Stabilized Mixed Finite Element Formulation

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Definitions

$\mathbf{F}$: Deformation gradient
$\mathbf{I}$: second-order unit tensor
$\boldsymbol{u}$: Displacement
J: determinant of the deformation gradient
$\mathbf{C}$: Right Cauchy-Green Strain Tensor
$\mathcal{W}(\mathbf{F})$: strain energy function
$\mathbf{P}$: first Piola-Kirchhoff stress tensor
$\mathbf{S}$: second PK stress tensor

α: cracks are represented by a scalar phase-field variable
p: Lagrange multiplier, hydrostatic pressure field
κ: bulk modulus

$$\kappa = \frac{E}{3(1-2\nu)} \tag{0.1}$$

μ: shear modulus

$$\mu = \frac{E}{2(1+\nu)} \tag{0.2}$$

λ: Lamé modulus

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{0.3}$$

$\mathcal{E}_\ell$: potential energy functional $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume
c_w is a normalization constant

1 Hyperelastic Phase-Field Fracture Models

Deformation Gradient

$$\mathbf{F} = \mathbf{I} + \nabla \otimes \boldsymbol{u} \tag{1.1}$$

where $J = \det \mathbf{F}$ and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

The strain energy function $\mathcal{W}(\mathbf{F})$ is defined per unit reference volume such that the first PK and second PK

$$\mathbf{P} = \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \tag{1.2a}$$

$$\mathbf{S} = 2\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{C}} \tag{1.2b}$$

where $\mathbf{P} = \mathbf{F}\mathbf{S}$.

1.1 Phase-Field Fracture Model

Non-modified strain energy function is the compressible Neo-Hookean:

$$\mathcal{W}(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2\ln J) \tag{1.3}$$

For incompressible hyperelastic materials, the strain energy function is defined as

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p\,(J-1)\,, \tag{1.4}$$

We instead consider a damage dependent relaxation of the incompressibility constraint and the phase-field couples to the strain energy through the modified function

$$\widetilde{\mathcal{W}}(\mathbf{F},\alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa\,(J-1)^2 \tag{1.5}$$

where the decreasing stiffness modulation function and $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

$$a(\alpha) = (1-\alpha)^2 \quad w(\alpha) = \alpha \tag{1.6}$$

In the code we have the following definition

$$b(\alpha) = (1-\alpha)^3 = \sqrt{a^3(\alpha)}$$

To circumvent numerical difficulties, we resort to the classical mixed formulation and introduce the pressure-like field

$$p = -\sqrt{a^3(\alpha)}\kappa\,(J-1)\,, \tag{1.7}$$

as an independent variable along with the displacement field.

Lastly, the normalization constant is defined as:

$$c_w = \int_0^1 \sqrt{w(\alpha)}d\alpha \tag{1.8}$$

The first Piola-Kirchhoff stress tensor is given:

$$\begin{aligned}
\mathbf{P} &= \frac{\partial \tilde{\mathcal{W}}(\mathbf{F},\alpha)}{\partial \mathbf{F}} \\
&= \frac{\partial}{\partial \mathbf{F}}\left[a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa\,(J-1)^2\right] \\
&= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\frac{1}{2}\kappa\frac{\partial (J-1)^2}{\partial \mathbf{F}} \\
\mathbf{P} &= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa\,(J-1)\,\frac{\partial J}{\partial \mathbf{F}} \quad \text{where } \frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T} \\
\mathbf{P} &= a(\alpha)\frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa(J-1)J\mathbf{F}^{-T}
\end{aligned} \tag{1.9}$$

Note that we DO NOT consider this form of the modified functional; therefore, we do not consider this PK stress.

1.1.1 According to the rough draft

Therefore our modified function:

$$
\begin{aligned}
\widetilde{\mathcal{W}}(\mathbf{F},\alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J-1)^2 \\
&= a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\frac{-p}{\sqrt{a^3(\alpha)}}(J-1) \\
&= a(\alpha)\mathcal{W}(\mathbf{F}) - \sqrt{a^3(\alpha)}\frac{p}{2}(J-1) \\
\widetilde{\mathcal{W}}(\mathbf{F},\alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)\frac{p}{2}(J-1)
\end{aligned}
$$

In the rough draft of the paper we have Eq. 5: energy functional of a possibly fractured elastic body with isotropic surface energy

$$
\begin{aligned}
\mathcal{E}_\ell(\boldsymbol{u},\alpha) &= \int_\Omega \widetilde{\mathcal{W}}(\mathbf{F},\alpha)d\Omega + \frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_{\partial_N\Omega}\tilde{\mathbf{g}}_{\mathbf{0}}\cdot\mathbf{u}dA \\
\mathcal{E}_\ell(\boldsymbol{u},p,\alpha) &= \int_\Omega\left[a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)\frac{p}{2}(J-1)\right]d\Omega + \frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega \\
\mathcal{E}_\ell(\boldsymbol{u},p,\alpha) &= \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega - \int_\Omega b(\alpha)\frac{p}{2}(J-1)d\Omega + \frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega
\end{aligned}
$$

Note that we are missing one term.

1.1.2 According to the code

In the code we have the following for the energy functional of the energy problem

$$
\widetilde{W}(\mathbf{F},\alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}
$$

Energy functional, where we ignore the surface term

$$
\begin{aligned}
\mathcal{E}_\ell(\boldsymbol{u},\alpha) &= \int_\Omega \widetilde{\mathcal{W}}(\mathbf{F},\alpha)d\Omega + \frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_{\partial_N\Omega}\tilde{\mathbf{g}}_{\mathbf{0}}\cdot\mathbf{u}dA \\
&= \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega - \int_\Omega b(\alpha)p(J-1)d\Omega - \int_\Omega\frac{p^2}{2\lambda}d\Omega + \frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega
\end{aligned}
$$

1.1.3 According to a derivation from Bin2020 Paper

Versus Eq. 21 where we drop λ_b which is not a consideration in this formulation

$$
\mathcal{E}_\ell(\boldsymbol{u},\alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F},\alpha)d\Omega + \frac{G_c}{c_w}\int_\Omega\left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega \tag{1.10}
$$

Starting from Eq. 25 in the 2020 Li and Bouklas paper where κ is the bulk modulus

$$\begin{aligned}
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \Lambda, \alpha\right) &= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega \Lambda(p + \sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \quad \Lambda = -p/\kappa \\
&= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega -\frac{p}{\kappa}(p + \sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \\
&= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \frac{p^2}{\kappa} d\Omega - \int_\Omega \frac{p}{\kappa}\sqrt{a^3(\alpha)}\kappa(J-1))d\Omega \\
\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) &= \mathcal{E}_\ell(\boldsymbol{u}, \alpha) - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1))d\Omega
\end{aligned} \tag{1.11}$$

Substitute in $\mathcal{E}_\ell(\boldsymbol{u}, \alpha)$ and substitute Eq. 1.3

$$\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega$$

After taking the directional derivative of the prior equation, we can introduce the stabilization term

$$-\frac{\varpi h^2}{2\mu}\sqrt{a^3(\alpha)}\sum_{e=1}^{n_{el}}\int_{\Omega^e} J\mathbf{C}^{-1} : \left(\nabla p \cdot \nabla q\right) dV = 0$$

1.2 Changes for 2D Plane-Stress Models

Following the code, we have the following energy functional of the energy problem

$$\begin{aligned}
\widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda} \\
\widetilde{W}(\mathbf{F}, \alpha) &= \left(a(\alpha) + k_\ell\right)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}
\end{aligned}$$

Changes to relieve the residual stresses

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - (\alpha^2 + k_\ell)\frac{p^2}{2\lambda} \tag{1.12}$$

Where we know k_ℓ is a modeling parameter, so we can list the energy functional as:

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \alpha^2\frac{p^2}{2\lambda} \tag{1.13}$$

Derive the 1st PK stress, the change to the last term does not affect this derivation:

$$\begin{aligned}
\mathbf{P} &= \frac{\partial\widetilde{W}(\mathbf{F})}{\partial\mathbf{F}} \\
&= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T}) - b(\alpha)p\frac{\partial \det\mathbf{F}}{\partial\mathbf{F}} \\
\mathbf{P} &= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T}) - b(\alpha)pJ\mathbf{F}^{-T}
\end{aligned}$$

Taking the third component to be zero, in the plane stress case

$$\begin{aligned}
\mathbf{P}_{33} = a(\alpha)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
(1-\alpha)^2\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1-\alpha)^3 pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1-\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mathbf{F}_{33} - \mathbf{F}_{33}^{-1} - \frac{(1-\alpha)pJ}{\mu}\mathbf{F}_{33}^{-1} &= 0 \\
\mathbf{F}_{33}\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}\mathbf{F}_{33} - \frac{(1-\alpha)pJ}{\mu}\mathbf{F}_{33}^{-1}\mathbf{F}_{33} &= 0 \\
\mathbf{F}_{33}^2 - 1 - \frac{(1-\alpha)pJ}{\mu} &= 0
\end{aligned}$$

This can be multiplied by its associated test function to obtain the weak form

$$\int_\Omega \left(\mathbf{F}_{33}^2 - 1 - \frac{(1-\alpha)pJ}{\mu}\right) v_{F_{33}} dV = 0$$

1.2.1 Changes for 2D Discrete Crack Model

If we are considering a discrete fracture method

$$\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega$$

$$\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha) = \int_\Omega \mathcal{W}(\mathbf{F})d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega p(J-1)d\Omega$$

with the stabilization term and plane stress in the weak form

$$\begin{aligned}
-\frac{\varpi h^2}{2\mu}\int_\Omega J\mathbf{C}^{-1} : \left(\nabla p \cdot \nabla q\right) dV &= 0 \\
\int_\Omega \left(\mathbf{F}_{33}^2 - 1 - \frac{pJ}{\mu}\right) v_{F_{33}} dV &= 0
\end{aligned}$$

where we have assumed for the energy functional

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = \frac{\mu}{2}(I_c - 3 - 2\ln J) - p(J-1) - \frac{p^2}{2\lambda}$$

Therefore, we can calculate the 1st Piola Kirchoff Stress as:

$$\begin{aligned}
\mathbf{P} &= \frac{\mu}{2}(2\mathbf{F} - \frac{2}{J}J\mathbf{F}^{-T}) - pJ\mathbf{F}^{-T} \\
&= \mu(\mathbf{F} - \mathbf{F}^{-T}) - pJ\mathbf{F}^{-T}
\end{aligned}$$

Taking the third component to be zero

$$\begin{aligned}
P_{33} &= \mu(F_{33} - F_{33}^{-1}) - pJF_{33}^{-1} = 0 \\
&= F_{33} - F_{33}^{-1} - \frac{pJ}{\mu}F_{33}^{-1} = 0 \\
P_{33} &= F_{33}^2 - 1 - \frac{pJ}{\mu} = 0
\end{aligned}$$

1.2.2 Changes for 2D displacement formulation

Removing pressure terms

$$\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega$$
$$\mathcal{E}_\ell\left(\boldsymbol{u}, p, \alpha\right) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega$$

with plane stress in the weak form (no need for stabilization terms)

$$\int_\Omega \left(a(\alpha)\mu(F_{33} - F_{33}^{-1})\right)v_{F_{33}}dV = 0$$

We have assumed the modified energy functional

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) \tag{1.14}$$

Therefore, we can calculate the 1st Piola Kirchoff Stress as:

$$\begin{aligned}\mathbf{P} &= a(\alpha)\frac{\mu}{2}(2\mathbf{F} - \frac{2}{J}J\mathbf{F}^{-T}) \\ &= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-T})\end{aligned}$$

Taking the third component to be zero

$$P_{33} = a(\alpha)\mu(F_{33} - F_{33}^{-1}) = 0$$

2 Following Borden: Derivations of Analytical Phase Field

Note the full potential energy functional, which can also be called the lagrangian

$$\mathcal{E}_\ell(\boldsymbol{u}, p, \alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega - \int_\Omega \frac{p^2}{2\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1)d\Omega + \frac{G_c}{c_w}\int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2\right)d\Omega$$

We can use the Euler-Lagrange equations to arrive at the equations of motion by taking the derivative with respect to displacement, pressure, and the scalar damage field. Starting with displacement:

$$\frac{\partial\mathcal{E}_\ell}{\partial\boldsymbol{u}} = \int_\Omega a(\alpha)\frac{\partial\mathcal{W}(\mathbf{F})}{\partial\mathbf{u}}d\Omega$$

$$\frac{\partial\mathcal{E}_\ell}{\partial p} = -\int_\Omega \frac{p}{\kappa}d\Omega - \int_\Omega \sqrt{a^3(\alpha)}(J-1)d\Omega$$

$$\frac{\partial\mathcal{E}_\ell}{\partial\alpha} = -\int_\Omega 2(1-\alpha)\,\mathcal{W}(\mathbf{F})\,d\Omega + \int_\Omega 3p(1-\alpha)^2(J-1)d\Omega + \frac{G_c}{c_w}\int_\Omega \left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right]d\Omega$$

Therefore we have three equations:
First should be mechanical eq, second is a an equation for pressure,

$$
\begin{aligned}
-\frac{p}{\kappa} - \sqrt{a^3(\alpha)}(J-1) = 0 \\
-\frac{p}{\kappa} - (1-\alpha)^3(J-1) = 0 \\
-\kappa(J-1)(1-\alpha)^3 = p
\end{aligned}
$$

Lastly,

$$
-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) + 3p(1-\alpha)^2(J-1) + \frac{G_c}{c_w}\left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right] = 0
$$

Substitute second equation into third

$$
-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) - 3\kappa(1-\alpha)^5(J-1)^2 + \frac{G_c}{c_w}\left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right] = 0
$$

2.1 Homogeneous Solution

Therefore, we can study the homogeneous solution by ignoring spatial derivatives of α

$$
-2(1-\alpha_h)\,\mathcal{W}(\mathbf{F}) - 3\kappa(1-\alpha_h)^5(J-1)^2 + \frac{G_c}{c_w\ell} = 0
$$

We can expand and arrange

$$
\begin{aligned}
-2\mathcal{W}(\mathbf{F}) + 2\alpha_h\mathcal{W}(\mathbf{F}) - 3\kappa(J-1)^2(1 - 5\alpha_h + 10\alpha_h^2 - 10\alpha_h^3 + 5\alpha_h^4 - \alpha_h^5) + \frac{G_c}{c_w\ell} = 0 \\
-2\mathcal{W}(\mathbf{F}) + 2\alpha_h\mathcal{W}(\mathbf{F}) + \frac{G_c}{c_w\ell} \\
-3\kappa(J-1)^2 + 15\kappa(J-1)^2\alpha_h - 30\kappa(J-1)^2\alpha_h^2 + 30\kappa(J-1)^2\alpha_h^3 - 15\kappa(J-1)^2\alpha_h^4 + \kappa(J-1)^2\alpha_h^5 = 0 \\
-2\mathcal{W}(\mathbf{F}) - 3\kappa(J-1)^2 + \frac{G_c}{c_w\ell} + \left[2\mathcal{W}(\mathbf{F}) + 15\kappa(J-1)^2\right]\alpha_h \\
-30\kappa(J-1)^2\alpha_h^2 + 30\kappa(J-1)^2\alpha_h^3 - 15\kappa(J-1)^2\alpha_h^4 + \kappa(J-1)^2\alpha_h^5 = 0
\end{aligned}
$$

2.2 Non-Homogeneous Solution

Now for the Non-homogenous solution, we have the following

$$
-2(1-\alpha)\,\mathcal{W}(\mathbf{F}) - 3\kappa(1-\alpha)^5(J-1)^2 + \frac{G_c}{c_w}\left[\frac{1}{\ell} + 2\ell\nabla^2\alpha\right] = 0
$$

$$
\tag{2.1}
$$