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Elastic surface–substrate interactions

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A theory for three-dimensional finite deformations of elastic solids with conforming elastic films attached to their bounding surfaces is described. The Gurtin–Murdoch theory incorporating elastic resistance of the film to strain is generalized to account for the effects of intrinsic flexural resistance. This modification yields a model that can be used to describe equilibrium deformations in the presence of compressive-surface stress fields. An associated variational theory is given and material symmetry considerations are discussed. The theory is illustrated by examples.

Keywords: thin films; nonlinear elasticity; quasi-convexity; shell theory; surface stress; finite deformations

1. Introduction

Gurtin & Murdoch (1975) presented a rigorous theory of the mechanics of surface-stressed solids based on the idea of a two-dimensional membrane bonded to the surface of a bulk substrate material. Their work generalizes the classical notion of surface tension in solids and allows for the systematic theoretical description of general states of residual surface stress. In principle, this theory may be used to describe the mechanical behaviour of a bulk substrate material coated with a thin film of a different substance. It is also sufficiently general to accommodate inelastic response and dynamical effects.

The Gurtin–Murdoch theory was motivated in part by empirical observations pointing to the presence of compressive surface stress in certain types of crystals. It was later used by Andreussi & Gurtin (1977) to model wrinkling at the free surface of a solid as a bifurcation of the undeformed configuration induced by compressive residual surface stress. Steigmann & Ogden (1997*a*) have reconsidered the latter problem in the context of a plane-strain theory for elastic solids with surface energies that incorporate local elastic resistance to flexure in addition to the strain resistance included in the Gurtin–Murdoch theory. The findings reveal a marked departure from the earlier predictions, including the presence of dispersion due to an intrinsic length-scale associated with the constitutive relations of the surface. A variational argument was also given demonstrating the impossibility of compressive surface stress in energy-minimizing configurations of the entire film–substrate combination. Results like these are to be expected in view of the fact that flexural resistance singularly perturbs the membrane model equations and regularizes the associated variational problem. From the physical viewpoint it may therefore be argued that a surface model that does not account for flexural resistance cannot be used to simulate local surface features engendered by the response of solids to compressive surface stress of any magnitude.

In the present work, we consider a theory for the coupled three-dimensional deformations of elastic solids with elastic films attached to their bounding surfaces. Following Gurtin & Murdoch (1975), we model the surface film as a two-dimensional elastic continuum without accounting explicitly for three-dimensional effects. However, we generalize their work by incorporating elastic resistance to flexure of the film at the constitutive level. Thus, the film may be regarded as a shell on an elastic foundation. Conventional treatments of film–substrate combinations are usually based on shell theories derived from the three-dimensional elastic properties of the material of which the film is composed. In this regard we note the substantial progress achieved in the justification of such theories through the use of formal asymptotic expansions (see, for example, Fox *et al*. 1993), or, alternatively, through the use of Γ-convergence theory (De Giorgi & Dal Maso 1983) to characterize the limit of a sequence of variational problems for a thin body with thickness tending to zero (Le Dret & Raoult 1995, 1996). Here, however, we have in mind potential applications to systems in which the film may not occur naturally in bulk, as in the condensation, cooling, and solidification of films from the vapour phase of a particular substance. In such circumstances the properties of the solid film may differ markedly from the bulk properties of the same substance. The suitability of a theory derived from three-dimensional considerations would then be open to question.

The model of the surface may be based on a number of direct two-dimensional shell theories in common use. We adopt the simplest possible description valid for finite deformations and compatible with general mechanical principles. Thus, we consider the film to be an elastic surface that resists changes in its metric and the curvature associated with its embeddings in three-dimensional space. Such a model is well known in principle, being similar to theories put forth by Sanders (1963), Cohen & DeSilva (1966) and Pietraskiewicz (1989). It is also roughly equivalent to a special constrained theory of Cosserat surfaces (Naghdi 1972). Apart from the inclusion of mechanical coupling with elastic substrates, the contribution of the present work to this subject consists primarily of the derivation of certain necessary conditions for energy-minimizing configurations of the surface in the presence of the substrate material and the adaptation of a general theory of material symmetry for elastic surfaces due to Murdoch & Cohen (1979) to derive new canonical forms for the constitutive equations.

Section 2 contains a brief summary of the kinematics of surfaces convected by the deformation of a substrate material. These results are used in § 3 to introduce the basic constitutive hypotheses of the theory and in § 4 to derive the Euler equations for a class of variational problems. There we discuss the generalization to curved surfaces of the formalism adopted by Hilgers & Pipkin (1992*a*) for flat plates. In § 5 pointwise algebraic necessary conditions for energy-minimizing film–substrate systems that are analogous to the well-known Legendre–Hadamard condition of conventional elasticity are given. The latter are preceded by derivations of integral inequalities of the quasi-convexity type (Ball 1977) that apply to the surface film and the substrate material separately. The various inequalities extend those obtained by Steigmann & Ogden (1997*a*) for the corresponding plane-strain problem and, in particular, indicate that compressive surface stress in pure-membrane films without flexural resistance is incompatible with the requirement that a configuration be energy minimizing. By contrast, no such restrictions are found if flexural resistance is taken into account.

In § 6 the extension of Noll's (1958) theory of material symmetry to elastic surfaces, as presented by Murdoch & Cohen (1979), is summarized and used to obtain new constitutive equations for films having *hemitropic* symmetry. Our view is that the Murdoch–Cohen theory is the most logically appealing among the various alternative symmetry theories for elastic surfaces that have been advanced, and that it furnishes a particularly satisfactory framework for the description of the response of film–substrate systems. To our knowledge this theory has not been used in the subsequent literature on developments in shell theory. This may be due to the use of a notation that, while direct, is non-standard, and perhaps not readily accessible to shell theorists. For this reason we outline the main features of the theory in standard notation and present a number of supplementary remarks pertaining to its interpretation and use. Comparisons with alternative formulations are also given. Finally, in § 7, we discuss the application of the theory to the solution of simple finite-deformation problems, the detailed analyses of which will be given elsewhere.

2. Kinematics of surface–substrate interactions

Let $\boldsymbol{X}$ be the position of a typical particle of the bulk solid in a reference configuration Ω in three-dimensional space. We take the boundary $\partial\Omega$ of the region Ω to be piecewise smooth. The deformation of the solid is defined by the continuous, continuously differentiable and invertible map $\boldsymbol{\chi}$, the values of which are the positions $\boldsymbol{x}$ of the particles under the deformation

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}), \qquad \boldsymbol{X} \in \Omega. \tag{2.1}$$

In this work we assume that $\boldsymbol{\chi}$ is at least twice continuously differentiable in $\Omega \cup \partial\Omega$.

The gradient of the deformation is the second-order tensor $\boldsymbol{A}(\boldsymbol{X})$ defined by $\mathrm{d}\boldsymbol{\chi} = \boldsymbol{A}\mathrm{d}\boldsymbol{X}$. Thus,

$$\boldsymbol{A}(\boldsymbol{X}) = \operatorname{grad}\boldsymbol{\chi}(\boldsymbol{X}) = A_{ia}(\boldsymbol{X})\boldsymbol{e}_i \otimes \boldsymbol{e}_a, \qquad A_{ia} = \frac{\partial \chi_i}{\partial X_a}, \qquad i, a \in \{1, 2, 3\}, \tag{2.2}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a fixed orthonormal basis for 3-space, and $X_a = \boldsymbol{X} \cdot \boldsymbol{e}_a$, $\chi_i = \boldsymbol{\chi} \cdot \boldsymbol{e}_i$ are the Cartesian coordinates of a particle in its reference position and of the deformation function, respectively. Here, the notation $\boldsymbol{a} \otimes \boldsymbol{b}$ is used to denote the standard tensor product defined by $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$ for arbitrary vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.

We consider the bulk solid to be coated with an elastic surface on a smooth part P of its boundary $\partial\Omega$, which is parametrized locally by coordinates θ^α, $\alpha \in \{1, 2\}$, covering an open neighbourhood of a point. In general the surface may be identified with the union of the closures of such neighbourhoods (Whitney 1957). Then, the position function $\boldsymbol{Y}$ on the coated surface is identified with the restriction of $\boldsymbol{X}$ to P. Thus,

$$\boldsymbol{Y}(\theta^1, \theta^2) = \boldsymbol{X}|_P. \tag{2.3}$$

This surface is assumed to be convected by the deformation of the bulk solid so that its image $\boldsymbol{\chi}(P)$ under the deformation admits the local parametrization,

$$\boldsymbol{y}(\theta^1, \theta^2) = \boldsymbol{\chi}(\boldsymbol{Y}(\theta^1, \theta^2)), \tag{2.4}$$

in which the coordinates θ^α are regarded as identifying the same particle before and after deformation. These coordinates induce the tangent vectors,

$$\boldsymbol{G}_\alpha = \boldsymbol{Y}_{,\alpha} \in T_P, \qquad \boldsymbol{g}_\alpha = \boldsymbol{y}_{,\alpha} \in T_{\boldsymbol{\chi}(P)}, \qquad \alpha \in \{1,2\}, \tag{2.5}$$

where T_P and $T_{\boldsymbol{\chi}(P)}$ are the tangent planes to the surfaces P and $\boldsymbol{\chi}(P)$, respectively, at the particle in question, and $_{,\alpha}$ is used to denote the partial derivative with respect to θ^α.

It follows from (2.2), (2.4) and (2.5) that

$$\boldsymbol{g}_\alpha = \boldsymbol{A}(\boldsymbol{Y})\boldsymbol{G}_\alpha, \qquad \boldsymbol{g}_{\alpha,\beta} = \boldsymbol{K}(\boldsymbol{Y})[\boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\beta] + \boldsymbol{A}(\boldsymbol{Y})\boldsymbol{G}_{\alpha,\beta}, \tag{2.6}$$

where

$$\boldsymbol{K}(\boldsymbol{X}) = \left(\frac{\partial A_{ia}}{\partial X_b}\right)\boldsymbol{e}_i \otimes \boldsymbol{e}_a \otimes \boldsymbol{e}_b, \tag{2.7}$$

a third-order tensor field, is the gradient of $\boldsymbol{A}(\boldsymbol{X})$. Such tensors have the property

$$(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c})\boldsymbol{d} = (\boldsymbol{c} \cdot \boldsymbol{d})\boldsymbol{a} \otimes \boldsymbol{b}, \tag{2.8}$$

wherein the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are arbitrary, and in (2.6) we use the bilinear operation defined by

$$(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c})[\boldsymbol{d} \otimes \boldsymbol{e}] = (\boldsymbol{b} \cdot \boldsymbol{d})(\boldsymbol{c} \cdot \boldsymbol{e})\boldsymbol{a}. \tag{2.9}$$

Alternatively, the derivatives of the tangent vectors are given by the Gauss formulae,

$$\boldsymbol{g}_{\alpha,\beta} = \Gamma^\gamma_{\alpha\beta}\boldsymbol{g}_\gamma + b_{\alpha\beta}\boldsymbol{n}, \qquad \boldsymbol{G}_{\alpha,\beta} = \bar{\Gamma}^\gamma_{\alpha\beta}\boldsymbol{G}_\gamma + B_{\alpha\beta}\boldsymbol{N}, \tag{2.10}$$

where

$$\Gamma^\gamma_{\alpha\beta} = \boldsymbol{g}^\gamma \cdot \boldsymbol{g}_{\alpha,\beta}, \qquad \bar{\Gamma}^\gamma_{\alpha\beta} = \boldsymbol{G}^\gamma \cdot \boldsymbol{G}_{\alpha,\beta} \tag{2.11}$$

are the Christoffel symbols on the deformed and reference surfaces,

$$b_{\alpha\beta} = \boldsymbol{n} \cdot \boldsymbol{g}_{\alpha,\beta}, \qquad B_{\alpha\beta} = \boldsymbol{N} \cdot \boldsymbol{G}_{\alpha,\beta} \tag{2.12}$$

are the normal curvatures associated with the embedding of the surfaces in the underlying three-dimensional space, $\{\boldsymbol{g}^\alpha\}$ and $\{\boldsymbol{G}^\alpha\}$ are the dual tangent vectors induced by the coordinates θ^α, and $\boldsymbol{n}$ and $\boldsymbol{N}$ are the oriented unit normals to $T_{\boldsymbol{\chi}(P)}$ and T_P, respectively. These are given by

$$\boldsymbol{g}^\alpha = g^{\alpha\beta}\boldsymbol{g}_\beta, \qquad \boldsymbol{G}^\alpha = G^{\alpha\beta}\boldsymbol{G}_\beta \tag{2.13}$$

and

$$\boldsymbol{n} = \tfrac{1}{2}\varepsilon^{\alpha\beta}\boldsymbol{g}_\alpha \times \boldsymbol{g}_\beta, \qquad \boldsymbol{N} = \tfrac{1}{2}\mu^{\alpha\beta}\boldsymbol{G}_\alpha \times \boldsymbol{G}_\beta, \tag{2.14}$$

where

$$\left.\begin{array}{c} g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha_\beta = G^{\alpha\gamma}G_{\gamma\beta}, \qquad g_{\alpha\beta} = \boldsymbol{g}_\alpha \cdot \boldsymbol{g}_\beta, \qquad G_{\alpha\beta} = \boldsymbol{G}_\alpha \cdot \boldsymbol{G}_\beta, \\ \varepsilon^{\alpha\beta}\sqrt{g} = e^{\alpha\beta} = \mu^{\alpha\beta}\sqrt{G}, \qquad \varepsilon_{\alpha\beta}/\sqrt{g} = e_{\alpha\beta} = \mu_{\alpha\beta}/\sqrt{G}. \end{array}\right\} \tag{2.15}$$

In (2.15), $g = \det(g_{\alpha\beta})$, $G = \det(G_{\alpha\beta})$, δ^α_β is the Kronecker delta, and $e^{\alpha\beta} = e_{\alpha\beta}$ is the unit alternator ($e^{12} = -e^{21} = 1$, $e^{11} = e^{22} = 0$). It is evident that g and G are non-negative, and that the foregoing variables are uniquely defined if $g > 0$ and $G > 0$.

The relationships amongst the various coefficients associated with P and $\boldsymbol{\chi}(P)$ in (2.10) may be inferred upon substitution into (2.6). It is also useful to have a description of the deformation of P that does not involve the deformation of the bulk solid. For example, it follows from (2.5) that

$$\boldsymbol{g}_\alpha = F_{i\alpha}\boldsymbol{e}_i, \tag{2.16}$$

where $F_{i\alpha} = y_{i,\alpha}$ and $y_i(\theta^1, \theta^2) = \boldsymbol{e}_i \cdot \boldsymbol{y}$ are the Cartesian coordinates of a point on $\boldsymbol{\chi}(P)$. By writing $F_{i\alpha} = F_{i\beta}\delta^\beta_\alpha$ and $\delta^\beta_\alpha = \boldsymbol{G}^\beta \cdot \boldsymbol{G}_\alpha$, the latter following from the second equalities in (2.13) and (2.15), we obtain

$$\boldsymbol{g}_\alpha = \boldsymbol{F}\boldsymbol{G}_\alpha, \tag{2.17}$$

where

$$\boldsymbol{F} = F_{i\beta}\boldsymbol{e}_i \otimes \boldsymbol{G}^\beta = \boldsymbol{g}_\beta \otimes \boldsymbol{G}^\beta \tag{2.18}$$

is the surface-deformation gradient. This maps T_P to $T_{\boldsymbol{\chi}(P)}$. Thus, $\mathrm{d}\boldsymbol{y} = \boldsymbol{F}\mathrm{d}\boldsymbol{Y}$, where $\mathrm{d}\boldsymbol{y} = \boldsymbol{g}_\alpha \mathrm{d}\theta^\alpha$ and $\mathrm{d}\boldsymbol{Y} = \boldsymbol{G}_\alpha \mathrm{d}\theta^\alpha$. We emphasize that although $\boldsymbol{F}$ and $\boldsymbol{A}(\boldsymbol{Y})$ map vectors on T_P in the same way, they have different domains and ranges and are therefore not the same function.

The deformation tensor associated with $\boldsymbol{F}$ is the surface tensor on T_P given by

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = g_{\alpha\beta}\boldsymbol{G}^\alpha \otimes \boldsymbol{G}^\beta, \tag{2.19}$$

where the superscript 'T' is used to denote transposition.

The surface gradient of $\boldsymbol{F}$ is the third-order tensor $\nabla\boldsymbol{F}$, with domain T_P, defined by $\mathrm{d}\boldsymbol{F} = (\nabla\boldsymbol{F})\,\mathrm{d}\boldsymbol{Y}$, where $\mathrm{d}\boldsymbol{F} = \boldsymbol{F}_{,\alpha}\,\mathrm{d}\theta^\alpha = \boldsymbol{F}_{,\alpha}(\boldsymbol{G}^\alpha \cdot \mathrm{d}\boldsymbol{Y})$. Thus, $\nabla\boldsymbol{F} = \boldsymbol{F}_{,\alpha} \otimes \boldsymbol{G}^\alpha$. This may be related to $\boldsymbol{A}(\boldsymbol{Y})$ and $\boldsymbol{K}(\boldsymbol{Y})$ by substitution of (2.17) and use of (2.6). Alternatively, equations (2.17), (2.10) and (2.11) may be combined to obtain

$$\nabla\boldsymbol{F} = \boldsymbol{F}(S^\gamma_{\beta\alpha}\boldsymbol{G}_\gamma \otimes \boldsymbol{G}^\beta \otimes \boldsymbol{G}^\alpha + B_{\beta\alpha}\boldsymbol{G}^\beta \otimes \boldsymbol{N} \otimes \boldsymbol{G}^\alpha) - \boldsymbol{n} \otimes \boldsymbol{\kappa}, \tag{2.20}$$

where

$$S^\gamma_{\alpha\beta} = \mathit{\Gamma}^\gamma_{\alpha\beta} - \bar{\mathit{\Gamma}}^\gamma_{\alpha\beta} \tag{2.21}$$

and the *relative curvature* $\boldsymbol{\kappa}$ is defined by

$$\boldsymbol{\kappa} = \kappa_{\alpha\beta}\boldsymbol{G}^\alpha \otimes \boldsymbol{G}^\beta, \qquad \kappa_{\alpha\beta} = -b_{\alpha\beta}. \tag{2.22}$$

Here, we use the convention that $\boldsymbol{F}(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c}) = (\boldsymbol{F}\boldsymbol{a}) \otimes \boldsymbol{b} \otimes \boldsymbol{c}$ for arbitrary vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ with $\boldsymbol{a} \in T_P$, and the minus sign in (2.22) is inserted to facilitate later comparison with certain formulae in Naghdi (1972). In the course of deriving (2.20) we used the Gauss equation

$$\boldsymbol{G}^\alpha_{,\beta} = -\bar{\mathit{\Gamma}}^\alpha_{\gamma\beta}\boldsymbol{G}^\gamma + B^\alpha_\beta\boldsymbol{N}, \tag{2.23}$$

where $B^\alpha_\beta = G^{\alpha\gamma}B_{\gamma\beta}$.

Finally, it is useful to note that

$$\boldsymbol{\kappa} = -\boldsymbol{F}^{\mathrm{T}}\boldsymbol{b}\boldsymbol{F}, \tag{2.24}$$

where

$$\boldsymbol{b} = b_{\alpha\beta}\boldsymbol{g}^\alpha \otimes \boldsymbol{g}^\beta \tag{2.25}$$

is the tensor curvature of the surface $\boldsymbol{\chi}(P)$. We may interpret $\boldsymbol{\kappa}$ in physical terms by considering a curve on P defined by the arclength parametrization $\theta^\alpha(S)$. The positions of points of the curve along its image on the deformed surface $\boldsymbol{\chi}(P)$ are the values of the function $\hat{\boldsymbol{y}}(S) = \boldsymbol{y}(\theta^\alpha(S))$. If $\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}$ are the unit tangents to the curve on P and $\boldsymbol{\chi}(P)$, respectively, then $\boldsymbol{\lambda}_0 = (\mathrm{d}\theta^\alpha/\mathrm{d}S)\boldsymbol{G}_\alpha$ and

$$\mu\boldsymbol{\lambda} = \hat{\boldsymbol{y}}'(S) = \boldsymbol{y}_{,\alpha}\left(\frac{\mathrm{d}\theta^\alpha}{\mathrm{d}S}\right) = \boldsymbol{g}_\alpha(\boldsymbol{G}^\alpha \cdot \boldsymbol{\lambda}_0) = \boldsymbol{F}\boldsymbol{\lambda}_0, \tag{2.26}$$

where

$$\mu = |\hat{\boldsymbol{y}}'(S)| = (\boldsymbol{\lambda}_0 \cdot \boldsymbol{C}\boldsymbol{\lambda}_0)^{1/2} \tag{2.27}$$

is the *stretch* of the curve induced by the deformation. The normal curvature of the curve on $\boldsymbol{\chi}(P)$ is $c = \boldsymbol{\lambda} \cdot \boldsymbol{b}\boldsymbol{\lambda}$. According to (2.24) and (2.26), this may be written

$$c = -\mu^{-2}\boldsymbol{\lambda}_0 \cdot \boldsymbol{\kappa}\boldsymbol{\lambda}_0. \tag{2.28}$$

Thus, $\boldsymbol{\kappa}$ furnishes the curvature of a convected curve scaled by the square of its stretch.

3. Strain energy

(*a*) *Basic hypotheses*

To describe the mechanical response of the coated solid we assign strain-energy functions W, per unit volume of Ω, and U, per unit area of $P \subset \partial\Omega$, respectively. These are taken to depend on appropriate kinematic measures that characterize the local properties of the function $\boldsymbol{\chi}(\cdot)$ in Ω and on P. The total strain energy of the coated solid is the functional of $\boldsymbol{\chi}$ defined by

$$S[\boldsymbol{\chi}] = \int_\Omega W\,\mathrm{d}V + \int_P U\,\mathrm{d}A. \tag{3.1}$$

We adopt the usual assumption that the bulk solid is *simple* in the sense that W depends on the local deformation only through its gradient $\boldsymbol{A}$. It is well known that W is then invariant under superposed rigid-body motions if and only if the variable $\boldsymbol{A}$ appears in the combination $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

The surface analogue of the simple material furnishes a model of the mechanical behaviour of ideal membranes (see, for example, Stoker 1964; Cohen & Wang 1984). These are perfectly flexible surfaces that respond to variations in the surface-deformation gradient $\boldsymbol{F}$ but are entirely insensitive to other kinematic measures that characterize the local geometry of the deformation, such as strain gradients and variations in curvature. The latter measures are determined by $\nabla\boldsymbol{F}$. Their inclusion in the list of arguments of U renders the surface non-simple. The study of the general theory of non-simple deformable surfaces, as distinct from the so-called *director* or Cosserat theories (Naghdi 1972), was initiated by Balaban *et al*. (1967), although a restricted form of the model for elastic surfaces in which strain-gradient effects were suppressed was developed earlier by Cohen & DeSilva (1966). The latter authors subsequently generalized their model to incorporate these effects (Cohen & DeSilva 1968). The development of the subject was then largely suspended until the associated theory of material symmetry was presented by Murdoch & Cohen (1979). An alternative treatment of the constitutive theory for elastic surfaces with resistance to strain and flexure may be found in Simmonds (1985, § 2). More recently, the subject has been taken up independently by Hilgers & Pipkin (1992*a*, *b*, 1993, 1996) for the special case of elastic surfaces with reference configurations that are flat or developable onto the plane. This series of papers together comprises a thorough account of a number of mathematical and physical aspects of the subject. Dynamic effects have been examined by Hilgers (1997).

In the present work, we seek to model non-simple effects in elastic surfaces only to the extent of incorporating elastic resistance to flexure in addition to the strain resistance associated with membrane behaviour. Our reasons for this are (1) we envisage applications of the theory in which the effects of flexure may reasonably be assumed to play the more important role, as in the flexure-dominated bifurcation response of coated solids; (2) the restricted theory is far more tractable than the general theory in which gradient effects are taken into account; and (3) the constitutive aspects of the gradient theory involve material-symmetry considerations that are extremely complex (Murdoch & Cohen 1979; Murdoch 1979; Elzanowski & Epstein 1992). The latter considerations seem to have been overlooked in most of the literature on strain-gradient effects in elasticity.

A further appreciation of the difficulties attending the use of the full gradient theory may be gained from inspection of expression (2.20) above for $\nabla \boldsymbol{F}$. This expression renders explicit equation (4.4) of Murdoch & Cohen (1979) in terms of familiar variables from the conventional differential geometry of surfaces, albeit at the expense of foregoing a *direct* notation, and furnishes the appropriate tensorial generalization of the kinematic measure used by Hilgers & Pipkin (1992*a*), who restricted their development to Cartesian coordinates. It is evident that the correlation with experiment of a surface theory in which general dependence on $\nabla \boldsymbol{F}$ is included would require the experimenter to gain simultaneous control over a prohibitively large number of tensor variables associated with the reference and deformed geometries of the surface. Having said this, a theory in which a *restricted* dependence is included, the form of which is guided by physical principle, may prove useful in the modelling of a number of non-standard effects. These might include, for example, the geodesic bending of fibrous surfaces (Wang & Pipkin 1986) and the mechanics of disclinations in liquid-crystal films.

For our present purposes, then, it suffices to consider a special version of the theory, substantially equivalent to that originally proposed by Cohen & DeSilva (1966), in which the strain energy per unit area of P is given by a function

$$U(\boldsymbol{C}, \boldsymbol{\kappa}; \theta^1, \theta^2). \tag{3.2}$$

This is automatically insensitive to rigid motions superposed on the configuration $\boldsymbol{\chi}(P)$. In general, this function, like $W(\boldsymbol{A})$, may depend explicitly on the particle coordinates, although we normally suppress reference to such dependence for the sake of notational simplicity. Some reasons for this explicit dependence are discussed in § 6 below.

(*b*) *Incremental surface energy*

As a prelude to the discussion of the variational theory of § 4 we turn now to the derivation of formulae for the increment of the surface strain energy induced by incremental changes in the various kinematic quantities on which it depends. These in turn are expressed in terms of the incremental displacements and their gradients.

Consider a one-parameter family of placements $\boldsymbol{\chi}(\boldsymbol{X}; \epsilon)$, with $\epsilon \in (-\epsilon_0, \epsilon_0)$ for some constant $\epsilon_0 > 0$. For each value of the parameter in this interval the deformation of points on P is given by $\boldsymbol{y}(\theta^1,\theta^2;\epsilon) = \boldsymbol{\chi}(\boldsymbol{Y}(\theta^1,\theta^2);\epsilon)$, and the attendant strain and curvature are determined from (2.19) and (2.22), in which ϵ enters through the components $g_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. Thus,

$$\dot{U} = \frac{\partial U}{\partial g_{\alpha\beta}}\dot{g}_{\alpha\beta} + \frac{\partial U}{\partial \kappa_{\alpha\beta}}\dot{\kappa}_{\alpha\beta}, \tag{3.3}$$

where the superposed dot is used to denote the value of the derivative with respect to ϵ at some particular value, $\epsilon = 0$ say, and the partial derivatives are evaluated at the associated values of $\boldsymbol{C}$ and $\boldsymbol{\kappa}$.

Let $\boldsymbol{u}(\theta^1,\theta^2) = \dot{\boldsymbol{y}}$. Then, according to (2.15),

$$\dot{g}_{\alpha\beta} = \boldsymbol{g}_\alpha \cdot \boldsymbol{u}_{,\beta} + \boldsymbol{g}_\beta \cdot \boldsymbol{u}_{,\alpha} \tag{3.4}$$

and the first term in (3.3) may be written

$$\frac{\partial U}{\partial g_{\alpha\beta}}\dot{g}_{\alpha\beta} = J\sigma^{\alpha\beta}\boldsymbol{g}_\beta \cdot \boldsymbol{u}_{,\alpha}, \tag{3.5}$$

where

$$J\sigma^{\alpha\beta} \equiv \frac{\partial U}{\partial g_{\alpha\beta}} + \frac{\partial U}{\partial g_{\beta\alpha}} \tag{3.6}$$

and the factor

$$J \equiv (g/G)^{1/2} = (\det \boldsymbol{C})^{1/2} \tag{3.7}$$

has been inserted for the sake of later convenience. This factor furnishes the local ratio $\mathrm{d}a/\mathrm{d}A$ of elemental material areas on the deformed and reference surfaces, the former corresponding to the configuration $\epsilon = 0$.

The reduction of the second term in (3.3) proceeds by combining (2.12), (2.14), (2.22) and (3.7) to obtain

$$\dot{\kappa}_{\alpha\beta} = (\dot{J}/J)b_{\alpha\beta} - J^{-1}(Jb_{\alpha\beta})^{\cdot}, \tag{3.8}$$

where

$$\frac{\dot{J}}{J} = \frac{1}{2g}\frac{\partial g}{\partial g_{\alpha\beta}}\dot{g}_{\alpha\beta} = \tfrac{1}{2}g^{\alpha\beta}\dot{g}_{\alpha\beta} = \boldsymbol{g}^\alpha \cdot \boldsymbol{u}_{,\alpha}, \tag{3.9}$$

$$(Jb_{\alpha\beta})^{\cdot} = (J\boldsymbol{n})^{\cdot} \cdot \boldsymbol{y}_{,\alpha\beta} + J\boldsymbol{n} \cdot \boldsymbol{u}_{,\alpha\beta} \tag{3.10}$$

and, in the last expression, the derivative of $J\boldsymbol{n}$ is

$$(J\boldsymbol{n})^{\cdot} = J\varepsilon^{\alpha\beta}\boldsymbol{u}_{,\alpha} \times \boldsymbol{g}_\beta. \tag{3.11}$$

The combination of these results with the identity $\varepsilon^{\alpha\beta}\varepsilon_{\lambda\beta} = \delta^\alpha_\lambda$ and the Gauss equations (2.10), together with the cyclic symmetry of scalar triple products, yields

$$\dot{\kappa}_{\alpha\beta} = -\boldsymbol{n} \cdot \boldsymbol{u}_{;\alpha\beta}, \tag{3.12}$$

where

$$\boldsymbol{u}_{;\alpha\beta} = \boldsymbol{u}_{,\alpha\beta} - \Gamma^\lambda_{\alpha\beta}\boldsymbol{u}_{,\lambda} \tag{3.13}$$

is the covariant derivative of $\boldsymbol{u}_{,\alpha}$ on $\boldsymbol{\chi}(P)$.

The incremental energy (3.3) may now be written in the form

$$\dot{U} = J\sigma^{\alpha\beta}\boldsymbol{g}_\beta \cdot \boldsymbol{u}_{,\alpha} - Jm^{\alpha\beta}\boldsymbol{n} \cdot \boldsymbol{u}_{;\alpha\beta}, \tag{3.14}$$

where

$$Jm^{\alpha\beta} \equiv \frac{\partial U}{\partial \kappa_{\alpha\beta}}. \tag{3.15}$$

The variables $\sigma^{\alpha\beta}$ and $m^{\alpha\beta}$ defined by (3.6) and (3.15) are interpreted in terms of edge tractions and couples in § 4.

It is apparent from (3.3) or (3.14) that only the symmetric part of $m^{\alpha\beta}$ contributes to the incremental strain energy. Since all the equations that describe the response of the elastic surface are derived from this energy, we may, without loss of generality, identify $m^{\alpha\beta}$ with its symmetric part (Cohen & DeSilva 1966; Naghdi 1972; Hilgers & Pipkin 1992*a*) and thus regard the right-hand side of (3.15) as being equivalent to

$$\tfrac{1}{2}\left(\frac{\partial U}{\partial \kappa_{\alpha\beta}} + \frac{\partial U}{\partial \kappa_{\beta\alpha}}\right). \tag{3.16}$$

For the considerations of § 4 it is convenient to express $\dot{U}$ as a linear combination of the partial derivatives of $\boldsymbol{u}$ rather than its covariant derivatives. To this end we define

$$\boldsymbol{T}^{\alpha} \equiv J(\sigma^{\alpha\beta}\boldsymbol{g}_{\beta} + m^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta}\boldsymbol{n}), \qquad \boldsymbol{M}^{\alpha\beta} \equiv -Jm^{\alpha\beta}\boldsymbol{n}. \tag{3.17}$$

Thus,

$$\dot{U} = \boldsymbol{T}^{\alpha}\cdot\boldsymbol{u}_{,\alpha} + \boldsymbol{M}^{\alpha\beta}\cdot\boldsymbol{u}_{,\alpha\beta}. \tag{3.18}$$

It is of interest to observe that the latter formula implies that

$$\boldsymbol{T}^{\alpha} = \frac{\partial U}{\partial F_{i\alpha}}\boldsymbol{e}_i, \qquad \boldsymbol{M}^{\alpha\beta} = \frac{\partial U}{\partial F_{i\alpha,\beta}}\boldsymbol{e}_i, \tag{3.19}$$

where $F_{i\alpha}$ are the components of the surface-deformation gradient defined in (2.16). Here, U is regarded as a function of these components and their *partial* derivatives with respect to the surface coordinates θ^{α} in addition to parameters associated with the geometry of the fixed surface P. In view of the compatibility condition $F_{i\alpha,\beta} = F_{i\beta,\alpha}$, we regard the second equation in (3.19) as defining the symmetric part of $\boldsymbol{M}^{\alpha\beta}$. The skew part is assumed to vanish, without loss of generality, for the reasons indicated previously.

Equations (3.19) reduce to equations (4.1) in Hilgers & Pipkin (1992*a*), apart from minor differences in notation, upon specialization to the Cartesian coordinates used by those authors to parametrize their reference plane. It is apparent from (2.10) and (2.16) that the $F_{i\alpha,\beta}$ involve the Christoffel symbols on $\boldsymbol{\chi}(P)$ in such a way that they do not transform as components of surface tensors under general coordinate transformations. For this reason neither of the two collections of products in (3.18) is an absolute scalar, although their *sum*, the energy increment, *is* an absolute scalar. We draw attention to this point because our examination of the literature (Cohen & DeSilva 1966, 1968; Balaban *et al*. 1967) suggests that concerns about the tensorial properties of the individual terms in expressions for the incremental energy may have led previous investigators to forego analytically tractable decompositions such as (3.18) in favour of formulae like (3.14), with consequent increases in complexity elsewhere in the development of the theory. This issue does not arise in the work of Hilgers & Pipkin (1992*a*) because the Christoffel symbols associated with distortion of the embedded Cartesian coordinates transform as tensor components under *affine* coordinate transformations, that is under transformations from one Cartesian coordinate system to another. The appropriate tensorial generalization of their theory, with strain-gradient effects incorporated, would involve the Christoffel symbols in the combination $S^{\gamma}_{\alpha\beta}$ defined by (2.21). These observations notwithstanding, it is always possible to express the *total* energy increment in the form (3.18), provided that the dependence of U on $F_{i\alpha}$ and $F_{i\alpha,\beta}$ implied by (3.19) is interpreted appropriately.

The use of partial derivatives rather than covariant derivatives in the formulation of shell theory has also been advocated by Ericksen (1971) and Carrive-Bédouani *et al*. (1995).

4. Stationary energy and equilibrium

In order to obtain the equilibrium equations for the surface–substrate combination, we assume that equilibria are described by the Euler equations associated with the problem of minimizing a potential energy, E, defined by

$$E[\boldsymbol{\chi}] = S[\boldsymbol{\chi}] + L[\boldsymbol{\chi}], \tag{4.1}$$

where $S[\boldsymbol{\chi}]$ is the strain energy defined in (3.1) and $L[\boldsymbol{\chi}]$ is a load potential. This approach entails a restriction to conservative boundary-value problems, but nevertheless furnishes an appropriate form of the principle of virtual work whether or not the material is elastic or the boundary-value problem conservative. Postulated forms of the latter principle form the basis of earlier treatments of theories of the kind considered here (Cohen & DeSilva 1966, 1968; Balaban *et al*. 1967; Pietraszkiewicz 1989). The present approach has the advantage of identifying certain classes of problems as conservative at the outset. These in turn enjoy certain mathematical properties, such as self-adjointness of the associated differential operators, the *a priori* knowledge of which may be advantageous from the viewpoint of analysis. Such properties are particularly important in the study of potential instabilities associated with the bifurcation of equilibrium configurations, for example.

(*a*) *Euler equations*

To derive the Euler equations for (4.1) we first analyse the Gâteaux differential of the surface strain energy. Thus, let $\boldsymbol{y}(\theta^1, \theta^2; \epsilon)$ be a one-parameter family of deformations of the surface P, and let superposed dots denote the values of derivatives with respect to ϵ at $\epsilon = 0$. Then, according to (3.18),

$$\int_P \dot{U}\,\mathrm{d}A = \int_P (\boldsymbol{T}^{\alpha} \cdot \boldsymbol{u}_{,\alpha} + \boldsymbol{M}^{\alpha\beta} \cdot \boldsymbol{u}_{,\alpha\beta})\,\mathrm{d}A, \tag{4.2}$$

where $\boldsymbol{u}(\theta^1, \theta^2) = \dot{\boldsymbol{y}}$ is assumed to be at least twice differentiable.

To advance the analysis of (4.2) we use the Green–Stokes theorem in the form,

$$\int_P G^{-1/2}(G^{1/2}\psi^{\alpha})_{,\alpha}\,\mathrm{d}A = \oint_{\partial P} \psi^{\alpha}\nu_{\alpha}\,\mathrm{d}S, \tag{4.3}$$

where $\psi^{\alpha}(\theta^1, \theta^2)$, $\alpha \in \{1, 2\}$, are arbitrary smooth functions on P, S measures arclength on ∂P, $\nu_{\alpha} = \boldsymbol{\nu} \cdot \boldsymbol{G}_{\alpha}$, and

$$\boldsymbol{\nu} = \boldsymbol{\tau} \times \boldsymbol{N}, \qquad \boldsymbol{\tau} = \frac{\mathrm{d}}{\mathrm{d}S}\boldsymbol{Y}(\theta^{\alpha}(S)). \tag{4.4}$$

Here, $\boldsymbol{\tau}$ is the unit tangent to ∂P in the direction of increasing S and $\boldsymbol{\nu}$ is the unit normal to ∂P, defined such that $\{\boldsymbol{\nu}, \boldsymbol{\tau}, \boldsymbol{N}\}$ forms a right-handed triad. We assume ∂P to be a C^1 curve, so that $\nu_\alpha(S)$ and $\tau_\alpha(S)$ are continuous. Generalizations to piecewise-C^1 curves are possible and will be considered elsewhere. These are expected to give rise to singularities in the surface–substrate interaction of a non-standard type at the corners of ∂P where ν_α and τ_α are discontinuous.

Let $\psi^\alpha = \boldsymbol{T}^\alpha \cdot \boldsymbol{u}$. Then, the first term in (4.2) may be written

$$\int_P \boldsymbol{T}^\alpha \cdot \boldsymbol{u}_{,\alpha} \,\mathrm{d}A = \oint_{\partial P} \boldsymbol{u} \cdot \boldsymbol{T}^\alpha \nu_\alpha \,\mathrm{d}S - \int_P G^{-1/2}(G^{1/2}\boldsymbol{T}^\alpha)_{,\alpha} \cdot \boldsymbol{u} \,\mathrm{d}A. \tag{4.5}$$

Alternatively, let $\psi^\alpha = \boldsymbol{M}^{\alpha\beta} \cdot \boldsymbol{u}_{,\beta}$ and define

$$\boldsymbol{L}^\alpha = G^{-1/2}(G^{1/2}\boldsymbol{M}^{\beta\alpha})_{,\beta}. \tag{4.6}$$

We use this in (4.5), with $\boldsymbol{L}^\alpha$ substituted in place of $\boldsymbol{T}^\alpha$, to reduce the second term in (4.2) to

$$\int_P \boldsymbol{M}^{\alpha\beta} \cdot \boldsymbol{u}_{,\alpha\beta} \,\mathrm{d}A = \oint_{\partial P} (\boldsymbol{M}^{\alpha\beta} \cdot \boldsymbol{u}_{,\beta} - \boldsymbol{L}^\alpha \cdot \boldsymbol{u})\nu_\alpha \,\mathrm{d}S + \int_P G^{-1/2}(G^{1/2}\boldsymbol{L}^\alpha)_{,\alpha} \cdot \boldsymbol{u} \,\mathrm{d}A. \tag{4.7}$$

Then, (4.2) becomes

$$\int_P \dot{U} \,\mathrm{d}A = \oint_{\partial P} (\boldsymbol{P}^\alpha \cdot \boldsymbol{u} + \boldsymbol{M}^{\alpha\beta} \cdot \boldsymbol{u}_{,\beta})\nu_\alpha \,\mathrm{d}S - \int_P G^{-1/2}(G^{1/2}\boldsymbol{P}^\alpha)_{,\alpha} \cdot \boldsymbol{u} \,\mathrm{d}A, \tag{4.8}$$

where

$$\boldsymbol{P}^\alpha = \boldsymbol{T}^\alpha - \boldsymbol{L}^\alpha. \tag{4.9}$$

To invoke the fundamental lemma of variational calculus, we must first express the derivatives $\boldsymbol{u}_{,\alpha}$ on ∂P as combinations of terms that can be assigned independently, namely the tangential and normal derivatives of $\boldsymbol{u}$. To this end we use the orthonormality of $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ to write

$$\boldsymbol{G}_\alpha = \tau_\alpha \boldsymbol{\tau} + \nu_\alpha \boldsymbol{\nu}, \qquad \boldsymbol{G}^\alpha = \tau^\alpha \boldsymbol{\tau} + \nu^\alpha \boldsymbol{\nu}, \tag{4.10}$$

where

$$\tau_\alpha = \boldsymbol{\tau} \cdot \boldsymbol{G}_\alpha, \quad \nu_\alpha = \boldsymbol{\nu} \cdot \boldsymbol{G}_\alpha, \quad \tau^\alpha = \boldsymbol{\tau} \cdot \boldsymbol{G}^\alpha, \quad \nu^\alpha = \boldsymbol{\nu} \cdot \boldsymbol{G}^\alpha. \tag{4.11}$$

Then,

$$\boldsymbol{u}_{,\beta} \otimes \boldsymbol{G}^\beta = (\tau^\beta \boldsymbol{u}_{,\beta}) \otimes \boldsymbol{\tau} + (\nu^\beta \boldsymbol{u}_{,\beta}) \otimes \boldsymbol{\nu}. \tag{4.12}$$

It follows from (4.4) that $\tau^\alpha = \mathrm{d}\theta^\alpha/\mathrm{d}S$ on ∂P. Thus, $\tau^\alpha \boldsymbol{u}_{,\alpha}$ is the derivative of $\boldsymbol{u}(\theta^\alpha(S))$ with respect to S. We denote this by $\boldsymbol{u}'(S)$. Similarly, $\boldsymbol{u}_\nu(S) \equiv \nu^\alpha \boldsymbol{u}_{,\alpha}|_{\partial P}$ is the restriction of the *normal* derivative of $\boldsymbol{u}(\theta^\alpha)$ to ∂P, and we obtain

$$\boldsymbol{u}_{,\alpha} = (\boldsymbol{u}_{,\beta} \otimes \boldsymbol{G}^\beta)\boldsymbol{G}_\alpha = \tau_\alpha \boldsymbol{u}' + \nu_\alpha \boldsymbol{u}_\nu \qquad \text{on } \partial P. \tag{4.13}$$

With reference to (4.8) we now have

$$\boldsymbol{M}^{\alpha\beta} \cdot \boldsymbol{u}_{,\beta}\nu_\alpha = (\boldsymbol{M}^{\alpha\beta}\nu_\alpha\tau_\beta) \cdot \boldsymbol{u}' + (\boldsymbol{M}^{\alpha\beta}\nu_\alpha\nu_\beta) \cdot \boldsymbol{u}_\nu, \tag{4.14}$$

and

$$\int_P \dot{U}\,\mathrm{d}A = \oint_{\partial P}[\boldsymbol{P}^\alpha\nu_\alpha - (\boldsymbol{M}^{\alpha\beta}\nu_\alpha\tau_\beta)']\cdot\boldsymbol{u}\,\mathrm{d}S + \oint_{\partial P}(\boldsymbol{M}^{\alpha\beta}\nu_\alpha\nu_\beta)\cdot\boldsymbol{u}_\nu\,\mathrm{d}S - \int_P G^{-1/2}(G^{1/2}\boldsymbol{P}^\alpha)_{,\alpha}\cdot\boldsymbol{u}\,\mathrm{d}A, \quad (4.15)$$

wherein we have invoked the assumed continuity of $\boldsymbol{M}^{\alpha\beta}\nu_\alpha\tau_\beta$ on ∂P.

The Gâteaux differential of the strain energy of the substrate material may be written in the form

$$\int_\Omega \dot{W}\,\mathrm{d}V = \oint_{\partial\Omega}\boldsymbol{PN}\cdot\boldsymbol{u}\,\mathrm{d}A - \int_\Omega \boldsymbol{u}\cdot\operatorname{div}\boldsymbol{P}\,\mathrm{d}V, \quad (4.16)$$

where $\boldsymbol{u}(\boldsymbol{X}) = \dot{\boldsymbol{\chi}}$, $\boldsymbol{N}$ is the exterior unit normal to $\partial\Omega$,

$$\boldsymbol{P} = P_{ia}\boldsymbol{e}_i\otimes\boldsymbol{e}_a \quad (4.17)$$

is the Piola stress with components

$$P_{ia} = \frac{\partial W}{\partial A_{ia}}, \quad (4.18)$$

and

$$\operatorname{div}\boldsymbol{P} = \frac{\partial P_{ia}}{\partial X_a}\boldsymbol{e}_i \quad (4.19)$$

is its divergence with respect to position $\boldsymbol{X}\in\Omega$.

We assume that the Gâteaux differential of the load potential admits the representation

$$\dot{L} = -\int_{\partial\Omega_t}\boldsymbol{t}\cdot\boldsymbol{u}\,\mathrm{d}A, \quad (4.20)$$

where $\partial\Omega_t$ is a subset of $\partial\Omega$ and $\boldsymbol{t}$ is a prescribed vector-valued function. Examples include (a) dead loading, with $\boldsymbol{t}$ a function of $\boldsymbol{X}$ only, (b) applied pressure of intensity p, with $\boldsymbol{t} = -p\boldsymbol{A}^*\boldsymbol{N}$, where $\boldsymbol{A}^* = (\det\boldsymbol{A})(\boldsymbol{A}^{-1})^{\mathrm{T}}$ is the adjugate of $\boldsymbol{A}$, and (i) $\partial\Omega_t$ is a closed surface with p a function of $\boldsymbol{\chi}$, as in the case of a hydrostatic pressure, or of the enclosed volume (Fisher 1988; Steigmann 1991; Bufler & Schneider 1994), and (ii) p is fixed and position is assigned along the entire edge of $\partial\Omega_t$ (Pearson 1956; Hill 1962; Steigmann 1992). In the non-conservative case, the right-hand side of (4.20) is to be interpreted simply as the virtual work of the boundary tractions.

The foregoing formulae are substituted into the expression for the Gâteaux differential of the total energy E obtained from (3.1) and (4.1). The fundamental lemma then implies that E is stationary at a deformation $\boldsymbol{\chi}(\boldsymbol{X})$ for $\boldsymbol{X}\in\Omega\cup\partial\Omega$ if and only if

$$\left.\begin{aligned} \operatorname{div}\boldsymbol{P} &= \boldsymbol{0} && \text{in } \Omega,\\ \boldsymbol{PN} &= \boldsymbol{t} && \text{on } \partial\Omega_t,\\ \boldsymbol{PN} &= G^{-1/2}(G^{1/2}\boldsymbol{P}^\alpha)_{,\alpha} && \text{on } P,\\ \boldsymbol{P}^\alpha\nu_\alpha &= (\boldsymbol{M}^{\alpha\beta}\nu_\alpha\tau_\beta)' && \text{on } \partial P,\\ \boldsymbol{M}^{\alpha\beta}\nu_\alpha\nu_\beta &= \boldsymbol{0} && \text{on } \partial P. \end{aligned}\right\} \quad (4.21)$$

In the event that P and $\partial\Omega_t$ overlap, as is often the case in practice, the right-hand sides of the second or third equation are replaced by the sum of the right-hand sides of these two equations, the result then applying on $P \cap \partial\Omega_t$.

The first and second of equations (4.21) are the familiar equilibrium equation and traction boundary condition of conventional elasticity. The third equation couples the response of the elastic surface to that of the substrate. It may be interpreted as requiring that the traction transmitted by the surface to the substrate be equal and opposite to the distributed force exerted by the substrate on the surface. The fourth condition is equivalent to the statement that no forces are applied to the edge ∂P of the surface. It generalizes the corresponding Kirchhoff boundary condition of classical plate theory (Kirchhoff 1850; Pietraszkiewicz 1989; Hilgers & Pipkin 1992*a*).

The last of equations (4.21) requires that the bending couple vanishes on ∂P. To establish this we suppose for the sake of illustration that rigid rotations of the body are kinematically possible, and evaluate the Gâteaux differential of the total energy with respect to the one-parameter family of configurations defined by $\boldsymbol{\chi}(\boldsymbol{X};\epsilon) = \boldsymbol{Q}(\epsilon)\boldsymbol{\chi}(\boldsymbol{X})$, where $\boldsymbol{Q}$ is a rotation ($\det \boldsymbol{Q} = 1$, $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\mathrm{T}}$), and $\boldsymbol{Q}(0) = \boldsymbol{I}$, the identity in three dimensions. The ϵ-derivative of $\boldsymbol{\chi}$ at $\epsilon = 0$ is $\boldsymbol{u} = \dot{\boldsymbol{\chi}} = \boldsymbol{\omega} \times \boldsymbol{\chi}$, where $\boldsymbol{\omega}$ is the axial vector of the skew tensor $\dot{\boldsymbol{Q}}$.

Let $\boldsymbol{m}(S)$ be defined on ∂P such that the virtual work of the applied loads is

$$\dot{L} = -\int_{\partial\Omega_t} \boldsymbol{t} \cdot \boldsymbol{u} \, \mathrm{d}A - \oint_{\partial P} \boldsymbol{m} \cdot \boldsymbol{u}_\nu \, \mathrm{d}S \tag{4.22}$$

in place of (4.20), where $\boldsymbol{u}_\nu$ is the normal derivative of $\boldsymbol{u}$ on ∂P. Since the strain energies of the substrate and the surface are invariant under rotations, it follows that the total energy is stationary at $\epsilon = 0$ for arbitrary $\boldsymbol{\omega}$ only if

$$\int_{\partial\Omega_t} \boldsymbol{\chi} \times \boldsymbol{t} \, \mathrm{d}A + \oint_{\partial P} \boldsymbol{c} \, \mathrm{d}S = \boldsymbol{0}, \tag{4.23}$$

where

$$\boldsymbol{c} = \boldsymbol{y}_\nu \times \boldsymbol{m} \tag{4.24}$$

and $\boldsymbol{y} = \boldsymbol{\chi}(\boldsymbol{Y})$. Here, $\boldsymbol{y}_\nu = \nu^\alpha \boldsymbol{y}_{,\alpha}$ is the normal derivative of $\boldsymbol{y}$ on ∂P. This result is equivalent to the statement that the net moment on the body vanishes. It is automatically satisfied by any solution of the equilibrium equations. Moreover, it is evident that $\boldsymbol{c}$ is the couple per unit length of ∂P acting on $\boldsymbol{\chi}(\partial P)$.

A repetition of the argument leading to (4.21), with (4.22) substituted in place of (4.20), yields

$$\boldsymbol{M}^{\alpha\beta}\nu_\alpha\nu_\beta = \boldsymbol{m} \qquad \text{on } \partial P \tag{4.25}$$

in place of the last of equations (4.21). The latter equation may thus be interpreted as implying that the couple vanishes on ∂P (Hilgers & Pipkin 1992*a*).

We note that for (4.25) to apply in general $\boldsymbol{m}$ must be directed along the normal $\boldsymbol{n}$ to the deformed surface in accordance with (3.17). The couple $\boldsymbol{c}$ then lies in the deformed tangent plane at ∂P and thus generates a pure-bending couple. For prescribed (non-zero) $\boldsymbol{m}$ this amounts to a restriction on the orientation of the deformed surface along its edge such that the bending couple $\boldsymbol{c}$, which is *not* prescribed, is conservative. Such a restriction may be difficult to achieve in practice. A similar situation exists regarding the application of conservative moments to spatial rods (Ziegler 1956; Steigmann & Faulkner 1993). In the present context the special case associated with the prescription $\boldsymbol{m} = \boldsymbol{0}$ presents no such difficulties.

A comprehensive analysis of conservative edge loads for shell theories of the type considered here has been given by Makowski & Pietraszkiewicz (1989), whose formulation incorporates the fourth and fifth of equations (4.21) and (4.25) as special cases. This subject is also discussed at length in Libai & Simmonds (1998).

(b) *Component forms*

To provide for the further interpretation of the various terms in (4.21) and to facilitate comparison with alternative formulations we represent the vectors $\boldsymbol{P}^\alpha$ and $\boldsymbol{M}^{\alpha\beta}$ in terms of components relative to the basis $\{\boldsymbol{g}_\alpha, \boldsymbol{n}\}$. To this end let

$$\boldsymbol{p}^\alpha = J^{-1}\boldsymbol{P}^\alpha, \tag{4.26}$$

where J is defined in (3.7). Then

$$G^{-1/2}(G^{1/2}\boldsymbol{P}^\alpha)_{,\alpha} = Jg^{-1/2}(g^{1/2}\boldsymbol{p}^\alpha)_{,\alpha} = J\boldsymbol{p}^\alpha_{;\alpha}, \tag{4.27}$$

where

$$\boldsymbol{p}^\alpha_{;\beta} = \boldsymbol{p}^\alpha_{,\beta} + \boldsymbol{p}^\lambda \Gamma^\alpha_{\lambda\beta} \tag{4.28}$$

is the covariant derivative of $\boldsymbol{p}^\alpha$ on $\boldsymbol{\chi}(P)$ and we have made use of the standard identity $\Gamma^\alpha_{\lambda\alpha} = (\sqrt{g})_{,\lambda}/\sqrt{g}$.

Next, we use (4.6) and the second of equations (3.17) to write

$$\boldsymbol{L}^\alpha = -J(m^{\beta\alpha}_{,\beta} + m^{\beta\alpha}\Gamma^\lambda_{\lambda\beta})\boldsymbol{n} + Jm^{\beta\alpha}b_{\beta\lambda}\boldsymbol{g}^\lambda, \tag{4.29}$$

wherein we have used the result $\boldsymbol{n}_{,\beta} = -b_{\beta\lambda}\boldsymbol{g}^\lambda$, which follows from the first of (2.12) and the orthogonality of $\{\boldsymbol{g}_\alpha\}$ and $\boldsymbol{n}$. Substitution into (4.9) and (4.26) yields

$$\boldsymbol{p}^\alpha = p^{\alpha\beta}\boldsymbol{g}_\beta + q^\alpha\boldsymbol{n} \tag{4.30}$$

with

$$p^{\alpha\beta} \equiv \sigma^{\alpha\beta} - m^{\lambda\alpha}b^\beta_\lambda, \qquad q^\alpha \equiv m^{\beta\alpha}_{;\beta}, \tag{4.31}$$

where $b^\lambda_\beta = g^{\lambda\alpha}b_{\alpha\beta}$,

$$m^{\alpha\beta}_{;\gamma} = m^{\alpha\beta}_{,\gamma} + m^{\lambda\beta}\Gamma^\alpha_{\lambda\gamma} + m^{\alpha\lambda}\Gamma^\beta_{\lambda\gamma}, \tag{4.32}$$

is the covariant derivative of $m^{\alpha\beta}$ and the first of equations (3.17) has been used.

On combining the third of equations (4.21) with (4.27) and (4.30) and simplifying the result with the aid of the formulae

$$\boldsymbol{g}_{\beta;\alpha} \equiv \boldsymbol{g}_{\beta,\alpha} - \Gamma^\lambda_{\beta\alpha}\boldsymbol{g}_\lambda = b_{\beta\alpha}\boldsymbol{n}, \qquad \boldsymbol{n}_{;\alpha} = \boldsymbol{n}_{,\alpha} = -b^\beta_\alpha\boldsymbol{g}_\beta, \tag{4.33}$$

we obtain

$$\boldsymbol{PN} = J\boldsymbol{p}^\alpha_{;\alpha} \qquad \text{on} \quad P, \tag{4.34}$$

where

$$\boldsymbol{p}^\alpha_{;\alpha} = (p^{\alpha\beta}_{;\alpha} - q^\alpha b^\beta_\alpha)\boldsymbol{g}_\beta + (q^\alpha_{;\alpha} + p^{\alpha\beta}b_{\alpha\beta})\boldsymbol{n}, \tag{4.35}$$

in which the covariant derivatives appearing on the right-hand side are defined as in (4.28) and (4.32).

The pure-membrane formulation of Gurtin & Murdoch (1975) is recovered by taking q^α and $m^{\alpha\beta}$ to be identically zero and the surface strain energy to be independent of $\boldsymbol{\kappa}$. In this case the foregoing variational theory generalizes that of Podio-Guidugli & Vergara Caffarelli (1990), who restricted their considerations to membranes of the classical surface-tension type. In the context of the present theory their model is recovered from the surface energy $U = cJ$, with c a positive constant. The corresponding membrane stress, obtained from (3.6), (3.7) and the formula for the derivative of the determinant of a matrix with respect to its components is $\sigma^{\alpha\beta} = cg^{\alpha\beta}$. This is the isotropic stress state associated with classical soap-film theory in which the constant c plays the role of surface tension.

To compare the foregoing with alternative formulations of shell theory we first note that $\boldsymbol{P}^\alpha \nu_\alpha \mathrm{d}S$ represents a force transmitted across an infinitesimal arc $\mathrm{d}S$ of a curve on P (see (4.15)). Let $\mathrm{d}s$ be the element of arc associated with the image of the curve on $\boldsymbol{\chi}(P)$, and let $\hat{\boldsymbol{\nu}} = \hat{\nu}_\alpha \boldsymbol{g}^\alpha$ be the associated unit normal. Then $\hat{\nu}_\alpha \mathrm{d}s = J\nu_\alpha \mathrm{d}S$ (Steigmann & Pipkin 1991), and from (4.26) it follows that the same force per unit length of the deformed curve is

$$\boldsymbol{p}^\alpha \hat{\nu}_\alpha = p^{\beta\alpha} \hat{\nu}_\beta \boldsymbol{g}_\alpha + q^\alpha \hat{\nu}_\alpha \boldsymbol{n}. \tag{4.36}$$

We conclude that $p^{\alpha\beta}$ and q^α furnish, respectively, the tangential and shear components of the curve force per unit length.

In the case of no surface–substrate interaction, equations (4.34) and (4.35) require that $\boldsymbol{p}^\alpha_{;\alpha} = \boldsymbol{0}$, or, equivalently,

$$p^{\alpha\beta}_{;\alpha} - q^\alpha b^\beta_\alpha = 0, \qquad q^\alpha_{;\alpha} + p^{\alpha\beta} b_{\alpha\beta} = 0 \qquad \text{on} \quad \boldsymbol{\chi}(P). \tag{4.37}$$

With the variables interpreted as indicated the component formulae are identical, apart from notational differences, to respectively the first and second of equations (9.47) of Naghdi (1972) when the latter are specialized to the case of equilibrium of a shell under a vanishing distributed load. They also coincide with corresponding formulae in Sanders (1963). Naghdi's equations describe the response of a directed surface with the director constrained to coincide with the surface normal. Furthermore, elimination of q^α using the second of (4.31) reduces equations (4.37) to equations (10.22) of Naghdi (1972). We note that Naghdi does not discuss a boundary condition corresponding to the fourth equation in (4.21), except in the context of the linear theory (Naghdi 1972, § 20), whereas the discussion of Sanders (1963) is limited to the case of small strains with moderate rotations. General boundary conditions of this type in the presence of large displacements and strains are discussed by Makowski & Pietraszkiewicz (1989) and Pietraszkiewicz (1989).

5. Minimum energy configurations and necessary conditions of the quasi-convexity type

In this section, we extend the well-known quasi-convexity condition of conventional finite elasticity (Morrey 1952; Ball 1977) to the surface–substrate interaction problem. We thus generate necessary conditions for smooth energy-minimizing configurations that apply to the substrate and surface materials separately. These correspond to inequalities of the quasi-convexity type in the two-dimensional theory (Steigmann & Ogden 1997*a*). Additional pointwise necessary conditions analogous to the rank-one convexity (or Legendre–Hadamard) condition of three-dimensional elasticity are then discussed in the context of the present theory.

The inequalities derived here are unrelated to the necessary condition of *quasi-convexity at the boundary* introduced by Ball & Marsden (1984) in the context of conventional finite elasticity. The latter condition involves the strain energy of the bulk solid and assigned traction data jointly. In the present context this restriction applies only to traction data specified on uncoated parts of the boundary $\partial\Omega$. It does not apply to tractions specified on the elastically coated subset $P \subset \partial\Omega$. These statements are proved in Steigmann & Ogden (1997*a*) for the plane problem and are also valid in the present three-dimensional setting.

In Steigmann & Ogden (1997*a*) we used the pertinent necessary conditions for plane deformations to prove that elastic surfaces of the pure-membrane type do not support compressive stresses in minimizing configurations despite the fact that the surface is bonded to the substrate. Thus, equilibrium states obtained from surface–substrate interaction models of the kind proposed by Gurtin & Murdoch (1975) are compatible with the energy criterion of elastic stability only if restricted to tensile states of surface stress. This restriction applies to the pure-membrane model in which no flexural resistance is attributed to the surface. Such restrictions do not apply to membrane *states*, if any, associated with the model in which bending effects are incorporated, nor to other equilibrium states associated with the latter theory. We remark that compressive surface stresses in certain thin-film–substrate systems have been widely documented in the experimental literature (see, for example, Krulevitch *et al*. 1992).

We first outline the derivation of the stated conditions for the pure-membrane model of surface stress. The argument closely parallels that of conventional elasticity (Ball 1977). For the problem in which flexural resistance is taken into account we indicate only those aspects of the derivation that differ from the pure-membrane problem. We state without proof that the necessary conditions obtained remain valid in the presence of conservative loadings of the kind discussed in § 4 (Steigmann & Ogden 1997*a*). This is due to well-known properties of *null Lagrangians* associated with the load potentials (Ball 1977; Podio-Guidugli & Vergara Caffarelli 1990; Steigmann 1991).

(*a*) *Pure-membrane surfaces*

The pure-membrane model is obtained from (3.2) by requiring the surface energy to be independent of $\boldsymbol{\kappa}$. The strain energy U is then some function of the surface-deformation tensor $\boldsymbol{C}$, equal to some other function of the surface-deformation gradient $\boldsymbol{F}$ (cf. (2.17), (2.19)). To emphasize the coupling with the substrate material we use the first of equations (2.6) to write this dependence in the form,

$$U = U(\boldsymbol{A}(\boldsymbol{Y})\mathbf{1}_Y), \tag{5.1}$$

where $\mathbf{1}_Y = \boldsymbol{G}_\alpha(\boldsymbol{Y}) \otimes \boldsymbol{G}^\alpha(\boldsymbol{Y})$ is the two-dimensional unit tensor on the tangent plane to the surface P at the point with position $\boldsymbol{Y}$. Here, we have suppressed the possible explicit dependence of U on the coordinates θ^α and used the same symbol U to denote the various functional forms of the strain energy. Modulo load potentials the total energy may then be written in the form,

$$E = \int_\Omega W(\boldsymbol{A}(\boldsymbol{X}))\,\mathrm{d}V + \int_P U(\boldsymbol{A}(\boldsymbol{Y})\mathbf{1}_Y)\,\mathrm{d}A. \tag{5.2}$$

Consider the perturbation,

$$\boldsymbol{\chi}_\epsilon(\boldsymbol{X}) = \boldsymbol{\chi}(\boldsymbol{X}) + \epsilon\boldsymbol{\phi}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi}(\boldsymbol{X}) = \epsilon^{-1}(\boldsymbol{X} - \boldsymbol{X}_0), \tag{5.3}$$

where $\boldsymbol{\chi}$ is an energy minimizer, $\boldsymbol{X}_0$ is a point of $\Omega \cup \partial\Omega$, ϵ is a positive constant, and $\boldsymbol{\phi}$ is a smooth vector-valued function compactly supported in a region D containing the point $\boldsymbol{\xi} = \boldsymbol{0}$. Then

$$\int_{\Omega'} W(\boldsymbol{A}(\boldsymbol{X}))\,\mathrm{d}V_X + \int_{P'} U(\boldsymbol{A}(\boldsymbol{Y})\boldsymbol{1}_Y)\,\mathrm{d}A_Y \leqslant \int_{\Omega'} W(\boldsymbol{A}_\epsilon(\boldsymbol{X}))\,\mathrm{d}V_X + \int_{P'} U(\boldsymbol{A}_\epsilon(\boldsymbol{Y})\boldsymbol{1}_Y)\,\mathrm{d}A_Y, \tag{5.4}$$

where

$$\boldsymbol{A}_\epsilon(\boldsymbol{X}) = \boldsymbol{A}(\boldsymbol{X}) + \nabla\boldsymbol{\phi}(\boldsymbol{\xi}) \tag{5.5}$$

is the gradient of $\boldsymbol{\chi}_\epsilon$ with respect to $\boldsymbol{X}$, $\nabla(\cdot)$ is the gradient with respect to $\boldsymbol{\xi}$, and $\mathrm{d}V_X$, $\mathrm{d}A_Y$ are the volume and area measures based on $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. Here, the domains of integration are

$$\Omega' = \Omega \cap (X_0 + \epsilon D), \qquad P' = P \cap (X_0 + \epsilon D), \tag{5.6}$$

where $X_0 + \epsilon D$ is the support of $\boldsymbol{\phi}(\boldsymbol{\xi}(\boldsymbol{X}))$.

We now change variables in accordance with the second equation in (5.3) and evaluate the limit of the inequality (5.4) for small ϵ with D fixed. For $\boldsymbol{X}_0$ belonging to the interior of Ω, P' is empty for sufficiently small ϵ, and the change of variable introduces the multiplicative factor ϵ^3 in the volume integrals. On dividing the resulting inequality by ϵ^3, letting $\epsilon \to 0^+$, and invoking Lebesgue's dominated convergence theorem, we obtain the well-known quasi-convexity condition

$$\int_D [W(\boldsymbol{A}(\boldsymbol{X}_0) + \nabla\boldsymbol{\phi}(\boldsymbol{\xi})) - W(\boldsymbol{A}(\boldsymbol{X}_0))]\,\mathrm{d}V_\xi \geqslant 0 \tag{5.7}$$

(Ball 1977), where $\mathrm{d}V_\xi$ is the volume measure based on the variable $\boldsymbol{\xi}$. Necessary for this is the algebraic rank-one convexity condition (Graves 1939), which holds pointwise in Ω. The linearized version of the latter condition is the Legendre–Hadamard inequality (the *strong ellipticity* condition in the case of a strict minimum of the energy). We refer to Ball (1977) for a discussion of the relationships among these inequalities in a rigorous function space setting.

If we now consider the case $\boldsymbol{X}_0 \in P$, we find that the surface integrals are multiplied by ϵ^2. After division by ϵ^2 the volume integrals contribute terms of order $O(\epsilon)$, and in the limit we obtain the quasi-convexity condition for the film:

$$\int_{P''} [U(\boldsymbol{A}_0\boldsymbol{1}_0 + \nabla\boldsymbol{\phi}(\boldsymbol{\eta})\boldsymbol{1}_0) - U(\boldsymbol{A}_0\boldsymbol{1}_0)]\,\mathrm{d}A_\eta \geqslant 0, \tag{5.8}$$

where $P'' = \boldsymbol{\xi}(P')$, $\boldsymbol{\eta} = \boldsymbol{\xi}(\boldsymbol{Y})$, and $\mathrm{d}A_\eta$ is the associated area measure. Here, $\boldsymbol{A}_0$ and $\boldsymbol{1}_0$ are the values of the deformation gradient and the surficial unit tensor respectively at the point $\boldsymbol{X}_0 \in P$.

To reduce this to a more standard form we introduce *affine* coordinates u^α on the tangent plane $T_{P_0} = T_P(\boldsymbol{X}_0)$, such that

$$\frac{\partial\boldsymbol{\eta}}{\partial u^\alpha} = \frac{\partial\eta_b}{\partial u^\alpha}\boldsymbol{e}_b = \hat{\boldsymbol{G}}_\alpha, \tag{5.9}$$

where the notation $(\hat{\ })$ is used to denote the values of functions at $\boldsymbol{X}_0$. The first equation of (5.3) then yields

$$\nabla\boldsymbol{\phi}(\boldsymbol{\eta})\mathbf{1}_0 = [\nabla\boldsymbol{\phi}(\boldsymbol{\eta})\hat{\boldsymbol{G}}_\alpha] \otimes \hat{\boldsymbol{G}}^\alpha = \boldsymbol{\phi}_{,\alpha} \otimes \hat{\boldsymbol{G}}^\alpha, \tag{5.10}$$

where $\boldsymbol{\phi}_{,\alpha} = \partial\boldsymbol{\phi}/\partial u^\alpha$. From the first equation in (2.6) we also have $\boldsymbol{A}_0\mathbf{1}_0 = \hat{\boldsymbol{g}}_\alpha \otimes \hat{\boldsymbol{G}}^\alpha$. Omitting reference to the fixed vectors $\hat{\boldsymbol{G}}^\alpha$, which play the role of parameters, we thus reduce (5.8) to the form

$$\int_{P^*} [U(\hat{\boldsymbol{g}}_\alpha + \boldsymbol{\phi}_{,\alpha}) - U(\hat{\boldsymbol{g}}_\alpha)]\, \mathrm{d}A \geqslant 0, \tag{5.11}$$

where $P^* = P'' \cap T_{P_0}$.

This form of the inequality was studied by Graves (1939), who showed that it is satisfied only if the function of θ defined by $U(\hat{\boldsymbol{g}}_\alpha + \theta\boldsymbol{a}v_\alpha)$, with $\boldsymbol{a}$ and v_α fixed, is convex at $\theta = 0$. This is the rank-one convexity condition in the present context. It necessarily holds at every point of P since the point $\boldsymbol{X}_0 \in P$ may be chosen arbitrarily. Linearization of the rank-one convexity condition furnishes an inequality that corresponds to the usual Legendre–Hadamard condition (Steigmann 1986, 1990), namely

$$\boldsymbol{a} \cdot (\hat{\boldsymbol{E}}^{\alpha\beta} v_\alpha v_\beta)\boldsymbol{a} \geqslant 0 \qquad \text{for all} \quad \boldsymbol{a}v_\alpha, \tag{5.12}$$

where

$$\boldsymbol{E}^{\alpha\beta} = \frac{\partial^2 U}{\partial F_{i\alpha}\partial F_{j\beta}} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = 4\frac{\partial^2 U}{\partial g_{\alpha\gamma}\partial g_{\beta\lambda}} \boldsymbol{g}_\gamma \otimes \boldsymbol{g}_\lambda + J\sigma^{\alpha\beta}\boldsymbol{I}, \tag{5.13}$$

where $J\sigma^{\alpha\beta}$ is defined in (3.6) and $\boldsymbol{I}$ is the unit tensor in three dimensions. In the second equality U is presumed to be expressed as a symmetric function of $g_{\alpha\beta}$ and $g_{\beta\alpha}$. It is immediately evident that for $\boldsymbol{a}$ orthogonal to T_{P_0} inequality (5.12) reduces to $\hat{\sigma}^{\alpha\beta}v_\alpha v_\beta \geqslant 0$, and this implies that a configuration can be energy minimizing only if the surface stress is non-compressive at every point of P.

We remark that Graves's proof is based on functions $\boldsymbol{\phi}$ that do not possess the degree of regularity assumed here. To rectify this one may use the method of *mollifiers*; alternatively, (5.12) may be derived directly by linearizing (5.11) and taking the restriction of $\boldsymbol{\phi}$ to P^* to be an oscillatory C^∞ function (see, for example, Giaquinta & Hildebrandt 1996).

(*b*) *Surfaces with flexural resistance*

We now consider the perturbation,

$$\boldsymbol{\chi}_\epsilon(\boldsymbol{X}) = \boldsymbol{\chi}(\boldsymbol{X}) + \epsilon^2\boldsymbol{\phi}(\boldsymbol{\xi}), \tag{5.14}$$

in place of the first expression in (5.3). Here the factor ϵ^2 is used instead of ϵ to ensure that the energy remains bounded as $\epsilon \to 0$ in the presence of flexural resistance.

A straightforward modification of the argument leading to (5.11), as in the development given by Steigmann & Ogden (1997*a*) for the plane strain problem, now yields the quasi-convexity condition in the form,

$$\int_{P^*} \{U(\hat{\boldsymbol{g}}_\alpha,\, \hat{\boldsymbol{g}}_{\alpha,\beta} + \nabla^2\boldsymbol{\phi}[\hat{\boldsymbol{G}}_\alpha \otimes \hat{\boldsymbol{G}}_\beta]) - U(\hat{\boldsymbol{g}}_\alpha,\, \hat{\boldsymbol{g}}_{\alpha,\beta})\}\, \mathrm{d}A \geqslant 0, \tag{5.15}$$

where

$$\nabla^2\boldsymbol{\phi} = \frac{\partial^2\phi_i}{\partial\eta_a\partial\eta_b}\boldsymbol{e}_i\otimes\boldsymbol{e}_a\otimes\boldsymbol{e}_b \tag{5.16}$$

and the remaining terms in the arguments of the function U are evaluated at the point $\boldsymbol{X}_0$. Here, we use the notation defined by (2.9). In particular,

$$\nabla^2\boldsymbol{\phi}[\hat{\boldsymbol{G}}_\alpha\otimes\hat{\boldsymbol{G}}_\beta] = \frac{\partial^2\phi_i}{\partial\eta_a\partial\eta_b}\frac{\partial\eta_a}{\partial u^\alpha}\frac{\partial\eta_b}{\partial u^\beta}\boldsymbol{e}_i = \frac{\partial^2\boldsymbol{\phi}}{\partial u^\alpha\partial u^\beta}, \tag{5.17}$$

where, in the second equality, we have used $\partial^2\eta_a/\partial u^\alpha\partial u^\beta = 0$, which follows from (5.9).

The combination of (5.15) and (5.17) furnishes an inequality belonging to a class studied by Meyers (1965). Hilgers & Pipkin (1993) adapted Meyers's work to a nonlinear theory of elastic plates and showed that the appropriate modification to Graves's necessary condition is the requirement that $F(\theta)\equiv U(\hat{\boldsymbol{g}}_\alpha,\, \hat{\boldsymbol{g}}_{\alpha,\beta}+\theta\boldsymbol{a}v_\alpha v_\beta)$ be convex at $\theta=0$. The local form of the latter condition furnishes the Legendre–Hadamard inequality in the present context (Meyers 1965).

To reduce this condition to a more transparent form we note that U depends on $\boldsymbol{g}_{\alpha,\beta}$ only through the components $\kappa_{\alpha\beta}$. In particular, equations (2.12) and (2.22) give $\kappa_{\alpha\beta} = -\boldsymbol{n}\cdot\boldsymbol{g}_{\alpha,\beta}$, where $\boldsymbol{n}$ is the unit normal to the surface $\boldsymbol{\chi}(P)$. According to (2.14) the normal is determined by the vectors $\boldsymbol{g}_\alpha$ and thus takes the fixed value $\hat{\boldsymbol{n}}$ in the Graves condition. The convexity of $F(\theta)$ at $\theta=0$ then yields

$$(\boldsymbol{a}\cdot\hat{\boldsymbol{n}})^2\frac{\partial^2 U}{\partial\kappa_{\alpha\beta}\partial\kappa_{\gamma\delta}}v_\alpha v_\beta v_\gamma v_\delta \geqslant 0, \tag{5.18}$$

where the derivatives are evaluated in the minimizing configuration at the particle $\boldsymbol{X}_0\in P$. This must hold for arbitrary $\boldsymbol{a}$ and is thus equivalent to the statement that the function of θ defined by $U(\hat{\boldsymbol{C}},\, \hat{\boldsymbol{\kappa}}+\theta\boldsymbol{v}\otimes\boldsymbol{v})$ is convex at $\theta=0$ for arbitrary $\boldsymbol{v}=v_\alpha\hat{\boldsymbol{G}}^\alpha$.

For deformations restricted to a fixed plane this result reduces to a necessary condition obtained previously in the context of a two-dimensional specialization of the theory considered here (Steigmann & Ogden 1997*a*). The latter condition has been shown to belong to a family of inequalities involving the second derivatives of the surface energy with respect to curvature and strain jointly (Steigmann & Ogden 1997*b*). This was proved by using perturbations having a structure more general than the two-dimensional counterpart of (5.14). For this reason, we conjecture that a similar generalization is possible in the present three-dimensional context. In any case, for surfaces with flexural resistance, minimum-energy considerations of the type discussed yield no *a priori* conditions restricting the signs of the tractions.

6. Material symmetry

The formulation of a complete theory for surface–substrate interactions requires the specification of detailed constitutive equations for each component separately. This in turn necessitates the consideration of material symmetry. Here, we discuss material symmetry for surfaces only; the associated theory for bulk substrate materials is well known. Following Noll (1958), we view symmetries in terms of local mappings of the reference configuration that leave the strain energy invariant in a given deformation.

This concept was extended to non-simple elastic surfaces by Murdoch & Cohen (1979). We adopt this theory in the present work.

We remark that the notion of symmetry adopted here is not equivalent to that used by Naghdi (1972); his theory presumes the *form invariance* of the response functions of the surface under certain coordinate transformations. This requires that the strain energy, written as a function of tensor components referred to a particular coordinate system, be the same function of the components obtained by a specific transformation of coordinates that reflects the underlying notion of symmetry. It appears that the two theories yield equivalent results for models of simple materials and surfaces (membranes), but not for models that incorporate curvature or strain-gradient effects.

Our preference for Noll's framework derives from its accessibility to empirical test and its coordinate invariance. However, coordinate parametrizations are freely used in its description. We first summarize the theory and then use it to describe the response of surfaces that possess a particular type of symmetry. This is followed by the derivation of the associated constitutive equations.

(*a*) *Noll–Murdoch–Cohen theory*

Following Murdoch & Cohen (1979), we use the invariance of the surface strain energy with respect to translations and rotations to conclude that, insofar as the local constitutive response of the surface is concerned, no generality is lost by taking T_P to coincide with $T_{\boldsymbol{\chi}}(\boldsymbol{P})$ and $\boldsymbol{Y}$ to coincide with $\boldsymbol{\chi}(\boldsymbol{Y})$ at a particular particle, where $P:\boldsymbol{Y}(\theta^1,\theta^2)$ is a reference surface. Consider another reference surface $P^*:\boldsymbol{Y}^*(\theta^1,\theta^2)$, parametrized by the same embedded coordinates (θ^1,θ^2). The foregoing conclusion applies equally to P^* and thus we set $\boldsymbol{Y}=\boldsymbol{Y}^*$ and $T_P=T_{P^*}$ at the particle in question. It follows that there is a non-singular tensor

$$\boldsymbol{H}=H^{\alpha}_{\cdot\lambda}\boldsymbol{G}^*_{\alpha}\otimes\boldsymbol{G}^{*\lambda} \tag{6.1}$$

such that

$$\boldsymbol{G}_{\alpha}=\boldsymbol{H}\boldsymbol{G}^*_{\alpha}=H^{\beta}_{\cdot\alpha}\boldsymbol{G}^*_{\beta}, \tag{6.2}$$

where $\boldsymbol{G}_{\alpha}=\boldsymbol{Y}_{,\alpha}$, $\boldsymbol{G}^*_{\alpha}=\boldsymbol{Y}^*_{,\alpha}$ and $\boldsymbol{G}^{\alpha}$, $\boldsymbol{G}^{*\alpha}$ are the dual vectors induced by the coordinates on T_P and T_{P^*}. Similarly, there is a non-singular tensor

$$\boldsymbol{R}=R_{\alpha}^{\cdot\lambda}\boldsymbol{G}^{*\alpha}\otimes\boldsymbol{G}^*_{\lambda} \tag{6.3}$$

such that

$$\boldsymbol{G}^{\alpha}=\boldsymbol{R}\boldsymbol{G}^{*\alpha}=R_{\beta}^{\cdot\alpha}\boldsymbol{G}^{*\beta}. \tag{6.4}$$

The unit tensors on T_P and T_{P^*} are

$$\mathbf{1}_Y=\boldsymbol{I}-\boldsymbol{N}\otimes\boldsymbol{N},\qquad \mathbf{1}_{Y^*}=\boldsymbol{I}-\boldsymbol{N}^*\otimes\boldsymbol{N}^* \tag{6.5}$$

respectively, where $\boldsymbol{N}=\pm\boldsymbol{N}^*$ according to $\det\boldsymbol{H}$ and $\det\boldsymbol{R}$ are positive or negative. Thus,

$$\mathbf{1}_{Y^*}=\mathbf{1}_Y=\boldsymbol{H}\boldsymbol{G}^*_{\alpha}\otimes\boldsymbol{R}\boldsymbol{G}^{*\alpha}=\boldsymbol{H}\mathbf{1}_{Y^*}\boldsymbol{R}^{\mathrm{T}},\qquad \delta^{\alpha}_{\beta}=H^{\alpha}_{\cdot\lambda}R_{\beta}^{\cdot\lambda}, \tag{6.6}$$

so that

$$\boldsymbol{R}^{\mathrm{T}}=\boldsymbol{H}^{-1}. \tag{6.7}$$

Let $\boldsymbol{y}(\theta^1, \theta^2)$ be a parametrization of a configuration of the surface. The deformation $\boldsymbol{Y} \to \boldsymbol{y}$ has gradient

$$\boldsymbol{F} = \boldsymbol{g}_\alpha \otimes \boldsymbol{G}^\alpha = \boldsymbol{g}_\alpha \otimes \boldsymbol{R}\boldsymbol{G}^{*\alpha} = \boldsymbol{F}^*\boldsymbol{R}^{\mathrm{T}}, \tag{6.8}$$

where $\boldsymbol{F}^*$ is the gradient of the deformation $\boldsymbol{Y}^* \to \boldsymbol{y}$. The associated deformation tensors are related by

$$\boldsymbol{C} = \boldsymbol{R}\boldsymbol{C}^*\boldsymbol{R}^{\mathrm{T}}. \tag{6.9}$$

Similarly,

$$\boldsymbol{\kappa} = \boldsymbol{R}\boldsymbol{\kappa}^*\boldsymbol{R}^{\mathrm{T}}. \tag{6.10}$$

This follows from (2.24), (2.25), (6.8) and the fact that $\boldsymbol{b} = b_{\alpha\beta}\boldsymbol{g}^\alpha \otimes \boldsymbol{g}^\beta$ is independent of the choice of reference configuration.

Let a particle on the surface $\boldsymbol{y}(\cdot)$ be in a given state characterized by a specific value of the strain energy per unit mass $\Psi = \rho^{-1}U$, where ρ is the mass per unit reference area. Let this energy be the value delivered by constitutive functions Ψ and Ψ_* associated with the reference configurations P and P^*, respectively, and let N and N^* be open neighbourhoods of the particle on P and P^*. Then,

$$\Psi_*(\boldsymbol{C}^*, \boldsymbol{\kappa}^*) = \Psi(\boldsymbol{C}, \boldsymbol{\kappa}) = \Psi(\boldsymbol{R}\boldsymbol{C}^*\boldsymbol{R}^{\mathrm{T}}, \boldsymbol{R}\boldsymbol{\kappa}^*\boldsymbol{R}^{\mathrm{T}}). \tag{6.11}$$

We have suppressed the dependence of the energy functions on the particle θ^α.

The extension of Noll's theory developed by Murdoch & Cohen (1979) is based on the requirement that for N and N^* to be symmetry related, their mechanical responses to a given deformation $\boldsymbol{\chi}(\cdot)$ must be identical. Here, N and N^* are regarded as being convected by $\boldsymbol{\chi}(\cdot)$. This formalizes the idea that experiments to characterize the response of a surface necessarily involve immersions of the surface in 3-space. It is a particularly appropriate hypothesis in the present context, since we assume surface films to be convected by the deformation of the bulk materials to which they are attached.

The kinematic formulae required to give mathematical expression to this concept have already been given in § 2. On combining (2.6), (2.12), and (2.22), we obtain

$$-\kappa_{\alpha\beta} = \boldsymbol{n} \cdot \boldsymbol{K}[\boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\beta] + \boldsymbol{n} \cdot \boldsymbol{A}\boldsymbol{G}_{\alpha,\beta}, \tag{6.12}$$

where $\boldsymbol{n}$ is the unit normal to the surface after deformation, and $\boldsymbol{A}$ and $\boldsymbol{K}$, respectively, are the first and second gradients of $\boldsymbol{\chi}$. We use (2.6) and (2.10) to reduce the second term on the right-hand side to

$$\boldsymbol{n} \cdot \boldsymbol{A}\boldsymbol{G}_{\alpha,\beta} = B_{\alpha\beta}\boldsymbol{n} \cdot \boldsymbol{A}\boldsymbol{N}, \tag{6.13}$$

where $B_{\alpha\beta}$ are the components of the reference curvature.

Following Murdoch & Cohen (1979), we now invoke the insensitivity of the strain energy to rigid motions to assert that no generality is lost in the description of the response of the surface by taking $\boldsymbol{n} = \boldsymbol{N}$ at the particle in question. We thus obtain

$$-\boldsymbol{\kappa} = (\boldsymbol{N} \cdot \boldsymbol{K}[\boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\beta])\boldsymbol{G}^\alpha \otimes \boldsymbol{G}^\beta + (\boldsymbol{A}\boldsymbol{N} \cdot \boldsymbol{N})\boldsymbol{B}, \tag{6.14}$$

where

$$\boldsymbol{B} = B_{\alpha\beta}\boldsymbol{G}^\alpha \otimes \boldsymbol{G}^\beta \tag{6.15}$$

is the curvature tensor of the reference surface. Equation (6.14) delivers the value of $\boldsymbol{\kappa}$ induced by the deformation $\boldsymbol{\chi}(\cdot)$. Here, the values of $\boldsymbol{A}$ and $\boldsymbol{K}$ at the particle are fixed by the particular function $\boldsymbol{\chi}(\cdot)$ considered. From (6.2), (6.4) and (6.6) it is then straightforward to show that the first term in (6.14) is the same whether N or N^* is used to compute it, provided that $\boldsymbol{N} = \boldsymbol{N}^*$ ($\det \boldsymbol{R} > 0$) at the particle in question. By contrast, the second term depends on the embedding geometry of the neighbourhood, and may therefore take different values in N and N^*. Let $\boldsymbol{B}$ be the curvature tensor of N. Then (6.14) gives the relative curvature $\boldsymbol{\kappa}$ of N induced by the deformation. The relative curvature of N^* induced by the *same* deformation is $\bar{\boldsymbol{\kappa}} = \boldsymbol{\kappa} + (\boldsymbol{AN} \cdot \boldsymbol{N})(\boldsymbol{B} - \boldsymbol{B}^*)$, where $\boldsymbol{B}^*$ is the curvature of N^*. In addition, (2.6) and (2.19) may be used to show that $\boldsymbol{C} = \bar{\boldsymbol{C}}$, where $\boldsymbol{C}$ and $\bar{\boldsymbol{C}}$, respectively, are the surface-deformation tensors of N and N^* induced by $\boldsymbol{\chi}$.

The two neighbourhoods of the particle are symmetry related if the values of Ψ and Ψ_*, generated in response to a prescribed deformation, coincide. Thus,

$$\Psi(\boldsymbol{C}, \boldsymbol{\kappa}) = \Psi_*(\bar{\boldsymbol{C}}, \bar{\boldsymbol{\kappa}}). \tag{6.16}$$

Combining this with (6.11) yields the condition to be satisfied for all $\boldsymbol{C}$ and $\boldsymbol{\kappa}$ in the domain of Ψ, and for all $(\boldsymbol{R}, \boldsymbol{B} - \boldsymbol{B}^*) \in \mathcal{G}$, the *symmetry set* of the surface at the particle in question. Thus,

$$\Psi(\boldsymbol{C}, \boldsymbol{\kappa}) = \Psi(\boldsymbol{R}\bar{\boldsymbol{C}}\boldsymbol{R}^{\mathrm{T}}, \boldsymbol{R}\bar{\boldsymbol{\kappa}}\boldsymbol{R}^{\mathrm{T}}) = \Psi(\boldsymbol{RCR}^{\mathrm{T}}, \boldsymbol{R}[\boldsymbol{\kappa} + (\boldsymbol{AN} \cdot \boldsymbol{N})(\boldsymbol{B} - \boldsymbol{B}^*)]\boldsymbol{R}^{\mathrm{T}}), \tag{6.17}$$

this being operative in the case $\det \boldsymbol{R} > 0$. Clearly $\mathcal{G}$ contains the element $(\mathbf{1}, \mathbf{0})$ ($\det \boldsymbol{R} = 1$). If this is the only member of $\mathcal{G}$, then the symmetry set is said to be trivial.

Suppose that the orientation of N^* is opposite that of N ($\boldsymbol{n}^* = \boldsymbol{N}^* = -\boldsymbol{N}$). It follows that $\bar{\boldsymbol{\kappa}} = -[\boldsymbol{\kappa} + (\boldsymbol{AN} \cdot \boldsymbol{N})(\boldsymbol{B} + \boldsymbol{B}^*)]$, and the considerations leading to (6.17) now yield the restriction

$$\Psi(\boldsymbol{C}, \boldsymbol{\kappa}) = \Psi(\boldsymbol{RCR}^{\mathrm{T}}, -\boldsymbol{R}[\boldsymbol{\kappa} + (\boldsymbol{AN} \cdot \boldsymbol{N})(\boldsymbol{B} + \boldsymbol{B}^*)]\boldsymbol{R}^{\mathrm{T}}) \tag{6.18}$$

for $(\boldsymbol{R}, \boldsymbol{B} + \boldsymbol{B}^*) \in \mathcal{G}$ and $\det \boldsymbol{R} < 0$, provided that N^* and N are symmetry related.

Remark 6.1. Equations (6.17) and (6.18) have counterparts among the formulae obtained by Murdoch & Cohen (1979, eqns (5.26) and (5.27)). It is apparent that the dependence of the function Ψ on its second argument generates a non-standard symmetry condition involving the geometry of the reference surface. This has no analogue in Noll's theory of simple materials, nor in the concept of form invariance, but is nevertheless readily understood. To illustrate this, let $\det \boldsymbol{R} > 0$ and consider a local configuration having the geometry of a circular cylinder. A mapping by the two-dimensional rotation $\boldsymbol{R} = -\mathbf{1}$ ($\det \boldsymbol{R} = 1$) may be regarded as the projection onto the tangent plane of a three-dimensional rotation through the angle π about the normal to the surface. This mapping preserves generators and latitudinal curves on the cylinder, and thus yields $\boldsymbol{B}^* = \boldsymbol{B}$. If the response of the surface to *strain* is unaffected by such a transformation, then intuitively one would expect the original and the mapped neighbourhoods to respond identically to a given deformation. However, a mapping $\boldsymbol{R}$ corresponding to an arbitrary rotation about the normal yields a neighbourhood that may not be congruent to the original, and there is no reason to expect that the two neighbourhoods would now respond identically to the same deformation. Thus, the presence of $\boldsymbol{B} - \boldsymbol{B}^*$ in (6.17) is plausible on physical grounds. In the foregoing example we have $\{\pm\mathbf{1}, \mathbf{0}\} \subset \mathcal{G}$; the associated specialization of (6.17) is an identity and thus yields no restrictions on the form of the function Ψ.

Remark 6.2. In an addendum to their work, Murdoch & Cohen (1981) propose the use of a limited class of mappings $\boldsymbol{\chi}(\boldsymbol{X})$ in the definition of the symmetry set. This is motivated by the observation (cf. (2.4)) that the arguments $\boldsymbol{C}$ and $\boldsymbol{\kappa}$ of the surface-strain-energy function involve only the restriction $\boldsymbol{\chi}|_P$ of $\boldsymbol{\chi}$ to the surface P. If this restriction is chosen to be the identity transformation on P, then the induced deformation tensors of the neighbourhoods $N \subset P$ and $N^* \subset P^*$ are both equal to $\mathbf{1}$, the unit tensor on T_P and T_{P^*} at the particle in question. Further, the component $\boldsymbol{AN} \cdot \boldsymbol{N}$ of the gradient of the function $\boldsymbol{\chi}$ is unaffected by the specification of $\boldsymbol{\chi}|_P$. It then follows from (6.16) that $\Psi(\mathbf{1}, -\boldsymbol{B}) = \Psi_*(\mathbf{1}, \pm(\boldsymbol{AN}\cdot\boldsymbol{N} - 1)\boldsymbol{B} - (\boldsymbol{AN}\cdot\boldsymbol{N})\boldsymbol{B}^*)$ whenever N and N^* are symmetry related, with the choice of sign depending on the orientation of N^* relative to N. Since infinitely many values of $\boldsymbol{AN} \cdot \boldsymbol{N}$ may be associated with $\boldsymbol{\chi}|_P$, we conclude that $\Psi_*(\mathbf{1}, \bar{\boldsymbol{\kappa}})$ takes the same value at infinitely many $\bar{\boldsymbol{\kappa}}$. This is clearly undesirable. Exceptionally, if $\boldsymbol{B}^* = \pm\boldsymbol{B}$, as appropriate, or if $\boldsymbol{\chi}(\boldsymbol{X})$ is chosen such that $\boldsymbol{AN} \cdot \boldsymbol{N} = 1$ on P, then the symmetry condition yields $\Psi(\mathbf{1}, -\boldsymbol{B}) = \Psi_*(\mathbf{1}, -\boldsymbol{B}^*)$ and the foregoing conclusion is avoided. If neither alternative applies, then it is likely that the symmetry set is trivial.

In Murdoch & Cohen (1981), a constraint equivalent to $\boldsymbol{AN} \cdot \boldsymbol{N} = 1$ is adopted to allow for the existence of non-trivial symmetry sets without restrictions on the embedding geometry of N^* relative to that of N. In their framework, such a constraint may be imposed without loss of generality since the function $\boldsymbol{\chi}$ merely plays the role of an orientation-preserving diffeomorphism of the enveloping 3-space, and it is only the function $\boldsymbol{\chi}|_P$ that is operative in the associated constitutive theory. Murdoch & Cohen impose the constraint for arbitrary deformations of the surface. However, in the present context the response of the film–substrate system is affected by the properties of $\boldsymbol{\chi}(\boldsymbol{X})$ in a three-dimensional neighbourhood of a point on P. Thus it is more natural to regard the map $\boldsymbol{\chi}(\boldsymbol{X}) = \boldsymbol{X}$ as the appropriate identity transformation in the present theory. This choice automatically satisfies the Murdoch–Cohen constraint for the trivial deformation. However, the imposition of the constraint in an arbitrary deformation of the surface is unnatural to the extent that it imposes undue restrictions on the substrate deformation. It is then likely that the symmetry set is trivial for arbitrary film–substrate deformations unless $\boldsymbol{B}^* = \pm\boldsymbol{B}$.

Remark 6.3. Murdoch & Cohen (1979) have established the group properties of the sets $\mathcal{G}$ associated with (6.17) and (6.18). The line of reasoning used in conventional elasticity (Gurtin & Williams 1966) to restrict symmetry groups to subgroups of the unimodular group thus applies in the present context, with the added proviso that surface mass density should now be uniform in a neighbourhood of the particle (θ^1, θ^2) (see Murdoch & Cohen 1979, p. 85). We therefore impose the constraints $|\det \boldsymbol{R}| = |\det \boldsymbol{H}| = 1$. This implies that the mass density is unaffected by symmetry transformations, and thus that (6.16), (6.17) and (6.18) apply to the function U as well as to Ψ. Accordingly, we henceforth replace Ψ by U in our further considerations of material symmetry.

Continuing the analogy with the conventional elasticity of solids, we assume that distortion of the surface consumes energy, and therefore that $\boldsymbol{R}$ is an orthogonal transformation on T_P. Then $\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{-1}$, $\boldsymbol{H}^{\mathrm{T}} = \boldsymbol{H}^{-1}$ and (6.7) requires that $\boldsymbol{H} = \boldsymbol{R}$.

Remark 6.4. For those surfaces having $\mathcal{G} \supset \{\boldsymbol{R}, \mathbf{0}\}$ with $\det \boldsymbol{R} = 1$, (6.17) reduces to the functional equation $U(\boldsymbol{C}, \boldsymbol{\kappa}) = U(\boldsymbol{RCR}^{\mathrm{T}}, \boldsymbol{R\kappa R}^{\mathrm{T}})$, which is amenable to analysis by available representation theory. In particular, it is then equivalent to the form invariance of U. Planes are obvious examples of this, since all neighbourhoods of a point on a plane have zero curvature. In the course of seeking other examples we are led to re-examine the nature of the mappings between the neighbourhoods N^* and N discussed previously. In particular, we note that $\boldsymbol{H}$ in (6.1) is the value of the gradient of the mapping at a particular point (θ^1, θ^2). Consider a three-dimensional neighbourhood of the point whose intersection with the surface is N^*, and let the neighbourhood undergo an affine three-dimensional rotation about the normal, with N^* embedded, the projection of which onto the tangent plane is $\boldsymbol{H}$ at the particle in question. Then, N is the image of a rigid mapping of N^*, and it follows that the curvatures $\boldsymbol{B}$ and $\boldsymbol{B}^*$ of the two neighbourhoods are such that $\boldsymbol{B} = \boldsymbol{R}\boldsymbol{B}^*\boldsymbol{R}^{\mathrm{T}}$, where $\boldsymbol{R} = \boldsymbol{H}^{-\mathrm{T}}$ (Naghdi 1972, eqn (5.54)). To ensure that $\boldsymbol{B} = \boldsymbol{B}^*$ (such that $\mathcal{G} \supset \{\boldsymbol{R}, \boldsymbol{0}\}$ with $\det \boldsymbol{R} = 1$) we then require that $\boldsymbol{R}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{R}$. If this should hold for *arbitrary* rotations $\boldsymbol{R}$, then it is necessary and sufficient that $\boldsymbol{B}$ be a scalar multiple of the surficial unit tensor. In this case the local embedding geometry is that of a plane or a sphere. Of course, it is possible to consider other kinds of symmetry in the same framework. However, if non-affine transformations are admitted then symmetry considerations are further complicated by the fact that $\boldsymbol{B} \neq \boldsymbol{R}\boldsymbol{B}^*\boldsymbol{R}^{\mathrm{T}}$.

Remark 6.5. It is to be expected that the strain energy will be affected by the curvature of the reference surface for reasons other than those implied in (6.17) and (6.18), and that this will generally be manifested by an explicit coordinate dependence in the function U. An illustrative example is furnished by a coiled cylinder. The tightness of the coil, as measured by the reference curvature tensor, varies from one generator to the next; thus, the local response to a prescribed deformation is expected to vary as the orthogonal trajectories are traversed, even if the corresponding *membrane* energy, obtained by suppressing the curvature dependence of U, is uniform. The presence of this type of non-uniformity, reflected in the θ^α-dependence of $U(\boldsymbol{C}, \boldsymbol{\kappa}; \theta^\alpha)$, is thus to be expected in surfaces with flexural resistance.

We note that it is common practice in shell theory to suppose that the strain energy depends parametrically on reference curvature (see, for example, Simmonds 1985). The foregoing example furnishes motivation for such an assumption. Typically this is introduced via a *curvature strain* $\boldsymbol{\varDelta} = \boldsymbol{\kappa} + \boldsymbol{B}$ (Naghdi 1972, § 13; Pietraszkiewicz 1989) that vanishes in the reference configuration, where $\boldsymbol{\kappa} = -\boldsymbol{B}$. In the context of the present theory $\boldsymbol{\varDelta}$ may be introduced as a constitutive variable through an explicit dependence of the function U on coordinates of the form $U(\boldsymbol{C}, \boldsymbol{\kappa}; \theta^\alpha) = \hat{U}(\boldsymbol{C}, \boldsymbol{\kappa} + \boldsymbol{B}(\theta^\alpha))$. This modification does not alter the foregoing conclusions regarding symmetry. Thus, examination of the symmetry restrictions given in the works cited reveals that they are generally incompatible with (6.17) and (6.18). Exceptionally, if the constraint $\boldsymbol{A}\boldsymbol{N}\cdot\boldsymbol{N} = 1$ is imposed, and if $\boldsymbol{B} = \boldsymbol{R}\boldsymbol{B}^*\boldsymbol{R}^{\mathrm{T}}$, then the argument leading from (6.16) to (6.17) and (6.18) may be used to derive the restriction $\hat{U}(\boldsymbol{C}, \boldsymbol{\varDelta}) = \hat{U}(\boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^{\mathrm{T}}, \pm\boldsymbol{R}\boldsymbol{\varDelta}\boldsymbol{R}^{\mathrm{T}})$, with the sign chosen in accordance with that of $\det \boldsymbol{R}$. The present theory then reduces to the alternatives cited when $\det \boldsymbol{R} > 0$.

We remark that the concept of symmetry advanced by Carroll & Naghdi (1972) reflects the view that reference curvature should enter into the local constitutive response in a fundamental way. However, they furnished no primitive notion of symmetry upon which to base a deductive line of reasoning leading to a characterization of the nature of this dependence. The Murdoch–Cohen generalization of Noll's theory yields a precise and readily understood restriction on the strain energy that reveals the role of reference curvature explicitly.

(*b*) *Hemitropic planes and spheres*

We specialize the general theory to *hemitropic* films. These are defined to be surfaces for which a reference configuration can be found such that $\mathcal{G} = \{\boldsymbol{R}, \boldsymbol{0}\}$ with $\boldsymbol{R}$ proper orthogonal ($\det \boldsymbol{R} = +1$) but otherwise arbitrary at every point. We take $P \subset \partial\Omega$ to be such a configuration. In light of the fourth of the foregoing remarks, planes and spherical sectors furnish examples for which the characterization of hemitropy does not involve the non-standard terms in (6.17) associated with the curvature of the reference surface. Thus, we seek to characterize those functions U with the property,

$$U(\boldsymbol{C}, \boldsymbol{\kappa}) = U(\boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^{\mathrm{T}}, \boldsymbol{R}\boldsymbol{\kappa}\boldsymbol{R}^{\mathrm{T}}), \qquad \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{1}, \quad \det \boldsymbol{R} = 1. \tag{6.19}$$

According to standard representation theory, this is satisfied for all such $\boldsymbol{R}$ if and only if U is expressible as a function of the elements of the functional basis consisting of the invariants (see Zheng 1993, table 2)

$$\left.\begin{aligned} I_1 &= \operatorname{tr} \boldsymbol{C} = C_{\alpha\beta} G^{\alpha\beta}, \\ I_2 &= \det \boldsymbol{C} = g/G, \\ I_3 &= \operatorname{tr} \boldsymbol{\kappa} = \kappa_{\alpha\beta} G^{\alpha\beta}, \\ I_4 &= \det \boldsymbol{\kappa} = \tfrac{1}{2}\mu^{\alpha\beta}\mu^{\lambda\gamma}\kappa_{\alpha\lambda}\kappa_{\beta\gamma}, \\ I_5 &= \operatorname{tr}(\boldsymbol{C}\boldsymbol{\kappa}) = C_{\alpha\beta}\kappa^{\alpha\beta} = C^{\alpha\beta}\kappa_{\alpha\beta}, \\ I_6 &= \operatorname{tr}(\boldsymbol{C}\boldsymbol{\kappa}\boldsymbol{\mu}) = C_{\alpha\beta} D^{\alpha\beta} = \kappa_{\alpha\beta} E^{\alpha\beta}. \end{aligned}\right\} \tag{6.20}$$

(The two-dimensional Cayley–Hamilton theorem may be used to establish the equivalence of this list of invariants to that given by Zheng.) Here, we use the notation defined by (2.15), (2.19), (2.24) and (2.25), together with the definitions

$$\left.\begin{gathered} C_{\alpha\beta} = g_{\alpha\beta}, \qquad C^{\alpha\beta} = G^{\alpha\gamma} G^{\beta\delta} C_{\gamma\delta}, \qquad \kappa^{\alpha\beta} = G^{\alpha\gamma} G^{\beta\delta} \kappa_{\gamma\delta}, \\ D^{\alpha\beta} = \kappa^{\alpha\sigma}\mu^{\gamma\beta} G_{\gamma\sigma}, \qquad E^{\alpha\beta} = \mu^{\alpha\gamma} C^{\sigma\beta} G_{\sigma\gamma}, \end{gathered}\right\} \tag{6.21}$$

and $\boldsymbol{\mu} = \mu^{\alpha\beta} \boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\beta$ is the permutation tensor-density on T_P induced by the components $\mu^{\alpha\beta}$ defined in (2.15).

The constitutive relations for the surface follow from

$$\left.\begin{gathered} \frac{\partial I_1}{\partial C_{\alpha\beta}} = G^{\alpha\beta}, \qquad \frac{\partial I_2}{\partial C_{\alpha\beta}} = G^{-1}\frac{\partial g}{\partial g_{\alpha\beta}} = I_2 g^{\alpha\beta} = \tilde{C}^{\alpha\beta}, \quad \frac{\partial I_3}{\partial \kappa_{\alpha\beta}} = G^{\alpha\beta}, \\ \frac{\partial I_4}{\partial \kappa_{\alpha\beta}} = \mu^{\alpha\gamma}\mu^{\beta\lambda}\kappa_{\gamma\lambda} = \tilde{\kappa}^{\alpha\beta}, \qquad \frac{\partial I_5}{\partial C_{\alpha\beta}} = \kappa^{\alpha\beta}, \qquad \frac{\partial I_5}{\partial \kappa_{\alpha\beta}} = C^{\alpha\beta}, \\ \frac{\partial I_6}{\partial C_{\alpha\beta}} = D^{\alpha\beta}, \qquad \frac{\partial I_6}{\partial \kappa_{\alpha\beta}} = E^{\alpha\beta}, \end{gathered}\right\} \tag{6.22}$$

wherein the notation $(\tilde{\cdot})$ is used to denote the adjugate of a symmetric surface tensor. An alternative formula for the latter may be obtained from the Cayley–Hamilton theorem. Thus, for any such tensor $\boldsymbol{T}$, the adjugate is given by

$$\tilde{\boldsymbol{T}} = (\operatorname{tr} \boldsymbol{T})\boldsymbol{1} - \boldsymbol{T}, \tag{6.23}$$

which is valid whether or not the determinant of $\boldsymbol{T}$ vanishes. Equations (3.6), (3.15) and (3.16) then yield the constitutive equations

$$\left.\begin{aligned} \tfrac{1}{2}J\sigma^{\alpha\beta} &= \frac{\partial U}{\partial I_1}G^{\alpha\beta} + \frac{\partial U}{\partial I_2}\tilde{C}^{\alpha\beta} + \frac{\partial U}{\partial I_5}\kappa^{\alpha\beta} + \frac{1}{2}\frac{\partial U}{\partial I_6}(D^{\alpha\beta} + D^{\beta\alpha}), \\ Jm^{\alpha\beta} &= \frac{\partial U}{\partial I_3}G^{\alpha\beta} + \frac{\partial U}{\partial I_4}\tilde{\kappa}^{\alpha\beta} + \frac{\partial U}{\partial I_5}C^{\alpha\beta} + \frac{1}{2}\frac{\partial U}{\partial I_6}(E^{\alpha\beta} + E^{\beta\alpha}). \end{aligned}\right\} \tag{6.24}$$

The dependence of the energy on the invariants I_5 and I_6 allows for the modelling of thin three-dimensional bodies composed of lamellae having different properties. These generate coupling of the local (two-dimensional) response to strain and flexure. Related issues are discussed in Libai & Simmonds (1998).

(*c*) *Natural configurations and residual states*

It is frequently of interest in applications to determine conditions under which a reference configuration is *natural* in the sense that the associated values of the functions $\sigma^{\alpha\beta}$ and $m^{\alpha\beta}$ vanish identically. Such configurations exist only if the strain energy function satisfies certain restrictions. To derive them we set equations (6.24) to zero at $\boldsymbol{C} = \boldsymbol{1}$ and $\boldsymbol{\kappa} = -\boldsymbol{B}$ and obtain

$$K_1G^{\alpha\beta} - K_2B^{\alpha\beta} - K_3(B^\alpha_\gamma\mu^{\gamma\beta} + B^\beta_\gamma\mu^{\gamma\alpha}) = 0, \qquad (K_2 + K_4)G^{\alpha\beta} - K_5\tilde{B}^{\alpha\beta} = 0, \tag{6.25}$$

where

$$K_1 = \frac{\partial U}{\partial I_1} + \frac{\partial U}{\partial I_2}, \qquad K_2 = \frac{\partial U}{\partial I_5}, \qquad K_3 = \frac{1}{2}\frac{\partial U}{\partial I_6}, \quad K_4 = \frac{\partial U}{\partial I_3}, \qquad K_5 = \frac{\partial U}{\partial I_4}, \tag{6.26}$$

and the derivatives are evaluated at

$$I_1 = 2, \qquad I_2 = 1, \qquad I_3 = I_5 = -2H, \qquad I_4 = \kappa, \qquad I_6 = 0. \tag{6.27}$$

Here, H and κ respectively are the mean and Gaussian curvatures of the reference surface.

Necessary and sufficient conditions for (6.25) may be obtained by introducing an orthogonal decomposition of $\boldsymbol{B}$ into circular and deviatoric parts. We denote the latter by $\acute{\boldsymbol{B}}$ and use (6.23) to obtain the corresponding decomposition of $\tilde{\boldsymbol{B}}$. Thus,

$$B^{\alpha\beta} = \acute{B}^{\alpha\beta} + HG^{\alpha\beta}, \qquad B^\alpha_\beta = \acute{B}^\alpha_\beta + H\delta^\alpha_\beta, \qquad \tilde{B}^{\alpha\beta} = HG^{\alpha\beta} - \acute{B}^{\alpha\beta}. \tag{6.28}$$

Noting that $\acute{\boldsymbol{B}} \equiv \boldsymbol{0}$ for the types of reference surface under consideration we find that such surfaces are natural configurations if and only if

$$K_1 = HK_2, \qquad K_2 + K_4 = HK_5. \tag{6.29}$$

Of equal interest are the residual states compatible with a given symmetry class that can be supported in a reference configuration. For the case of hemitropy with $\acute{\boldsymbol{B}} = \boldsymbol{0}$ these are given by (6.24), again evaluated at $\boldsymbol{C} = \boldsymbol{1}$ and $\boldsymbol{\kappa} = -\boldsymbol{B}$. We obtain $(\sigma^{\alpha\beta})_0 = \sigma_0G^{\alpha\beta}$ and $(m^{\alpha\beta})_0 = m_0G^{\alpha\beta}$, where the notation $(\cdot)_0$ refers to values in the reference configuration,

$$\sigma_0/2 = K_1 - HK_2, \qquad m_0 = K_2 + K_4 - HK_5, \tag{6.30}$$

and equations (6.29) no longer apply.

(*d*) *Legendre–Hadamard inequality*

We recall that the Legendre–Hadamard condition reduces, in the present context, to the requirement that the function $F(\theta) \equiv U(\boldsymbol{C}, \boldsymbol{\kappa} + \theta \boldsymbol{v} \otimes \boldsymbol{v})$ be convex at $\theta = 0$ for arbitrary $\boldsymbol{v} \in T_P$, where $\boldsymbol{C}$ and $\boldsymbol{\kappa}$ are associated with an energy-minimizing configuration. For hemitropic surfaces, U depends on the parameter θ through the invariants I_3 to I_6, and it is straightforward to show that this dependence is linear. Thus, the convexity condition is equivalent to

$$\sum_{i,j=3}^{6} U_{ij} I_i' I_j' \geqslant 0, \tag{6.31}$$

where the $U_{ij} \equiv \partial^2 U / \partial I_i \partial I_j$ are evaluated at $\theta = 0$, and

$$I_3' = |\boldsymbol{v}|^2, \qquad I_4' = \boldsymbol{v} \cdot \tilde{\boldsymbol{\kappa}} \boldsymbol{v}, \qquad I_5' = \boldsymbol{v} \cdot \boldsymbol{C} \boldsymbol{v}, \qquad I_6' = \mathrm{tr}(\boldsymbol{C}(\boldsymbol{v} \otimes \boldsymbol{v})\boldsymbol{\mu}). \tag{6.32}$$

We do not present a detailed analysis of this inequality here, except to note that *sufficient* conditions are furnished by the non-negativity of the principal minors of the matrix U_{ij}. The latter conditions are not necessary because it is not possible to specify the I_i' independently.

7. Examples

In this final section, we illustrate the theory by solving two problems of technical interest involving isotropic substrate materials with hemitropic or isotropic films attached to their surfaces. We assume that the three-dimensional substrate material is composed of a general *incompressible* isotropic elastic solid. The modifications to the constitutive equation (4.18) required to account for incompressibility and the requisite modifications to the variational theory of § 4 are well known. Furthermore, the necessary conditions (5.12) and (5.18) continue to apply provided that the function $\boldsymbol{\phi}(\boldsymbol{\xi})$ in (5.3) and (5.14) is suitably restricted. In particular, incompressibility requires the determinant of $\boldsymbol{I} + \epsilon(\boldsymbol{A}(\boldsymbol{X}))^{-1} \nabla \boldsymbol{\phi}(\boldsymbol{\xi})$ to be unity if the film possesses flexural resistance, where ϵ is the small parameter introduced in § 5. For fixed $\boldsymbol{\xi}$, the small ϵ-limit yields an identity and the conclusions of § 5 *b* follow as before. For films without flexural resistance, (5.3) applies and the conclusions of § 5 *a* remain valid provided that $\det[\boldsymbol{I} + \boldsymbol{A}_0^{-1} \nabla \boldsymbol{\phi}(\boldsymbol{\xi})] = 1$, where $\boldsymbol{A}_0$ is the value of the deformation gradient at the point $\boldsymbol{X}_0$ involved in the second of equations (5.3).

(*a*) *Radial compression of a coated spherical shell*

In the first example, we consider the spherically symmetric deformation of a thick-walled spherical shell of incompressible isotropic elastic material. We suppose that the material is homogeneous and, for illustration, we take the shell to be coated with a hemitropic film on its *inner* boundary, which is also taken to be traction free. The outer boundary is uncoated but subject to a pressure, $\mathcal{P}$ say.

Let the shell be centred on the origin and defined by

$$A \leqslant R \leqslant B, \tag{7.1}$$

where $R = |\boldsymbol{X}|$. We represent the deformation of the bulk material in the form

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}) = \lambda(R)\boldsymbol{X}, \tag{7.2}$$

with

$$\lambda(R) = r(R)/R > 0 \tag{7.3}$$

being the (equibiaxial) stretch in the plane normal to the radial direction, where $r = |\boldsymbol{x}|$ and $\boldsymbol{x}$ is the current position of the material point initially at $\boldsymbol{X}$.

Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 = \boldsymbol{k}$ be orthogonal unit Cartesian basis vectors and let

$$\boldsymbol{e}_r = \cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \boldsymbol{e}_2 \tag{7.4}$$

be the radial unit vector normal to $\boldsymbol{k}$, where θ is the (plane) polar angle in the plane normal to $\boldsymbol{k}$. Let ϕ be the longitudinal angle measured from the latter plane. Then,

$$\boldsymbol{e}_\rho = \cos\phi\, \boldsymbol{e}_r + \sin\phi\, \boldsymbol{k} \tag{7.5}$$

is a unit vector in the radial direction and the position vector $\boldsymbol{X}$ may be written

$$\boldsymbol{X} = R\boldsymbol{e}_\rho. \tag{7.6}$$

(i) *Deformation of the coating*

For the coating on the surface $R = A$, we have $\boldsymbol{X} = \boldsymbol{Y} = A\boldsymbol{e}_\rho$. Let $\theta^1 = \phi$, $\theta^2 = \theta$ so that, in the notation of § 2, $\boldsymbol{G}_1 \times \boldsymbol{G}_2$ points towards the origin. Specifically,

$$\boldsymbol{G}_1 = \boldsymbol{Y}_{,\phi} = A\boldsymbol{e}_\phi, \qquad \boldsymbol{G}_2 = \boldsymbol{Y}_{,\theta} = A\cos\phi\, \boldsymbol{e}_\theta, \tag{7.7}$$

where $\boldsymbol{e}_\phi, \boldsymbol{e}_\theta$ are the unit basis vectors associated with ϕ, θ respectively. Then, $\boldsymbol{N} = \boldsymbol{e}_\phi \times \boldsymbol{e}_\theta = -\boldsymbol{e}_\rho$.

It then follows from the definitions in (2.15) and the paragraph that follows it that

$$(G_{\alpha\beta}) = A^2 \operatorname{diag}(1, \cos^2\phi), \qquad (G^{\alpha\beta}) = A^{-2} \operatorname{diag}(1, \sec^2\phi) \tag{7.8}$$

and $\sqrt{G} = A^2\cos\phi$.

Also, we obtain

$$\boldsymbol{G}_{1,1} = -A\boldsymbol{e}_\rho, \qquad \boldsymbol{G}_{1,2} = \boldsymbol{G}_{2,1} = -A\sin\phi\, \boldsymbol{e}_\theta, \qquad \boldsymbol{G}_{2,2} = -A\cos\phi\, \boldsymbol{e}_r, \tag{7.9}$$

and hence, from the second equation in (2.12),

$$B_{11} = A, \qquad B_{12} = B_{21} = 0, \qquad B_{22} = A\cos^2\phi. \tag{7.10}$$

Comparison with (7.8) shows that $B_{\alpha\beta} = A^{-1}G_{\alpha\beta}$, or

$$\boldsymbol{B} = A^{-1}\mathbf{1}, \tag{7.11}$$

where $\mathbf{1}$ is the unit tensor on the reference surface.

Let $\lambda_a = a/A$, where $a = r(A)$. Then, the deformed position of the coating on $R = A$ is simply $\boldsymbol{y} = \lambda_a \boldsymbol{Y} = a\boldsymbol{e}_\rho$, and we easily obtain

$$g_{\alpha\beta} = \lambda_a^2 G_{\alpha\beta}, \qquad g^{\alpha\beta} = \lambda_a^{-2} G^{\alpha\beta}. \tag{7.12}$$

Also,

$$b_{\alpha\beta} = a^{-1} g_{\alpha\beta} = A^{-1}\lambda_a G_{\alpha\beta} \tag{7.13}$$

and

$$J = \sqrt{g/G} = \lambda_a^2. \tag{7.14}$$

From the definition (2.19), we derive

$$\boldsymbol{C} = \lambda_a^2 \mathbf{1} \tag{7.15}$$

and hence

$$C^{\alpha\beta} = \lambda_a^2 G^{\alpha\beta} = \lambda_a^4 g^{\alpha\beta}. \tag{7.16}$$

Similarly, from (2.22) with (7.13),

$$\boldsymbol{\kappa} = -A^{-1}\lambda_a \mathbf{1} \tag{7.17}$$

and

$$\kappa^{\alpha\beta} = -A^{-1}\lambda_a G^{\alpha\beta} = -A^{-1}\lambda_a^3 g^{\alpha\beta}. \tag{7.18}$$

Further, from (7.15) and (7.17) it follows that

$$\tilde{\boldsymbol{C}} = \boldsymbol{C}, \qquad \tilde{\boldsymbol{\kappa}} = \boldsymbol{\kappa}. \tag{7.19}$$

(ii) *Response of the coating*

With reference to the constitutive equations (6.24) and the definitions of $D^{\alpha\beta}$ and $E^{\alpha\beta}$ in (6.21) it is easy to show using the definition of $\mu^{\alpha\beta}$ in (2.15) that $D^{\alpha\beta}+D^{\beta\alpha} = 0$ and $E^{\alpha\beta} + E^{\beta\alpha} = 0$ for the present class of deformations.

The invariants specialize to

$$\left.\begin{aligned} I_1 = 2\lambda_a^2, \qquad I_2 = \lambda_a^4, \qquad I_3 = -2A^{-1}\lambda_a, \\ I_4 = A^{-2}\lambda_a^2, \qquad I_5 = -2A^{-1}\lambda_a^3, \qquad I_6 = 0. \end{aligned}\right\} \tag{7.20}$$

Since these are constants it follows that for a homogeneous material the coefficients in (6.24) are also constants.

The response functions may now be expressed simply in the forms,

$$\sigma^{\alpha\beta} = \sigma g^{\alpha\beta}, \qquad m^{\alpha\beta} = m g^{\alpha\beta}, \tag{7.21}$$

where σ and m are functions of λ_a, which can be related to the strain-energy function by using (6.24), (7.14) and (7.16) to (7.19).

(iii) *The coupling equation*

The coupling condition is given by (4.34) evaluated at the inner surface with J given by (7.14). At the outer surface the pressure boundary condition is

$$\boldsymbol{P}\boldsymbol{N} = -\mathcal{P}\boldsymbol{A}^*\boldsymbol{N}, \tag{7.22}$$

where $\boldsymbol{A}^* = (\det \boldsymbol{A})\boldsymbol{A}^{-\mathrm{T}}$ is the adjugate of $\boldsymbol{A}$.

Recalling (4.31), we see that for the present problem (7.21) and (7.22) yield

$$p^{\alpha\beta} = \Sigma g^{\alpha\beta}, \qquad \Sigma \equiv \sigma - A^{-1}\lambda_a^{-1} m. \tag{7.23}$$

The identity $g^{\alpha\beta}_{;\gamma} = 0$ implies that $p^{\alpha\beta}_{;\alpha} = 0$ and $q^{\alpha} = 0$ on the inner surface. From (4.35) and (7.14) we then have $J\boldsymbol{p}^{\alpha}_{;\alpha} = 2A^{-1}\lambda_a \Sigma \boldsymbol{n}$, where $\boldsymbol{n} = -\boldsymbol{e}_\rho$, and the coupling condition (4.34) becomes

$$\boldsymbol{P}\boldsymbol{N} = 2A^{-1}\lambda_a \Sigma \boldsymbol{n} \tag{7.24}$$

on $R = A$.

If $\boldsymbol{\sigma}$ is the Cauchy stress operative in the bulk material, then

$$\boldsymbol{P} = \boldsymbol{\sigma}\boldsymbol{A}^*, \tag{7.25}$$

and from Nanson's formula,

$$\boldsymbol{A}^*\boldsymbol{N} = J\boldsymbol{n}, \tag{7.26}$$

we obtain $\boldsymbol{P}\boldsymbol{N} = \lambda_a^2\boldsymbol{\sigma}\boldsymbol{n}$. We may thus reduce the coupling condition (7.24) to

$$\boldsymbol{\sigma}\boldsymbol{n} = 2A^{-1}\lambda_a^{-1}\Sigma\boldsymbol{n} \qquad \text{on} \quad R = A. \tag{7.27}$$

Similarly, the boundary condition (7.22) may be written

$$\boldsymbol{\sigma}\boldsymbol{n} = -\mathcal{P}\boldsymbol{n} \qquad \text{on} \quad R = B. \tag{7.28}$$

(iv) *Overall response of the coated shell*

The problem of radial deformation of a thick-walled incompressible isotropic elastic spherical shell has been studied by various authors. Here, we follow the analysis presented in Ogden (1984). Thus, we have

$$\lambda_a^3 - 1 = (R/A)^3(\lambda^3 - 1) = (B/A)^3(\lambda_b^3 - 1), \tag{7.29}$$

where λ is defined by (7.3), $\lambda_a = a/A$ has already been introduced, and $\lambda_b = b/B$ with $b = r(B)$. Assuming the displacement to be directed inward we therefore have

$$\lambda_a \leqslant \lambda \leqslant \lambda_b \leqslant 1, \tag{7.30}$$

with equality holding if and only if $\lambda \equiv 1$.

The only component of the equilibrium equation not satisfied trivially is the radial equation,

$$\frac{\mathrm{d}\sigma_1}{\mathrm{d}r} + \frac{2}{r}(\sigma_1 - \sigma_2) = 0, \tag{7.31}$$

where σ_1 is the principal Cauchy stress in the radial direction and σ_2 the (equibiaxial) principal stress associated with the θ and ϕ directions.

The coupling and boundary conditions (7.27) and (7.28) may be put simply as

$$\sigma_1 = 2A^{-1}\lambda_a^{-1}\Sigma \qquad \text{on} \quad R = A, \qquad \sigma_1 = -\mathcal{P} \qquad \text{on} \quad R = B. \tag{7.32}$$

From the incompressibility condition $\lambda_1\lambda_2\lambda_3 = 1$, where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches associated with the deformation of the bulk material, we have $\lambda_2 = \lambda_3 = \lambda$ and $\lambda_1 = \lambda^{-2}$. Writing the strain-energy function $W(\lambda_1, \lambda_2, \lambda_3)$ in the form,

$$\hat{W}(\lambda) = W(\lambda^{-2}, \lambda, \lambda), \tag{7.33}$$

we then have (Ogden 1984)

$$\hat{W}'(\lambda) = 2(\sigma_2 - \sigma_1). \tag{7.34}$$

Following Ogden (1984, § 5.3), integration of (7.31) using λ as the independent variable in place of r yields, via (7.3) and (7.29),

$$\mathcal{P} = \int_{\lambda_a}^{\lambda_b} \frac{\hat{W}'(\lambda)}{\lambda^3 - 1}\,\mathrm{d}\lambda - 2A^{-1}\lambda_a^{-1}\Sigma. \tag{7.35}$$

Note that (7.35) gives $\mathcal{P}$ as a function of λ_a since, by (7.29),

$$\lambda_b = [1 + \eta^{-1}(\lambda_a^3 - 1)]^{1/3}, \tag{7.36}$$

where $\eta = (B/A)^3 > 1$.

In the absence of the bulk material and with the external pressure replaced by an internal pressure, Σ may be interpreted as a surface tension. In the present context we may regard Σ as an effective surface compression ($\Sigma < 0$ for $\lambda_a < 1$). Further discussion of this point will be provided below.

With reference to (7.34), for equibiaxial stress ($\sigma_2 = \sigma_3$) with σ_1 taken to be zero (since (7.34) is insensitive to a superposed hydrostatic stress) and with associated equibiaxial stretch $\lambda_2 = \lambda_3 = \lambda$, we may assume that the inequality $\hat{W}'(\lambda) > 0$ (< 0) holds for $\lambda > 1$ (< 1). (This is also a consequence of strong ellipticity.) Thus, the integrand in (7.35) is positive. For fixed $\lambda_a < 1$ the integral is fixed and positive. Hence, if $\Sigma < 0$ for $\lambda_a < 1$, as assumed, then the pressure $\mathcal{P}$ required to maintain the *same* value of λ_a in the presence of the coating exceeds that needed when there is no coating. The coating therefore has a stiffening effect. Alternatively, we note that for given $\mathcal{P}$, λ_a is closer to unity in the presence of coating so that the coating reduces the strain (and the strain gradient) in the bulk material.

In this discussion the possibility of bifurcation is disregarded, but we remark that bifurcation may be promoted by the presence of the coating, as happens in some circumstances in the analogous two-dimensional problem of an externally pressurized circular cylindrical tube (Ogden *et al.* 1997). Bifurcation analysis for the problem considered here will be discussed elsewhere.

(v) *Equatorial traction and bending moment*

With reference to the integrand in the first integral on the right-hand side of (4.15) it may be inferred that the traction across an equatorial circle per unit reference length is $\boldsymbol{f}$, say, where

$$\boldsymbol{f} = \boldsymbol{P}^{\alpha}\nu_{\alpha} - (\boldsymbol{M}^{\alpha\beta}\nu_{\alpha}\tau_{\beta})', \tag{7.37}$$

and, for the current problem, $\boldsymbol{\tau} = \boldsymbol{e}_{\theta}$ is tangent to the equator, $\boldsymbol{\nu} = \boldsymbol{e}_{\phi}$ is normal to the equator in the tangent plane and the prime denotes $A^{-1}\mathrm{d}/\mathrm{d}\theta$.

Recalling (4.26), (4.30) and (7.24), we have

$$\boldsymbol{P}^{\alpha} = \lambda_a^2 \Sigma \boldsymbol{g}^{\alpha}, \tag{7.38}$$

while, with reference to (3.17) and the second equation in (7.21) and on use of the second equation in (7.12) with $\boldsymbol{n} = -\boldsymbol{e}_{\phi}$, we obtain

$$\boldsymbol{M}^{\alpha\beta} = mG^{\alpha\beta}\boldsymbol{e}_{\rho}. \tag{7.39}$$

It follows that $\boldsymbol{M}^{\alpha\beta}\nu_{\alpha}\tau_{\beta} = \boldsymbol{0}$ so that there is no twisting couple on the equator, and hence

$$\boldsymbol{f} = \lambda_a^2 \Sigma \boldsymbol{g}^{\alpha}\nu_{\alpha}. \tag{7.40}$$

Using the connection $\hat{\boldsymbol{f}}\mathrm{d}s = \boldsymbol{f}\mathrm{d}S$, where $\hat{\boldsymbol{f}}$ is the traction across the equator per unit *current* length, it is then straightforward to show that

$$\hat{\boldsymbol{f}} = \Sigma\hat{\boldsymbol{\nu}}, \tag{7.41}$$

where $\hat{\boldsymbol{\nu}} = \boldsymbol{e}_\phi$ is the current (and also reference) normal to the equator in the tangent plane. Because of the spherical symmetry there is therefore, as expected, no shear traction on the equator.

Since the spherical shell is subject to an *external* pressure, it is to be expected that the traction on the equator is compressive, that is $\Sigma < 0$ for $\lambda_a < 1$. This furnishes an interpretation of Σ, and it is reasonable to take this condition as a constitutive assumption.

From (4.24) the moment per unit reference length on the equator is given by $\boldsymbol{c} = \boldsymbol{y}_\nu \times \boldsymbol{m}$. For the present problem we deduce from (7.39) with (4.25) that $\boldsymbol{m} = m\boldsymbol{e}_\rho$. With the results $\nu^\alpha = \boldsymbol{\nu}\cdot\boldsymbol{G}^\alpha = \boldsymbol{e}_\phi\cdot\boldsymbol{G}^\alpha = G^{\alpha\beta}\boldsymbol{e}_\phi\cdot\boldsymbol{G}_\beta$ and (7.7) and (7.8) we obtain $\boldsymbol{y}_\nu = \nu^\alpha \boldsymbol{y}_{,\alpha} = \lambda_a \boldsymbol{e}_\phi$. Hence,

$$\boldsymbol{c} = \lambda_a m \boldsymbol{e}_\theta. \tag{7.42}$$

The corresponding moment per unit *deformed* length of the equator is $m\boldsymbol{e}_\theta$. Since the curvature of the surface is expected to increase under compression, it is natural to assume that $\boldsymbol{c}$ is directed along $-\boldsymbol{e}_\theta$, corresponding to $m < 0$ for $\lambda_a < 1$.

(*b*) *Tension/compression of a coated solid circular cylinder*

We consider the problem of axial extension/contraction of an incompressible isotropic elastic solid circular cylinder bonded at its lateral surface to an elastic film. We assume the film to be isotropic, in a sense to be defined, with respect to a certain plane configuration. The film is first mapped isometrically to the lateral surface of the undeformed cylinder and the combination is then extended or contracted axially. This induces radial contraction or expansion of the film respectively. This problem differs from that considered in (*a*) above in that the material symmetries of the film and substrate are specified with respect to configurations that are not compatible. In addition, we impose further restrictions on the strain energy of the film such that its response is consistent with that of a conventional elastic plate. We first discuss this modification and then proceed to study the film–substrate interaction.

(i) *Strain-energy function*

We define the plane film to be isotropic if the symmetry set $\mathcal{G} = \{\boldsymbol{R}, \boldsymbol{0}\}$ for all two-dimensional orthogonal $\boldsymbol{R}$. According to (6.17) and (6.18) we then have

$$U(\boldsymbol{C}, \boldsymbol{\kappa}) = \begin{cases} U(\boldsymbol{RCR}^{\mathrm{T}},\, \boldsymbol{R\kappa R}^{\mathrm{T}}), & \det \boldsymbol{R} = +1, \\ U(\boldsymbol{RCR}^{\mathrm{T}}, -\boldsymbol{R\kappa R}^{\mathrm{T}}), & \det \boldsymbol{R} = -1. \end{cases} \tag{7.43}$$

This does not conform to the standard definition of isotropy, according to which the first branch of (7.43) applies with $\det \boldsymbol{R} = \pm 1$. The conventional definition is incompatible with the theory of symmetry adopted in this work unless U happens to be insensitive to curvature. The latter specialization furnishes a model of ideal membranes.

We further assume the strain energy to be an even function of curvature since this implies that the couples $m^{\alpha\beta}$ vanish at $\boldsymbol{\kappa} = \boldsymbol{0}$ provided that U is differentiable, and this in turn requires that plane deformations be supported without bending moments. This restriction is consistent with the behaviour of conventional isotropic plates, regarded as thin, uniform, three-dimensional prismatic bodies.

The functional basis for isotropic response differs from that for hemitropy. To construct it we note, with reference to (6.20), that for arbitrary orthogonal $\boldsymbol{R}$ the scalars I_1 to I_5 are invariant under replacement of $\boldsymbol{C}$ and $\boldsymbol{\kappa}$ by $\boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^{\mathrm{T}}$ and $\boldsymbol{R}\boldsymbol{\kappa}\boldsymbol{R}^{\mathrm{T}}$, respectively, whereas $I_6 \to (\det \boldsymbol{R}) I_6$. Among these, only I_3 and I_5 are altered (to the extent of a change of sign) if $\boldsymbol{\kappa}$ is replaced by $-\boldsymbol{R}\boldsymbol{\kappa}\boldsymbol{R}^{\mathrm{T}}$, while I_6 is unaffected by this replacement provided that $\det \boldsymbol{R} = -1$. The functional basis for isotropic response under the additional restriction that U be an even function of curvature thus consists of the independent invariants

$$I_1,\ I_2,\ I_3^2,\ I_4,\ I_5^2,\ I_6^2,\ I_7 \equiv I_3 I_5. \tag{7.44}$$

All but I_6^2 have counterparts among the invariants given by Simmonds (1985).

The response functions are now given by

$$\left.\begin{aligned}
\tfrac{1}{2} J\sigma^{\alpha\beta} &= \frac{\partial U}{\partial I_1} G^{\alpha\beta} + \frac{\partial U}{\partial I_2} \tilde{C}^{\alpha\beta} + \left(2I_5 \frac{\partial U}{\partial I_5^2} + I_3 \frac{\partial U}{\partial I_7}\right)\kappa^{\alpha\beta} + I_6 \frac{\partial U}{\partial I_6^2}(D^{\alpha\beta} + D^{\beta\alpha}),\\
Jm^{\alpha\beta} &= \left(2I_3 \frac{\partial U}{\partial I_3^2} + I_5 \frac{\partial U}{\partial I_7}\right) G^{\alpha\beta} + \frac{\partial U}{\partial I_4}\tilde{\kappa}^{\alpha\beta} + \left(2I_5 \frac{\partial U}{\partial I_5^2} + I_3 \frac{\partial U}{\partial I_7}\right) C^{\alpha\beta}\\
&\quad + I_6 \frac{\partial U}{\partial I_6^2}(E^{\alpha\beta} + E^{\beta\alpha}).
\end{aligned}\right\} \tag{7.45}$$

(ii) *Response of the coated cylinder*

In the reference configuration of the solid circular cylinder a material particle has position vector,

$$\boldsymbol{X} = R\boldsymbol{e}_r(\theta) + Z\boldsymbol{k}, \qquad 0 \leqslant R \leqslant A, \quad 0 \leqslant \theta < 2\pi, \quad 0 \leqslant Z \leqslant L, \tag{7.46}$$

where (R, θ, Z) are the cylindrical polar coordinates of the particle, $\boldsymbol{e}_r$ and $\boldsymbol{k}$ are the associated radial and axial unit vectors, and A, L are the radius and length of the cylinder, respectively. We consider the isochoric deformation defined by

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}) = \lambda^{-1/2} R\boldsymbol{e}_r(\theta) + \lambda Z\boldsymbol{k}, \qquad \lambda > 0, \tag{7.47}$$

where λ is the axial stretch. The lateral surface of the reference cylinder is defined by (7.46), evaluated at $R = A$. We take this to be the reference configuration of the film for the purpose of analysing the coupled response of the film and cylinder. Thus,

$$\left.\begin{aligned}
\boldsymbol{Y}(\theta^1, \theta^2) &= A\boldsymbol{e}_r(\theta) + Z\boldsymbol{k},\\
\boldsymbol{y}(\theta^1, \theta^2) &= \lambda^{-1/2} A\boldsymbol{e}_r(\theta) + \lambda Z\boldsymbol{k}, \qquad \theta^1 = A\theta, \quad \theta^2 = Z.
\end{aligned}\right\} \tag{7.48}$$

The configuration used in the specification of the response functions is a rectangle of dimensions $2\pi A \times L$ lying in the plane spanned by a fixed orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\,(=\boldsymbol{k})\}$. In this configuration the particle with coordinates (θ^1, θ^2) has position

$$\boldsymbol{Y}_0(\theta^1, \theta^2) = \theta^\alpha \boldsymbol{e}_\alpha. \tag{7.49}$$

The natural basis and metric components on the tangent plane of the stretched cylinder are obtained from (2.5) and (2.15) in the forms

$$\boldsymbol{g}_1 = \lambda^{-1/2}\boldsymbol{e}_\theta(\theta), \quad \boldsymbol{g}_2 = \lambda\boldsymbol{e}_2, \quad (g_{\alpha\beta}) = \operatorname{diag}(\lambda^{-1}, \lambda^2), \quad (g^{\alpha\beta}) = \operatorname{diag}(\lambda, \lambda^{-2}), \tag{7.50}$$

where $\boldsymbol{e}_\theta = \boldsymbol{e}_r'(\theta)$ is the azimuthal unit vector. The corresponding quantities on the reference cylinder are obtained by setting $\lambda = 1$. The deformation tensor of the film induced by the deformation $\boldsymbol{Y}_0 \to \boldsymbol{y}$ is

$$\boldsymbol{C} = \lambda^{-1}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda^2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \tag{7.51}$$

and its adjugate is

$$\tilde{\boldsymbol{C}} = \lambda^2 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda^{-1}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2. \tag{7.52}$$

Thus, the local area stretch is $J = \sqrt{\lambda}$. Using (7.50) and (2.12) with $\boldsymbol{n} = \boldsymbol{e}_r$ and $\boldsymbol{e}_\theta' = -\boldsymbol{e}_r$ we derive $b_{11} = -(A\sqrt{\lambda})^{-1}$, and all other $b_{\alpha\beta} = 0$. Then, the curvature of the deformed film relative to the reference plane and its adjugate, respectively, are given by

$$\boldsymbol{\kappa} = (A\sqrt{\lambda})^{-1}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \qquad \tilde{\boldsymbol{\kappa}} = (A\sqrt{\lambda})^{-1}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2. \tag{7.53}$$

The invariants are

$$\left.\begin{aligned}
I_1 &= \operatorname{tr}\boldsymbol{C} = \lambda^2 + \lambda^{-1},\\
I_2 &= \det\boldsymbol{C} = \lambda,\\
I_3^2 &= (\operatorname{tr}\boldsymbol{\kappa})^2 = (A^2\lambda)^{-1},\\
I_4 &= \det\boldsymbol{\kappa} = 0,\\
I_5^2 &= (\operatorname{tr}(\boldsymbol{C\kappa}))^2 = (A^2\lambda^3)^{-1},\\
I_6^2 &= (\operatorname{tr}(\boldsymbol{C\kappa\mu}))^2 = 0,\\
I_7 &= (\operatorname{tr}\boldsymbol{\kappa})\operatorname{tr}(\boldsymbol{C\kappa}) = (A^2\lambda^2)^{-1},
\end{aligned}\right\} \tag{7.54}$$

wherein $\boldsymbol{\mu} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1$ has been used to evaluate I_6.

The response of the film to the deformation $\boldsymbol{Y}_0 \to \boldsymbol{y}$ may be determined from (7.51) to (7.54) and (7.45), the latter equations involving contravariant components of the tensors in (7.51) to (7.53) based on the Cartesian metric associated with (7.49). The non-zero values of the response functions are σ^{11}, σ^{22}, m^{11} and m^{22}, and these are constants throughout the film provided that the strain energy is not an explicit function of coordinates. The specific relationships between these quantities and the deformation parameters are not needed here. We remark that the non-standard terms involving $D^{\alpha\beta}$ and $E^{\alpha\beta}$ vanish in the present class of deformations because $I_6 \equiv 0$. The latter variables generate non-zero σ^{12} and m^{12} in the case of hemitropic symmetry unless suitable restrictions are imposed on the strain energy.

Coupling of the response of the film and cylinder is described by (4.34) and (4.35), in which $p^{\alpha\beta}$ and q^α are determined from (4.31) and covariant derivatives are based on the metric $g_{\alpha\beta}$. In particular, the associated Christoffel symbols vanish identically, and the uniformity of the $m^{\alpha\beta}$ then implies that $q^\alpha = 0$, so that no transverse shear forces are required to support the deformation. Likewise, the covariant derivatives of the $p^{\alpha\beta}$ vanish and we obtain $\boldsymbol{p}^\alpha_{;\alpha} = \mathcal{P}\boldsymbol{n}$, where $\mathcal{P} = p^{\alpha\beta}b_{\alpha\beta} = p^{11}b_{11}$ is the pressure transmitted by the film to the cylinder. To compute it we use (4.31), in which $b_1^1 = -\sqrt{\lambda}/A$, and all other b_β^α vanish. We thus derive $p^{11} = \sigma^{11} + (\sqrt{\lambda}/A)m^{11}$, $p^{22} = \sigma^{22}$, $p^{12} = 0$, $p^{21} = 0$ and

$$\mathcal{P} = -(\sigma^{11} + \sqrt{\lambda}m^{11}/A)/A\sqrt{\lambda}. \tag{7.55}$$

Consider next the edge traction $\boldsymbol{f}_\theta$ per unit reference length transmitted across a generator of the cylinder with normal $\boldsymbol{\nu} = \boldsymbol{e}_\theta$. This is given by the expression resulting from the subtraction of the right-hand side from the left in the fourth of equations (4.21). Here, $\nu_1 = \tau_2 = 1$, $\nu_2 = \tau_1 = 0$ and we obtain $\boldsymbol{M}^{\alpha\beta}\nu_\alpha\tau_\beta = -Jm^{12}\boldsymbol{n} = \boldsymbol{0}$. Thus, with $\boldsymbol{P}^1 = J\boldsymbol{p}^1 = \sqrt{\lambda}p^{11}\boldsymbol{g}_1$, we derive $\boldsymbol{f}_\theta = f_\theta \boldsymbol{e}_\theta$, where

$$f_\theta = \sigma^{11} + \sqrt{\lambda}m^{11}/A = -\mathcal{P}A\sqrt{\lambda} \tag{7.56}$$

is the hoop traction. The relation between hoop traction and pressure is identical to that furnished by elementary statics in which $-\mathcal{P}$ is the pressure acting on the film. The sign of the hoop traction depends on the value of λ and the details of the constitutive relations. In particular, we assume the existence of an interval of λ values larger than unity for which $f_\theta < 0$. With reference to the analysis of § 5, we note that such deformations cannot be energy minimizers in the absence of flexural resistance. According to the energy criterion they would then be unstable with respect to arbitrary disturbances, however small. While this restriction does not obtain in the presence of flexural resistance, compressive hoop traction of sufficient magnitude would nevertheless be expected to promote bifurcation buckling, thus making the cylinder susceptible to a source of potential instability that would not exist in the absence of the film. An analysis of the associated bifurcation criteria based on a theory for infinitesimal deformations has been given by Bert & Birman (1993).

The bending couple $\boldsymbol{c}_\theta$ per unit reference length of the generator is obtained from (4.24) and (4.25) in which $\boldsymbol{y}_\nu = \partial\boldsymbol{y}/\partial\theta^1 = \boldsymbol{e}_\theta/\sqrt{\lambda}$ and $\boldsymbol{m} = \boldsymbol{M}^{11} = -\sqrt{\lambda}m^{11}\boldsymbol{n}$. Thus, $\boldsymbol{c}_\theta = m^{11}\boldsymbol{k}$. If we adopt the mild assumption that $\operatorname{sgn}(\partial U/\partial\kappa_{11}) = \operatorname{sgn}\kappa_{11}$, then $\boldsymbol{c}_\theta$ is directed along $\boldsymbol{k}$ in accordance with a prediction of elementary plate theory.

The traction and couple on a circle of latitude are obtained by setting $\nu_2 = \tau_1 = 1$ and $\nu_1 = \tau_2 = 0$ in (4.21). Thus, $\boldsymbol{M}^{\alpha\beta}\nu_\alpha\tau_\beta = -Jm^{21}\boldsymbol{n} = \boldsymbol{0}$ and the edge traction per unit reference length is $\boldsymbol{f}_z = \boldsymbol{P}^2 = f_z\boldsymbol{k}$, where $f_z = \lambda^{3/2}\sigma^{22}$. The couple per unit reference length is $\boldsymbol{c}_z = -\lambda^{3/2}m^{22}\boldsymbol{e}_\theta$. This is non-zero due to the suppression of anticlastic bending in the present class of deformations.

To compute the response of the film–cylinder combination we assume the solid cylinder to be composed of a homogeneous incompressible isotropic elastic solid. The general constitutive relation for the Piola stress is

$$\boldsymbol{P} = -p\boldsymbol{A}^{-\mathrm{T}} + 2\boldsymbol{A}\left[\left(\frac{\partial W}{\partial J_1} + J_1\frac{\partial W}{\partial J_2}\right)\boldsymbol{I} - \frac{\partial W}{\partial J_2}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right], \tag{7.57}$$

where $\boldsymbol{A}$ is the deformation gradient, $\boldsymbol{I}$ is the unit tensor for 3-space, p is the kinematically indeterminate constraint pressure associated with incompressibility, and $W(J_1, J_2)$ is the strain energy per unit reference volume, a function of the isotropic invariants

$$J_1 = \operatorname{tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}), \qquad J_2 = \tfrac{1}{2}[(\operatorname{tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}))^2 - \operatorname{tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^2]. \tag{7.58}$$

In the present example, we have

$$\boldsymbol{A} = \lambda\boldsymbol{k}\otimes\boldsymbol{k} + \lambda^{-1/2}(\boldsymbol{I} - \boldsymbol{k}\otimes\boldsymbol{k}), \qquad J_1 = \lambda^2 + 2\lambda^{-1}, \qquad J_2 = \lambda^{-2} + 2\lambda \tag{7.59}$$

and the equilibrium equation (the first of equations (4.21)) is satisfied provided that p is uniform. Its value is determined by substituting (7.58) and (7.59) into (4.34) with $\boldsymbol{N} = \boldsymbol{e}_r$. Thus,

$$p = \frac{2}{\lambda}\left(\frac{\partial W}{\partial J_1} + J_1\frac{\partial W}{\partial J_2}\right) - \frac{2}{\lambda^2}\frac{\partial W}{\partial J_2} - \mathcal{P}, \tag{7.60}$$

where $\mathcal{P}$ is the interaction pressure between the cylinder and the film. It is now a straightforward exercise to generate the overall axial force–extension relation for the coated cylinder.

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