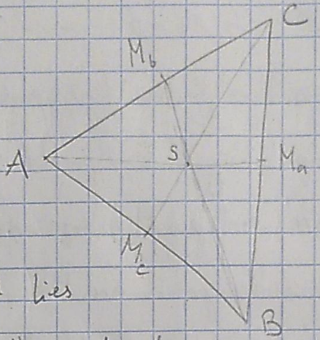
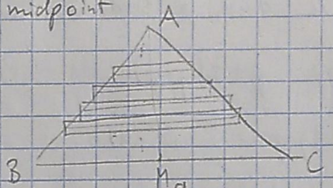
CAGD Ü1

1) Zeige, dass der Schwerpunkt eines Dreiecks eine Konvexkombination der Ecken ist.



Let the triangle be given by points A, B and C.

First we show that the centroid lies on the line segment $\overrightarrow{AM_a}$ where M_a is the midpoint between B and C.



By approximating the triangle with rectangles as shown on the right we can see that in this approximation each rectangle has equal amount of "mass" on each side of the $\overrightarrow{AM_a}$ line. By taking a better and better approximation (smaller height of rectangles) this property stays the same. Therefore the centroid is on the line.

The same holds for the lines BM_b and CM_c.

The line segments on which S lies are

I $A + x(M_a - A) = (1-x)A + \frac{1}{2}xB + \frac{1}{2}xC$ with $x \in [0,1]$

II $B + y(M_b - B) = (1-y)B + \frac{1}{2}yA + \frac{1}{2}yC$ with $y \in [0,1]$

III $C + z(M_c - C) = (1-z)C + \frac{1}{2}zA + \frac{1}{2}zB$ with $z \in [0,1]$

For $x = y = z = \frac{2}{3} \in [0,1]$ we get

I $(1-\frac{2}{3})A + \frac{1}{2}\frac{2}{3}B + \frac{1}{2}\cdot\frac{2}{3}C = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$ II $\ldots = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$

III $\ldots = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$

Thus the point $\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$ is an intersection of the three line segments, as is S. Since two lines, that are not the same (assume $A \neq B, B \neq C, A \neq C$ otherwise we can argue about the definition of triangle) have at most one intersection we can derive that $S = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$ which is indeed a convex combination ($\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, $\frac{1}{3} \geq 0$, finite sum). $\square$

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2) Zeige, dass für Bernsteinpolynome $B_i^n(t) = B_{n-i}^n(1-t)$ gilt.

$$B_{n-i}^n(1-t) = \binom{n}{n-i}(1-t)^{n-i}\underbrace{(1-(1-t))}_{=t}^{\overbrace{n-(n-i)}^{=i}} = \binom{n}{n-i}(1-t)^{n-i}t^i \overset{*}{=} \binom{n}{i}t^i(1-t)^{n-i} = B_i^n(t)$$

*show: $\binom{n}{n-i} = \binom{n}{i}$ either symmetry of the pascal triangle or

$$\binom{n}{n-i} = \frac{n!}{(n-i)!\,(n-\underbrace{(n-i)}_{=i})!} = \frac{n!}{(n-i)!\,i!} = \binom{n}{i}$$

$\square$

CAGD Ü1

6) Zeige, dass das Bernsteinpolynom $B_i^n(t)$ im Intervall $[0,1]$ an der Stelle $t=\frac{i}{n}$ ein Maximum annimmt.

$$\frac{d}{dt}B_i^n(t) = \frac{d}{dt}\binom{n}{i}t^i(1-t)^{n-i} = \binom{n}{i}\left(i\,t^{i-1}(1-t)^{n-i} - (n-i)t^i(1-t)^{n-i-1}\right)$$

$$\xrightarrow{t=\frac{i}{n}} \binom{n}{i}\left(i\frac{(i)^{i-1}}{(n)^{i-1}}\frac{(n-i)^{n-i}}{(n)^{n-i}} - (n-i)\frac{(i)^i}{(n)^i}\frac{(n-i)^{n-i-1}}{(n)^{n-i-1}}\right)$$

$$= \binom{n}{i}\left(\frac{(i)^i(n-i)^{n-i}}{(n)^{n-1}} - \frac{(i)^i(n-i)^{n-i}}{(n)^{n-1}}\right) = 0$$

$$\frac{d^2}{dt^2}B_i^n(t) = \frac{d}{dt}\binom{n}{i}\left(i\,t^{i-1}(1-t)^{n-i} - (n-i)t^i(1-t)^{n-i-1}\right)$$

$$= \binom{n}{i}\left(i(i-1)t^{i-2}(1-t)^{n-i} - i\,t^{i-1}(n-i)(1-t)^{n-i-1} - (n-i)i\,t^{i-1}(1-t)^{n-i-1} + (n-i)t^i(n-i-1)\cdot(1-t)^{n-i-2}\right)$$

$$= \binom{n}{i}t^{i-2}(1-t)^{n-i-2}\left(i(i-1)(1-t)^2 - it(n-i)(1-t) - it(n-i)(1-t) + (n-i)t^2(n-i-1)\right)$$

$$= \binom{n}{i}t^{i-2}(1-t)^{n-i-2}\left(i(i-1)(1-t)^2 - 2it(n-i)(1-t) + (n-i)t^2(n-i-1)\right)$$

$$\xrightarrow{t=\frac{i}{n}} \binom{n}{i}\frac{i^{i-2}}{n^{i-2}}\frac{(n-i)^{n-i-2}}{n^{n-i-2}}\left(i(i-1)\frac{(n-i)^2}{n^2} - 2\frac{i^2}{n}\frac{(n-i)^2}{n} + (n-i)\frac{i^2}{n^2}(n-i-1)\right)$$

$$= \binom{n}{i}\frac{i^{i-2}(n-i)^{n-i-2}}{n^{n-4}}\frac{i(n-i)}{n^2}\left((i-1)(n-i) - 2i(n-i) + i(n-i-1)\right)$$

$$= \binom{n}{i}\frac{i^{i-1}(n-i)^{n-i-1}}{n^{n-2}}\left(ni - i^2 - n + i - 2ni + 2i^2 + ni - i^2 - i\right)$$

$$= \text{---}\quad '' \quad \text{---}\quad (-n)$$

$$= -\binom{n}{i}\frac{i^{i-1}(n-i)^{n-i-1}}{n^{n-3}} < 0 \text{ for } i<n$$ ($i, n \in \mathbb{N}_0$ for well-definedness of $B_i^n(t)$)

For $i=n$:

$$B_n^n(t) = \binom{n}{n}t^n(1-t)^{n-n} = t^n$$

$$\frac{d}{dt}B_n^n(t) = nt^{n-1} \xrightarrow{t=\frac{n}{n}=1} n \neq 0 \quad \text{if } n\neq 0 \Rightarrow \text{no max.}$$

For $i>n$:

$$B_i^n(t) \equiv 0 \quad \text{so every point is a max.}$$

$\square$

CAGD Ü1

7) Seien $p_0, p_t, p_1 \in \mathbb{R}^2$ und $t \in (0,1)$. Entwerfe einen Algorithmus, der die Kontrollpunkte der quadratischen Bezierkurve $b(t)$ mit $b(0)=p_0$, $b(t)=p_t$, $b(1)=p_1$, berechnet.

As bézier curves are endpoint interpolating we choose

$b_0 := p_0$ and $b_2 := p_1$.

First method: Using algorithm of de Casteljau

$b_0^0(t) := p_0, \quad b_1^0(t) := x \quad b_2^0(t) := p_1$

$b_0^1(t) = (1-t) b_0^0(t) + t\, b_1^0(t) = (1-t)p_0 + tx \qquad b_1^1(t) = (1-t) b_1^0(t) + t\, b_2^0(t) = (1-t)x + tp_1$

$b_0^2(t) = (1-t) b_0^1(t) + t\, b_1^1(t) = (1-t)^2 p_0 + t(1-t)x + t(1-t)x + t^2 p_1 = (1-t)^2 p_0 + 2t(1-t)x + t^2 p_1$

Solve for x: $p_t = b(t) = b_0^2(t) = (1-t)^2 p_0 + 2t(1-t)x + t^2 p_1$

$$\Rightarrow x = \frac{p_t - (1-t)^2 p_0 - t^2 p_1}{2t(1-t)} \qquad \Rightarrow b_1 := \frac{p_t - (1-t)^2 p_0 - t^2 p_1}{2t(1-t)}$$

Second method: Using Bernstein polynomials

$$p_t = b(t) = \sum_{l=0}^{2} B_l^2(t)\, b_l = B_0^2(t) \underbrace{b_0}_{=p_0} + B_1^2(t) \underbrace{b_1}_{x:=} + B_2^2(t) \underbrace{b_2}_{=p_1}$$

$$= \underbrace{\binom{2}{0}}_{=1} t^0 (1-t)^{2-0} p_0 + \underbrace{\binom{2}{1}}_{=2} t^1 (1-t)^{2-1} x + \underbrace{\binom{2}{2}}_{=1} t^2 (1-t)^{2-2} p_1$$

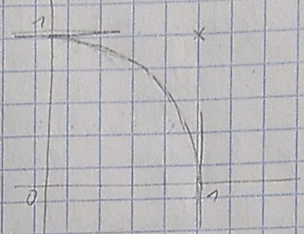
$$= (1-t)^2 p_0 + 2t(1-t)x + t^2 p_1$$

As above we get that $\quad b_1 = x = \dfrac{p_t - (1-t)^2 p_0 - t^2 p_1}{2t(1-t)}$

$\square$

CAGD Ü1

9) Approximiere einen Viertelkreis durch eine Bézierkurve vom Grad 2, sodass sich die Tangenten des Kreises und der Bézierkurve im Anfangs- und Endpunkt berühren.



Consider the circle segment on the right.
It is given by $\{(x,y)^T : x,y \in [0,1], x^2+y^2=1\}$.

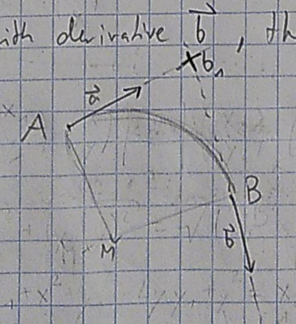
Obviously we have to have $b_0 := \begin{pmatrix}0\\1\end{pmatrix}$ and $b_2 := \begin{pmatrix}1\\0\end{pmatrix}$.
We want to find b_1, such that the resulting bezier curve $b(t)$ has the two properties: $b'(0) = \alpha\begin{pmatrix}1\\0\end{pmatrix}$ and $b'(1) = \beta\begin{pmatrix}0\\1\end{pmatrix}$

We know that $b'(0) = \underbrace{2}_{n}(b_1 - b_0)$ and $b'(1) = \underbrace{2}_{n}(\underbrace{b_2}_{b_n} - \underbrace{b_1}_{b_{n-1}})$

$$\left.\begin{aligned} \Rightarrow \alpha\begin{pmatrix}1\\0\end{pmatrix} &= b'(0) = 2\begin{pmatrix}x-0\\y-1\end{pmatrix} = \begin{pmatrix}2x\\2(y-1)\end{pmatrix} \\ \beta\begin{pmatrix}0\\1\end{pmatrix} &= b'(1) = 2\begin{pmatrix}1-x\\0-y\end{pmatrix} = \begin{pmatrix}2(1-x)\\-2y\end{pmatrix}\end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= 2x \\ 0 &= 2y-2 \\ 0 &= 2-2x \\ \beta &= -2y \end{aligned} \Rightarrow \begin{aligned} y&=1 \\ x&=1\end{aligned}$$

$$\Rightarrow b_1 = \begin{pmatrix}1\\1\end{pmatrix}$$

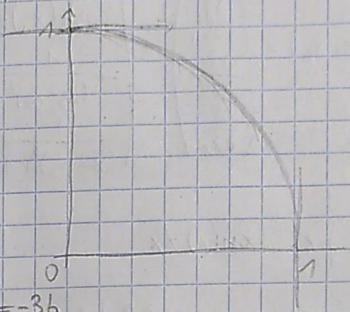
Scaling and rotating obviously does not change the fact that b_1 lies on the intersection of the two lines given by the derivative of the start and end point.
e.g. if the quarter circle starts at point A with derivative $\vec{a}$ and ends at point B with derivative $\vec{b}$, then $b_1 = A + t\vec{a} = B + u\vec{b}$ for some $t, u \in \mathbb{R}$.



$\square$

CAGD Ü1

10) Approximiere einen Viertelkreis durch eine Bézierkurve $b(t)$ vom Grad 3, sodass sich die Tangenten des Kreises und der Bézierkurve im Anfangs- und Endpunkt berühren und $b(\frac{1}{2})$ auf dem Kreis liegt.



$$b_0 = \begin{pmatrix}0\\1\end{pmatrix}, \quad b_1 = \begin{pmatrix}x\\y\end{pmatrix}, b_2 = \begin{pmatrix}a\\b\end{pmatrix}, b_3 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\alpha\begin{pmatrix}1\\0\end{pmatrix} = b'(0) = 3(b_1 - b_0) = \begin{pmatrix}3x\\3y-3\end{pmatrix}$$

$$\beta\begin{pmatrix}0\\1\end{pmatrix} = b'(1) = 3(b_3 - b_2) = \begin{pmatrix}3-3a\\-3b\end{pmatrix}$$

$$\Rightarrow \alpha = 3x, \quad 0 = 3y-3, \quad 0 = 3-3a, \quad \beta = -3b$$

$$\Rightarrow x = \frac{\alpha}{3}, \quad y = 1, \quad a = 1, \quad b = -\frac{\beta}{3}$$

$b(\frac{1}{2})$ lies on the circle $\Rightarrow b(\frac{1}{2}) \in \{\begin{pmatrix}x\\y\end{pmatrix} : x,y \in [0,1], x^2+y^2 = 1\}$

$$b(\tfrac{1}{2}) = b_0^3(\tfrac{1}{2}) = \frac{1}{2}\left(b_0^2(\tfrac{1}{2}) + b_1^2(\tfrac{1}{2})\right) = \frac{1}{4}\left(b_0^1(\tfrac{1}{2}) + b_1^1(\tfrac{1}{2}) + b_1^1(\tfrac{1}{2}) + b_2^1(\tfrac{1}{2})\right)$$

$$= \frac{1}{8}\left(b_0^0(\tfrac{1}{2}) + b_1^0(\tfrac{1}{2}) + b_1^0(\tfrac{1}{2}) + b_2^0(\tfrac{1}{2}) + b_1^0(\tfrac{1}{2}) + b_2^0(\tfrac{1}{2}) + b_2^0(\tfrac{1}{2}) + b_3^0(\tfrac{1}{2})\right)$$

$$= \frac{1}{8}\left(\begin{pmatrix}0\\1\end{pmatrix} + 3\begin{pmatrix}x\\y\end{pmatrix} + 3\begin{pmatrix}a\\b\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\right) = \frac{1}{8}\begin{pmatrix}3x+3+1\\1+3+3b\end{pmatrix} = \frac{1}{8}\begin{pmatrix}3x+4\\3b+4\end{pmatrix}$$

$$1 = \left(\frac{3}{8}x + \frac{1}{2}\right)^2 + \left(\frac{3}{8}b + \frac{1}{2}\right)^2$$

Let us guess that $b(\frac{1}{2})$ is halfway between b_0 and b_3 in other words in the middle of the quarter circle.

circle seg $= \{\begin{pmatrix}\cos(t)\\\sin(t)\end{pmatrix} : t \in [0, \frac{\pi}{2}]\}$ halfway point $= \begin{pmatrix}\cos(\frac{\pi}{4})\\\sin(\frac{\pi}{4})\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}\cos(\frac{\pi}{4})\\\sin(\frac{\pi}{4})\end{pmatrix} = \begin{pmatrix}\frac{3}{8}x + \frac{1}{2}\\\frac{3}{8}b + \frac{1}{2}\end{pmatrix} \quad \Rightarrow x = \frac{8}{3}\left(\cos(\tfrac{\pi}{4}) - \tfrac{1}{2}\right) \quad b = \frac{8}{3}\left(\sin(\tfrac{\pi}{4}) - \tfrac{1}{2}\right)$$

$$\approx 0{,}5523 \qquad\qquad \approx 0{,}5523$$

$$\left(\frac{3}{8}x + \frac{1}{2}\right)^2 + \left(\frac{3}{8}b + \frac{1}{2}\right)^2 = \left(\frac{3}{8}\frac{8}{3}\left(\cos(\tfrac{\pi}{4}) - \tfrac{1}{2}\right) + \tfrac{1}{2}\right)^2 + \left(\frac{3}{8}\frac{8}{3}\left(\sin(\tfrac{\pi}{4}) - \tfrac{1}{2}\right)\right)^2$$

$$= \cos^2(\tfrac{\pi}{4}) + \sin^2(\tfrac{\pi}{4}) = 1$$

$$\Rightarrow b_0 = \begin{pmatrix}0\\1\end{pmatrix} \quad b_1 = \begin{pmatrix}\frac{8}{3}(\cos(\frac{\pi}{4}) - \frac{1}{2})\\1\end{pmatrix} \quad b_2 = \begin{pmatrix}1\\\frac{8}{3}(\sin(\frac{\pi}{4}) - \frac{1}{2})\end{pmatrix} \quad b_3 = \begin{pmatrix}1\\0\end{pmatrix}$$

$\square$