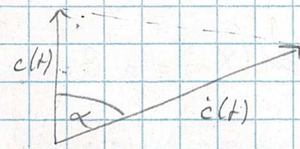
CAGD Ü2

11) $c(t) := e^t(\cos t, \sin t)$ ges: Winkel zw. $\dot{c}(t)$ und $c(t)$

$$\dot{c}(t) = \begin{pmatrix} \frac{d}{dt} e^t \cos t \\ \frac{d}{dt} e^t \sin t \end{pmatrix}^T = \begin{pmatrix} e^t \cos t + e^t(-\sin t) \\ e^t \sin t + e^t \cos t \end{pmatrix}^T = e^t \begin{pmatrix} -\sin t + \cos t \\ \sin t + \cos t \end{pmatrix}^T$$

$$\cos(\alpha) = \frac{c(t) \cdot \dot{c}(t)}{\|c(t)\| \cdot \|\dot{c}(t)\|}$$



$$c(t) \cdot \dot{c}(t) = (e^t \cos t, e^t \sin t) \begin{pmatrix} e^t \cos t - e^t \sin t \\ e^t \sin t + e^t \cos t \end{pmatrix}$$

$$= e^{2t}(\cancel{\cos^2 t} - \cancel{\sin t \cos t} + \cancel{\sin^2 t} + \cancel{\sin t \cos t}) = e^{2t}$$

$$\|c(t)\| = \sqrt{e^{2t}(\cancel{\cos^2 t + \sin^2 t})} = e^t$$

$$\|\dot{c}(t)\| = \sqrt{e^{2t}((-\sin t + \cos t)^2 + (\sin t + \cos t)^2)} = e^t \sqrt{\underbrace{\sin^2 t - \cancel{2 \sin t \cos t} + \cos^2 t}_{1} + \underbrace{\sin^2 t + \cancel{2 \sin t \cos t} + \cos^2 t}_{1}}$$

$$= e^t \sqrt{2}$$

$$\Rightarrow \cos(\alpha) = \frac{e^{2t}}{e^{2t}\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

unabhängig von t! $\square$

CAGD Ü2

12) $c(t) = e^t(\cos t, \sin t)$ ges: Krümmung

$$K(t) := \frac{\det(\dot{c}(t); \ddot{c}(t))}{\|\dot{c}(t)\|^3}$$

$$\dot{c}(t) = e^t \begin{pmatrix} -\sin t + \cos t \\ \sin t + \cos t \end{pmatrix}$$

$$\ddot{c}(t) = e^t \begin{pmatrix} -\sin t + \cos t \\ \sin t + \cos t \end{pmatrix} + e^t \begin{pmatrix} -\cos t - \sin t \\ \cos t - \sin t \end{pmatrix} = e^t \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} = 2e^t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

$$\det(\dot{c}(t), \ddot{c}(t)) = \det \begin{pmatrix} e^t(-\sin t + \cos t) & -2e^t \sin t \\ e^t(\sin t + \cos t) & 2e^t \cos t \end{pmatrix} = 2e^{2t}(-\sin t \cos t + \cos^2 t) + 2e^{2t}(\sin^2 t + \sin t \cos t)$$

$$= 2e^{2t}(\sin^2 t + \cos^2 t) = 2e^{2t}$$

$$\|\dot{c}(t)\| = e^t \sqrt{2}$$

$$\Rightarrow K(t) = \frac{2e^{2t}}{e^{3t} 2\sqrt{2}} = \frac{1}{e^t \sqrt{2}}$$

$\square$

CAGD Ü2

13) $c: \mathbb{R} \to \mathbb{R}^2$ ... Parametrisierung von $f: \mathbb{R} \to \mathbb{R}$ also $c(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$

ges: Krümmung von c

$$\kappa(t) := \frac{\det(\dot{c}(t), \ddot{c}(t))}{\|\dot{c}(t)\|^3} = \frac{\det\begin{pmatrix} 1 & 0 \\ f'(t) & f''(t) \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ f'(t) \end{pmatrix} \right\|^3}$$

$$= \frac{f''(t)}{\left(\sqrt{1+(f'(t))^2}\right)^3} \left(= \frac{f''(t)}{(1+f'(t)^2)\sqrt{1+f'(t)^2}} \right)$$

$\square$

CAGD Ü2

14) zz: Kreis kann nicht durch Bezierkurve parametrisiert werden

Wir wissen, dass Bezierkurven Polynome sind.

Kreis = $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r\}$ Wir wählen $r = 1$.

Angenommen es gäbe eine Bezierkurve, die den Kreis ergibt.
Nennen wir sie $(x(t), y(t))$ für $t \in [0,1]$.

$\Rightarrow x^2(t) + y^2(t) = 1 \quad \forall t \in [0,1]$

also können nicht $x \equiv 0 \equiv y$. Sei d die max Ordnung der beiden.

$\Rightarrow x^2(t) + y^2(t)$ ist ein Polynom von Ordnung $2d$. mit positivem führenden Koeffizienten.

Da $x^2(t) + y^2(t)$ auf $[0,1]$ konstant sein soll muss aber der führende Koeffizient gleich 0 sein ↯ zu Ordnung $2d$.

$\Rightarrow$ Der Kreis kann nicht als Polynom und somit schon gar nicht als Bezierkurve dargestellt werden.

$\square$

CAGD Ü2

15) ges: Bezierkurve mit singulärem Punkt

$b(t) \equiv \binom{0}{0}$ ist eine Bezierkurve $\qquad \dot{b}(t) \equiv \binom{0}{0}$

$$b(t) = B_0^2(t)\, b_0 + B_1^2(t)\, b_1 + B_2^2(t)\, b_0$$
$$= 1\, t^0 (1-t)^2 b_0 + 2 t^1 (1-t)^1 b_1 + 1\, t^2 (1-t)^0 b_0$$
$$= (1-2t+t^2)\, b_0 + (2t-2t^2) b_1 + t^2 b_0$$
$$= (1-2t+2t^2)\, b_0 + (2t-2t^2) b_1$$
$$b'(t) = (-2+4t)\, b_0 + (2-4t)\, b_1$$

$$b'(\tfrac{1}{2}) = (-2+\tfrac{4}{2})\, b_0 + (2-\tfrac{4}{2}) b_1 = 0 \cdot b_0 + 0 \cdot b_1 = 0$$

$\Rightarrow b(\frac{1}{2})$ ist ein singulärer Punkt.

$\square$

CACD Ü2

16) $y(x) = 2(x-1)^2 + 1 = 2(x^2 - 2x + 1) + 1 = 2x^2 - 4x + 3$

ges: Darstellung als Bezierkurve

Polynomdarstellung ist eindeutig $\Rightarrow n = 2$

$$b(t) = B_0^2(t)\,b_0 + B_1^2(t)\,b_1 + B_2^2(t)\,b_2 = 1\cdot t^0(1-t)^2 b_0 + 2t(1-t)b_1 + 1t^2(1-t)^0 b_2$$

$$= (1-2t+t^2)b_0 + (2t-2t^2)b_1 + t^2 b_2 = b_0 + t(-2b_0 + 2b_1) + t^2(b_0 - 2b_1 + b_2)$$

$$\Rightarrow \begin{pmatrix} x \\ 2x^2-4x+3 \end{pmatrix} = (b_0 - 2b_1 + b_2)t^2 + (-2b_0 + 2b_1)t + b_0 \qquad (x = t)$$

$$\Rightarrow \begin{pmatrix} 0 \\ 2 \end{pmatrix} = b_0 - 2b_1 + b_2 \qquad \begin{pmatrix} 1 \\ -4 \end{pmatrix} = -2b_0 + 2b_1 \qquad \begin{pmatrix} 0 \\ 3 \end{pmatrix} = b_0$$

$$\Rightarrow \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} + 2b_1 \quad \Rightarrow 2b_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \Rightarrow b_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0-1 \\ 3-2 \end{pmatrix} + b_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + b_2 \quad \Rightarrow b_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow b(t) = B_0^2(t)\begin{pmatrix} 0 \\ 3 \end{pmatrix} + B_1^2(t)\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + B_2^2(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 2t^2 - 4t + 3 \end{pmatrix}$$

$\square$

CAGD ÜR

17) $b_0, b_1, b_2 \in \mathbb{R}^2$

$$b(t) = \sum_{i=0}^{2} B_i^n(t)\, b_i = (1-t)^2 b_0 + 2t(1-t) b_1 + t^2 b_2$$
$$= (1-2t+t^2) b_0 + (2t-2t^2) b_1 + t^2 b_2$$

$$\dot b(t) = (-2+2t) b_0 + (2-4t) b_1 + (2t) b_2$$

$$v = b_0 - 2b_1 + b_2 \qquad \begin{pmatrix} b_0^x - 2b_1^x + b_2^x \\ b_0^y - 2b_1^y + b_2^y \end{pmatrix}$$

$$0 = \dot b(t) \cdot v = \left((-2+2t)b_0^x + (2-4t)b_1^x + 2t b_2^x,\ (-2+2t)b_0^y + (2-4t)b_1^y + 2t b_2^y\right)$$

$$= (-2+2t){b_0^x}^2 - 2(-2+2t)b_0^x b_1^x + (-2+2t)b_0^x b_2^x + (2-4t)b_0^x b_1^x - 2(2-4t){b_1^x}^2 + (2-4t)b_1^x b_2^x + 2t b_0^x b_2^x - 4t b_1^x b_2^x + 2t {b_2^x}^2 +$$
$$(-2+2t){b_0^y}^2 - 2(-2+2t)b_0^y b_1^y + (-2+2t)b_0^y b_2^y + (2-4t)b_0^y b_1^y - 2(2-4t){b_1^y}^2 + (2-4t)b_1^y b_2^y + 2t b_0^y b_2^y - 4t b_1^y b_2^y + 2t {b_2^y}^2$$

$$= 2\left((-1+t)b_0 + (1-2t)b_1 + t b_2\right) \cdot (b_0 - 2b_1 + b_2)$$

$$= 2\left((-1+t)b_0 \cdot b_0 - 2(-1+t)b_0 \cdot b_1 + (-1+t)b_0 b_2 + (1-2t)b_0 \cdot b_1 - 2(1-2t)b_1 \cdot b_1 + (1-2t)b_1 \cdot b_2 + t b_0 b_2 - 2t b_1 b_2 + t b_2 \cdot b_2\right)$$

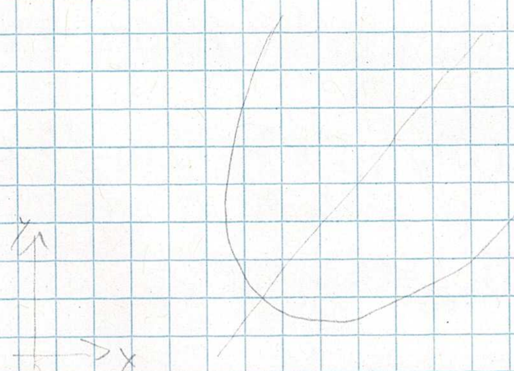
$$= 2\left(t(b_0 \cdot b_0 - 2b_0 b_1 + b_0 b_2 - 2b_0 b_1 + 4b_1 \cdot b_1 - 2b_1 \cdot b_2 + b_0 b_2 - 2b_1 b_2 + b_2 \cdot b_2) + \right.$$
$$\left.(-b_0 \cdot b_0 + 2b_0 b_1 - b_0 b_2 + b_0 b_1 - 2b_1 b_1 + b_1 \cdot b_2)\right) =$$

$$= 2\left(t(b_0 \cdot b_0 - 4b_0 \cdot b_1 + 2b_0 \cdot b_2 + 4b_1 \cdot b_1 - 4b_1 \cdot b_2 + b_2 \cdot b_2) + (-b_0 \cdot b_0 + 3b_0 \cdot b_1 - b_0 \cdot b_2 - 2b_1 \cdot b_1 + b_1 \cdot b_2)\right)$$

$$\Rightarrow \quad t = \frac{b_0 \cdot b_0 - 3b_0 \cdot b_1 + b_0 \cdot b_2 + 2b_1 \cdot b_1 - b_1 \cdot b_2}{b_0 \cdot b_0 - 4b_0 \cdot b_1 + 2b_0 \cdot b_2 + 4b_1 \cdot b_1 - 4b_1 \cdot b_2 + b_2 \cdot b_2}$$

$\boxed{?}$

$$b(t) = (b_0 - 2b_1 + b_2)t^2 + (-2b_0 + 2b_1)t + b_0$$



CAGD Ü3

18) $pr: \mathbb{R}^3 \to \mathbb{R}^2 \qquad (x,y,z) \mapsto (x,y)$

$$b(t) = \sum_{i=0}^{n} B_i^n(t)\, b_i \qquad b_0, \dots, b_n \in \mathbb{R}^3$$

$$pr(b(t)) = \left(\sum_{i=0}^{n} B_i^n(t)\, b_i^x ,\ \sum_{i=0}^{n} B_i^n(t)\, b_i^y \right)$$

$$\tilde{b}(t) := \sum_{i=0}^{n} B_i^n(t)\, pr(b_i) = \left(\sum_{i=0}^{n} B_i^n(t)\, b_i^x ,\ \sum_{i=0}^{n} B_i^n(t)\, b_i^y \right) = pr(b(t))$$

$\square$

CAGD Ü3

19)

$$b(t) = \sum_{i=0}^{3} B_i^3(t) b_i$$

$b_0^0 = b_0, \quad b_1^0 = b_1, \quad b_2^0 = b_2, \quad b_3^0 = b_3$

$b_0^1 = (1-t) b_0^0 + t b_1^0 = b_0 - t b_0 + t b_1 \qquad b_1^1 = (1-t) b_1^0 + t b_2^0 = b_1 - t b_1 + t b_2$

$b_2^1 = (1-t) b_2^0 + t b_3^0 = b_2 - t b_2 + t b_3$

$b_0^2 = (1-t) b_0^1 + t b_1^1 = b_0 - t b_0 + t b_1 - t b_0 + t^2 b_0 - t^2 b_1 + t b_1 - t^2 b_1 + t^2 b_2 = b_0 - 2t b_0 + 2t b_1 + t^2 b_0 - 2t^2 b_1 + t^2 b_2$

$b_1^2 = (1-t) b_1^1 + t b_2^1 = b_1 - t b_1 + t b_2 - t b_1 + t^2 b_1 - t^2 b_2 + t b_2 - t^2 b_2 + t^2 b_3 = b_1 - 2t b_1 + 2t b_2 + t^2 b_1 - 2t^2 b_2 + t^2 b_3$

$b_0^3 = (1-t) b_0^2 + t b_1^2 = b_0 - 2t b_0 + 2t b_1 + t^2 b_0 - 2t^2 b_1 + t^2 b_2 - t b_0 + 2t^2 b_0 - 2t^2 b_1 - t^3 b_0 + 2t^3 b_1 - t^3 b_2 +$

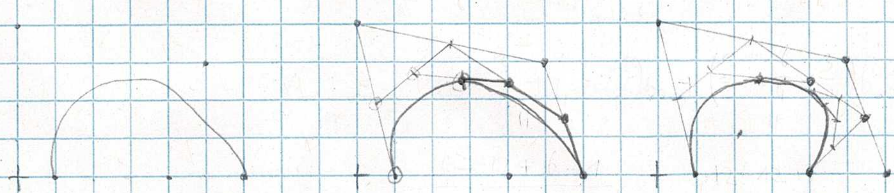
$+ t b_1 - 2t^2 b_1 + 2t^2 b_2 + t^3 b_1 - 2t^3 b_2 + t^3 b_3$

$= b_0 - 3t b_0 + 3t b_1 + 3t^2 b_0 - 6t^2 b_1 + 3t^2 b_2 - t^3 b_0 + 3t^3 b_1 - 3t^3 b_2 + t^3 b_3$

$c(t) := b(\tfrac{1}{2} t) = \sum_{i=0}^{3} B_i^3(t)\, b_0^i(\tfrac{1}{2}) \qquad d(t) := b((1-\tfrac{1}{2})t + \tfrac{1}{2}) = b(\tfrac{1}{2}t + \tfrac{1}{2}) = \sum_{i=0}^{3} B_i^3(t)\, b_i^{n-i}(\tfrac{1}{2})$

$c(t) = B_0^3(t) b_0 + B_1^3(t)\left(\frac{b_0 + b_1}{2}\right) + B_2^3(t)\left(\frac{b_0 + 2b_1 + b_2}{4}\right) + B_3^3(t)\left(\frac{b_0 + 3b_1 - 3b_2 + b_3}{8}\right)$

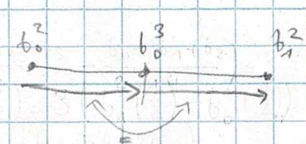
$\left(b_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad b_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad b_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad b_3 = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad b_4 = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right)$



$d(t) = B_0^3(t)\, b_0^3(\tfrac{1}{2}) + B_1^3(t)\, b_1^2(\tfrac{1}{2}) + B_2^3(t)\, b_2^1(\tfrac{1}{2}) + B_3^3(t)\, b_3^0(\tfrac{1}{2})$

stetig? $\qquad c(1) = b(\tfrac{1}{2}) = d(0) \quad \checkmark$

stetig diffbar? $\qquad \dot{c}(1) = 3(b_0^3(\tfrac{1}{2}) - b_0^2(\tfrac{1}{2}))$



$\checkmark \qquad \dot{d}(0) = 3(b_1^2(\tfrac{1}{2}) - b_0^3(\tfrac{1}{2}))$

$b_0^3(\tfrac{1}{2}) - b_0^2(\tfrac{1}{2}) = b_0 - \tfrac{3}{2} b_0 + \tfrac{3}{2} b_1 + \tfrac{3}{4} b_0 - \tfrac{6}{4} b_1 + \tfrac{3}{4} b_2 - \tfrac{1}{8} b_0 + \tfrac{3}{8} b_1 - \tfrac{3}{8} b_2 + \tfrac{1}{8} b_3 - (b_0 - \tfrac{2}{2} b_0 + \tfrac{2}{2} b_1 + \tfrac{1}{4} b_0 - \tfrac{2}{4} b_1 + \tfrac{1}{4} b_2) =$

$= (1 - \tfrac{3}{2} + \tfrac{3}{4} - \tfrac{1}{8} - 1 + \tfrac{2}{2} - \tfrac{1}{4}) b_0 + (\tfrac{3}{2} - \tfrac{6}{4} + \tfrac{3}{8} - \tfrac{2}{2} + \tfrac{2}{4}) b_1 + (\tfrac{3}{4} - \tfrac{3}{8} - \tfrac{1}{4}) b_2 + \tfrac{1}{8} b_3 = -\frac{b_0}{8} - \frac{b_1}{8} + \frac{b_2}{8} + \frac{b_3}{8}$

$b_1^2(\tfrac{1}{2}) - b_0^3(\tfrac{1}{2}) = b_1 - \tfrac{2}{2} b_1 + \tfrac{2}{2} b_2 + \tfrac{1}{4} b_1 - \tfrac{2}{4} b_2 + \tfrac{1}{4} b_3 - (b_0 - \tfrac{3}{2} b_0 + \tfrac{3}{2} b_1 + \tfrac{3}{4} b_0 - \tfrac{6}{4} b_1 + \tfrac{3}{4} b_2 - \tfrac{1}{8} b_0 + \tfrac{3}{8} b_1 - \tfrac{3}{8} b_2 + \tfrac{1}{8} b_3) =$

$= (-1 + \tfrac{3}{2} - \tfrac{3}{4} + \tfrac{1}{8}) b_0 + (1 - 1 + \tfrac{1}{4} - \tfrac{3}{2} + \tfrac{6}{4} - \tfrac{3}{8}) b_1 + (1 - \tfrac{2}{4} - \tfrac{3}{4} + \tfrac{3}{8}) b_2 + (\tfrac{1}{4} - \tfrac{1}{8}) b_3 = -\frac{b_0}{8} - \frac{b_1}{8} + \frac{b_2}{8} + \frac{b_3}{8}$

...