.7)$$

5. Temperature relation, stress relation, dissipation inequality

Our postulate states that the inequality (4.7) must hold for *every* admissible process. We consider now processes that are defined by *homogeneous deformations* and *homogeneous entropy distributions*. For such processes, \mathbf{F} and η are independent of the material point and depend on only the time. By (3.2), the temperature distribution ϑ is also homogeneous, and hence $\text{grad } \vartheta = 0$. The inequality (4.7) then becomes, after substitution of (3.2) and (3.3),

$$\begin{aligned} \varrho [\hat{\vartheta}(\mathbf{F}, \eta) - \hat{\varepsilon}_{\eta}(\mathbf{F}, \eta)] \dot{\eta} + \text{tr} \{ [\hat{\mathbf{T}}(\mathbf{F}, \eta) - \varrho \mathbf{F} \hat{\varepsilon}_{\mathbf{F}}(\mathbf{F}, \eta)] \mathbf{L} \} + \\ + \text{tr} \{ \mathbf{I}(\mathbf{F}, \eta) [\mathbf{L}] \mathbf{L} \} \geq 0. \end{aligned} \quad (5.1)$$

As functions of time, \mathbf{F} and η can be chosen arbitrarily. It follows that for a *particular* time, the values of \mathbf{F} , η , $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, and $\dot{\eta}$ can be assigned arbitrarily. Taking $\mathbf{L} = 0$ in (5.1) shows that $[\hat{\vartheta}(\mathbf{F}, \eta) - \hat{\varepsilon}_{\eta}(\mathbf{F}, \eta)] \dot{\eta} \geq 0$ must hold for *every* value of $\dot{\eta}$; this can be the case only if the *temperature relation*

$$\vartheta = \hat{\vartheta}(\mathbf{F}, \eta) = \hat{\varepsilon}_{\eta}(\mathbf{F}, \eta) \quad (5.2)$$

holds.

Choosing $\dot{\eta} = 0$ and substituting $\alpha \mathbf{L}$ for \mathbf{L} in (5.1), where α is an arbitrary number, we obtain

$$\alpha \text{tr} \{ [\hat{\mathbf{T}}(\mathbf{F}, \eta) - \varrho \mathbf{F} \hat{\varepsilon}_{\mathbf{F}}(\mathbf{F}, \eta)] \mathbf{L} \} + \alpha^2 \text{tr} \{ \mathbf{L} \mathbf{I}(\mathbf{F}, \eta) [\mathbf{L}] \} \geq 0. \quad (5.3)$$

¹ The definition of the gradient used here is that given in reference [7], equations (C. 11) and (C. 11c). In components: $[\hat{\varepsilon}_{\mathbf{F}}(\mathbf{F}, \eta)]_{ij} = \frac{\partial \hat{\varepsilon}}{\partial F_{ji}}$.

This polynomial inequality in α must be satisfied for all values of α when \mathbf{F} , η , and \mathbf{L} are kept fixed. It is clear that this is the case if and only if the coefficient of α vanishes and the coefficient of α^2 is non-negative. Thus we obtain the *stress relation*

$$\hat{\mathbf{T}}(\mathbf{F}, \eta) = \varrho \mathbf{F} \hat{\mathbf{e}}_{\mathbf{F}}(\mathbf{F}, \eta) \quad (5.4)$$

and the *dissipation inequality*

$$\text{tr}\{\mathbf{L}\mathbf{I}(\mathbf{F}, \eta)[\mathbf{L}]\} \geq 0 \quad (5.5)$$

which must hold for all tensors \mathbf{L} .

Let $\mathbf{F} = \mathbf{R}\mathbf{U}$, where \mathbf{R} is the rotation tensor and \mathbf{U} the right stretch tensor. Writing (3.7)₄ with the choice $\mathbf{Q} = \mathbf{R}^T$ and $\mathbf{W} = \frac{1}{2}\mathbf{R}^T(\mathbf{L}^T - \mathbf{L})\mathbf{R}$, multiplying the result by $\mathbf{R}^T\mathbf{L}\mathbf{R}$, and then taking the trace, we obtain

$$\text{tr}\{\mathbf{L}\mathbf{I}(\mathbf{F}, \eta)[\mathbf{L}]\} = \text{tr}\{\mathbf{S}\mathbf{I}(\mathbf{U}, \eta)[\mathbf{S}]\}, \quad (5.6)$$

where $\mathbf{S} = \frac{1}{2}\mathbf{R}^T(\mathbf{L} + \mathbf{L}^T)\mathbf{R}$. It is evident that \mathbf{L} can always be chosen so that \mathbf{S} is an arbitrarily prescribed symmetric tensor. Therefore, by (5.6), the dissipation inequality (5.5) is equivalent to the requirement that

$$\text{tr}\{\mathbf{S}\mathbf{I}(\mathbf{U}, \eta)[\mathbf{S}]\} \geq 0 \quad (5.7)$$

hold for all symmetric tensors \mathbf{S} .

6. Heat conduction inequality. Summary

Let us assume now that the temperature relation (5.2) can be solved for the entropy, so that (5.2) is equivalent to

$$\eta = \bar{\eta}(\mathbf{F}, \vartheta). \quad (6.1)$$

The invertibility of (5.2) is assured, in particular, when for each fixed \mathbf{F} , ε is a convex function of η .

Continuing to keep the homogeneous reference configuration fixed, let us consider deformations χ that are both homogeneous and constant in time. Each such deformation is characterised by a unique value of \mathbf{F} , and, of course, we have $\mathbf{L} = \mathbf{0}$. Let us also consider time-independent temperature distributions $\vartheta = \vartheta(X)$. By (6.1), for each pair $\vartheta = \vartheta(X)$, $\mathbf{F} = \text{const.}$, a corresponding time-independent entropy distribution $\eta = \eta(X)$ is uniquely determined. Thus, by the last paragraph of Section 2, a prescribed constant \mathbf{F} and a prescribed function $\vartheta(X)$ uniquely determine an admissible process. For such a process, the inequality (4.7) reduces to

$$-\mathbf{q} \cdot \text{grad } \vartheta \geq 0. \quad (6.2)$$

Since $\vartheta = \vartheta(X)$ can be so chosen that, at a particular material point, both ϑ and $\text{grad } \vartheta$ have arbitrarily assigned values, it follows from (6.2) and (3.4) that the *heat conduction inequality*

$$\mathbf{v} \cdot \hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v}) \leq 0 \quad (6.3)$$

must hold for all vectors \mathbf{v} , all tensors \mathbf{F} and all scalars η .

Keeping \mathbf{F} and η fixed, let us define a scalar function of the vector \mathbf{v} by

$$\varphi(\mathbf{v}) = \mathbf{v} \cdot \hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v}). \quad (6.4)$$

The heat conduction inequality (6.3) states that $\varphi(\mathbf{v})$ has the maximum 0 at $\mathbf{v} = \mathbf{0}$. A rule of calculus tells us then that the gradient of $\varphi(\mathbf{v})$ with respect to \mathbf{v} must vanish for $\mathbf{v} = \mathbf{0}$. Using the rules for the differentiation of products, we find from (6.4) that

$$\{[\mathbf{v} \cdot \hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v})]^T \mathbf{v} + \hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v})\}|_{\mathbf{v}=\mathbf{0}} = \hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{0}) = \mathbf{0}. \quad (6.5)$$

Therefore, the condition (3.15), expressing the non-existence of a piezo-caloric effect, is also a consequence of the heat conduction inequality and hence can be expected to hold even if $-\mathbf{I}$ does not belong to the isotropy group of the material. This is, in fact, a result already obtained by PIPKIN & RIVLIN [8], § 18.

We summarize the results obtained as a consequence of our postulate: *An elastic material with heat conduction and viscosity is determined by the three response functions $\hat{\varepsilon}$, \mathbf{I} , and $\hat{\mathbf{q}}$. The temperature and stress are given by the relations (5.2) and (5.4). The response function \mathbf{I} is restricted by the dissipation inequality (5.5). If the temperature relation is invertible, the response function $\hat{\mathbf{q}}$ is restricted by the heat conduction inequality (6.3) and hence also by (3.15).*

Using (5.2), (3.3), and (5.4), we can rewrite the inequality (4.5) as follows:

$$\varrho \vartheta \gamma = \text{tr}\{\mathbf{L}\mathbf{I}(\mathbf{F}, \eta)[\mathbf{L}]\} - \frac{1}{\vartheta} \mathbf{q} \cdot \text{grad } \vartheta \geq 0. \quad (6.6)$$

It is clear from (6.6) that the temperature relation (5.2), the stress relation (5.4), the dissipation inequality (5.5), and the heat conduction inequality (6.3) are *sufficient* that the Clausius-Duhem inequality hold for all admissible processes.

7. Linearly viscous fluids

We say that a material is a *fluid* if its isotropy group \mathcal{J} is the group \mathcal{U} of all unimodular tensors¹. If $\mathcal{J} = \mathcal{U}$ for one reference configuration, then $\mathcal{J} = \mathcal{U}$ for all reference configurations of the same material; hence, the property of being a fluid is intrinsic to the material.

For an *elastic fluid with heat conduction and viscosity* we have the identities (3.12) for all unimodular tensors \mathbf{H} , and, therefore, $\hat{\varepsilon}(\mathbf{F}, \eta)$, $\mathbf{I}(\mathbf{F}, \eta)$ and $\hat{\mathbf{q}}(\mathbf{F}, \eta, \mathbf{v})$ can depend on \mathbf{F} only through $|\det \mathbf{F}|$ or, equivalently, through the specific volume $v = \frac{1}{\varrho}$. The temperature and stress relations now reduce to

$$\vartheta = \hat{\varepsilon}_\eta(v, \eta), \quad \mathbf{T}(v, \eta) = \hat{\varepsilon}_v(v, \eta) \mathbf{I}. \quad (7.1)$$

The identity (3.7)₄ implies that

$$\mathbf{Q} \mathbf{I}(v, \eta) [\mathbf{D}] \mathbf{Q}^T = \mathbf{I}(v, \eta) [\mathbf{Q} \mathbf{D} \mathbf{Q}^T] \quad (7.2)$$

holds for all symmetric tensors \mathbf{D} and all orthogonal tensors \mathbf{Q} . It is well known² that the identity (7.2) holds if and only if \mathbf{I} is of the form

$$\mathbf{I}(v, \eta) [\mathbf{D}] = 2\mu \mathbf{D} + \lambda (\text{tr } \mathbf{D}) \mathbf{I}, \quad (7.3)$$

¹ Cf. [2].

² Cf. [4], § 298.

where μ and λ are scalar functions of v and η . Substituting (7.1)₂ and (7.3) into (3.8)₃, we find

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} + \lambda(\text{tr}\mathbf{D})\mathbf{I} \quad (7.4)$$

where

$$p = -\hat{\varepsilon}_v(v, \eta). \quad (7.5)$$

Of course, (7.4) is the familiar constitutive equation for linearly viscous fluids. It is significant, however, that the pressure p is given by the same formula (7.5) which holds for perfect (inviscid) fluids. The dissipation inequality (5.7) states that

$$2\mu \text{tr}\mathbf{S}^2 + \lambda(\text{tr}\mathbf{S})^2 \geq 0 \quad (7.6)$$

for all symmetric tensors \mathbf{S} . It is known² that the inequalities

$$\mu \geq 0, \quad \lambda + \frac{2}{3}\mu \geq 0 \quad (7.7)$$

are equivalent to (7.6). They state that the shear viscosity μ and the bulk viscosity $\lambda + \frac{2}{3}\mu$ must both be non-negative.

For fluids, the identity (3.7)₅ reduces to

$$\mathbf{Q} \hat{\mathbf{q}}(v, \eta, \mathbf{v}) = \hat{\mathbf{q}}(v, \eta, \mathbf{Q}\mathbf{v}). \quad (7.8)$$

This identity holds if and only if $\hat{\mathbf{q}}$ is of the form

$$\hat{\mathbf{q}}(v, \eta, \mathbf{v}) = -\kappa \mathbf{v} \quad (7.9)$$

where κ is a scalar function of v, η , and $\mathbf{v} \cdot \mathbf{v}$. By (7.9), the heat conduction inequality (6.3) reduces to

$$\kappa \geq 0, \quad (7.10)$$

which states that the heat conductivity κ cannot be negative.

Acknowledgements. We are grateful to Drs. J. L. ERICKSEN, A. E. GREEN, R. A. TOUPIN and C. TRUESDELL for stimulating discussions.

This research was supported by the Air Force Office of Scientific Research under Contract AF 49(638)541 with Mellon Institute and by the National Science Foundation under Grant G 11316 to The Johns Hopkins University.

References

- [1] NOLL, W.: The foundations of classical mechanics in the light of recent advances in continuum mechanics. In: *The Axiomatic Method with Special Reference to Geometry and Physics*. Pp. 266—281. Amsterdam: North Holland Co. 1959.
- [2] NOLL, W.: A mathematical theory of the mechanical behavior of continuous media. *Arch. Rational Mech. Anal.* **2**, 197—226 (1958).
- [3] NOLL, W.: La mécanique classique, basée sur un axiome d'objectivité. A paper read at the Colloque Internationale sur la Méthode Axiomatique in *Mécanique Classique et Moderne*, Paris, 1959 (to be published by Gauthier-Villars, Paris).
- [4] TRUESDELL, C., & R. A. TOUPIN: The Classical Field Theories. In: *Encyclopedia of Physics*, Vol. III/1, edited by S. FLÜGGE. Berlin-Göttingen-Heidelberg: Springer 1960.
- [5] GREEN, A. E., & R. S. RIVLIN: The mechanics of non-linear materials with memory. *Arch. Rational Mech. Anal.* **1**, 1—34 (1957).
- [6] COLEMAN, B. D., & W. NOLL: An approximation theorem for functionals, with applications in continuum mechanics. *Arch. Rational Mech. Anal.* **6**, 355—370 (1960).

- [7] NOLL, W.: On the continuity of the solid and fluid states. J. Rational Mech. Anal. **4**, 3—81 (1955).
- [8] PIPKIN, A. C., & R. S. RIVLIN: The formulation of constitutive equations in continuum physics. Technical Report DA 4531/4 to the Department of the Army, Ordnance Corps (1958).
- [9] GREEN, A. E., & J. E. ADKINS: Large Elastic Deformations and Non-Linear Continuum Mechanics. Oxford: Clarendon Press 1960.
- [10] COLEMAN, B. D., & W. NOLL: Material symmetry and thermostatic inequalities in finite elastic deformations, forthcoming.

Mellon Institute
Pittsburgh, Pennsylvania
and
The Johns Hopkins University
Baltimore, Maryland

(Received February 11, 1963)