Stabilized Mixed Finite Element Formulation

Personal Notes by Ida Ang

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Definitions

$\mathbf{F}$: Deformation gradient
$\mathbf{I}$: second-order unit tensor
$\boldsymbol{u}$: Displacement
J: determinant of the deformation gradient
$\mathbf{C}$: Right Cauchy-Green Strain Tensor
$\mathcal{W}(\mathbf{F})$: strain energy function
$\mathbf{P}$: first Piola-Kirchhoff stress tensor
$\mathbf{S}$: second PK stress tensor

α: cracks are represented by a scalar phase-field variable
p: Lagrange multiplier, hydrostatic pressure field
κ: bulk modulus

$$\kappa = \frac{E}{3(1-2\nu)} \tag{0.1}$$

μ: shear modulus

$$\mu = \frac{E}{2(1+\nu)} \tag{0.2}$$

λ: Lamé modulus

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{0.3}$$

For Plane Stress

$$\kappa = \frac{3-\nu}{1+\nu}, \quad \lambda = \frac{E\nu}{(1-\nu)^2} \tag{0.4}$$

$\mathcal{E}_\ell$: potential energy functional
$a(\alpha)$ is the decreasing stiffness modulation function
$w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume
c_w is a normalization constant

1 Hyperelastic Phase-Field Fracture Models

Deformation Gradient

$$\mathbf{F} = \mathbf{I} + \nabla \otimes \boldsymbol{u} \tag{1.1}$$

where $J = \det \mathbf{F}$ and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

The strain energy function $\mathcal{W}(\mathbf{F})$ is defined per unit reference volume such that the first PK and second PK

$$\mathbf{P} = \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \tag{1.2a}$$

$$\mathbf{S} = 2\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{C}} \tag{1.2b}$$

where $\mathbf{P} = \mathbf{FS}$.

Non-modified strain energy function is the compressible Neo-Hookean:

$$\mathcal{W}(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2\ln J) \tag{1.3}$$

For incompressible hyperelastic materials, the strain energy function is defined using the Lagrangian formulation

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p\,(J-1)\,, \tag{1.4}$$

If we consider the perturbed lagrangian formulation

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p\,(J-1) - \frac{p^2}{2\kappa}, \tag{1.5}$$

Decreasing stiffness modulation function is $a(\alpha)$ and $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

$$a(\alpha) = (1-\alpha)^2 \quad w(\alpha) = \alpha \tag{1.6}$$

In the code, we have the following definition

$$b(\alpha) = (1-\alpha)^3 = \sqrt{a^3(\alpha)}$$

The normalization constant is defined as:

$$c_w = \int_0^1 \sqrt{w(\alpha)}d\alpha \tag{1.7}$$

For presention

$$\begin{aligned}\widetilde{\mathcal{W}}(\mathbf{F},\alpha) &= a(\alpha)\,\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{\kappa}{2}\,(J-1)^2\,,\\ \widetilde{\mathcal{W}}(\mathbf{F},\alpha) &= a(\alpha)\,\frac{\mu}{2}(I_1 - 3 - 2\ln J) + a^3(\alpha)\frac{\kappa}{2}\,(J-1)^2\,,\end{aligned} \tag{1.8}$$

1.1 Derivation from 2020 Li and Bouklas Paper

Here, unlike Eq. 21 from Bin2020, we drop λ_b which is not a consideration in this formulation

$$\mathcal{E}_\ell(\boldsymbol{u},\alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F},\alpha)d\Omega + \frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega \tag{1.9}$$

We want to enforce the following relationship for pressure with a Lagrange multiplier

$$p = -\sqrt{a^3(\alpha)}\kappa\,(J-1) \tag{1.10}$$

Giving us Eq. 25 in the 2020 Li and Bouklas paper where κ is the bulk modulus

$$\mathcal{E}_\ell\,(\boldsymbol{u}, p, \Lambda, \alpha) = \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega \Lambda(p + \sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \tag{1.11}$$

Identify the stationary point of the energy functional with respect to pressure (not Λ)

$$\begin{aligned}\frac{\partial \mathcal{E}_\ell}{\partial p} &= \int_\Omega \frac{p}{\kappa} d\Omega + \int_\Omega \Lambda d\Omega \\ 0 &= \frac{p}{\kappa} + \Lambda \rightarrow \Lambda = -p/\kappa\end{aligned}$$

Substituting this relationship into the energy functional yields:

$$\begin{aligned}\mathcal{E}_\ell\,(\boldsymbol{u}, p, \alpha) &= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega -\frac{p}{\kappa}(p + \sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \\ &= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \frac{p^2}{\kappa} d\Omega - \int_\Omega \frac{p}{\kappa}\sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \\ \mathcal{E}_\ell\,(\boldsymbol{u}, p, \alpha) &= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1))d\Omega\end{aligned} \tag{1.12}$$

Substitute in $\mathcal{E}_\ell(\boldsymbol{u}, \alpha)$ and substitute Eq. 1.3

$$\mathcal{E}_\ell\,(\boldsymbol{u}, p, \alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega$$

The prior equation includes the full weak form, unless we want to consider linear interpolation of all fields. In that case, we can introduce the stabilization term

$$-\frac{\varpi h^2}{2\mu}\sqrt{a^3(\alpha)}\sum_{e=1}^{n_{el}}\int_{\Omega^e} J\mathbf{C}^{-1} : \left(\nabla p \cdot \nabla q\right) dV = 0$$

1.2 Summary

Therefore the modified strain energy functional

$$\begin{aligned}\widetilde{W}(\mathbf{F}, \alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\kappa(J-1)^2 - \frac{p^2}{2\kappa} \\ \widetilde{W}(\mathbf{F}, \alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) - \sqrt{a^3(\alpha)}p(J-1) - \frac{p^2}{2\kappa}\end{aligned}$$

Following the code, we have a small number for numerical purposes

$$\widetilde{W}(\mathbf{F}, \alpha) = \left(a(\alpha) + k_\ell\right)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\kappa}$$

The first Piola-Kirchhoff stress tensor is given:

$$
\begin{aligned}
\mathbf{P} &= \frac{\partial \widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{\partial \mathbf{F}} \\
&= \frac{\partial}{\partial \mathbf{F}}\left[a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa\,(J-1)^2\right] \\
&= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\frac{1}{2}\kappa\frac{\partial (J-1)^2}{\partial \mathbf{F}} \\
&= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa\,(J-1)\,\frac{\partial J}{\partial \mathbf{F}} \quad \text{where } \frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T} \\
&= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa(J-1)J\mathbf{F}^{-T} \quad \text{substituting in pressure equation} \\
\mathbf{P} &= a(\alpha)\mu(\mathbf{F}-\mathbf{F}^{-T}) - b(\alpha)pJ\mathbf{F}^{-T}
\end{aligned}
\tag{1.13}
$$

1.3 Changes for 2D Plane-Stress Models

Recalling the 1st PK stress in Eq. 1.13.

$$\mathbf{P} = a(\alpha)\mu(\mathbf{F}-\mathbf{F}^{-T}) - b(\alpha)pJ\mathbf{F}^{-T}$$

In a plane-stress case, the $\mathbf{P_{33}}$ component is zero:

$$P_{33} = a(\alpha)\mu(F_{33} - F_{33}^{-1}) - b(\alpha)pJF_{33}^{-1} = 0$$

This can be multiplied by its associated test function to obtain the weak form

$$\int_\Omega \left(a(\alpha)\mu(F_{33} - F_{33}^{-1}) - b(\alpha)pJF_{33}^{-1}\right) v_{F_{33}} dV = 0$$

In the FEniCS code, we expand the solution space to include displacement, pressure, and a component of the deformation gradient $\mathbf{F_{33}}$. Therefore, we include a change to the invariants of the deformation tensors:

```
J = det(F)*F33
Ic = tr(C) + F33**2
```

Together with the weak form from above:

```
F_u = derivative(elastic_potential, w_p, v_q) \
        + (a(alpha)*mu*(F33 - 1/F33) - b(alpha)*p*J/F33)*v_F33*dx
```

1.3.1 Changes for 2D Discrete Crack Model

If we are considering a discrete fracture method

$$\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega$$

$$\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) = \int_\Omega \mathcal{W}(\mathbf{F})d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega p(J-1)d\Omega$$

where we have assumed for the energy functional

$$\widetilde{\mathcal{W}}(\mathbf{F},\alpha) = \frac{\mu}{2}(I_c - 3 - 2\ln J) - p(J-1) - \frac{p^2}{2\lambda}$$

Therefore, we can calculate the 1st Piola Kirchoff Stress as:

$$\begin{aligned}\mathbf{P} &= \frac{\mu}{2}(2\mathbf{F} - \frac{2}{J}J\mathbf{F}^{-T}) - pJ\mathbf{F}^{-T} \\ &= \mu(\mathbf{F} - \mathbf{F}^{-T}) - pJ\mathbf{F}^{-T}\end{aligned}$$

Taking the third component to be zero

$$\begin{aligned}P_{33} &= \mu(F_{33} - F_{33}^{-1}) - pJF_{33}^{-1} = 0 \\ &= F_{33} - F_{33}^{-1} - \frac{pJ}{\mu}F_{33}^{-1} = 0 \\ P_{33} &= F_{33}^2 - 1 - \frac{pJ}{\mu} = 0\end{aligned}$$

with the stabilization term and plane stress in the weak form

$$\begin{aligned}-\frac{\varpi h^2}{2\mu}\int_\Omega J\mathbf{C}^{-1} : \big(\nabla p \cdot \nabla q\big)\, dV = 0 \\ \int_\Omega \left(\mathbf{F}_{33}^2 - 1 - \frac{pJ}{\mu}\right) v_{F_{33}} dV = 0\end{aligned}$$

1.3.2 Changes for 2D displacement formulation

Removing pressure terms

$$\begin{aligned}\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha) &= \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right) d\Omega - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1) d\Omega \\ \mathcal{E}_\ell(\boldsymbol{u}, \alpha) &= \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F}) d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right) d\Omega\end{aligned}$$

with plane stress in the weak form (no need for stabilization terms)

$$\int_\Omega \big(a(\alpha)\mu(F_{33} - F_{33}^{-1})\big) v_{F_{33}} dV = 0$$

We have assumed the modified energy functional

$$\widetilde{W}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) \tag{1.14}$$

Therefore, we can calculate the 1st Piola Kirchoff Stress as:

$$\begin{aligned}\mathbf{P} &= a(\alpha)\frac{\mu}{2}(2\mathbf{F} - \frac{2}{J}J\mathbf{F}^{-T}) \\ &= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T})\end{aligned}$$

Taking the third component to be zero

$$P_{33} = a(\alpha)\mu(F_{33} - F_{33}^{-1}) = 0$$

2 Obtaining the Critical Stretch

Assuming a Neo-Hookean energy where μ is the shear modulus

$$
\begin{aligned}
W(I_1, I_2) &= \beta_1(I_1 - 3) + \beta_2(I_2 - 3) \quad \text{where } \beta_1 = \frac{\mu}{2},\ \beta_2 = 0 \\
W(I_1, I_2) &= \frac{\mu}{2}(I_1 - 3) \quad \text{where } I_1 = I_2 = \lambda_A^2 + \lambda_A^{-2} + 1 \\
W(I_1, I_2) &= \frac{\mu}{2}\left(\lambda_A^2 + \frac{1}{\lambda_A^2} - 2\right) \\
W(I_1, I_2) &= \frac{\mu}{2}\left(\lambda_A - \frac{1}{\lambda_A}\right)^2
\end{aligned}
$$

The J-integral for a pure shear strip geometry can be calculated as:

$$
\begin{aligned}
J &= 2h_0 W(I_1, I_2) \\
J &= h_0\mu\left(\lambda_A - \frac{1}{\lambda_A}\right)^2
\end{aligned} \tag{2.1}
$$

where for the stretch:

$$\lambda_A = 1 + \frac{\Delta}{h_0} \tag{2.2}$$

where the total height of the strip is $2h_0$ and Δ is the loading

Theoretically, the critical condition for crack initiation is where the fracture energy is equivalent to the energy release rate

$$G_c = J$$

Determining whether the length of the strip is long enough:

1. Choose height of strip, h_0, shear modulus, μ, and critical fracture energy G_c
2. Use Matlab to calculate the critical stretch λ_c

$$G_c = h_0\mu\left(\lambda_c - \frac{1}{\lambda_c}\right)^2$$

3. Calculate the critical displacement Δ_c using eq. 2.2

$$\Delta_c = h_0(\lambda_c - 1)$$

4. Run two simulations using 2D-planestress-TH-BL.py
   (a) Assign a displacement slightly below the predicted Δ_c
   (b) Change P3 point to either $2hsize$ before or after the crack center to calculate an energy max and energy min

(c) Calculate J numerical

$$J_n = -\frac{E_{max} - E_{min}}{4hsize}$$

(d) Calculate percentage error with J analytical from Eq (2.1)

$$\%Error = \left(\frac{J_a - J_n}{J_a}\right)100\%$$

Note that there is an effective critical energy release rate

$$G_c^e = G_c\left(1 + \frac{3hsize}{8\ell}\right) \tag{2.3}$$

where hsize is the element size and ℓ is the width of the phase-field

Once a length is determined, we can then run both a phase field and discrete trial and determine where the crack initiaties

1. The strip has a length of 6 and a total width of 1.0 where $h_0 = 0.5$
2. hsize $= 0.002$, $\ell = 0.01$
3. Used an exponential function to ramp the displacement in order to obtain a close agreement

3 Strain energy decomposition

The Heaviside function is defined as

$$H(x) = \frac{x + |x|}{2x} = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

The Macaulay bracket is defined

$$M(x) = \frac{x + |x|}{2} = \begin{cases} x, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

3.1 Following Ye 2020 and Tang 2019

In the Ye 2020 paper, the internal energy is expressed as:

$$W_{int}(\mathbf{F}, \alpha, \nabla\alpha) = [a(\alpha) + k_\ell]W_{act} + W_{pas} + G_c\left(\frac{\alpha^2}{2\ell} + \frac{\ell}{2}|\nabla\alpha|^2\right)$$

Now in section 3.2.3 of Ye 2020, is stated the decomposition for a Mooney Rivlin constitutive law:

$$\begin{aligned} W_{MR}(I_1, I_2) &= C_1(I_1 - 3) + C_2(I_2 - 3) \\ W_{MR}(\lambda_1, \lambda_1, \lambda_3) &= C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3) \\ &= C_1\sum_{i=1}^{3}(\lambda_i^2 - 1) + C_2\sum_{i=1}^{3}(\lambda_i^{-2} - 1) \end{aligned}$$

Which can be decomposed to active and passive internal energy terms. First we can rewrite:

$$
\begin{aligned}
W_{MR}(\lambda_1, \lambda_1, \lambda_3) = C_1 \sum_{i=1}^{3} H(\lambda_i - 1)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} H(\lambda_i - 1)(\lambda_i^{-2} - 1) \\
+ C_1 \sum_{i=1}^{3} H(1 - \lambda_i)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} H(1 - \lambda_i)(\lambda_i^{-2} - 1)
\end{aligned}
$$

The active and passive terms can be stated as follows where the active part represents the crack-driven energy.

$$
\begin{aligned}
W_{act} &= C_1 \sum_{i=1}^{3} H(\lambda_i - 1)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} H(\lambda_i - 1)(\lambda_i^{-2} - 1) \\
W_{pas} &= C_1 \sum_{i=1}^{3} H(1 - \lambda_i)(\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} H(1 - \lambda_i)(\lambda_i^{-2} - 1)
\end{aligned}
$$

One way to better understand these is to consider some cases 1) triaxial tension $\lambda_i > 1$ 2) other stress states where $\lambda_i < 1$:

$$
\begin{aligned}
&\text{For } \lambda_i > 1: \\
&\qquad W_{act} = C_1 \sum_{i=1}^{3} (\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} (\lambda_i^{-2} - 1) \\
&\qquad W_{pas} = 0 \\
&\text{For } \lambda_i < 1: \\
&\qquad W_{act} = 0 \\
&\qquad W_{pas} = C_1 \sum_{i=1}^{3} (\lambda_i^2 - 1) + C_2 \sum_{i=1}^{3} (\lambda_i^{-2} - 1)
\end{aligned}
$$

For this second case, we end up with a negative energy component (first term of the passive energy).

We can also note the definitions within Tang 2019 for Model M_I. In this model, we consider the free energy density of a neo-Hookean constitutive law:

$$
W(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2\ln J) + \frac{\kappa}{2}(\ln J)^2
$$

which can also be rephrased in terms of stretches

$$
\begin{aligned}
W(\lambda_1, \lambda_2, \lambda_3) &= W_1 + W_2 \\
&= \frac{\mu}{2} \sum_{i=1}^{3} (\lambda_i^2 - 1 - 2\ln \lambda_i) + \frac{\kappa}{2}(\ln J)^2
\end{aligned}
$$

where we note that W_1 is a linear function of $\ln \lambda_i$ and W_2 is a nonlinear function of $\ln J$
The free energy is stated as

$$
G_{rub} = [(1 - K)\alpha^2 + K]W^+ + W^-
$$

Note that K is not κ. No definition is provided in the paper. This is another way of coupling the damage to the free energy density, and I believe we can rewrite our own version where:

$$G_{rub} = [a(\alpha) + k_\ell]W^+ + W^-$$

Now we turn to the definition of W^+ and W^- which refers to the energy with tensile stretching

$$W^+ = W(\lambda_i^+, J^+) = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) + \frac{\kappa}{2}(\ln J^+)^2$$

and the energy with compression respectively.

$$W^- = W(\lambda_i^-, J^-) = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) + \frac{\kappa}{2}(\ln J^-)^2$$

The definitions for these superscript + and - terms gives us

$$\lambda_i^+ = \begin{cases} \lambda_i, & \lambda_i > 1, \\ 1, & \lambda_i \le 1 \end{cases} \qquad J^+ = \begin{cases} J, & J > 1, \\ 1, & J \le 1 \end{cases}$$
$$\lambda_i^- = \begin{cases} \lambda_i, & \lambda_i < 1, \\ 1, & \lambda_i \ge 1 \end{cases} \qquad J^- = \begin{cases} J, & J < 1, \\ 1, & J \ge 1 \end{cases}$$

This isn't the definition for the heaviside function, but it could be a shifted Macaulay bracket

$$M_s(x) = \frac{x - 1 + |x-1|}{2} + 1 = \begin{cases} x, & x > 1, \\ 1, & x \le 1 \end{cases}$$

Now we can consider some examples.

$$\text{For } J > 1:$$
$$W^+ = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) + \frac{\kappa}{2}(\ln J)^2$$
$$W^- = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-)$$
$$\text{For } J < 1:$$
$$W^+ = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+)$$
$$W^- = \frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) + \frac{\kappa}{2}(\ln J)^2$$

If we consider the same stress states as in Ye2020, 1) triaxial tension $\lambda_i > 1$ 2) all other stress states $\lambda_i < 1$:

$$
\begin{aligned}
&\text{For } J > 1,\ \lambda_i > 1: \\
&\qquad W^+ = \frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}(\ln J)^2 \\
&\qquad W^- = 0 \\
&\text{For } J < 1,\ \lambda_i > 1: \\
&\qquad W^+ = \frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) \\
&\qquad W^- = \frac{\kappa}{2}(\ln J)^2
\end{aligned}
$$

Now for all other stress states:

$$
\begin{aligned}
&\text{For } J > 1,\ \lambda_i < 1: \\
&\qquad W^+ = \frac{\kappa}{2}(\ln J)^2 \\
&\qquad W^- = \frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) \\
&\text{For } J < 1,\ \lambda_i < 1: \\
&\qquad W^+ = 0 \\
&\qquad W^- = \frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}(\ln J)^2
\end{aligned}
$$

This should be roughly equivalent to the considerations in the Ye2020 paper.

3.2 Our strain energy decomposition

Following the section above, we can consider our modified strain energy

$$
\begin{aligned}
\widetilde{W}(\mathbf{F},\alpha) &= a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\kappa} \quad \text{where } p = -b(\alpha)\kappa(J-1) \\
\widetilde{W}(\mathbf{F},\alpha) &= a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) + \frac{\kappa}{2}a(\alpha)^3(J-1)^2
\end{aligned}
$$

We can also rewrite the first term with regards to stretches

$$
\widetilde{W}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}a(\alpha)^3(J-1)^2
$$

Following Tang 2019 we rewrite the strain energy as

$$
\widetilde{W}(\mathbf{F},\alpha) = \widetilde{W}_{\text{act}}(\mathbf{F},\alpha) + \widetilde{W}_{\text{pas}}(\mathbf{F},\alpha) \tag{3.1}
$$

where the active and passive parts of the strain energy can be written as:

$$\widetilde{W}_{act}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) + \frac{\kappa}{2}a(\alpha)^3(J^+ - 1)^2$$

$$\widetilde{W}_{pas}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) + \frac{\kappa}{2}a(\alpha)^3(J^- - 1)^2$$

where the definitions of the superscript + and - terms remain the same as in Tang2020:

$$\begin{aligned}
&\text{For } J > 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) + \frac{\kappa}{2}a(\alpha)^3(J - 1)^2 \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) \\
&\text{For } J \le 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^+)^2 - 1 - 2\ln\lambda_i^+) \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}((\lambda_i^-)^2 - 1 - 2\ln\lambda_i^-) + \frac{\kappa}{2}a(\alpha)^3(J - 1)^2
\end{aligned} \tag{3.2}$$

Now considering the same two cases of, triaxial tension and

$$\begin{aligned}
&\text{For } J > 1,\ \lambda_i > 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}a(\alpha)^3(J - 1)^2 \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = 0 \\
&\text{For } J \le 1,\ \lambda_i > 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = \frac{\kappa}{2}a(\alpha)^3(J - 1)^2
\end{aligned}$$

all other cases:

$$\begin{aligned}
&\text{For } J > 1,\ \lambda_i < 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = \frac{\kappa}{2}a(\alpha)^3(J - 1)^2 \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) \\
&\text{For } J \le 1,\ \lambda_i < 1: \\
&\qquad \widetilde{W}_{act}(\mathbf{F},\alpha) = 0 \\
&\qquad \widetilde{W}_{pas}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1 - 2\ln\lambda_i) + \frac{\kappa}{2}a(\alpha)^3(J - 1)^2
\end{aligned}$$

These can be concisely summarized with the following expressions, where the active part of the strain energy is

$$\widetilde{W}_{\text{act}}\left(\mathbf{F},\alpha\right) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3} H(\lambda_i - 1)\left(\lambda_i^2 - 1 - 2\ln\lambda_i\right) + a^3(\alpha)H(J-1)\frac{1}{2}\kappa\left(J-1\right)^2, \tag{3.3}$$

and the passive part of the strain energy is

$$\widetilde{W}_{\text{pas}}\left(\mathbf{F},\alpha\right) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3} H(1 - \lambda_i)\left(\lambda_i^2 - 1 - 2\ln\lambda_i\right) + a^3(\alpha)H(1-J)\frac{1}{2}\kappa\left(J-1\right)^2, \tag{3.4}$$

3.2.1 Compute the principal stretches λ_i

The eigenvalues of Cauchy-Green strain tensor $\mathbf{C}$ are λ_i^2, $i = 1, 2, 3$. With following definitions

$$d = \frac{Tr\mathbf{C}}{3}, \quad e = \sqrt{\frac{Tr(\mathbf{C} - d\mathbf{I})^2}{6}}, \quad f = \frac{1}{e}\left(\mathbf{C} - d\mathbf{I}\right), \quad g = \frac{\det f}{2}, \tag{3.5}$$

and assuming the eigenvalues satisfying $\lambda_3^2 \leq \lambda_2 \leq \lambda_1$, we could obtain (**?**)

$$\lambda_1^2 = d + 2e\cos\left(\frac{\arccos g}{3}\right), \quad \lambda_3^2 = d + 2e\cos\left(\frac{\arccos g}{3} + \frac{2\pi}{3}\right), \quad \lambda_2^2 = 3d - \lambda_1^2 - \lambda_3^3. \tag{3.6}$$

3.3 Hybrid Formulation

The principal stretches can be computed as shown above, but for spherical stretch ($\mathbf{C} = constant\mathbf{I}$) leading to NaN error. This means that for 3D strain decomposition using the explicit eigenvalue formulation, the computation of the first variation and second variation are nontrivial. FEniCS auto-differential function cannot detect these special cases.

The workaround is to consider the Hybrid model in Ambati 2015: A review on phase-field models of brittle fracture and a new fast hybrid formulation.

$$\sigma(\mathbf{u},\alpha) = (1-\alpha)^2\frac{\partial W(\epsilon)}{\partial \epsilon}$$
$$-l^2\nabla^2\alpha + \alpha = \frac{2l}{G_c}(1-\alpha)\mathcal{H}^+$$

Again, we consider our modified strain energy

$$\widetilde{W}(\mathbf{F},\alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) + \frac{\kappa}{2}a(\alpha)^3(J-1)^2$$

Then the active and

$$\widetilde{W}_{\text{act}}\left(\mathbf{F},\alpha\right) = a(\alpha)\frac{\mu}{2}\sum_{i=1}^{3} H(\lambda_i - 1)\left(\lambda_i^2 - 1 - 2\ln\lambda_i\right) + a^3(\alpha)H(J-1)\frac{1}{2}\kappa\left(J-1\right)^2, \tag{3.7}$$

the passive part of the strain energy is

$$\widetilde{W}_{\text{pas}}\left(\mathbf{F},\alpha\right) = \frac{\mu}{2}\sum_{i=1}^{3} H(1 - \lambda_i)\left(\lambda_i^2 - 1 - 2\ln\lambda_i\right) + H(1-J)\frac{1}{2}\kappa\left(J-1\right)^2, \tag{3.8}$$

4 Gateaux Derivative

The total potential energy functional:

$$\mathcal{E}_\ell\left(\boldsymbol{u},p,\alpha\right)=\int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega+\frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell}+\ell\|\nabla\alpha\|^2\right)d\Omega-\int_\Omega\frac{p^2}{2\kappa}d\Omega-\int_\Omega\sqrt{a^3(\alpha)}p(J-1)d\Omega$$

The Gateaux derivative with respect to $(\boldsymbol{u},p,\alpha)$ in direction $(\boldsymbol{v},q,\beta)$ under the irreversibility condition $\dot{\alpha}\geq 0$.

$$d\mathcal{E}_\ell\left(\boldsymbol{u},p,\alpha;\boldsymbol{v},q,\beta\right)\geq 0. \tag{4.1}$$

Calculation of the Gateaux derivative

$$\begin{aligned}
d\mathcal{E}_\ell(\boldsymbol{u},\boldsymbol{v})(p,q)(\alpha,\beta)&=\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}+\delta\boldsymbol{v},p+\delta q,\alpha+\delta\beta)\big|_{\delta=0}\\
d\mathcal{E}_\ell(\boldsymbol{u},\boldsymbol{v})(p,q)(\alpha,\beta)&=\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}+\delta\boldsymbol{v},\alpha)\big|_{\delta=0}+\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u},p+\delta q,\alpha)\big|_{\delta=0}+\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u},\alpha+\delta\beta)\big|_{\delta=0}
\end{aligned}$$

Starting with the first term:

$$\begin{aligned}
\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}+\delta\boldsymbol{v},p,\alpha)\big|_{\delta=0}&=\frac{d}{d\delta}\left[\int_\Omega\widetilde{\mathcal{W}}(\mathbf{I}+\nabla(\boldsymbol{u}+\delta\boldsymbol{v}),\alpha)\,d\Omega-\int_{\partial_N\Omega}\tilde{\boldsymbol{g}}_0\cdot(\boldsymbol{u}+\delta\boldsymbol{v})\,dA\right]\Bigg|_{\delta=0}\\
&=\left[\int_\Omega\frac{d\widetilde{\mathcal{W}}(\mathbf{I}+\nabla(\boldsymbol{u}+\delta\boldsymbol{v}),\alpha)}{d\delta}\,d\Omega-\int_{\partial_N\Omega}\tilde{\boldsymbol{g}}_0\cdot\frac{d(\boldsymbol{u}+\delta\boldsymbol{v})}{d\delta}\,dA\right]\Bigg|_{\delta=0}\quad\text{chain rule}\\
&=\left[\int_\Omega\frac{d\widetilde{\mathcal{W}}(\mathbf{I}+\nabla(\boldsymbol{u}+\delta\boldsymbol{v}),\alpha)}{d(\mathbf{I}+\nabla(\boldsymbol{u}+\delta\boldsymbol{v})}\frac{d(\mathbf{I}+\nabla(\boldsymbol{u}+\delta\boldsymbol{v}))}{d\delta}\,d\Omega-\int_{\partial_N\Omega}\tilde{\boldsymbol{g}}_0\cdot\frac{d(\boldsymbol{u}+\delta\boldsymbol{v})}{d\delta}\,dA\right]\Bigg|_{\delta=0}\\
&=\left[\int_\Omega\frac{d\widetilde{\mathcal{W}}(\mathbf{I}+\nabla\boldsymbol{u}+\delta\nabla\boldsymbol{v},\alpha)}{d(\mathbf{I}+\nabla\boldsymbol{u}+\delta\nabla\boldsymbol{v})}\frac{d(\mathbf{I}+\nabla\boldsymbol{u}+\delta\nabla\boldsymbol{v})}{d\delta}\,d\Omega-\int_{\partial_N\Omega}\tilde{\boldsymbol{g}}_0\cdot\boldsymbol{v}\,dA\right]\Bigg|_{\delta=0}\\
&=\left[\int_\Omega\frac{d\widetilde{\mathcal{W}}(\mathbf{I}+\nabla\boldsymbol{u}+\delta\nabla\boldsymbol{v},\alpha)}{d(\mathbf{I}+\nabla\boldsymbol{u}+\delta\nabla\boldsymbol{v})}\nabla\boldsymbol{v}\,d\Omega-\int_{\partial_N\Omega}\tilde{\boldsymbol{g}}_0\cdot\boldsymbol{v}\,dA\right]\Bigg|_{\delta=0}\\
&=\int_\Omega\frac{d\widetilde{\mathcal{W}}(\mathbf{I}+\nabla\boldsymbol{u},\alpha)}{d(\mathbf{I}+\nabla\boldsymbol{u})}\nabla\boldsymbol{v}\,d\Omega-\int_{\partial_N\Omega}\tilde{\boldsymbol{g}}_0\cdot\boldsymbol{v}\,dA
\end{aligned}$$

First equation

$$\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}+\delta\boldsymbol{v},\alpha)\big|_{\delta=0}=\int_\Omega\frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\mathbf{F}}\nabla\boldsymbol{v}\,d\Omega-\int_{\partial_N\Omega}\tilde{\boldsymbol{g}}_0\cdot\boldsymbol{v}\,dA \tag{4.2}$$

Second term:

$$\begin{aligned}
\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u},p+\delta q,\alpha)\big|_{\delta=0}&=\frac{\partial}{\partial\delta}\left[-\int_\Omega\frac{(p+\delta q)^2}{2\kappa}d\Omega-\int_\Omega\sqrt{a^3(\alpha)}(p+\delta q)(J-1)d\Omega\right]\\
&=\left[-\int_\Omega\frac{1}{2\kappa}\frac{d(p+\delta q)^2}{d\delta}d\Omega-\int_\Omega\sqrt{a^3(\alpha)}\frac{d(p+\delta q)}{d\delta}(J-1)d\Omega\right]\Bigg|_{\delta=0}\\
&=\left[-\int_\Omega\frac{2(p+\delta q)q}{2\kappa}d\Omega-\int_\Omega\sqrt{a^3(\alpha)}q(J-1)d\Omega\right]\Bigg|_{\delta=0}\\
&=-\int_\Omega\frac{p}{\kappa}q\,d\Omega-\int_\Omega\sqrt{a^3(\alpha)}q(J-1)d\Omega
\end{aligned}$$

Second equation

$$\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}, p+\delta q, \alpha)\big|_{\delta=0} = \int_\Omega \left(-\frac{p}{\kappa} - \sqrt{a^3(\alpha)}q(J-1)\right) q\, d\Omega \tag{4.3}$$

Third term:

$$\begin{aligned}
&\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha+\delta\beta)\big|_{\delta=0} \\
&= \frac{d}{d\delta}\Bigg[\int_\Omega \widetilde{\mathcal{W}}(\mathbf{F}, \alpha+\delta\beta)\, d\Omega + \frac{\mathcal{G}_c}{c_w}\int_\Omega \left(\frac{w(\alpha+\delta\beta)}{\ell} + \ell\|\nabla(\alpha+\delta\beta)\|^2\right) dV\Bigg]\Bigg|_{\delta=0} \\
&= \Bigg[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha+\delta\beta)}{d\delta}\, d\Omega + \frac{\mathcal{G}_c}{c_w}\frac{d}{d\delta}\int_\Omega \left(\frac{w(\alpha+\delta\beta)}{\ell} + \ell\|\nabla(\alpha+\delta\beta)\|^2\right) dV\Bigg]\Bigg|_{\delta=0} \\
&= \Bigg[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha+\delta\beta)}{d(\alpha+\delta\beta)}\frac{d(\alpha+\delta\beta)}{d\delta}\, d\Omega + \frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega \frac{dw(\alpha+\delta\beta)}{d\delta}\, dV + \frac{\mathcal{G}_c}{c_w}\ell\int_\Omega \frac{\|\nabla(\alpha+\delta\beta)\|^2}{d\delta}\, dV\Bigg]\Bigg|_{\delta=0} \\
&= \Bigg[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha+\delta\beta)}{d(\alpha+\delta\beta)}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega \frac{dw(\alpha+\delta\beta)}{d(\alpha+\delta\beta)}\frac{d(\alpha+\delta\beta)}{d\delta}\, dV + \frac{\mathcal{G}_c}{c_w}\ell\int_\Omega 2\nabla(\alpha+\delta\beta)\frac{\nabla(\alpha+\delta\beta)}{d\delta}\, dV\Bigg]\Bigg|_{\delta=0} \\
&= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha}\beta\, d\Omega + \Bigg[\frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega \frac{dw(\alpha+\delta\beta)}{d(\alpha+\delta\beta)}\beta\, dV + \frac{\mathcal{G}_c}{c_w}\ell\int_\Omega 2\nabla(\alpha+\delta\beta)\nabla\beta\, dV\Bigg]\Bigg|_{\delta=0} \\
&= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w}\frac{1}{\ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta\, dV + 2\ell\frac{\mathcal{G}_c}{c_w}\int_\Omega \nabla\alpha\cdot\nabla\beta\, dV \\
&= \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega \left[\frac{dw(\alpha)}{d\alpha}\beta + 2\ell^2(\nabla\alpha\cdot\nabla\beta)\right] dV
\end{aligned}$$

Giving the final equation

$$\frac{d}{d\delta}\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha+\delta\beta)\big|_{\delta=0} = \int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega \left[\frac{dw(\alpha)}{d\alpha}\beta + 2\ell^2(\nabla\alpha\cdot\nabla\beta)\right] dV \tag{4.4}$$

Therefore we can obtain the weak form by combining Eq. 4.2, 4.3, and 4.4

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\mathbf{F}}\nabla\boldsymbol{v}\, d\Omega - \int_{\partial_N\Omega}\tilde{\boldsymbol{g}}_0\cdot\boldsymbol{v}\, dA = 0 \tag{4.5a}$$

$$\int_\Omega \left(-\sqrt{a^3(\alpha)}(J-1) - \frac{p}{\kappa}\right) q dV = 0 \tag{4.5b}$$

$$\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w\ell}\int_\Omega \left[\frac{dw(\alpha)}{d\alpha}\beta + 2\ell^2(\nabla\alpha\cdot\nabla\beta)\right] dV \geq 0 \tag{4.5c}$$

The strong form

$$\operatorname{Div}\mathbf{P} = 0 \quad \text{in} \quad \Omega \tag{4.6a}$$

$$\mathbf{u} = \tilde{\mathbf{u}}_0 \quad \text{in} \quad \partial_D\Omega \tag{4.6b}$$

$$[\mathbf{FS}]\,\boldsymbol{n} = \tilde{\boldsymbol{g}}_0 \quad \text{on} \quad \partial_N\Omega, \tag{4.6c}$$

where from Eq. 1.13 we can substitute Eq. 1.10

$$
\begin{aligned}
\mathbf{P} &= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa\,(J-1)\,\frac{\partial J}{\partial \mathbf{F}} \quad \text{where}\, p = -\sqrt{a^3(\alpha)}\kappa\,(J-1) \\
\mathbf{P} &= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} - p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial \mathbf{F}}
\end{aligned}
$$

and write the mechanical equilibrium equation in Eq. 4.6:

$$
\operatorname{Div}\left[a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} - p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial \mathbf{F}}\right] = 0 \tag{4.7}
$$

Derivation of the KKT condition equations where $\nabla\beta \cdot \nabla\alpha = \nabla(\beta\nabla\alpha) - \beta\Delta\alpha$

$$
\begin{aligned}
\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \left[\frac{dw(\alpha)}{d\alpha}\beta + 2\ell^2(\nabla\alpha\cdot\nabla\beta)\right] dV \geq 0 \\
\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta dV + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega 2\ell^2(\nabla\alpha\cdot\nabla\beta)\, dV \geq 0 \\
\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta dV + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega 2\ell^2(\nabla(\beta\nabla\alpha))\, dV - \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega 2\ell^2(\beta\Delta\alpha)\, dV \geq 0 \\
\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\beta\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \frac{dw(\alpha)}{d\alpha}\beta dV - \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega 2\ell^2(\beta\Delta\alpha)\, dV \geq 0 \\
\left[\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right) dV\right]\beta \geq 0 \\
\int_\Omega \frac{d\widetilde{\mathcal{W}}(\mathbf{F},\alpha)}{d\alpha}\, d\Omega + \frac{\mathcal{G}_c}{c_w \ell}\int_\Omega \left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2\Delta\alpha\right) dV \geq 0
\end{aligned}
$$

Grouping terms, we obtain

$$
\begin{aligned}
\dot{\alpha} \geq 0 \quad \text{in} \quad \Omega_0, \\
\frac{d\widetilde{\mathcal{W}}(\mathbf{F},p,\alpha)}{d\alpha} + \frac{\mathcal{G}_c}{c_w \ell}\left(\frac{\partial w(\alpha)}{\partial\alpha} - \ell^2\Delta\alpha\right) \geq 0 \quad \text{in} \quad \Omega_0, \\
\dot{\alpha}\left[\frac{d\widetilde{\mathcal{W}}(\mathbf{F},p,\alpha)}{d\alpha} + \frac{\mathcal{G}_c}{c_w \ell}\left(\frac{\partial w(\alpha)}{\partial\alpha} - \ell^2\Delta\alpha\right)\right] = 0 \quad \text{in} \quad \Omega_0,
\end{aligned} \tag{4.8}
$$

Lastly, we have the following boundary conditions

$$
\frac{\partial\alpha}{\partial\mathbf{n}} \geq 0 \quad \text{and} \quad \dot{\alpha}\frac{\partial\alpha}{\partial\mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega_0 \tag{4.9}
$$

Multiply Eq. 4.7 with weighting function $\boldsymbol{v} + (Ih^2)/(2\mu)\mathbf{F}^{-T}\nabla q$

$$\int_\Omega \text{Div}\mathbf{P} \cdot \left[\boldsymbol{v} + \frac{\varpi h^2}{2\mu}\mathbf{F}^{-T}\nabla q\right] dV = 0$$

$$\int_\Omega \text{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV + \int_\Omega \text{Div}\mathbf{P} \cdot \left[\frac{\varpi h^2}{2\mu}\mathbf{F}^{-T}\nabla q\right] dV = 0$$

$$\int_\Omega \text{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV + \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \text{Div}\mathbf{P} \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV = 0$$

$$\int_\Omega \text{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV$$

$$+\frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \text{Div}\left[a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}}\right] \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \text{Div}\left[p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial \mathbf{F}}\right] \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV = 0$$

$$'''' + '''' - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \nabla\left(p\sqrt{a^3(\alpha)}\right)J\mathbf{F}^{-T} \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV = 0$$

$$'''' + '''' - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \nabla\left(p\sqrt{a^3(\alpha)}\right)J\mathbf{F}^{-1}\mathbf{F}^{-T} \cdot \left(\mathbf{F}^{-1}\mathbf{F}^{-T}\nabla q\right)\, dV = 0$$

$$'''' + '''' - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \nabla\left(p\sqrt{a^3(\alpha)}\right)J \cdot \left(\mathbf{C}^{-1}\nabla q\right)\, dV = 0$$

$$'''' + '''' - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} J\mathbf{C}^{-1} : \left[\nabla\left(p\sqrt{a^3(\alpha)}\right) \cdot \nabla q\right] dV = 0$$

where $\mathbf{P} = a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} - p\sqrt{a^3(\alpha)}\frac{\partial J}{\partial \mathbf{F}}$

We also want to deal with the first term where $(fg)' = f'g + fg'$

$$\int_\Omega \text{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV = \int_\Omega (\mathbf{F} \cdot \boldsymbol{v})_{,X}\, dV - \int_\Omega \mathbf{P} \cdot \frac{\partial \boldsymbol{v}}{\partial X}\, dV$$

Leaving

$$\int_\Omega \text{Div}\mathbf{P} \cdot \boldsymbol{v}\, dV + \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} \text{Div}\left[a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}}\right] \cdot \left(\mathbf{F}^{-T}\nabla q\right)\, dV - \frac{\varpi h^2}{2\mu}\sum_{e=1}^{n_{el}}\int_{\Omega^e} J\mathbf{C}^{-1} : \left[\nabla\left(p\sqrt{a^3(\alpha)}\right) \cdot \nabla q\right] dV \tag{4.10}$$

5 Following Borden: Derivations of Analytical Phase Field

Note the full potential energy functional, which can also be called the lagrangian

$$\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega$$

We can use the Euler-Lagrange equations to arrive at the equations of motion by taking the derivative with respect to displacement, pressure, and the scalar damage field. Starting with displacement:

$$\frac{\partial \mathcal{E}_\ell}{\partial \boldsymbol{u}} = \int_\Omega a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{u}} d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial p} = -\int_\Omega \frac{p}{\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}(J-1) d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial \alpha} = -\int_\Omega 2(1-\alpha)\,\mathcal{W}(\mathbf{F})\,d\Omega + \int_\Omega 3p(1-\alpha)^2(J-1)d\Omega + \frac{G_c}{c_w}\int_\Omega \left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right] d\Omega$$

Therefore we have three equations:
First is mechanical eq,

$$\frac{\partial \mathcal{W}(\mathbf{F})}{\partial u_i} = 0$$
$$\frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{W}}{\partial \epsilon_{ij}}\right) = 0$$
$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

Second is an equation for pressure,

$$-\frac{p}{\kappa} - \sqrt{a^3(\alpha)}(J-1) = 0$$
$$-\frac{p}{\kappa} - (1-\alpha)^3(J-1) = 0$$
$$-\kappa(J-1)(1-\alpha)^3 = p$$

Lastly,

$$-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) + 3p(1-\alpha)^2(J-1) + \frac{G_c}{c_w}\left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right] = 0$$

Substitute second equation into third

$$-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) - 3\kappa(1-\alpha)^5(J-1)^2 + \frac{G_c}{c_w}\left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right] = 0$$

5.1 Homogeneous Solution

We can study the homogeneous solution by ignoring spatial derivatives of α. If we don't substitute p:

$$-2(1-\alpha_h)\,\mathcal{W}(\mathbf{F}) + 3p(1-\alpha_h)2(J-1) + \frac{G_c}{c_w\ell} = 0$$

or if we substitute pressure

$$-2(1-\alpha_h)\,\mathcal{W}(\mathbf{F}) - 3\kappa(1-\alpha_h)^5(J-1)^2 + \frac{G_c}{c_w\ell} = 0$$

5.2 Non-Homogeneous Solution

Now for the Non-homogenous solution, we have the following

$$-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) + 3p(1-\alpha)^2(J-1) + \frac{G_c}{c_w}\left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right] = 0$$

Multiply by $d\alpha/dx$

$$\begin{aligned}
\frac{d\alpha}{dx}\Bigg[-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) + 3p(1-\alpha)^2(J-1) + \frac{G_c}{c_w}\left(\frac{1}{\ell} + 2\ell\nabla^2\alpha\right)\Bigg] &= 0\\
\frac{d}{dx}\int\Bigg[-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) + 3p(1-\alpha)^2(J-1) + \frac{G_c}{c_w}\left(\frac{1}{\ell} + 2\ell\nabla^2\alpha\right)\Bigg]d\alpha &= 0\\
\frac{d}{dx}\Bigg[(1-\alpha)^2\,\mathcal{W}(\mathbf{F}) - p(1-\alpha)^3(J-1) + \frac{G_c}{c_w}\left(\frac{\alpha}{\ell} + 2\ell\nabla^2\alpha\right)\Bigg] &= 0
\end{aligned}$$

now integrate from x to infinity

$$\begin{aligned}
\Bigg[(1-\alpha)^2\,\mathcal{W}(\mathbf{F}) - p(1-\alpha)^3(J-1) + \frac{G_c}{c_w}\left(\frac{\alpha}{\ell} + 2\ell\nabla^2\alpha\right)\Bigg]\Bigg|_0^\infty &= 0\\
(1-\alpha)^2\,\mathcal{W}(\mathbf{F}) - p(1-\alpha)^3(J-1) + \frac{G_c}{c_w}\left(\frac{\alpha}{\ell} + 2\ell\nabla^2\alpha\right)&\\
-\Bigg[(1-\alpha_h)^2\,\mathcal{W}(\mathbf{F}) - p(1-\alpha_h)^3(J-1) + \alpha_h\frac{G_c}{c_w\ell}\Bigg] &= 0
\end{aligned}$$

with some rearrangement we can call the bracketed section

$$a_{hom} = (1-\alpha_h)^2\,\mathcal{W}(\mathbf{F}) - p(1-\alpha_h)^3(J-1) + \alpha_h\frac{G_c}{c_w\ell} \tag{5.1}$$

which can yield an expression that can solve for the phase field profile

$$\begin{aligned}
-(1-\alpha)^2\,\mathcal{W}(\mathbf{F}) + p(1-\alpha)^3(J-1) - \frac{G_c}{c_w}\frac{\alpha}{\ell} + \Big[a_{hom}\Big] &= 2\ell\nabla^2\alpha\frac{G_c}{c_w}\\
\frac{c_w}{2\ell G_c}\Big[-(1-\alpha)^2\,\mathcal{W}(\mathbf{F}) + p(1-\alpha)^3(J-1)\Big] - \frac{\alpha}{2\ell^2} + \frac{c_w}{2\ell G_c}\Big[a_{hom}\Big] &= \frac{d^2\alpha}{dx^2}
\end{aligned}$$

This expression needs to be non-dimensionalized accurately in order to be plotted