ANA Ü8

1) $\Omega, \Omega' \subseteq \mathbb{R}^n$ ... offene Mengen $K \subseteq \Omega$ ... kompakt $\Phi: \Omega \to \Omega'$ ... Diffeomorphismus

zz: Hausdorffdimension von K ist invariant unter Φ, d.h. $\dim(K) = \dim(\Phi(K))$

Def Φ... Diffeomorphismus $\Leftrightarrow$ Φ... bijektiv, stetig diffbar $\wedge$ Φ^{-1}... stetig diffbar

Es reicht zu zeigen $\dim(\Phi(K)) \leq \dim(K)$ für Φ... bijektiv, stetig diffbar, da dann auch Φ^{-1}... bijektiv, stetig diffbar und somit $\dim(\Phi^{-1}(\Phi(K))) = \dim(K) \leq \dim(\Phi(K))$

zz: $\dim(\Phi(K)) \leq \dim(K)$

Wir wollen Bsp 7.7. verwenden f... Hölder-Stetig $\Rightarrow \alpha \dim(f(A)) \leq \dim(A)$

Φ... Hölder-Stetig mit $\alpha = 1$?

[Lemma 4.2.7. $\Phi: \Omega \to \mathbb{R}^n$, Ω... offen in $\mathbb{R}^n$ Φ... inj. C^1 $d\Phi$ hat auf K... kompakt Rang n

$\Rightarrow \exists C\ \forall x, y \in K: |x-y| \leq C|\Phi(x) - \Phi(y)|$]

$d\Phi^{-1}$ hat auf $\Phi(K)$ vollen Rang n, da Φ^{-1} dort ableitbar ist. $\Phi(K)$... kompakt (da K kompakt $\wedge$ Φ stetig)

$\Rightarrow \exists C\ \forall a, b \in \Phi(K): |a-b| \leq C|\Phi^{-1}(a) - \Phi^{-1}(b)|$

$\Rightarrow \forall x, y \in K: |\Phi(x) - \Phi(y)| \leq C|\Phi^{-1}(\Phi(x)) - \Phi^{-1}(\Phi(y))| = C|x-y|$

$\Rightarrow$ Hölder stetig mit $\alpha = 1$!

$\Rightarrow \dim(\Phi(K)) \leq \dim(K)$

$\Rightarrow \dim(K) = \dim(\Phi(K))$

$\square$

ANA U8

3) $Q := \{(x,y) \in \mathbb{R}^2 : x>0, y>0\}$ $\Phi: Q \to \Phi(Q)$ $(x,y) \mapsto (x^2-y^2, xy)^T$

(i) zz: Φ ist Diffeomorphismus wobei $\Phi(Q) = \{(a,b) \in \mathbb{R}^2 : b>0\}$

- $\Phi(Q) \subseteq \{(a,b)\in\mathbb{R}^2 : b>0\}$, da $x>0, y>0 \Rightarrow xy>0$ ✓
- $\{(a,b)\in\mathbb{R}^2: b>0\} \subseteq \Phi(Q)$, da $a\in\mathbb{R}, b>0$ bel. $\Rightarrow$

1. Fall: $0 \leq |a| < \sqrt{a^2+4b^2}$: $\Rightarrow -a \leq |a| < \sqrt{a^2+4b^2} \Rightarrow a+\sqrt{a^2+4b^2} > 0$

$$x := \frac{\sqrt{a+\sqrt{a^2+4b^2}}}{\sqrt{2}} \quad y := \frac{b}{x} = \frac{\sqrt{2}\,b}{\sqrt{a+\sqrt{a^2+4b^2}}}$$

$$\Phi(x,y) = \left(\frac{a+\sqrt{a^2+4b^2}}{2} - \frac{2b^2}{a+\sqrt{a^2+4b^2}}, \frac{\sqrt{a+\sqrt{a^2+4b^2}}}{\sqrt{2}} \frac{\sqrt{2}\,b}{\sqrt{a+\sqrt{a^2+4b^2}}}\right)$$

$$= \left(\frac{(a+\sqrt{a^2+4b^2})^2-4b^2}{2(a+\sqrt{a^2+4b^2})}, b\right) = \left(\frac{a^2+2a\sqrt{a^2+4b^2}+a^2+4b^2-4b^2}{2(a+\sqrt{a^2+4b^2})}, b\right)$$

$$= \left(\frac{a^2+a\sqrt{a^2+4b^2}}{a+\sqrt{a^2+4b^2}}, b\right) = (a,b)$$

2. Fall $0 < \sqrt{a^2+4b^2} < |a|$ o.B.d.A. $a=|a|$ sonst abjelzt überall $|a|$ $\Rightarrow a-\sqrt{a^2+4b^2}>0$

$$x := \frac{\sqrt{a-\sqrt{a^2+4b^2}}}{\sqrt{2}} \quad y := \frac{b}{x} = \frac{\sqrt{2}\,b}{\sqrt{a-\sqrt{a^2+4b^2}}}$$

$$\Phi(x,y) = \left(\frac{a-\sqrt{a^2+4b^2}}{2} - \frac{2b^2}{a-\sqrt{a^2+4b^2}}, \frac{\sqrt{a-\sqrt{a^2+4b^2}}}{\sqrt{2}} \frac{\sqrt{2}\,b}{\sqrt{a-\sqrt{a^2+4b^2}}}\right)$$

$$= \left(\frac{a^2-2a\sqrt{a^2+4b^2}+a^2+4b^2-4b^2}{2(a-\sqrt{a^2+4b^2})}, b\right) = \left(\frac{2a(a-\sqrt{a^2+4b^2})}{2(a-\sqrt{a^2+4b^2})}, b\right) = (a,b)$$

- bijektiv, da surjektiv nach ↗ und injektiv, da $\Phi(x,y) = (0,0) \Rightarrow x\cdot y = 0$

$\Rightarrow$ 1. Fall $x=0 \Rightarrow x^2-y^2 = -y^2 = 0 \Rightarrow y=0$ 2. Fall $y=0 \Rightarrow x^2-y^2 = x^2 = 0 \Rightarrow x=0$

- $d\Phi$ stetig, da $d\Phi(x,y) = \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}$ ... stetig
- $d\Phi^{-1}$ stetig, da $d\Phi^{-1}(a,b) = d\begin{pmatrix} \frac{\sqrt{a\pm\sqrt{a^2+4b^2}}}{\sqrt{2}} \\ \frac{\sqrt{2}\,b}{\sqrt{a\pm\sqrt{a^2+4b^2}}} \end{pmatrix} = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$ durch nachrechnen stetig

$\Rightarrow$ Diffeomorphismus

(ii) $\Omega := \{(x,y) \in Q : 1 < xy < 3, 5 < x^2-y^2 < 9\}$ ges: $\int_\Omega x^2+y^2 \, dx\,dy$

$\Omega = \Phi^{-1}((5,9)\times(1,3))$ $\det d\Phi = 2x^2+2y^2 = 2(x^2+y^2)$

$$\int_\Omega x^2+y^2\,dx\,dy = \frac{1}{2}\int_\Omega |\det d\Phi(x,y)|\,d\lambda^2(x,y) = \frac{1}{2}\int_{\Phi(\Omega)} 1\,d\lambda^2(x,y) = \frac{1}{2}\lambda^2(\Phi(\Omega))$$

$$= \frac{1}{2}\lambda^2((5,9)\times(1,3)) = \frac{1}{2}(9-5)(3-1) = \frac{4\cdot 2}{2} = 4$$

ANA 08

4) $\Omega = \{(x, y, 1-x^2-y^2) \in \mathbb{R}^3 : x^2+y^2 \leq 1\}$ ges: Flächenmaß von Ω

$\Phi : (0,1) \times (0, 2\pi) \longrightarrow \mathbb{R}^3$

$(r, \theta) \longmapsto (r\cos\theta,\ r\sin\theta,\ 1-r^2\cos^2\theta - r^2\sin^2\theta) = (r\cos\theta, r\sin\theta, 1-r^2)$

$$d\Phi = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ -2r & 0 \end{pmatrix} \qquad d\Phi^T d\Phi = \begin{pmatrix} \cos^2\theta + \sin^2\theta + 4r^2 & -r\sin\theta\cos\theta + r\sin\theta\cos\theta \\ -r\sin\theta\cos\theta + r\sin\theta\cos\theta & r^2\sin^2\theta + r^2\cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 4r^2+1 & 0 \\ 0 & r^2 \end{pmatrix} \text{ hat det } r^2(4r^2+1)$$

Φ ist injektiv, da $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ schon auf $(0,1)\times(0,2\pi)$ inj. ist.

$$\mathcal{H}^2(\Omega) = \mathcal{H}^2(\Phi((0,1)\times(0,2\pi))) = \int_{(0,1)\times(0,2\pi)} \sqrt{\det d\Phi^T d\Phi}\, d\lambda^2 = \int_{(0,1)\times(0,2\pi)} r\sqrt{4r^2+1}\, d\lambda^2$$

$$= \int_0^{2\pi} d\lambda\theta \int_0^1 r\sqrt{4r^2+1}\, dr = 2\pi \int_1^5 \frac{r}{8r} \sqrt{u}\, du \qquad \left[u = 4r^2+1 \quad \frac{du}{dr} = 8r \right.$$

$$= \frac{\pi}{4} \int_1^5 \sqrt{u}\, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^5 = \frac{\pi}{6}\left(5^{\frac{3}{2}} - 1\right) = \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

ANA Ü8

5) $a<b \qquad r,z \in C^1(a,b)\ ,\ r(t)>0\ \forall t\in(a,b)$

$\Phi: (a,b)\times[0,2\pi) \to \mathbb{R}^3,\quad \Phi(t,\theta) := (r(t)\cos(\theta), r(t)\sin(\theta), z(t))^T$ ... inj.

$F := \Phi((a,b)\times[0,2\pi)) \subseteq \mathbb{R}^3$

zz: $\mathcal{H}^2(F) = 2\pi \int_{(a,b)} r(t)\sqrt{r'(t)^2+z'(t)^2}\,dt$

$$d\varphi(t,\theta) = \begin{pmatrix} r'(t)\cos(\theta) & -r(t)\sin(\theta) \\ r'(t)\sin(\theta) & r(t)\cos(\theta) \\ z'(t) & 0 \end{pmatrix}$$ ... stetig, da $r,z\in C^1(a,b)$

$\Rightarrow \varphi$ ... inj. C^1 von $B_b \setminus B_a \subseteq B_b$ ... offen $\quad B_b\setminus B_a$ ... messbar

Nach Satz 4.2.10. Flächenformel für C^1-Abbildungen gilt nun

$$\mathcal{H}^2(F) = \mathcal{H}^2(\Phi((a,b)\times[0,2\pi))) = \int_{(a,b)\times[0,2\pi)} \sqrt{\det(d\varphi^T d\varphi)}\, d\lambda^n(t,\theta)$$

$$\det d\varphi^T d\varphi = \det\begin{pmatrix} (r'(t))^2\cos^2\theta + (r'(t))^2\sin^2\theta + (z'(t))^2 & -r'(t)r(t)\sin\theta\cos\theta + r'(t)r(t)\sin\theta\cos(\theta) \\ -r'(t)r(t)\sin\theta\cos\theta + r'(t)r(t)\sin\theta\cos\theta & (r(t))^2\sin^2\theta + (r(t))^2\cos^2(\theta) \end{pmatrix}$$

$$= \det\begin{pmatrix} (r'(t))^2 + (z'(t))^2 & 0 \\ 0 & (r(t))^2 \end{pmatrix} = (r'(t)^2 + z'(t)^2)\, r(t)^2$$

$$\Rightarrow \mathcal{H}^2(F) = \int_{(a,b)\times[0,2\pi)} r(t)\sqrt{r'(t)^2+z'(t)^2}\, d\lambda^2(t,\theta) =$$

$$= \int_{(a,b)} r(t)\sqrt{r'(t)^2+z'(t)^2}\, d\lambda(t) \cdot \int_{[0,2\pi)} 1\, d\theta = 2\pi \int_{(a,b)} r(t)\sqrt{r'(t)^2+z'(t)^2}\, dt$$

ANA Ü8

7) Def $S \in \mathbb{R}^d$ heißt Schwerpunkt eines Objektes $A \subseteq \mathbb{R}^d$ mit Dichtefunktion

$$\rho: A \to \mathbb{R} \iff S := \left(\int_A \rho(x)\, d\lambda^d(x)\right)^{-1} \int_A x\, \rho(x)\, d\lambda^d(x) \quad , x = (x_1, \dots, x_d)^T$$

$\rho \equiv 1$ $\quad R > 0$ $\quad$ Halbkugel mit Radius R ist $A := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq R^2,\ z \geq 0\}$

ges: Schwerpunkt der Halbkugel mit Radius R (mithilfe von Zylinderkoordinaten)

$\varphi: [0,R) \times [0,2\pi) \times [0,R)$ ... Zylinderkoordinaten

$(r,\theta,z) \mapsto (r\cos\theta, r\sin\theta, z)$ $\qquad x^2+y^2+z^2 = r^2+z^2 \leq R^2 \Rightarrow 0 \leq z \leq \sqrt{R^2-r^2}$

$$d\varphi = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \det d\varphi = r\cos^2\theta + r\sin^2\theta = r$$

$$A = \varphi\underbrace{\{(r,\theta,z) : r \in (0,R),\ \theta \in (0,2\pi),\ z \in (0,R),\ r^2+z^2 \leq R^2\}}_{D}$$

$$\int_A d\lambda^3 = \int_{\varphi(D)} |\det d\varphi|\, d\lambda^3 = \int_{\varphi(D)} r\, d\lambda^3 = \int_0^R \int_0^{\sqrt{R^2-r^2}} r\, d\lambda(z)\, d\lambda(r) \int_0^{2\pi} d\theta$$

$$= 2\pi \int_0^R \frac{R^2}{2}\Big|_0^{\sqrt{R^2-r^2}} dr = \pi \int_0^R R^2 - r^2\, dr = \pi\left(R^2\int_0^R dr - \int_0^R r^2\, dr\right) = \pi\left(R^3 - \frac{R^3}{3}\right) = \frac{2\pi}{3}R^3$$

$$\int_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} d\lambda^3 = \int_D \varphi(r,\theta,z)\, |\det d\varphi|\, d\lambda^3 = \int_D \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} r\, d\lambda^3 = \int_D \begin{pmatrix} r^2\cos\theta \\ r^2\sin\theta \\ rz \end{pmatrix} d\lambda^3$$

$$= \int_0^R \int_0^{2\pi} \int_0^{\sqrt{R^2-z^2}} \begin{pmatrix} r^2\cos\theta \\ r^2\sin\theta \\ rz \end{pmatrix} dr\, d\theta\, dz = \int_0^R \int_0^{2\pi} \begin{pmatrix} \cos\theta \frac{1}{3}\left(\sqrt{R^2-z^2}\right)^3 \\ \sin\theta \frac{1}{3}\left(\sqrt{R^2-z^2}\right)^3 \\ z\frac{1}{2}\left(\sqrt{R^2-z^2}\right)^2 \end{pmatrix} d\theta\, dz = \int_0^R \int_0^{2\pi} \begin{pmatrix} \frac{1}{3}(R^2-z^2)^{\frac{3}{2}}\cos\theta \\ \frac{1}{3}(R^2-z^2)^{\frac{3}{2}}\sin\theta \\ \frac{1}{2}(R^2-z^2)\, z \end{pmatrix} d\theta\, dz$$

$$= \int_0^R \begin{pmatrix} \frac{1}{3}(R^2-z^2)^{3/2} \cdot 0 \\ \frac{1}{3}(R^2-z^2)^{3/2} \cdot 0 \\ \frac{1}{2}(R^2-z^2)\, z\, 2\pi \end{pmatrix} dz = \int_0^R \begin{pmatrix} 0 \\ 0 \\ \pi(R^2 z - z^3) \end{pmatrix} dz = \begin{pmatrix} 0 \\ 0 \\ \pi\left(\frac{1}{2}R^2 z^2 - \frac{z^4}{4}\right) \end{pmatrix}\Bigg|_0^R = \begin{pmatrix} 0 \\ 0 \\ \pi\left(\frac{R^4}{2} - \frac{R^4}{4}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\pi}{4}R^4 \end{pmatrix}$$

$$\Rightarrow S = \left(\frac{2\pi}{3}R^3\right)^{-1} \begin{pmatrix} 0 \\ 0 \\ \frac{\pi}{4}R^4 \end{pmatrix} = \frac{3}{2\pi}\frac{1}{R^3} \begin{pmatrix} 0 \\ 0 \\ \frac{\pi}{4}R^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{8}R \end{pmatrix}$$

ANA Ü8

8) ges: Schwerpunkt einer inhomogenen ($\rho(x) := |x|^2$, euklidische Norm) Halbkugel mit Radius $R > 0$ (mithilfe von Kugelkoordinaten).

$\Phi : [0,R) \times [0,2\pi) \times (0,\frac{\pi}{2}) \to \mathbb{R}^3$ ... Kugelkoordinaten

$(r,\varphi,\theta) \mapsto (r\cos\varphi\cos\theta,\ r\sin\varphi\cos\theta,\ r\sin\theta)^T$

$H := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq R^2,\ z \geq 0\}$

$$S = \left(\int_H |(x,y,z)|^2 \, d\lambda^3(x,y,z)\right)^{-1} \left(\int_H \begin{pmatrix} x \\ y \\ z \end{pmatrix} |(x,y,z)|^2 \, d\lambda^3(x,y,z)\right)$$

$\det d\Phi = r^2\cos\theta$ nach Skript

$$\int_H x^2+y^2+z^2 \, d\lambda^3 = \int_{\Phi(D)} f(x,y,z)\, d\lambda^3 = \int_D f(\Phi(r,\varphi,\theta)) \, |\det d\Phi| \, d\lambda^3$$

$$= \int_D (r^2\cos^2\varphi\cos^2\theta + r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\theta)\, r^2\cos\theta \, d\lambda^3 = \int_D (r^2\cos^2\theta + r^2\sin^2\theta) r^2\cos\theta \, d\lambda^3$$

$$= \int_D r^2 r^2 \cos\theta \, d\lambda^3 = \int_0^R \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r^4 \cos\theta \, d\theta \, d\varphi \, dr = \int_0^R r^4 dr \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta$$

$$= \frac{R^5}{5} 2\pi \sin\left(\frac{\pi}{2}\right) = \frac{2}{5}\pi R^5$$

$$\int_H \begin{pmatrix} x \\ y \\ z \end{pmatrix} x^2+y^2+z^2 \, d\lambda^3 = \int_D \begin{pmatrix} r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta \\ r\sin\theta \end{pmatrix} r^2 r^2 \cos\theta \, d\lambda^3 = \int_D \begin{pmatrix} r^5\cos\varphi\cos^2\theta \\ r^5\sin\varphi\cos^2\theta \\ r^5\sin\theta\cos\theta \end{pmatrix} d\lambda^3$$

$$= \int_0^R \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (\ldots)\, d\theta\, d\varphi\, dr = \begin{pmatrix} \int_0^R r^5 dr \underbrace{\int_0^{2\pi} \cos\varphi \, d\varphi}_{=0} \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta \\ \int_0^R r^5 dr \underbrace{\int_0^{2\pi} \sin\varphi \, d\varphi}_{=0} \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta \\ \int_0^R r^5 dr \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \, d\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{R^6}{6} 2\pi \frac{1}{2}\sin^2\theta \Big|_0^{\frac{\pi}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{\pi}{6} R^6 \end{pmatrix}$$

$$\Rightarrow S = \frac{1}{\frac{2}{5}\pi R^5} \begin{pmatrix} 0 \\ 0 \\ \frac{\pi}{6} R^6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{5}{2}\frac{1}{\cancel{\pi}}\frac{1}{\cancel{R^5}}\frac{\cancel{\pi}}{6}R^{\cancel{6}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{5}{12} R \end{pmatrix}$$