

Notes for Contact Methods
Linear Elasticity and Hyperelasticity

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1 Problem Definition

These codes were created to troubleshoot the more advanced code incorporating mass transport, large deformations, and contact. For the contact methods, I recall that Lagrange methods were not sufficient so I do not have much documentation on them. I follow Wriggers closely.

2 Formulation

Equation for strain under assumptions of linear elasticity

$$\begin{aligned}\epsilon_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ \epsilon &= \frac{1}{2} (\nabla u + \nabla u^T)\end{aligned}\tag{2.1}$$

General expression of the linear elastic isotropic constitutive relationship

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}\tag{2.2}$$

This can be inverted to be strain in terms of stress:

$$\epsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}\tag{2.3}$$

where the Lamé coefficients are given by:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \mu = \frac{E}{2(1+\nu)}\tag{2.4}$$

Starting from the mechanical equilibrium equation:

$$\begin{aligned}-\nabla \cdot \sigma &= f \quad \text{Convert to indicial notation} \\ -\frac{\partial}{\partial x_k} \mathbf{e}_k \cdot \sigma_{ij} (\mathbf{e}_i \otimes \mathbf{e}_j) &= f_k \mathbf{e}_k \\ -\frac{\partial \sigma_{ij}}{\partial x_k} \delta_{ki} \mathbf{e}_j &= f_k \mathbf{e}_k \\ -\frac{\partial \sigma_{ij}}{\partial x_i} \mathbf{e}_j &= f_k \mathbf{e}_k \quad \text{Multiply by a test function} \\ -\frac{\partial \sigma_{ij}}{\partial x_i} \mathbf{e}_j \cdot v_p \mathbf{e}_p &= f_k \mathbf{e}_k \cdot v_p \mathbf{e}_p \\ -\frac{\partial \sigma_{ij}}{\partial x_i} v_p \delta_{jp} &= f_k v_p \delta_{kp} \\ -\frac{\partial \sigma_{ij}}{\partial x_i} v_j &= f_k v_k \quad \text{Integrate over domain} \\ -\int_{\Omega} \frac{\partial \sigma_{ij}}{\partial x_i} v_j dx &= \int_{\Omega} f_k v_k dx\end{aligned}$$

Integration by parts on the LHS

$$\begin{aligned}(fg)' &= f'g + fg' \rightarrow f'g = (fg)' - fg' \\ \frac{\partial \sigma_{ij}}{\partial x_i} v_j &= (\sigma_{ij} v_j)_{,i} - \sigma_{ij} \frac{\partial v_j}{\partial x_i}\end{aligned}$$

Substitute the result from integration by parts:

$$\begin{aligned}
& - \int_{\Omega} (\sigma_{ij} v_j)_{,i} dx + \int_{\Omega} \sigma_{ij} \frac{\partial v_j}{\partial x_i} dx = \int_{\Omega} f_k v_k dx \quad \text{Use the divergence theorem} \\
& - \int_{\partial\Omega} \sigma_{ij} v_j n_i ds + \int_{\Omega} \sigma_{ij} \frac{\partial v_j}{\partial x_i} dx = \int_{\Omega} f_k v_k dx \quad \text{Recognize the traction term} \\
& - \int_{\partial\Omega} t_i v_j ds + \int_{\Omega} \sigma_{ij} \frac{\partial v_j}{\partial x_i} dx = \int_{\Omega} f_k v_k dx \quad \text{Rearrange} \\
& \int_{\Omega} \sigma_{ij} \frac{\partial v_j}{\partial x_i} dx = \int_{\Omega} f_k v_k dx + \int_{\partial\Omega} t_i v_j ds
\end{aligned}$$

The principle of virtual work states that:

$$\int_{\Omega} \delta W dV = \int_{\Omega} \sigma_{ij} \delta \epsilon_{ij} dV = \int_{\partial\Omega} t_i \delta u_i dS + \int_{\Omega} b_i \delta u_i dV \quad (2.5)$$

Applying this principle to, where \mathbf{f} is the body force, \mathbf{b} :

$$\begin{aligned}
& \int_{\Omega} \sigma_{ij} \frac{\partial v_j}{\partial x_i} dx = \int_{\Omega} f_k v_k dx + \int_{\partial\Omega} t_i v_j ds \\
& \int_{\Omega} \sigma_{ij} \epsilon_{ij} dV = \int_{\Omega} f_k v_k dx + \int_{\partial\Omega} t_i v_j ds \quad \text{no traction applied} \\
& \int_{\Omega} \sigma_{ij} \epsilon_{ij} dV = \int_{\Omega} f_k v_k dx
\end{aligned}$$

Writing in direct notation we have the variational (weak) formulation. Find $\mathbf{u} \in V$ such that:

$$\int_{\Omega} \sigma(\mathbf{u}) : \epsilon(\mathbf{v}) d\Omega = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\Omega \quad \forall \mathbf{v} \in V \quad (2.6)$$

3 Contact Methods

The rigid indenter with a spherical surface can be approximated by a parabolic equation instead of explicitly modeled and meshed. Consider the indenter radius, R , to be sufficiently large with respect to the contact region characteristic size ($R \gg a$). This relationship, $R \gg a$, allows the spherical surface to be approximated by a parabola.

$$\begin{aligned}
h(x, z) &= -h_o + \frac{1}{2R}(x^2 + z^2) \quad \text{about origin} \\
h(x, z) &= -h_o + \frac{1}{2R}[(x - 0.5)^2 + (z - 0.5)^2] \quad \text{about center point (0.5, 0.5)}
\end{aligned} \quad (3.1)$$

The definition of the MacKauley bracket:

$$\begin{aligned}
\langle x \rangle &= \frac{x + |x|}{2} \\
&= \begin{cases} 0 & \text{for } x \leq 0, \\ x & \text{for } x > 0 \end{cases}
\end{aligned} \quad (3.2)$$

3.1 Penalty Approach

3.1.1 Theory

Adding a penalty term to the energy, Π .

$$\Pi_c^P = \frac{1}{2} \int_{\Gamma_c} \epsilon_N (g_N)^2 dA \quad \epsilon_N > 0 \quad (3.3)$$

Take the variation of Eq. 3.3, this holds for pure stick. This is the addition to the weak form:

$$C_c^P = \int_{\Gamma_c} (\epsilon_N g_N \delta g_N) dA$$

3.1.2 Incorporating into weak form

$$\int_{\Omega} \sigma(\mathbf{u}) : \epsilon(\mathbf{v}) d\Omega - \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\Omega + k_{pen} \int_{\partial\Omega} \langle u_1 - h(x, z) \rangle ds = 0 \quad \text{No body force} \quad (3.4)$$

$$\int_{\Omega} \sigma(\mathbf{u}) : \epsilon(\mathbf{v}) d\Omega + k_{pen} \int_{\partial\Omega} \langle u_1 - h(x, z) \rangle v_1 ds = 0 \quad (3.5)$$

3.2 Augmented Lagrange Methods

The main idea of this method is combining the penalty method with Lagrange multiplier methods.

3.2.1 Theory

The augmented Lagrange functional is introduced for normal contact:

$$l_n = \lambda g + \frac{\epsilon}{2} g^2 - \frac{1}{2\epsilon} \langle \lambda + \epsilon g \rangle^2 \quad (3.6)$$

Take the variation of Eq. 3.6

$$\begin{aligned} \Pi_N^{AM} &= \delta \lambda g + \lambda \delta g + \epsilon g \delta g - \frac{1}{\epsilon} \langle \lambda + \epsilon g \rangle \delta \lambda - \langle \lambda + \epsilon g \rangle \delta g \\ &= g \delta \lambda + (\lambda + \epsilon g) \delta g - \frac{1}{\epsilon} \langle \lambda + \epsilon g \rangle \delta \lambda - \langle \lambda + \epsilon g \rangle \delta g \\ &= g \delta \lambda + \hat{\lambda} \delta g - \frac{1}{\epsilon} \langle \hat{\lambda} \rangle \delta \lambda - \langle \hat{\lambda} \rangle \delta g \end{aligned}$$

We introduce the following augmented lagrangian term.

$$\hat{\lambda} = \lambda + \epsilon g$$

Where for $\hat{\lambda} \leq 0$, we are in contact or penetrating the surface and the gap is 0 or negative. For this case, everything within a Mackauley bracket in the variation goes to zero.

$$\begin{aligned} C_N^{AM} &= g \delta \lambda + (\lambda + \epsilon g) \delta g \\ &= g \delta \lambda + \hat{\lambda} \delta g \end{aligned}$$

Next, for $\hat{\lambda} > 0$, we are not in contact, and the gap is positive.

$$\begin{aligned} C_N^{AM} &= g \delta \lambda + (\lambda + \epsilon g) \delta g - \frac{1}{\epsilon} (\lambda + \epsilon g) \delta \lambda - (\lambda + \epsilon g) \delta g \\ &= g \delta \lambda - \frac{1}{\epsilon} \lambda \delta \lambda - g \delta \lambda \\ &= -\frac{1}{\epsilon} \lambda \delta \lambda \end{aligned}$$

Therefore, summarizing the two cases:

$$C_N^{AM} = \begin{cases} \int_{\Gamma_c} \left(g\delta\lambda + \hat{\lambda}\delta g \right) d\Gamma & \text{for } \hat{\lambda}_N \leq 0, \\ \int_{\Gamma_c} -\frac{1}{\epsilon} \lambda\delta\lambda d\Gamma & \text{for } \hat{\lambda}_N > 0 \end{cases} \quad (3.7)$$

where $\hat{\lambda}_N \leq 0$ indicates that the gap is closed and $\hat{\lambda}_N > 0$ means the gap is open.

3.2.2 Incorporating into weak form