Computer Aided Geometric Design Compendium WS2023

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Orginazation

Lecture each Thursday 12:00 to 14:00 (full 2 hours).
Oral exam. Write email to fix date and time.
Problem session each Thursday 14:00 to 16:00. Mandatory attendance!
Kreuzerlübung.

1 Bezier curves

Example 1. ***Linear combination*** *$\lambda a + \mu b$*
TODO image
Affine combination *$\lambda a + \mu b$ and $\lambda + \mu = 1$*
TODO image
What is μ so that $\lambda a + \mu B$ is on the line?

$$\lambda a + \mu b = a + t(b-a) \implies a(\underbrace{\lambda - 1 + t}_{=0}) + b(\underbrace{\mu - t}_{=0}) = 0$$

If a, b are linearly independent $\implies \mu = t \wedge \lambda + \mu = 1$
Convex combination *$\lambda a + \mu b$ and $\lambda + \mu = 1$ and $\lambda, \mu \geq 0$*
TODO image
Line is $a + t(b-a)$ with $t \in [0,1] \implies \mu, \lambda \in [0,1]$

Definition 1 (combinations)**.** *linear combination $\sum_{i=1}^n \lambda_i v_i$ with $v_1, ..., v_n \in \mathbb{R}^d, \lambda_1, ..., \lambda_n \in \mathbb{R}$*
affine combination $\sum_{i=1}^n \lambda_i v_i$ with $\sum_{i=1}^n \lambda_i = 1$
convex combination $\sum_{i=1}^n \lambda_i v_i$ with $\sum_{i=1}^n \lambda_i = 1$ and $\forall i : \lambda_i \geq 0$

Algorithm 1 (of de Casteljou, Bezier curve)**.** *Given: $b_0, ..., b_n \in \mathbb{R}^d$ (called control points / Kontrollpunkte), $t \in \mathbb{R}$*
Recursion: $b_i^0(t) := b_i$
$b_i^j(t) := (1-t)b_i^{j-1}(t) + tb_{i+1}^{j-1}(t)$ for $j = 1, ..., n$ and $i = 0, ..., n-j$
Result: $b(t) := b_0^n(t)$ (called Bezier curve)

Remark 1. *In the algorithm above often we choose $t \in [0,1]$.*

Example 2. *TODO images*

Remark 2. *In this course $\mathbb{N} = \{1, 2, 3, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$*

Recap 1. *$0! := 1, n! := n(n-1)(n-2)\cdots 1$ for $n \geq 1$.*

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!} & , n \geq k \geq 0 \\ 0 & , k > n \end{cases} \quad \textit{for } n, k \in \mathbb{N}_0$$

Definition 2 (Bernstein polynomials)**.** *For $n, i \in \mathbb{N}_0$ we define $B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i} \in \mathbb{R}[t]$*

Remark 3. *Special cases of Bernstein polynomials*

$$i > n \implies B_i^n(t) = 0 \qquad B_i^n(0) = \begin{cases} 0, i \neq 0 \\ 1, i = 0 \end{cases}$$

$$B_i^n(1) = \begin{cases} 0, i \neq n \\ 1, i = n \end{cases} \qquad B_0^0(t) = 1$$

Theorem 1. $b_i^j(t) = \sum_{l=0}^{j} B_l^j(t) b_{i+l}$

Proof. Induction over j: $j=0$:

$$j = 0: \qquad b_i^0(t) := b_i = 1 \cdot b_i = B_0^0(t) \cdot b_i \qquad \checkmark$$

$$j-1 \to j: \qquad b_i^j(t) := (1-t) b_i^{j-1}(t) + t b_{i+1}^{j-1}(t) \overset{\text{IA}}{=} (1-t) \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+l} + t \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+1+l} =$$

$$(1-t) \sum_{l=0}^{\overset{j}{\cancel{j-1}}} B_l^{j-1}(t) b_{i+l} + t \sum_{l=\underset{0}{\cancel{1}}}^{j} B_{l-1}^{j-1}(t) b_{i+l} = \sum_{l=0}^{j} (\underbrace{(1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t)}_{= B_l^j(t) \text{ using the following lemma}}) b_{i+l} =$$

$$\sum_{l=0}^{j} B_l^j(t) b_{i+l} \qquad \checkmark$$

$\square$

Corollary 1. *The Bezier curve equals* $b(t) = b_0^n(t) = \sum_{l=0}^{n} B_l^j(t) b_{i+l}$, *which is called the Bernstein representation of the Bezier curve.*

Remark 4. *As* $b(t) = \sum_{l=0}^{n} B_l^n(t) b_l \in C^\infty$ *it is a polynomial curve of degree* n, *which is in* C^∞ *and therefore "very smooth".*

Lemma 1. $B_l^j(t) = (1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t)$

Proof.

$$(1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t) = (1-t) \binom{j-1}{l} t^l (1-t)^{j-1-l} + t \binom{j-1}{l-1} t^{l-1} (1-t)^{j-1-l+1} =$$

$$\binom{j-1}{l} t^l (1-t)^{j-l} + \binom{j-1}{l-1} t^l (1-t)^{j-l} = \left(\binom{j-1}{l} + \binom{j-1}{l-1} \right) t^l (1-t)^{j-l} = \binom{j}{l} t^l (1-t)^{j-l} = B_l^j(t)$$

$\square$

Remark 5. *What is* $b(0)$*?* $\qquad b(0) = \sum_{i=0}^{n} B_i^n(0) b_i = b_0 + 0 + 0 + \cdots + 0 = b_0$
What is $b(1)$*?* $\qquad b(1) = \sum_{i=0}^{n} B_i^n(1) b_i = 0 + \cdots + 0 + b_n = b_n$

Definition 3 (end-point-interpolating)**.** *Curves which pass through the first and last point are called end-point-interpolating (Endpunktinterpolierend).*

Remark 6. *Bezier curves are end-point-interpolating.*

Remark 7. *How many intersection points are there between a planar (i.e. in* $\mathbb{R}^2$*) Bezier curve and a straight line?*

Straight line: $p + t(q-p)$ $\qquad$ *Bezier curve:* $b(t) = \sum_{i=0}^{n} B_i^n(t) \underbrace{b_i}_{\in \mathbb{R}^2}$

Solving $p + t(q-p) = \sum_{i=0}^{n} B_i^n(t) b_i$ *results in at most* n *solutions.*

Lemma 2. $\frac{d}{dt} B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$

Proof.

$$\frac{d}{dt} B_i^n(t) = \frac{d}{dt} \binom{n}{i} t^i (1-t)^{n-i} = \binom{n}{i} i t^{i-1} (1-t)^{n-i} - \binom{n}{i} t^i (n-i)(1-t)^{n-i-1} =$$

$$\frac{n!}{\underset{(i-1)!}{\cancel{i!}}\,(n-i)!} \cancel{i} t^{i-1} (1-t)^{n-i} - \frac{n!}{i! \underset{(n-i-1)!}{\cancel{(n-i)!}}} \cancel{(n-i)} t^i (1-t)^{n-i-1} =$$

$$n \left(\frac{(n-1)!}{(i-1)!(n-i)!} t^{i-1} (1-t)^{n-i} - \frac{(n-1)!}{i!(n-i-1)!} t^i (1-t)^{n-i-1} \right) =$$

$$n \left(\binom{n-1}{i-1} t^{i-1} (1-t)^{n-i} - \binom{n-1}{i} t^i (1-t)^{n-i-1} \right) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$$

$\square$

Theorem 2. $\dot{b}(t) := \frac{d}{dt}b(t) = n\sum_{i=0}^{n-1} B_i^{n-1}(t)(b_{i+1} - b_i) = n(b_1^{n-1}(t) - b_0^{n-1}(t))$

Corollary 2.

- $\dot{b}(0) = n(b_1 - b_0)$
- $\dot{b}(1) = n(b_n - b_{n-1})$
- *The last segment in the algorithm of de Casteljou is the tangent of the Bezier curve in $b(t)$.*
- *The derivative of a bezier curve of degree n is a bezier curve of degree $n-1$ with control points $(b_1, b_0), (b_2 - b_1), \cdots, (b_n - b_{n-1})$.*

Remark 8. *Different applications using these curves are Rhino, OpenSCAD, Autocad, Geogebra, ...*