Notes on Research

Updated on _

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Contents

1 neo-Hookean Model

The deformation can be decomposed into elastic and plastic (growth) terms

$$\mathbf{F} = \mathbf{F_e}\mathbf{F_g} \tag{1.1}$$

where $\mathbf{F_e}$ is connected to the elastic stress response of the material and $\mathbf{F_g}$ is the growth tensor

$$\begin{aligned} J &= \det \mathbf{F} \\ &= \det(\mathbf{F_e}\mathbf{F_g}) \quad \text{property of determinant} \\ &= \det \mathbf{F_e} \det \mathbf{F_g} \\ J &= J_e J_g \end{aligned} \tag{1.2}$$

where

$$I_e = \operatorname{tr} \mathbf{C_e} \tag{1.3}$$

$$J_e = \det \mathbf{F_e} \quad \text{and} \quad J_g = \det \mathbf{F_g} \tag{1.4}$$

where the right Cauchy Green Tensor is $\mathbf{C_e} = \mathbf{F_e}^T\mathbf{F_e}$

Proof of the derivative $\frac{\partial \operatorname{tr}(\mathbf{F^T F})}{\partial \mathbf{F}}$ in indicial:

$$\begin{aligned} \frac{\partial \operatorname{tr}(\mathbf{F^T F})}{\partial \mathbf{F}} &= \frac{\partial F_{kI} F_{kI}}{\partial F_{pQ}} \quad \text{Product Rule} \\ &= \frac{\partial F_{kI}}{\partial F_{pQ}} F_{kI} + F_{kI} \frac{\partial F_{kI}}{\partial F_{pQ}} \\ &= \delta_{kp}\delta_{IQ} F_{kI} + F_{kI}\delta_{kp}\delta_{IQ} \\ &= F_{pQ} + F_{pQ} = 2F_{pQ} \end{aligned}$$

Therefore, we have the following relationship:

$$\frac{\partial \operatorname{tr}(\mathbf{F^T F})}{\partial \mathbf{F}} = 2\mathbf{F} \tag{1.5}$$

Second proof:

$$\frac{\partial \det \mathbf{F}}{\partial \mathbf{F}} = \det \mathbf{F}\mathbf{F^{-T}} = J\mathbf{F}^{-T}$$

1.1 Compressible

If the compressible neo-Hookean model is as follows:

$$W_e = \frac{\mu}{2}(I_e - 3 - 2\ln J_e) + \frac{\lambda}{2}(\ln J_e)^2 \tag{1.6}$$

The nominal stress tensor is defined as:

$$\mathbf{P} = \frac{\partial W_e}{\partial \mathbf{F_e}} \tag{1.7}$$

substituting in Eq. 1.6 and using the chain rule:

$$
\begin{aligned}
\mathbf{P} &= \frac{\partial W_e}{\partial I_e}\frac{\partial I_e}{\partial \mathbf{F_e}} + \frac{\partial W_e}{\partial J_e}\frac{\partial J_e}{\partial \mathbf{F_e}} \\
&= \frac{\mu}{2}2\mathbf{F_e} + \left[\frac{\mu}{2}\left(-2\frac{\partial \ln J_e}{\partial J_e}\right) + \frac{\lambda}{2}\frac{\partial (lnJ_e)^2}{\partial J_e}\right] \det \mathbf{F_e}{\mathbf{F_e}}^{-T} \\
&= \mu\mathbf{F_e} + \left[-\mu\frac{\partial lnJ_e}{\partial J_e} + \frac{\lambda}{2}\frac{\partial (\ln J_e)^2}{\partial J_e}\right] J_e{\mathbf{F_e}}^{-T} \\
&= \mu\mathbf{F_e} + \left[-\mu\frac{1}{J_e} + \lambda \ln J_e \frac{1}{J_e}\right] J_e{\mathbf{F_e}}^{-T} \\
\mathbf{P} &= \mu\mathbf{F_e} + (\lambda \ln J_e - \mu){\mathbf{F_e}}^{-T}
\end{aligned}
$$

1.2 Incompressible

For an incompressible material, $J_e = \det \mathbf{F_e} = 1$; therefore, Eq. 1.6 can be modified to:

$$
\begin{aligned}
W_e &= \frac{\mu}{2}(I_e - 3 - 2\ln(1)) + \frac{\lambda}{2}(\ln(1))^2 \\
W_e &= \frac{\mu}{2}(I_e - 3)
\end{aligned}
$$

To enforce incompressibility, introduce a lagrange multiplier, p, which acts like a stress term.

$$
\begin{aligned}
W_e(\mathbf{F_e}) &= \frac{\mu}{2}(I_e - 3) + \mathrm{p}(J_e - 1) \\
W_e(\mathbf{F_e}) &= \frac{\mu}{2}(\mathrm{tr}\big({\mathbf{F_e}}^T\mathbf{F_e}\big) - 3) + \mathrm{p}(\det \mathbf{F_e} - 1)
\end{aligned}
\tag{1.8}
$$

Use the chain rule, just as in the last case.

$$
\begin{aligned}
\mathbf{P} &= \frac{\partial W_e}{\partial I_e}\frac{\partial I_e}{\partial \mathbf{F_e}} + \frac{\partial W_e}{\partial J_e}\frac{\partial J_e}{\partial \mathbf{F_e}} \\
&= \frac{\partial W_e}{\partial \,\mathrm{tr}\big({\mathbf{F_e}}^T\mathbf{F_e}\big)}\frac{\partial \,\mathrm{tr}\big({\mathbf{F_e}}^T\mathbf{F_e}\big)}{\partial \mathbf{F_e}} + \frac{\partial W_e}{\partial \det \mathbf{F_e}}\frac{\partial \det \mathbf{F_e}}{\partial \mathbf{F_e}} \\
&= \frac{\mu}{2}2\mathbf{F_e} + pJ_e{\mathbf{F_e}}^{-T} \\
&= \mu\mathbf{F_e} + pJ_e{\mathbf{F_e}}^{-T}
\end{aligned}
$$

2 Incompressible formulation based on Alawiye paper

The deformation still has a multiplicative decomposition. Eq. 3.2

$$
\mathbf{F} = \mathbf{F_e}\mathbf{F_g}
$$

but we assume that the material is incompressible and the only local change in volume of the material comes from the growth process.

$$
\det \mathbf{F_e} = 1 \tag{2.1}
$$

For a hyperelastic material, we can define an augmented energy density functional

$$
\overline{W}(\mathbf{F}, \mathbf{F_g}) = (\det \mathbf{F_g})W_e(\mathbf{F_e}) - p(\det \mathbf{F_e} - 1) \tag{2.2}
$$

where $\mathbf{F_e} = \mathbf{F}\mathbf{F_g}^{-1}$.

$$W_e(\mathbf{F_e}) = \frac{\mu}{2}(\mathrm{tr}\,\mathbf{F_e}\mathbf{F_e}^T - 3) \tag{2.3}$$

We can calculate

$$\begin{aligned}
\frac{\partial W_e}{\partial \mathbf{F_e}} &= \frac{\mu}{2}\frac{\partial}{\partial \mathbf{F_e}}\big(\mathrm{tr}\,\mathbf{F_e}\mathbf{F_e}^T - 3\big) \\
&= \frac{\mu}{2}\frac{\partial(\mathrm{tr}\,\mathbf{F_e}\mathbf{F_e}^T)}{\partial \mathbf{F_e}} \\
\frac{\partial W_e}{\partial \mathbf{F_e}} &= \mu\mathbf{F_e}
\end{aligned}$$

The nominal stress tensor can be found as

$$\mathbf{P} = \frac{\partial \overline{W}}{\partial \mathbf{F}}$$

Using the chain rule

$$\begin{aligned}
\frac{\partial \overline{W}}{\partial \mathbf{F}} &= \frac{\partial \overline{W}}{\partial \mathbf{F_e}}\frac{\partial \mathbf{F_e}}{\partial \mathbf{F}} \\
&= \frac{\partial}{\partial \mathbf{F_e}}\Big[(\det \mathbf{F_g})W_e(\mathbf{F_e}) - p(\det \mathbf{F_e} - 1)\Big]\frac{\partial \mathbf{F_e}}{\partial \mathbf{F}} \\
&= \left[\det \mathbf{F_g}\frac{\partial W_e(\mathbf{F_e})}{\partial \mathbf{F_e}} - p\frac{\partial(\det \mathbf{F_e} - 1)}{\partial \mathbf{F_e}}\right]\frac{\partial(\mathbf{F}\mathbf{F_g}^{-1})}{\partial \mathbf{F}} \\
&= \left[\det \mathbf{F_g}\frac{\partial W_e(\mathbf{F_e})}{\partial \mathbf{F_e}} - p\frac{\partial \det \mathbf{F_e}}{\partial \mathbf{F_e}}\right]\mathbf{F_g}^{-1} \quad \text{where } \det \mathbf{F} = \det \mathbf{F_g} \\
&= \left[\det \mathbf{F}\frac{\partial W_e(\mathbf{F_e})}{\partial \mathbf{F_e}} - p\det \mathbf{F_e}\mathbf{F_e}^{-T}\right]\mathbf{F_g}^{-1} \quad \text{substitute} \\
\frac{\partial \overline{W}}{\partial \mathbf{F}} &= \left[\mu J\mathbf{F_e} - pJ_e\mathbf{F_e}^{-T}\right]\mathbf{F_g}^{-1}
\end{aligned}$$

This disagrees with Eq. 5 in Ciarletta paper

$$\begin{aligned}
\text{Ours:}\,\mathbf{P} = \frac{\partial \overline{W}}{\partial \mathbf{F}} &= \mathbf{F_g}^{-1}\left[J\frac{\partial W_e}{\partial \mathbf{F_e}} - pJ_e\mathbf{F_e}^{-T}\right] \\
\text{Ciarletta et. al.:}\,\mathbf{P} = \frac{\partial \overline{W}}{\partial \mathbf{F}} &= J\mathbf{F_g}^{-1}\left[\frac{\partial W_e}{\partial \mathbf{F_e}} - p\mathbf{F_e}^{-1}\right]
\end{aligned}$$

2.1 Weak Form

Finally, in order to enforce incompressibility we use Eq. **??** where:

$$\det F_e = 1 \rightarrow \det F_e - 1 = 0 \rightarrow J_e - 1 = 0$$

This can be multiplied by a test function and integrated over the domain Ω_o

$$\int_{\Omega_o}(J_e - 1)\tau dV = 0 \tag{2.4}$$

We have the bilinear form and linear form where we have two test functions

$$a\big((\sigma, u), (\tau, v)\big) = L\big((\tau, v)\big) \qquad \forall(\tau, v) \in \Sigma_0 \times V$$

Therefore, we have the bilinear and linear forms:

$$a\big((\sigma, u), (\tau, v)\big) = \int_{\Omega_o} \big(P : \mathrm{Grad}(v) + (J - 1) \cdot \tau\big) dV \tag{2.5}$$

$$L\big((\tau, v)\big) = \int_{\Omega_o} bvdV + \int_{\partial\Omega_o} TvdS \tag{2.6}$$

The bilinear and linear form (Eq. 2.5 and 2.6)can also be combined into one statement:

$$a\big((\sigma, u), (\tau, v)\big) - L\big((\tau, v)\big) = \int_{\Omega_o} \big(P : \mathrm{Grad}(v) + (J_e - 1) \cdot \tau\big) dV - \int_{\Omega_o} bvdV - \int_{\partial\Omega_o} TvdS = 0 \tag{2.7}$$

Solving for two unknowns, displacement (u) and hydrostatic pressure (p), which will allow us to obtain the nominal stress field (P)

3 Formulation based on Ambrosi and Mollika Paper

Deformation gradient:

$$\mathbf{F} = \frac{\partial \chi}{\partial \mathbf{X}} \tag{3.1}$$

The deformation can be decomposed into elastic and plastic (growth) terms

$$\mathbf{F} = \mathbf{F_e F_g} \tag{3.2}$$

where $\mathbf{F_e}$ is connected to the stress response of the material and $\mathbf{F_g}$ is the growth tensor

$$J = J_e J_g \tag{3.3}$$

The third invariant of the elastic deformation tensor where ϱ is the density field

$$J_e = \det \mathbf{F_e} = \frac{\varrho_o}{\varrho} \tag{3.4}$$

The third invariant of the growth tensor shows whether a particle is growing ($J_g > 1$)or resorbing ($J_g < 1$)

$$J_g = \det \mathbf{F_g} \tag{3.5}$$

3.1 Strain Energy Density and Stress Equations

Strain Energy Density for General Blatz-Ko Material

$$\begin{aligned} W_e &= \frac{vf}{2}\left[(\mathrm{I}_e - 3) - \frac{2}{q}\left(\mathrm{III}_e^{q/2} - 1\right)\right] + \frac{v(1-f)}{2}\left[\left(\frac{\mathrm{II}_e}{\mathrm{III}_e} - 3\right) - \frac{2}{q}\left(\mathrm{III}_e^{-(q/2)} - 1\right)\right] \quad \text{where } f = 1 \\ W_e &= \frac{v}{2}\left[(\mathrm{I}_e - 3) - \frac{2}{q}\left(\mathrm{III}_e^{q/2} - 1\right)\right] \end{aligned} \tag{3.6}$$

The first and third invariant are calculated as follows:

$$\mathrm{I}_e = \mathrm{tr}\, \mathbf{C_e} = \mathrm{tr}\, \mathbf{F_e^T F_e} \qquad \mathrm{III}_e = \det \mathbf{F_e}$$

The Cauchy Stress Tensor

$$\mathbf{T} = \varrho \mathbf{F}_e \left(\frac{\partial W_e}{\partial \mathbf{F}_e}\right)^{\mathrm{T}} \tag{3.7}$$

First Piola Stress Tensor

$$\mathbf{P} = J\mathbf{T}\mathbf{F}^{-\mathrm{T}} \tag{3.8}$$

Second Piola Stress Tensor

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\mathbf{T}} \tag{3.9}$$

3.2 Stress Calculation

$$\frac{\partial W_e}{\partial \mathbf{F_e}} = \frac{\partial W_e}{\partial \mathrm{I}_e}\frac{\partial \mathrm{I}_e}{\partial \mathbf{F_e}} + \frac{\partial W_e}{\partial \mathrm{III}_e}\frac{\partial \mathrm{III}_e}{\partial \mathbf{F_e}} \tag{3.10}$$

Invariant derivative relationships

$$\begin{aligned} \frac{\partial \mathrm{I}_e}{\partial \mathbf{F_e}} &= 2\mathbf{F_e} \\ \frac{\partial \mathrm{I}_e}{\partial \mathbf{F_e}} &= \det \mathbf{F_e} \mathbf{F_e^{-T}} = \mathrm{III}_e \mathbf{F_e^{-T}} \end{aligned}$$

Calculate partials of strain energy density with respect to the invariants:

$$\begin{aligned}\frac{\partial W_e}{\partial \mathrm{I}_e} &= \frac{v}{2}\frac{\partial(\mathrm{I}_e - 3)}{\partial \mathrm{I}_e} \\ &= \frac{v}{2}\end{aligned}$$

$$\begin{aligned}\frac{\partial W_e}{\partial \mathrm{III}_e} &= \frac{v}{2}\bigg(-\frac{2}{q}\bigg)\frac{\partial(\mathrm{III}_e^{q/2} - 1)}{\partial \mathrm{III}_e} \\ &= -\frac{v}{q}\frac{q}{2}\mathrm{III}_e^{q/2-1} \\ &= -\frac{v}{2}\mathrm{III}_e^{q/2-1}\end{aligned}$$

Therefore, substituting this into Eq. 3.10

$$\begin{aligned}\frac{\partial W_e}{\partial \mathbf{F_e}} &= \frac{\partial W_e}{\partial \mathrm{I}_e}\frac{\partial \mathrm{I}_e}{\partial \mathbf{F_e}} + \frac{\partial W_e}{\partial \mathrm{III}_e}\frac{\partial \mathrm{III}_e}{\partial \mathbf{F_e}} \\ &= \frac{v}{2}2\mathbf{F_e} - \frac{v}{2}\mathrm{III}_e^{q/2-1}\mathrm{III}_e\mathbf{F_e^{-T}} \\ \frac{\partial W_e}{\partial \mathbf{F_e}} &= v\mathbf{F_e} - \frac{v}{2}\mathrm{III}_e^{q/2}\mathbf{F_e^{-T}}\end{aligned} \tag{3.11}$$

3.2.1 Cauchy Stress Tensor

The Cauchy Stress Tensor can now be calculated using Eq. 3.7

$$\begin{aligned}\mathbf{T} &= \varrho\mathbf{F}_e\bigg(\frac{\partial W_e}{\partial \mathbf{F}_e}\bigg)^{\mathrm{T}} \\ &= \varrho\mathbf{F}_e\big(v\mathbf{F_e} - \frac{v}{2}\mathrm{III}_e^{q/2}\mathbf{F_e^{-T}}\big)^{\mathrm{T}} \\ &= \varrho\mathbf{F}_e\big(v\mathbf{F_e}^T - \frac{v}{2}\mathrm{III}_e^{q/2}\mathbf{F_e^{-1}}\big) \\ &= \varrho v\big(\mathbf{F}_e\mathbf{F_e}^T - \frac{1}{2}\mathrm{III}_e^{q/2}\mathbf{F}_e\mathbf{F_e^{-1}}\big) \\ \mathbf{T} &= \varrho v\big(\mathbf{F}_e\mathbf{F_e}^T - \frac{1}{2}\mathrm{III}_e^{q/2}\big)\end{aligned}$$

Rewrite this term where

$$\begin{aligned}\mathbf{T} &= \varrho v\big(\mathbf{F}_e\mathbf{F_e}^T - \frac{1}{2}\mathrm{III}_e^{q/2}\big) \quad \text{where } \varrho = \frac{\varrho_o}{J_e} \\ &= \frac{\varrho_o v}{J_e}\big(\mathbf{F}_e\mathbf{F_e}^T - \frac{1}{2}\mathrm{III}_e^{q/2}\big) \quad \text{where } \mu = \varrho_o v \\ \mathbf{T} &= \frac{\mu}{J_e}\big(\mathbf{F}_e\mathbf{F_e}^T - \frac{1}{2}\mathrm{III}_e^{q/2}\big)\end{aligned} \tag{3.12}$$

3.2.2 First Piola-Kirchoff Stress Tensor

The first PK stress tensor can be calculated using Eq. 3.8 First Piola Stress Tensor

$$\begin{aligned}
\mathbf{P} &= J\mathbf{T}\mathbf{F}^{-\mathrm{T}} \quad \text{Substitute Eq. 3.12}\\
&= J\frac{\mu}{J_e}\big(\mathbf{F}_e\mathbf{F_e}^T - \frac{1}{2}\mathrm{III}_e^{q/2}\big)\mathbf{F}^{-\mathrm{T}} \quad \text{Use equation 3.2 and 3.3}\\
&= J_eJ_g\frac{\mu}{J_e}\big(\mathbf{F}_e\mathbf{F_e}^T - \frac{1}{2}\mathrm{III}_e^{q/2}\big)(\mathbf{F_e}\mathbf{F_g})^{-\mathrm{T}}\\
&= \mu J_g\big[\mathbf{F}_e\mathbf{F_e}^T\mathbf{F_e}^{-T} - \frac{1}{2}\mathrm{III}_e^{q/2}\mathbf{F_e}^{-T}\big]\mathbf{F_g}^{-\mathrm{T}}\\
&= \mu J_g\big[\mathbf{F}_e(\mathbf{F_e}^{-1}\mathbf{F_e})^T - \frac{1}{2}\mathrm{III}_e^{q/2}\mathbf{F_e}^{-T}\big]\mathbf{F_g}^{-\mathrm{T}} \quad \text{Recognize identity}\\
\mathbf{P} &= \mu J_g\big[\mathbf{F}_e - \frac{1}{2}J_e^{q/2}\mathbf{F_e}^{-T}\big]\mathbf{F_g}^{-\mathrm{T}}
\end{aligned}$$

3.3 Summary

$$W_e = \frac{v}{2}\left[(\mathrm{I}_e - 3) - \frac{2}{q}\left(\mathrm{III}_e^{q/2} - 1\right)\right] \tag{3.13}$$

Discrepancy shown below:

$$\begin{aligned}
\mathbf{P} &= \mu J_g\big[\mathbf{F}_e - \frac{1}{2}J_e^{q/2}\mathbf{F_e}^{-T}\big]\mathbf{F_g}^{-\mathrm{T}}\\
\mathbf{P} &= \mu J_g\big[\mathbf{F}_e - J_e^{q}\mathbf{F_e}^{-T}\big]\mathbf{F_g}^{-\mathrm{T}}
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\alpha &= \frac{\lambda}{2\mu} = \frac{\nu}{1-2\nu}\\
-\alpha &= \frac{q}{2}
\end{aligned}$$

Therefore:

$$\mathbf{P} = \mu J_g\big[\mathbf{F}_e - \frac{1}{2}J_e^{-\alpha}\mathbf{F_e}^{-T}\big]\mathbf{F_g}^{-\mathrm{T}} \tag{3.15}$$

3.4 Weak Form

$$\frac{\partial P_{iK}}{\partial X_K} = -B_i \tag{3.16}$$

Rearrange and multiply Eq. 4.1 with a test function ξ_i

$$\begin{aligned}
\frac{\partial P_{iK}}{\partial X_K}\xi_i &= -B_i\xi_i \quad \text{Integrate over domain}\\
\int \frac{\partial P_{iK}}{\partial X_K}\xi_i dV &= -\int B_i\xi_i dV\\
\int (P_{iK}\xi_i)_{,K}dV - \int P_{iK}\frac{\partial \xi_i}{\partial X_K}dV &= -\int B_i\xi_i dV\\
\int (P_{iK}\xi_i)_{,K}dV - \int P_{iK}\frac{\partial \xi_i}{\partial X_K}dV &= -\int B_i\xi_i dV \quad \text{Use divergence theorem}\\
\int P_{iK}N_K\xi_i dA - \int P_{iK}\frac{\partial \xi_i}{\partial X_K}dV &= -\int B_i\xi_i dV \quad \text{Recognize } P_{iK}N_K = T_i\\
\int T_i\xi_i dA - \int P_{iK}\frac{\partial \xi_i}{\partial X_K}dV &= -\int B_i\xi_i dV
\end{aligned}$$

Rearrange

$$\int P_{iK} \frac{\partial \xi_i}{\partial X_K} dV - \int T_i \xi_i dA - \int B_i \xi_i dV = 0 \tag{3.17}$$