Notes on Miehe 2010

Updated on _

Ida Ang

Contents

1 Definitions

Young's Modulus: E
Poisson's Ratio: ν
Shear Modulus or first Lamé Parameter: μ

$$\mu = \frac{E}{2(1+\nu)} \tag{1.1}$$

Second Lamé Parameter: λ

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{1.2}$$

Bulk Modulus: κ

$$\kappa = \lambda + \frac{2}{3}\mu \tag{1.3}$$

Invariants of strain tensor $\boldsymbol{\varepsilon}$

$$\begin{aligned} E_1 &= \varepsilon_{ii} = \operatorname{tr}\boldsymbol{\varepsilon} \\ E_2 &= \varepsilon_{ij}\varepsilon_{ij} = \operatorname{tr}\boldsymbol{\varepsilon}^2 \\ E_3 &= \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} = \operatorname{tr}\varepsilon^3 \end{aligned} \tag{1.4}$$

The derivative of the invariants with respect to the strain tensor:

$$\begin{array}{ll} \partial E_1/\partial\varepsilon_{ij} = \delta_{ij} & \partial E_1/\partial\boldsymbol{\varepsilon} = \mathrm{I} \\ \partial E_2/\partial\varepsilon_{ij} = 2\varepsilon_{ij} & \partial E_2/\partial\boldsymbol{\varepsilon} = 2\boldsymbol{\varepsilon} \\ \partial E_3/\partial\varepsilon_{ij} = 3\varepsilon_{ik}\varepsilon_{kj} & \partial E_3/\partial\boldsymbol{\varepsilon} = 3\boldsymbol{\varepsilon}^2 \end{array} \tag{1.5}$$

Deformation

$$\mathbf{F} = \nabla\mathbf{u} + \mathbf{I} \tag{1.6}$$

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.7}$$

Invariants of deformation gradient

$$\begin{aligned} I_1 &= \operatorname{tr}\mathbf{F} = \lambda_1 + \lambda_2 + \lambda_3 \\ I_2 &= \frac{1}{2}\left[(\operatorname{tr}\mathbf{F})^2 - \operatorname{tr}(\mathbf{F}^2)\right] \\ I_3 &= \det\mathbf{F} = \lambda_1\lambda_2\lambda_3 \end{aligned} \tag{1.8}$$

Strain

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u}) \tag{1.9}$$

$$= \frac{1}{2}\left[\mathbf{F} - \mathbf{I} + \mathbf{F}^T - \mathbf{I}\right] \tag{1.10}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \mathbf{I} \tag{1.11}$$

which can also be written as:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \mathbf{I} \tag{1.12}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix} \tag{1.13}$$

Therefore we can write certain invariant in terms of stretches

$$
\begin{aligned}
E_1 &= \operatorname{tr} \boldsymbol{\varepsilon} \\
&= \lambda_1 + \lambda_2 + \lambda_3 - 3 \\
E_1 &= \lambda_i - 3
\end{aligned}
\tag{1.14}
$$

$$
\begin{aligned}
E_2 &= \operatorname{tr} \boldsymbol{\varepsilon}^2 \\
&= (\lambda_1 - 1)^2 + (\lambda_2 - 1)^2 + (\lambda_3 - 1)^2 \\
&= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2(\lambda_1 + \lambda_2 + \lambda_3) + 3 \\
E_2 &= \lambda_i^2 - 2\lambda_i + 3
\end{aligned}
\tag{1.15}
$$

This one is an invariant relationship

$$
\begin{aligned}
E_1^2 &= \left(\operatorname{tr} \boldsymbol{\varepsilon}\right)^2 \\
&= (\lambda_1 + \lambda_2 + \lambda_3 - 3)(\lambda_1 + \lambda_2 + \lambda_3 - 3) \\
&= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - 3(\lambda_1 + \lambda_2 + \lambda_3) + 9 \\
E_1^2 &= \lambda_i^2 + \frac{J}{\lambda_i} - 3\lambda_i + 9
\end{aligned}
\tag{1.16}
$$

Right Cauchy-Green (CG) definition

$$
\mathbf{C} = \mathbf{F}^T \mathbf{F}
\tag{1.17}
$$

Invariants of right CG

$$
\begin{aligned}
I_1^{\mathbf{C}} &= \operatorname{tr} \mathbf{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
I_2^{\mathbf{C}} &= \frac{1}{2}\left[(\operatorname{tr} \mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)\right] \\
I_3^{\mathbf{C}} &= \det \mathbf{C} = \lambda_1^2\lambda_2^2\lambda_3^2
\end{aligned}
\tag{1.18}
$$

Lastly we have the following useful identites

$$
\begin{aligned}
\frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} &= 2\mathbf{F} \\
\frac{\partial I_3}{\partial \mathbf{F}} &= I_3 \mathbf{F}^{-T}
\end{aligned}
\tag{1.19}
$$

1.1 Stresses

Pull-back operation

$$
\begin{aligned}
\boldsymbol{\sigma} &= \frac{1}{J}\frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T \\
\boldsymbol{\sigma} &= \frac{1}{J}\mathbf{P}\mathbf{F}^T \rightarrow \mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^T
\end{aligned}
\tag{1.20}
$$

where for convenience we are using W instead of ψ_0, which is Miehe's notation

$$
\begin{aligned}
\boldsymbol{\sigma} &= \frac{\partial \psi_0}{\partial \boldsymbol{\varepsilon}} \\
&= \frac{\partial \psi_0}{\partial E_1}\frac{\partial E_1}{\partial \boldsymbol{\varepsilon}} + \frac{\partial \psi_0}{\partial E_2}\frac{\partial E_2}{\partial \boldsymbol{\varepsilon}} + \frac{\partial \psi_0}{\partial E_3}\frac{\partial E_3}{\partial \boldsymbol{\varepsilon}} \\
\boldsymbol{\sigma} &= \frac{\partial \psi_0}{\partial E_1}\mathrm{I} + 2\frac{\partial \psi_0}{\partial E_2}\boldsymbol{\varepsilon} + 3\frac{\partial \psi_0}{\partial E_3}\boldsymbol{\varepsilon}^2
\end{aligned}
\tag{1.21}
$$

and the nominal stress is defined as

$$\begin{aligned}\mathbf{P} &= \frac{\partial W}{\partial \mathbf{F}} \\ &= \frac{\partial W}{\partial I_1^{\mathbf{C}}}\frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{F}} \\ \mathbf{P} &= 2\frac{\partial W}{\partial I_1^{\mathbf{C}}}\mathbf{F} + I_3\frac{\partial W}{\partial I_3}\mathbf{F}^{-T}\end{aligned} \tag{1.22}$$

2 Isotropic elasticity

Standard free energy

$$\begin{aligned}\psi_0(\boldsymbol{\varepsilon}) &= \frac{\lambda}{2}\operatorname{tr}^2[\boldsymbol{\varepsilon}] + \mu \operatorname{tr}\left[\boldsymbol{\varepsilon}^2\right] \\ \psi_0(\boldsymbol{\varepsilon}) &= \frac{\lambda}{2}E_1^2 + \mu E_2\end{aligned} \tag{2.1}$$

Obtain stress

$$\begin{aligned}\boldsymbol{\sigma}_\mathbf{0} &= \frac{\partial \psi_0}{\partial E_1}\mathrm{I} + 2\frac{\partial \psi_0}{\partial E_2}\boldsymbol{\varepsilon} \\ &= \lambda E_1 \mathrm{I} + 2\mu\boldsymbol{\varepsilon} \\ \boldsymbol{\sigma}_\mathbf{0} &= \lambda\big(\operatorname{tr}\boldsymbol{\varepsilon}\big)\mathrm{I} + 2\mu\boldsymbol{\varepsilon}\end{aligned} \tag{2.2}$$

Additive decomposition form of stored energy

$$\psi(\boldsymbol{\varepsilon}, d) = [g(d) + k]\psi_0^+(\boldsymbol{\varepsilon}) + \psi_0^-(\boldsymbol{\varepsilon}) \tag{2.3}$$

The stress associated with this form

$$\begin{aligned}\boldsymbol{\sigma} &= \partial_{\boldsymbol{\varepsilon}}\psi \\ &= \left[(1-d)^2 + k\right]\frac{\partial \Psi_0^+(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} + \frac{\partial \Psi_0^-(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \\ \boldsymbol{\sigma} &= \left[(1-d)^2 + k\right]\boldsymbol{\sigma}_0^+ - \boldsymbol{\sigma}_0^-\end{aligned} \tag{2.4}$$

where

$$g(d) = (1-d)^2 \tag{2.5}$$

3 Hyperelasticity

3.1 Compressible neo-Hookean: Case I

Strain energy function

$$W(\mathbf{F}) = \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \frac{\kappa}{2}(I_3 - 1)^2 \tag{3.1}$$

Calculation of the 1st PK

$$\begin{aligned}\mathbf{P} &= 2\frac{\partial W}{\partial I_1^{\mathbf{C}}}\mathbf{F} + I_3\frac{\partial W}{\partial I_3}\mathbf{F}^{-T} \\ &= 2\frac{\mu}{2}\mathbf{F} + I_3\left[\mu\frac{1}{I_3} + \frac{\kappa}{2}2(I_3 - 1)\right]\mathbf{F}^{-T} \\ &= \mu\mathbf{F} + \mu\mathbf{F}^{-T} + \kappa I_3(I_3 - 1)\mathbf{F}^{-T} \\ \mathbf{P} &= \mu(\mathbf{F} + \mathbf{F}^{-T}) + \kappa I_3(I_3 - 1)\mathbf{F}^{-T}\end{aligned} \tag{3.2}$$

Therefore, we can also calculate the Cauchy Stress

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{1}{J}\mathbf{P}\mathbf{F}^T \\ &= \frac{1}{I_3}\left[\mu(\mathbf{F}+\mathbf{F}^{-T}) + \kappa I_3(I_3-1)\mathbf{F}^{-T}\right]\mathbf{F}^T \\ &= \frac{1}{I_3}\mu(\mathbf{F}\mathbf{F}^T + \mathbf{F}^{-T}\mathbf{F}^T) + \kappa(I_3-1)\mathbf{F}^{-T}\mathbf{F}^T \\ \boldsymbol{\sigma} &= \frac{\mu}{I_3}(\mathbf{b}+\mathbf{I}) + \kappa(I_3-1)\mathbf{I}\end{aligned} \tag{3.3}$$

3.2 Incompressible neo-Hookean: Case I

$$W(\mathbf{F}) = \frac{\mu}{2}(I_1^{\mathbf{C}} - 3) + \mathrm{p}(I_3 - 1) \tag{3.4}$$

Calculation of the 1st PK

$$\begin{aligned}\mathbf{P} &= 2\frac{\partial W}{\partial I_1^{\mathbf{C}}}\mathbf{F} + I_3\frac{\partial W}{\partial I_3}\mathbf{F}^{-T} \\ &= 2\frac{\mu}{2}\mathbf{F} + I_3 p\mathbf{F}^{-T} \\ \mathbf{P} &= \mu\mathbf{F} + pI_3\mathbf{F}^{-T}\end{aligned} \tag{3.5}$$

Therefore, we can also calculate the Cauchy Stress

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{1}{I_3}\left[\mu\mathbf{F} + pI_3\mathbf{F}^{-T}\right]\mathbf{F}^T \\ &= \frac{1}{I_3}\left[\mu\mathbf{b} + pI_3\mathbf{I}\right] \\ \boldsymbol{\sigma} &= \frac{\mu}{I_3}\mathbf{b} + p\mathbf{I}\end{aligned} \tag{3.6}$$

If $\det\mathbf{F} = 1$, then

$$\boldsymbol{\sigma} = \mu\mathbf{b} + p\mathbf{I} \tag{3.7}$$

3.3 Summary

Compressible neo-Hookean, case 1

$$\begin{aligned}W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \frac{\kappa}{2}(I_3-1)^2 \\ \mathbf{P} &= \mu(\mathbf{F} - \mathbf{F}^{-T}) + \kappa I_3(I_3-1)\mathbf{F}^{-T}\end{aligned} \tag{3.8}$$

Compressible neo-Hookean, case 2

$$\begin{aligned}W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \frac{\lambda}{2}(\ln I_3)^2 \\ \mathbf{P} &= \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda I_3(\ln I_3)\mathbf{F}^{-T}\end{aligned} \tag{3.9}$$

Incompressible neo-Hookean, case 1

$$\begin{aligned}W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3) + \mathrm{p}(I_3 - 1) \\ \mathbf{P} &= \mu\mathbf{F} + pI_3\mathbf{F}^{-T}\end{aligned} \tag{3.10}$$

Incompressible neo-Hookean, case 2

$$
\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \mathrm{p}(I_3 - 1) \\
\mathbf{P} &= \mu(\mathbf{F} - \mathbf{F}^{-T}) + pI_3\mathbf{F}^{-T}
\end{aligned}
\tag{3.11}
$$