

Stabilized Mixed Finite Element Formulation

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Definitions

\mathbf{F} : Deformation gradient

\mathbf{I} : second-order unit tensor

\mathbf{u} : Displacement

J : determinant of the deformation gradient

\mathbf{C} : Right Cauchy-Green Strain Tensor

$\mathcal{W}(\mathbf{F})$: strain energy function

\mathbf{P} : first Piola-Kirchhoff stress tensor

\mathbf{S} : second PK stress tensor

α : cracks are represented by a scalar phase-field variable

p : Lagrange multiplier, hydrostatic pressure field

κ : bulk modulus

$$\kappa = \frac{E}{3(1 - 2\nu)} \quad (0.1)$$

μ : shear modulus

$$\mu = \frac{E}{2(1 + \nu)} \quad (0.2)$$

λ : Lamé modulus

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad (0.3)$$

\mathcal{E}_ℓ : potential energy functional $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

c_w is a normalization constant

1 Hyperelastic Phase-Field Fracture Models

Deformation Gradient

$$\mathbf{F} = \mathbf{I} + \nabla \otimes \mathbf{u} \quad (1.1)$$

where $J = \det \mathbf{F}$ and $\mathbf{C} = \mathbf{F}^T \mathbf{F}$.

The strain energy function $\mathcal{W}(\mathbf{F})$ is defined per unit reference volume such that the first PK and second PK

$$\mathbf{P} = \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \quad (1.2a)$$

$$\mathbf{S} = 2 \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{C}} \quad (1.2b)$$

where $\mathbf{P} = \mathbf{F}\mathbf{S}$.

1.1 Phase-Field Fracture Model

Non-modified strain energy function is the compressible Neo-Hookean:

$$\mathcal{W}(\mathbf{F}) = \frac{\mu}{2}(I_1 - 3 - 2 \ln J) \quad (1.3)$$

For incompressible hyperelastic materials, the strain energy function is defined as

$$\widetilde{\mathcal{W}}(\mathbf{F}) = \mathcal{W}(\mathbf{F}) + p(J - 1), \quad (1.4)$$

We instead consider a damage dependent relaxation of the incompressibility constraint and the phase-field couples to the strain energy through the modified function

$$\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J - 1)^2 \quad (1.5)$$

where the decreasing stiffness modulation function and $w(\alpha)$ is an increasing function representing the specific energy dissipation per unit of volume

$$a(\alpha) = (1 - \alpha)^2 \quad w(\alpha) = \alpha \quad (1.6)$$

In the code we have the following definition

$$b(\alpha) = (1 - \alpha)^3 = \sqrt{a^3(\alpha)}$$

To circumvent numerical difficulties, we resort to the classical mixed formulation and introduce the pressure-like field

$$p = -\sqrt{a^3(\alpha)}\kappa(J - 1), \quad (1.7)$$

as an independent variable along with the displacement field.

Lastly, the normalization constant is defined as:

$$c_w = \int_0^1 \sqrt{w(\alpha)} d\alpha \quad (1.8)$$

The first Piola-Kirchhoff stress tensor is given:

$$\begin{aligned} \mathbf{P} &= \frac{\partial \widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{\partial \mathbf{F}} \\ &= \frac{\partial}{\partial \mathbf{F}} \left[a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J - 1)^2 \right] \\ &= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\frac{1}{2}\kappa\frac{\partial (J - 1)^2}{\partial \mathbf{F}} \\ \mathbf{P} &= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa(J - 1)\frac{\partial J}{\partial \mathbf{F}} \quad \text{where } \frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T} \\ \mathbf{P} &= a(\alpha)\frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha)\kappa(J - 1)J\mathbf{F}^{-T} \end{aligned} \quad (1.9)$$

Note that we DO NOT consider this form of the modified functional; therefore, we do not consider this PK stress.

1.1.1 According to the rough draft

Therefore our modified function:

$$\begin{aligned}\widetilde{\mathcal{W}}(\mathbf{F}, \alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\kappa(J-1)^2 \\ &= a(\alpha)\mathcal{W}(\mathbf{F}) + a^3(\alpha)\frac{1}{2}\frac{-p}{\sqrt{a^3(\alpha)}}(J-1) \\ &= a(\alpha)\mathcal{W}(\mathbf{F}) - \sqrt{a^3(\alpha)}\frac{p}{2}(J-1) \\ \widetilde{\mathcal{W}}(\mathbf{F}, \alpha) &= a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)\frac{p}{2}(J-1)\end{aligned}$$

In the rough draft of the paper we have Eq. 5: energy functional of a possibly fractured elastic body with isotropic surface energy

$$\begin{aligned}\mathcal{E}_\ell(\mathbf{u}, \alpha) &= \int_{\Omega} \widetilde{\mathcal{W}}(\mathbf{F}, \alpha) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \mathbf{u} dA \\ \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \int_{\Omega} \left[a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)\frac{p}{2}(J-1) \right] d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega \\ \mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \int_{\Omega} a(\alpha)\mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} b(\alpha)\frac{p}{2}(J-1) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega\end{aligned}$$

Note that we are missing one term.

1.1.2 According to the code

In the code we have the following for the energy functional of the energy problem

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}$$

Energy functional, where we ignore the surface term

$$\begin{aligned}\mathcal{E}_\ell(\mathbf{u}, \alpha) &= \int_{\Omega} \widetilde{\mathcal{W}}(\mathbf{F}, \alpha) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \mathbf{u} dA \\ &= \int_{\Omega} a(\alpha)\mathcal{W}(\mathbf{F}) d\Omega - \int_{\Omega} b(\alpha)p(J-1) d\Omega - \int_{\Omega} \frac{p^2}{2\lambda} d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega\end{aligned}$$

1.1.3 According to a derivation from Bin2020 Paper

Versus Eq. 21 where we drop λ_b which is not a consideration in this formulation

$$\mathcal{E}_\ell(\mathbf{u}, \alpha) = \int_{\Omega} a(\alpha)\mathcal{W}(\mathbf{F}, \alpha) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell \|\nabla \alpha\|^2 \right) d\Omega \quad (1.10)$$

Starting from Eq. 25 in the 2020 Li and Bouklas paper where κ is the bulk modulus

$$\begin{aligned}
\mathcal{E}_\ell(\mathbf{u}, p, \Lambda, \alpha) &= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega \Lambda(p + \sqrt{a^3(\alpha)}\kappa(J-1)) d\Omega \quad \Lambda = -p/\kappa \\
&= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega + \int_\Omega -\frac{p}{\kappa}(p + \sqrt{a^3(\alpha)}\kappa(J-1)) d\Omega \\
&= \mathcal{E}_\ell(\mathbf{u}, \alpha) + \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \frac{p^2}{\kappa} d\Omega - \int_\Omega \frac{p}{\kappa} \sqrt{a^3(\alpha)}\kappa(J-1) d\Omega \\
\mathcal{E}_\ell(\mathbf{u}, p, \alpha) &= \mathcal{E}_\ell(\mathbf{u}, \alpha) - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1) d\Omega
\end{aligned} \tag{1.11}$$

Substitute in $\mathcal{E}_\ell(\mathbf{u}, \alpha)$ and substitute Eq. 1.3

$$\mathcal{E}_\ell(\mathbf{u}, p, \alpha) = \int_\Omega a(\alpha)\mathcal{W}(\mathbf{F})d\Omega + \frac{G_c}{c_w} \int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2 \right) d\Omega - \int_\Omega \frac{p^2}{2\kappa} d\Omega - \int_\Omega \sqrt{a^3(\alpha)}p(J-1) d\Omega$$

1.2 Changes for 2D Code Version

Following the code, we have the following energy functional of the energy problem

$$\begin{aligned}
\widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\mathcal{W}(\mathbf{F}) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda} \\
\widetilde{W}(\mathbf{F}, \alpha) &= (a(\alpha) + k_\ell)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda}
\end{aligned}$$

Where we know k_ℓ is a modeling parameter, so we can list the energy functional as:

$$\widetilde{W}(\mathbf{F}, \alpha) = a(\alpha)\frac{\mu}{2}(I_c - 3 - 2\ln J) - b(\alpha)p(J-1) - \frac{p^2}{2\lambda} \tag{1.12}$$

Derive the 1st PK stress

$$\begin{aligned}
\mathbf{P} &= \frac{\partial \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}} \\
&= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-1}) - b(\alpha)p\frac{\partial \det \mathbf{F}}{\partial \mathbf{F}} \\
\mathbf{P} &= a(\alpha)\mu(\mathbf{F} - \mathbf{F}^{-1}) - b(\alpha)pJ\mathbf{F}^{-1}
\end{aligned}$$

Taking the third component to be zero, in the plane stress case

$$\begin{aligned}
\mathbf{P}_{33} &= a(\alpha)\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - b(\alpha)pJ\mathbf{F}_{33}^{-1} = 0 \\
(1-\alpha)^2\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1-\alpha)^3pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1-\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu\mathbf{F}_{33} - \mu\mathbf{F}_{33}^{-1} - (1-\alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\
\mu\mathbf{F}_{33} &= \mu\mathbf{F}_{33}^{-1} + (1-\alpha)pJ\mathbf{F}_{33}^{-1} \\
\mu\mathbf{F}_{33} &= [\mu + (1-\alpha)pJ]\mathbf{F}_{33}^{-1} \\
\mathbf{F}_{33}^2 &= \frac{\mu + (1-\alpha)pJ}{\mu} \\
\mathbf{C}_{33} &= 1 + \frac{(1-\alpha)pJ}{\mu}
\end{aligned}$$

Treating F_{33} as an independent unknown, we can state the governing equation

$$\begin{aligned}\mu(\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}) - (1 - \alpha)pJ\mathbf{F}_{33}^{-1} &= 0 \\ \mathbf{F}_{33} - \mathbf{F}_{33}^{-1} - \frac{(1 - \alpha)pJ}{\mu}\mathbf{F}_{33}^{-1} &= 0 \\ \mathbf{F}_{33}\mathbf{F}_{33} - \mathbf{F}_{33}^{-1}\mathbf{F}_{33} - \frac{(1 - \alpha)pJ}{\mu}\mathbf{F}_{33}^{-1}\mathbf{F}_{33} &= 0 \\ \mathbf{F}_{33}^2 - 1 - \frac{(1 - \alpha)pJ}{\mu} &= 0\end{aligned}$$

This can be multiplied by its associated test function to obtain the weak form

2 Following Borden: Derivations of Analytical Phase Field

Note the full potential energy functional, which can also be called the lagrangian

$$\mathcal{E}_\ell(\mathbf{u}, p, \alpha) = \int_{\Omega} a(\alpha)\mathcal{W}(\mathbf{F})d\Omega - \int_{\Omega} \frac{p^2}{2\kappa}d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)}p(J - 1)d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha)}{\ell} + \ell\|\nabla\alpha\|^2 \right) d\Omega$$

We can use the Euler-Lagrange equations to arrive at the equations of motion by taking the derivative with respect to displacement, pressure, and the scalar damage field. Starting with displacement:

$$\frac{\partial \mathcal{E}_\ell}{\partial \mathbf{u}} = \int_{\Omega} a(\alpha) \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{u}} d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial p} = - \int_{\Omega} \frac{p}{\kappa} d\Omega - \int_{\Omega} \sqrt{a^3(\alpha)}(J - 1) d\Omega$$

$$\frac{\partial \mathcal{E}_\ell}{\partial \alpha} = - \int_{\Omega} 2(1 - \alpha) \mathcal{W}(\mathbf{F}) d\Omega + \int_{\Omega} 3p(1 - \alpha)^2(J - 1) d\Omega + \frac{G_c}{c_w} \int_{\Omega} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] d\Omega$$

Therefore we have three equations:

First should be mechanical eq

$$-\frac{p}{\kappa} - \sqrt{a^3(\alpha)}(J - 1) = 0$$

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) + 3p(1 - \alpha)^2(J - 1) + \frac{G_c}{c_w} \left[\frac{1}{\ell} + 2\ell \nabla^2 \alpha \right] = 0$$

$$-2(1 - \alpha) \mathcal{W}(\mathbf{F}) + 3p a(\alpha)(J - 1) + \frac{G_c}{c_w \ell} \left[1 + 2\ell^2 \nabla^2 \alpha \right] = 0$$

Obtain for the final equation

$$\begin{aligned}\frac{G_c}{c_2 \ell} - 2\mathcal{W}(\mathbf{F}) + 3p(J - 1) + \\ 2\alpha \mathcal{W}(\mathbf{F}) - 6p\alpha(J - 1) + \\ 3p(J - 1)\alpha^2 + \frac{G_c}{c_w} 2\ell \nabla^2 \alpha &= 0\end{aligned}$$

3 Stabilized Finite Element Method

From this point, the notes haven't been revised in a long time

3.1 Gateaux Derivative

The Gateaux derivative with respect to (\mathbf{u}, α) in direction (\mathbf{v}, β) under the irreversibility condition $\dot{\alpha} \geq 0$.

$$d\mathcal{E}_\ell(\mathbf{u}, \alpha; \mathbf{v}, \beta) \geq 0. \quad (3.1)$$

Calculation of the Gateaux derivative

$$\begin{aligned} d\mathcal{E}_\ell(\mathbf{u}, \mathbf{v})(\alpha, \beta) &= \frac{d}{d\delta} \mathcal{E}_\ell(\mathbf{u} + \delta \mathbf{v}, \alpha + \delta \beta) \Big|_{\delta=0} \\ &= \frac{d}{d\delta} \mathcal{E}_\ell(\mathbf{u} + \delta \mathbf{v}, \alpha) \Big|_{\delta=0} + \frac{d}{d\delta} \mathcal{E}_\ell(\mathbf{u}, \alpha + \delta \beta) \Big|_{\delta=0} \end{aligned}$$

Starting with the first term:

$$\begin{aligned} \frac{d}{d\delta} \mathcal{E}_\ell(\mathbf{u} + \delta \mathbf{v}, \alpha) \Big|_{\delta=0} &= \frac{d}{d\delta} \left[\int_{\Omega} \widetilde{\mathcal{W}}(\mathbf{I} + \nabla(\mathbf{u} + \delta \mathbf{v}), \alpha) d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot (\mathbf{u} + \delta \mathbf{v}) dA \right] \Big|_{\delta=0} \\ &= \left[\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla(\mathbf{u} + \delta \mathbf{v}), \alpha)}{d\delta} d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \frac{d(\mathbf{u} + \delta \mathbf{v})}{d\delta} dA \right] \Big|_{\delta=0} \quad \text{chain rule} \\ &= \left[\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla(\mathbf{u} + \delta \mathbf{v}), \alpha)}{d(\mathbf{I} + \nabla(\mathbf{u} + \delta \mathbf{v}))} \frac{d(\mathbf{I} + \nabla(\mathbf{u} + \delta \mathbf{v}))}{d\delta} d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \frac{d(\mathbf{u} + \delta \mathbf{v})}{d\delta} dA \right] \Big|_{\delta=0} \\ &= \left[\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla \mathbf{u} + \delta \nabla \mathbf{v}, \alpha)}{d(\mathbf{I} + \nabla \mathbf{u} + \delta \nabla \mathbf{v})} \frac{d(\mathbf{I} + \nabla \mathbf{u} + \delta \nabla \mathbf{v})}{d\delta} d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \mathbf{v} dA \right] \Big|_{\delta=0} \\ &= \left[\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla \mathbf{u} + \delta \nabla \mathbf{v}, \alpha)}{d(\mathbf{I} + \nabla \mathbf{u} + \delta \nabla \mathbf{v})} \nabla \mathbf{v} d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \mathbf{v} dA \right] \Big|_{\delta=0} \\ &= \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{I} + \nabla \mathbf{u}, \alpha)}{d(\mathbf{I} + \nabla \mathbf{u})} \nabla \mathbf{v} d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \mathbf{v} dA \\ \frac{d}{d\delta} \mathcal{E}_\ell(\mathbf{u} + \delta \mathbf{v}, \alpha) \Big|_{\delta=0} &= \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\mathbf{F}} \nabla \mathbf{v} d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \mathbf{v} dA \end{aligned}$$

Second term:

$$\begin{aligned}
& \frac{d}{d\delta} \mathcal{E}_\ell(\mathbf{u}, \alpha + \delta\beta) \Big|_{\delta=0} \\
&= \frac{d}{d\delta} \left[\int_{\Omega} \widetilde{\mathcal{W}}(\mathbf{F}, \alpha + \delta\beta) d\Omega + \frac{\mathcal{G}_c}{c_w} \int_{\Omega} \left(\frac{w(\alpha + \delta\beta)}{\ell} + \ell \|\nabla(\alpha + \delta\beta)\|^2 \right) dV \right] \Big|_{\delta=0} \\
&= \left[\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha + \delta\beta)}{d\delta} d\Omega + \frac{\mathcal{G}_c}{c_w} \frac{d}{d\delta} \int_{\Omega} \left(\frac{w(\alpha + \delta\beta)}{\ell} + \ell \|\nabla(\alpha + \delta\beta)\|^2 \right) dV \right] \Big|_{\delta=0} \\
&= \left[\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha + \delta\beta)}{d(\alpha + \delta\beta)} \frac{d(\alpha + \delta\beta)}{d\delta} d\Omega + \frac{\mathcal{G}_c}{c_w} \frac{1}{\ell} \int_{\Omega} \frac{dw(\alpha + \delta\beta)}{d\delta} dV + \frac{\mathcal{G}_c}{c_w} \ell \int_{\Omega} \frac{\|\nabla(\alpha + \delta\beta)\|^2}{d\delta} dV \right] \Big|_{\delta=0} \\
&= \left[\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha + \delta\beta)}{d(\alpha + \delta\beta)} \beta d\Omega + \frac{\mathcal{G}_c}{c_w} \frac{1}{\ell} \int_{\Omega} \frac{dw(\alpha + \delta\beta)}{d(\alpha + \delta\beta)} \frac{d(\alpha + \delta\beta)}{d\delta} dV + \frac{\mathcal{G}_c}{c_w} \ell \int_{\Omega} 2\nabla(\alpha + \delta\beta) \frac{\nabla(\alpha + \delta\beta)}{d\delta} dV \right] \Big|_{\delta=0} \\
&= \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} \beta d\Omega + \left[\frac{\mathcal{G}_c}{c_w} \frac{1}{\ell} \int_{\Omega} \frac{dw(\alpha + \delta\beta)}{d(\alpha + \delta\beta)} \beta dV + \frac{\mathcal{G}_c}{c_w} \ell \int_{\Omega} 2\nabla(\alpha + \delta\beta) \nabla\beta dV \right] \Big|_{\delta=0} \\
&= \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} \beta d\Omega + \frac{\mathcal{G}_c}{c_w} \frac{1}{\ell} \int_{\Omega} \frac{dw(\alpha)}{d\alpha} \beta dV + 2\ell \frac{\mathcal{G}_c}{c_w} \int_{\Omega} \nabla\alpha \cdot \nabla\beta dV \\
&= \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} \beta d\Omega + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} \left[\frac{dw(\alpha)}{d\alpha} \beta + 2\ell^2 (\nabla\alpha \cdot \nabla\beta) \right] dV
\end{aligned}$$

First, consider Eq. 1.7

$$\begin{aligned}
p &= -\sqrt{a^3(\alpha)} \kappa (J - 1) \\
\frac{p}{\kappa} &= -\sqrt{a^3(\alpha)} (J - 1) \\
0 &= -\sqrt{a^3(\alpha)} (J - 1) - \frac{p}{\kappa}
\end{aligned}$$

Multiplying this by test function q and integrating over volume, we obtain an equation that can be combined with the equations from the Gateaux Derivative.

$$\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\mathbf{F}} \nabla \mathbf{v} d\Omega - \int_{\partial_N \Omega} \tilde{\mathbf{g}}_0 \cdot \mathbf{v} dA = 0 \quad (3.2a)$$

$$\int_{\Omega} \left(-\sqrt{a^3(\alpha)} (J - 1) - \frac{p}{\kappa} \right) q dV = 0 \quad (3.2b)$$

$$\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} \beta d\Omega + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} \left[\frac{dw(\alpha)}{d\alpha} \beta + 2\ell^2 (\nabla\alpha \cdot \nabla\beta) \right] dV \geq 0 \quad (3.2c)$$

The strong form

$$\text{Div } \mathbf{P} = 0 \quad \text{in } \Omega \quad (3.3a)$$

$$\mathbf{u} = \tilde{\mathbf{u}}_0 \quad \text{in } \partial_D \Omega \quad (3.3b)$$

$$[\mathbf{FS}] \mathbf{n} = \tilde{\mathbf{g}}_0 \quad \text{on } \partial_N \Omega, \quad (3.3c)$$

where from Eq. 1.9 we can substitute Eq. 1.7

$$\begin{aligned}
\mathbf{P} &= a(\alpha) \frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + a^3(\alpha) \kappa (J - 1) \frac{\partial J}{\partial \mathbf{F}} \quad \text{where } p = -\sqrt{a^3(\alpha)} \kappa (J - 1) \\
\mathbf{P} &= a(\alpha) \frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} - p \sqrt{a^3(\alpha)} \frac{\partial J}{\partial \mathbf{F}}
\end{aligned}$$

and write the mechanical equilibrium equation in Eq. 3.3:

$$\text{Div} \left[a(\alpha) \frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} - p \sqrt{a^3(\alpha)} \frac{\partial J}{\partial \mathbf{F}} \right] = 0 \quad (3.4)$$

Derivation of the KKT condition equations where $\nabla \beta \cdot \nabla \alpha = \nabla(\beta \nabla \alpha) - \beta \Delta \alpha$

$$\begin{aligned} \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} \beta d\Omega + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} \left[\frac{dw(\alpha)}{d\alpha} \beta + 2\ell^2 (\nabla \alpha \cdot \nabla \beta) \right] dV &\geq 0 \\ \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} \beta d\Omega + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} \frac{dw(\alpha)}{d\alpha} \beta dV + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} 2\ell^2 (\nabla \alpha \cdot \nabla \beta) dV &\geq 0 \\ \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} \beta d\Omega + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} \frac{dw(\alpha)}{d\alpha} \beta dV + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} 2\ell^2 (\nabla(\beta \nabla \alpha)) dV - \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} 2\ell^2 (\beta \Delta \alpha) dV &\geq 0 \\ \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} \beta d\Omega + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} \frac{dw(\alpha)}{d\alpha} \beta dV - \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} 2\ell^2 (\beta \Delta \alpha) dV &\geq 0 \\ \left[\int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} d\Omega + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} \left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2 \Delta \alpha \right) dV \right] \beta &\geq 0 \\ \int_{\Omega} \frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} d\Omega + \frac{\mathcal{G}_c}{c_w \ell} \int_{\Omega} \left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2 \Delta \alpha \right) dV &\geq 0 \end{aligned}$$

Grouping terms, we obtain

$$\frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} + \frac{\mathcal{G}_c}{c_w \ell} \left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2 \Delta \alpha \right) \geq 0 \quad \text{in } \Omega \quad (3.5a)$$

$$\dot{\alpha} \geq 0 \quad \text{in } \Omega \quad (3.5b)$$

$$\dot{\alpha} \left[\frac{d\widetilde{\mathcal{W}}(\mathbf{F}, \alpha)}{d\alpha} + \frac{\mathcal{G}_c}{c_w \ell} \left(\frac{dw(\alpha)}{d\alpha} - 2\ell^2 \Delta \alpha \right) \right] \geq 0 \quad \text{in } \Omega \quad (3.5c)$$

$$(3.5d)$$

Lastly, we have the following, (Neumann?)

$$\frac{\partial \alpha}{\partial \mathbf{n}} \geq 0 \quad \text{and} \quad \dot{\alpha} \frac{\partial \alpha}{\partial \mathbf{n}} = 0 \quad \text{on } \partial \Omega \quad (3.6)$$

Multiply Eq. 3.4 with weighting function $\mathbf{v} + (\varpi h^2)/(2\mu)\mathbf{F}^{-T}\nabla q$

$$\begin{aligned}
& \int_{\Omega} \text{Div} \mathbf{P} \cdot \left[\mathbf{v} + \frac{\varpi h^2}{2\mu} \mathbf{F}^{-T} \nabla q \right] dV = 0 \\
& \int_{\Omega} \text{Div} \mathbf{P} \cdot \mathbf{v} dV + \int_{\Omega} \text{Div} \mathbf{P} \cdot \left[\frac{\varpi h^2}{2\mu} \mathbf{F}^{-T} \nabla q \right] dV = 0 \\
& \int_{\Omega} \text{Div} \mathbf{P} \cdot \mathbf{v} dV + \frac{\varpi h^2}{2\mu} \sum_{e=1}^{n_{el}} \int_{\Omega^e} \text{Div} \mathbf{P} \cdot (\mathbf{F}^{-T} \nabla q) dV = 0 \\
& \int_{\Omega} \text{Div} \mathbf{P} \cdot \mathbf{v} dV \\
& + \frac{\varpi h^2}{2\mu} \sum_{e=1}^{n_{el}} \int_{\Omega^e} \text{Div} \left[a(\alpha) \frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \right] \cdot (\mathbf{F}^{-T} \nabla q) dV - \frac{\varpi h^2}{2\mu} \sum_{e=1}^{n_{el}} \int_{\Omega^e} \text{Div} \left[p \sqrt{a^3(\alpha)} \frac{\partial J}{\partial \mathbf{F}} \right] \cdot (\mathbf{F}^{-T} \nabla q) dV = 0 \\
& \quad \text{""} + \text{""} - \frac{\varpi h^2}{2\mu} \sum_{e=1}^{n_{el}} \int_{\Omega^e} \nabla (p \sqrt{a^3(\alpha)}) J \mathbf{F}^{-T} \cdot (\mathbf{F}^{-T} \nabla q) dV = 0 \\
& \quad \text{""} + \text{""} - \frac{\varpi h^2}{2\mu} \sum_{e=1}^{n_{el}} \int_{\Omega^e} \nabla (p \sqrt{a^3(\alpha)}) J \mathbf{F}^{-1} \mathbf{F}^{-T} \cdot (\mathbf{F}^{-1} \mathbf{F}^{-T} \nabla q) dV = 0 \\
& \quad \text{""} + \text{""} - \frac{\varpi h^2}{2\mu} \sum_{e=1}^{n_{el}} \int_{\Omega^e} \nabla (p \sqrt{a^3(\alpha)}) J \cdot (\mathbf{C}^{-1} \nabla q) dV = 0 \\
& \quad \text{""} + \text{""} - \frac{\varpi h^2}{2\mu} \sum_{e=1}^{n_{el}} \int_{\Omega^e} J \mathbf{C}^{-1} : \left[\nabla (p \sqrt{a^3(\alpha)}) \cdot \nabla q \right] dV = 0
\end{aligned}$$

where $\mathbf{P} = a(\alpha) \frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} - p \sqrt{a^3(\alpha)} \frac{\partial J}{\partial \mathbf{F}}$

We also want to deal with the first term where $(fg)' = f'g + fg'$

$$\int_{\Omega} \text{Div} \mathbf{P} \cdot \mathbf{v} dV = \int_{\Omega} (\mathbf{F} \cdot \mathbf{v})_{,X} dV - \int_{\Omega} \mathbf{P} \cdot \frac{\partial \mathbf{v}}{\partial X} dV$$

Leaving

$$\int_{\Omega} \text{Div} \mathbf{P} \cdot \mathbf{v} dV + \frac{\varpi h^2}{2\mu} \sum_{e=1}^{n_{el}} \int_{\Omega^e} \text{Div} \left[a(\alpha) \frac{\mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} \right] \cdot (\mathbf{F}^{-T} \nabla q) dV - \frac{\varpi h^2}{2\mu} \sum_{e=1}^{n_{el}} \int_{\Omega^e} J \mathbf{C}^{-1} : \left[\nabla (p \sqrt{a^3(\alpha)}) \cdot \nabla q \right] dV \quad (3.7)$$

(3.8)