

Stabilized Mixed Finite Element Formulation

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Definitions

F: Deformation gradient

I: second-order unit tensor

u: Displacement

J: determinant of the deformation gradient

C: Right Cauchy-Green Strain Tensor

$\mathcal{W}(\mathbf{F})$: strain energy function

P: first Piola-Kirchhoff stress tensor

S: second PK stress tensor

α : cracks are represented by a scalar phase-field variable

p: Lagrange multiplier, hydrostatic pressure field

κ : bulk modulus

$$\kappa = \frac{E}{3(1 - 2\nu)} \quad (0.1)$$

μ : shear modulus

$$\mu = \frac{E}{2(1 + \nu)} \quad (0.2)$$

λ : Lamé modulus

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad (0.3)$$

For Plane Stress

$$\kappa = \frac{3 - \nu}{1 + \nu}, \quad \lambda = \frac{E\nu}{(1 - \nu)^2} \quad (0.4)$$

\mathcal{E}_ℓ : potential energy functional

$a(\alpha)$ is the decreasing stiffness modulation function

$w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

c_w is a normalization constant

1 Hyperelastic Phase-Field Fracture Models

Deformation Gradient

$$\mathbf{F} = \mathbf{I} + \nabla \otimes \mathbf{u} \quad (1.1)$$

where $J = \det \mathbf{F}$ and $\mathbf{C} = \mathbf{F}^T \mathbf{F}$.

The strain energy function $\mathcal{W}(\mathbf{F})$ is defined per unit reference volume such that the first PK and second PK

$$\mathbf{P} = \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \quad (1.2a)$$

$$\mathbf{S} = 2 \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{C}} \quad (1.2b)$$

where $\mathbf{P} = \mathbf{F}\mathbf{S}$.

1.1 Phase-Field Fracture Model

Non-modified strain energy function is the compressible Neo-Hookean:

$$\mathcal{W}(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2 \ln J) \quad (1.3)$$

For incompressible hyperelastic materials, the strain energy function is defined as

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p(J - 1), \quad (1.4)$$

We instead consider a damage dependent relaxation of the incompressibility constraint and the phase-field couples to the strain energy through the modified function

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J - 1)^2 \quad (1.5)$$

where the decreasing stiffness modulation function is $a(\alpha)$ and $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

$$a(\alpha) = (1 - \alpha)^2 \quad w(\alpha) = \alpha \quad (1.6)$$

In the code, we have the following definition

$$b(\alpha) = (1 - \alpha)^3 = \sqrt{a^3(\alpha)}$$

To circumvent numerical difficulties, we resort to the classical mixed formulation and introduce the pressure-like field

$$\begin{aligned} p &= -\sqrt{a^3(\alpha)}\kappa(J - 1) \\ p &= -b(\alpha)\kappa(J - 1) \end{aligned} \quad (1.7)$$

as an independent variable along with the displacement field.

Following the code, we have the following energy functional of the energy problem

$$\begin{aligned} \widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J - 1) - \frac{p^2}{2\lambda} \\ \widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\frac{\mu}{2}(I_1 - 3 - 2 \ln J) - b(\alpha)p(J - 1) - \frac{p^2}{2\lambda} \end{aligned}$$

Changes to relieve the residual stresses seen in the plane stress model:

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\frac{\mu}{2}(I_1 - 3 - 2 \ln J) - b(\alpha)p(J - 1) - (\alpha^2 + k_\ell)\frac{p^2}{2\lambda} \quad (1.8)$$

Where we know k_ℓ is a modeling parameter, so we can list the final energy functional as:

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha) \frac{\mu}{2} (I_1 - 3 - 2 \ln J) - b(\alpha) p (J - 1) - \alpha^2 \frac{p^2}{2\lambda} \quad (1.9)$$

The first Piola-Kirchhoff stress tensor is given:

$$\begin{aligned} \mathbf{P} &= \frac{\partial \widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{\partial \mathbf{F}} \\ &= \frac{\partial}{\partial \mathbf{F}} \left[a(\alpha) \mathcal{W}(\mathbf{F}) + a^3(\alpha) \frac{1}{2} \kappa (J - 1)^2 \right] \\ &= a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha) \frac{1}{2} \kappa \frac{\partial (J - 1)^2}{\partial \mathbf{F}} \\ &= a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha) \kappa (J - 1) \frac{\partial J}{\partial \mathbf{F}} \quad \text{where } \frac{\partial J}{\partial \mathbf{F}} = J \mathbf{F}^{-T} \\ &= a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha) \kappa (J - 1) J \mathbf{F}^{-T} \quad \text{substituting in pressure equation} \\ \mathbf{P} &= a(\alpha) \mu (\mathbf{F} - \mathbf{F}^{-T}) - b(\alpha) p J \mathbf{F}^{-T} \end{aligned} \quad (1.10)$$

1.1.1 Derivation from Bin2020 Paper

Here, unlike Eq. 21 from Bin2020, we drop λ_b which is not a consideration in this formulation

$$\mathcal{E}_\ell(\mathbf{u}, \alpha) = \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}, \alpha) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega \quad (1.11)$$

The normalization constant is defined as:

$$c_w = \int_0^1 \sqrt{w(\alpha)} d\alpha \quad (1.12)$$

Starting from Eq. 25 in the 2020 Li and Bouklas paper where κ is the bulk modulus

$$\mathcal{E}_\ell(\mathbf{u}, p, \Lambda, \alpha) = \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_{\Omega} \frac{p^2}{2\kappa} d\Omega + \int_{\Omega} \Lambda (p + \sqrt{a^3(\alpha)} \kappa (J - 1)) d\Omega \quad (1.13)$$

Identify the stationary point of the energy functional with respect to **pressure** (not Λ)

$$\begin{aligned} \frac{\partial \mathcal{E}_\ell}{\partial p} &= \int_{\Omega} \frac{p}{\kappa} d\Omega + \int_{\Omega} \Lambda d\Omega \\ 0 &= \frac{p}{\kappa} + \Lambda \rightarrow \Lambda = -p/\kappa \end{aligned}$$

Substituting this relationship into the energy functional yields:

$$\begin{aligned} \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_{\Omega} \frac{p^2}{2\kappa} d\Omega + \int_{\Omega} -\frac{p}{\kappa} (p + \sqrt{a^3(\alpha)} \kappa (J - 1)) d\Omega \\ &= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \frac{p^2}{\kappa} d\Omega - \int_{\Omega} \frac{p}{\kappa} \sqrt{a^3(\alpha)} \kappa (J - 1) d\Omega \\ \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \mathcal{E}_\ell(\mathbf{u}, \alpha) - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega \end{aligned} \quad (1.14)$$

Substitute in $\mathcal{E}_\ell(\mathbf{u}, \alpha)$ and substitute Eq. 1.3

$$\mathcal{E}_\ell(\mathbf{u}, p, \alpha) = \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega$$

The prior equation includes the full weak form, unless we want to consider linear interpolation of all fields. In that case, we can introduce the stabilization term

$$-\frac{\varpi h^2}{2\mu} \sqrt{a^3(\alpha)} \sum_{e=1}^{n_{el}} \int_{\Omega^e} J \mathbf{C}^{-1} : (\nabla p \cdot \nabla q) dV = 0$$

1.2 Changes for 2D Plane-Stress Models

Recalling the 1st PK stress in Eq. 1.10.

$$\mathbf{P} = a(\alpha) \mu (\mathbf{F} - \mathbf{F}^{-T}) - b(\alpha) p J \mathbf{F}^{-T}$$

In a plane-stress case, the \mathbf{P}_{33} component is zero:

$$\mathbf{P}_{33} = a(\alpha) \mu (\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha) p J \mathbf{F}_{33}^{-1} = 0$$

This can be multiplied by its associated test function to obtain the weak form

$$\int_{\Omega} \left(a(\alpha) \mu (\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha) p J \mathbf{F}_{33}^{-1} \right) v_{F33} dV = 0$$

In the FEniCS code, we expand the solution space to include displacement, pressure, and a component of the deformation gradient \mathbf{F}_{33} . Therefore, we include a change to the invariants of the deformation tensors:

$$\begin{aligned} J &= \det(\mathbf{F}) * F33 \\ I_c &= \text{tr}(\mathbf{C}) + F33 ** 2 \end{aligned}$$

Together with the weak form from above:

$$\begin{aligned} F_u &= \text{derivative}(\text{elastic_potential}, w_p, v_q) \setminus \\ &+ (a(\alpha) * \mu * (F33 - 1/F33) - b(\alpha) * p * J / F33) * v_F33 * dx \end{aligned}$$

1.2.1 Changes for 2D Discrete Crack Model

If we are considering a discrete fracture method

$$\begin{aligned} \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega \\ \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \int_{\Omega} \mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} p (J - 1) d\Omega \end{aligned}$$

where we have assumed for the energy functional

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = \frac{\mu}{2} (I_c - 3 - 2 \ln J) - p (J - 1) - \frac{p^2}{2\lambda}$$

Therefore, we can calculate the 1st Piola Kirchoff Stress as:

$$\begin{aligned}\mathbf{P} &= \frac{\mu}{2}(2\mathbf{F} - \frac{2}{J}J\mathbf{F}^{-T}) - pJ\mathbf{F}^{-T} \\ &= \mu(\mathbf{F} - \mathbf{F}^{-T}) - pJ\mathbf{F}^{-T}\end{aligned}$$

Taking the third component to be zero

$$\begin{aligned}P_{33} &= \mu(F_{33} - F_{33}^{-1}) - pJF_{33}^{-1} = 0 \\ &= F_{33} - F_{33}^{-1} - \frac{pJ}{\mu}F_{33}^{-1} = 0 \\ P_{33} &= F_{33}^2 - 1 - \frac{pJ}{\mu} = 0\end{aligned}$$

with the stabilization term and plane stress in the weak form

$$\begin{aligned}-\frac{\varpi h^2}{2\mu} \int_{\Omega} J\mathbf{C}^{-1} : (\nabla p \cdot \nabla q) dV &= 0 \\ \int_{\Omega} \left(\mathbf{F}_{33}^2 - 1 - \frac{pJ}{\mu} \right) v_{F_{33}} dV &= 0\end{aligned}$$

1.2.2 Changes for 2D displacement formulation

Removing pressure terms

$$\begin{aligned}\mathcal{E}_{\ell}(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2 \right) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)}p(J-1)d\Omega \\ \mathcal{E}_{\ell}(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2 \right) d\Omega\end{aligned}$$

with plane stress in the weak form (no need for stabilization terms)

$$\int_{\Omega} (a(\alpha)\mu(F_{33} - F_{33}^{-1}))v_{F_{33}}dV = 0$$

We have assumed the modified energy functional

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) \quad (1.15)$$

Therefore, we can calculate the 1st Piola Kirchoff Stress as:

$$\begin{aligned}\mathbf{P} &= a(\alpha)\frac{\mu}{2}(2\mathbf{F} - \frac{2}{J}J\mathbf{F}^{-T}) \\ &= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T})\end{aligned}$$

Taking the third component to be zero

$$P_{33} = a(\alpha)\mu(F_{33} - F_{33}^{-1}) = 0$$

2 Following Borden: Derivations of Analytical Phase Field

Note the full potential energy functional, which can also be called the lagrangian

$$\mathcal{E}_\ell(\mathbf{u}, p, \alpha) = \int_{\Omega} a(\alpha) \mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} \frac{p^2}{2\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} p (J - 1) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega$$

We can use the Euler-Lagrange equations to arrive at the equations of motion by taking the derivative with respect to displacement, pressure, and the scalar damage field. Starting with displacement:

$$\frac{\partial \mathcal{E}_\ell}{\partial \mathbf{u}} = \int_{\Omega} a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{u}} d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial p} = - \int_{\Omega} \frac{p}{\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)} (J - 1) d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial \alpha} = - \int_{\Omega} 2(1 - \alpha) \mathcal{W}(\mathbf{F}) d\Omega + \int_{\Omega} 3p(1 - \alpha)^2 (J - 1) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] d\Omega$$

Therefore we have three equations:

First should be mechanical eq, second is a an equation for pressure,

$$\begin{aligned} -\frac{p}{\kappa} - \sqrt{a^3(\alpha)} (J - 1) &= 0 \\ -\frac{p}{\kappa} - (1 - \alpha)^3 (J - 1) &= 0 \\ -\kappa (J - 1) (1 - \alpha)^3 &= p \end{aligned}$$

Lastly,

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) + 3p(1 - \alpha)^2 (J - 1) + \frac{G_c}{c_w} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] = 0$$

Substitute second equation into third

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) - 3\kappa(1 - \alpha)^5 (J - 1)^2 + \frac{G_c}{c_w} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] = 0$$

2.1 Homogeneous Solution

Therefore, we can study the homogeneous solution by ignoring spatial derivatives of α

$$-2(1 - \alpha_h) \mathcal{W}(\mathbf{F}) - 3\kappa(1 - \alpha_h)^5 (J - 1)^2 + \frac{G_c}{c_w \ell} = 0$$

We can expand and arrange

$$\begin{aligned} -2\mathcal{W}(\mathbf{F}) + 2\alpha_h \mathcal{W}(\mathbf{F}) - 3\kappa(J - 1)^2(1 - 5\alpha_h + 10\alpha_h^2 - 10\alpha_h^3 + 5\alpha_h^4 - \alpha_h^5) + \frac{G_c}{c_w \ell} &= 0 \\ -2\mathcal{W}(\mathbf{F}) + 2\alpha_h \mathcal{W}(\mathbf{F}) + \frac{G_c}{c_w \ell} &= 0 \\ -3\kappa(J - 1)^2 + 15\kappa(J - 1)^2 \alpha_h - 30\kappa(J - 1)^2 \alpha_h^2 + 30\kappa(J - 1)^2 \alpha_h^3 - 15\kappa(J - 1)^2 \alpha_h^4 + \kappa(J - 1)^2 \alpha_h^5 &= 0 \\ -2\mathcal{W}(\mathbf{F}) - 3\kappa(J - 1)^2 + \frac{G_c}{c_w \ell} + [2\mathcal{W}(\mathbf{F}) + 15\kappa(J - 1)^2] \alpha_h &= 0 \\ -30\kappa(J - 1)^2 \alpha_h^2 + 30\kappa(J - 1)^2 \alpha_h^3 - 15\kappa(J - 1)^2 \alpha_h^4 + \kappa(J - 1)^2 \alpha_h^5 &= 0 \end{aligned}$$

2.2 Non-Homogeneous Solution

Now for the Non-homogenous solution, we have the following

$$-2(1-\alpha)\mathcal{W}(\mathbf{F})-3\kappa(1-\alpha)^5(J-1)^2+\frac{G_c}{c_w}\left[\frac{1}{\ell}+2\ell\nabla^2\alpha\right]=0$$
(2.1)