PDGL Ü7

1) $\Omega \subseteq \mathbb{R}^n$ ... beschränktes Gebiet $f: \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ ... stetig als $f(\cdot, x)$ und C^1 als $f(u, \cdot)$ mit
$\forall (u,x) \in \mathbb{R} \times \Omega: f_u(u,x) \geq 0$

i) $\Delta u = f(u,x)$ in Ω, $u = \varphi$ auf $\partial\Omega$ zz: es gibt höchstens eine klassische Lsg

Sei u_1, u_2 klassische Lösungen. $\Rightarrow u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

$\Delta(u_1 - u_2) = \Delta u_1 - \Delta u_2 = f(u_1, x) - f(u_2, x)$ und $u_1 - u_2 = \varphi - \varphi = 0$ auf $\partial\Omega$

$f'(\cdot, x) \geq 0$ also ist $f(\cdot, x)$ monoton wachsend

$\Rightarrow u_1(x) \leq u_2(x) \Leftrightarrow f(u_1, x) \leq f(u_2, x)$ und $u_1(x) \geq u_2(x) \Leftrightarrow f(u_1,x) \geq f(u_2, x)$

Betrachte die Menge $\Omega_1 := \{x \in \Omega : u_1(x) < u_2(x)\}$.

Für $x \in \partial\Omega_1$ gilt entweder $x \in \partial\Omega$ also $u_1(x) = u_2(x)$ oder direkt $u_1(x) = u_2(x)$, da die u_1, u_2 stetig sind.

$L(u) := -\Delta u$

Für $u_1 - u_2$ und $x \in \Omega_1$ gilt: $L(u_1 - u_2) = -\Delta(u_1 - u_2) = -f(u_1, x) + f(u_2, x) \geq 0$

Mit dem schwachen Maximumprinzip folgt $\inf\limits_{x \in \Omega_1} u_1(x) - u_2(x) = \inf\limits_{x \in \partial\Omega_1} u_1(x) - u_2(x) = 0$

$\Rightarrow u_1(x) \geq u_2(x) \; \forall x \in \Omega_1 = \{x \in \Omega : u_1(x) < u_2(x)\}$

$\Rightarrow \Omega_1 = \emptyset$

Analog für $\Omega_2 = \{x \in \Omega : u_1(x) > u_2(x)\}$

$\Rightarrow \Omega = \{x \in \Omega : u_1(x) = u_2(x)\}$ also stimmen u_1 und u_2 überein

ii) $n = 1$ ges: Intervall $\Omega \subseteq \mathbb{R}$ $f \in C^1(\mathbb{R})$ mit $f'(u) < 0 \; \forall u \in \mathbb{R}$ sodass
$u'' = f(u)$ in Ω, $u = 0$ auf $\partial\Omega$ nicht eindeutig lösbar ist.

$\Omega := [0, \pi]$ $f(u) := -u \in C^1(\mathbb{R})$ $f'(u) = -1 < 0 \; \forall u \in \mathbb{R}$

$u_1(x) := \sin(x)$ $u_2(x) := -\sin(x)$

$\Rightarrow u_1''(x) = \sin''(x) = -\sin(x) = f(u_1)$ $u_2''(x) = \ldots = f(u_2)$ $u_1(0) = u_1(\pi) = 0 = u_2(0) = u_2(\pi)$

$u_1 \neq u_2$ trivialerweise

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2) $\Omega := (0,a)\times(0,b) \subseteq \mathbb{R}^2$ mit $a,b>0$

$-\Delta u = \lambda u$ in Ω $\quad u=0$ auf $\partial\Omega$ $\quad$ ges: Lsg mittels Separation der Variablen

$u(x,y) = X(x)\cdot Y(y)$

$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = -X''\cdot Y - XY'' \overset{!}{=} \lambda X\cdot Y = \lambda u$

$\Rightarrow X(\lambda Y + Y'') = -X''Y \quad \Rightarrow \lambda\frac{Y}{Y} + \frac{Y''}{Y} = -\frac{X''}{X} \quad \Rightarrow \lambda + \frac{Y''}{Y} = -\frac{X''}{X} = c \in \mathbb{R}$

$\Rightarrow X'' = -cX \qquad Y'' = (c-\lambda)Y$

lösen ergibt $X(x) = k_1 \sin(\sqrt{c}\,x) + k_2\cos(\sqrt{c}\,x) \qquad Y(y) = k_3 e^{y\sqrt{c-\lambda}} + k_4 e^{-y\sqrt{c-\lambda}}$

Randwerte $\Rightarrow X(0)=0=X(a) \qquad Y(0)=0=Y(b)$ also

$0 = X(0) = k_2 \qquad 0 = Y(0) = k_3 + k_4 \quad \Rightarrow k_2 = 0 \quad k_4 = -k_3$

$X(x) = k_1\sin(\sqrt{c}\,x) \qquad Y(y) = k_3\left(e^{y\sqrt{c-\lambda}} - e^{-y(\sqrt{c-\lambda})}\right) = k_3\, 2\sinh(y\sqrt{c-\lambda})$

$0 = X(a) = k_1 \sin(\sqrt{c}\,x) \Rightarrow c = \frac{\pi^2 n^2}{a^2}$ mit $n\in\mathbb{N}$

$0 = Y(b) = 2k_3\sinh\left(b\sqrt{\frac{\pi^2n^2}{a^2}-\lambda}\right) \iff b\sqrt{\frac{\pi^2n^2}{a^2}-\lambda} \in i\pi\mathbb{Z} \overset{*}{\Rightarrow} \lambda = \pi^2\left(\frac{n^2}{a^2}+\frac{k^2}{b^2}\right)$

Insgesamt $u(x,y) = X(x)\cdot Y(y) = k_1\sin\left(\frac{\pi n}{a}x\right)\cdot 2k_3\sinh\left(y\sqrt{\frac{\pi^2n^2}{a^2} - \pi^2\left(\frac{n^2}{a^2}+\frac{k^2}{b^2}\right)}\right) =$

$= 2k_1k_3\sin\left(\frac{\pi n}{a}x\right)\sinh\left(y\pi\underbrace{\sqrt{\frac{n^2}{a^2}-\frac{n^2}{a^2}-\frac{k^2}{b^2}}}_{=\sqrt{-\frac{k^2}{b^2}}}\right) =$

$= 2k_1k_3\sin\left(\frac{\pi n}{a}x\right)\sinh\left(\frac{i\pi k}{b}y\right)$

Probe: $u_x(x,y) = 2k_1k_3\cos\left(\frac{\pi n}{a}x\right)\frac{\pi n}{a}\sinh\left(\frac{i\pi k}{b}y\right)$

$u_{xx}(x,y) = -2k_1k_3\sin\left(\frac{\pi n}{a}x\right)\frac{\pi^2n^2}{a^2}\sinh\left(\frac{i\pi k}{b}y\right) = -\frac{\pi^2n^2}{a^2}u$

$u_y(x,y) = 2k_1k_3\sin\left(\frac{\pi n}{a}x\right)\cosh\left(\frac{i\pi k}{b}y\right)\frac{i\pi k}{b}$

$u_{yy}(x,y) = 2k_1k_3\sin\left(\frac{\pi n}{a}x\right)\sinh\left(\frac{i\pi k}{b}y\right)\frac{-\pi^2k^2}{b^2} = -\frac{\pi^2k^2}{b^2}u$

$\Rightarrow -\Delta u = \frac{\pi^2n^2}{a^2}u + \frac{\pi^2k^2}{b^2}u = \pi^2\left(\frac{n^2}{a^2}+\frac{k^2}{b^2}\right)u = \lambda u$ ✓

$u(0,y)=0 \qquad u(x,0)=0 \qquad u(a,y)=0 \qquad u(x,b)=0$ ✓

$\square$

$* \quad b\sqrt{\frac{\pi^2n^2}{a^2}-\lambda} = i\pi k \iff b^2\left(\frac{\pi^2n^2}{a^2}-\lambda\right) = -\pi^2k^2 \iff \frac{\pi^2n^2b^2}{a^2} - \lambda b^2 = -\pi^2k^2 \iff$

$\lambda b^2 = \frac{\pi^2n^2b^2}{a^2} + \pi^2k^2 \iff \lambda = \frac{\pi^2n^2}{a^2} + \frac{\pi k^2}{b^2} = \pi^2\left(\frac{n^2}{a^2}+\frac{k^2}{b^2}\right)$

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3) Lösen Sie $u_t - u_{xx} = \sin(t)\sin(2x)$ $x\in(0,\pi), t>0$ $u(0,t)=0=u(\pi,t)$, $t>0$

$u(x,0)=\sin(x)+\frac{\sin(10x)}{1000}$ $x\in(0,\pi)$

Wir rechnen sie später aus, aber seien λ_k, Φ_k die EW und EF von $-\partial^2_{xx}$ (also normiert, orthogonal) wobei $k\in\mathbb{N}$

Ansatz $u(x,t)=\sum_{k=1}^{\infty} a_k(t)\Phi_k(x)$ $\quad -\partial^2_{xx}\Phi_k = \lambda_k\Phi_k$

$\Rightarrow \sin(t)\sin(2x) \overset{!}{=} u_t - u_{xx} = \sum_{k=1}^{\infty} a_k'\Phi_k - \sum_{k=1}^{\infty} a_k\Phi_k'' = \sum_{k=1}^{\infty}(a_k'\Phi_k - a_k(-\lambda_k\Phi_k)) = \sum_{k=1}^{\infty}(a_k'+\lambda_k a_k)\Phi_k$

Für $l\in\mathbb{N}$ bel. gilt

$<\sin(t)\sin(2x), \Phi_l(x)> \overset{!}{=} <u_t-u_{xx}, \Phi_l(x)> = <\sum_{k=1}^{\infty}(a_k'(t)+\lambda_k a_k(t))\Phi_k(x), \Phi_l(x)> =$

$= \sum_{k=1}^{\infty}(a_k'(t)+\lambda_k a_k(t))\underbrace{<\Phi_k,\Phi_l>}_{=0 \text{ für } k\neq l} = a_l' + \lambda_l a_l \underbrace{<\Phi_l,\Phi_l>}_{=1} = a_l' + \lambda_l a_l$

$\sin(x)+\frac{\sin(10x)}{1000} = u(x,0) = \sum_{k=1}^{\infty} a_k(0)\Phi_k(x)$

Nun zur Berechnung von λ_k, Φ_k:

$-\partial^2_{xx}\Phi_k(x) = \lambda_k\Phi_k(x) \Leftrightarrow \Phi_k'' = -\lambda_k\Phi_k \quad \Rightarrow \Phi_k(x) = c_1 e^{\sqrt{-\lambda_k}x} + c_2 e^{-\sqrt{-\lambda_k}x}$

$\Phi_k(0) = c_1 + c_2 \overset{!}{=} 0 \quad \Rightarrow c_2 = -c_1 \quad \Rightarrow \Phi_k(x) = c_1(e^{i\sqrt{\lambda_k}x} - e^{-i\sqrt{\lambda_k}x}) = c_1 2i\sin(\sqrt{\lambda_k}x)$

$\Phi_k(\pi) = 2ic_1\sin(\sqrt{\lambda_k}\pi) \overset{!}{=} 0 \quad \Rightarrow \sqrt{\lambda_k}\in\mathbb{Z} \quad \Rightarrow \lambda_k = k^2 \in \mathbb{Z}^2$ und $\Phi_k(x) = 2ic_k\sin(kx)$

$\Phi_{-k}(x) = 2ic_{-k}\sin(-kx) = -2ic_{-k}\sin(kx)$ ist l.a. von $\Phi_k(x)$ also nur $k\in\mathbb{N}$

Normieren: $<\Phi_k,\Phi_k> = \int_0^{\pi}(2ic_k\sin(kx))^2dx = -4c_k^2\int_0^{\pi}\sin^2(kx)dx = -4c_k^2\frac{\pi}{2} = -2\pi c_k^2 \overset{!}{=} 1$

$\Rightarrow c_k = \sqrt{-\frac{1}{2\pi}} = i\frac{1}{\sqrt{2\pi}}$ und somit $\Phi_k(x) := -\sqrt{\frac{2}{\pi}}\sin(kx)$

Es gilt tatsächlich $<\Phi_k,\Phi_l> = \begin{cases}0, & k\neq l\\ 1, & k=l\end{cases}$

Wieder oben weiterrechnen: $\sin(t)\cdot\sin(2x) = -\sqrt{\frac{\pi}{2}}\sin(t)\Phi_2(x)$

$\sin(x)+\frac{1}{1000}\sin(10x) = -\sqrt{\frac{\pi}{2}}\Phi_1(x) - \sqrt{\frac{\pi}{2}}\frac{1}{1000}\Phi_{10}(x)$

$\Rightarrow \forall l: a_l' + l^2 a_l = <\sin(t)\sin(2x), \Phi_l(x)> = -\sqrt{\frac{\pi}{2}}\sin(t)<\Phi_2(x),\Phi_l(x)> = \begin{cases}-\sqrt{\frac{\pi}{2}}\sin(t), & l=2\\ 0, & l\neq 2\end{cases}$

$\sum_{k=1}^{\infty} a_k(0)\Phi_k(x) = \sin(x)+\frac{\sin(10x)}{1000} = -\sqrt{\frac{\pi}{2}}\Phi_1 - \sqrt{\frac{\pi}{2}}\frac{1}{1000}\Phi_{10} \quad \Rightarrow a_1(0) = -\sqrt{\frac{\pi}{2}},\ a_{10}(0) = -\sqrt{\frac{\pi}{2}}\frac{1}{1000}$

$\forall l\notin\{1,10\}: a_l(0)=0$

...

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3) ... $a_1: \quad a_1' + 1^2 a_1 = 0 \qquad a_1(0) = -\sqrt{\frac{\pi}{2}} \quad \Rightarrow a_1(t) = -\sqrt{\frac{\pi}{2}}\, e^{-t}$

$a_2: \quad a_2' + 4a_2 = -\sqrt{\frac{\pi}{2}} \sin(t) \qquad a_2(0) = 0 \quad \Rightarrow a_2(t) = \frac{1}{17}\sqrt{\frac{\pi}{2}}\, e^{-4t}\left(-4e^{4t}\sin(t) + e^{4t}\cos(t) - 1\right)$

$a_{10}: \quad a_{10}' + 100 a_{10} = 0 \qquad a_{10}(0) = -\sqrt{\frac{\pi}{2}}\frac{1}{1000} \quad \Rightarrow a_{10}(t) = -\frac{1}{1000}\sqrt{\frac{\pi}{2}}\, e^{-100t}$

$\ell \neq 1,2,10: a_\ell: \quad a_\ell' + \ell^2 a_\ell = 0 \qquad a_\ell(0) = 0 \quad \Rightarrow a_\ell(t) = 0$

$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} a_k(t)\phi_k(x) = a_1(t)\phi_1(x) + a_2(t)\phi_2(x) + a_{10}(t)\phi_{10}(x) =$

$= \left(-\sqrt{\frac{\pi}{2}}\, e^{-t}\right)\left(-\sqrt{\frac{2}{\pi}}\sin(x)\right) + \left(\frac{1}{17}\sqrt{\frac{\pi}{2}}\, e^{-4t}\left(-4e^{4t}\sin(t) + e^{4t}\cos(t) - 1\right)\right)\left(-\sqrt{\frac{2}{\pi}}\sin(2x)\right) +$

$\left(-\frac{1}{1000}\sqrt{\frac{\pi}{2}}\, e^{-100t}\right)\left(-\sqrt{\frac{2}{\pi}}\sin(10x)\right) =$

$= e^{-t}\sin(x) - \frac{1}{17}e^{-4t}\sin(2x)\left(-4e^{4t}\sin(t) + e^{4t}\cos(t) - 1\right) + \frac{1}{1000}e^{-100t}\sin(10x) =$

$= e^{-t}\sin(x) + \frac{4}{17}\sin(t)\sin(2x) - \frac{1}{17}\cos(t)\sin(2x) + \frac{1}{17}e^{-4t}\sin(2x) + \frac{1}{1000}e^{-100t}\sin(10x)$

Probe: $u(0,t) = 0 \qquad u(\pi,t) = 0$

$u(x,0) = \sin(x) - \frac{1}{17}\sin(2x) + \frac{1}{17}\sin(2x) + \frac{1}{1000}\sin(10x) = \sin(x) + \frac{\sin(10x)}{1000}$

$u_t = -e^{-t}\sin(x) + \frac{4}{17}\cos(t)\sin(2x) + \frac{1}{17}\sin(t)\sin(2x) - \frac{4}{17}e^{-4t}\sin(2x) - \frac{1}{10}e^{-100t}\sin(10x)$

$u_x = e^{-t}\cos(x) + \frac{8}{17}\sin(t)\cos(2x) - \frac{2}{17}\cos(t)\cos(2x) + \frac{2}{17}e^{-4t}\cos(2x) + \frac{1}{100}e^{-100t}\cos(10x)$

$u_{xx} = -e^{-t}\sin(x) - \frac{16}{17}\sin(t)\sin(2x) + \frac{4}{17}\cos(t)\sin(2x) - \frac{4}{17}e^{-4t}\sin(2x) - \frac{1}{10}e^{-100t}\sin(10x)$

$u_t - u_{xx} = 0 + 0 + \left(\frac{1}{17} + \frac{16}{17}\right)\sin(t)\sin(2x) + 0 + 0 = \sin(t)\sin(2x)$

✓

$\square$

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4) $\Omega \subseteq \mathbb{R}^2$ beschränktes Gebiet, $\partial\Omega \in C^1$, $\partial\Omega = \Gamma_1 \cup \Gamma_2$ mit $\Gamma_1 \cap \Gamma_2 = \emptyset$

$p, h \in L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ $\quad q, r \in L^\infty(\Omega)$ $\quad p \geq p_0 > 0$, $q, r, h > 0$ f.ü. in Ω

$h > 0$ f.ü. in $\partial\Omega$

$-\operatorname{div}(p \nabla u) + q u_x + r u_y = f$ in Ω $\quad u = 0$ auf Γ_1 $\quad \nabla u \cdot n + hu = g$ auf Γ_2

a) schwache Formulierung

Gauß: $\int_\Omega (\operatorname{div} v) w \, dx = -\int_\Omega v \nabla w \, dx + \int_{\partial\Omega} (v \cdot n) w \, ds$

$$\int_\Omega f \phi \, dx = \int_\Omega -\operatorname{div}(p \nabla u) \phi \, dx + \int_\Omega q u_x \phi \, dx + \int_\Omega r u_y \phi \, dx =$$

$$\overset{\text{Gauß}}{=} \int_\Omega p \nabla u \cdot \nabla \phi \, dx - \int_{\partial\Omega} (p \nabla u \cdot n) \phi \, ds + \int_\Omega q u_x \phi \, dx + \int_\Omega r u_y \phi \, dx =$$

$$= \int_\Omega p \nabla u \nabla \phi \, dx - \int_{\Gamma_1} (p \underbrace{\nabla u \cdot n) \phi}_{=0} \, ds - \int_{\Gamma_2} (p \underbrace{\nabla u \cdot n}_{= g - hu}) \phi \, ds + \int_\Omega q u_x \phi \, dx + \int_\Omega r u_y \phi \, dx =$$

$$= \int_\Omega p \nabla u \nabla \phi + (q u_x + r u_y) \phi \, dx - \int_{\Gamma_2} p g \phi \, ds + \int_{\Gamma_2} p h u \phi \, ds$$

$$\Rightarrow \int_\Omega p \nabla u \nabla \phi + (q, r) \cdot \nabla u \, \phi \, dx + \int_{\Gamma_2} p h u \phi \, ds = \int_\Omega f \phi \, dx + \int_{\Gamma_2} p g \phi \, ds$$

Also wir suchen nach $u \in H^1_{\Gamma}(\Omega)$ mit $\forall \phi \in H^1_{\Gamma}(\Omega)$ gilt obige Gleichung.

b) ZZ: $\exists$ schwache Lsg

$F(\phi) := \int_\Omega f \phi \, dx + \int_{\Gamma_2} p g \phi \, ds$ $\qquad \tilde{a}(u, \phi) := \int_\Omega p \nabla u \nabla \phi + (q, r) \nabla u \phi \, dx + \int_{\Gamma_2} p h u \phi \, ds$

Lax Milgram

F lin. klar $\quad$ F stetig: $|F(\phi)| \leq |\int_\Omega \overset{\in L_2}{f \phi} \, dx| + |\int_{\Gamma_2} p g \phi \, ds| \leq \|f\|_{L_2} \|\phi\|_{L_2} + \|T p g \phi\|_{L_1}$

$$\leq \|f\|_{L_2} \|\phi\|_{L_2} + \|T\| \|p\|_\infty \|g\|_{L_2} \|\phi\|_{L_2} = \underbrace{(\|f\|_{L_2} + \|T\| \|p\|_\infty \|g\|_{L_2})}_{K} \|\phi\|_{L_2}$$

a stetig $|a(u, v)| \leq |\int_\Omega p \nabla u \nabla v \, dx| + |\int_\Omega (q, r) \nabla u \, v \, dx| + |\int_{\Gamma_2} p h u v \, ds|$

$$\leq \|p\|_\infty \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \max(\|q\|_\infty, \|r\|_\infty) \|\nabla u\|_{L^2} \|v\|_{L^2} + \|T\| \|p\|_\infty \|h\|_\infty \|u\|_{L^2} \|v\|_{L^2}$$

$$\leq (\|p\|_\infty + \max(\|q\|_\infty, \|r\|_\infty) + \|T\| \|p\|_\infty \|h\|_\infty) \|u\|_{H^1_{\Gamma_1}} \|v\|_{H^1_{\Gamma_1}}$$

a koerziv $a(u, u) = \int_\Omega p \nabla u \nabla u + (q, r) \nabla u \, u \, dx + \underbrace{\int_{\Gamma_2} p h u^2 \, ds}_{\geq 0 \text{ f.ü.}} \geq$

Poincare $\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}$

$$\geq p_0 \|\nabla u\|_{L^2}^2 - \max(\|q\|_{L^\infty}, \|r\|_{L^\infty}) \|\nabla u\|_{L^2} \|u\|_{L^2}$$

$$\geq p_0 \|\nabla u\|_{L^2}^2 - \max(\|q\|_{L^\infty}, \|r\|_{L^\infty}) C \|\nabla u\|_{L^2}^2 = \underbrace{(p_0 - \max(\|q\|_{L^\infty}, \|r\|_{L^\infty}) C)}_{=: K} \|\nabla u\|_{L^2}^2$$

$$\geq K \frac{1}{(C+1)^2} \|u\|_{H^1_{\Gamma_1}}^2 \quad \text{wobei } K > 0 \text{ und } \frac{K}{(C+1)^2} > 0$$

folgt aus $K > 0$

$\|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}$

$\leq (C+1) \|\nabla u\|_{L^2}$

Mit Lax Milgram $\Rightarrow \exists!$ Lsg

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$$a(u,v) = \int_\Omega p \nabla v \nabla v + (q,r)\cdot\nabla v\, v\, dx - \int_{\Gamma_1} p \nabla v\, n\, v\, ds - \int_{\Gamma_2} p h v^2 ds$$

$$\geq p_0 \|\nabla v\|_{L^2}^2 + \int_\Omega (q v_x + r v_y) v\, dx - \|p\|_\infty \int_{\Gamma} \nabla v \cdot n\, v\, ds - \|p\|_\infty \|h\|_\infty \|v\|_{L^2}^2 \|T\|$$

$$\geq p_0 \|\nabla v\|_{L^2}^2 + \int_\Omega (q,r)\nabla v\, v\, dx - \|p\|_\infty \|T\| \|h\|_\infty \|v\|_{L^2}^2 - \|p\|_\infty \int_{\Gamma_1} \nabla v \cdot \underbrace{n v}_{=0} ds$$

$$\geq \frac{p_0}{C_p} \|v\|_{L^2}^2 - \|p\|_\infty \|T\| \|h\|_\infty \|v\|_{L^2}^2 + \int_\Omega (q v_x + r v_y) v\, dx$$

$$\int_\Omega (q v_x + r v_y) v\, dx = \int_\Omega (f + \operatorname{div}(p\nabla v)) dx = \int_\Omega f v\, dx + \int_\Omega \operatorname{div}(p\nabla v) v\, dx$$

$$= \int_\Omega f v\, dx - \int_\Omega p \nabla v \cdot \nabla v\, dx + \int_{\partial\Omega} p \nabla v \cdot v\, ds$$

$$\int_\Omega (q,r)\nabla v\, v\, dx \overset{?}{\geq} \alpha \|v\|_{L^2}^2$$

$$\|v\|_{H_1}^2 = \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \Rightarrow \|v\|_{H_1}^2 \geq \|v\|_{L^2}^2$$

$$\|v\|_{H_1}^2 \geq \|\nabla v\|_{L^2}^2$$

$$\|p\|_\infty \|T\| \|\nabla v\|_{L^2} \|v\|_{L^2} \leq \|p\|_\infty \|T\| \|\nabla v\|_{L^2} C_p \|\nabla v\|_{L^2}$$

4)... Forderungen:

- $f \in L^2$, $g \in L^2$ (aus F lin.)
- $p_0 \geq \max(\|q\|_{L^\infty}, \|r\|_{L^\infty}) \cdot C$ (aus a koerziv) wobei C... Poincare Konstante

$\square$