Neo-Hookean Strain Energy Density Functions

Updated on _

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1 Definitions

Young's Modulus: E
Poisson's Ratio: ν
Shear Modulus or first Lamé Parameter: μ
Second Lamé Parameter: λ

$$\mu = \frac{E}{2(1+\nu)} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{1.1}$$

Bulk Modulus: κ

$$\kappa = \lambda + \frac{2}{3}\mu \tag{1.2}$$

Deformation Gradient: $\mathbf{F}$
Right Cauchy Green Tensor: $\mathbf{C} = \mathbf{F}^T\mathbf{F}$
1st Piola Kirchoff Stress: $\mathbf{P}$
The third invariant of the deformation gradient

$$I_3 = J = \det \mathbf{F} \tag{1.3}$$

The first invariant of the right Cauchy Green Tensor:

$$I_1^{\mathbf{C}} = \operatorname{tr} \mathbf{C} \tag{1.4}$$

Lastly, we note a relationship between pressure and the bulk modulus

$$p = -\kappa I_3(I_3 - 1) \tag{1.5}$$

Proof of the derivative $\frac{\partial \operatorname{tr}(\mathbf{F^T F})}{\partial \mathbf{F}}$ in indicial:

$$\begin{aligned}
\frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} &= \frac{\partial \operatorname{tr}(\mathbf{F^T F})}{\partial \mathbf{F}} \\
&= \frac{\partial F_{kI} F_{kI}}{\partial F_{pQ}} \quad \text{Product Rule} \\
&= \frac{\partial F_{kI}}{\partial F_{pQ}} F_{kI} + F_{kI} \frac{\partial F_{kI}}{\partial F_{pQ}} \\
&= \delta_{kp}\delta_{IQ} F_{kI} + F_{kI}\delta_{kp}\delta_{IQ} \\
&= F_{pQ} + F_{pQ} = 2F_{pQ}
\end{aligned}$$

Therefore, we have the following relationships:

$$\begin{aligned}
\frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} &= 2\mathbf{F} \\
\frac{\partial I_3}{\partial \mathbf{F}} &= I_3 \mathbf{F}^{-T}
\end{aligned} \tag{1.6}$$

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2 Compressible Cases

There are several variations for the strain energy density function of a compressible neo-Hookean material

$$\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \frac{\kappa}{2}(I_3 - 1)^2 \\
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \frac{\lambda}{2}(\ln I_3)^2
\end{aligned} \tag{2.1}$$

We can calculate the first PK for the first expression using Eq. 1.6

$$
\begin{aligned}
\mathbf{P} &= \frac{\partial W}{\partial \mathbf{F}} \quad \text{using chain rule}\\
&= \frac{\partial W}{\partial I_1^{\mathbf{C}}}\frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{F}}\\
&= \frac{\mu}{2}\big(2\mathbf{F}\big) + \left[\frac{\mu}{2}\Big(-2\frac{\partial \ln I_3}{\partial I_3}\Big) + \frac{\kappa}{2}\frac{\partial (I_3-1)^2}{\partial I_3}\right] I_3\mathbf{F}^{-T}\\
&= \mu\mathbf{F} + \left[-\frac{\mu}{I_3} + \kappa(I_3-1)\right] I_3\mathbf{F}^{-T}\\
&= \mu\mathbf{F} - \mu\mathbf{F}^{-T} + \kappa I_3(I_3-1)\mathbf{F}^{-T}\\
\mathbf{P} &= \mu(\mathbf{F}-\mathbf{F}^{-T}) + \kappa I_3(I_3-1)\mathbf{F}^{-T}
\end{aligned}
\tag{2.2}
$$

For the second expression

$$
\begin{aligned}
\mathbf{P} &= \frac{\partial W}{\partial I_1^{\mathbf{C}}}\frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{F}}\\
&= \frac{\mu}{2}\big(2\mathbf{F}\big) + \left[\frac{\mu}{2}\Big(-2\frac{\partial \ln I_3}{\partial I_3}\Big) + \frac{\lambda}{2}\frac{\partial (\ln I_3)^2}{\partial I_3}\right] I_3\mathbf{F}^{-T}\\
&= \mu\mathbf{F} + \left[-\frac{\mu}{I_3} + \lambda \ln I_3\right] I_3\mathbf{F}^{-T}\\
&= \mu\mathbf{F} - \mu\mathbf{F}^{-T} + \lambda I_3(\ln I_3)\mathbf{F}^{-T}\\
\mathbf{P} &= \mu(\mathbf{F}-\mathbf{F}^{-T}) + \lambda I_3(\ln I_3)\mathbf{F}^{-T}
\end{aligned}
\tag{2.3}
$$

3 Incompressible Cases

For an incompressible material, $I_1 = \det(\mathbf{F}) = 1$; therefore, Eq. 3.1 can be modified,

$$
\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln(1)) + \frac{\kappa}{2}(1-1)^2\\
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln(1)) + \frac{\lambda}{2}(\ln(1))^2
\end{aligned}
$$

Therefore, we can write

$$
W(\mathbf{F}) = \frac{\mu}{2}(I_1^{\mathbf{C}} - 3) \tag{3.1}
$$

To enforce incompressibility, we can utilize the Lagrange multiplier method where the lagrange multiplier, p, enforces the constraint that J=1:

$$
\begin{aligned}
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3) + \mathrm{p}(I_3 - 1)\\
W(\mathbf{F}) &= \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \mathrm{p}(I_3 - 1)
\end{aligned}
\tag{3.2}
$$

The 1st PK of Eq. 3.2 the first expression

$$
\begin{aligned}
\mathbf{P} &= \frac{\partial W}{\partial I_1^{\mathbf{C}}}\frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{F}}\\
&= \frac{\mu}{2}\big(2\mathbf{F}\big) + pI_3\mathbf{F}^{-T}\\
\mathbf{P} &= \mu\mathbf{F} + pI_3\mathbf{F}^{-T}
\end{aligned}
$$

The 1st PK of the second expression

$$
\begin{aligned}
\mathbf{P} &= \frac{\partial W}{\partial I_1^{\mathbf{C}}}\frac{\partial I_1^{\mathbf{C}}}{\partial \mathbf{F}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{F}} \\
&= \frac{\mu}{2}(2\mathbf{F}) + \left[\frac{\mu}{2}\big(-2\frac{\partial \ln I_3}{\partial I_3}\big) + p\right] I_3\mathbf{F}^{-T} \\
&= \mu\mathbf{F} + \left[-\frac{\mu}{I_3} + p\right] I_3\mathbf{F}^{-T} \\
&= \mu\mathbf{F} - \mu\mathbf{F}^{-T} + pI_3\mathbf{F}^{-T} \\
\mathbf{P} &= \mu(\mathbf{F} - \mathbf{F}^{-T}) + pI_3\mathbf{F}^{-T}
\end{aligned} \tag{3.3}
$$

4 Summary

Compressible neo-Hookean, case 1

$$
\begin{gathered}
W(\mathbf{F}) = \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \frac{\kappa}{2}(I_3 - 1)^2 \\
\mathbf{P} = \mu(\mathbf{F} - \mathbf{F}^{-T}) + \kappa I_3(I_3 - 1)\mathbf{F}^{-T}
\end{gathered} \tag{4.1}
$$

Compressible neo-Hookean, case 2

$$
\begin{gathered}
W(\mathbf{F}) = \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \frac{\lambda}{2}(\ln I_3)^2 \\
\mathbf{P} = \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda I_3(\ln I_3)\mathbf{F}^{-T}
\end{gathered} \tag{4.2}
$$

Incompressible neo-Hookean, case 1

$$
\begin{gathered}
W(\mathbf{F}) = \frac{\mu}{2}(I_1^{\mathbf{C}} - 3) + \mathrm{p}(I_3 - 1) \\
\mathbf{P} = \mu\mathbf{F} + pI_3\mathbf{F}^{-T}
\end{gathered} \tag{4.3}
$$

Incompressible neo-Hookean, case 2

$$
\begin{gathered}
W(\mathbf{F}) = \frac{\mu}{2}(I_1^{\mathbf{C}} - 3 - 2\ln I_3) + \mathrm{p}(I_3 - 1) \\
\mathbf{P} = \mu(\mathbf{F} - \mathbf{F}^{-T}) + pI_3\mathbf{F}^{-T}
\end{gathered} \tag{4.4}
$$