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Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods☆

Thomas J.R. Hughes[1]

Division of Applied Mechanics, Durand Building, Stanford University, Stanford, CA 94305, USA

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Abstract

An approach is developed for deriving variational methods capable of representing multiscale phenomena. The ideas are first illustrated on the exterior problem for the Helmholtz equation. This leads to the well-known Dirichlet-to-Neumann formulation. Next, a class of subgrid scale models is developed and the relationships to 'bubble function' methods and stabilized methods are established. It is shown that both the latter methods are approximate subgrid scale models. The identification for stabilized methods leads to an analytical formula for τ, the 'intrinsic time scale', whose origins have been a mystery heretofore.

1. Introduction

The motivation behind the present study was to better understand the origins of stabilized methods. In particular, it was endeavored to show that stabilized methods could be *derived* from a firm theoretical foundation and, in the process, a *precise definition* of τ, the 'intrinsic time scale', would emerge. This is accomplished in the present work. It turns out that the origins of stabilized methods emanate from a particular class of subgrid scale models. These may be thought of as methods for dealing with multiscale phenomena which are pervasive in physics and engineering. A more general interest in multiscale phenomena led to the development of an abstraction of the method used to create the subgrid scale models. The abstract approach is the basis of the present study and its applicability is illustrated in a few different situations involving linear boundary value problems.

We begin with the exterior problem for the Helmholtz equation (i.e. the complex-valued, time-harmonic, wave equation). The viewpoint adopted is that there are two sets of scales present, one associated with the near field and one associated with the far field. The near-field scales are viewed as those of the exact solution exterior to the body, but within an enclosing simple surface, such as, for example, a sphere. The enclosing surface is not part of the specification of the boundary value problem, but rather it is specified by the analyst. The near-field scales are viewed as numerically 'resolvable' in this case. They may also be thought of as local or small scales. The scales associated with the solution exterior to the sphere (far field) are the global or large scales and are viewed as numerically 'unresolvable' in the sense that the infinite domain of the far field cannot be dealt with by conventional bounded-domain discretization methods. The solution of the original problem is decomposed into

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[1] Professor of Mechanical Engineering.

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non-overlapping near-field and far-field components, and the far-field component is exactly solved for in terms of the exterior Green's function satisfying homogeneous Dirichlet boundary conditions on the sphere. (Shapes other than a sphere are admissible, and useful in particular cases, but for each shape one must be able to solve the exterior Green's function problem in order to determine the far-field solution.) The far-field component of the solution is then eliminated from the problem for the near field. This results in a well-known variational formulation on a bounded domain which exactly characterizes the near-field component of the original problem. It is referred to as the Dirichlet-to-Neumann, or DtN, formulation because of the form of the boundary condition on the sphere in the problem on the bounded domain [1–5]. The so-called DtN boundary condition is non-local in the sense that it involves an integral operator coupling all points on the sphere. Non-locality is a typical ingredient in formulations of multiscale phenomena.

We next turn our attention to a problem for a general non-symmetric operator defined on a bounded domain. We envision both large and small scales present in the solution to this problem. We assume a discretization of the domain into an unstructured grid of non-overlapping subdomains. We prefer to think of these as finite elements, but this interpretation is not necessary. The characteristic length scale of the grid is much larger than the small scales, hence we refer to them as subgrid scales. On the other hand, the characteristic length scale of the grid is assumed small compared with the large scales. This problem has opposite traits when compared with the previous one, namely, here the small scales are viewed as unresolvable, whereas the large scales are viewed as resolvable. Another differentiating feature is that small and large scales overlap, whereas before they were disjoint. There is no loss of generality up to this point, but now we make the rather strong assumption that the subgrid scales vanish identically on the boundaries of the element domains. This assumption has the effect of localizing calculations for the subgrid scales in the sense that the problems are uncoupled from element to element. The subgrid scales are represented in terms of element Green's functions for the adjoint operator with homogeneous Dirichlet boundary conditions. Elimination of the subgrid scales from the variational problem for the resolvable scales leads to a method which exactly represents the effect of the subgrid scales on the resolvable scales, up to the assumption that the subgrid scales are zero on element boundaries. The effect of the subgrid scales is non-local within individual elements. This is an interesting formulation in its own right but our prime interest in it here is that it seems to represent a paradigm for some existing methods.

This theme is first explored in the context of 'bubbles' which are functions defined on the interiors of finite elements which vanish on element boundaries. The degrees-of-freedom associated with bubbles are eliminated by the well-known technique of 'static condensation.' Obviously, the element Green's function of the subgrid problem represents the 'ultimate bubble.' It follows that bubbles must somehow represent an approximation to the element Green's function. An explicit formula is derived which makes this idea precise. In multiscale problems in which the structure of the subgrid scales is not well characterized by typical finite element polynomials, bubbles are not effective. For example, in the case of advection-dominated diffusion phenomena, the subgrid scales consist of thin layers with steep gradients. These simply cannot be represented by typical polynomial bubbles. These observations are made precise in [6–8]. For recent developments in the theory of bubbles, see [9–11].

Stabilized methods have been successful in calculating the resolved scales for advective–diffusive phenomena, but a key ingredient in the formulation has appeared to be unmotivated and, despite the widespread success of stabilized methods, stabilized methods have never been derived from fundamental principles. For this reason, stabilized methods have been open to criticism. A sampling of works on stabilized methods is contained in [12–26].

The class of subgrid scale models derived herein has a very similar structure to stabilized methods. (This is not an accident.) It is observed that stabilized methods are related to the subgrid scale models and that τ is derivable from the element Green's function. Typically, a simple mean value of the Green's function suffices. In the case of higher-order elements, τ needs to be a polynomial agreeing with moments of the Green's function over the element. The main results obtained are: (1) stabilized methods are approximate subgrid scale models and (2) an explicit formula for evaluating τ emanates from fundamental principles. These results appear to be the denouement of stabilized methods. This brings to a close one chapter in the development of stabilized methods and simultaneously opens another: the derivation of stabilized methods from subgrid scale modeling concepts and the evaluation of τ from a firm theoretical basis.

The procedures developed herein appear to have many other fruitful applications which we intend to pursue in future work.

2. Point of departure: The Dirichlet-to-Neumann formulation for the Helmholtz operator

We consider the exterior problem for the Helmholtz operator. Let $\Omega \subset \mathbb{R}^d$ be an exterior domain, where d is the number of space dimensions (see Fig. 1). The boundary of Ω is denoted by Γ and admits the decomposition

$$\Gamma = \overline{\Gamma_g \cup \Gamma_h} \tag{1}$$

$$\emptyset = \overline{\Gamma_g \cap \Gamma_h} \tag{2}$$

where Γ_g and Γ_h are subsets of Γ. The unit outward normal vector to Γ is denoted by $\boldsymbol{n}$. The boundary-value problem consists of finding a function $u: \Omega \to \mathbb{C}$, such that for given functions $f: \Omega \to \mathbb{C}$, $g: \Gamma_g \to \mathbb{C}$ and $h: \Gamma_h \to \mathbb{C}$, the following equations are satisfied

$$\mathscr{L}u = f \qquad \text{in } \Omega \tag{3}$$

$$u = g \qquad \text{on } \Gamma_g \tag{4}$$

$$u_{,n} = ikh \qquad \text{on } \Gamma_h \tag{5}$$

$$\lim_{r\to\infty} r^{(d-1)/2}(u_{,r} - iku) = 0 \quad \text{(Sommerfeld radiation condition)} \tag{6}$$

where

$$-\mathscr{L} = \Delta + k^2 \quad \text{(Helmholtz operator)} \tag{7}$$

and $k \in \mathbb{C}$ is the wave number, $i = \sqrt{-1}$, and Δ is the Laplacian operator. The radial coordinate is denoted by r and a comma denotes partial differentiation.

Next, we consider a decomposition of the domain Ω into a bounded domain $\bar{\Omega}$ and an exterior region Ω'. The boundary which separates $\bar{\Omega}$ and Ω' is denoted Γ_R. It is assumed to have a simple shape (e.g. spherical) (see Fig. 2). The decomposition of Ω, and a corresponding decomposition of the solution of the boundary-value problem, are expressed analytically as follows

$$\Omega = \overline{\bar{\Omega} \cup \Omega'} \tag{8}$$

$$\emptyset = \bar{\Omega} \cap \Omega' \tag{9}$$

$$u = \bar{u} + u' \quad \text{(sum decomposition)} \tag{10}$$

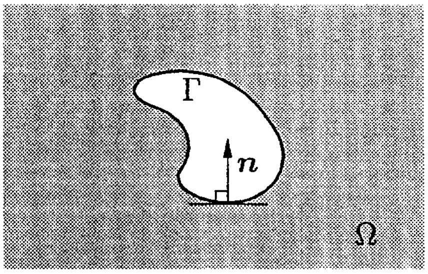

Fig. 1. An exterior domain.

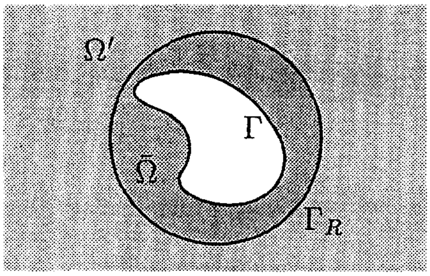

Fig. 2. Decomposition of Ω into a bounded domain $\bar{\Omega}$ and exterior domain Ω'.

$$\left.\begin{array}{l} \bar{u}|_{\Omega'} = 0 \\ u'|_{\bar{\Omega}} = 0 \end{array}\right\} \quad \text{(disjoint decomposition)} \tag{11}$$

$$u = \begin{cases} \bar{u} & \text{on } \bar{\Omega} \\ u' & \text{on } \Omega' \end{cases} \tag{12}$$

We think of $\bar{u}$ as the near-field solution and u' as the far-field solution.

2.1. Exterior Dirichlet problem for u'

We now focus our attention on the problem in the domain Ω' exterior to Γ_R. The unit outward normal vector on Γ_R (with respect to Ω') is denoted $\boldsymbol{n}'$ (see Fig. 3). We shall assume that f vanishes in the far field, i.e.

$$f = 0 \quad \text{on } \Omega' \tag{13}$$

The exterior Dirichlet problem consists of finding a function $u' : \Omega \to \mathbb{C}$ such that

$$\mathscr{L}u' = 0 \quad \text{in } \Omega' \tag{14}$$

$$u' = \bar{u} \quad \text{on } \Gamma_R \tag{15}$$

$$\lim_{r \to \infty} r^{(d-1)/2}(u'_{,r} - iku') = 0 \tag{16}$$

Note that the boundary condition (15) follows from the continuity of u across Γ_R.

2.1. Green's function for the exterior Dirichlet problem

The solution of the exterior Dirichlet problem can be expressed in terms of a Green's function g satisfying

$$\mathscr{L}\mathrm{g} = \delta \quad \text{in } \Omega' \tag{17}$$

$$\mathrm{g} = 0 \quad \text{on } \Gamma_R \tag{18}$$

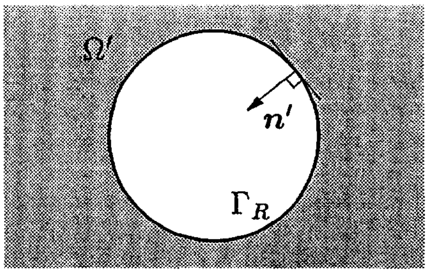

Fig. 3. Domain for the far-field problem.

$$\lim_{r\to\infty} r^{(d-1)/2}(g_{,r} - ikg) = 0 \tag{19}$$

From Green's identity, we have

$$u'(y) = -\int_{\Gamma_R} g_{,n'_x}(x, y)u'(x)\, d\Gamma_x \tag{20}$$

The so-called DtN map is obtained from (20) by differentiation with respect to $\boldsymbol{n}'$, viz.

$$u'_{,n'}(y) = -\int_{\Gamma_R} g_{,n'_x n'_y}(x, y)u'(x)\, d\Gamma_x \tag{21}$$

$$\Leftrightarrow \quad u'_{,n'} = Mu' \tag{22}$$

The DtN map is used to develop a formulation for $\bar{u}$ on the bounded domain $\bar{\Omega}$. In this way the far-field phenomena are incorporated in the problem for the near field. We shall develop the Dirichlet-to-Neumann formulation for $\bar{u}$ by way of a variational argument.

2.3. Variational formulation

Let Π denote the potential energy for the original boundary-value problem, namely

$$\begin{aligned}\Pi(u) &= \Pi(\bar{u} + u') \\ &= \frac{1}{2} a(\bar{u}, \bar{u}) + \frac{1}{2} a(u', u') - (\bar{u}, f) - (\bar{u}, ikh)_\Gamma\end{aligned} \tag{23}$$

where

$$a(w, u) = \int_\Omega (\nabla w \cdot \nabla u - k^2 wu)\, d\Omega \tag{24}$$

$$(w, f) = \int_\Omega wf\, d\Omega \tag{25}$$

$$(w, ikh)_\Gamma = \int_{\Gamma_h} w\, ikh\, d\Gamma \tag{26}$$

We consider a one-parameter family of variations of u, i.e.

$$(\bar{u} + u') + \varepsilon(\bar{w} + w') \tag{27}$$

where $\varepsilon \in \mathbb{R}$ is a parameter, subject to the following *continuity* constraints

$$\bar{u} = u' \quad \text{on } \Gamma_R \tag{28}$$

$$\bar{w} = w' \quad \text{on } \Gamma_R \tag{29}$$

We calculate the first variation of Π as follows

$$\begin{aligned}0 &= D\Pi(\bar{u} + u') \cdot (\bar{w} + w') \\ &= a(\bar{w}, \bar{u}) + a(w', u') - (\bar{w}, f) - (\bar{w}, ikh)_\Gamma \\ &= a(\bar{w}, \bar{u}) + (w', \mathscr{L}u') + (w', u'_{,n'})_{\Gamma_R} - (\bar{w}, f) - (\bar{w}, ikh)_\Gamma \\ &= a(\bar{w}, \bar{u}) + 0 + (\bar{w}, M\bar{u})_{\Gamma_R} - (\bar{w}, f) - (\bar{w}, ikh)_\Gamma\end{aligned} \tag{30}$$

where

$$(\bar{w}, M\bar{u})_{\Gamma_R} = \int_{\Gamma_R}\int_{\Gamma_R} \bar{w}(y) g_{,\bar{n}_x \bar{n}_y}(x, y)\bar{u}(x)\, d\Gamma_x\, d\Gamma_y \tag{31}$$

In obtaining (30) we have used (22) and the continuity conditions (28) and (29). Note that in (31) we have differentiated with respect to $\bar{\boldsymbol{n}} = -\boldsymbol{n}'$.

Eq. (30) can be written concisely as

$$\boxed{B(\bar{w}, \bar{u}; \mathrm{g}) = L(\bar{w})} \tag{32}$$

where

$$B(\bar{w}, \bar{u}; \mathrm{g}) = a(\bar{w}, \bar{u}) + (\bar{w}, M\bar{u})_{\Gamma_R} \tag{33}$$

$$L(\bar{w}) = (\bar{w}, f) + (\bar{w}, ikh)_{\Gamma} \tag{34}$$

REMARK 1. (32) is an *exact* characterization of $\bar{u}$.

REMARK 2. The effect of u' on the problem for $\bar{u}$ is *non-local*.

REMARK 3. (32) is the basis of *numerical* approximations, viz.

$$B(\bar{w}^h, \bar{u}^h; \mathrm{g}) = L(\bar{w}^h) \tag{35}$$

where $\bar{w}^h$ and $\bar{u}^h$ are finite-dimensional approximations of $\bar{w}$ and $\bar{u}$, respectively.

REMARK 4. In practice, M (or equivalently g) is also *approximated* by way of truncated series, differential operators, etc. Thus, in practice, we work with

$$B(\bar{w}^h, \bar{u}^h; \tilde{\mathrm{g}}) = L(\bar{w}^h) \tag{36}$$

where

$$\tilde{\mathrm{g}} \approx \mathrm{g} \tag{37}$$

2.4. Bounded domain problem for $\bar{u}$

The Euler–Lagrange equations of the variational formulation give rise to the boundary-value problem for $\bar{u}$ on the bounded domain $\bar{\Omega}$ (see Fig. 4), i.e.

$$\mathscr{L}\bar{u} = f \quad \text{in } \bar{\Omega} \tag{38}$$

$$\bar{u} = g \quad \text{on } \Gamma_g \tag{39}$$

$$\bar{u}_{,\bar{n}} = ikh \quad \text{on } \Gamma_h \tag{40}$$

$$\bar{u}_{,\bar{n}} = -M\bar{u} \quad \text{on } \Gamma_R \tag{41}$$

The preceding developments may be summarized in the following statements:
(1) $u = \bar{u} + u'$ (disjoint sum decomposition).
(2) u' is determined analytically.
(3) u' is eliminated, resulting in a formulation for $\bar{u}$ which is the basis of numerical approximations.
(4) The effect of u' is non-local in the problem for $\bar{u}$.

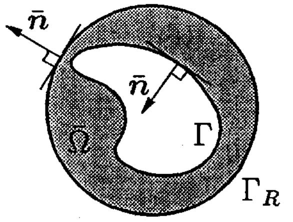

Fig. 4. Bounded domain for the near-field problem.

(5) Interpreted as a multiscale problem, u' represents the large scales of the far field, whereas $\bar{u}$ represents the small scales of the near field.

3. Subgrid scale models

We consider a bounded domain discretized into element subdomains Ω^e, with boundary Γ^e, $e = 1, 2, \ldots, n_{\text{el}}$, where n_{el} is the number of elements (see Fig. 5).

Let

$$\Omega' = \bigcup_{e=1}^{n_{\text{el}}} \Omega^e \quad \text{(element interiors)} \tag{42}$$

$$\Gamma' = \bigcup_{e=1}^{n_{\text{el}}} \Gamma^e \quad \text{(element boundaries)} \tag{43}$$

$$\Omega = \bar{\Omega} = \text{closure}(\Omega') \tag{44}$$

As before, we assume $u = \bar{u} + u'$, but this time we assume the sum decomposition is overlapping. We wish to think of $\bar{u}$ representing the resolvable scales, and u' representing the unresolvable, or subgrid, scales (see Figs. 6 and 7).

3.1. Abstract Dirichlet problem

We consider an abstract Dirichlet problem,

$$\mathscr{L}u = f \quad \text{in } \Omega \tag{45}$$

$$u = g \quad \text{on } \Gamma \tag{46}$$

where $\mathscr{L}$ may be non-symmetric. A variational formulation for (45), (46) may be developed as follows: Let

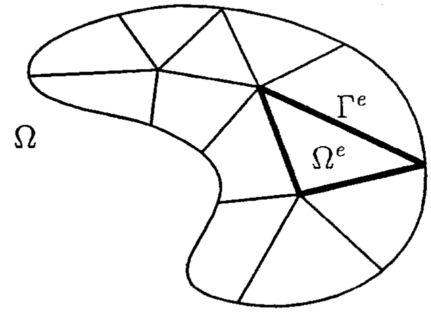

Fig. 5. Bounded domain discretized into element subdomains.

Fig. 6. Resolvable scales.

Fig. 7. Unresolvable, or subgrid, scales.

$$a(w, u) = (w, \mathscr{L}u) = (\mathscr{L}^* w, u) \tag{47}$$

for all sufficiently smooth w, u such that

$$w = u = 0 \quad \text{on } \Gamma \tag{48}$$

where $\mathscr{L}^*$ is the adjoint of $\mathscr{L}$.

Let

$$u = \bar{u} + u' \tag{49}$$

$$w = \bar{w} + w' \tag{50}$$

where we assume

$$u' = w' = 0 \quad \text{on } \Gamma' \tag{51}$$

REMARK. This is a rather strong assumption. We shall return to it later on.

With these, the variational equation may be written as

$$a(w, u) = (w, f) \tag{52}$$

or

$$a(\bar{w} + w', \bar{u} + u') = (\bar{w} + w', f) \tag{53}$$

Eq. (53) leads to two subproblems

$$(1) \quad \begin{cases} a(\bar{w}, \bar{u}) + a(\bar{w}, u') = (\bar{w}, f) \\ a(\bar{w}, \bar{u}) + (\mathscr{L}^* \bar{w}, u') = (\bar{w}, f) \end{cases} \tag{54}$$

$$(2) \quad \begin{cases} a(w', \bar{u}) + a(w', u') = (w', f) \\ (w', \mathscr{L}\bar{u}) + (w', \mathscr{L}u') = (w', f) \end{cases} \tag{55}$$

The Euler–Lagrange equations of the second subproblem are

$$\left.\begin{array}{ll} \mathscr{L}u' = -(\mathscr{L}\bar{u} - f) & \text{in } \Omega^e \\ u' = 0 & \text{on } \Gamma^e \end{array}\right\} \quad e = 1, 2, \ldots, n_{\text{el}} \tag{56}$$

3.2. Corresponding Green's function problem

The relevant Green's function problem corresponding to (56) is

$$\left.\begin{array}{ll} \mathscr{L}^* \mathrm{g} = \delta & \text{in } \Omega^e \\ \mathrm{g} = 0 & \text{on } \Gamma^e \end{array}\right\} \quad e = 1, 2, \ldots, n_{\text{el}} \tag{57}$$

Thus

$$u'(y) = -\int_{\Omega'} \mathrm{g}(x, y)(\mathscr{L}\bar{u} - f)(x)\, \mathrm{d}\Omega_x \tag{58}$$

$$u' = M(\mathscr{L}\bar{u} - f) \tag{59}$$

REMARK 1. $\mathscr{L}\bar{u} - f$ is the *residual* of the resolved scales.

REMARK 2. The subgrid scales u' are *driven* by the residual of the resolved scales.

We now wish to use (59) to eliminate the explicit dependence in (54) on the subgrid scales. Substituting (59) into (54), we arrive at

$$\boxed{a(\bar{w}, \bar{u}) + (\mathscr{L}^* \bar{w}, M(\mathscr{L}\bar{u} - f)) = (\bar{w}, f)} \tag{60}$$

where

$$(\mathscr{L}^*\bar{w}, M(\mathscr{L}\bar{u} - f)) = -\int_{\Omega'}\int_{\Omega'} (\mathscr{L}^*\bar{w})(y)g(x, y)(\mathscr{L}\bar{u} - f)(x)\, d\Omega_x\, d\Omega_y \tag{61}$$

and

$$\int_{\Omega'} = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tag{62}$$

REMARK 1. The effect of the unresolved scales on the resolved scales is *exactly* accounted for (up to the assumption (51)).

REMARK 2. The *non-local* effect of the unresolved scales on the resolved scales is confined within individual elements (due to the assumption (51)).

Eq. (60) may be written concisely as

$$\boxed{B(\bar{w}, \bar{u}; g) = L(\bar{w}; g)} \tag{63}$$

where

$$B(\bar{w}, \bar{u}; g) = a(\bar{w}, \bar{u}) + (\mathscr{L}^*\bar{w}, M(\mathscr{L}\bar{u})) \tag{64}$$

$$L(\bar{w}; g) = (\bar{w}, f) + (\mathscr{L}^*\bar{w}, Mf) \tag{65}$$

Eq. (63) is the basis of numerical approximations, namely

$$\boxed{B(\bar{w}^h, \bar{u}^h; \tilde{g}) = L(\bar{w}^h; \tilde{g})} \tag{66}$$

As before, $\bar{w}^h$ and $\bar{u}^h$ are finite dimensional approximations of $\bar{w}$ and $\bar{u}$, respectively, and $\tilde{g}$ is an approximate Green's function.

The development of the subgrid scale model is summarized by the following:

(1) $u = \bar{u} + u'$ (overlapping sum decomposition).
(2) u' is determined analytically on each element.
(3) The effect of u' is non-local within each element.
(4) The resulting problem for $\bar{u}$ can be solved numerically.
(5) The multiscale interpretation amounts to assuming that unresolvable, fine-scale behavior exists within each element, but not on element boundaries. Up to this assumption (i.e. (51)), the effect of the unresolved, fine scales on the resolvable coarse-scale behavior is exactly accounted for. The assumption (51) gives rise to a class of subgrid scale models which are sufficiently general for our purposes herein. However, there will be many cases of practical interest in which this assumption will be invalid. We hope to pursue this subject in subsequent work.

4. Bubbles

Bubble functions are typically higher-order polynomials which vanish on element boundaries. Practical and theoretical interest in bubbles has increased in recent years. Advocates of the Galerkin finite element method have resorted to bubbles in cases where the classical Galerkin method fails. We begin with a short development of the bubble function method and then we compare it to the class of subgrid scale models derived in the previous section. Let

$$\bar{u}^h = \sum_{A=1}^{n_{\text{nodes}}} N_A \bar{u}_A \quad (\text{likewise } \bar{w}^h) \tag{67}$$

where N_A is a standard finite element shape function, u_A is the corresponding nodal value, and n_{nodes} is the number of nodal points. Bubbles are represented by

$$u' = \sum_{A=1}^{n_{\text{bubbles}}} N'_A u'_A \quad (\text{likewise } w') \tag{68}$$

where N'_A is a bubble function, u'_A is the corresponding generalized coordinate, and n_{bubbles} is the number of bubble functions (see Fig. 8).

We substitute (67) and (68) into (54)–(55), and eliminate u'_A by *static condensation*:

$$\boxed{B(\bar{w}^h, \bar{u}^h; \tilde{g}) = L(\bar{w}^h; \tilde{g})} \tag{69}$$

where

$$B(\bar{w}^h, \bar{u}^h; \tilde{g}) = a(\bar{w}^h, \bar{u}^h) + (\mathscr{L}^*\bar{w}^h, \tilde{M}(\mathscr{L}\bar{u}^h)) \tag{70}$$

$$L(\bar{w}^h; \tilde{g}) = (\bar{w}^h, f) + (\mathscr{L}^*\bar{w}^h, \tilde{M}f) \tag{71}$$

and

$$(\mathscr{L}^*\bar{w}^h, \tilde{M}(\mathscr{L}\bar{u}^h)) = -\int_{\Omega'}\int_{\Omega'} (\mathscr{L}^*\bar{w}^h)(y)\tilde{g}(x, y)(\mathscr{L}\bar{u}^h)(x)\, \mathrm{d}\Omega_x\, \mathrm{d}\Omega_y \tag{72}$$

$$\boxed{\tilde{g}(x, y) = \sum_{A,B=1}^{n_{\text{bubbles}}} N'_A(x)[a(N'_B, N'_A)]^{-1} N'_B(y)} \tag{73}$$

REMARK 1. Comparing (69)–(71) to (63)–(65), respectively, we see that bubbles give rise to an approximate subgrid scale model.

REMARK 2. From (73), we se that bubbles generate an *approximate* element Green's function, $\tilde{g} \approx g$.

REMARK 3. In practice, the quality of the approximation is very often poor. The reason for this is

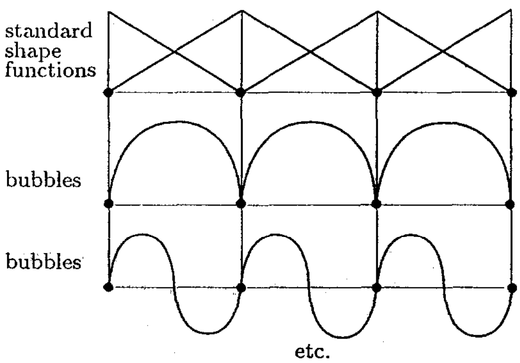

Fig. 8. Finite element shape functions and 'bubbles'.

that polynomial bubbles are usually not a good representation of the subgrid scale phenomena of interest (e.g. sharp layers, discontinuities, etc.).

5. Stabilized methods

The most common stabilized methods are generalized Galerkin methods of the form

$$a(\bar{w}^h, \bar{u}^h) + (\mathbb{L}\bar{w}^h, \tau(\mathscr{L}\bar{u}^h - f))_{\Omega'} = (\bar{w}^h, f) \tag{74}$$

REMARK 1. $\mathbb{L}$ is typically a *differential operator*. Examples are:

$\mathbb{L} = +\mathscr{L}$ Galerkin/least-squares (GLS) [20]

$\mathbb{L} = +\mathscr{L}_{\text{adv}}$ SUPG [12, 20]

$\mathbb{L} = -\mathscr{L}^*$ Franca et al. [26]

$\mathscr{L}_{\text{adv}}$ denotes the advective part of the operator $\mathscr{L}$.

REMARK 2. τ is typically an *algebraic operator*.

REMARK 3. The origin of τ and its precise definition have been unsolved mysteries.

5.1. *Relationship of stabilized methods with subgrid scale models*

By comparing (74) with (60), we see that a stabilized method of the type introduced by Franca et al. [26] is an approximate subgrid scale model in which the algebraic operator τ approximates the exact integral operator M. Thus

$$\tau = -\tilde{M} \approx -M \tag{75}$$

Equivalently,

$$\tau \cdot \delta(y - x) = \tilde{g}(x, y) \approx g(x, y) , \tag{76}$$

which may be concluded from the following calculation

$$\begin{aligned} &\int_{\Omega'} \int_{\Omega'} (-\mathscr{L}^*\bar{w}^h)(y)\tilde{g}(x, y)(\mathscr{L}\bar{u}^h - f)(x)\, d\Omega_x\, d\Omega_y \\ &\quad = \int_{\Omega'} \int_{\Omega'} (-\mathscr{L}^*\bar{w}^h)(y)\tau \cdot \delta(y - x)(\mathscr{L}\bar{u}^h - f)(x)\, d\Omega_x\, d\Omega_y \\ &\quad = \int_{\Omega'} (-\mathscr{L}^*\bar{w}^h)(x)\tau \cdot (\mathscr{L}\bar{u}^h - f)(x)\, d\Omega_x \end{aligned} \tag{77}$$

5.2. *A formula for τ*

Eq. (76) gives rise to a formula for τ in terms of the element Green's function, viz.,

$$\begin{aligned} \int_{\Omega^e} \int_{\Omega^e} \tau \cdot \delta(y - x)\, d\Omega_x\, d\Omega_y &= \int_{\Omega^e} \int_{\Omega^e} \tilde{g}(x, y)\, d\Omega_x\, d\Omega_y \\ &\approx \int_{\Omega^e} \int_{\Omega^e} g(x, y)\, d\Omega_x\, d\Omega_y \end{aligned} \tag{78}$$

$$\boxed{\tau = \frac{1}{\text{meas}(\Omega^e)} \int_{\Omega^e} \int_{\Omega^e} \mathrm{g}(x, y)\, \mathrm{d}\Omega_x\, \mathrm{d}\Omega_y} \tag{79}$$

REMARK 1. From (79) we see that the element *mean value* of the Green's function provides a definition of τ.

REMARK 2. (79) is adequate for low-order methods (h-adaptivity). For high-order methods (p-adaptivity), accounting for variation of τ over an element may be required. In this case, we may assume $\tau = \tau(x,y)$ is a polynomial of sufficiently high degree, and express τ in terms of the moments of the Green's function.

REMARK 3. Note that from (77) τ has the effect of converting two-point (non-local) correlations to single-point (local) correlations.

REMARK 4. It is interesting to observe that it may be possible to directly calculate the mean value of the Green's function without knowing the Green's function in detail.

5.3. Example: The advection–diffusion equation in one dimension

Let

$$\mathcal{L} = \mathcal{L}_{\text{adv}} + \mathcal{L}_{\text{diff}} \tag{80}$$

where

$$\mathcal{L}_{\text{adv}} = a \frac{\mathrm{d}}{\mathrm{d}x} \tag{81}$$

$$\mathcal{L}_{\text{diff}} = -\kappa \frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{82}$$

and a and κ are assumed to be positive constants. The homogeneous Dirichlet problem on an interval of length L consists of the following equations:

$$\mathcal{L}u = f \quad \text{in } \Omega = [0, L] \tag{83}$$

$$u = 0 \quad \text{on } \Gamma = \{0, L\} \tag{84}$$

The corresponding element Green's function problem is

$$\left.\begin{array}{ll} \mathcal{L}^*\mathrm{g} = \delta & \text{in } \Omega^e \\ \mathrm{g} = 0 & \text{on } \Gamma^e \end{array}\right\} \quad e = 1, 2, \ldots, n_{\text{el}} \tag{85}$$

where

$$\mathcal{L}^* = \mathcal{L}^*_{\text{adv}} + \mathcal{L}^*_{\text{diff}} \tag{86}$$

$$= -\mathcal{L}_{\text{adv}} + \mathcal{L}_{\text{diff}} \tag{87}$$

The solution of the Green's function problem is schematically illustrated in Fig. 9. Once g is obtained, we may calculate τ from (79):

$$\begin{aligned} \tau &= \frac{1}{\text{meas}(\Omega^e)} \int_{\Omega^e} \int_{\Omega^e} g(x, y)\, d\Omega_x\, d\Omega_y \\ &= \frac{h}{2a} (\coth \alpha - 1/\alpha) \end{aligned} \tag{88}$$

REMARK 1. The subgrid scale model is *pointwise exact* for any

$$\bar{u} = \sum_{A=1}^{n_{\text{nodes}}} N_A u_A \tag{89}$$

REMARK 2. The formula for τ is well known from the theory of stabilized methods.

REMARK 3. This τ results in a *nodally exact* stabilized method for piecewise linear N_A's and element-wise constant f. Element lengths may be unequal. Note that, for this case, all stabilized methods are identical.

REMARK 4. If we employ piecewise quadratic N_A's and element-wise linear f, τ needs to be assumed to be a linear function of x and y and must agree with the element Green's function through linear moments, etc.

5.4. Final review of the method

We summarize for the final time the essential features of the multiscale approach adopted herein:

(1) We look at sum decompositions of the form $u = \bar{u} + u'$, where $\bar{u}$ is viewed as resolvable and u' is viewed as unresolvable.
(2) u' is determined analytically in terms of a Green's function.
(3) u' is eliminated, resulting in a problem for $\bar{u}$.
(4) The effect of u' on $\bar{u}$ is non-local.
(5) $\bar{u}$ is determined numerically.

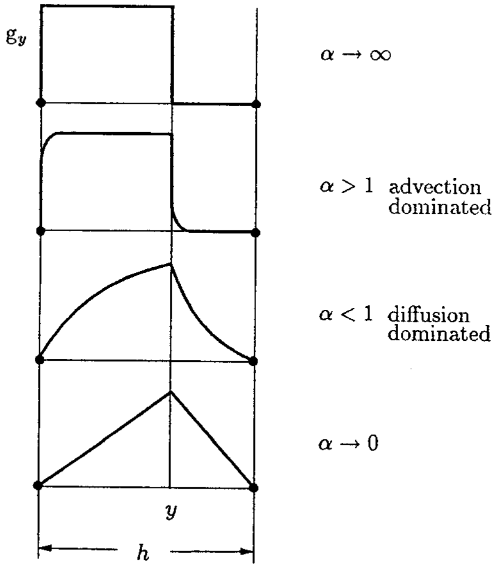

Fig. 9. Element Green's function for the adjoint advection–diffusion operator, where $\alpha = (ah)/2\kappa$ is the element Peclet number, and $h = \text{meas}(\Omega^e)$ is the element length.

6. Concluding remarks

A method for studying multiscale phenomena has been presented herein. The following results were obtained:

(1) The approach generates the DtN formulation by way of a disjoint sum decomposition $u = \bar{u} + u'$.
(2) The approach generates a class of subgrid scale models by way of an overlapping sum decomposition $u = \bar{u} + u'$ in which $u' = 0$ on element boundaries.
(3) Bubble functions give rise to an *approximate* Green's function in the subgrid scale model.
(4) Stabilized methods are identified as *approximate* subgrid scale models.
(5) A formula for τ in terms of the element Green's function emanates from this identification, namely

$$\tau = \frac{1}{\text{meas}(\Omega^e)} \int_{\Omega^e} \int_{\Omega^e} g(x, y)\, \mathrm{d}\Omega_x\, \mathrm{d}\Omega_y$$

(6) This opens the way to improved representations of τ in stabilized methods.
(7) The theoretical foundations of stabilized methods have been identified (at last!).

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