**Abstract**

**Introduction (Bouklas and Ang)**

Fracture of elastomers

Phase Field Method

Stabilized

For elastomers

In the limit of Incompressbillity

Summary

Hyperelastic phase-field fracture model (Li)

**Plane Stress (Ang)**

Hybrid Formulation (Li?)

Stabilized Finite Element Method (Li)

**Strain Energy Decomposition (Li)**

**Numerical Examples (Ang)**

**Plane Stress**

**3D Decomposition**

Studying the fracture behavior of soft materials such as hydrogel and elastomers has been of interest in various fields as these materials see increased usage in a wide variety of technologies, from robotics to biomedical applications. Elastomers are made up of a network of polymer chains which can then be cross-linked to achieve a material which can display a high degree of incompressibility and a degree of elasticity such that large deformations occur. Both elastomers and hydrogel can display time-dependent visco- and poro-elastic effects, which complicates fracture analysis. Furthermore, crack propagation in soft materials is highly rate-dependent and cracks can exhibit blunting before propagation.

To model damage and fracture, we utilize the phase-field formulation, a regularized approach to fracture based on energy minimization. A scalar damage field is introduced which transitions smoothly between a damaged and undamaged state and approximates the sharp crack discontinuity over a region defined by a length-scale. Phase-field modeling has several advantages, the first being that it can allow for accurate crack growth without re-meshing as long as the intrinsic length-scale of the model is resolved by the finite element mesh. Furthermore, it overcomes the inability of classical Griffith’s theory to predict crack initiation, crack kinking and branching angles as well as more complex fracture phenomena.

The formulation of the regularized approximation of brittle fracture in incompressible materials can exhibit severe volumetric locking issues near the incompressible limit. At this limit, traditional mixed displacement and pressure formulations coupled with the damage field can lead to the development of a “pressure bubble”, applied as a traction boundary condition on the newly created fracture surface. To circumvent this unphysical feature, we present a mixed formulation, in which we introduce a damage dependent relaxation of the incompressibility constraint to account for crack opening or microcrack growth. We utilize a combination of a pure Lagrangian formulation for enforcing incompressibility in the undamaged material and a perturbed Lagrangian formulation for the damaged material to diminish the effects of residual pressure fields on the diffused crack faces.

A stabilization technique, formerly utilized for the Stokes flow and incompressible linear/finite elasticity, based on the residuals of the Euler-Lagrange equations multiplied with a differential operator acting on the weight space, and evaluated elementwise is utilized. The stabilization of the mixed finite element method allows for linear interpolation of all field variables that is beneficial to large simulations.

We utilize a staggered solution scheme in FEniCS where the hyperelastic problem (displacements and pressure) is solved for followed by the damage problem (\alpha) using the variational inequality solver distributed in the PETSc/TAO libraries respectively.

**Phase Field method and advantages**

In these approaches, discontinuities are not introduced into the solid. Instead, the fracture surface is approximated by a phase-field, which smoothes the boundary of the crack over a small region. The major advantage of using a phase-field is that the evolution of fracture surfaces follows from the solution of a coupled system of partial differential equations.

In contrast to discrete descriptions of fracture, phase-field descriptions do not require numerical tracking of discontinuities in the displacement field. This greatly reduces implementation complexity.

As will be detailed in the following sections, all phase-field models introduce a regularization parameter, which is related to the diffusive approximation of the sharp crack. In the models from the mechanics community, this parameter has dimensions of a length and can be interpreted as the width of the regularized crack. An important feature of some of these models is that they are provably convergent (in the sense of the so-called Γ-convergence) to Griffith’s theory of brittle fracture as the length scale parameter tends to zero.

Within a finite element formulation, the size of the elements must be sufficiently small to adequately resolve the length scale parameter. This typically leads to the need for very fine meshes, and thus to a significant computational cost, unless an appropriate adaptive local mesh refinement strategy is implemented.

Algorithmically, phase-field approaches may be implemented with monolithic or with staggered schemes, where, respectively, the displacement and the crack phase-field are computed simultaneously or alternately. While monolithic schemes are more efficient, as they only need one loop of (typically Newton–Raphson) iterations, staggered implementations have proved to lead to a significantly better robustness [16] and thus will be considered in this paper.

**Phase Field History**

A general concept in the Griffith’s-type brittle fracture models is that upon the violation of the critical energy release rate a fully opened crack is nucleated or propagated. As a consequence, the process zone, i.e., the zone in which the material transitions from the undamaged to the damaged state is lumped into a single point at the crack tip.

All of these approaches represent cracks as discrete discontinuities, either by inserting discontinuity lines by means of remeshing strategies, or by enriching the displacement field with discontinuities using the partition of unity method of Babuška and Melenk. Tracing the evolution of complex fracture surfaces has, however, proven to be a tedious task, particularly in three dimensions.

This leads to a significant advantage over the discrete fracture description, whose numerical implementation requires explicit (in the classical finite element setting [1, 2]) or implicit (within extended finite element methods [3]) handling of the discontinuities.

A phase-field model for quasi-static brittle fracture emanated from the work of Francfort and co-workers on the variational formulation for Griffith’s-type fracture models. The genesis of the approach is due to Francfort and Marigo [21], and that of the phase-field implementation to Bourdin et al. [13] (see [14] for a comprehensive overview). The variational formulation for quasi- static brittle fracture leads to an energy functional that closely resembles the potential presented by Mumford and Shah [41], which is encountered in image segmentation. A phase-field approximation of the Mumford-Shah potential, based on the theory of C-convergence, was presented by Ambrosio and Tortorelli. An alternative quasi-static formulation of this phase-field approximation has been presented in the recent work of Miehe et al. [37] and Miehe et al. [38]. In this formulation, the phase-field approximation follows from continuum mechanics and thermodynamic arguments.

Within the formulations stemming from Griffith’s theory, we focus on quasi-static models featuring a tension-compression split, which prevent cracking in compression and interpenetration of the crack faces upon closure, and on the staggered algorithmic implementation due to its proved robustness.

The dynamic models developed within the physics community are derived by adapting the phase transition formalism of Landau and Ginzburg [18]. In contrast, the models proposed within the mechanics community are based on the variational formulation of brittle fracture by Francfort and Marigo [19], regularized by Bourdin et al. [10], which extends the classical Griffith’s theory of fracture.

Within the models related to Griffith’s theory, in this paper we will in particular focus on those featuring a tension- compression split. In absence of such split, such as in the basic formulations in [10–13] (herein referred to as *isotropic* models), cracking may arise also under compressive load states, thus leading to unphysical crack propagation patterns.

Finite Strain Crack Tip Fields in Soft Incompressible Elastic Solids

Krishnan, Hui, Long

2008

* Fracture of soft materials such as gels has recently been of interest to physicists and chemists. The physical structure of these materials consists of a dense network of polymer chains in a liquid matrix giving rise to a high degree of elasticity, incompressibility, and strain hardening.
* The nature of cracks in soft elastic materials is qualitatively very different from those in metals and other commonly known brittle materials such as polymer glass and ceramics, in that soft materials exhibit very large deformation and a high degree of strain hardening
* However, theoretical models of the stress and deformation fields near the tips of cracks in soft materials are mostly based on the assumption of small deformations and linear elastic behavior.
* In this paper, we address the nonlinear deformation of the crack tip fields in soft elastic solids in a way that is accessible to experimentalists who are interested in the fracture of soft materials.
* One may question our choice of studying a neo-Hookean solid since real gels or rubber exhibits finite extensibility and strain hardening much more than this idealized model. There are several reasons to study the neo-Hookean solid. First, it is one of the few elastic materials which have universal appeal and where an exact closed form solution of the crack tip fields is known.

The 1/r singularity in weakly nonlinear fracture mechanics

Bouchbinder, Livne, and Fineberg

2009

* The most developed theoretical approach to fracture dynamics, linear elastic fracture mechanics (LEFM) (Freund, 1998; Broberg, 1999), is based on the assumption that deviations from a linear elastic constitutive material behavior takes place only in a small region near the tip of a crack.
* The major prediction of this theoretical framework is that outside of this small nonlinear near-tip region, but not very far from it, the displacement-gradient (and stress) fields are characterized by a 1= r singularity, where r is measured from the crack’s tip.

Gradient Damage Models: Towards full-scale computations

Lorentz and Godard

2011

* Continuum Damage Mechanics seems an appealing approach to describe the different steps of structural failure within a single framework [1]: damage initiation, micro-crack growth, coales- cence and macro-crack propagation up to the ultimate failure of the structure.
* On a physical ground, damage localises in narrow bands so that the length scales of macroscopic fields and of material microstructures are no more separated, hence hindering the valid- ity of macroscopic local laws.

A phase-field description of dynamic brittle fracture

Borden, Verhoosel, Scott, Hughes, and Landis

2012

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* An alternative quasi-static formulation of this phase-field approximation has been presented in the recent work of Miehe et al. [37] and Miehe et al. [38]. In this formulation, the phase-field approximation follows from continuum mechanics and thermodynamic arguments.

An augmented-Lagrangian method for the phase-field approach for pressurized fractures

Wheeler, Wick and Wollner

2014

* Here, the crack propagates if the rate of elastic energy decrease per unit surface area of the increment step is equal to the quasi-static critical energy release rate Gc. The crack does not move if the elastic energy release rate is less than Gc. On the contrary, it is unstable if Gc exceeds the critical rate. Griffith found that Gc is related to the crack surface energy increase.
* The major advantages of using phase-field modeling for crack propagation are fourfold. First, it is a fixed-mesh approach in which remeshing is avoided. Second, the model is purely based on energy minimization and therefore, crack nucleation, propagation and the path are automatically determined (avoiding calculation of additional components such as stress intensity factors). Third, multiple joining and branching of cracks do not require any additional techniques. Consequently, phase- field modeling allows simple handling of large and complex fracture networks. Fourth, crack growth in heterogeneous media does not require any modification in the framework.

A review on phase-field models of brittle fracture and a new fast hybrid formulation

Ambati, Gerasimov, and Lorenzis

2014

* Within the formulations stemming from Griffith’s theory, we focus on quasi-static models featuring a tension-compression split, which prevent cracking in compression and interpenetration of the crack faces upon closure, and on the staggered algorithmic implementation due to its proved robustness.
* In general, the *phase-field* approach to model systems with sharp interfaces consists in incorporating a continuous field variable– -the field *order parameter*—which differentiates between multiple physical phases within a given system through a smooth transition. In the context of fracture, such order parameter describes the smooth transition between the fully broken and intact material phases, thus approximating the sharp crack discontinuity, and is, therefore, referred to as the *crack field*.
* This leads to a significant advantage over the discrete fracture description, whose numerical implementation requires explicit (in the classical finite element setting [1, 2]) or implicit (within extended finite element methods [3]) handling of the discontinuities.
* The dynamic models developed within the physics community are derived by adapting the phase transition formalism of Landau and Ginzburg [18]. In contrast, the models proposed within the mechanics community are based on the variational formulation of brittle fracture by Francfort and Marigo [19], regularized by Bourdin et al. [10], which extends the classical Griffith’s theory of fracture.
* As will be detailed in the following sections, all phase-field models introduce a regularization parameter, which is related to the diffusive approximation of the sharp crack. In the models from the mechanics community, this parameter has dimensions of a length and can be interpreted as the width of the regularized crack. An important feature of some of these models is that they are provably convergent (in the sense of the so-called Γ-convergence) to Griffith’s theory of brittle fracture as the length scale parameter tends to zero.
* Within a finite element formulation, the size of the elements must be sufficiently small to adequately resolve the length scale parameter. This typically leads to the need for very fine meshes, and thus to a significant computational cost, unless an appropriate adaptive local mesh refinement strategy is implemented.
* Algorithmically, phase-field approaches may be implemented with monolithic or with staggered schemes, where, respectively, the displacement and the crack phase-field are computed simultaneously or alternately. While monolithic schemes are more efficient, as they only need one loop of (typically Newton–Raphson) iterations, staggered implementations have proved to lead to a significantly better robustness [16] and thus will be considered in this paper.
* Within the models related to Griffith’s theory, in this paper we will in particular focus on those featuring a tension- compression split. In absence of such split, such as in the basic formulations in [10–13] (herein referred to as *isotropic* models), cracking may arise also under compressive load states, thus leading to unphysical crack propagation patterns.
* On the other hand, we propose and test what we term a hybrid formulation. This formulation formally comprises features from both the isotropic and anisotropic models. It is shown to lead to results very similar to those of the anisotropic model by Miehe et al. [15,16]. However, within a staggered implementation it leads to an incrementally linear problem, which enables a significant reduction of computational cost—about one order of magnitude—with respect to the available anisotropic models. The only seeming disadvantage of the hybrid formulation is its variational inconsistency. In order to deepen our understanding of this new model, we show how it can be derived through a non- variational procedure, inspired by conceptual and structural similarities with gradient-enhanced continuum damage formulations.

Crack tip fields in soft elastic solids subjected to large quasi-static deformation – A review

Long and Hui

2015

* We review analytical solutions for the asymptotic deformation and stress fields near the tip of a crack in soft elastic solids. These solutions are based on finite strain elastostatics and hyperelastic material models, and exhibit significantly different characteristics than the classical crack tip field solutions in linear elastic fracture mechanics.
* Based on this finding, they proposed an energy based fracture criterion where the onset of growth of a pre-existing crack occurs when the applied energy release rate is equal to a characteristic energy/area which in mod- ern terms is called fracture energy — a characteristic of the material
* The most common way to test the fracture behaviors of soft materials is to first introduce a sharp crack in a specimen and then mechanically load the specimen until the crack propagates forward. Compared to other fracture tests which relies on the presence of initial defects to in- duce local failure mechanisms such as cavitation [17–19], the crack geometry provides a well-defined initial configuration where the stress distribution near the crack tip is similar, independent of geometry.
* However, unlike stiff materials such as metal and ceramics, cracks in soft and tough materials can become highly deformed and blunted before propagation [20]. This can be further complicated by time- dependent effects such as viscoelasticity [6] and poroelas- ticity [21,22]. Therefore, the classical solutions for crack tip fields in linear elastic fracture mechanics (LEFM), or the K-field [23], which were based on the assumption of in- finitesimal deformation, are not sufficient to describe the local crack tip stress field in soft materials.
* For thin specimens, e.g. a plate with thickness much smaller than all other relevant geometrical dimensions, it is common to use plane stress theory to compute the crack tip fields. In this theory, the out of plane stresses σ3i are taken to be zero, and the out of plane variation of the in- plane displacement and stress fields are averaged in the thickness direction.
* Section 3.3.3