Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of the Mechanics and Physics of Solids
53 (2005) 2284–2319

JOURNAL OF THE MECHANICS AND PHYSICS OF SOLIDS

www.elsevier.com/locate/jmps

Growth and instability in elastic tissues

Martine Ben Amar[a], Alain Goriely[a,b,*]

[a]Laboratoire de Physique Statistique de l'Ecole Normale Supérieure, 24 rue Lhomond, F75231 Paris Cedex 05, France

[b]Department of Mathematics and Program in Applied Mathematics, University of Arizona, AZ 85721, USA

Received 17 November 2004; received in revised form 21 April 2005; accepted 25 April 2005

Abstract

The effect of growth in the stability of elastic materials is studied. From a stability perspective, growth and resorption have two main effects. First a change of mass modifies the geometry of the system and possibly the critical lengths involved in stability thresholds. Second, growth may depend on stress but also it may induce residual stresses in the material. These stresses change the effective loads and they may both stabilize or destabilize the material. To discuss the stability of growing elastic materials, the theory of finite elasticity is used as a general framework for the mechanical description of elastic properties and growth is taken into account through the multiplicative decomposition of the deformation gradient. The formalism of incremental deformation is adapted to include growth effects. As an application of the formalism, the stability of a growing neo-Hookean incompressible spherical shell under external pressure is analyzed. Numerical and analytical methods are combined to obtain explicit stability results and to identify the role of mechanical and geometric effects. The importance of residual stress is established by showing that under large anisotropic growth a spherical shell can become spontaneously unstable without any external loading.

Keywords: Morphoelasticity; Finite elasticity; Instability; Soft tissues

*Corresponding author. Department of Mathematics and Program in Applied Mathematics, University of Arizona, AZ 85721, USA. Tel.: +1 5206212559; fax: +1 5206218322.

E-mail addresses: benamar@lps.ens.fr (M. Ben Amar), goriely@math.arizona.edu (A. Goriely).

URL: http://www.math.arizona.edu/~goriely.

0022-5096/$ - see front matter
doi:10.1016/j.jmps.2005.04.008

1. Introduction

Growth describes the process by which a material increases in size by addition of mass. In physics, growth may refer to processes such as epitaxial growth where material is fed in the system and reorganize on the substrate but mostly refers to phenomena associated with phase transition: the interface between the phases evolves in time to produce a crystal (Langer, 1980; Ben Amar and Pomeau, 1986; Kessler et al., 1988). In these free boundary problems controlled by diffusion, the interface is a line of discontinuity with no particular material property. By opposition, most non-diffusive growth processes are the hallmark of life and are generated by biological systems (with the notable exception of swelling gels (Onuki, 1989; Sudipto et al., 2002)). Increase in size in biological systems fills many purposes and functions and accordingly it takes place in many different ways and guises. Growth can be mostly restricted to precise locations such as in *tip growth*, first described by Reinhardt (1892). Tip growth takes place in most microscopic filamentary systems such as fungi, filamentary bacteria or root hair (Gooday and Trinci, 1980; Howard and Valent, 1996; Harold, 1997; Money, 1997; Goriely and Tabor, 2003). *Surface growth* describes mechanisms such as accretion and deposition in hard tissues found for instance in the formation of teeth, seashells, horns and, to a lesser extent, bones (Thomspon, 1992; Skalak and Hoger, 1997). *Volumetric growth* takes place in the bulk of the material and is typical of many developmental, physiological or pathological processes. Volumetric growth has been particularly well documented in specific systems such as arteries, heart, muscles, and solid tumors (Taber, 1995; Humphrey, 2003; Cowin, 2004).

Continuum mechanics provides a natural framework for the description of soft tissues. In particular, the theory of elasticity has been used since the nineteenth century for the modeling of mechanical properties resulting from the interaction between the physical environment and biological structures and functions such as in the theory of Wolff for bone growth or Woods for the regulation of heart wall stress (Taber, 1995). However, it was not before the work of Fung (1990, 1993) and Skalak et al. (1973) that it was fully appreciated that soft tissues are complex mechanical materials with typical nonlinear, anisotropic, inhomogeneous behaviors which are often subject to large strains and stresses (Humphrey, 2003). Therefore, their mechanical properties are best described within the theory of finite elasticity and much work has been done to establish constitutive relationships for general biological materials with particular symmetry or for specific physiological materials.

There have been many attempts to include the effect of growth and remodeling in biological materials. The general idea is that growth can be taken into account by considering that the deformations of a body can be due to both change of mass and to elastic deformation (Hsu, 1968; Cowin and Hegedus, 1976; Skalak, 1981; Entov, 1983; Drozdov, 1990; Stein, 1995). These lines of thoughts culminated to a general statement of growth (Rodriguez et al., 1994) in terms of a multiplicative decomposition of deformation gradient similar to the ones used in elasto-plasticity (Lee, 1969; Casey and Naghdi, 1980). The fundamental idea of Rodriguez et al. (1994) is that the geometric deformation tensor is decomposed as the product of a growth tensor describing the local addition of a material and an elastic tensor characterizing the reorganization of the body needed to ensure compatibility and integrity. Moreover, the stresses generated through growth depend only on the elastic part of the deformation tensor. This early seminal work on growth has since been amply discussed (Drozdov and Khanina, 1997; Cowin, 2004; Humphrey, 2003) and put on a more rigorous foundation (Epstein and Maugin, 2000; DiCarlo and Quiligotti, 2002; Lubarda and Hoger, 2002). This kinematic theory is now a starting point of most theoretical analyses in which other effects are included (Cowin, 1996; Klisch et al., 2001; Garikipati et al., 2004) and to study specific models involving growth of soft tissues such as the heart (Lin and Taber, 1995), arteries (Taber and Humphrey, 2001), the aorta (Taber and Eggers, 1996; Taber, 1998b), smooth muscles (Taber, 1998a), and cartilage (Klisch et al., 2003). Despite progresses in this field it has been commented by many authors (Taber, 1998a; Ambrosi and Mollica, 2004) that one of the central problems is the development of a constitutive theory for the relationship between growth and mechanical variables such as stresses and strains. Furthermore, the multiplicative decomposition is mostly a geometric modeling of growth and does not address the process by which growth occurs. Humphrey and Rajagopal (2002) recently argued that growth should be modeled within a mixture theory by following mass production, resorption and remodeling of individual constituents. Growth is then represented by the evolution of different natural configurations and the differences between these configurations lead to residual stress.

One of the most important mechanical features associated with growth is the generation of residual stress (Skalak et al., 1996). These stresses are due to the incompatibility of growth and have been shown to play an important role in the function of soft tissues such as peak stress regulation in arteries (Humphrey, 2003) or in the growth of solid tumor (Ambrosi and Mollica, 2004). Essentially, as growth takes place locally, parts of the body needs to be stretched or compressed to ensure integrity (no cavitation) and compatibility (no overlap) of the body. In turn, these strains are associated with stresses referred to as *residual stresses* (Skalak, 1981; Hoger, 1986; Skalak et al., 1996). It is well-known that elastic materials under external loads can develop instabilities such as buckling or wrinkling and a natural question is whether growth itself can generate sufficient stress as to destabilize the body. This is not obvious since the geometric effect of growth is to change the different lengths associated with the body (such as thickness) and this may have a stabilizing effect (typically, stubby bodies are more stable than slender ones). Therefore, geometric and mechanical effects can both help stabilize or destabilize the body and there is an interesting interplay between these effects to be studied.

In this paper, we study instabilities related to growth or resorption in elastic materials. Elastic materials are represented by hyperelastic materials in the theory of finite elasticity and growth is described through the decomposition of the deformation gradient. Stability is defined here as a bifurcation, that is the existence of new solutions to the incremental deformation problem superimposed on the bifurcation of a finite strain solution. The theory of incremental deformation (Green and Adkins, 1970; Green and Zerna, 1992; Ogden, 1984) is adopted and adapted to allow for the addition of growth.

To illustrate the formalism we study the instability due to constant anisotropic growth with respect to the reference configuration in a simple problem. Specifically, we consider the anisotropic but spherically symmetric constant growth of an incompressible spherical shell made out of a neo-Hookean material. The geometry of this problem is relevant in many similar problems (Totafurno and Bjerknes, 1983; Chen et al., 2001) and its exhaustive analysis gives us some insight into the possible effects of growth on stability. We compute the residual stress due to growth and consider the bifurcation to axisymmetric solutions. We combine analytic techniques available for high-mode numbers and thin shells together with numerical results to study the relative effects of geometry and mechanics. Finally, we compute the critical anisotropic growth necessary to destabilize shells of arbitrary thickness.

2. General setup

Consider an elastic body $\mathscr{B} \in \mathbb{R}^3$ described in the *reference configuration* by the material coordinates $\mathbf{X}$. After deformation, the body is described by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The description of the body in these coordinates is referred to as the *current configuration*.[1] We assume that this deformation is caused by two effects, either an elastic response (a *pure response* of the material) or due to growth. These contributions are embedded in the deformation tensor. Let $\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} \equiv \mathrm{Grad}(\boldsymbol{\chi})$ be the *geometric deformation tensor*, then the basic assumption of elastic growth is that $\mathbf{F}$ is decomposed as follows (Rodriguez et al., 1994):

$$\mathbf{F} = \mathbf{A} \cdot \mathbf{G}, \tag{1}$$

where $\mathbf{A}$ is the *elastic deformation tensor* describing pure deformation resulting from stresses and $\mathbf{G}$ is the *growth tensor*, describing locally the generation (or removal) of new material points (see Fig. 1 for a diagram). If the material is *elastically incompressible*, we have $\det(\mathbf{A}) = 1$ and $J = \det(\mathbf{F}) = \det(\mathbf{G})$ describes the change in volume due to growth.

The body is assumed to be hyperelastic, that is, there is a strain energy function $W = W(\mathbf{A})$ and the *nominal stress-deformation relation* for unconstrained material associated with the deformation gradient tensor $\mathbf{F}$ is given by:

$$\mathbf{S} = J \frac{\partial W}{\partial \mathbf{F}}(\mathbf{A}) \tag{2}$$

$$= J\mathbf{G}^{-1} \cdot W_{\mathbf{A}}, \tag{3}$$

where S is the *nominal stress tensor* and $W_{\mathbf{A}}$ denotes the derivative of $W(\mathbf{A})$ with respect to the tensor $\mathbf{A}$. The extra factor J is due to the fact that elastic strains are computed from the grown state and not from the reference state. We have used Eq. (95) in Appendix A to express the derivative of W as a function of $\mathbf{A}$. If the material is *elastically constrained* (that is, there is one or more scalar relations, $C(\mathbf{A}) = 0$,

[1]Other terminologies for the reference and current configurations include, respectively, *Lagrangian* and *Eulerian* or *material* and *instantaneous*.

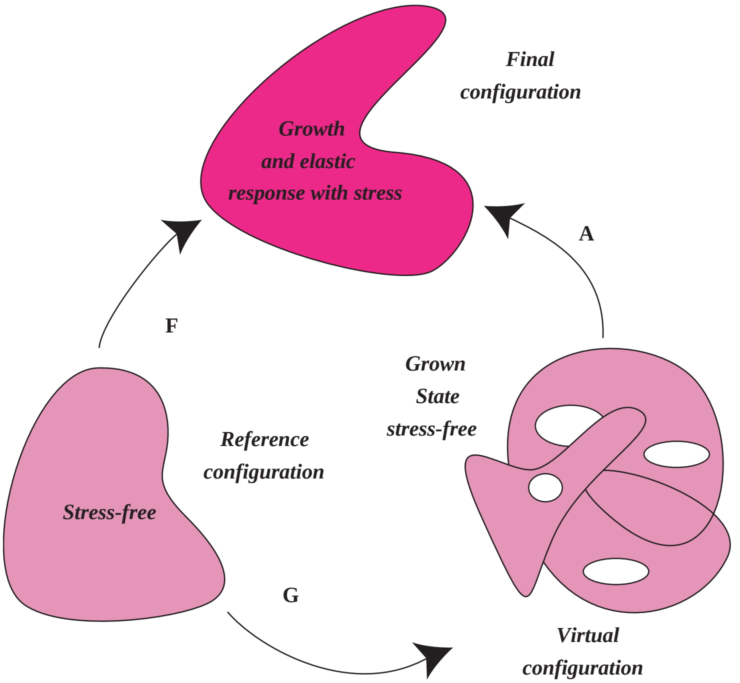

Fig. 1. Decomposition of the geometric deformation tensor into growth G and elastic A deformation tensors.

between the elastic strains), a Lagrange multiplier q has to be introduced and the nominal stress becomes

$$\begin{aligned}\mathbf{S} &= J\frac{\partial W}{\partial \mathbf{F}}(\mathbf{A}) - qJ\frac{\partial C}{\partial \mathbf{F}}(\mathbf{A}) &(4)\\ &= J\mathbf{G}^{-1}\cdot W_{\mathbf{A}} - qJ\mathbf{G}^{-1}\cdot C_{\mathbf{A}}. &(5)\end{aligned}$$

In particular, the condition for volumetric incompressibility is $C = \det(\mathbf{A}) - 1$ and the nominal stress is (using Eq. (96)):

$$\mathbf{S} = J\mathbf{G}^{-1}\cdot W_{\mathbf{A}} - qJ\mathbf{G}^{-1}\cdot \mathbf{A}^{-1}, \tag{6}$$

where $q = q(\mathbf{X}, t)$ is related to the hydrostatic pressure. Once the nominal stress tensor is known, the *Cauchy stress tensor* $\mathbf{T}$ expressing the actual stress in the body after deformation can be obtained from the geometric connection:

$$\mathbf{T}^{\mathrm{T}} = J^{-1}\mathbf{F}\cdot\mathbf{S}. \tag{7}$$

That is,

$$\begin{aligned}\mathbf{T}^{\mathrm{T}} &= JJ^{-1}\mathbf{A}\cdot\mathbf{G}\cdot\mathbf{G}^{-1}\cdot(W_{\mathbf{A}} - qC_{\mathbf{A}})\\ &= \mathbf{A}\cdot(W_{\mathbf{A}} - qC_{\mathbf{A}}). &(8)\end{aligned}$$

The Cauchy stress tensor is expressed in the current configuration variables $\mathbf{x}$. From objectivity considerations, we have $\mathbf{A} \cdot W_{\mathbf{A}} = (W_{\mathbf{A}})^{\mathrm{T}} \cdot \mathbf{A}^{\mathrm{T}}$ and $\mathbf{A} \cdot C_{\mathbf{A}} = (C_{\mathbf{A}})^{\mathrm{T}} \cdot \mathbf{A}^{\mathrm{T}}$. Therefore, we have $\mathbf{T}^{\mathrm{T}} = \mathbf{T}$ which is the local form of Cauchy's second law on the balance of rotational momentum and the expression for the Cauchy stress tensor becomes

$$\mathbf{T} = \mathbf{A} \cdot (W_{\mathbf{A}} - qC_{\mathbf{A}}). \tag{9}$$

In the incompressible case, this last relation becomes

$$\mathbf{T} = \mathbf{A} \cdot W_{\mathbf{A}} - q\mathbf{1} \tag{10}$$

and q is the hydrostatic pressure necessary to ensure incompressibility. The equation for mechanical equilibrium in the absence of body forces is

$$\mathrm{Div}(\mathbf{S}) = 0, \tag{11}$$

or,

$$\mathrm{div}(\mathbf{T}) = 0, \tag{12}$$

where Div (resp. div.) denotes the divergence in the reference (resp. current) configuration. Eq. (11) (or (12)) provides, through the constitutive relationship (6) (or (8)) a system of three equations for the strains $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. If the strains are known, the stresses are obtained from (6).

As usual, conditions on the boundary can be imposed through dead-loading (that is prescribing the normal components of the stresses $\mathbf{T}$ or $\mathbf{S}$ at the boundary), rigid-loading (that is by fixing the deformations allowed at the boundary of the body) or mixed-loading (fixing deformations on some part of the body and stresses on the remainder). As an example and of particular interest for our analysis is the loading by fluid pressure P. The traction on the body is given by the Cauchy stress in the normal direction $\mathbf{n}$ of the boundary:

$$\mathbf{T} \cdot \mathbf{n} = -P\mathbf{n}. \tag{13}$$

The corresponding boundary condition on the nominal stress reads:

$$\mathbf{S}^{\mathrm{T}} \cdot \mathbf{N} = -PJ(\mathbf{F}^{\mathrm{T}})^{-1} \cdot \mathbf{N}. \tag{14}$$

3. Incremental deformation

We consider the case where the growth tensor $\mathbf{G}$ is constant in space and time. This corresponds to a case where growth is homogeneous but possibly anisotropic. This is the simplest type of growth tensor leading to residual stress. Here growth is completely driven from outside and it is not influenced by the state of the system such as stresses or strains. Therefore, at this level of description growth tensor can be seen as the control parameter that changes the geometry of the body and its residual stress. Once, a particular reference configuration has been identified with prescribed boundary conditions and allowable deformations, the deformations and stresses can be computed, in principle, by solving (12). This finite-deformation solution represents a mechanical equilibrium whose stability can be established by considering a wider class of deformation. For instance, in Section 4, we compute the radially symmetric deformation of a growing shell under pressure. Once the change in radius due to growth and applied stress has been established, one can study the stability of this configuration with respect to non-radial deformations. The general procedure to test the stability is to implement a perturbation expansion in which the new deformations are assumed to be infinitesimal (or "incremental"). Namely, we assume that the nominal stress $\mathbf{S}^{(0)}$ and deformations $\boldsymbol{\chi}^{(0)}$ are known for given boundary conditions. Then, we introduce incremental deformations as follows:

$$\boldsymbol{\chi} = \boldsymbol{\chi}^{(0)} + \varepsilon\boldsymbol{\chi}^{(1)}, \tag{15}$$

where ε has been introduced as a small parameter (the term $\boldsymbol{\chi}^{(1)}$ is then of order one) that characterizes the size of the perturbation superimposed on the finite deformation. Accordingly, we have

$$\mathbf{F} = \mathrm{Grad}(\boldsymbol{\chi}) = (\mathbf{1} + \varepsilon\mathbf{F}^{(1)}) \cdot \mathbf{F}^{(0)}, \tag{16}$$

and, since the growth tensor $\mathbf{G}$ is assumed to be constant, the elastic deformation tensor is simply

$$\mathbf{A} = (\mathbf{1} + \varepsilon\mathbf{A}^{(1)}) \cdot \mathbf{A}^{(0)}, \tag{17}$$

where $\mathbf{A}^{(0)} = \mathbf{F}^{(0)} \cdot \mathbf{G}^{-1}$ and $\mathbf{A}^{(1)} = \mathbf{F}^{(1)}$. The peculiar form of the expansions for $\mathbf{F}$ and $\mathbf{A}$ with $\mathbf{F}^{(0)}$ as a factor is chosen so that the incremental deformation is expressed in the current configuration rather than the reference configuration. Note that since $\det(\mathbf{F}) = \det(\mathbf{F}^{(0)}) = J$, we have

$$\mathrm{tr}(\mathbf{F}^{(1)}) = 0. \tag{18}$$

Since both the incremental deformation $\mathbf{F}^{(1)}$ and the boundary conditions are expressed in the current configuration we only consider the Cauchy stress tensor and assume it can be expanded in ε:

$$\mathbf{T} = \mathbf{T}^{(0)} + \varepsilon\mathbf{T}^{(1)} + \mathrm{O}(\varepsilon^2). \tag{19}$$

We can now relate the incremental stresses to the incremental strains by the constitutive relationship. In terms of the nominal stress tensor, relation (8) to zeroth and first-order reads

$$\mathbf{T}^{(0)} = \mathbf{A}^{(0)} \cdot (W_{\mathbf{A}}^{(0)} - q^{(0)} C_{\mathbf{A}}^{(0)}), \tag{20}$$

and

$$\mathbf{T}^{(1)} = \mathbf{F}^{(1)} \cdot \mathbf{T}^{(0)} - q^{(1)}\mathbf{A}^{(0)} \cdot C_{\mathbf{A}}^{(0)} + \mathscr{L}{:}\,\mathbf{F}^{(1)} - q^{(0)}\mathscr{C} : \mathbf{F}^{(1)}, \tag{21}$$

where $q = q^{(0)} + \varepsilon q^{(1)}$, and $\mathscr{L}$, $\mathscr{C}$ are the fourth-order tensors defined by

$$\mathscr{L} : \mathbf{F}^{(1)} = \mathbf{A}^{(0)} \cdot W_{\mathbf{AA}}^{(0)} : \mathbf{F}^{(1)} \cdot \mathbf{A}^{(0)}, \tag{22}$$

$$\mathscr{C} : \mathbf{F}^{(1)} = \mathbf{A}^{(0)} \cdot C_{\mathbf{AA}}^{(0)} : \mathbf{F}^{(1)} \cdot \mathbf{A}^{(0)}, \tag{23}$$

and $W_{\mathbf{A}}^{(0)}$, $W_{\mathbf{AA}}^{(0)}$ are the first and second derivatives of W with respect to $\mathbf{A}$ evaluated on $\mathbf{A}^{(0)}$. The components of the fourth-order tensor $\mathscr{L}$ are the *instantaneous elastic moduli*. Their explicit form is given in Appendix C. In the incompressible case, $C = \det(\mathbf{A}) - 1$, so that (20) now reads

$$\mathbf{T}^{(0)} = \mathbf{A}^{(0)} \cdot W_{\mathbf{A}}^{(0)} - q^{(0)}\mathbf{1}, \tag{24}$$

and we can use Eq. (97) of Appendix A to show that

$$\mathscr{C} : \mathbf{F}^{(1)} = -\mathbf{F}^{(1)}. \tag{25}$$

The incremental Cauchy stress tensor then becomes

$$\mathbf{T}^{(1)} = \mathscr{L} : \mathbf{F}^{(1)} + \mathbf{F}^{(1)} \cdot \mathbf{A}^{(0)} \cdot W_{\mathbf{A}}^{(0)} - p^{(1)}\mathbf{1}. \tag{26}$$

The stability analysis proceeds by expanding the equation for mechanical equilibrium (Eqs. (11) or (12)) to first-order in epsilon. That is:

$$\operatorname{div}(\mathbf{T}^{(1)}) = 0. \tag{27}$$

This last equation takes the form (in the incompressible case):

$$\operatorname{div}(\mathscr{L} : \mathbf{F}^{(1)}) + (\mathbf{F}^{(1)})^{\mathrm{T}} \cdot \operatorname{grad}(q^{(0)}) - \operatorname{grad}(q^{(1)}) = 0, \tag{28}$$

where we have used $(\mathbf{T}^{(\mathbf{i})})^{\mathrm{T}} = \mathbf{T}^{(\mathbf{i})}$, $\operatorname{div}(\mathbf{T}^{(0)}) = 0$ and

$$\operatorname{div}(\mathbf{F}^{(1)}) = 0. \tag{29}$$

This last relation comes from the application of Nanson's formula applied to the transformation of a volume element so that

$$\operatorname{div}(J^{-1}\mathbf{F}) = 0, \tag{30}$$

which implies both $\operatorname{div}(J^{-1}\mathbf{F}^{(0)}) = 0$ and (29).

The boundary conditions are prescribed by fixing $\boldsymbol{\chi}^{(1)}$ on some part of the boundary and $\mathbf{n}.(\mathbf{T}^{(1)})$ on the remainder of the boundary (where $\mathbf{n}$ is the unit outward vector normal to the current boundary).

Relation (28) is the central linear equation for the stability analysis of the configuration associated with the deformation $\boldsymbol{\chi}^{(0)}$. For given boundary conditions, the onset of solutions to this system of equations indicates the possibility of an instability.

4. The growing shell

4.1. Spherically-symmetric deformations

We consider the symmetric growth and deformation of a spherical shell with border radii $R = A$ and $R = B$ in the reference configuration. Under deformation, the shell expands and a sphere located at R in the reference configuration is now a sphere located at a distance $r = r(R)$ (see Fig. 2). The geometric deformation

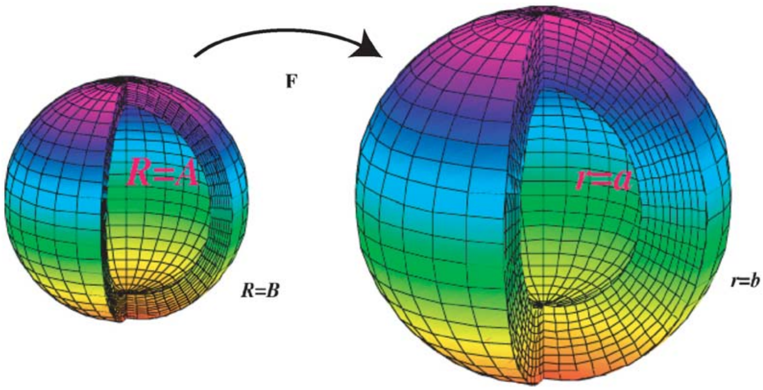

Fig. 2. Radial deformation of a shell with inner and outer radii A and B to a shell with radii a and b.

gradient tensor is then given by

$$\mathbf{F} = \mathrm{diag}(r', r/R, r/R), \tag{31}$$

where the primes denote derivatives with respect to R. The elastic strain tensor is

$$\mathbf{A} = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_2). \tag{32}$$

The effect of symmetric growth is contained in the growth tensor:

$$\mathbf{G} = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_2), \tag{33}$$

where γ_1 and γ_2 are functions of R.

Isotropic growth is achieved when $\gamma_1 = \gamma_2$. If $\gamma_1 = 1$ and $\gamma_2 > 1$, growth is purely radial and, following (Klisch et al., 2001) is referred to as *circumferential growth*, that is each sphere inside the shell increases (or decreases) its surface area (see Fig. 3). If $\gamma_1 = 1$ and $\gamma_2 < 1$ material is removed in the same manner and this process is referred to as *circumferential resorption*. If $\gamma_2 = 1$ and $\gamma_1 \neq 1$, growth and resorption take place in the radial direction and each radial fiber in the shell increases (or decreases). This case is referred to as *radial growth* (or *radial resorption* if mass is lost). In many respects, the important parameter is the ratio $\gamma = \gamma_1/\gamma_2$ and mechanical solutions with radial growth will be qualitatively similar to solutions with circumferential resorption ($\gamma > 1$) up to an isotropic growth.

Growth is usually the result of a continuous process on time scales varying from minutes to years depending on the specific system. After each incremental growth, relaxation and remodeling take place on time scales associated with elastic and viscous times smaller than one second (depending on the specific viscous model). Therefore, we only consider here the case where the two time-scales decouple (see discussion in Cowin, 2004) and the process is represented by a series of small quasi-static incremental growth steps of an unloaded configuration followed by elastic relaxation at each step (Klisch et al., 2001). Here, growth is defined with respect to the initial reference configuration and we assume, for simplicity, that the cumulative effect of the incremental growth process can be represented by a tensor $\mathbf{G}$. The

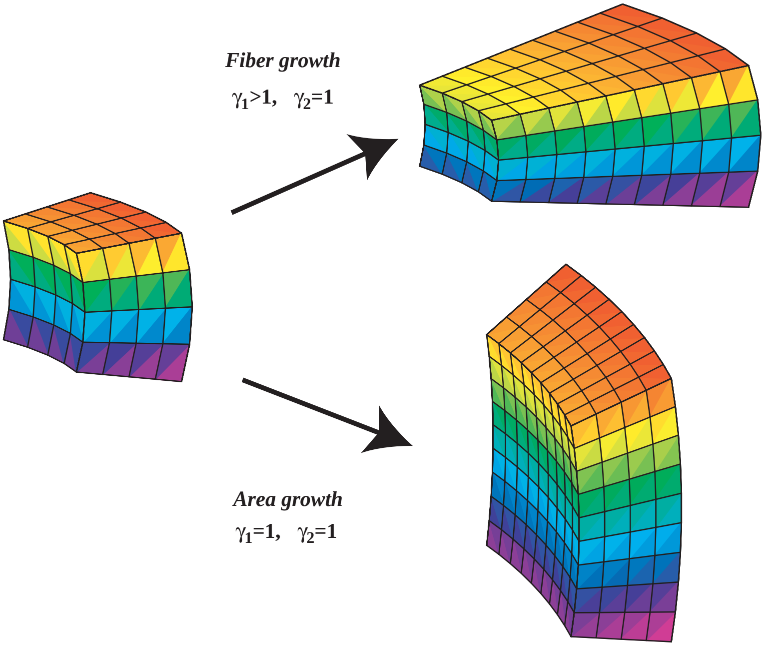

Fig. 3. Transformation of a spherical volume element under radial and circumferential growth.

incompressibility condition $\det(\mathbf{A}) = 1$ together with the relation $\mathbf{F} = \mathbf{A} \cdot \mathbf{G}$ has two consequences. First, $\alpha_2 = \alpha$, $\alpha_1 = \alpha^{-2}$ and $r' = \alpha_1\gamma_1$, $r/R = \alpha_2\gamma_2$. Second, the deformation is completely determined by $r'r^2/R^2 = \gamma_1\gamma_2^2$, that is, after integration

$$r^3 - a^3 = 3\int_A^R \gamma_1(R)\gamma_2^2(R)R^2\,\mathrm{d}R, \tag{34}$$

where $a = r(A)$ and $b = r(B)$ denote the inner and outer radii of the deformed shell. Let $s_1 = S_{11}$ and $s_2 = S_{22} = S_{33}$ be the non-vanishing components of the nominal stress tensor $\mathbf{S}$. Then the stress–strain relation (6) reads

$$s_1 = \gamma_2^2(W_1 - q\alpha_1^{-1}), \tag{35}$$

$$s_2 = \gamma_1\gamma_2(W_2 - q\alpha_2^{-1}), \tag{36}$$

where $W_i = \frac{\partial W}{\partial \alpha_i}$. The only nonvanishing equation for mechanical equilibrium (11) is

$$\frac{\partial s_1}{\partial R} + \frac{2}{R}(s_1 - s_2) = 0. \tag{37}$$

A closed equation for s_1 is obtained by eliminating q between (35) and (36) and introducing the auxiliary function $\widehat{W} = W(\alpha^{-2}, \alpha, \alpha)$ so that, we have

$$\frac{\partial s_1}{\partial R} = \frac{2s_1}{R}(\gamma\alpha^{-3} - 1) + \gamma_1\gamma_2\frac{\partial_\alpha \widehat{W}}{R}, \tag{38}$$

where

$$\alpha(R) = \frac{(a^3 + 3\int_A^R \gamma_1(R)\gamma_2^2(R)R^2\,\mathrm{d}R)^{1/3}}{R\gamma_2(R)}. \tag{39}$$

An equation for s_1 as a function of the strain α can be obtained from (38) and $\alpha = r/(R\gamma_2)$. It reads

$$\frac{\partial s_1}{\partial \alpha} = \frac{2s_1(\gamma_1\alpha^{-3} - \gamma_2) + \gamma_1\gamma_2^2\partial_\alpha \widehat{W}}{\alpha\left(\gamma_1\alpha^{-3} - \gamma_2 - R\frac{\mathrm{d}\gamma_2}{\mathrm{d}R}\right)}. \tag{40}$$

Note that a closed equation for s_1 as a function of α can only be obtained by either inverting relation (39) or by considering the special case where γ_1 and γ_2 are independent of R.

We can also compute the stresses in terms of the Cauchy stress tensors and we denote the non-vanishing components of $\mathbf{T}$ by $t_1 = T_{11}$ (*radial stress*), and $t_2 = T_{22} = T_{33}$ (*hoop stress*). Then the stress–strain relation (8) reads

$$t_1 = \alpha_1 W_1 - q, \tag{41}$$

$$t_2 = \alpha_2 W_2 - q. \tag{42}$$

The only nonvanishing equation for mechanical equilibrium (11) in the current configuration is

$$\frac{\partial t_1}{\partial r} + \frac{2}{r}(t_1 - t_2) = 0, \tag{43}$$

and a closed equation for t_1 is obtained:

$$\frac{\partial t_1}{\partial r} = \frac{\alpha}{r}\partial_\alpha \widehat{W}. \tag{44}$$

As a function of the strain α we have

$$\frac{\partial t_1}{\partial \alpha} = \frac{\gamma_1\partial_\alpha \widehat{W}}{[\gamma_1 - \alpha^3(\gamma_2 + R\gamma_2')]}, \tag{45}$$

and

$$t_2 = t_1 + \frac{\alpha}{2}\partial_\alpha \widehat{W}. \tag{46}$$

We now study the deformation of a growing shell under external pressure, the case without growth has been considered by many authors (See Section 5.5.1 for a review of existing results) and it is particularly interesting to understand the effect of growth in such context. The boundary conditions are then simply given by $t_1(r(A)) = 0$ and $t_1(r(B)) = -P$.

4.2. Constant growth

Before we study the stability of thin shells under nonradially symmetric perturbations, it is of interest to understand the stresses and the radial deformations that can be created through growth. To do so, we look at the particular case of constant growth (constant γ_1 and γ_2) and applied load $P = t_1(A) - t_1(B)$. Without loss of generality we take $t_1(A) = 0$ so that $P = -t_1(B)$. If $P > 0$, the shell is subject to an external hydrostatic pressure P and if $P < 0$ the shell is inflated with an internal pressure $-P$. In this case, Eq. (39) takes the form:

$$\alpha^3 = \gamma\left(1 - \frac{A^3}{R^3}\right) + \frac{a^3}{\gamma_2^3 R^3}. \tag{47}$$

Defining $\alpha_a = \alpha(A) = a/(A\gamma_2)$ and $\alpha_b = \alpha(B) = b/(B\gamma_2)$, we have

$$\alpha_b^3 = \gamma\left[1 - \left(\frac{A}{B}\right)^3\right] + \alpha_a^3\left(\frac{A}{B}\right)^3. \tag{48}$$

Furthermore, Eq. (45) is a closed equation for $t_1(\alpha)$ which can be readily integrated:

$$t_1(\alpha) = \gamma \int_{\alpha_a}^{\alpha} \frac{\partial_\alpha \widehat{W}}{(\gamma - \alpha^3)}\,\mathrm{d}\alpha. \tag{49}$$

The boundary condition $t_1(\alpha_b) = -P$ leads to

$$-P = \gamma \int_{\alpha_a}^{\alpha_b} \frac{\partial_\alpha \widehat{W}}{(\gamma - \alpha^3)}\,\mathrm{d}\alpha \tag{50}$$

and since, α_b is related to α_a by (48), this last relation is an equation for P as a function of α_a. For a given P, one can invert this relation to find the strain at the boundary α_a caused by the traction. In turn, once α_a is known, the position of the inner radius is known and so is the deformation $r = r(R)$ through (47). In Fig. 4, we show the radial stress as a function of α_a for a neo-Hookean material without growth ($\gamma_1 = 1, \gamma_2 = 1$). Neo-Hookean materials are characterized by a strain energy function $W = \mu_1(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 3)$ with $\mu_1 > 0$, that is $\widehat{W} = \mu_1(\alpha^{-4} + 2\alpha^2 - 3)$ and $t_1 = \mu_1(4\alpha_b^{-1} + \alpha_b^{-4} - 4\alpha_a^{-1} - \alpha_a^{-4})$.

4.3. Residual stress

In the absence of external stress, growth can still create stress, this so-called *residual stress* is due to the anisotropic packing or removal of material (Skalak, 1981; Hoger, 1986; Skalak et al., 1996). In Figs. 5 and 6, we show explicitly the deformation and stresses created by radial and circumferential growth for an isotropic neo-Hookean shell with no external loading ($P = 0$). Both radial growth and circumferential resorption create a compressive residual stress (i.e. a negative radial stress) in the material whereas circumferential growth and radial resorption create a tensile radial stress (i.e. a positive radial stress).

For constant growth, the radial stress vanishes only at the boundaries and its sign depends on whether γ is less than or greater than one. If we consider non-constant growth, the effect of the term γ_2' in (45) can be sufficiently important as to create regions with compressive stresses and other regions with tensile stresses in the case of circumferential growth as noted in Chen and Hoger (2000).

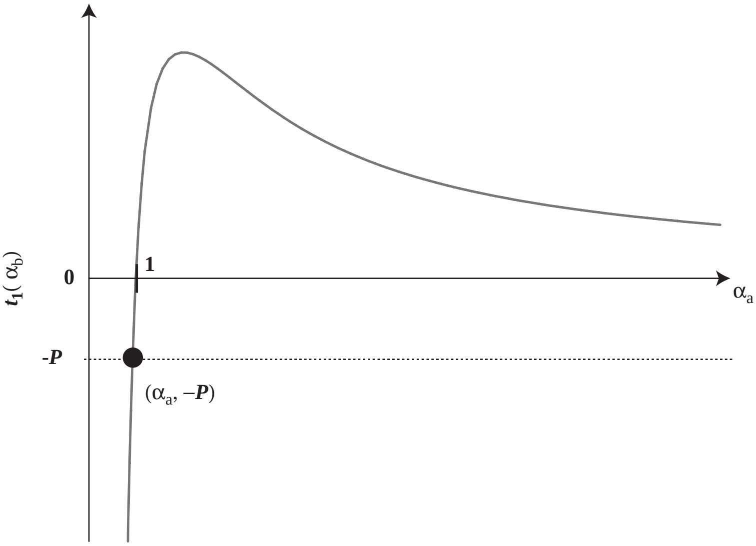

Fig. 4. The radial stress for a non-growing neo-Hookean material as a function of the inner boundary strain α_a. For a given P, the strain $\alpha(R)$ and the deformation $r = r(R)$ is specified by finding the value of α_a such that $t_1(\alpha_b) = -P$.

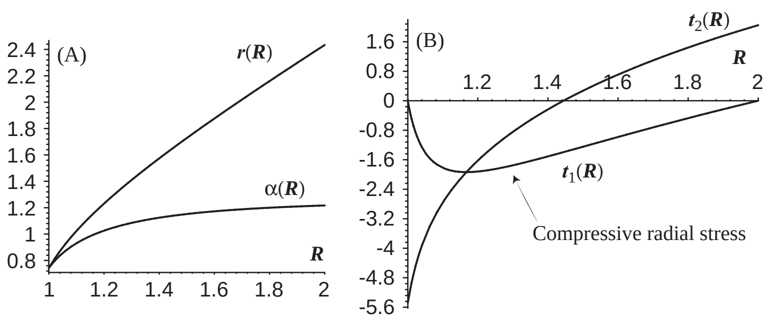

Fig. 5. Radial growth ($\gamma_1 = 2, \gamma_2 = 1$) of a neo-Hookean shell $\mu_1 = 1$, $A = 1$, $B = 2$. A. Strains (Note that the radii ratio a/b after growth is now close to 1/3 smaller than $A/B = 1/2$), B. Radial and hoop residual stresses. Radial growth creates a compressive radial stress inside the shell which can be understood from the fact that usually the stress is opposed to the direction of growth. The combined geometric effect (effective thickening) and mechanical effect (residual stress) compete for the buckling instability. The hoop stress is tensile on the outer radius and compressive inside.

4.4. Limit-point instability and inflation jump

The classical theory of rubber materials predicts that for particular choices of strain-energy functions and parameters, a limit-point instability may occur in spherical shell as the internal pressure is increased (Adkins and Rivlin, 1952;

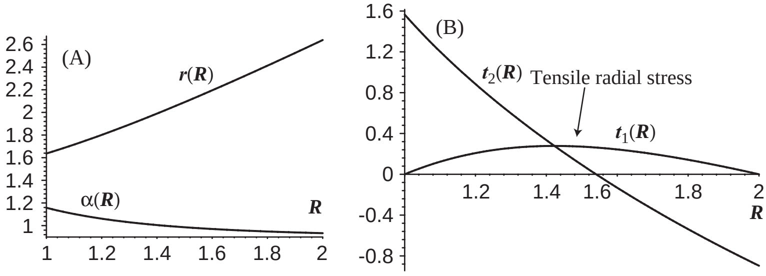

Fig. 6. Circumferential growth ($\gamma_1 = 1, \gamma_2 = \sqrt{2}$) for a neo-Hookean shell $\mu_1 = 1$, $A = 1$, $B = 2$. A. Strains, B. Radial and hoop residual stresses. The situation is reverse from the previous case and equivalent to fiber resorption. Here, growth creates tensile stress inside the shell which tends to stabilize the shell under hydrostatic pressure but the shell is effectively thinner (with $a/b > A/B$) and more prone to mechanical buckling.

Alexander, 1971; Ogden, 1972; Chen and Healey, 1991; Gent, 1999; Müller and Struchtrup, 2002). This effect is triggered by the loss of monotonicity of the function $t_1(\alpha_b)$ as a function of α_a, that is the pressure-stretch curve has a local maximum (see Fig. 4) and the resulting instability is known as a *limit-point instability*. For instance, in the limit of thin shell, a neo-Hookean membrane becomes unstable for $\alpha_a = 7^{1/6}$ independently of the initial radius or the modulus. Past this critical value, the membrane continues stretching for reduced pressure. For a material undergoing isotropic constant growth, the critical value of the stretch where such an instability occurs is given by

$$\begin{aligned}\alpha_a &= 7^{1/6}\left\{1 + \frac{\delta[415513 - 157039\sqrt{7} + (113960 - 43079\sqrt{7})(\gamma - 1)]}{14(55\sqrt{7} - 148)^2}\right. \\ &\quad \left. + \mathrm{O}(\delta^2, (\gamma - 1)^2)\right\} \\ &\approx 1.383\{1 + \delta[0.311 - 0.189(\gamma - 1)]\} + \mathrm{O}(\delta^2, (\gamma - 1)^2), \end{aligned} \tag{51}$$

where $\delta = \frac{B-A}{A}$ measures the thickness of the shell. For other materials such as the ones described by Mooney–Rivlin functions, the pressure stretch curve may present a maximum followed by a minimum at finite stretch. Therefore, under controlled pressure, the stretch may jump for increasing pressure and present a hysterisis when the pressure is reduced leading to an *inflation jump*. For instance, if we consider a Mooney–Rivlin function

$$W = \mu_1(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 3) + \mu_2(\alpha_1^{-2} + \alpha_2^{-2} + \alpha_3^{-2} - 3), \tag{52}$$

then for proper choice of the parameters $\mu_{1,2}$ and shell radii, the curve $t_1(\alpha_b)$ presents both a minimum and a maximum (see Fig. 7) and allows for the possibility of an inflation jump. This jump is more pronounced in thinner shells and eventually disappears for thicker shells.

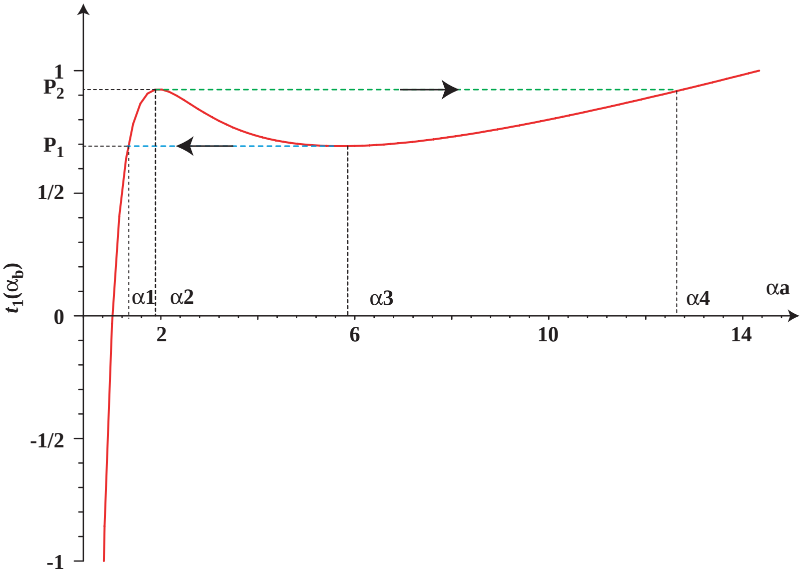

Fig. 7. Inflation jump: for a Mooney–Rivlin material with $\mu_1 = 1$, $\mu_2 = -0.03$, the curve $t_1(\alpha_b)$ as a function of α_1 has a minimum and a maximum ($A = 1$, $B = 2, \gamma_1 = \gamma_2 = 1$). If the internal pressure of the shell is raised to P_2, a sudden inflation occurs through a jump from α_2 to α_4. When pressure is decreased to P_1, a second jump from α_3 to α_1 occurs.

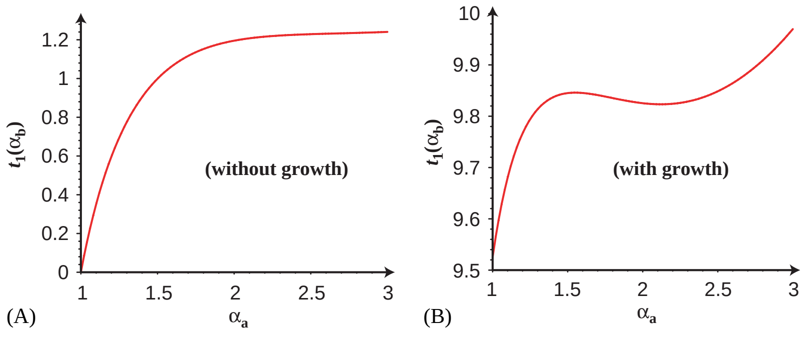

Fig. 8. Limit-point instability for growing material: for a Mooney–Rivlin material with $\mu_1 = 1$, $\mu_2 = -0.1$, $A = 1$, $B = 2$. A. The curve $t_1(\alpha_b)$ is monotonic for a non-growing material ($\gamma_1 = \gamma_2 = 1$). B. For a growing material ($\gamma_1 = 8$, $\gamma_2 = 1$) the curve $t_1(\alpha_b)$ has a minimum and a maximum and the strain becomes a discontinuous function of the pressure.

The effect of growth can be such that it effectively changes the property of the material as to allow for the possibility of an inflation jump. As an example, consider a non-growing material and values of the parameters where there is no instability, that is the strain is a continuous function of the applied pressure. Now, for the same elastic material with radial growth (see Fig. 8), an inflation jump occurs if γ is large enough. To understand how growth, geometry and elasticity control this effect, we compute from (50) and (48) the derivative of the pressure as a function of α_a:

$$\left(\frac{\alpha_a}{\gamma} - \alpha_a^{-2}\right)\frac{\mathrm{d}P}{\mathrm{d}\alpha_a} = \frac{\partial_\alpha \widehat{W}(\alpha_b)}{\alpha_b^2} - \frac{\partial_\alpha \widehat{W}(\alpha_a)}{\alpha_a^2}. \tag{53}$$

We conclude that limit point instabilities are controlled by the function $\frac{\partial_\alpha \widehat{W}(\alpha)}{\alpha^2}$ as found in the classical case (Ogden, 1984, p. 285) and that the only dependence to growth enters in the strain through relation (48). Therefore, the effect of constant radial or circumferential growth is to effectively change the geometry of the shell; that is a thick shell with radial growth effectively behaves as a classical thin shell.

5. Bifurcation of a shell

The general form for the first-order deformation $\boldsymbol{\chi}^{(1)}$ describing the bifurcation of a spherical shell is

$$\boldsymbol{\chi}^{(1)} = [u(r,\theta,\varphi), v(r,\theta,\varphi), w(r,\theta,\varphi)]^{\mathrm{T}}, \tag{54}$$

where (r,θ,φ) are the usual spherical coordinates. The standard procedure is then to expand the functions (u,v,w) in spherical harmonics and solve Eq. (28) for each mode. However, in the determination of the values of the parameter where a bifurcation occurs, it has been established in the classical case (Wesolowski, 1967; Wang and Ertepinar, 1972) and verified in the case of a growing shell that the equations for w decouples so that w does not play a role in the bifurcation. Furthermore, the equations for each order of the spherical harmonics are identical so the analysis of the order 0, corresponding to Legendre polynomials, is sufficient to identify the bifurcation point. We conclude that the bifurcation analysis of the spherical shell reduces to the analysis of axisymmetric deformations, that is,

$$\boldsymbol{\chi}^{(1)} = [u, v, 0]^{\mathrm{T}}, \tag{55}$$

with u, v independent of φ. Then,

$$\mathbf{F}^{(1)} = \begin{bmatrix} \partial_r u & (\partial_\theta u - v)/r & 0 \\ \partial_r v & (\partial_\theta v + u)/r & 0 \\ 0 & 0 & (u + v\cot\theta)/r \end{bmatrix}. \tag{56}$$

The condition (18) gives

$$r\partial_r u + 2u + \partial_\theta v + v\cot\theta = 0. \tag{57}$$

The third component of Eq. (28) written in spherical coordinates (see Eqs. (103)–(106) in Appendix B for the gradient and divergence of a tensor in spherical coordinates) vanishes identically. Using connections (110)–(115) the two remaining equations can be written:

$$c_1\frac{\partial^2 u}{\partial r^2} + c_2\frac{\partial u}{\partial r} + c_3\frac{\partial^2 u}{\partial \theta^2} + c_4\frac{\partial u}{\partial \theta} + c_5 u = \frac{\partial}{\partial r}q^{(1)}, \tag{58}$$

$$c_6 \frac{\partial^2 v}{\partial r^2} + c_7 \frac{\partial v}{\partial r} + c_8 \frac{\partial^2 u}{\partial r \partial \theta} + c_9 \left(\frac{\partial u}{\partial \theta} - v \right) = \frac{1}{r} \frac{\partial}{\partial \theta} q^{(1)}, \tag{59}$$

with

$$c_1 = \mathscr{L}_{1111} - \mathscr{L}_{1122} - \mathscr{L}_{1212} + \alpha_1 W_1, \tag{60}$$

$$c_2 = \frac{1}{r} \left[r^{-1} \frac{\partial}{\partial r} (r^2 \mathscr{L}_{1111} - r^2 \mathscr{L}_{1122}) + \mathscr{L}_{2222} - 2\mathscr{L}_{1122} - 2\mathscr{L}_{1212} + \mathscr{L}_{2233} \right.$$
$$\left. + 2\alpha_1 W_1 - \alpha_2 W_2 + r^{-1} \frac{\partial}{\partial r} (r^2 \alpha_1 W_1) \right], \tag{61}$$

$$c_3 = r^{-2} \mathscr{L}_{2121}, \quad c_4 = r^{-2} \cot(\theta) \mathscr{L}_{2121}, \quad c_5 = 2r^{-2} \mathscr{L}_{2121}, \tag{63}$$

$$c_6 = \mathscr{L}_{1212}, \quad c_7 = r^{-2} \frac{\partial}{\partial r} (r^2 \mathscr{L}_{1212}), \quad c_8 = \frac{1}{r} (\mathscr{L}_{1122} - \mathscr{L}_{2222} + \mathscr{L}_{1212} - \alpha_1 W_1), \tag{64}$$

$$c_9 = r^{-2} \left[r^{-1} \frac{\partial}{\partial r} (r^2 \mathscr{L}_{1212}) - \mathscr{L}_{2222} + \mathscr{L}_{2233} - \alpha_2 W_2 \right]. \tag{65}$$

Eqs. (57)–(59) form a system of three linear partial differential equations for $(u, v, q^{(1)})$. The standard method to solve Eqs. (58)–(59) is to express all variables of r and θ into products of a function of r alone and a Legendre polynomial in $\cos(\theta)$. That is, we write

$$u(r, \theta) = U_n(r) P_n(\cos\theta), \quad v(r, \theta) = V_n(r) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n(\cos\theta),$$
$$q^{(1)}(r, \theta) = Q_n(r) P_n(\cos\theta), \tag{66}$$

where $P_n(\cos\theta)$ are the Legendre polynomials for which the following equality holds:

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} P_n(\cos\theta) + \cot\theta \frac{\mathrm{d}}{\mathrm{d}\theta} P_n(\cos\theta) + n(n+1) P_n(\cos\theta) = 0. \tag{67}$$

By using substitution (66) into Eqs. (57)–(59), together with identity (67), one can express V_n and Q_n as a function of U_n and its derivatives and obtain a closed fourth-order linear ordinary differential equation for U_n

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(C_3 \frac{\mathrm{d}^3 U_n}{\mathrm{d}r^3} + C_2 \frac{\mathrm{d}^2 U_n}{\mathrm{d}r^2} + C_1 \frac{\mathrm{d} U_n}{\mathrm{d}r} \right) + C_0 U_n = 0, \tag{68}$$

where

$$C_3 = r^4 \mathscr{L}_{1212}, \tag{69}$$

$$C_2 = r^4 \frac{\mathrm{d}}{\mathrm{d}r} \mathscr{L}_{1212} + 4r^3 \mathscr{L}_{1212}, \tag{70}$$

$$C_1 = r^3\left(2\frac{\mathrm{d}}{\mathrm{d}r}\mathscr{L}_{1212} + t_1\right) + r^2[(2n^2+2n-1)\mathscr{L}_{1221} + 2n(n+1)\mathscr{L}_{1122} - n(n+1)\mathscr{L}_{1111} - (n^2+n-1)\mathscr{L}_{2222} - \mathscr{L}_{2233} - \mathscr{L}_{2121}], \tag{71}$$

$$C_0 = (n+2)(n-1)\left[r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\mathscr{L}_{1212} + r\frac{\mathrm{d}}{\mathrm{d}r}(2\mathscr{L}_{1212} + \mathscr{L}_{1221} - \mathscr{L}_{2121} - \mathscr{L}_{2222} + \mathscr{L}_{2233})(n^2+n+1)\mathscr{L}_{2121} - \mathscr{L}_{1221} - 2\mathscr{L}_{1212} + \mathscr{L}_{2222} - \mathscr{L}_{2233}\right]. \tag{72}$$

The boundary conditions are

$$\mathbf{T}^{(1)}\cdot\mathbf{n} = 0 \quad \text{on } r = a, \tag{73}$$

$$\mathbf{T}^{(1)}\cdot\mathbf{n} = -P^{(1)}\mathbf{n} \quad \text{on } r = b. \tag{74}$$

With the help of (26), the boundary condition for the incremental deformation takes the form

$$\mathbf{T}^{(1)}\cdot\mathbf{n} = (\mathscr{L} : \mathbf{F}^{(1)} + \mathbf{F}^{(1)}\cdot\mathbf{A}^{(0)}\cdot W_{\mathbf{A}} - q^{(1)}\mathbf{1})\cdot\mathbf{n}. \tag{75}$$

In the particular case of the growing shell the radial and hoop components of $\mathbf{T}^{(1)}\cdot\mathbf{n}$ give

$$(\mathbf{T}^{(1)}\cdot\mathbf{n})\cdot\mathbf{n} = (\mathscr{L}_{1111} - \mathscr{L}_{1122} + \alpha_1 W_1)\frac{\partial u}{\partial r} - q^{(1)}, \tag{76}$$

and

$$\frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial\theta} - v\right) = 0 \quad \text{on } r = a, b. \tag{77}$$

Eq. (68) and boundary conditions (76)–(77) seem to be independent of the growth parameters γ. However, the coefficients of (68) depend on α and its derivatives through the relation

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = \frac{\alpha}{r} - \frac{\alpha^4}{r\gamma}. \tag{78}$$

In the case of constant growth and a neo-Hookean strain energy function, Eq. (68) takes the explicit form

$$\frac{\mathrm{d}^4 U_n}{\mathrm{d}r^4} + C_3\frac{\mathrm{d}^3 U_n}{\mathrm{d}r^3} + C_2\frac{\mathrm{d}^2 U_n}{\mathrm{d}r^2} + C_1\frac{\mathrm{d}U_n}{\mathrm{d}r} + C_0 U_n = 0, \tag{79}$$

with

$$C_3 = \frac{8\alpha^3}{r\gamma}, \tag{80}$$

$$C_2 = \frac{(4-\gamma^2(n^2+n-2))\alpha^6 + 16\gamma\alpha^3 - \gamma^2(n^2+n+10)}{r^2\gamma^2}, \tag{81}$$

$$C_1 = \frac{2\gamma(n^2+n-2)\alpha^9 - 4(\gamma^2(n^2+n-2)-2)\alpha^6 - 4\gamma(n^2+n+8)\alpha^3 + 2\gamma^2(n^2+n+10)}{r^3\gamma^2}, \tag{82}$$

$$C_0 = \frac{(n^2+n-2)[4\gamma\alpha^9 + (\gamma^2(n^2+n-2)+4)\alpha^6 - 16\gamma\alpha^3 + 10\,\gamma^2]}{r^4\gamma^2}, \tag{83}$$

with boundary conditions at $r = a, b$ given by

$$\frac{\mathrm{d}^3 U_n}{\mathrm{d}r^3} + \frac{2\gamma + 4\alpha^3}{r\gamma}\frac{\mathrm{d}^2 U_n}{\mathrm{d}r^2} + \frac{8\alpha^3 - \gamma(\alpha^6-2)(n^2+n-2)}{r^2\gamma}\frac{\mathrm{d}U_n}{\mathrm{d}r} + \frac{(-2\gamma\alpha^6 + 4\alpha^3 - 2\gamma)(n^2+n-2)}{r^3\gamma}U_n = 0, \tag{84}$$

$$\frac{\mathrm{d}^2 U_n}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}U_n}{\mathrm{d}r} + \frac{(n^2+n-2)}{r^2}U = 0. \tag{85}$$

The integration of Eq. (79) takes place between $r = a$ and $r = b$ and the central problem of solving this equation lies in finding, for a given initial thickness A/B, the value of a such that the boundary conditions are satisfied (the outer radius b being a function of a). In the case of a growing shell, the outer radius b depends not only on the initial thickness A/B but also on the parameters $\gamma_{1,2}$ that describe growth through Eq. (48). The stability equations for the growing shell (with constant growth) and the boundary conditions are similar to the well-known equations for the mechanical shell under pressure and most of the analytical and numerical results can be adapted to the growing shell. Our strategy is as follows: First, we summarize the basic stability results for the shell without growth. Second, we see how these basic results are changed by the addition of constant growth.

5.1. A summary of stability results for the incompressible elastic shell

Before we proceed with the effect of anisotropic growth on the stability of shells, it is of interest to summarize the basic results related to stability properties of classical shells under pressure. Stability computations for this problem have a long history going back to the work of Wesolowski (1967), and Wang and Ertepinar (1972) on the neo-Hookean shell. Later on, Hill (1976), Haughton and Ogden (1978), Ogden (1984) and Chen and Healey (1991) obtained general results for general strain-energy functions and thin pressurized shells. More recent work on asymptotic limits, relationship with other stability criterion and the eversion problem include Fu (1998), Haughton and Kirkinis (2003), Haughton (2004) and Haughton and Chen (1999).

The stability of a pressurized shell and the critical mode (the first unstable mode) depends on whether the pressure is applied internally ($P<0$) or externally ($P>0$), its thickness and the choice of strain-energy functions. The mode $n = 0$ corresponds to a symmetric increase in shell radius and its existence does not correspond to a true axisymmetric bifurcation but a limit-point instability as discussed in Section 4.4.4. Therefore, it will not be discussed here.

5.1.1. Results for internally pressurized shells ($P<0$)

If the pressure is applied internally, a bifurcation analysis (Ogden, 1984) shows that the mode that can be excited depends on the choice of strain-energy functions and the thickness of the shell. Analytical results can be obtained for thin shells (membrane) and extended numerically for a variety of materials. Various analyses show that

(i) The critical mode depends on the nonlinearity of the strain-energy functions and that neo-Hookean materials are stable under axisymmetric perturbation;
(ii) Unstable modes are ordered, in the sense that as strain is increased under pressure the various values α_k at which mode n_k first becomes unstable are ordered ($\alpha_k < \alpha_{k+1}$), and the α_k are distinct even in the membrane limit;
(iii) Depending on the nonlinearity, not all modes may become unstable (so that only modes $n_1 \geqslant 1$ to n_k (possibly infinite) may become unstable).

Since the neo-Hookean shell is always stable, the case of internally pressurized shells will not be further discussed in this work.

5.1.2. Results for externally pressurized shells ($P>0$)

The situation is quite different for a shell under external pressure. Again, a number of exact asymptotic results can be obtained in the membrane limit or for high mode numbers. To discuss possible bifurcations, a typical plot of the strain α_a required to excite different modes for various thickness is shown in Fig. 9 for a neo-Hookean material. Once the critical strain α_a is known for the nth mode, α_b can be computed from (48) and the critical pressure necessary to excite the mode n is given by (50). For further discussion on the growing shell, the following features are of importance (Note that results (iv)–(vi) are new and have been developed specifically for this paper) (Fig. 10):

(i) Thin shells are more likely to be unstable than thicker shells since the critical strain increases with thickness;
(ii) Unstable modes are not ordered, in particular, as thickness decreases, higher modes are first excited;
(iii) For thick shells (that is $A/B-1=\mathrm{O}(1)$) and high modes, the critical strain curve $\alpha_a^{(n)}$ for the nth mode is given asymptotically by (see Appendix F)

$$\alpha_a^{(n)} = 0.6661 + 0.5906/n - 10.2679/n^2. \tag{86}$$

(iv) In the limit $A/B \to 1$, the exact critical strain $\alpha_0(n)$ for the nth mode is given by (see Appendix E)

$$\alpha_0(n) = \left[\frac{-(n^2+n+7)+\sqrt{4n^4+8\,n^3+12\,n^2+8\,n+49}}{n^2+n-2}\right]^{1/6}, \tag{87}$$

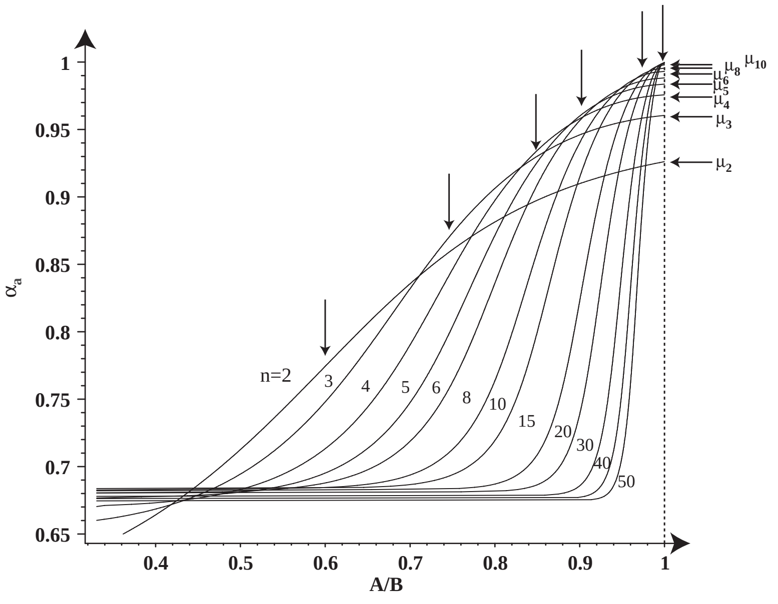

Fig. 9. Critical strains α_a for the nth mode of a neo-Hookean shells as a function of the thickness A/B. Only the modes $n = 2, 3, 4, 5, 6, 8, 10, 15, 20, 30, 40, 50$ are shown. For a given thickness A/B, as the external pressure is increased, the inner boundary shell moves inward and the strain α_a decreases until it reaches a critical curve. The critical mode is the first excited mode. For instance at $A/B = 6/10$, the critical mode is $n = 2$ whereas for a thinner shell at $A/B = 3/4$, the critical mode is $n = 3$. The corresponding shapes are shown in the next figure.

(v) For thin shells, a Taylor expansion in $\delta = \frac{B-A}{A}$ for the nth mode critical curve valid to order $\mathrm{O}(\delta^4)$ is given by

$$\alpha_a^{(n)} = \alpha_0(n) + \alpha_1(1,n)\delta + \alpha_2(1,n)\delta^2 + \alpha_3(1,n)\delta^3 + \mathrm{O}(\delta^4), \tag{88}$$

where the coefficients are given in Appendix F and the comparison with numerical curves is shown in Fig. 11.

(vi) For thin shells, an expression for the envelope of the critical curves is given by (see Appendix F and Fig. 11)

$$\alpha_a = 1 - \frac{1}{3}\delta - \frac{7}{36}\,\delta^2 + \frac{235}{864}\,\delta^3 + \mathrm{O}(\delta^3). \tag{89}$$

5.2. Stability of the growing shell

We now consider the case of a neo-Hookean elastic shell with externally applied pressure and study the effect of anisotropic growth on stability. We focus our attention on radial growth and resorption. The effect of circumferential growth is equivalent to radial resorption up to an isotropic change of volume.

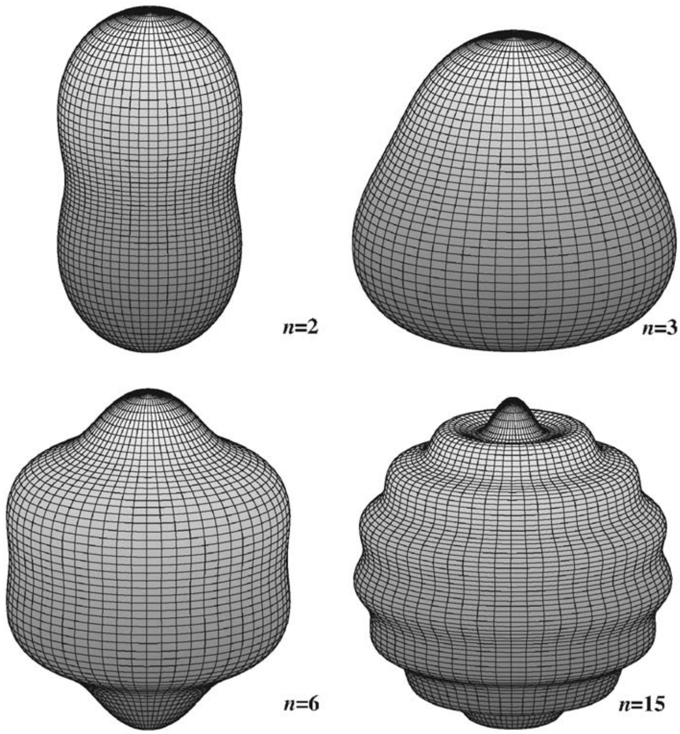

Fig. 10. Examples of shell deformation after the bifurcation. Note that the amplitude of the mode has been chosen to show the structure of the solution and is not related to the mechanical problem at hand. Since the analysis in this paper is linear there is no information on the mode amplitude or its sign.

5.2.1. Mechanical versus geometric effects

Two possible growth effects are involved in stability. The first effect is due to *geometry*. The increase or decrease in volume and hence shell thickness modifies the stability properties of the shell by changing its effective geometry. For instance at a given pressure, a reduction in thickness after resorption tends to destabilize the shell (since thinner shells are more unstable than thicker ones). Likewise, an increase in the thickness after growth stabilizes the shell. The second effect is related to *mechanics* through changes in residual stress. Radial growth induces a compressive radial stress (see Fig. 5) which tends to further destabilize the shell under pressure. Similarly, radial resorption (or circumferential growth) induces a tensile radial stress (see Fig. 5) that stabilizes the shell. To illustrate these two opposing effects, we show in Fig. 12 the critical pressure P_{crit} for the mode $n = 10$ as a function of thickness and for small values of radial growth and resorption. For thin shells (A/B close to 1), radial growth is stabilizing. This can be understood as follows. For thin shells, a small change in thickness has a direct effect on the strain needed for instability (see Fig. 9 in the thin shell regime). For thick shells, a change in thickness does not

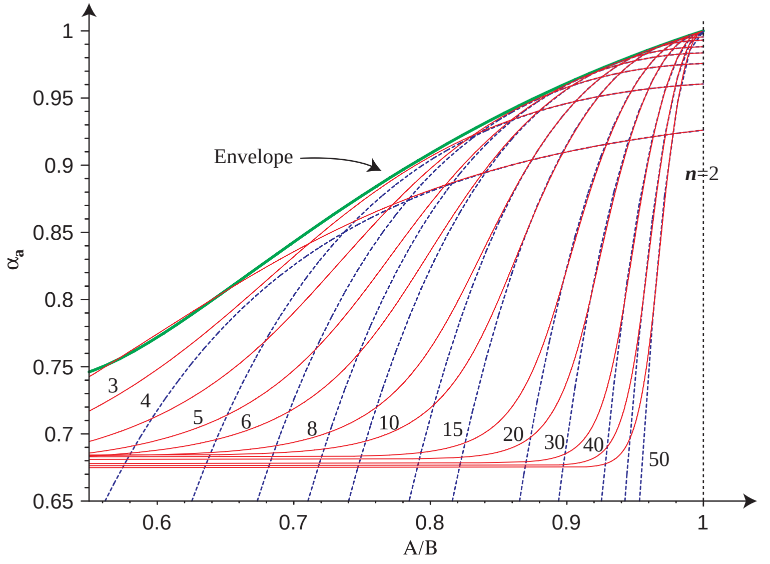

Fig. 11. Blow-up of the critical curves (solid lines) of Fig. 9 together with the asymptotic curves (dotted lines) valid for thin shells and the critical envelope given by Eq. (89).

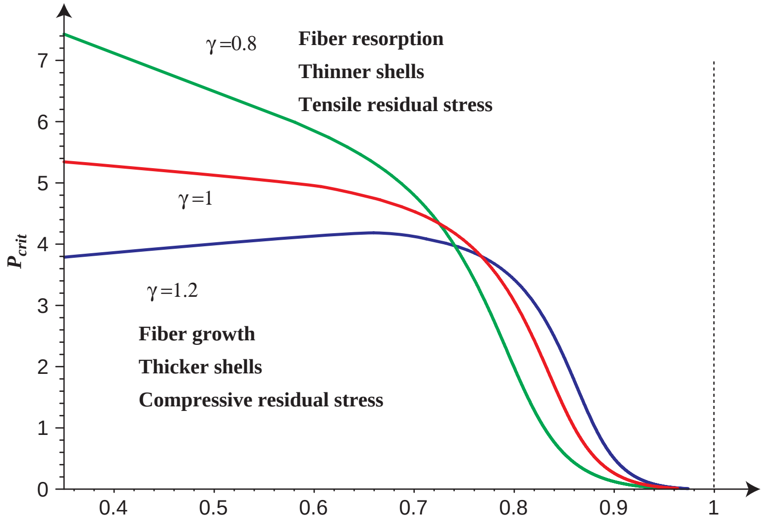

Fig. 12. Critical pressure for mode $n = 10$ as a function of initial thickness for radial growth and resorption. Geometric effect of growth: for thin shells, radial growth is stabilizing (larger critical pressure) and radial resorption is destabilizing (smaller critical pressure). Mechanical effect of growth: for thick shells, radial growth is destabilizing and radial resorption is stabilizing.

significantly change the critical strain needed for the existence of an axisymmetric solution (see Fig. 9 for $A/B<1/2$).

The effect of growth on thin shells can be explicitly computed by using the small shell approximation and computing asymptotically the critical displacement for each mode:

$$\alpha_a^{(n)} = \alpha_0(n) + \alpha_1(\gamma, n)\delta + \alpha_2(\gamma, n)\delta^2 + \alpha_3(\gamma, n)\delta^3 + \mathrm{O}(\delta^4), \tag{90}$$

where $\delta = \frac{B-A}{A}$ and the coefficients $\alpha_i(\gamma, n)$ are explicitly given in Appendix E. The critical mode number is the first unstable mode number, it is given by the solution of $\frac{\partial \alpha_a}{\partial n} = 0$. Once the critical mode number n_{crit} is known, the critical envelope giving the critical value of the strain for instability can be computed explicitly as $\alpha_a(n_{\mathrm{crit}})$ and is given by

$$\begin{aligned}\alpha_a = 1 &+ \left(\frac{1}{2} - \frac{5}{6}\gamma\right)\delta + \left(\frac{5}{24} - \frac{3\gamma}{4} + \frac{25}{72}\gamma^2\right)\delta^2 \\ &- \frac{\gamma}{864}(21 - 168\,\gamma - 88\gamma^2)\delta^3 + \mathrm{O}(\delta^4).\end{aligned} \tag{91}$$

Once the critical envelope is known, the critical pressure can be calculated explicitly by integrating (50) for a neo-Hookean energy:

$$P_{\mathrm{crit}} = 8(\gamma\delta)^2 + \frac{2}{3}(\gamma\delta)^3 + \mathrm{O}(\delta^4). \tag{92}$$

And since,

$$\frac{\mathrm{d}P}{\mathrm{d}\gamma_i}(\gamma_1 = 1, \gamma_2 = 1) = (-1)^{i+1}(16\delta^2 + 2\delta^3 + \mathrm{O}(\delta^4)), \quad i = 1, 2, \tag{93}$$

we conclude that for thin shells, radial growth or circumferential resorption has a stabilizing effect whereas radial resorption or circumferential growth is destabilizing.

5.2.2. *Spontaneous instabilities*

Another feature of interest can be seen in Fig. 12. For thick shells and radial growth, the pressure necessary to destabilize the shell decreases with the thickness. This is due to the residual compressive stress associated with radial growth and it suggests that for sufficient growth, the shell might become unstable without any external applied stress. To test this possibility, we consider thick shells and high mode numbers so that the WKB expansion given in Appendix F is valid. In this regime, one can compute the critical value of γ leading to an instability at zero pressure. In Fig. 13, we show the critical radial growth γ_{crit} as a function of the initial thickness for high mode number (including $n = \infty$ obtained by taking $\alpha_a = \alpha_0$ in the WKB expansion (138)). The exact threshold for an instability is not known since the WKB expansion is only valid for high mode number and the first instability occurs around mode $n = 10$. Nevertheless, the value of γ_{crit}, provides an upper bound for the instability.

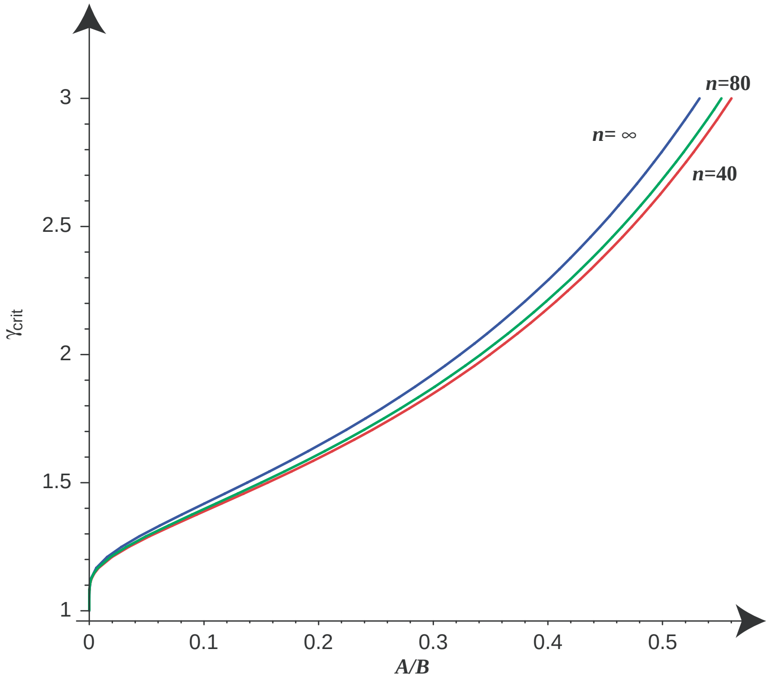

Fig. 13. The critical value of radial growth necessary to induce a spontaneous instability in a shell of initial thickness A/B for the modes $n = 40, 80, \infty$. Note that the WKB expansion is not valid for low modes so that the value of γ_{crit} given by the mode $n = \infty$ is a upper bound for the instability.

6. Conclusions

Biological growth is a problem of considerable complexity with many experimental and theoretical challenges. The theory of growth for biological soft tissues has been developed over the years to include general effects and interactions with other constituents and is now reaching maturity. In this paper we have focused our attention on the consequence of growth on the stability properties of a neo-Hookean elastic material. We have considered the stability of a growing incompressible shell under constant but anisotropic growth. This is the simplest model for growth that can be considered but it already leads to interesting behaviors and consideration regarding the effects of growth on stability. In particular we have identified two regimes where geometric and mechanical effects play different role and show that anisotropic growth can be the source of spontaneous instabilities. For thin shells, geometric effects dominate and radial growth (or circumferential resorption) has a stabilizing effect. This is due to the fact that in these cases, shell thickness increases with radial growth. Conversely, radial resorption (or circumferential growth) has a destabilizing effect since the shell thickness decreases and thin shells are more unstable. For thick shells, mechanical effects dominate as residual stress builds up. In radial growth, thick shells are more unstable due to the buildup of compressive residual stress. Conversely, circumferential growth develops tensile residual stress and are more stable.

Soft tissues in large deformations are not well characterized by rubber-like energy functionals and some of the instabilities shown here may not be present for more realistic choices of constitutive functions. For instance, for Fung-type materials, the instability found for half-space under bi-axial compression disappears when the deviation from neo-Hookean behavior increases. The dependence of stability threshold for growing elastic materials on the particular choice of the constitutive function and parameters represents an interesting field of study that could be explored by the methods given here.

Most tissues react and adapt during growth and the development of a material cannot be effectively modeled by a tensor with slowly increasing components and accordingly a proper constitutive relationship for the growth rate which include adaptation should be used. Nevertheless, the growing shell can be used as a paradigm to understand the effect of anisotropic growth but also to study two important effects namely, differential growth and stress-dependent growth. Differential growth refers to growth dependent on position in the reference body. Some part of the body experiences faster growth than others which naturally leads to change of shape and possibly to morphogenesis. Within the theory of exact elasticity here, it corresponds to a growth tensor with explicit dependence on the position. The effect of differential growth for spherical elastic shells is discussed in Goriely and Ben Amar (2005) where it is shown that an isotropic resorption with a linear profile leads to compressive stresses that can produce buckling. The second effect is stress-mediated growth. It is now well documented that growth is partly regulated by the stress applied to the body (Taber, 1995; Humphrey, 2003; Cowin, 2004). Epstein and Maugin (2000) argue on mechanical grounds that the natural way to introduce this effect (and other anisotropic effects) is to relate the growth rate to the Mandel stress in the intermediate configuration obtained after application of the growth tensor (see Imatani and Maugin, 2002 for an implementation of these ideas). The general effect of stress-induced growth on stability properties remains an open problem of fundamental importance.

Elastic instabilities due to growth have been proposed by different authors as a fundamental mechanism for morphogenesis (Odell et al., 1981; Steele, 2000; Dumais et al., 2000; Keller et al., 2003; Shipman and Newell, 2004). The basic idea is that in a morphogenetic event, the wavelengths selected for new emerging shapes are selected by a buckling-type instabilities arising through growth-induced stress. However, most of these models decouple the two problems (growth and instability) by assuming either that the effect of growth is to create residual-stress in the material (seen as an elastic shell on an elastic foundation) in which buckling occur or to change the effective metric (Audoly and Boudaoud, 2003) on which relative strains are computed (thereby, introducing residual strains in the problem). The theory of growing elastic soft-tissue could be used to describe both growth and instability on the same footing, thereby providing a unified framework to describe morphogenesis.

Acknowledgements

We would like to thank M. Tabor, Y.-B. Fu, M. Destrade, and G. Maugin for various discussions on finite elasticity. A.G. is funded under DMS grant #0307427 and this work was made possible by a visiting position from the Université Pierre et Marie Curie, the CNRS and a Fellowship from La Ville de Paris.

Appendix A. Useful rules about tensors

In the following, let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be second-order tensors with cartesian components A_{ij}, B_{ij}, C_{ij} and $F = F(\mathbf{A})$ a scalar function of $\mathbf{A}$. We denote by $(\cdot)$ the contraction, in Cartesian components, over one repeated index and by $(:)$ a contraction on two repeated indices. We recall some elementary but useful rules about tensor differentiation.

The derivative of the scalar function F with respect to the tensor $\mathbf{A}$ is a tensor whose cartesian components are

$$\left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}}\right)_{ij} = \frac{\partial F(\mathbf{A})}{\partial A_{ji}}. \tag{94}$$

Now let $\mathbf{A} = \mathbf{B} \cdot \mathbf{C}$ and consider the derivative of F with respect to $\mathbf{B}$:

$$\frac{\partial F}{\partial \mathbf{B}}(\mathbf{A}) = \mathbf{C} \cdot \frac{\partial F}{\partial \mathbf{A}}(\mathbf{A}). \tag{95}$$

Other useful identities are Jacobi's relations for the first and second derivatives of a nonvanishing determinant,

$$\frac{\partial}{\partial \mathbf{A}}(\det(\mathbf{A})) = \det(\mathbf{A})\mathbf{A}^{-1}, \tag{96}$$

$$\frac{\partial}{\partial \mathbf{A}}\frac{\partial}{\partial \mathbf{A}}(\det(\mathbf{A})) : \mathbf{B} = \det(\mathbf{A})[(\mathbf{A}^{-1} : \mathbf{B})\mathbf{A}^{-1} - \mathbf{A}^{-1} \cdot \mathbf{B} \cdot \mathbf{A}^{-1}], \tag{97}$$

where we have used the derivative of the inverse of a tensor

$$\left(\frac{\partial}{\partial \mathbf{A}}\mathbf{A}^{-1}\right) : \mathbf{B} = -\mathbf{A}^{-1} \cdot \mathbf{B} \cdot \mathbf{A}^{-1}. \tag{98}$$

Appendix B. Spherical coordinates

Let (r, θ, φ) be the spherical coordinates defined with respect to the Cartesian coordinates (x, y, z) by

$$x = r\cos\theta\sin\varphi, \tag{99}$$

$$y = r\sin\theta\sin\varphi, \tag{100}$$

$$z = r\cos\varphi, \tag{101}$$

where $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, $\varphi \in [0, \pi]$ and $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$ the associated basis. Let $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_\varphi\mathbf{e}_\varphi$ then

$$\operatorname{div}(\mathbf{v}) = \frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta[(\sin\theta)v_\theta] + \frac{1}{r\sin\theta}\partial_\varphi v_\varphi. \tag{102}$$

Let $f = f(r, \theta, \varphi)$, then

$$\operatorname{grad}(f) = (\partial_r f)\mathbf{e}_r + \frac{1}{r}(\partial_\theta f)\mathbf{e}_\theta + \frac{1}{r\sin\theta}(\partial_\varphi f)\mathbf{e}_\varphi. \tag{103}$$

Let $\mathbf{A}$ be a tensor, then the components of the divergence of $\mathbf{A}$ are

$$[\operatorname{div}(\mathbf{A})]_r = \frac{1}{r^2}\partial_r(r^2\mathbf{A}_{rr}) + \frac{1}{r\sin\theta}\partial_\theta[(\sin\theta)\mathbf{A}_{\theta r}] + \frac{1}{r\sin\theta}\partial_\varphi\mathbf{A}_{\varphi r} - \frac{\mathbf{A}_{\theta\theta} + \mathbf{A}_{\varphi\varphi}}{r}, \tag{104}$$

$$[\operatorname{div}(\mathbf{A})]_\theta = \frac{1}{r^2}\partial_r(r^2\mathbf{A}_{r\theta}) + \frac{1}{r\sin\theta}\partial_\theta[(\sin\theta)\mathbf{A}_{\theta\theta}] + \frac{1}{r\sin\theta}\partial_\varphi\mathbf{A}_{\varphi\theta} + \frac{\mathbf{A}_{\theta r} - \cot\theta\mathbf{A}_{\varphi\varphi}}{r}, \tag{105}$$

$$[\operatorname{div}(\mathbf{A})]_\varphi = \frac{1}{r^2}\partial_r(r^2\mathbf{A}_{r\varphi}) + \frac{1}{r\sin\theta}\partial_\theta[(\sin\theta)\mathbf{A}_{\theta\varphi}] + \frac{1}{r\sin\theta}\partial_\varphi\mathbf{A}_{\varphi\varphi} + \frac{\mathbf{A}_{\varphi r} + \cot\theta\mathbf{A}_{\varphi\theta}}{r}. \tag{106}$$

Appendix C. The elastic moduli

Let $W = W(\mathbf{A})$ be the elastic energy and $\mathscr{A} = \frac{\partial^2 W}{\partial\mathbf{A}\partial\mathbf{A}}(\mathbf{A}^{(0)})$ be the elastic moduli evaluated in the reference configuration. The *instantaneous elastic moduli* are the elements of the fourth-order tensor $\mathscr{L}$ defined by the relation

$$\mathscr{L} : \mathbf{F}^{(1)} = \mathbf{A}^{(0)} \cdot (\mathscr{A} : \mathbf{F}^{(1)}) \cdot \mathbf{A}^{(0)}. \tag{107}$$

In Cartesian components, the elastic moduli are:

$$\mathscr{A}_{ijkl} = \frac{\partial W}{\partial A_{ji}\partial A_{lk}}, \tag{108}$$

and

$$\mathscr{L}_{ijkl} = A^{(0)}_{im}\mathscr{A}_{mjkn}A^{(0)}_{nl}. \tag{109}$$

In general, let α_i be the principal values of $\mathbf{A}$, then for an isotropic hyper-elastic materials the components of $\mathscr{L}$ on the Eulerian principal axes associated with **A** are (Ogden, 1984, p. 412):

$$\mathscr{L}_{iijj} = \alpha_i\alpha_j \frac{\partial^2 W}{\partial\alpha_i \partial\alpha_j}, \tag{110}$$

$$\mathscr{L}_{ijij} = \left[\alpha_i \frac{\partial W}{\partial \alpha_i} - \alpha_j \frac{\partial W}{\partial \alpha_j}\right] \frac{\alpha_i^2}{(\alpha_i^2 - \alpha_j^2)}, \quad i \neq j, \tag{111}$$

$$\mathscr{L}_{ijji} = \mathscr{L}_{jiij} = \mathscr{L}_{ijij} - \alpha_i \frac{\partial W}{\partial \alpha_i}, \quad i \neq j. \tag{112}$$

If $\alpha_i = \alpha_j$ for some $i \neq j$, then we have

$$\mathscr{L}_{ijij} = \frac{1}{2}\left(\mathscr{L}_{iiii} - \mathscr{L}_{iijj} + \alpha_i \frac{\partial W}{\partial \alpha_i}\right), \tag{113}$$

$$\mathscr{L}_{iiii} = \mathscr{L}_{jjjj}, \quad \mathscr{L}_{iikk} = \mathscr{L}_{jjkk}, \tag{114}$$

$$\mathscr{L}_{ikik} = \mathscr{L}_{jkjk}, \quad \mathscr{L}_{ikki} = \mathscr{L}_{jkkj}. \tag{115}$$

Appendix D. Numerical integration

We give here a brief presentation of the method used to solve Eq. (68) with boundary values given by Eqs. (76)–(77). The method that we used is based on the general determinantal method but is implemented differently to avoid the typical issues associated with the evaluation of determinants. The general problem is to find the values of a parameter (say, λ) for which there is a solution of a fourth-order equation with mixed boundary values. Let $\mathbf{y} = [U, U', U'', U''']^{\mathrm{T}}$ be the vector of variable and derivatives. The fourth-order problem can be written as a system of four first-order equations that we write

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}; r), \quad \mathbf{y} \in R^4, \tag{116}$$

where r is the independent variable. The boundary value conditions are given by a set of four linear functions $c_{1,2,3,4}(\mathbf{y}(r); r)$ which vanish at one of the boundaries, namely

$$c_{1,2}(\mathbf{y}(a); a) = 0, \quad c_{3,4}(\mathbf{y}(b); b) = 0. \tag{117}$$

In order to find values of the parameter for which a solution of the equation with proper boundary conditions exists, the general strategy of the determinantal method (Haughton and Ogden, 1979) is to consider two copies of system (116), that is

$$\mathbf{y}^{(1)'} = \mathbf{f}(\mathbf{y}^{(1)}; r), \quad \mathbf{y}^{(2)'} = \mathbf{f}(\mathbf{y}^{(2)}; r). \tag{118}$$

The parameter λ in the system is given by the relationship between the endpoints a and b through relations such as (48) (in our case $\lambda = A/B$). For a first guess λ_1, we integrate System (118) up to $r = b$ with linearly independent conditions $y^{(1)}(a)$, $y^{(2)}(a)$ that satisfy both conditions at $r = a$ (that is $c_{1,2}(y^{(1,2)}(a); a) = 0$). Then at $r = b$, one evaluates the determinant

$$\Delta(r) = \begin{vmatrix} c_3(\mathbf{y}^{(1)}(r); r) & c_3(\mathbf{y}^{(1)}(r); r) \\ c_4(\mathbf{y}^{(2)}(r); r) & c_4(\mathbf{y}^{(2)}(r); r) \end{vmatrix}. \tag{119}$$

Since the system is linear, if $\Delta(b) = 0$ there is a linear combination of $y^{(1)}$ and $y^{(2)}$ that satisfies System (116) with boundary values (117). If $\Delta(b) \neq 0$, then one iterates on λ_1 until the determinant vanishes. However, in many applications numerical instabilities are present due to the typical difficulties of evaluating the determinant Δ. To circumvent this problem, the determinantal method is usually replaced by the compound matrix equations where a suitable basis for the evaluation of the determinant is introduced and a larger system of equations is integrated (Haughton and Orr, 1997).

Here we propose a slightly different strategy that does not require an iteration on the parameter λ or the direct evaluation of a determinant. Since the boundary points a and b are related by λ through the strain, the problem can be rephrased as follows. For a given value of a, consider System (118) with an extra equation for the determinant

$$\Delta'(r) = \sum_{i=1,2} \frac{\partial \Delta(r)}{\partial \mathbf{y}^{(i)}} . \mathbf{f}(\mathbf{y}^{(i)}; r, \lambda). \tag{120}$$

The new system for $\mathbf{z} = [\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \Delta]^{\mathrm{T}}$ is integrated on r with a stop condition on the vanishing of the last component. This provides the value of $r = b$ from which the value of λ is found (For the problem at hand, we integrate from a and once b is known, we deduce the value of A/B from relation (48)). The two main advantages of this approach is that the evaluation of the determinant is performed by the integration routine that can be chosen appropriately if the problem is stiff and that there is no iteration on the parameter. The entire problem is reduced to the integration of a system of nine equations with a stop condition.

Appendix E. Thin shell analysis

The numerical procedure for the computation of critical curves given in the previous appendix can be adapted for the computation of series solutions in the thin shell limit. For thin shells, one can obtain analytic results by introducing the small parameter $\varepsilon = (b - a)$ and the stretched variable (Fu, 1998)

$$\rho = \frac{r - a}{\varepsilon}. \tag{121}$$

System (118) now reads

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}\rho} = \mathbf{g}(\mathbf{z}; \rho; \varepsilon), \tag{122}$$

where $\mathbf{z} = [\mathbf{y}^{(1)}, \mathbf{y}^{(2)}]^{\mathrm{T}}$, $\mathbf{g} = [\mathbf{f}(\mathbf{y}^{(1)}), \mathbf{f}(\mathbf{y}^{(2)})]^{\mathrm{T}}$, and the integration takes place between $\rho = 0$ and $\rho = 1$. We seek solutions of this system of eight equations with initial values $\mathbf{z}(\rho=0;\varepsilon)=[\mathbf{y}_0^{(1)},\mathbf{y}_0^{(2)}]^{\mathrm{T}}=\mathbf{z}_0(\varepsilon)$ where we choose $\mathbf{y}_0^{(1)}$ and $\mathbf{y}_0^{(2)}$ to be linearly independent and such that the conditions

$$c_{1,2}(\mathbf{y}_0^{(1)};\rho=0;\varepsilon),\quad c_{1,2}(\mathbf{y}_0^{(2)};\rho=0;\varepsilon), \tag{123}$$

given by (117) are identically satisfied. We look for solutions analytic in ε

$$z=\sum_{i=0}^{k}\mathbf{z}_i\varepsilon^i+\mathrm{O}(\varepsilon^{k+1}) \tag{124}$$

that satisfy

$$\frac{\mathrm{d}\mathbf{z}_i}{d\rho}=\mathbf{g}_i(\mathbf{z_1},\mathbf{z_2},\ldots,\mathbf{z_i}),\quad \mathbf{z(0)}_i=\mathbf{z}_{0,i}, \tag{125}$$

where the linear vector fields $\mathbf{g}_i$ and the initial values $\mathbf{z}_{0,i}$ are, respectively, the coefficients of the expansions $\mathbf{g}=\sum_i\mathbf{g}_i\varepsilon^i$ and $\mathbf{z}_0=\sum_i\mathbf{z}_{0,i}\varepsilon^i$. These linear systems can be integrated explicitly and the solution $\mathbf{z}_{1,i}$ at $\rho=1$ computed. The solution at the outer boundary $\mathbf{z}_1=\sum\mathbf{z}_{1,i}\varepsilon^i$ is a function of the unknown parameter a (or, equivalently, the parameter α_a) and the condition for the existence of a solution of the original problem is the vanishing of the determinant

$$\Delta=\begin{vmatrix} c_3(\mathbf{y^{(1)}}(1)) & c_3(\mathbf{y^{(1)}}(1)) \\ c_4(\mathbf{y^{(2)}}(1)) & c_4(\mathbf{y^{(2)}}(1)) \end{vmatrix}, \tag{126}$$

where $[\mathbf{y}^{(1)}(1),\mathbf{y}^{(2)}(1)]^{\mathrm{T}}=\mathbf{z}_1$. We expand α_a in powers of ε and solve $\Delta=0$ to each order. To order 3 the solution reads

$$\alpha_a^{(n)}=\alpha_0(n)+\tilde{\alpha}_1(\gamma,n)\varepsilon+\tilde{\alpha}_2(\gamma,n)\varepsilon^2+\mathrm{O}(\varepsilon^3), \tag{127}$$

where α_0 is the first positive root of

$$(n+2)(n-1)\alpha_0^{12}+2(n^2+n+7)\alpha_0^6-3n(n+1)=0, \tag{128}$$

and

$$\tilde{\alpha}_1=\frac{\alpha_0^3}{2\gamma\gamma_2^3}-\frac{1}{2}, \tag{129}$$

$$\begin{aligned}\tilde{\alpha}_2=&-\frac{1}{72\gamma_2^3\gamma^2}[-462\alpha_0^6\gamma^2\gamma_2^3-342n^2\gamma\alpha_0^9+33n^2\gamma^2\gamma_2^9+98n\gamma^2\gamma_2^9+6n^4\alpha_0^9\gamma\\&-330\gamma_2^3n\alpha_0^6+90n^4\gamma\alpha_0^3\gamma_2^6-41n^4\gamma^2\gamma_2^9+672\gamma\alpha_0^9+24n^5\gamma^2\gamma_2^9-1470\gamma_2^3\alpha_0^6\\&-180\gamma n\alpha_0^3\gamma_2^6-348n\gamma\alpha_0^9+345n\gamma^2\alpha_0^6\gamma_2^3-57n^4\alpha_0^6\gamma_2^3\gamma^2-114n^3\alpha_0^6\gamma_2^3\gamma^2\\&+288\alpha_0^6\gamma_2^3n^2\gamma^2+180n^3\gamma\alpha_0^3\gamma_2^6-90n^2\gamma\alpha_0^3\gamma_2^6-122n^3\gamma^2\gamma_2^9+360\gamma_2^9n^2\\&+90\gamma_2^9n^3+12\alpha_0^9n^3\gamma+45\gamma_2^9n^4+315\gamma_2^9n-150\gamma_2^3n^3\alpha_0^6-405\gamma_2^3n^2\alpha_0^6\\&-75\gamma_2^3n^4\alpha_0^6+8n^6\gamma^2\gamma_2^9]\\&\times[-3\gamma_2^6n^2-\alpha_0^6n^4-2\alpha_0^6n^3+14\alpha_0^6-6n^2\alpha_0^6-5n\alpha_0^6-6\gamma_2^6n\\&+6n^3\gamma_2^6+3n^4\gamma_2^6]^{-1}.\end{aligned} \tag{130}$$

Higher order coefficients can be computed in a similar manner. To obtain the critical curves in terms of the parameter $\delta = (B - A)/A$, we use the expression for the volume of the deformed shell:

$$b^3 - a^3 = \gamma\gamma_2^3(B^3 - A^3). \tag{131}$$

To second-order it reads $\varepsilon = \frac{\gamma\gamma_2^3}{\alpha_0^2}\delta$ and the expansion of the critical curves in terms of δ reads

$$\alpha_a^{(n)} = \alpha_0(n) + \alpha_1(\gamma, n)\delta + \alpha_2(\gamma, n)\delta^2 + \mathrm{O}(\delta^3), \tag{132}$$

where $\alpha_i = \tilde{\alpha}_i\left(\frac{\gamma\gamma_2^3}{\alpha_0^2}\right)^i$ for $i > 0$.

In Fig. 11, the comparison between the numerical curves and the expansion for thin shells to third-order is given. The thin shell expansion gives an excellent approximation of the numerical curves. Moreover, the limiting value corresponding to $\varepsilon = \delta = 0$ is the exact critical strain.

For high mode number n and thin shells a simpler expression for the critical curve can be obtained:

$$\begin{aligned}\alpha_a = {} & \frac{4n^4 - 2n^2 + 2n - 1}{4n^4} \\ & - \frac{(5\gamma n + 4n^5\gamma - 4n^2\gamma - 4n^5 + n - 6\gamma + 2n^3 - 2n^2 + 4n^3\gamma)\delta}{8n^5} \\ & - \frac{1}{864n^5}(-276n^3\gamma^2 - 396n^3\gamma + 90n^3 + 48n^6\gamma^2 + 1640n\gamma^2 - 1467\gamma n \\ & + 45n + 48\gamma^2 n^7 + 276n^2\gamma^2 + 396n^2\gamma - 90n^2 - 420n^5\gamma^2 + 576n^5\gamma \\ & - 180n^5 + 2538\gamma - 3556\gamma^2)\delta^2 + \mathrm{O}\left(\delta^3, \frac{1}{n^6}\right).\end{aligned} \tag{133}$$

Appendix F. High mode analysis

Eq. (79) with boundary conditions (84) and (85) also admits a WKB expansion valid in the large n limit. Such solution has been computed in the classical case in (Fu, 1998). This expansion can be captured by setting

$$U(r) = \exp\left(\int_a^r S(r)\,\mathrm{d}r\right), \tag{134}$$

with

$$S = n\sum_{i=0}^{\infty} S_i n^{-i}. \tag{135}$$

The functions S_i can be obtained by substituting Eq. (134) in Eq. (79) and solving to each order in n. Since the problem is of fourth-order, the equation to order n^{-4} has four distinct solutions given by

$$S_0^{(1,2)} = \pm\frac{1}{r}, \quad S_0^{(3,4)} = \pm\frac{\alpha^3}{r}. \tag{136}$$

Each of these solutions gives rise to a solution

$$U^{(k)} = \exp\left(\int_a^r S^{(k)}\,\mathrm{d}r\right) = \exp\left[\int_a^r \left(n\sum_{i=0}^{\infty} S_i^{(k)} n^{-i}\right)\mathrm{d}r\right]. \tag{137}$$

The equations for $S_i^{(k)}$ with $i>0$ is linear with coefficients depending only on $\{S_j^{(k)};\ j=0,\dots,i-1\}$. We test the existence of a solution to the boundary value problem by looking for a linear combination of the four fundamental solutions satisfying the boundary conditions. There exists such a solution when the determinant of the matrix built on the four boundary conditions (84)–(85) evaluated on the four solutions vanish identically. Let $c_{1,2}(U)$ and $c_{3,4}(U)$ be the boundary conditions (taken respectively at $r=a$ and $r=b$) and $M_{ij}=c_i(U^{(j)}),\ i,j=1,\dots,4$. Then the condition for the existence of a solution to the boundary value problem in the large n limit is simply given by the vanishing of $\Delta = \det(M_{ij})$. However, the values of the parameters at which this determinant vanishes have to be evaluated numerically.

Further theoretical progress can be achieved for thick shells, that is for shells such that $(B-A)/A = \mathrm{O}(1)$. In such case, the expansion of the determinant is itself an asymptotic series and an excellent approximation of the neutral curves can be obtained by finding the roots of the coefficient in front of the exponentially dominant term. This analysis leads to

$$\alpha_a = \alpha_0 + \alpha_1\frac{1}{n} + \alpha_2\frac{1}{n^2} + \alpha_3\frac{1}{n^3} + \mathrm{O}\left(\frac{1}{n^4}\right), \tag{138}$$

where α_0 is the first positive root of

$$\begin{aligned}\alpha_0 = {} & \frac{1}{12}\left[576\left(26+6\sqrt{33}\right)^{1/3} + 234\left(26+6\sqrt{33}\right)^{2/3}\right.\\ & \left.-576 - 54\left(26+6\sqrt{33}\right)^{2/3}\sqrt{33}\right]^{1/3}\\ \approx {} & .6661423391\end{aligned} \tag{139}$$

and

$$\alpha_1 = \frac{-(3\gamma+11)\alpha_0^6 - (32+4\gamma)\alpha_0^3 + 9 + 5\gamma}{12\gamma\alpha_0^5(\alpha_0^3+2)}, \tag{140}$$

$$\alpha_2 = \frac{(-70752\gamma - 31349 - 3111\gamma^2)\alpha_0^6 - (164574 + 73724\gamma - 45474\gamma^2)\alpha_0^3 - 13163\gamma^2 + 27972\gamma + 51387}{18(\alpha_0^3+2)^3\gamma^2\alpha_0^{14}(\alpha_0-1)^3(\alpha_0+1)^2(\alpha_0^2+1+\alpha_0)^3(\alpha_0^2+1-\alpha_0)^2}. \tag{141}$$

For $\gamma = 1$, we recover the classical expansion (86). Note also that the limit from thick shell to the complete sphere can be obtained by taking the limit $A \to 0$ at fixed B or, alternatively, by taking A constant and $B \to \infty$. This last limit corresponds to a half-space under biaxial compression and the limiting value $\alpha_a = 0.6661$ is the classical value obtained for such an instability (Green and Zerna, 1992).

References

Adkins, J.E., Rivlin, R.S., 1952. Large elastic deformations of isotropic materials. IX. The deformation of thin shells. Phil. Trans. Roy. Soc. A 244, 505–531.

Alexander, H., 1971. Tensile instability of initially spherical balloons. Int. J. Engng. Sci 9, 151–162.

Ambrosi, D., Mollica, F., 2004. The role of stress in the growth of a multicell spheroid. J. Math. Biol. 48, 477–499.

Audoly, B., Boudaoud, A., 2003. Self-similar structures near boundaries in strained systems. Phys. Rev. Lett. 91, 086105.

Ben Amar, M., Pomeau, Y., 1986. Theory of dendritic growth in a weakly undercooled melt. Europhys. Lett. 2, 307–314.

Casey, J., Naghdi, P.M., 1980. A remark on the use of the decomposition $F = F_e F_p$ in plasticity. Trans. ASME 47, 672–675.

Chen, C.Y.M., Byrne, H., King, J.R., 2001. The influence of growth-induced stress from the surrounding medium on the development of multicell spheroids. J. Math. Biol. 43, 191–220.

Chen, Y.C., Healey, T.J., 1991. Bifurcation to pear-shaped equilibria of pressurized spherical membranes. Int. J. Non-Linear Mechanics 26, 279–291.

Chen, Y., Hoger, A., 2000. Constitutive functions of elastic materials in finite growth and deformation. J. Elasticity 59, 175–193.

Cowin, S.C., 1996. Strain or deformation rate dependent finite growth in soft tissues. J. Biomechanics 29, 647–649.

Cowin, S.C., 2004. Tissue growth and remodeling. Annu. Rev. Biomed. Eng. 6, 77–107.

Cowin, S.C., Hegedus, D.M., 1976. Bone remodeling I: A theory of adaptive elasticity. J. Elasticity 6, 313–325.

DiCarlo, A., Quiligotti, S., 2002. Growth and balance. Mech. Res. Comm. 29, 449–456.

Drozdov, A.D., 1990. Volumetric growth of viscoleastic solids. Mech. Solids 25, 99–106.

Drozdov, A.D., Khanina, H., 1997. A model for the volumetric growth of a soft tissue. Math. Comput. Modelling 25, 11–29.

Dumais, J., Steele, C.R., Rennich, S.C., 2000. New evidence for the role of mechanical forces in the shoot of apical meristem. J. Plant Growth Regul. 19, 7–18.

Entov, V.M., 1983. Mechanical model of scoliosis. IMech. Solids 18, 199–206.

Epstein, M., Maugin, G., 2000. Thermomechanics of volumetric growth in uniform bodies. Int. J. Plasticity 16, 951–978.

Fu, Y., 1998. Some asymptotic results concerning the buckling of a spherical shell of arbitrary thickness. Int. J. Non-linear Mech. 33, 1111–1122.

Fung, Y.C., 1990. Biomechanics: Motion, Flow, Stress, and Growth. Springer, New York.

Fung, Y.C., 1993. Biomechanics: Material Properties of Living Tissues. Springer, New York.

Garikipati, K., Arruda, E.M., Grosh, K., Narayanan, H., Clave, S., 2004. A continuum treatment of growth in biological tissue: the coupling of mass transport and mechanics. J. Nonlinear Mech. Phys. 52, 1595–1625.

Gent, A., 1999. Elastic instabilities of inflated rubber shells. Rubber Chem. Technol. 72, 263–268.

Gooday, G.W., Trinci, A.P.J., 1980. Wall structure and biosynthesis in fungi. In: Gooday, G.W., Lloyd, D., Trinci, A.P.J. (Eds.), The Eukaryote Microbial Cell. Cambridge University Press, London, pp. 207–251.

Goriely, A., Ben Amar, M., 2005. Differential growth and instability in elastic shells. Phys. Rev. Lett.

Goriely, A., Tabor, M., 2003. Biomechanical models of hyphal growth in actinomycetes. J. Theor. Biol. 222, 211–218.

Green, A.E., Adkins, J.E., 1970. Large Elastic Deformations. Clarendon Press, Oxford.

Green, A.E., Zerna, W., 1992. Theoretical Elasticity. Dover, New York.

Harold, F.M., 1997. How hyphae grow: morphogenesis explained. Protoplasm 197, 137–147.

Haughton, D.M., 2004. On non-linear stability in unconstrained non-linear elasticity. Int. J. Non-linear Mech. 39, 1181–1192.

Haughton, D.M., Chen, Y., 1999. On the eversion of incompressible elastic spherical shells. Z. Angew. Math. Phys. 50, 312–326.

Haughton, D.M., Kirkinis, E., 2003. A comparison of stability and bifurcation criteria for inflated spherical elastic shells. Math. Mech. Solids 8, 561–572.

Haughton, D.M., Ogden, R.W., 1978. On the incremental equations in non-linear elasticity—II. Bifurcation of pressurized spherical shells. J. Mech. Phys. Solids 26, 111–138.

Haughton, D.M., Ogden, R.W., 1979. Bifurcation of inflated circular cylinders of elastic material under axial loading—I. Membrane theory for thin-walled tubes. J. Mech. Phys. Solids 27, 489–512.

Haughton, D.M., Orr, A., 1997. On the eversion of compressible elastic cylinders. Int. J. Solids Struct. 34, 1893–1914.

Hill, J.M., 1976. Closed form solutions for small deformations superimposed upon the symmetrical expansion of a spherical shell. J. Elasticity 6, 125–136.

Hoger, A., 1986. On the determination of residual stress in an elastic body. J. Elasticity 16, 303–324.

Howard, R.J., Valent, B., 1996. Breaking and entering: host penetration by the fungal rice blast pathogen, *Magnaporthe grisea*. Annu. Rev. Microbiol. 50, 491–512.

Hsu, F.H., 1968. The influences of mechanical loads on the form of a growing elastic body. J. Biomech. 1, 303–311.

Humphrey, J.D., 2003. Continuum biomechanics of soft biological tissues. Proc. Roy. Soc. Lond. A 459, 3–46.

Humphrey, J.D., Rajagopal, K.R., 2002. A constrained mixture model for growth and remodeling of soft tissues. Math. Models. Meth. Appl. Sci. 12, 407–430.

Imatani, S., Maugin, G.A., 2002. A constitutive model for material growth and its application to three-dimensional finite element analysis. Mech. Res. Comm. 29, 477–483.

Keller, R., Davidson, L.A., Shook, D.R., 2003. How we are shaped: the biomechanics of gastrulation. Differentiation 71, 171–205.

Kessler, D.A., Koplik, J., Levine, H., 1988. Pattern selection in fingered growth phenomena. Adv. Phys. 37, 255–339.

Klisch, S.M., Van Dyke, T.J., Hoger, A., 2001. A theory of volumetric growth for compressible elastic biological materials. Math. Mech. Solids 6, 551–575.

Klisch, S.M., Chen, S.S., Sah, R.S., Hoger, A., 2003. A growth mixture theory for cartilage with application to growth-related experiments on cartilage explants. J. Biomech. Eng. 125, 169–179.

Langer, J.S., 1980. Instabilities and pattern-formation in crystal-growth. Rev. Mod. Phys. 52, 1–28.

Lee, E.H., 1969. Elastic–plastic deformation at finite strains. J. Appl. Mech. 36, 1–8.

Lin, L.E., Taber, L., 1995. A model for stress-induced growth in the developing heart. J. Biomech. Eng. 117, 343–349.

Lubarda, V.A., Hoger, A., 2002. On the mechanics of solids with a growing mass. Int. J. Solids Struct. 39, 4627–4664.

Money, N.P., 1997. Wishful thinking of turgor revisited: the mechanics of fungal growth. Fungal Genetics and Biol. 21, 173–187.

Müller, I., Struchtrup, H., 2002. Inflating a rubber balloon. Math. Mech. Solids 7, 569–577.

Odell, G., Oster, G., Alberch, P., Burnside, B., 1981. The mechanical basis of morphogenesis. Dev. Biol. 85, 446–462.

Ogden, R.W., 1972. Large deformation isotropic elasticity—On the correlation of theory and experiment for incompressible rubberlike solids. Proc. Roy. Soc. Lond. A 326, 565–584.

Ogden, R.W., 1984. Non-linear Elastic Deformation. Dover, New york.

Onuki, A., 1989. Theory of pattern formation in gels: surface folding in highly compressible elastic bodies. Phys. Rev. A 39, 5932–5950.

Reinhardt, M.O., 1892. Das wachsthum der Pilzhyphen. Jahrbücher für wissenschaftliche botanik 23, 479–566.

Rodriguez, E.K., Hoger, A., McCulloch, A., 1994. Stress-dependent finite growth in soft elastic tissue. J. Biomech. 27, 455–467.

Shipman, P., Newell, A., 2004. Phyllotactic patterns on plants. Phys. Rev. Lett. 92, 168102.

Skalak, R., 1981. Growth as a finite displacement field. In: Carlson, D.E., Shield, R.T. (Eds.), Proceedings of the IUTAM Symposium on Finite Elasticity. Martinus Nijhoff, The Hague, pp. 347–355.

Skalak, R., Hoger, A., 1997. Kinematics of surface growth. J. Math. Biol. 35, 869–907.

Skalak, R., Tozeren, A., Zarda, R.P., Chien, S., 1973. Strain energy function of red blood cell membranes. Biophys. J. 13, 245–264.

Skalak, R., Zargaryan, S., Jain, R.K., Netti, P.A., Hoger, A., 1996. Compatibility and the genesis of residual stress by volumetric growth. J. Math. Biol. 34, 889–914.

Steele, C.R., 2000. Shell stability related to pattern formation in plants. J. Appl. Mech. 67, 237–247.

Stein, A.A., 1995. The deformation of a rod of growing biological material under longitudinal compression. J. Appl. Math. Mech. 59, 139–146.

Sudipto, K.D., Aluru, N.R., Johnson, B., Crone, W.C., Beebe, D.J., Moorwe, J., 2002. Equilibrium swelling and kinetics of pH-responsive hydrogels: models, experiments, and simulations. J. Microelectromech. Syst. 11, 544–555.

Taber, L., 1995. Biomechanics of growth, remodeling and morphogenesis. Appl. Mech. Rev. 48, 487–545.

Taber, L., 1998a. Biomechanical growth laws for muscle tissue. J. Theor. Biol. 193, 201–213.

Taber, L., 1998b. A model of aortic growth based on fluid shear and fiber stresses. J. Biomech. Eng. 120, 348–354.

Taber, L., Eggers, D.W., 1996. Theoretical study of stress-modulated growth in the aorta. J. Theor. Biol. 180, 343–357.

Taber, L.A., Humphrey, J.D., 2001. Stress-modulated growth, residual stress, and vascular heterogeneity. J. Biomech. Eng. 123, 528–535.

Thomspon, D.W., 1992. On Growth and Form: The Complete Revised Edition. Dover, New York.

Totafurno, J., Bjerknes, M., 1983. Morphogenesis and mechanical instability of a prestressed tissue. Biochem. Cell. Biol., 565–574.

Wang, A.S.D., Ertepinar, A., 1972. Stability and vibrations of elastic thick-walled cylindrical and spherical shells subjected to pressure. Int. J. Non-linear Mech. 7, 539–555.

Wesolowski, Z., 1967. Stability of an elastic, thick-walled spherical shell loaded by an external pressure. Arch. Mech. Stosow. 19, 3–23.