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Damage functions of the internal variables for soft biological fibred tissues

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ABSTRACT

Several typical equations for damage variables are proposed in the literature in the context of Continuum Damage Mechanics for fibred materials such as soft biological tissues. All these proposed functions are defined for the same or similar soft tissues. The applicability of some of them, however, is not clear. The paper compares a series of damage functions given in the literature for the description of softening in soft biological tissues. In addition, a new damage function is introduced. The difference in the response of this function compared to those previously proposed in the literature is that the first derivative is zero at the initial damage state. This represents a potential advantage with respect to numerical calculations since smooth derivatives are obtained.

1. Introduction

There is increasing interest in the mechanics of soft biological tissues in damaged regions as a fundamental part of the study of non-physiological states during diseases or surgery. The description of the constitutive behavior of fibred soft tissue materials in the damage region is usually modelled using the continuum damage mechanics (CDM) approach from which stress–strain relations and local elasticity tensors are derived by internal variables (Simo, 1987). The models that were initially developed to model elastomers have been extended to soft biological tissues in recent years. Some of these models have been restricted to isotropic damage (Hokanson and Yazdami, 1997; Arnoux et al., 2002; Schechtman and Bader, 2002) or damage of the collagenous component only (Natali et al., 2005; Balzani et al., 2006). Only Natali et al. (2003), Calvo et al. (2007), Rodríguez et al. (2008), and Peña et al. (2008, 2009b) consider different damage behavior for matrix and fibres. For other approaches to modelling softening and damage of soft biological tissues see Peña and Doblare (2009) and Volokh (2007).

There are several common equations for the damage variables proposed in the literature for fibred materials such as soft biological tissues (Natali et al., 2003; Balzani et al., 2006; Calvo et al., 2007; Rodríguez et al., 2008; Peña et al., 2008, 2009b). All these proposed functions are applied to similar soft tissues, however, the applicability of some of them is not clear. The main objective of this work is to analyze these damage functions and indicate some of their advantages and disadvantages. The monotonicity and differentiable properties, the number of parameters and the physical meaning of each function are considered. Finally, we propose a new simple damage equation that verifies all these requisites. The paper is organized as follows. In Section 2, a brief summary of the basic results in CDM is introduced. Section 3 presents and analyzes the equations for the internal damage variables for soft biological fibred tissues. Section 4 is devoted to some concluding remarks.

2. Basic results in continuum damage mechanics

Consider a continuum body with reference configuration $\mathcal{B}_0 \subset \mathbb{R}^3$ at the initial reference time $t = 0$. Then, a motion $\boldsymbol{\chi} : \mathcal{B}_0 \times \mathcal{T} \rightarrow \mathcal{B}_t \subset \mathbb{R}^3$ maps this configuration to the current configuration $\mathcal{B}_t$ at each time $t \in \mathcal{T} \subset \mathbb{R}$. Hence, a point $\mathbf{X} \in \mathcal{B}_0$ transforms to a point $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \in \mathcal{B}_t$, where $\mathbf{X}$ and $\mathbf{x}$ define the respective positions of a particle in the reference and current configurations relative to a fixed set of axes. The preferential direction associated to a fibre at a point $\mathbf{X}$ is defined by a unit vector field $\mathbf{m}_0(\mathbf{X})$, $|\mathbf{m}_0| = 1$. It is usually assumed that, under deformation, the fibre moves with the material points of the continuum body, that is, it follows an affine deformation. Therefore, the stretch λ of the fibre defined as the ratio between its lengths at the deformed and reference configurations can be expressed as

$$\lambda \mathbf{m}(\mathbf{x}, t) = \mathbf{F}(\mathbf{X}, t)\mathbf{m}_0(\mathbf{X}) \quad \text{and} \quad \lambda^2 = \mathbf{m}_0 \cdot \mathbf{F}^T\mathbf{F}\mathbf{m}_0 = \mathbf{m}_0 \cdot \mathbf{C}\mathbf{m}_0 \qquad (1)$$

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where $\mathbf{m}$ is the unit vector of the fibre in the deformed configuration. In (1), $\mathbf{F}=\nabla\boldsymbol{\chi}$ and $\mathbf{C}=\mathbf{F}^T\mathbf{F}$ are the standard deformation gradient and the right Cauchy-Green deformation measure. The kinematics introduced for one family of fibres can be applied to other fibre families in an analogous manner. We shall denote a second preferred fibre orientation by the unit vector field $\mathbf{n}_0(\mathbf{X})$. A multiplicative decomposition of $\mathbf{F}=J^{1/3}\bar{\mathbf{F}}$ and $\mathbf{C}=J^{2/3}\bar{\mathbf{C}}$ into *volume-changing (dilational)* and *volume-preserving (distortional)* parts is usually established as in (Flory, 1961), $J=\det\mathbf{F}$ being the Jacobian of the motion.

To characterize isothermal processes, we postulate the existence of a unique decoupled representation of the strain-energy density function Ψ. Because of the directional dependence on the deformation, we require that the function Ψ explicitly depends on both the right Cauchy-Green tensor $\mathbf{C}$ and the fibre directions $\mathbf{m}_0$ and $\mathbf{n}_0$ in the reference configuration. Since the sign of $\mathbf{m}_0$ and $\mathbf{n}_0$ is not significant, Ψ must be an even function of $\mathbf{m}_0$ and $\mathbf{n}_0$ and so it may be expressed by $\Psi=\Psi(\mathbf{C},\mathbf{M},\mathbf{N})$ where $\mathbf{M}=\mathbf{m}_0\otimes\mathbf{m}_0$ and $\mathbf{N}=\mathbf{n}_0\otimes\mathbf{n}_0$ are structural tensors (Spencer, 1971). Based on the kinematic description, the free energy can be rewritten in decoupled form as

$$\Psi(\mathbf{C},\mathbf{M},\mathbf{N})=\Psi_{\text{vol}}(J)+\Psi_{\text{ich}}(\bar{\mathbf{C}},\mathbf{M},\mathbf{N}) \tag{2}$$

where $\Psi_{\text{vol}}(J)$ and $\Psi_{\text{ich}}(\bar{\mathbf{C}},\mathbf{M},\mathbf{N})$ are given scalar-valued functions of J, $\bar{\mathbf{C}}$, $\mathbf{m}_0$ and $\mathbf{n}_0$ respectively that describe the volumetric and isochoric responses of the material (Holzapfel, 2000). In terms of the strain invariants Ψ can be written as

$$\Psi=\Psi_{\text{vol}}(J)+\Psi_{\text{ich}}(\bar{I}_1,\bar{I}_2,\bar{I}_4,\bar{I}_5,\bar{I}_6,\bar{I}_7,\bar{I}_8,\bar{I}_9) \tag{3}$$

with $\bar{I}_1$ and $\bar{I}_2$ being the first two modified strain invariants of the symmetric modified right Cauchy-Green tensor $\bar{\mathbf{C}}$ (note that $I_3=J^2$). Finally, the pseudo-invariants $\bar{I}_4,\ldots,\bar{I}_9$ characterize the constitutive response of the fibres (Spencer, 1971):

$$\begin{aligned}&\bar{I}_4=\bar{\mathbf{C}}:\mathbf{M}=\bar{\lambda}_m^{2}, \quad \bar{I}_5=\bar{\mathbf{C}}^2:\mathbf{M}, \quad \bar{I}_6=\bar{\mathbf{C}}:\mathbf{N}=\bar{\lambda}_n^{2}\\ &\bar{I}_7=\bar{\mathbf{C}}^2:\mathbf{N}, \quad \bar{I}_8=[\mathbf{m}_0\cdot\mathbf{n}_0]\mathbf{m}_0\cdot\bar{\mathbf{C}}\mathbf{n}_0, \quad \bar{I}_9=[\mathbf{m}_0\cdot\mathbf{n}_0]^2\end{aligned} \tag{4}$$

To introduce the damage mechanism let us consider the following Helmholtz free energy function

$$\begin{aligned}\Psi(\mathbf{C},\mathbf{M},\mathbf{N},D_k)&=\Psi_{\text{vol}}(J)+\Psi_{\text{ich}}(\bar{\mathbf{C}},\mathbf{M},\mathbf{N},D_k)\\ &=\Psi_{\text{vol}}(J)+\sum_{k=m,f_1,f_2}[1-D_k]\Psi^0_{\text{ich(k)}}(\bar{\mathbf{C}},\mathbf{M},\mathbf{N})\end{aligned} \tag{5}$$

where $\Psi^0_{\text{ich(k)}}$ is a purely isochoric contribution of the perfectly elastic material defined in (2) being the contributions of the matrix ($k=m$) and the two families of fibres ($k=f_1,f_2$). $[1-D_k]$ are known as the reduction factors, being the internal variables $D_k\in[0,1]$ referred to as the damage variables for the matrix (D_m) and the two families of fibres (D_{f_1} and D_{f_2}) respectively (Calvo et al., 2007).

Standard arguments based on the Clausius–Duhem inequality $\mathcal{D}_{int}=-\dot{\Psi}+(1/2)\mathbf{S}:\dot{\mathbf{C}}\geq 0$, lead to the representation

$$\begin{aligned}\mathbf{S}&=2\frac{\partial\Psi(\mathbf{C},\mathbf{M},\mathbf{N},D_k)}{\partial\mathbf{C}}=2\left[\frac{\partial\Psi_{\text{vol}}(J)}{\partial J}\frac{\partial J}{\partial\mathbf{C}}+\frac{\partial\Psi_{\text{ich}}(\bar{\mathbf{C}},\mathbf{M},\mathbf{N},D_k)}{\partial\bar{\mathbf{C}}}\frac{\partial\bar{\mathbf{C}}}{\partial\mathbf{C}}\right]\\ &=\mathbf{S}_{\text{vol}}+\mathbf{S}_{\text{ich}}\end{aligned} \tag{6}$$

where the second Piola–Kirchhoff stress $\mathbf{S}$ consists of a purely volumetric contribution ($\mathbf{S}_{\text{vol}}$) and a purely isochoric one ($\mathbf{S}_{\text{ich}}$). Moreover, one obtains the following noticeable relations $\partial_{\mathbf{C}}J=(1/2)J\mathbf{C}^{-1}$ and $\mathbf{P}=\partial_{\mathbf{C}}\bar{\mathbf{C}}=J^{-2/3}[\mathbf{I}-(1/3)\mathbf{C}\otimes\mathbf{C}^{-1}]$ that is the fourth-order projection tensor and $\mathbf{I}$ denotes the fourth-order unit tensor, which, in index notation, has the form $\mathrm{I}_{IJKL}=(1/2)[\delta_{IK}\delta_{JL}+\delta_{IL}\delta_{JK}]$. Application of the fourth-order projection tensor $\mathbf{P}$ furnishes the physically correct deviatoric operator in the Lagrangian description, so that $[\boldsymbol{P}:(\cdot)]:\mathbf{C}=\mathbf{0}$.

After some manipulations, we obtain

$$\begin{aligned}\mathbf{S}&=Jp\mathrm{C}^{-\mathbf{1}}+\sum_{k=m,f_1,f_2}2[1-D_k]\left[\boldsymbol{P}:\frac{\partial\Psi^0_{\text{ich(k)}}(\bar{\mathbf{C}},\mathbf{M},\mathbf{N})}{\partial\bar{\mathbf{C}}}\right]\\ &=\mathbf{S}_{\text{vol}}+\sum_{k=m,f_1,f_2}[1-D_k]\mathbf{S}^0_{\text{ich(k)}}\end{aligned} \tag{7}$$

where $\mathbf{S}^0_{\text{ich(k)}}$ is a purely isochoric stress contribution of the perfectly elastic material (Bonet and Wood, 2008), and

$$\mathcal{D}_{int}=\sum_{k=m,f_1,f_2}-\frac{\partial\Psi_k}{\partial D_k}\dot{D}_k=\sum_{k=m,f_1,f_2}f_k\dot{D}_k\geq 0 \tag{8}$$

with f_k conjugate state functions of the internal variables D_k defined as

$$f_m=\Psi^0_{\text{ich(m)}}(\bar{\mathbf{C}});\quad f_{f_1}=\Psi^0_{\text{ich(f1)}}(\bar{\mathbf{C}},\mathbf{M});\quad f_{f_2}=\Psi^0_{\text{ich(f2)}}(\bar{\mathbf{C}},\mathbf{N})\geq 0 \tag{9}$$

Therefore, we can conclude that the undamaged energy function $\Psi^0_{\text{ich(k)}}$ is the thermodynamic force (damage energy release rate) conjugate to the damage variable D_k. The damage mechanism is assumed to be associated to the distortional or isochoric energy and independent of the hydrostatic pressure.

The damage criterion is usually defined in the strain space by the condition that, at any time t of the loading process, the following expression is fulfilled (Simo, 1987)

$$\Phi_k(\mathbf{C}(t),r_{k_t})=\Xi_k-r_{k_t}\leq 0 \tag{10}$$

where Ξ_k is the damaged release rate variable at time $t\in\mathbb{R}_+$ and the parameter r_{k_t} signifies the damage threshold at current time t (i.e. the radius of the damage surface) for matrix and fibres. The equation $\Phi_k(\mathbf{C}(t),r_{k_t})=0$ defines a damage surface in the strain space. The definition of Ξ_k and r_{k_t} depends on the damage model (Natali et al., 2003; Calvo et al., 2007; Balzani et al., 2006; Peña, in press).

Finally, denoting the normal to the damage surface in the strain space by $\mathbf{N}_k=\partial_{\bar{\mathbf{C}}}\Phi_k$, the evolution of the damage parameters D_k is characterized by an irreversible equation of evolution (Simo, 1987; Calvo et al., 2007)

$$\dot{D}_k=\begin{cases}\dot{D}_k & \text{if}\quad \Phi_k=0\quad\text{and}\quad \mathbf{N}_k:\dot{\bar{\mathbf{C}}}>0\\ 0 & \text{otherwise}\end{cases} \tag{11}$$

3. Equations for the internal damage variables

To complete the constitutive formulation of anisotropic finite strain elasticity with damage-caused energy-based softening effects, we have to determine the equation for the internal damage variables. These equations have some important prerequisites:

- The function must be a monotonically increasing function.
- The damage evolution must be dependent on the thermodynamic force associated to the damage.

Additionally, non-necessary but recommendable properties of the damage equations are:

- The equation should ideally be relatively simple and involve only a few material parameters.
- The ideal function should represent physical meaning or interpretation.

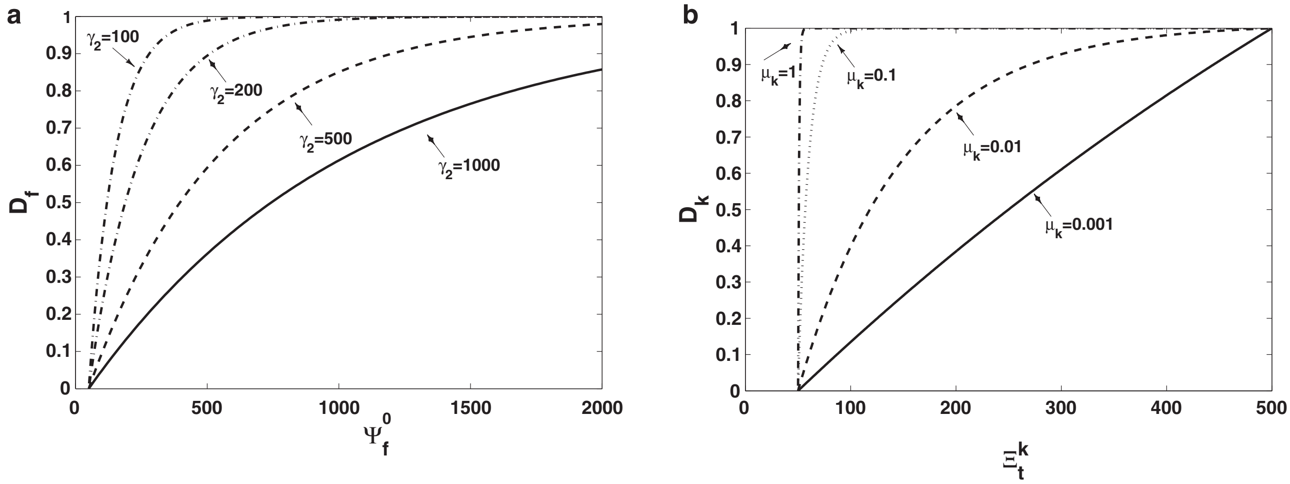

Fig. 1. Damage evolution for the exponential type functions for several values of the parameters γ_2 and μ_k for Eqs. (13) and (14) respectively.

3.1. Exponential type functions

The first CMD phenomenological approach to modelling damage in fibred tissues was proposed by Natali et al. (2003). The proposed damage softening function was

$$D_k(\lambda_{max}, N_f) = 1 - \frac{1 - \exp(\beta_k[\lambda^4_{max}(\alpha(t)) - \lambda^4_{max}(\alpha(1))])}{1 - \exp(\beta_k[\lambda^4_{max}(\alpha(0)) - \lambda^4_{max}(\alpha(1))])} \quad (12)$$

where $\alpha(t) = N_f/N$, N is the density of collagen fibrils on the surface orthogonal to $\mathbf{m}_0$, N_f the density of collagen fibrils that have reached failure and β_k is a scalar parameter that is set on the basis of experimental data. This damage model raises several concerns. First, the damage equation (12) is proposed in stretch λ and not in strain energy Ψ^0 (the thermodynamic force associated to the damage), so it results in non-symmetric algorithmic tangent moduli with a high computational cost (Simo, 1987). Second, we cannot control damage initiation since the functions are defined over the entire stretch range.

Balzani et al. (2006) and Calvo et al. (2007) proposed continuum damage models for fibred biological tissues. Balzani et al. (2006) proposed damage in fibre components only using the following equation

$$D_f(\Psi^0_f) = \gamma_1 \left[1 - \exp\left(\frac{-\beta_f(\Psi^0_f)}{\gamma_2}\right)\right] \quad (13)$$

with the internal variable $\beta_f(\Psi^0_f) = \sup_{0 \le s \le t} \langle \Psi^0_{f(s)} - \Psi^0_{f(ini)} \rangle$, $\Psi^0_{f(ini)}$ characterizes the effective transversely isotropic energy at an initial damage state, γ_1 and γ_2 are material parameters where γ_1 describes the maximum possible discontinuous damage. Similar saturated functions were used by Hokanson and Yazdami (1997) and Ehret and Itskov (2009) for isotropic and anisotropic damage of arteries respectively and Peña et al. (2009b) for a continuous damage variable function.

Remark 3.1. The basic contribution in the original model of Balzani et al. (2006) is the introduction of a modified internal variable and damage criterion rather than the definition of the damage function. The effective transversely isotropic energy at an initial damage state $\Psi^0_{f(ini)}$ is not a material parameter identified by experiments. It is obtained numerically as a result of solving inhomogeneous boundary value problems and is different at each integration point. This was introduced in order to take into account that no damage should occur for the physiological loading at each

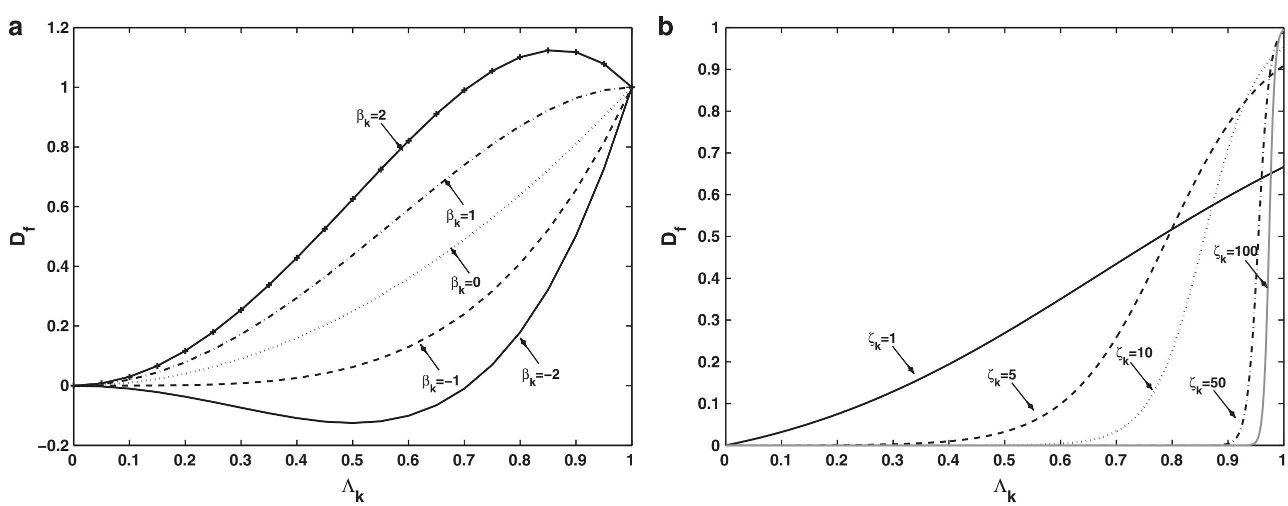

Fig. 2. Damage evolution for the polynomial (a) and sigmoidal (b) functions for several values of the parameters η_k and ζ_k for Eqs. (15) and (17) respectively.

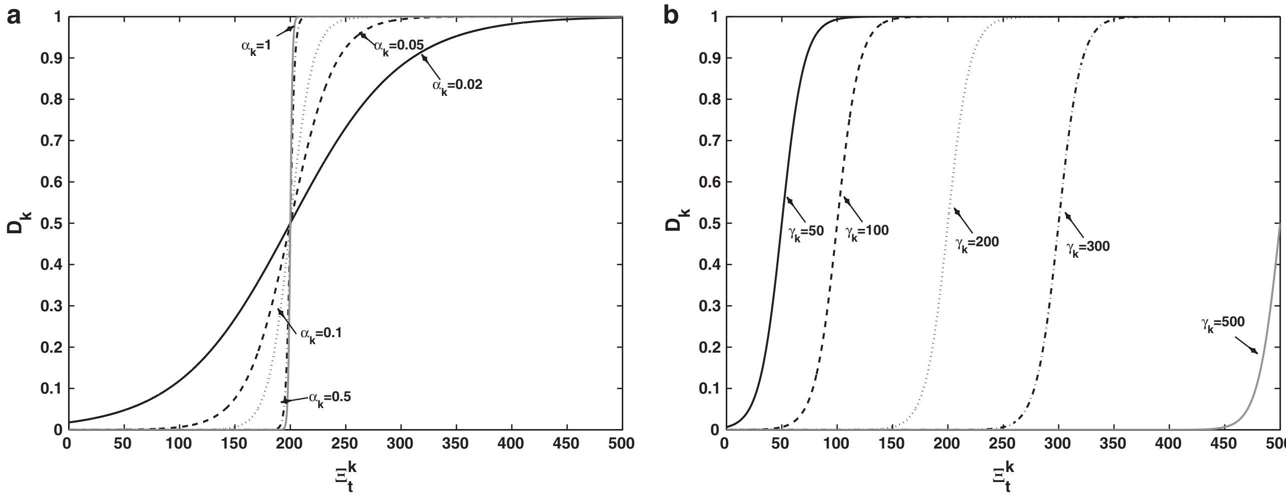

Fig. 3. Damage evolution for the sigmoidal function for several values of the parameter (a) α_k with $\gamma_k = 200$ and (b) γ_k with $\alpha_k = 0.1$ for Eq. (18).

individual point. In addition, the first parameter γ_1 is not a real material parameter since from a physical point of view it should be always equal to 1. However, due to numerical reasons it is usually chosen close to 1 and not adjusted to experimental data. Therefore, Eq. (13) basically considers only one real material parameter.

Calvo et al. (2007) proposed the following damage softening function for matrix and fibres

$$D_k(\Xi_{k_t}) = \begin{cases} 0 & \text{if} \quad \Xi_{k_t} < \Xi_{min_k} \\ 1 - \dfrac{1 - \exp(\mu_k[\Xi_t^k - \Xi_{max_k}])}{1 - \exp(\mu_k[\Xi_{min_k} - \Xi_{max_k}])} & \text{if} \quad \Xi_{min_k} \le \Xi_{k_t} \le \Xi_{max_k} \\ 1 & \text{if} \quad \Xi_{k_t} > \Xi_{max_k} \end{cases} \tag{14}$$

where Ξ_{min_k} and Ξ_{max_k} are the variables associated to the strain energies (10) at initial and total damage for matrix and fibres respectively and $\mu_k \geq 0$ is a material parameter.

Fig. 1 represents Eqs. (13) and (14) for the same Ξ_{min_k}. Both equations reproduce similar damage curves using two (γ_1 and γ_2) or three parameters (μ_k, Ξ_{min_k} and Ξ_{max_k}) respectively. It can be shown that Eq. (13) presents a soft path when damage is close to the asymptotic value of 1 without the need to define Ξ_{max_k}. However, both equations have non-smooth derivatives when the damage starts, and this is a potential disadvantage with respect to numerical calculations. As these equations follow the Eyring theory of thermally activated processes (Ettema et al., 1936) that defines the rates of breakage of the matrix and collagen fibrils (Ciarletta and Ben-Amar, 2009), both equations, (13) and (14), have physical meaning.

3.2. Polynomial type functions

Peña et al. (2008) presented the following polynomial equation

$$D_k(\Xi_{k_t}) = \begin{cases} 0 & \text{if} \quad \Xi_{k_t} < \Xi_{min_k} \\ \Lambda_k^2[1 - \eta_k[\Lambda_k^2 - 1]] & \text{if} \quad \Xi_{min_k} \le \Xi_{k_t} \le \Xi_{max_k} \\ 1 & \text{if} \quad \Xi_{k_t} > \Xi_{max_k} \end{cases} \tag{15}$$

with $\Lambda_k = \frac{\Xi_{k_t} - \Xi_{min_k}}{\Xi_{max_k} - \Xi_{min_k}}$ being a dimensionless variable, and η_k a model exponential parameter. As presented in Peña et al. (2009b), to maintain a monotonic increasing function of D_k with Ξ_{k_t}, $\eta_k \in [-1.0, 1.0]$. This implies that the quality of the fitting of experimental data may be low when constants are restricted by stability considerations to values $\eta_k \in [-1.0, 1.0]$, Fig. 2a. Another important handicap of this function is that the number of material parameters is three (η_k, Ξ_{min_k} and Ξ_{max_k}) and there is no physical meaning.

3.3. Sigmoidal type functions

Another equation proposed in the literature to evolve the damage parameter was presented by Rodríguez et al. (2008)

$$D_k(\Xi_{k_t}) = \frac{1}{2}\left[1 + \frac{2\xi_k \Xi_{k_t} \exp(2\xi_k[[2\Xi_{k_t}/\rho_k] - 1]) - 1}{2\xi_k \Xi_{k_t} \exp(2\xi_k[[2\Xi_{k_t}/\rho_k] - 1]) + 1}\right] \tag{16}$$

with $\rho_k \geq 0$ material parameters. In (16) we cannot control damage initiation since the parameters Ξ_{min_k} and Ξ_{max_k} are not considered. With this in the mind, Peña et al. (2009b) considered the new damage equation

$$D_k(\Xi_{k_t}) = \begin{cases} 0 & \text{if} \quad \Xi_{k_t} < \Xi_{min_k} \\ \dfrac{1}{2}\left[1 + \dfrac{2\zeta_k \Lambda_k \exp(2\zeta_k[2\Lambda_k - 1]) - 1}{2\zeta_k \Lambda_k \exp(2\zeta_k[2\Lambda_k - 1]) + 1}\right] & \text{if} \quad \Xi_{min_k} \le \Xi_{k_t} \le \Xi_{max_k} \\ 1 & \text{if} \quad \Xi_{k_t} > \Xi_{max_k} \end{cases} \tag{17}$$

that is convex for $\zeta_k \geq 0$.

In Fig. 2b Eq. (17) has been presented. This figure shows that the equation presents a soft and continuous path and the function monotonically increases. It can be shown that depending on the parameter ζ_k when $\Lambda_k = 1$ the value of the damage is not always equal to 1, so the total damage function cannot be reached. Eq. (17) follows the Eyring theory and has a physical meaning. However, jumps of the damage function value may occur when switching from $\Xi_{k_t} = \Xi_{max_k}$ to $\Xi_{k_t} > \Xi_{max_k}$ if $D(\Xi_{k_t} = \Xi_{max_k}) \neq 1$ for arbitrary $\zeta_k \geq 0$. In addition, three material parameters (ζ_k, Ξ_{min_k} and Ξ_{max_k}) are needed to characterize the damage equation.

In order to solve these concerns arising from the previously damage equations, a simple sigmoid function is proposed. A sigmoid curve is produced by a mathematical function having a classical "S" shape.

$$D_k(\Xi_{k_t}) = \frac{1}{1 + \exp(-\alpha_k[\Xi_{k_t} - \gamma_k])} \tag{18}$$

where the damaged release rate is $\Xi_{k_t} = \sqrt{2\Psi^0_{\mathrm{ich(k)}}(\overline{\mathbf{C}}(t))}$. The parameter α_k controls the slope and γ_k defines the value Ξ_{k_t} such that $D_k(\Xi_{k_t}) = 0.5$.

Table 1
Material, damage and softening parameters for cava tissue. μ and k_1, k_3 are in MPa, γ_i is in MPa$^{1/2}$ and other parameter are dimensionless.

Exponential	c_1	c_2	c_3	I_{4_0}			
	0.11	6.89	0.0904	1.05			
	μ_m	Ξ_{min_m}	Ξ_{max_m}	μ_f	Ξ_{min_f}	Ξ_{max_f}	ε
	0.116	0.1981	2.964	0.0	0.270	4.1009	0.2697
Polynomial	c_1	c_2	c_3	I_{4_0}			
	0.11	7.01	0.0945	1.05			
	η_m	Ξ_{min_m}	Ξ_{max_m}	η_f	Ξ_{min_f}	Ξ_{max_f}	ε
	−0.274	0.0457	3.9125	1.0	0.0	4.9858	0.1999
Sigmoidal	c_1	c_2	c_3	I_{4_0}			
	0.2583	7.2901	0.0896	1.05			
	α_m	γ_m	α_f	γ_m			ε
	10.0	0.9351	2.016	2.483			0.0649

In general, a sigmoid function is real-valued and differentiable, having either a non-negative or non-positive first derivative which is bell shaped and there is also a pair of horizontal asymptotes in 0 and 1 (see Fig. 3). This means that it is possible to obtain a damage function without consideration of the parameters Ξ_{min_k} and Ξ_{max_k} that control the initial and total damage of the material. In Fig. 3, we can see the effect of the parameters α_k and γ_k that permit this property. The parameter α_k defines the damage growth rate, that is, when α_k increases, the damage evolves very fast and when the parameter α_k is very low the damage occurs more steadily (Fig. 3a). This means that for the same values of the elastic properties, high values of α_k produces lower maximum stress values before rupture and the rupture process is faster. On the other hand, γ_k controls the value of Ξ_{min_k} and Ξ_{max_k}, that is, when γ_k increases the energy needed to start the damage process increases – obviously the energy for total damage increases, see Fig. 3b. This means that for the same values of the elastic properties, for high values of γ_k the damage produces higher maximum stress values before rupture and the rupture process starts for high values of the stress.

Additionally to the convex and monotonic properties, the function follows the Eyring theory, having physical meaning, and requires only two parameters (α_k and γ_k) are needed. Function (18) is therefore a good candidate for a damage equation.

3.4. Experimental data fitting

To further assess the applicability of the different damage functions, the prediction of each kind of functions (exponential (14), polynomial (15) and sigmoidal (18)) was compared with the experimental data collected in our laboratory for muscular vaginal tissue (Calvo et al., 2009). The fitting of the experimental data was developed by using a Levenberg–Marquardt type minimization algorithm Marquardt (1963). The following fitting procedure was performed by defining the objective function $\chi^2 = \Sigma_{j=1}^{n}\left[\sigma_i - \sigma_i^{\tilde{\Psi}}\right]_j^2$. In this function, σ_i is the Cauchy stress data obtained from the uniaxial test (Calvo et al., 2009) and $\sigma_i^{\tilde{\Psi}}$ is the Cauchy stress for the jth point computed following Eq. (7), and n is the number of data points.

The modified expression of the SEF proposed by Natali et al. (2003) was used to define $\Psi^0_{\mathrm{ich(f)}}$ and only one family of fibre oriented in the loading direction was considered (Calvo et al., 2009)

$$\Psi^0_{\mathrm{ich(f)}} = c_1\left[\bar{I}_1 - 3\right] + \frac{c_2}{c_3}[\exp(c_3[\bar{I}_4 - \bar{I}_{4_0}]) - c_3[\bar{I}_4 - \bar{I}_{4_0}] - 1] \quad (19)$$

where c_1 is a stress-like parameter of the matrix (corresponding to a Neo-Hookean SEF with the condition $c_1 \geq 0$ (Ogden, 1996)), $c_2 > 0$ is a stress-like and $c_3 > 0$ a dimensionless parameters associated to the fibres. Moreover, it has been assumed that the strain energy corresponding to the anisotropic terms only contributes to the global mechanical response of the tissue when stretched, that is, $\bar{I}_4 > \bar{I}_{4_0}$. The goodness of the fitting was measured by computing the normalized mean square root error $\varepsilon = \left(\sqrt{\chi^2/(n-q)}\right)/\mu$. In this equation q is the number of parameters of the SEF, so $n - q$ is the number of degrees of freedom, and μ the mean stress already defined above.

The optimized parameters obtained are included in Table 1 and the experimental and numerical results are shown in Fig. 4. The best fit was obtained using the sigmoidal function (18) where ε is close to one. Similar results were obtained using the exponential (14) and polynomial (15) functions where the fit was worse in the final damage region and better than sigmoidal in the initial region. The fitting of the experimental tensile test on the muscular vaginal tissue shows an evident discrepancy between numerical and experimental data for the range of "moderate" stretches, where possibly damage phenomena are limited and the tissue behaves elastically, if viscous effects can be considered negligible (Peña et al., 2009a, 2010).

4. Discussion and summary

The analysis in this paper concerns equations for damage models in the context of CDM that contribute to the characterization of the response of fibred soft biological tissues such as vessels, ligaments and tendons. Several equations are analyzed and some of their advantages and disadvantages are discussed. Finally, we propose a new simple damage equation. The monotonic increase and differentiable properties, the number of parameters and the

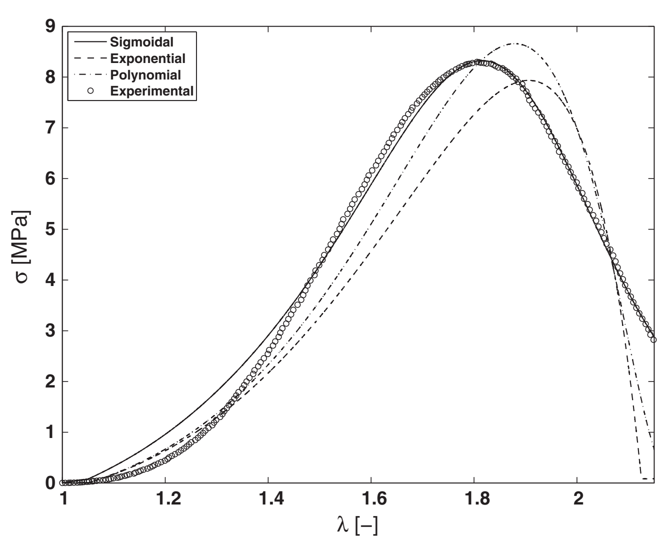

Fig. 4. Experimental and fitted curves for each kind of damage functions for vaginal tissue.

physical meaning of each function are all considered in order to identify a good candidate for a damage equation.

The proposed sigmoid function (18) has four main advantages. First, the first derivative of the sigmoid function is zero at the initial damage state. This represents a potential advantage over numerical calculations since smooth derivatives are obtained. This property is not verified by Eqs. (13) and (14). Second, there is also a pair of horizontal asymptotes in [0, 1], Fig. 3, so it is possible to obtain a damage function without consideration of additional parameters, unlike Eqs. (14), (15) and (17). Third, the sigmoid function is a monotonic increasing function for whole values of the energy and parameters. This important characteristic is not verified by Eq. (15). And finally, Eq. (18), like Eqs. (13) and (14), follows the Eyring theory showing a physical meaning. This fact is not verified by Eq. (15). However, it is important to note that the term "physical meaning" should be used with care since CDM theory represents only a constitutive assumption, mainly based on phenomenological considerations. This holds particularly for complex composite materials such as fibred tissues.

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