LINAG Ü10

4.3.x $V \ldots VR/K$ $\dim(V) = n < \infty$ $n \geq 2$

$B = (b_1, b_2, \ldots, b_n)$ $C = (c_1, \ldots, c_n)$ ... Basen von V $x \in K^x$

$B^* = (b_1^*, b_2^*, \ldots, b_n^*)$ $C^* = (c_1^*, \ldots, c_n^*)$ ... duale Basen zu B bzw C

a) $\forall i \geq 2 : b_i = c_i \Rightarrow \forall i \geq 2 : b_i^* = c_i^*$ falsch

$K = \mathbb{R}$ $V = \mathbb{R}^2$ $B = (\binom{1}{0}, \binom{0}{1})$ $C = (\binom{1}{1}, \binom{0}{1})$ $v = \binom{1}{1} \in V$

$b_2^*(v) = 1$ $c_2^*(v) = 0$

$\Rightarrow \exists i \geq 2 : b_i^* \neq c_i^*$

b) $b_1 = c_2 \wedge b_2 = c_1 \wedge \forall i \geq 3 : b_i = c_i \Rightarrow b_1^* = c_2^* \wedge b_2^* = c_1^* \wedge \forall i \geq 3 : b_i^* = c_i^*$

Sei $v \in V$ bel. $s_1 \cdot b_1 + \ldots + s_n \cdot b_n = v = t_1 \cdot c_1 + \ldots + t_n \cdot c_n$ wahr

Da $b_1 = c_2 \wedge b_2 = c_1 \wedge \forall i \geq 3 : b_i = c_i$ gilt: $v = t_1 \cdot b_2 + t_2 \cdot b_1 + \ldots + t_n \cdot b_n$

Da LK aus einer Basis eindeutig folgt $s_1 = t_2 \wedge s_2 = t_1 \wedge \forall i \geq 3 : s_i = t_i$

$b_1^*(v) = s_1$ $b_2^*(v) = s_2$ $\forall i \geq 3 : b_i^*(v) = s_i$

$c_1^*(v) = t_1 = s_2$ $c_2^*(v) = t_2 = s_1$ $\forall i \geq 3 : c_i^*(v) = t_i = s_i$

$\Rightarrow b_1^* = c_2^* \wedge b_2^* = c_1^* \wedge \forall i \geq 3 : b_i^* = c_i^*$

c) $\{b_1, \ldots, b_n\} = \{c_1, \ldots, c_n\} \Rightarrow \{b_1^*, \ldots, b_n^*\} = \{c_1^*, \ldots, c_n^*\}$ wahr

$\forall i \in \{1, \ldots, n\} : b_i \in \{c_1, \ldots, c_n\}$

$\Rightarrow b_i^* \in \{c_1^*, \ldots, c_n^*\}$ $\Rightarrow \{b_1^*, \ldots, b_n^*\} \subseteq \{c_1^*, \ldots, c_n^*\}$

umgekehrt genauso

d) $x b_1 = c_1 \wedge \forall i \geq 2 : b_i = c_1 \Rightarrow x \cdot b_1^* = c_1^* \wedge \forall i \geq 2 : b_i^* = c_i^*$

$K = \mathbb{R}$ $V = \mathbb{R}^2$ $B = (\binom{1}{0}, \binom{0}{1})$ $C = (\binom{2}{0}, \binom{0}{1})$ $x = 2$ falsch

Sei $v = \binom{2}{2}$. $b_1^*(v) = 2$ $c_1^*(v) = 1$

$x \cdot b_1^*(v) = 2 \cdot 2 = 4 \neq 1 = c_1^*(v)$