ANA Ü7

1.) (i) $n\left(1-\sqrt{1-\frac{1}{n}}\right) = n - n\sqrt{1-\frac{1}{n}} = n - \sqrt{n^2\left(1-\frac{1}{n}\right)}$

$$= n - \sqrt{n^2-n} = \frac{(n-\sqrt{n^2-n})(n+\sqrt{n^2-n})}{(n+\sqrt{n^2-n})} = \frac{n^2-(n^2-n)}{n+\sqrt{n^2-n}}$$

$$= \frac{n}{n+\sqrt{n^2-n}} = \frac{\frac{n}{n}}{\frac{n+\sqrt{n^2-n}}{n}} = \frac{1}{1+\frac{\sqrt{n^2-n}}{n}}$$

$$\frac{\sqrt{n^2-n}}{n} = \frac{\frac{\sqrt{n^2-n}}{n}}{\frac{n}{n}} = \sqrt{\frac{n^2}{n^2}-\frac{n}{n^2}} = \sqrt{1-\frac{1}{n}} \xrightarrow{n\to\infty} 1$$

$$\lim_{n\to\infty} \frac{1}{1+\frac{\sqrt{n^2-n}}{n}} = \frac{1}{1+\lim\left(\frac{\sqrt{n^2-n}}{n}\right)} = \frac{1}{1+1} = \frac{1}{2}$$

(ii) $\left[\frac{\sqrt{n^5+3}}{n^2-2n+6} = \frac{\sqrt{n^5+3}\ \sqrt{n^5+3}}{((n-1)^2+5)\cdot\sqrt{n^5+3}} = \frac{(\sqrt{n^5+3})^2}{((n-1)^2+5)\sqrt{n^5+3}}\right.$

$$\left.= \frac{n^5+3}{(n-1)^2\cdot\sqrt{n^5+3}+5\sqrt{n^5+3}}\right]$$

$$(a_n)^2 = \left(\frac{\sqrt{n^5+3}}{n^2-2n+6}\right)^2 = \frac{n^5+3}{(n^2-2n+6)(n^2-2n+6)} = \frac{n^5+3}{n^4-2n^3+6n^2-2n^3+4n^2-12n\ldots}$$

$$\ldots+6n^2-12n+36$$

$$= \frac{n^5+3}{n^4-4n^3+16n^2-24n+36} \xrightarrow{n\to\infty} +\infty$$

$$\Rightarrow a_n \longrightarrow +\infty$$

(iii) $\left(1+\frac{1}{n}\right)^{(n^2)} = \left(1+\frac{1}{n}\right)^{(n\cdot n)}$

$$\boxed{\sqrt[a\cdot b]{x} = x^{\frac{1}{a\cdot b}} = x^{\frac{1}{a}\cdot\frac{1}{b}} = \left(x^{\frac{1}{a}}\right)^{\frac{1}{b}} = \left(\sqrt[a]{x}\right)^{\frac{1}{b}} = \sqrt[b]{\sqrt[a]{x}}}$$

$$\sqrt[n\cdot n]{(a_n)} = \sqrt[n\cdot n]{\left(1+\frac{1}{n}\right)^{n\cdot n}} = 1+\frac{1}{n}$$

$$\lim_{n\to\infty} \sqrt[n^2]{(a_n)} = \lim_{n\to\infty} 1+\frac{1}{n} = 1 \quad \Rightarrow \lim_{n\to\infty}(a_n) = \left(\lim_{n\to\infty} \sqrt[n^2]{(a_n)}\right)^{(n^2)} = 1^{(n^2)} = 1$$

ANA Ü7

2.) (i) $a_n = (-1)^n (n - \sqrt{n^2+1})$

Die Folge $(-1)^n$ ist beschränkt (untere Schranke -1, obere Schranke 1).

$$n - \sqrt{n^2+1} = \frac{(n-\sqrt{n^2+1})(n+\sqrt{n^2+1})}{n+\sqrt{n^2+1}} = \frac{n^2 - n^2 + 1}{n+\sqrt{n^2+1}}$$

$$= \frac{\frac{1}{n}}{\frac{n}{n} + \sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}}}$$

$$\lim \frac{\frac{1}{n}}{1+\sqrt{1+\frac{1}{n^2}}} = \frac{0}{1+\sqrt{1+0}} = 0$$

$\Rightarrow a_n$ ist gegen 0 konvergent

(ii) $\frac{\sqrt[3]{2n^7+7n^3+1}}{3n^2-1}$ Behauptung: divergent

Beweis durch Widerspruch: Angenommen $\lim\limits_{n\to\infty}(a_n) = a \in \mathbb{R}$, dann muss gelten $\lim\limits_{n\to\infty}(a_n)^3 \to a^3$.

$$\left(\frac{\sqrt[3]{2n^7+7n^3+1}}{3n^2-1}\right)^3 = \frac{2n^7+7n^3+1}{(3n^2-1)^3} = \frac{2n^7+7n^3+1}{27n^6-27n^4+9n^2-1}$$

$$\lim_{n\to\infty} \frac{2n^7+7n^3+1}{27n^6-27n^4+9n^2-1} = +\infty$$ ↯ $\Rightarrow (a_n)$ divergiert

(iii) $\sqrt{n+\sqrt{n}} - \sqrt{n-\sqrt{n}}$ Behauptung: konvergent gegen 1

$$\sqrt{n+\sqrt{n}} - \sqrt{n-\sqrt{n}} = \frac{(\sqrt{n+\sqrt{n}} - \sqrt{n-\sqrt{n}})(\sqrt{n+\sqrt{n}} + \sqrt{n-\sqrt{n}})}{(\sqrt{n+\sqrt{n}} + \sqrt{n-\sqrt{n}})}$$

$$= \frac{n+\sqrt{n} - (n-\sqrt{n})}{\sqrt{n+\sqrt{n}} + \sqrt{n-\sqrt{n}}} = \frac{2\sqrt{n}}{\sqrt{n+\sqrt{n}} + \sqrt{n-\sqrt{n}}} = \frac{2\frac{\sqrt{n}}{\sqrt{n}}}{\sqrt{1+\frac{\sqrt{n}}{n}} + \sqrt{1-\frac{\sqrt{n}}{n}}}$$

$$= \frac{2}{\sqrt{1+\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\cdot\sqrt{n}}} + \sqrt{1-\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\cdot\sqrt{n}}}} = \frac{2}{\sqrt{1+\frac{1}{\sqrt{n}}} + \sqrt{1-\frac{1}{\sqrt{n}}}}$$

$$\lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{\sqrt{n}}} + \sqrt{1-\frac{1}{\sqrt{n}}}} = \frac{2}{\sqrt{1+0}+\sqrt{1+0}} = \frac{2}{1+1} = 1$$

ANA Ü7

3. (i) $\left(3^{-n}((-1)^n \cdot n),\ 3^{-n} \cdot (2^n+3),\ 3^{-n}\left(\frac{2}{n^2+1}\right)\right)_{n \in \mathbb{N}}$

$$\frac{1}{3^n} \cdot \frac{(-1)^n \cdot n}{1},\ \underbrace{\frac{1}{3^n} \cdot \frac{2^n+3}{1}}_{\text{konvergiert}},\ \underbrace{\frac{1}{3^n} \cdot \frac{2}{n^2+1}}_{\xrightarrow{n\to\infty} 0}$$

$$\frac{2^n+3}{3^n} > 0$$

$$\frac{2^n+3}{3^n} - \frac{2^{n+1}+3}{3^{n+1}} = \frac{(2^n+3)\cdot 3}{3\cdot 3^n} - \frac{2\cdot 2^n+3}{3\cdot 3^n} = \frac{3\cdot 2^n+9-2\cdot 2^n-3}{3\cdot 3^n}$$

$$= \frac{2^n+6}{3\cdot 3^n} > 0 \quad \Rightarrow \frac{2^n+3}{3^n} \text{ konvergiert}$$

Da $\sqrt[n]{\frac{n}{3^n}} = \frac{\sqrt[n]{n}}{\sqrt[n]{3^n}} = \frac{\sqrt[n]{n}}{3}$ gegen $\frac{1}{3}$ konvergiert, muss auch $\left(\sqrt[n]{\frac{n}{3^n}}\right)^n$ gegen $\left(\frac{1}{3}\right)^n = \frac{1}{3^n} \to 0$ konvergieren. (Satz 3.3.5 v)

$\Rightarrow$ konvergiert

(ii) $\left(\frac{-1}{3n+1},\ \frac{1}{3^{n+1}},\ (-1)^n \frac{1}{n}\right)$

$\left(\frac{-1}{3n+1}\right) = \left(-1 \cdot \frac{1}{3n+1}\right)$, da (-1) beschränkt und $\frac{1}{3n+1}$ Teilfolge von $\frac{1}{n} \xrightarrow{n\to\infty} 0 \Rightarrow \frac{-1}{3n+1} \xrightarrow{n\to\infty} 0$

$\frac{1}{3^{n+1}}$ ist auch Teilfolge von $\frac{1}{n} \xrightarrow{n\to\infty} 0 \Rightarrow \frac{1}{3^{n+1}} \xrightarrow{n\to\infty} 0$

$(-1)^n$ ist beschränkt; $\frac{1}{n} \xrightarrow{n\to\infty} 0 \Rightarrow (-1)^n \cdot \frac{1}{n} \xrightarrow{n\to\infty} 0$

$\Rightarrow$ konvergiert

ANA Ü7

4 iii) $a_n = \sqrt[n]{a^n + b^n}$, wobei $a, b \in \mathbb{R}^+$

1. Fall $a = b$: $a_n = \sqrt[n]{2a^n} = \sqrt[n]{2} \cdot a$

$$\lim \sqrt[n]{2} \cdot a = 1 \cdot a = a$$

2. Fall $a \neq b$: o.B.d.A $a < b$

$$b = \sqrt[n]{b^n} < \sqrt[n]{a^n + b^n} < \sqrt[n]{b^n + b^n} = \sqrt[n]{2b^n} = \sqrt[n]{2} \cdot b$$

$$\Rightarrow \underbrace{\lim_{n\to\infty} b}_{\to b} < \lim_{n\to\infty} \sqrt[n]{a^n + b^n} < \lim_{n\to\infty} \underbrace{\sqrt[n]{2}}_{\to 1} \cdot \underbrace{b}_{\to b}$$

$$\Rightarrow \lim \sqrt[n]{a^n + b^n} = b$$

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4 ii) $a_n = \left(1 + \frac{1}{n^2}\right)^{n^3} \geq 1 + \frac{1}{n^2} \cdot n^3 = 1 + n \xrightarrow{n\to\infty} +\infty$

$$\Rightarrow \left(1 + \frac{1}{n^2}\right)^{n^3} \xrightarrow{n\to\infty} +\infty$$

---

4 i) $a_n = \frac{1}{\sqrt[n]{n!}}$

$$(n!)^2 = n! \cdot n! = n \cdot n \cdot (n-1) \cdot (n-1) \cdot \ldots \cdot 2 \cdot 2 \cdot 1 \cdot 1$$

$$= (n \cdot 1) \cdot (n \cdot 1) \cdot ((n-1) \cdot 2) \cdot ((n-1) \cdot 2) \cdot ((n-2) \cdot 3)((n-2) \cdot 3) \cdot \ldots$$

$$= n^2 \cdot ((n-1) \cdot 2)^2 \cdot ((n-2) \cdot 3)^2 \cdot \ldots \cdot \left(\frac{n+1}{2}\right)^2$$

$$> \left(\frac{n}{2}\right)^2 \cdot \left(\frac{n}{2}\right)^2 \cdot \left(\frac{n}{2}\right)^2 \cdot \ldots \left(\frac{n}{2}\right)^2$$

$$= \left(\left(\frac{n}{2}\right)^2\right)^n$$

$$\Rightarrow n! > \left(\frac{n}{2}\right)^n$$

$$\Rightarrow \sqrt[n]{n!} > \frac{n}{2}$$

$$\Rightarrow \frac{1}{\sqrt[n]{n!}} < \frac{2}{n}$$

nach dem Sandwichsatz gilt:

$$\underbrace{0}_{\xrightarrow{n\to\infty} 0} < \frac{1}{\sqrt[n]{n!}} < \underbrace{\frac{2}{n}}_{\xrightarrow{n\to\infty} 0} \Rightarrow \frac{1}{\sqrt[n]{n!}} \xrightarrow{n\to\infty} 0$$

ANA 07

7.) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ Behauptung: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$

$$\frac{1}{n(n+1)(n+2)} = \frac{n+2-(n+1)}{n(n+1)(n+2)} = \frac{\cancel{n+2}}{n(n+1)\cancel{(n+2)}} - \frac{\cancel{n+1}}{n\cancel{(n+1)}(n+2)} = \frac{1}{n(n+1)} - \frac{1}{n(n+2)}$$

$$\frac{1}{n(n+2)} = \frac{1-n+n+2-2}{n(n+2)} = \frac{\cancel{n+2}}{n\cancel{(n+2)}} - \frac{\cancel{n}}{\cancel{n}(n+2)} - \frac{2-1}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2} - \frac{1}{n(n+2)}$$

$$\frac{1}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2} - \frac{1}{n(n+2)} \iff \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2} \iff \frac{1}{n(n+2)} = \frac{1}{2}\cdot\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} - \frac{1}{n(n+2)} = \sum_{n=1}^{\infty}\left(\left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{2}\cdot\left(\frac{1}{n} - \frac{1}{n+2}\right)\right)\right)$$

$$= \left(\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right) - \frac{1}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 - \frac{1}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\sum_{n=1}^{k}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \ldots + \cancel{\frac{1}{n-1}} - \frac{1}{n+1} + \cancel{\frac{1}{n}} - \frac{1}{n+2}$$

$$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \xrightarrow{n\to\infty} 1 + \frac{1}{2}$$

$$1 - \frac{1}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 - \frac{1}{2}\cdot\left(1 + \frac{1}{2}\right) = 1 - \frac{1}{2}\cdot\left(\frac{3}{2}\right) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n} + \frac{(-1)^n}{2^n}\right) = \sum_{k=0}^{\infty} \frac{2}{2^{2k}} = 2\cdot\sum_{k=0}^{\infty} \frac{1}{2^{2k}} = 2\cdot\sum_{k=0}^{\infty} \frac{1}{2^{2^k}} = 2\cdot\sum_{k=0}^{\infty} \frac{1}{4^k}$$

$$(b^n - a^n) = (b-a)(b^{n-1} + b^{n-2}a + \ldots + ba^{n-2} + a^{n-1}) \quad \text{(laut 2.9 Buch)}$$

bei $b = 1$ und $a = \frac{1}{4}$ $\quad \left(1^n - \left(\frac{1}{4}\right)^n\right) = \left(1 - \frac{1}{4}\right)\left(1^{n-1} + 1^{n-2}\cdot\frac{1}{4} + \ldots + 1\cdot\left(\frac{1}{4}\right)^{n-2} + \left(\frac{1}{4}\right)^{n-1}\right)$

$$1 - \frac{1}{4^n} = \left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \ldots + \left(\frac{1}{4}\right)^{n-1}\right) = \left(1 - \frac{1}{4}\right)\cdot\sum_{k=0}^{n-1}\left(\frac{1}{4}\right)^k$$

$$\Rightarrow \frac{1 - \frac{1}{4^n}}{1 - \frac{1}{4}} = \sum_{k=0}^{n-1}\left(\frac{1}{4}\right)^k \qquad \lim_{n\to\infty} \frac{1 - \frac{1}{4^n}}{\frac{3}{4}} = \frac{4}{3}\cdot\lim_{n\to\infty} 1 - \frac{1}{4^n} = \frac{4}{3}\cdot 1$$

$$\Rightarrow 2\cdot\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = 2\cdot\frac{4}{3} = \frac{8}{3}$$

ANA Ü7

8.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n^3+n}}$

$$\frac{1}{\sqrt[4]{n^3+n}} = \frac{\frac{1}{\sqrt[4]{n^3}}}{\sqrt[4]{\frac{n^3}{n^3}+\frac{n}{n^3}}} = \frac{\frac{1}{\sqrt[4]{n^3}}}{\sqrt[4]{1+\frac{1}{n^2}}} = \frac{\sqrt[4]{\left(\frac{1}{n}\right)^3}}{\sqrt[4]{1+\frac{1}{n^2}}}$$

$$\lim_{n\to\infty} \frac{\sqrt[4]{\left(\frac{1}{n}\right)^3}}{\sqrt[4]{1+\frac{1}{n^2}}} = \frac{\sqrt[4]{0^3}}{\sqrt[4]{1+0}} = \frac{0}{1} = 0$$

$$\frac{1}{\sqrt[4]{n^3+n}} - \frac{1}{\sqrt[4]{(n+1)^3+n+1}} = \frac{1}{\sqrt[4]{n^3+n}} - \frac{1}{\sqrt[4]{n^3+3n^2+3n+1+n+1}}$$

$$= \frac{1}{\sqrt[4]{n^3+n}} - \frac{1}{\sqrt[4]{n^3+3n^2+4n+2}} \qquad \boxed{\sqrt[4]{n^3+n} < \sqrt[4]{n^3+3n^2+4n+2}}$$

$$\Rightarrow \frac{1}{\sqrt[4]{n^3+n}} > \frac{1}{\sqrt[4]{n^3+3n^2+4n+2}} \Rightarrow \text{monoton fallend}$$

$\Rightarrow$ laut Leibnitz Kriterium konvergent

$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)n^{1+(-1)^n}}$ bei n gerade: $\frac{1}{(n+1)n^2}$ bei n ungerade: $\frac{-1}{n+1}$

$$\frac{\left|\frac{-1}{n+1}\right|}{\left|\frac{1}{(n+1)\cdot n^2}\right|} = \frac{(n+1)\cdot n^2}{n+1} = n^2$$

, d.h. für alle geraden n ist $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$

$\Rightarrow$ divergiert, laut Quotientenkriterium

ANA Ü7

9.) $\sum_{n=1}^{\infty} \frac{n!\,2^n}{(2n)!}$ Quotientenkriterium mit $q=\frac{1}{2}$

$$\frac{\left|\frac{(n+1)!\,2^{n+1}}{(2(n+1))!}\right|}{\left|\frac{n!\,2^n}{(2n)!}\right|} = \frac{\frac{n!\cdot(n+1)\cdot 2^n\cdot 2}{(2n+2)!}}{\frac{n!\cdot 2^n}{(2n)!}} = \frac{\cancel{n!}\cdot(n+1)\cdot \cancel{2^n}\cdot 2\cdot \cancel{(2n)!}}{\cancel{(2n)!}(2n+1)(2n+2)\cdot \cancel{n!}\cdot \cancel{2^n}}$$

$$= \frac{2n+2}{(2n+1)\cancel{(2n+2)}} = \frac{1}{2n+1} \leq \frac{1}{2} = q \qquad \Rightarrow \sum_{n=1}^{\infty} \frac{n!\,2^n}{(2n)!} \text{ konvergiert}$$

$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[4]{n^3}}$ Leibnitzkriterium

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[4]{n^3}} = \frac{\frac{1}{\sqrt[4]{n^3}}\cdot\sqrt{n+1} - \frac{1}{\sqrt[4]{n^3}}\cdot\sqrt{n}}{\frac{\sqrt[4]{n^3}}{\sqrt[4]{n^3}}} = \sqrt{\frac{n+1}{\sqrt{n^3}}} - \sqrt{\frac{n}{\sqrt{n^3}}}$$

$$= \sqrt{\frac{n}{\sqrt{n^3}} + \frac{1}{\sqrt{n^3}}} - \sqrt{\frac{n}{\sqrt{n^3}}} = \sqrt{\sqrt{\frac{n^2}{n^3}} + \sqrt{\frac{1}{n^3}}} - \sqrt{\sqrt{\frac{n^2}{n^3}}}$$

$$= \sqrt{\sqrt{\frac{1}{n}} + \sqrt{\frac{1}{n^3}}} - \sqrt{\sqrt{\frac{1}{n}}} \longrightarrow 0$$

$$\frac{1}{n^3} > \frac{1}{n^3+3n^2+3n+1} = \frac{1}{(n+1)^3}$$

$$\sqrt{\frac{1}{n^3}} > \sqrt{\frac{1}{(n+1)^3}}$$

$$\sqrt{\frac{1}{n}} + \sqrt{\frac{1}{n^3}} > \sqrt{\frac{1}{n+1}} + \sqrt{\frac{1}{(n+1)^3}}$$

$$\sqrt{\sqrt{\frac{1}{n}} + \sqrt{\frac{1}{n^3}}} > \sqrt{\sqrt{\frac{1}{n+1}} + \sqrt{\frac{1}{(n+1)^3}}}$$

$$\sqrt{\sqrt{\frac{1}{n}} + \sqrt{\frac{1}{n^3}}} - \sqrt{\sqrt{\frac{1}{n}}} > \sqrt{\sqrt{\frac{1}{n+1}} + \sqrt{\frac{1}{(n+1)^3}}} - \sqrt{\sqrt{\frac{1}{n+1}}}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[4]{n^3}} \text{ konvergiert}$$