DGL Ü4

1) a) $x' = \begin{pmatrix} 1 & 4 \\ 3 & 0 \end{pmatrix} x \qquad A = \begin{pmatrix} 1 & 4 \\ 3 & 0 \end{pmatrix}$

$\chi(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 4 \\ 3 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 4\cdot 3 = -\lambda + \lambda^2 - 12$

$\chi(\lambda) = 0 \iff \lambda^2 - \lambda - 12 = 0 \iff \lambda_{1,2} = +\frac{1}{2} \pm \sqrt{\frac{1}{4} + 12} = \frac{1}{2} \pm \frac{\sqrt{49}}{\sqrt{4}} = \frac{1 \pm 7}{2} = \begin{cases} -3 \\ 4 \end{cases}$

$\begin{pmatrix} 1 & 4 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + 4v_2 \\ 3v_1 \end{pmatrix} = \begin{pmatrix} -3v_1 \\ -3v_2 \end{pmatrix} = -3\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\Rightarrow 3v_1 = -3v_2 \Rightarrow v_1 = -v_2 \Rightarrow -3v_1 = v_1 + 4v_2 = v_1 - 4v_1 = -3v_1 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 4 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + 4v_2 \\ 3v_1 \end{pmatrix} = \begin{pmatrix} 4v_1 \\ 4v_2 \end{pmatrix} = 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\Rightarrow 3v_1 = 4v_2 \Rightarrow v_2 = \frac{3}{4}v_1 \Rightarrow v_1 + 4\frac{3}{4}v_1 = 4v_1 = 4v_1 \Rightarrow \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$\Rightarrow x(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad c_1, c_2 \in \mathbb{R}$

b) $x' = \begin{pmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} x$

$\chi(\lambda) = \det \begin{pmatrix} -1-\lambda & -2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{pmatrix} = (-1-\lambda)^2(-3-\lambda) + 4(-3-\lambda) = (-3-\lambda)(1 + 2\lambda + \lambda^2 + 4)$

$\chi(\lambda) = 0 \iff \lambda_1 = 3 \lor \lambda_{2,3} = -1 \pm \sqrt{1-5} = -1 \pm 2i$

$\begin{pmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_1 - 2v_2 \\ 2v_1 - v_2 \\ -3v_3 \end{pmatrix} = \begin{pmatrix} -3v_1 \\ -3v_2 \\ -3v_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -v_1 - 2v_2 \\ 2v_1 - v_2 \\ -3v_3 \end{pmatrix} = \begin{pmatrix} (-1+2i)v_1 \\ (-1+2i)v_2 \\ (-1+2i)v_3 \end{pmatrix} \Rightarrow \cancel{-v_1} - 2v_2 = \cancel{-v_1} + 2iv_1 \Rightarrow v_2 = -iv_1 \Rightarrow v_1 = iv_2$

$2v_1 - v_2 = 2iv_2 - v_2 = -v_2 + 2iv_2 \qquad \Rightarrow \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -v_1 - 2v_2 \\ 2v_1 - v_2 \\ -3v_3 \end{pmatrix} = \begin{pmatrix} (-1-2i)v_1 \\ (-1-2i)v_2 \\ (-1-2i)v_3 \end{pmatrix} \Rightarrow \cancel{-v_1} - 2v_2 = \cancel{-v_1} - 2iv_1 \Rightarrow v_2 = iv_1 \Rightarrow v_1 = -iv_2$

$2v_1 - v_2 = -2iv_2 - v_2 = -v_2 - 2iv_2 \qquad \Rightarrow \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow x^{\mathbb{C}}(t) = c_1 e^{-3t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{(-1+2i)t}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + c_3 e^{(-1-2i)t}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \qquad c_1, c_2, c_3 \in \mathbb{C}$

$Re(x^{\mathbb{C}}) = c_1 e^{-3t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-t}\cos(2t)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t}\cos(2t)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad c_1, c_2, c_3 \in \mathbb{R}$

$Im(x^{\mathbb{C}}) = c_2 e^{-t}\sin(2t)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t}\sin(2t)\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow x(t) = c_1 e^{-t}\sin(2t)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-t}\cos(2t)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-3t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad c_1, c_2, c_3 \in \mathbb{R}$

$= e^{-t}\begin{pmatrix} c_1 \sin(2t) \\ c_2 \cos(2t) \\ c_3 e^{3} \end{pmatrix}$

DGL Ü4

2) a) $f(t,x) = \sin(tx)$ $I=(a,b)$ $B=\mathbb{R}$

Satz 3.3. $G=(a,b)\times\mathbb{R} \subseteq \mathbb{R}^{1+1}$ ... für festes t konvex bzgl. x

$f: G\to\mathbb{R}^1 \equiv \sin(t,x)$ ist stetig und $\frac{\partial}{\partial x}\sin(tx) = t\cos(tx)$ ... stetig auf G

$\|\frac{\partial f}{\partial x}(t,x)\| = \|t\cos(tx)\| \leq (b-a)$.

$\Rightarrow f$ ist Lipschitz bzgl. x

b) $f(t,x)=\sin(tx)$ $I=\mathbb{R}$ $B=\mathbb{R}$

Satz 3.2. $G=\mathbb{R}\times\mathbb{R}\subseteq\mathbb{R}^2$ ... offen

$f: G\to\mathbb{R}^1 \equiv \sin(t,x)$ ... stetig und $\frac{\partial}{\partial x}\sin(tx) = t\cos(tx)$ ... stetig auf G

$\Rightarrow f$ ist lokal Lipschitz bzgl. x

c) $f(t,x) = x\sin(t)$ $I=\mathbb{R}$, $B=\mathbb{R}$

$\|f(t,x)-f(t,y)\| = \|x\sin(t) - y\sin(t)\| = |x-y|\underbrace{\|\sin(t)\|}_{\leq 1} \leq |x-y|$

$\Rightarrow f$ ist Lipschitz bzgl. x

d) $f(t,x) = A(t)x + g(t)$ $I=(a,b)$ $B=\mathbb{R}^n$ $A(t)$ ... stetige $n\times n$ Matrix auf (a,b), $f(t)$ ... auf (a,b) stetige vektorwertige Funktion

$\|f(t,x)-f(t,y)\| = \|A(t)x+g(t)-A(t)y-g(t)\| = \|A(t)x - A(t)y\|$

$= \|A(t)(x-y)\| \leq \underbrace{\|A(t)\|}_{:=L\in\mathbb{R}}\|x-y\|$

$\Rightarrow f$ ist Lipschitz bzgl. x

e) $f(t,x) = (x_1 e^{x_2}\cos(t), \sin(x_1x_2))$ $I=\mathbb{R}$ $B=[-2,5]\times[0,10]$

f ... stetig $\frac{\partial}{\partial x}f(t,x) = \begin{pmatrix} e^{x_2}\cos(t) & x_2\sin(x_1) \\ x_1e^{x_2}\cos(t) & x_1\sin(x_2)\end{pmatrix}$ ... stetig

$\|\frac{\partial}{\partial x}f(t,x)\| = \left\|\begin{matrix} e^{x_2}\cos(t) & x_2\sin(x_1) \\ x_1e^{x_2}\underbrace{\cos(t)}_{\leq 1} & x_1\sin(x_2)\end{matrix}\right\| < \infty$

$\Rightarrow f$ ist Lipschitz bzgl. x

DGL Ü4

3) I... Intervall $f: I\times\mathbb{R}\to\mathbb{R}$ ...stetig, bzgl. 2. Argument Lipschitz-stetig mit Konstante L

$x'(t) = f(t, x(t))$ $t\in I$ hat zwei Lösungen x_1, x_2

zz: $\forall t > t_0 \in I$: $|x_1(t) - x_2(t)| \leq |x_1(t_0) - x_2(t_0)|\, e^{L|t-t_0|}$

Da x_1, x_2 Lösungen von $x'(t) = f(t, x(t))$ sind gilt

$$x_1(t) = x_1(t_0) + \int_{t_0}^{t} f(s, x_1(s))\,ds; \quad x_2(t) = x_2(t_0) + \int_{t_0}^{t} f(s, x_2(s))\,ds$$

$$x_1(t) - x_2(t) = x_1(t_0) - x_2(t_0) + \int_{t_0}^{t} f(s, x_1(s))\,ds - \int_{t_0}^{t} f(s, x_2(s))\,ds$$

$$\|x_1(t) - x_2(t)\| = \|x_1(t_0) - x_2(t_0) + \int_{t_0}^{t} f(s, x_1(s)) - f(s, x_2(s))\,ds\,\|$$

$$\leq \|x_1(t_0) - x_2(t_0)\| + \int_{t_0}^{t} \|f(s, x_1(s)) - f(s, x_2(s))\|$$

$$\leq \|x_1(t_0) - x_2(t_0)\| + \int_{t_0}^{t} L\,\|x_1(s) - x_2(s)\|\,ds$$

Gronwall mit $K = |x_1(t_0) - x_2(t_0)|$ $\quad a(s) = L \quad x(s) = |x_1(s) - x_2(s)|$

$\Rightarrow \forall t \in [t_0, \infty)$: $x(t) \leq K\, e^{A(t)}$ mit $A(t) := \int_{t_0}^{t} a(s)\,ds$

$$A(t) = \int_{t_0}^{t} L\,ds = Lt\Big|_{t_0}^{t} = L(t - t_0)$$

$\Rightarrow |x_1(t) - x_2(t)| \leq |x_1(t_0) - x_2(t_0)|\, e^{L|t-t_0|}$

DGL Ü4

4) $G \subseteq \mathbb{R}^{n+1}$ ...kompakt $\quad f: G \to \mathbb{R}^n$ ...stetig und lokal Lipschitz bzgl. x

$\Rightarrow f$ ist Lipschitz bzgl. x

Beweis Indirekt. Angenommen f ist nicht Lipschitz bzgl. x

$\Rightarrow \forall L>0 \; \exists (t,x),(t,y) \in G : \|f(t,x)-f(t,y)\| > L\|x-y\|$

Wir definieren t_n, x_n, y_n durch

(1) $\forall n \in \mathbb{N} \; \exists (t_n, x_n),(t_n,y_n) \in G : \|f(t_n,x_n)-f(t_n,y_n)\| > n\|x_n - y_n\|$

Da G kompakt ist liegen $(t,x), (t,y) \in G$ mit

$\lim_{n\to\infty} t_{k(n)} = t \qquad \lim_{n\to\infty} x_{k(n)} = x \qquad \lim_{n\to\infty} y_{k(n)} = y$

Da G kompakt und f stetig ist folgt $f(G)$ ist beschränkt.

$\Rightarrow \exists m \in \mathbb{R} \; \forall (t,x) \in G: \|f(t,x)\| < m < \infty$

$$\|x_{k(n)} - y_{k(n)}\| \overset{(1)}{<} \frac{1}{k(n)} \|f(t_{k(n)}, x_{k(n)}) - f(t_{k(n)}, y_{k(n)})\| < \frac{2m}{k(n)} \to 0$$

$\Rightarrow x = \lim_{n\to\infty} x_{k(n)} = \lim_{n\to\infty} y_{k(n)} = y$

Da f lokal Lipschitz ist gilt

$\exists \varepsilon>0 \; \exists L>0 \; \forall (s,a),(s,b) \in B_\varepsilon((t,x)) : \|f(s,a)-f(s,b)\| \leq L\|a-b\|$

was ein Widerspruch zu (1) ist.

DGL Ü4

5) $u, \delta, L : I \to [0,\infty]$ ... stetig $\quad I = [t_0, t_1]$

$$\forall t \in I: u(t) \leq \delta(t) + \int_{t_0}^{t} L(x)\, u(x)\, dx$$

$$\Rightarrow \forall t \in I: u(t) \leq \delta(t) + \int_{t_0}^{t} \delta(x) L(x)\, e^{\int_x^t L(u)\,du}\, dx$$

$$y(t) := \int_{t_0}^{t} L(x)\, u(x)\, dx$$

$$\Rightarrow y'(t) = L(t)\, u(t) \leq L(t)\Big(\delta(t) + \underbrace{\int_{t_0}^{t} L(x) u(x)\, dx}_{=y(t)}\Big) \leq L\delta + Ly$$

(1) $$z(t) := y(t)\, e^{-\int_{t_0}^{t} L(x)\,dx}$$

$$\Rightarrow z'(t) = y'(t)\, e^{-\int_{t_0}^{t} L(x)\,dx} - y(t)\, e^{-\int_{t_0}^{t} L(x)\,dx} L(t)$$

$$\leq e^{-\int_{t_0}^{t} L\,dx} (L\delta + \cancel{Ly} - \cancel{Ly}) = L\delta\, e^{-\int_{t_0}^{t} L\,dx}$$

(2) $$\Rightarrow z(t) \leq \int_{t_0}^{t} \delta(s) L(s) \exp\Big(-\int_{t_0}^{s} L(u)\,du\Big)\, ds$$

$$\Rightarrow y(t) \overset{(1)}{=} z(t)\, e^{\int_{t_0}^{t} L(x)\,dx} \overset{(2)}{\leq} \int_{t_0}^{t} \delta(s) L(s) \exp\Big(\int_{t_0}^{t} L(u)\,du - \int_{t_0}^{s} L(u)\,du\Big)\, ds$$

$$= \int_{t_0}^{t} \delta(s) L(s)\, e^{\int_s^t L(u)\,du}\, ds$$

$$\Rightarrow u(t) \leq \delta(t) + y(t) \leq \delta(t) + \int_{t_0}^{t} \delta(x) L(x)\, e^{\int_x^t L(u)\,du}\, dx$$

DGL Ü4

6) $x'(t) = t^2(1-(x(t))^2) = f(t, x(t))$ ... stetig $\quad t \in [1,3]$

$x(2) = 1$

$G = (1,3) \times B_1(1)$ ... konvex, offen

$\| \frac{\partial}{\partial x} f(t,x) \| = \| -2t^2 x(t) \| = 2 \cdot 3^2 \cdot 2 = 36 =: L$

$\Rightarrow f$ ist Lipschitz mit Lipschitzkonstante 36

$(t_0, x_0) = (2, 1) \in G$

G...offen $\Rightarrow \alpha \in (0,1)$, $r \in (0,1)$, $L = 36 > 0$

$$R := [t_0 - \alpha, t_0 + \alpha] \times \overline{B}_r(x_0) \subseteq G$$

$$= [1-\alpha, 1+\alpha] \times \overline{B}_r(1) \subseteq G$$

$m := \max_{(t,x) \in R} \| f(t,x) \| = \max_{(t,x) \in R} \| t^2(1-x^2) \| = \| (1+\alpha)^2 (1-(1-r)^2) \|$

$\leq \| 2^2 (1-0^2) \| = 4$

$\delta < \min(\alpha, \frac{r}{m}, \frac{1}{L}) = \min(1, \frac{1}{4}, \frac{1}{36}) = \frac{1}{36}$

$J_\delta := [t_0 - \delta, t_0 + \delta] = [2 - \frac{1}{36}, 2 + \frac{1}{36}]$

Lösung ist $x(t) = 1$, da $x'(t) = 0$ und $t^2(1-(x(t))^2) = t^2(1-1) = 0$

und existiert somit auf ganz $(1,3)$

DGL Ü4

7) $G := (-10, 10) \times B_c(1)$ $\quad f: G \to \mathbb{R} \quad (t,x) \mapsto 1+x^2$

a) $x'(t) = f(t, x(t)) \quad x(0) = 1$

G…offen $\quad$ f…stetig $\quad \|\frac{\partial}{\partial x} f(t,x)\| = \|2x\| = 2(1+c) = 2c+2 =: L$

f… Lipschitz $\quad (t_0, x_0) = (0, 1) \in G$

$R := [t_0 - \alpha, t_0 + \alpha] \times \overline{B}_r(x_0) = [-\alpha, +\alpha] \times \overline{B}_r(1) \subseteq G$

$\alpha \in (0, 10) \quad r \in (0, c)$

$m := \max\limits_{(t,x)\in R} \|f(t,x)\| = \max\limits_{(t,x)\in R} \|1+x^2\| = 1 + (1+c)^2 = c^2+2c+2$

$\delta < \min(\alpha, \frac{r}{m}, \frac{1}{L}) \leq \min(10, \frac{c}{c^2+2c+2}, \frac{1}{2c+2})$

1. Fall $c < \sqrt{2}$: $\delta < \frac{c}{c^2+2c+2}$

2. Fall $c > \sqrt{2}$: $\delta < \frac{1}{2c+2}$

nach Picard-Lindelöf besitzt das AWP auf $[-\delta, \delta]$ genau eine Lösung.

b) $$\left.\begin{aligned} &\max_{c\in(0,\sqrt{2}]} \frac{c}{c^2+2c+2} = \frac{\sqrt{2}}{\sqrt{2}^2+2\sqrt{2}+2} = \frac{\sqrt{2}}{2\sqrt{2}+4} = \frac{\sqrt{2}}{2} - \frac{1}{2} \\ &\max_{c\in[\sqrt{2},\infty)} \frac{1}{2c+2} = \frac{1}{2\sqrt{2}+2} = \frac{\sqrt{2}}{2} - \frac{1}{2} \end{aligned}\right\} \Rightarrow c = \sqrt{2}$$

$\Rightarrow G = (-10, 10) \times B_{\sqrt{2}}(1)$ ergibt Intervalllänge $2\delta = \sqrt{2} - 1$

c) $x' = 1+x^2 \iff \frac{dx}{dt} = 1+x^2 \iff \frac{1}{1+x^2}\,dx = dt$

$\int \frac{1}{1+x^2}\,dx = \int 1\,dt \iff \arctan(x) = t + c$

$\iff x = \tan(t+c)$

AWP $x(0)=1 \Rightarrow 1 = \tan(0+c) = \tan(c) \iff c = \arctan(1) = \frac{\pi}{4}$

$\Rightarrow x(t) = \tan(t + \frac{\pi}{4})$

Die Lösung existiert auf $(-\frac{\pi}{2} - \frac{\pi}{4}, \frac{\pi}{2} - \frac{\pi}{4}) = (-\frac{3\pi}{4}, \frac{\pi}{4})$

$\sim (-2{,}356; 0{,}7854)$

$(-\frac{1}{2}(\sqrt{2}-1), \frac{1}{2}(\sqrt{2}-1)) \sim (-0{,}2071, 0{,}2071)$

DGL Ü4

8) $x'(t) = t - (x(t))^2 \qquad t \in \mathbb{R} \qquad x(0) = 1$

$$x_0(t) := x_0 = 1$$

$$x_1(t) := P(x_0)(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\,ds = 1 + \int_0^t s - 1\,ds = 1 + \frac{t^2}{2} - t$$

$$x_2(t) := P(x_1)(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\,ds = 1 + \int_0^t s - \left(\frac{s^2}{2} - s + 1\right)^2 ds$$

$$= 1 + \int_0^t \left(-\frac{s^4}{4} + s^3 - 2s^2 + 3s - 1\right) ds = 1 + \left(-\frac{s^5}{20} + \frac{s^4}{4} - \frac{2s^3}{3} + \frac{3s^2}{2} - s\right)\Big|_0^t$$

$$= -\frac{t^5}{20} + \frac{t^4}{4} - \frac{2t^3}{3} + \frac{3t^2}{2} - t + 1$$

$$x_3(t) := P(x_2)(t) = x_0 + \int_{t_0}^{t} f(s, x_2(s))\,ds$$

$$= 1 + \int_0^t s - (x_2(s))^2\,ds$$

$$= -\frac{t^{11}}{4400} + \frac{t^{10}}{400} - \frac{31t^9}{2160} + \frac{29t^8}{480} - \frac{233t^7}{1260} + \frac{13t^6}{30} - \frac{49t^5}{60} + \frac{13t^4}{12}$$

$$- \frac{4t^3}{3} + \frac{3t^2}{2} - t + 1$$

DGL Ü4

9) $x' = x^2 \quad x(0)=1$

$x_0(t) = x_0 = 1$

$x_1(t) = P(x_0)(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\,ds = 1 + \int_0^t 1\,ds = 1+t$

$x_2(t) = P(x_1)(t) = 1 + \int_0^t (1+s)^2\,ds = 1 + \int_0^t s^2+2s+1\,ds$

$= 1 + \frac{t^3}{3} + t^2 + t = \frac{1}{3}t^3 + t^2 + t + 1$

$x_3(t) = P(x_2)(t) = 1 + \int_0^t \left(\frac{1}{3}s^3 + s^2 + s + 1\right)^2 ds = 1 + \int_0^t \frac{s^6}{9} + \frac{2}{3}s^5 + \frac{5}{3}s^4 + \frac{8}{3}s^3 + 3s^2 + 2s + 1\,ds$

$= 1 + \frac{t^7}{63} + \frac{t^6}{9} + \frac{t^5}{3} + \frac{2}{3}t^4 + t^3 + t^2 + t$

$\Big(x_4(t) = 1 + \int_0^t \frac{1}{63}s^7 + \frac{1}{9}s^6 + \frac{1}{3}s^5 + \frac{2}{3}s^4 + s^3 + s^2 + s + 1\,ds$

$= \frac{1}{504}t^8 + \frac{1}{63}t^7 + \frac{1}{18}t^6 + \frac{2}{15}t^5 + \frac{1}{4}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 + t + 1 \Big)$

exakte Lösung berechnen:

$\frac{dx}{dt} = x^2 \iff \frac{1}{x^2}dx = 1\,dt \iff \int \frac{1}{x^2}dx = \int 1\,dt$

$\iff -\frac{1}{x} = t + c \iff x = -\frac{1}{t+c} \qquad c \in \mathbb{R}$

$x(0) = -\frac{1}{0+c} = -\frac{1}{c} = 1 \Rightarrow c = -1 \qquad \Rightarrow x(t) = -\frac{1}{t-1} = \frac{1}{1-t}$

$\sum_{k=0}^{\infty} t^k = \lim_{n\to\infty} \frac{t^{n+1}-1}{t-1} \begin{cases} = -\frac{1}{t-1} = \frac{1}{1-t} & \text{für } |t|<1 \\ = \infty & \text{sonst} \end{cases}$

$\Rightarrow$ für $t \in (-1, 1)$ konvergiert die Picarditeration gegen die exakte Lösung

für $\forall S \in (0,1): [-S, S]$ — " —