Stabilized Mixed Finite Element Formulation

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Definitions

$\mathbf{F}$: Deformation gradient
$\mathbf{I}$: second-order unit tensor
$\boldsymbol{u}$: Displacement
J: determinant of the deformation gradient
$\mathbf{C}$: Right Cauchy-Green Strain Tensor
$\mathcal{W}(\mathbf{F})$: strain energy function
$\mathbf{P}$: first Piola-Kirchhoff stress tensor
$\mathbf{S}$: second PK stress tensor

α: cracks are represented by a scalar phase-field variable
p: Lagrange multiplier, hydrostatic pressure field
κ: bulk modulus

$$\kappa = \frac{E}{3(1-2\nu)} \tag{0.1}$$

μ: shear modulus

$$\mu = \frac{E}{2(1+\nu)} \tag{0.2}$$

λ: Lamé modulus

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{0.3}$$

For Plane Stress

$$\kappa = \frac{3-\nu}{1+\nu}, \quad \lambda = \frac{E\nu}{(1-\nu)^2} \tag{0.4}$$

$\mathcal{E}_\ell$: potential energy functional
$a(\alpha)$ is the decreasing stiffness modulation function
$w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume
c_w is a normalization constant

1 Hyperelastic Phase-Field Fracture Models

Deformation Gradient

$$\mathbf{F} = \mathbf{I} + \nabla \otimes \boldsymbol{u} \tag{1.1}$$

where $J = \det \mathbf{F}$ and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

The strain energy function $\mathcal{W}(\mathbf{F})$ is defined per unit reference volume such that the first PK and second PK

$$\mathbf{P} = \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \tag{1.2a}$$

$$\mathbf{S} = 2\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{C}} \tag{1.2b}$$

where $\mathbf{P} = \mathbf{FS}$.

1.1 Phase-Field Fracture Model

Non-modified strain energy function is the compressible Neo-Hookean:

$$\mathcal{W}(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2\ln J) \tag{1.3}$$

For incompressible hyperelastic materials, the strain energy function is defined as

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p\,(J-1)\,, \tag{1.4}$$

We instead consider a damage dependent relaxation of the incompressibility constraint and the phase-field couples to the strain energy through the modified function

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa\,(J-1)^2 \tag{1.5}$$

where the decreasing stiffness modulation function is $a(\alpha)$ and $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

$$a(\alpha) = (1-\alpha)^2 \quad w(\alpha) = \alpha \tag{1.6}$$

In the code, we have the following definition

$$b(\alpha) = (1-\alpha)^3 = \sqrt{a^3(\alpha)}$$

To circumvent numerical difficulties, we resort to the classical mixed formulation and introduce the pressure-like field

$$\begin{aligned} p &= -\sqrt{a^3(\alpha)}\kappa\,(J-1) \\ p &= -b(\alpha)\kappa\,(J-1) \end{aligned} \tag{1.7}$$

as an independent variable along with the displacement field.

Following the code, we have the following energy functional of the energy problem

$$\widetilde{W}(\mathbf{F}, \alpha) = (a(\alpha) + k_\ell)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J-1) - \frac{p^2}{2\kappa}$$

$$\widetilde{W}(\mathbf{F}, \alpha) = \big(a(\alpha) + k_\ell\big)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\kappa}$$

The first Piola-Kirchhoff stress tensor is given:

$$\begin{aligned}
\mathbf{P} &= \frac{\partial \widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{\partial \mathbf{F}} \\
&= \frac{\partial}{\partial \mathbf{F}}\left[a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa\,(J-1)^2\right] \\
&= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\frac{1}{2}\kappa\frac{\partial (J-1)^2}{\partial \mathbf{F}} \\
&= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa\,(J-1)\,\frac{\partial J}{\partial \mathbf{F}} \quad \text{where } \frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T} \\
&= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa(J-1)J\mathbf{F}^{-T} \quad \text{substituting in pressure equation} \\
\mathbf{P} &= a(\alpha)\mu(\mathbf{F}-\mathbf{F}^{-T}) - b(\alpha)pJ\mathbf{F}^{-T}
\end{aligned} \tag{1.8}$$

1.1.1 Derivation from Bin2020 Paper

Here, unlike Eq. 21 from Bin2020, we drop λ_b which is not a consideration in this formulation

$$\mathcal{E}_\ell(\boldsymbol{u},\alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F},\alpha)d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega \tag{1.9}$$

The normalization constant is defined as:

$$c_w = \int_0^1 \sqrt{w(\alpha)}d\alpha \tag{1.10}$$

Starting from Eq. 25 in the 2020 Li and Bouklas paper where κ is the bulk modulus

$$\mathcal{E}_\ell\left(\boldsymbol{u},p,\Lambda,\alpha\right) = \mathcal{E}_\ell(\boldsymbol{u},\alpha) + \int_\Omega \frac{p^2}{2\kappa}d\Omega + \int_\Omega \Lambda(p + \sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \tag{1.11}$$

Identify the stationary point of the energy functional with respect to pressure (not Λ)

$$\begin{aligned}
\frac{\partial \mathcal{E}_\ell}{\partial p} &= \int_\Omega \frac{p}{\kappa}d\Omega + \int_\Omega \Lambda d\Omega \\
0 &= \frac{p}{\kappa} + \Lambda \rightarrow \Lambda = -p/\kappa
\end{aligned}$$

Substituting this relationship into the energy functional yields:

$$\begin{aligned}
\mathcal{E}_\ell\left(\boldsymbol{u},p,\alpha\right) &= \mathcal{E}_\ell(\boldsymbol{u},\alpha) + \int_\Omega \frac{p^2}{2\kappa}d\Omega + \int_\Omega -\frac{p}{\kappa}(p + \sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \\
&= \mathcal{E}_\ell(\boldsymbol{u},\alpha) + \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \frac{p^2}{\kappa}d\Omega - \int_\Omega \frac{p}{\kappa}\sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \\
\mathcal{E}_\ell\left(\boldsymbol{u},p,\alpha\right) &= \mathcal{E}_\ell(\boldsymbol{u},\alpha) - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1))d\Omega
\end{aligned} \tag{1.12}$$

Substitute in $\mathcal{E}_\ell(\boldsymbol{u},\alpha)$ and substitute Eq. 1.3

$$\mathcal{E}_\ell\left(\boldsymbol{u},p,\alpha\right) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega$$

The prior equation includes the full weak form, unless we want to consider linear interpolation of all fields. In that case, we can introduce the stabilization term

$$-\frac{\varpi h^2}{2\mu}\sqrt{a^3(\alpha)}\sum_{e=1}^{n_{el}}\int_{\Omega^e} J\mathbf{C}^{-1} : \big(\nabla p \cdot \nabla q\big)\, dV = 0$$

1.2 Changes for 2D Plane-Stress Models

Recalling the 1st PK stress in Eq. 1.8.

$$\mathbf{P} = a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T}) - b(\alpha)pJ\mathbf{F}^{-T}$$

In a plane-stress case, the $\mathbf{P_{33}}$ component is zero:

$$\mathbf{P}_{33} = a(\alpha)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha)pJ\mathbf{F}_{33}^{-1} = 0$$

This can be multiplied by its associated test function to obtain the weak form

$$\int_\Omega \left(a(\alpha)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha)pJ\mathbf{F}_{33}^{-1}\right) v_{F_{33}} dV = 0$$

In the FEniCS code, we expand the solution space to include displacement, pressure, and a component of the deformation gradient $\mathbf{F_{33}}$. Therefore, we include a change to the invariants of the deformation tensors:

```
J = det(F)*F33
Ic = tr(C) + F33**2
```

Together with the weak form from above:

```
F_u = derivative(elastic_potential, w_p, v_q) \
      + (a(alpha)*mu*(F33 - 1/F33) - b(alpha)*p*J/F33)*v_F33*dx
```

1.2.1 Changes for 2D Discrete Crack Model

If we are considering a discrete fracture method

$$\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right) d\Omega - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)} p(J-1) d\Omega$$

$$\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha) = \int_\Omega \mathcal{W}(\mathbf{F}) d\Omega - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega p(J-1) d\Omega$$

where we have assumed for the energy functional

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = \frac{\mu}{2}(I_c - 3 - 2\ln J) - p(J-1) - \frac{p^2}{2\lambda}$$

Therefore, we can calculate the 1st Piola Kirchoff Stress as:

$$\begin{aligned}\mathbf{P} &= \frac{\mu}{2}(2\mathbf{F} - \frac{2}{J}J\mathbf{F}^{-T}) - pJ\mathbf{F}^{-T} \\ &= \mu(\mathbf{F} - \mathbf{F}^{-T}) - pJ\mathbf{F}^{-T}\end{aligned}$$

Taking the third component to be zero

$$
\begin{aligned}
P_{33} &= \mu(F_{33} - F_{33}^{-1}) - pJF_{33}^{-1} = 0 \\
&= F_{33} - F_{33}^{-1} - \frac{pJ}{\mu}F_{33}^{-1} = 0 \\
P_{33} &= F_{33}^2 - 1 - \frac{pJ}{\mu} = 0
\end{aligned}
$$

with the stabilization term and plane stress in the weak form

$$
\begin{aligned}
-\frac{\varpi h^2}{2\mu}\int_\Omega J\mathbf{C}^{-1} : \big(\nabla p \cdot \nabla q\big)\, dV = 0 \\
\int_\Omega \left(\mathbf{F}_{33}^2 - 1 - \frac{pJ}{\mu}\right) v_{F_{33}} dV = 0
\end{aligned}
$$

1.2.2 Changes for 2D displacement formulation

Removing pressure terms

$$
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega
$$

$$
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega
$$

with plane stress in the weak form (no need for stabilization terms)

$$
\int_\Omega \big(a(\alpha)\mu(F_{33} - F_{33}^{-1})\big) v_{F_{33}} dV = 0
$$

We have assumed the modified energy functional

$$
\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) \tag{1.13}
$$

Therefore, we can calculate the 1st Piola Kirchoff Stress as:

$$
\begin{aligned}
\mathbf{P} &= a(\alpha)\frac{\mu}{2}(2\mathbf{F} - \frac{2}{J}J\mathbf{F}^{-T}) \\
&= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T})
\end{aligned}
$$

Taking the third component to be zero

$$
P_{33} = a(\alpha)\mu(F_{33} - F_{33}^{-1}) = 0
$$

2 Obtaining the Critical Stretch

In order to obtain the critical stretch of a particular specimen in Krishnan 2008, we can equate our dissipated energy or critical fracture energy density

$$\int_{\Gamma} G_c \, d\mathbf{S} = \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega$$

to Eq. 15 for the energy release rate of the strip geometry.

$$J = h\mu \left(\lambda_a - \frac{1}{\lambda_a} \right)^2$$

where the total height of the strip is $2h$ and μ is the shear modulus

3 Strain energy decomposition

The Heaviside function is defined as

$$H(x) = \frac{x + |x|}{2x} = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

The Macaulay bracket is defined

$$M(x) = \frac{x + |x|}{2} = \begin{cases} x, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

3.1 Following Ye 2020 and Tang 2019

In the Ye 2020 paper, the internal energy is expressed as:

$$W_{int}(\mathbf{F}, \alpha, \nabla \alpha) = [a(\alpha) + k_\ell] W_{act} + W_{pas} + G_c \left(\frac{\alpha^2}{2\ell} + \frac{\ell}{2} |\nabla \alpha|^2 \right)$$

Now in section 3.2.3 of Ye 2020, is stated the decomposition for a Mooney Rivlin constitutive law:

$$\begin{aligned} W_{MR}(I_1, I_2) &= C_1(I_1 - 3) + C_2(I_2 - 3) \\ W_{MR}(\lambda_1, \lambda_1, \lambda_3) &= C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3) \\ &= C_1 \sum_{i=1}^{3} (\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} (\lambda_i^{-2} - 1) \end{aligned}$$

Which can be decomposed to active and passive internal energy terms. First we can rewrite:

$$\begin{aligned} W_{MR}(\lambda_1, \lambda_1, \lambda_3) = C_1 \sum_{i=1}^{3} H(\lambda_i - 1)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} H(\lambda_i - 1)(\lambda_i^{-2} - 1) \\ + C_1 \sum_{i=1}^{3} H(1 - \lambda_i)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} H(1 - \lambda_i)(\lambda_i^{-2} - 1) \end{aligned}$$

The active and passive terms can be stated as follows where the active part represents the crack-driven energy.

$$W_{act} = C_1 \sum_{i=1}^{3} H(\lambda_i - 1)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} H(\lambda_i - 1)(\lambda_i^{-2} - 1)$$
$$W_{pas} = C_1 \sum_{i=1}^{3} H(1 - \lambda_i)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} H(1 - \lambda_i)(\lambda_i^{-2} - 1)$$

One way to better understand these is to consider some cases 1) triaxial tension $\lambda_i > 1$ 2) other stress states where $\lambda_i < 1$:

$$\begin{aligned}
&\text{For } \lambda_i > 1: \\
&\quad W_{act} = C_1 \sum_{i=1}^{3} (\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} (\lambda_i^{-2} - 1) \\
&\quad W_{pas} = 0 \\
&\text{For } \lambda_i < 1: \\
&\quad W_{act} = 0 \\
&\quad W_{pas} = C_1 \sum_{i=1}^{3} (\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} (\lambda_i^{-2} - 1)
\end{aligned}$$

For this second case, we end up with a negative energy component (first term of the passive energy).

We can also note the definitions within Tang 2019 for Model M_I. In this model, we consider the free energy density of a neo-Hookean constitutive law:

$$W(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2\ln J) + \frac{\kappa}{2}(\ln J)^2$$

which can also be rephrased in terms of stretches

$$\begin{aligned}
W(\lambda_1, \lambda_2, \lambda_3) &= W_1 + W_2 \\
&= \frac{\mu}{2} \sum_{i=1}^{3} (\lambda_i^2 - 1 - 2\ln \lambda_i) + \frac{\kappa}{2}(\ln J)^2
\end{aligned}$$

where we note that W_1 is a linear function of $\ln \lambda_i$ and W_2 is a nonlinear function of $\ln J$
The free energy is stated as

$$G_{rub} = [(1-K)\alpha^2 + K]W^+ + W^-$$

Note that K is not κ. No definition is provided in the paper. This is another way of coupling the damage to the free energy density, and I believe we can rewrite our own version where:

$$G_{rub} = [a(\alpha) + k_\ell]W^+ + W^-$$

Now we turn to the definition of W^+ and W^- which refers to the energy with tensile stretching

$$W^+ = W(\lambda_i^+, J^+) = \frac{\mu}{2} \sum_{i=1}^{3} ((\lambda_i^+)^2 - 1 - 2\ln \lambda_i^+) + \frac{\kappa}{2}(\ln J^+)^2$$

and the energy with compression respectively.

$$W^- = W(\lambda_i^-, J^-) = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) + \frac{\kappa}{2}(\ln J^-)^2$$

The definitions for these superscript + and - terms gives us

$$\lambda_i^+ = \begin{cases} \lambda_i, & \lambda_i > 1, \\ 1, & \lambda_i \le 1 \end{cases} \qquad J^+ = \begin{cases} J, & J > 1, \\ 1, & J \le 1 \end{cases}$$
$$\lambda_i^- = \begin{cases} \lambda_i, & \lambda_i < 1, \\ 1, & \lambda_i \ge 1 \end{cases} \qquad J^- = \begin{cases} J, & J < 1, \\ 1, & J \ge 1 \end{cases}$$

This isn't the definition for the heaviside function, but it could be a shifted Macaulay bracket

$$M_s(x) = \frac{x - 1 + |x-1|}{2} + 1 = \begin{cases} x, & x > 1, \\ 1, & x \le 1 \end{cases}$$

Now we can consider some examples.

$$\begin{aligned}
&\text{For } J > 1: \\
&\quad W^+ = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) + \frac{\kappa}{2}(\ln J)^2 \\
&\quad W^- = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) \\
&\text{For } J < 1: \\
&\quad W^+ = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) \\
&\quad W^- = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) + \frac{\kappa}{2}(\ln J)^2
\end{aligned}$$

If we consider the same stress states as in Ye2020, 1) triaxial tension $\lambda_i > 1$ 2) all other stress states $\lambda_i < 1$:

$$\begin{aligned}
&\text{For } J > 1,\ \lambda_i > 1: \\
&\quad W^+ = \frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}(\ln J)^2 \\
&\quad W^- = 0 \\
&\text{For } J < 1,\ \lambda_i > 1: \\
&\quad W^+ = \frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) \\
&\quad W^- = \frac{\kappa}{2}(\ln J)^2
\end{aligned}$$

Now for all other stress states:

$$\begin{aligned}
&\text{For } J > 1,\ \lambda_i < 1: \\
&\qquad W^+ = \frac{\kappa}{2}(\ln J)^2 \\
&\qquad W^- = \frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) \\
&\text{For } J < 1,\ \lambda_i < 1: \\
&\qquad W^+ = 0 \\
&\qquad W^- = \frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}(\ln J)^2
\end{aligned}$$

This should be roughly equivalent to the considerations in the Ye2020 paper.

3.2 Our strain energy decomposition

Following the section above, we can consider our modified strain energy

$$\begin{aligned}
\widetilde{W}(\mathbf{F},\alpha) &= a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\kappa} \quad \text{where } p = -b(\alpha)\kappa(J-1) \\
\widetilde{W}(\mathbf{F},\alpha) &= a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) + \frac{\kappa}{2}a(\alpha)^3(J-1)^2
\end{aligned}$$

We can also rewrite the first term with regards to stretches

$$\widetilde{W}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}a(\alpha)^3(J-1)^2$$

Following Tang 2019 we rewrite the strain energy as

$$\widetilde{W}(\mathbf{F},\alpha) = \widetilde{W}_{\text{act}}(\mathbf{F},\alpha) + \widetilde{W}_{\text{pas}}(\mathbf{F},\alpha) \tag{3.1}$$

where the active and passive parts of the strain energy can be written as:

$$\begin{aligned}
\widetilde{W}_{act}(\mathbf{F},\alpha) &= a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) + \frac{\kappa}{2}a(\alpha)^3(J^+ - 1)^2 \\
\widetilde{W}_{pas}(\mathbf{F},\alpha) &= a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) + \frac{\kappa}{2}a(\alpha)^3(J^- - 1)^2
\end{aligned}$$

where the definitions of the superscript + and - terms remain the same as in Tang2020:

$$
\begin{aligned}
&\text{For } J > 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) + \frac{\kappa}{2}a(\alpha)^3(J-1)^2 \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) \\
&\text{For } J \leq 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) + \frac{\kappa}{2}a(\alpha)^3(J-1)^2
\end{aligned}
\tag{3.2}
$$

Now considering the same two cases of, triaxial tension and

$$
\begin{aligned}
&\text{For } J > 1,\ \lambda_i > 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}a(\alpha)^3(J-1)^2 \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = 0 \\
&\text{For } J \leq 1,\ \lambda_i > 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = \frac{\kappa}{2}a(\alpha)^3(J-1)^2
\end{aligned}
$$

all other cases:

$$
\begin{aligned}
&\text{For } J > 1,\ \lambda_i < 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = \frac{\kappa}{2}a(\alpha)^3(J-1)^2 \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) \\
&\text{For } J \leq 1,\ \lambda_i < 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = 0 \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}a(\alpha)^3(J-1)^2
\end{aligned}
$$

These can be concisely summarized with the following expressions, where the active part of the strain energy is

$$
\widetilde{W}_{\text{act}}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3} H(\lambda_i - 1)\left(\lambda_i^2 - 1 - 2\ln\lambda_i\right) + a^3(\alpha)H(J-1)\frac{1}{2}\kappa\,(J-1)^2, \tag{3.3}
$$

and the passive part of the strain energy is

$$\widetilde{W}_{\text{pas}}\left(\mathbf{F},\alpha\right)=a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}H(1-\lambda_i)\left(\lambda_i^2-1-2\ln\lambda_i\right)+a^3(\alpha)H(1-J)\frac{1}{2}\kappa\left(J-1\right)^2, \tag{3.4}$$

3.2.1 Compute the principal stretches λ_i

The eigenvalues of Cauchy-Green strain tensor $\mathbf{C}$ are λ_i^2, $i=1,2,3$. With following definitions

$$d=\frac{Tr\mathbf{C}}{3},\quad e=\sqrt{\frac{Tr(\mathbf{C}-e\mathbf{I})^2}{6}},\quad f=\frac{1}{e}\left(\mathbf{C}-d\mathbf{I}\right),\quad g=\frac{\det f}{2}, \tag{3.5}$$

and assuming the eigenvalues satisfying $\lambda_3^2\leq\lambda_2\leq\lambda_1$, we could obtain (**?**)

$$\lambda_1^2=d+2e\cos\left(\frac{\arccos g}{3}\right),\quad \lambda_3^2=d+2e\cos\left(\frac{\arccos g}{3}+\frac{2\pi}{3}\right),\quad \lambda_2^2=3d-\lambda_1^2-\lambda_3^3. \tag{3.6}$$

4 Following Borden: Derivations of Analytical Phase Field

Note the full potential energy functional, which can also be called the lagrangian

$$\mathcal{E}_\ell(\boldsymbol{u},p,\alpha)=\int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega-\int_\Omega\frac{p^2}{2\kappa}d\Omega-\int_\Omega\sqrt{a^3(\alpha)}p(J-1)d\Omega+\frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell}+\ell\|\nabla\alpha\|^2\right)d\Omega$$

We can use the Euler-Lagrange equations to arrive at the equations of motion by taking the derivative with respect to displacement, pressure, and the scalar damage field. Starting with displacement:

$$\frac{\partial\mathcal{E}_\ell}{\partial\boldsymbol{u}}=\int_\Omega a(\alpha)\frac{\partial\mathcal{W}(\mathbf{F})}{\partial\mathbf{u}}d\Omega$$

$$\frac{\partial\mathcal{E}_\ell}{\partial p}=-\int_\Omega\frac{p}{\kappa}d\Omega-\int_\Omega\sqrt{a^3(\alpha)}(J-1)d\Omega$$

$$\frac{\partial\mathcal{E}_\ell}{\partial\alpha}=-\int_\Omega 2(1-\alpha)\,\mathcal{W}(\mathbf{F})\,d\Omega+\int_\Omega 3p(1-\alpha)^2(J-1)d\Omega+\frac{G_c}{c_w}\int_\Omega\left[\frac{1}{\ell}+2\ell\nabla^2\alpha\right]d\Omega$$

Therefore we have three equations:
First should be mechanical eq, second is a an equation for pressure,

$$\begin{aligned}-\frac{p}{\kappa}-\sqrt{a^3(\alpha)}(J-1)&=0\\ -\frac{p}{\kappa}-(1-\alpha)^3(J-1)&=0\\ -\kappa(J-1)(1-\alpha)^3&=p\end{aligned}$$

Lastly,

$$-2(1-\alpha)\,\mathcal{W}(\mathbf{F})+3p(1-\alpha)^2(J-1)+\frac{G_c}{c_w}\left[\frac{1}{\ell}+2\ell\nabla^2\alpha\right]=0$$

Substitute second equation into third

$$-2(1-\alpha)\,\mathcal{W}(\mathbf{F})-3\kappa(1-\alpha)^5(J-1)^2+\frac{G_c}{c_w}\left[\frac{1}{\ell}+2\ell\nabla^2\alpha\right]=0$$

4.1 Homogeneous Solution

Therefore, we can study the homogeneous solution by ignoring spatial derivatives of α

$$-2(1-\alpha_h)\,\mathcal{W}(\mathbf{F}) - 3\kappa(1-\alpha_h)^5(J-1)^2 + \frac{G_c}{c_w\ell} = 0$$

We can expand and arrange

$$\begin{aligned}
-2\mathcal{W}(\mathbf{F}) + 2\alpha_h\mathcal{W}(\mathbf{F}) - 3\kappa(J-1)^2(1-5\alpha_h+10\alpha_h^2-10\alpha_h^3+5\alpha_h^4-\alpha_h^5) + \frac{G_c}{c_w\ell} &= 0 \\
-2\mathcal{W}(\mathbf{F}) + 2\alpha_h\mathcal{W}(\mathbf{F}) + \frac{G_c}{c_w\ell} & \\
-3\kappa(J-1)^2 + 15\kappa(J-1)^2\alpha_h - 30\kappa(J-1)^2\alpha_h^2 + 30\kappa(J-1)^2\alpha_h^3 - 15\kappa(J-1)^2\alpha_h^4 + \kappa(J-1)^2\alpha_h^5 &= 0 \\
-2\mathcal{W}(\mathbf{F}) - 3\kappa(J-1)^2 + \frac{G_c}{c_w\ell} + \left[2\mathcal{W}(\mathbf{F}) + 15\kappa(J-1)^2\right]\alpha_h & \\
-30\kappa(J-1)^2\alpha_h^2 + 30\kappa(J-1)^2\alpha_h^3 - 15\kappa(J-1)^2\alpha_h^4 + \kappa(J-1)^2\alpha_h^5 &= 0
\end{aligned}$$

4.2 Non-Homogeneous Solution

Now for the Non-homogenous solution, we have the following

$$-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) - 3\kappa(1-\alpha)^5(J-1)^2 + \frac{G_c}{c_w}\left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right] = 0$$

$$\tag{4.1}$$