Hertzian Contact with a Rigid Indenter Using a Penalty Approach

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1 Problem Definition

Formulation of frictionless contact between the rigid surface (indenter) and an elastic domain, representing an infinite half-space. The elastic domain is approximated using a linear elastic isotropic constitutive relationship. Contact will be solved using a penalty formulation allowing a small amount of interpenetration between the solid and the indenter. The penalty method is a non-exact method of contact.

1.1 Weak Form

Starting from the equilibrium equation:

$$-\operatorname{Div}\boldsymbol{\sigma} = \boldsymbol{f} \rightarrow -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \boldsymbol{f} \tag{1}$$

Write in indicial and convert to weak form

$$\begin{aligned}
-\frac{\partial \sigma_{ij}}{\partial x_j}\mathbf{e_i} &= f_k\mathbf{e_k} \quad \text{Multiply by a test function}\\
-\frac{\partial \sigma_{ij}}{\partial x_j}\mathbf{e_i}\cdot v_p\mathbf{e_p} &= f_k\mathbf{e_k}\cdot v_p\mathbf{e_p}\\
-\frac{\partial \sigma_{ij}}{\partial x_j}v_p\delta_{ip} &= f_k v_p\delta_{kp}\\
-\frac{\partial \sigma_{ij}}{\partial x_j}v_i &= f_k v_k \quad \text{Integrate over the domain}\\
-\int_{\Omega_o}\frac{\partial \sigma_{ij}}{\partial x_j}v_i dV &= \int_{\Omega_o} f_k v_k dV
\end{aligned} \tag{2}$$

Integration by parts on the left hand side of Eq. 2

$$\begin{aligned}
(fg)' = f'g + fg' &\rightarrow f'g = (fg)' - fg'\\
\frac{\partial \sigma_{ij}}{\partial x_j}v_i &= (\sigma_{ij}v_i)_{,j} - \sigma_{ij}\frac{\partial v_i}{\partial x_j}
\end{aligned}$$

Substitute this into Eq. 2

$$\begin{aligned}
-\int_{\Omega_o}(\sigma_{ij}v_i)_{,j}dV + \int_{\Omega_o}\sigma_{ij}\frac{\partial v_i}{\partial x_j}dV &= \int_{\Omega_o} f_k v_k dV \quad \text{Use the divergence theorem}\\
-\int_{\partial\Omega_o}\sigma_{ij}v_i n_j dS + \int_{\Omega_o}\sigma_{ij}\frac{\partial v_i}{\partial x_j}dV &= \int_{\Omega_o} f_k v_k dV \quad \text{Recognize the traction term}\\
-\int_{\partial\Omega_o} t_i v_i dS + \int_{\Omega_o}\sigma_{ij}\frac{\partial v_i}{\partial x_j}dV &= \int_{\Omega_o} f_k v_k dV \quad \text{Rearrange}\\
\int_{\Omega_o}\sigma_{ij}\frac{\partial v_i}{\partial x_j}dV &= \int_{\Omega_o} f_k v_k dV + \int_{\partial\Omega_o} t_i v_i dS
\end{aligned}$$

The principle of virtual work states that:

$$\int_{\Omega} b_i\delta u_i dV + \int_{\partial\Omega} t_i\delta u_i dS = \int_{\Omega}\sigma_{ij}\delta\epsilon_{ij}dV = \int_V \delta W dV \tag{3}$$

Applying this principle to the LHS of our equation where f is equivalent to the body force, b:

$$\begin{aligned} \int_{\Omega_o} \sigma_{ij}\frac{\partial v_i}{\partial x_j}dV = \int_{\Omega_o} f_k v_k dV + \int_{\partial\Omega_o} t_i v_i dS = \int_{\Omega} \sigma_{ij}\epsilon_{ij}dV \quad \text{No traction or body forces} \\ 0 = \int_{\Omega} \sigma_{ij}\epsilon_{ij}\,dV \end{aligned} \tag{4}$$

In direct notation,

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\,d\Omega = 0 \tag{5}$$

1.2 Strain Energy Density

The strain energy density can be defined with the following equation,

$$\Psi = \lambda\,\mathrm{tr}\{\boldsymbol{\epsilon}\}\mathbf{I} + 2\mu\boldsymbol{\epsilon}$$

where the elasticity parameters are defined as the shear modulus, μ (Lame's second parameter) and Lame's first parameter λ. Note that `lambda` is a keyword in FEniCS so we define `lmbda`.

```
E, nu = 10.0, 0.3
mu = Constant(E/(2*(1 + nu)))
lmbda = Constant(E*nu/((1 + nu)*(1 - 2*nu)))
```

where E is the elasticity modulus and ν is Poisson's ratio, a measure of incompressibility.

2 Contact Formulation

The rigid indenter with a spherical surface can be approximated by a parobolic equation instead of explicitly modeled and meshed. Consider the indenter radius, R, to be sufficiently large with respect to the contact region characteristic size ($R >> a$). This relationship, $R >> a$, allows the spherical surface to be approximated by a parabola.

$$h(x, y) = -h_o + \frac{1}{2R}(x^2 + y^2) \tag{6}$$

where h_o is the initial gap between both surfaces at x = y = 0 (the center of the indenter).

The unilateral contact condition on the top surface Γ is known as the Hertz-Signorini-Moreau conditions for frictionless contact

$$g \geq 0, \quad p \leq 0, \quad g \cdot p = 0 \quad \text{on } \Gamma \tag{7}$$

The gap, g, is defined as follows,

$$g = u_z - h(x, y)$$

where we specify the displacement u_z, which indicates only normal contact is considered. $p = -\sigma_{zz}$ is the pressure.

2.1 Penalty Approach

The penalty approach is an inexact method, dependent on the magnitude of a constant known as the penalty parameter. The simplest way to solve this contact condition is to replace the previous complementary conditions by the following penalized condition:

$$\begin{aligned} p &= k_{pen}\langle g\rangle_+ \quad \text{where } g = u_z - h(x, y) \\ &= k_{pen}\langle u_z - h(x, y)\rangle_+ \end{aligned} \tag{8}$$

where k_{pen} is a large penalizing stiffness coefficient, and we have the definition of the Mackauley bracket

$$\langle x \rangle_+ = \frac{|x| + x}{2} \tag{9}$$

Therefore, we can add the penalty term in Eq. 8 to Eq. 4 after multiplying by the test function

$$\int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\, d\Omega + k_{pen} \int_\Gamma \langle g \rangle_+ v_z \, dS = 0$$

where it is important to note that dS represents the top surface that will experience contact.

An illustration of how the penalty formulation works is demonstrated in Fig. 1. A too high penalty parameter can lead to ill-conditioning of the stiffness matrix, but also correlates to better contact enforcement. As the penalty parameter increases, the gap function is more closely enforced at 0, resulting in a displacement that matches the indenter.

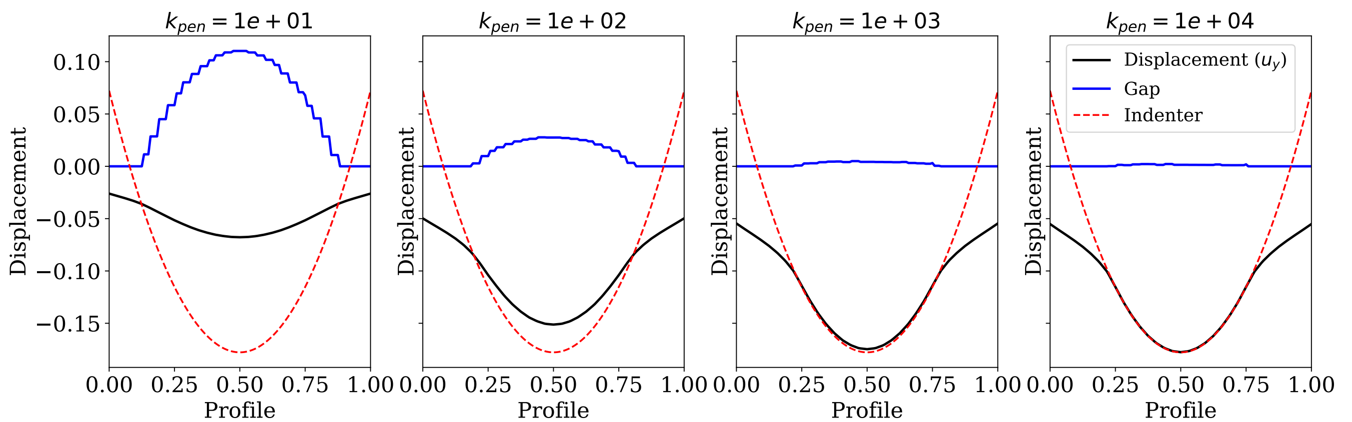

Figure 1: Displacement in the downwards direction u_y in black, gap function in blue, and indenter in red alongside the top surface. The same amount of contact is enforced in each case, as represented by the same indenter profile, where the penalty parameter is varying from a low to high value.

3 FEniCS Implementation

FEniCS implementation was originally provided in 3D, and changes were made to simplify the problem into a 2D problem.

3.1 Analytical Solution

The analytical problem gives the following:

- The contact area, a, is of circular shape (d = depth) and radius, R.

$$a = \sqrt{Rd}$$

- The force exerted by the indenter onto the surface is:

$$F = \frac{4}{3} \frac{E}{(1 + \nu^2)} ad$$

- The pressure distribution on the contact region is given by (p_o is the maximal pressure):

$$p(r) = p_o\sqrt{1 - (\frac{r}{a})^2} \qquad p_o = \frac{3F}{2\pi a^2}$$

These can be solved for in FEniCS

```
a = sqrt(R*d)
F = 4/3.*float(E)/(1-float(nu)**2)*a*d
p0 = 3*F/(2*pi*a**2)
```

Print the maximum pressure and applied force

```
print("Maximum pressure FE: {0:8.5f} Hertz:  {1:8.5f}"
     .format(max(np.abs(p.vector().get_local())), p0))
print("Applied force FE: {0:8.5f} Hertz:  {1:8.5f}"
     .format(4*assemble(p*ds(1)), F))
```