Notes for Contact Methods
Linear Elasticity and Hyperelasticity

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1 Problem Definition

These codes were created to troubleshoot the more advanced code incorporating mass transport, large deformations, and contact. For the contact methods, I recall that Lagrange methods were not sufficient so I do not have much documentation on them. I follow Wriggers closely.

2 Formulation

Equation for strain in terms of displacement

$$\begin{aligned}\epsilon_{ij} &= \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \\ \epsilon &= \frac{1}{2}(\nabla u + \nabla u^T)\end{aligned} \tag{2.1}$$

General expression of the linear elastic isotropic constitutive relationship

$$\sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \tag{2.2}$$

This can be inverted to be strain in terms of stress:

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij} \tag{2.3}$$

where the Lamé coefficients are given by:

$$\begin{aligned}\lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \\ \mu &= \frac{E}{2(1+\nu)}\end{aligned} \tag{2.4}$$

Starting from the equilibrium equation:

$$\begin{aligned}
-Div\sigma = f \quad &\rightarrow -\boldsymbol{\nabla}\cdot\sigma = f \quad \text{Convert to indicial notation} \\
-\frac{\partial}{\partial x_k}\mathbf{e_k}\cdot\sigma_{ij}(\mathbf{e_i}\otimes\mathbf{e_j}) &= f_k\mathbf{e_k} \\
-\frac{\partial\sigma_{ij}}{\partial x_k}\delta_{ki}\mathbf{e_j} &= f_k\mathbf{e_k} \\
-\frac{\partial\sigma_{ij}}{\partial x_i}\mathbf{e_j} &= f_k\mathbf{e_k} \quad \text{Multiply be a test function} \\
-\frac{\partial\sigma_{ij}}{\partial x_i}\mathbf{e_j}\cdot v_p\mathbf{e_p} &= f_k\mathbf{e_k}\cdot v_p\mathbf{e_p} \\
-\frac{\partial\sigma_{ij}}{\partial x_i}v_p\delta_{jp} &= f_k v_p\delta_{kp} \\
-\frac{\partial\sigma_{ij}}{\partial x_i}v_j &= f_k v_k \quad \text{Integrate over domain} \\
-\int_\Omega \frac{\partial\sigma_{ij}}{\partial x_i}v_j dx &= \int_\Omega f_k v_k dx
\end{aligned}$$

Integration by parts on the LHS

$$\begin{aligned}
(fg)' = f'g + fg' \rightarrow f'g &= (fg)' - fg' \\
\frac{\partial\sigma_{ij}}{\partial x_i}v_j &= (\sigma_{ij}v_j)_{,i} - \sigma_{ij}\frac{\partial v_j}{\partial x_i}
\end{aligned}$$

Substitute the result from integration by parts:

$$
\begin{aligned}
-\int_{\Omega}(\sigma_{ij}v_j)_{,i}dx + \int_{\Omega}\sigma_{ij}\frac{\partial v_j}{\partial x_i}dx &= \int_{\Omega} f_k v_k dx \quad \text{Use the divergence theorem} \\
-\int_{\partial\Omega}\sigma_{ij}v_j n_i ds + \int_{\Omega}\sigma_{ij}\frac{\partial v_j}{\partial x_i}dx &= \int_{\Omega} f_k v_k dx \quad \text{Recognize the traction term} \\
-\int_{\partial\Omega} t_i v_j ds + \int_{\Omega}\sigma_{ij}\frac{\partial v_j}{\partial x_i}dx &= \int_{\Omega} f_k v_k dx \quad \text{Rearrange} \\
\int_{\Omega}\sigma_{ij}\frac{\partial v_j}{\partial x_i}dx &= \int_{\Omega} f_k v_k dx + \int_{\partial\Omega} t_i v_j ds
\end{aligned}
$$

The principle of virtual work states that:

$$
\int_{\Omega}\delta W dV = \int_{\Omega}\sigma_{ij}\delta\epsilon_{ij}dV = \int_{\partial\Omega} t_i\delta u_i dS + \int_{\Omega} b_i\delta u_i dV \tag{2.5}
$$

Applying this principle to, where f is the body force, b:

$$
\begin{aligned}
\int_{\Omega}\sigma_{ij}\frac{\partial v_j}{\partial x_i}dx &= \int_{\Omega} f_k v_k dx + \int_{\partial\Omega} t_i v_j ds \\
\int_{\Omega}\sigma_{ij}\epsilon_{ij}dV &= \int_{\Omega} f_k v_k dx + \int_{\partial\Omega} t_i v_j ds \quad \text{no traction applied} \\
\int_{\Omega}\sigma_{ij}\epsilon_{ij}dV &= \int_{\Omega} f_k v_k dx
\end{aligned}
$$

Writing in direct notation we have the variational (weak) formulation. Find $\mathbf{u} \in V$ such that:

$$
\int_{\Omega}\sigma(\mathbf{u}) : \epsilon(\mathbf{v})d\Omega = \int_{\Omega}\mathbf{f}\cdot\mathbf{v}d\Omega \quad \forall v \in V \tag{2.6}
$$

3 Contact Methods

The rigid indenter with a spherical surface can be approximated by a parobolic equation instead of explicitly modeled and meshed. Consider the indenter radius, R, to be sufficiently large with respect to the contact region characteristic size (R >> a). This relationship, R >> a, allows the spherical surface to be approximated by a parabola.

$$
\begin{aligned}
h(x,z) &= -h_o + \frac{1}{2R}(x^2+z^2) \quad \text{about origin} \\
h(x,z) &= -h_o + \frac{1}{2R}\left[(x-0.5)^2+(z-0.5)^2\right] \quad \text{about point } (0.5, 0.5)
\end{aligned} \tag{3.1}
$$

The definition of the MacKauley bracket:

$$
\begin{aligned}
< x > &= \frac{x+|x|}{2} \\
&= \begin{cases} 0 \text{ for } x \le 0, \\ x \text{ for } x > 0 \end{cases}
\end{aligned} \tag{3.2}
$$

3.1 Penalty Approach

3.1.1 Theory

Adding a penalty term to the energy, Π.

$$\Pi_c^P = \frac{1}{2}\int_{\Gamma_c} \epsilon_N (g_N)^2 dA \quad \epsilon_N > 0 \tag{3.3}$$

Take the variation of Eq. 3.1, this holds for pure stick. This is the addition to the weak form:

$$C_c^P = \int_{\Gamma_c} \big(\epsilon_N g_N \delta g_N\big) dA$$

3.2 Augmented Lagrange Methods

The main idea of this method is combining the penalty method with Lagrange multiplier methods.

3.2.1 Theory

The augmented Lagrange functional is introduced for normal contact:

$$l_n = \lambda g + \frac{\epsilon}{2} g^2 - \frac{1}{2\epsilon} < \lambda + \epsilon g >^2 \tag{3.4}$$

Take the variation of Eq. 3.2

$$\begin{aligned}
\Pi_N^{AM} &= \delta\lambda g + \lambda \delta g + \epsilon g \delta g - \frac{1}{\epsilon} < \lambda + \epsilon g > \delta\lambda - < \lambda + \epsilon g > \delta g \\
&= g\delta\lambda + (\lambda + \epsilon g)\delta g - \frac{1}{\epsilon} < \lambda + \epsilon g > \delta\lambda - < \lambda + \epsilon g > \delta g \\
&= g\delta\lambda + \hat{\lambda}\delta g - \frac{1}{\epsilon} < \hat{\lambda} > \delta\lambda - < \hat{\lambda} > \delta g
\end{aligned}$$

We introduce the following augmented lagrangian term.

$$\hat{\lambda} = \lambda + \epsilon g$$

Where for $\hat{\lambda} \leq 0$, we are in contact or penetrating the surface and the gap is 0 or negative. For this case, everything within a Mackauley bracket in the variation goes to zero.

$$\begin{aligned}
C_N^{AM} &= g\delta\lambda + (\lambda + \epsilon g)\delta g \\
&= g\delta\lambda + \hat{\lambda}\delta g
\end{aligned}$$

Next, for $\hat{\lambda} > 0$, we are not in contact, and the gap is positive.

$$\begin{aligned}
C_N^{AM} &= g\delta\lambda + (\lambda + \epsilon g)\delta g - \frac{1}{\epsilon}(\lambda + \epsilon g)\delta\lambda - (\lambda + \epsilon g)\delta g \\
&= g\delta\lambda - \frac{1}{\epsilon}\lambda\delta\lambda - g\delta\lambda \\
&= -\frac{1}{\epsilon}\lambda\delta\lambda
\end{aligned}$$

Therefore, summarizing the two cases:

$$C_N^{AM} = \begin{cases} \displaystyle\int_{\Gamma_c} \Big(g\delta\lambda + \hat{\lambda}\delta g\Big) d\Gamma & \text{for } \hat{\lambda}_N \leq 0, \\ \displaystyle\int_{\Gamma_c} -\frac{1}{\epsilon}\lambda\delta\lambda d\Gamma & \text{for } \hat{\lambda}_N > 0 \end{cases} \tag{3.5}$$

where $\hat{\lambda}_N \leq 0$ indicates that the gap is closed and $\hat{\lambda}_N > 0$ means the gap is open.

4 Linear Elasticity Formulation

5 Hyperelasticity Formulation