Computer Aided Geometric Design Compendium WS2023

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Organization

Lecture each Thursday 12:00 to 14:00 (full 2 hours).
Oral exam. Write email to fix date and time.
Problem session each Thursday 14:00 to 16:00. Mandatory attendance!
Kreuzerlübung.

1 Bezier curves

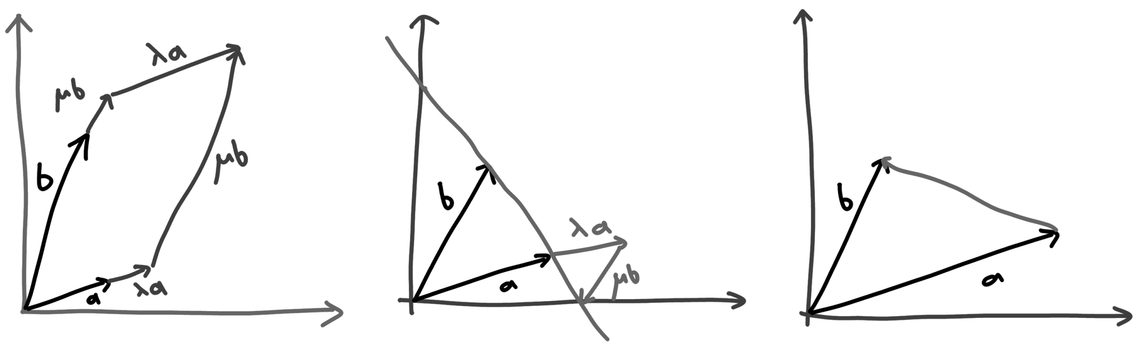

Figure 1: Linear combination, affine combination and convex combination

Example 1. ***Linear combination*** $\lambda a + \mu b$
Affine combination $\lambda a + \mu b$ *and* $\lambda + \mu = 1$
What is μ *so that* $\lambda a + \mu B$ *is on the line?*

$$\lambda a + \mu b = a + t(b - a) \implies a(\underbrace{\lambda - 1 + t}_{=0}) + b(\underbrace{\mu - t}_{=0}) = 0$$

If a, b *are linearly independent* $\implies \mu = t \wedge \lambda + \mu = 1$
Convex combination $\lambda a + \mu b$ *and* $\lambda + \mu = 1$ *and* $\lambda, \mu \geq 0$
Line is $a + t(b - a)$ *with* $t \in [0, 1] \implies \mu, \lambda \in [0, 1]$

Definition 1 (combinations). *linear combination* $\sum_{i=1}^{n} \lambda_i v_i$ *with* $v_1, ..., v_n \in \mathbb{R}^d, \lambda_1, ..., \lambda_n \in \mathbb{R}$
affine combination $\sum_{i=1}^{n} \lambda_i v_i$ *with* $\sum_{i=1}^{n} \lambda_i = 1$
convex combination $\sum_{i=1}^{n} \lambda_i v_i$ *with* $\sum_{i=1}^{n} \lambda_i = 1$ *and* $\forall i : \lambda_i \geq 0$

Algorithm 1 (of de Casteljou, Bezier curve). *Given:* $b_0, ..., b_n \in \mathbb{R}^d$ *(called control points / Kontrollpunkte),* $t \in \mathbb{R}$
Recursion: $b_i^0(t) := b_i$
$b_i^j(t) := (1 - t)b_i^{j-1}(t) + tb_{i+1}^{j-1}(t)$ *for* $j = 1, ..., n$ *and* $i = 0, ..., n - j$
Result: $b(t) := b_0^n(t)$ *(called Bezier curve)*

Remark 1. *In the algorithm above often we choose* $t \in [0, 1]$.

Example 2.

Remark 2. *In this course* $\mathbb{N} = \{1, 2, 3, ...\}$ *and* $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$

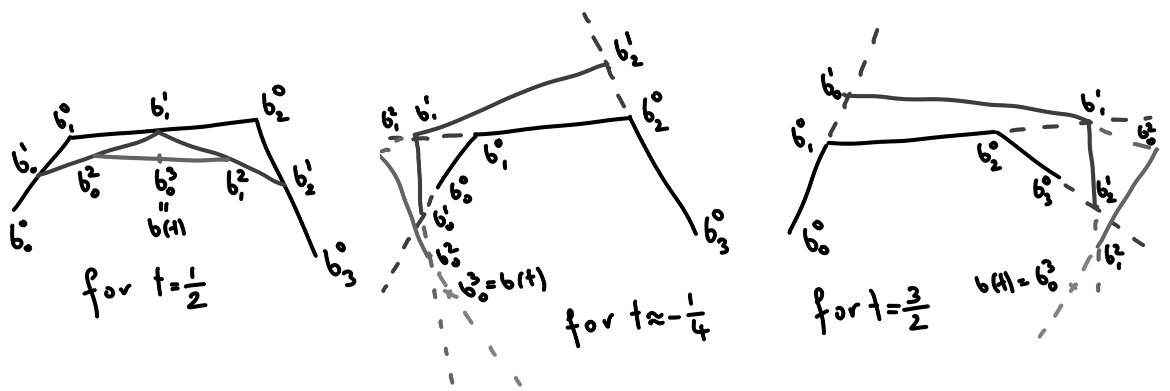

Figure 2: Examples of the de casteljou algorithm

Recap 1. $0! := 1, n! := n(n-1)(n-2)\cdots 1$ *for* $n \geq 1$.

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!} & , n \geq k \geq 0 \\ 0 & , k > n \end{cases} \quad \textit{for } n, k \in \mathbb{N}_0$$

Definition 2 (Bernstein polynomials)**.** *For* $n, i \in \mathbb{N}_0$ *we define* $B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i} \in \mathbb{R}[t]$

Remark 3. *Special cases of Bernstein polynomials*

$$i > n \implies B_i^n(t) = 0 \qquad B_i^n(0) = \begin{cases} 0, i \neq 0 \\ 1, i = 0 \end{cases}$$

$$B_i^n(1) = \begin{cases} 0, i \neq n \\ 1, i = n \end{cases} \qquad B_0^0(t) = 1$$

Theorem 1. $b_i^j(t) = \sum_{l=0}^{j} B_l^j(t) b_{i+l}$

Proof. Induction over j: $j = 0$:

$$j = 0: \qquad b_i^0(t) := b_i = 1 \cdot b_i = B_0^0(t) \cdot b_i \qquad \checkmark$$

$$j-1 \to j: \qquad b_i^j(t) := (1-t) b_i^{j-1}(t) + t b_{i+1}^{j-1}(t) \overset{\text{IA}}{=} (1-t) \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+l} + t \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+1+l} =$$

$$(1-t) \sum_{l=0}^{\overset{j}{\cancel{j-1}}} B_l^{j-1}(t) b_{i+l} + t \sum_{l=\cancel{1}\,0}^{j} B_{l-1}^{j-1}(t) b_{i+l} = \sum_{l=0}^{j} (\underbrace{(1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t)}_{= B_l^j(t) \text{ using the following lemma}}) b_{i+l} =$$

$$\sum_{l=0}^{j} B_l^j(t) b_{i+l} \qquad \checkmark$$

□

Corollary 1. *The Bezier curve equals* $b(t) = b_0^n(t) = \sum_{l=0}^{n} B_l^j(t) b_{i+l}$*, which is called the Bernstein representation of the Bezier curve.*

Remark 4. *As* $b(t) = \sum_{l=0}^{n} B_l^n(t) b_l \in C^\infty$ *it is a polynomial curve of degree* n*, which is in* C^∞ *and therefore "very smooth".*

Lemma 1. $B_l^j(t) = (1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t)$

Proof.

$$(1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t) = (1-t) \binom{j-1}{l} t^l (1-t)^{j-1-l} + t \binom{j-1}{l-1} t^{l-1} (1-t)^{j-1-l+1} =$$

$$\binom{j-1}{l} t^l (1-t)^{j-l} + \binom{j-1}{l-1} t^l (1-t)^{j-l} = \left(\binom{j-1}{l} + \binom{j-1}{l-1} \right) t^l (1-t)^{j-l} = \binom{j}{l} t^l (1-t)^{j-l} = B_l^j(t)$$

□

Remark 5. *What is $b(0)$?* $\quad b(0) = \sum_{i=0}^{n} B_i^n(0)b_i = b_0 + 0 + 0 + \cdots + 0 = b_0$
What is $b(1)$? $\quad b(1) = \sum_{i=0}^{n} B_i^n(1)b_i = 0 + \cdots + 0 + b_n = b_n$

Definition 3 (end-point-interpolating)**.** *Curves which pass through the first and last point are called end-point-interpolating (Endpunktinterpolierend).*

Remark 6. *Bezier curves are end-point-interpolating.*

Remark 7. *How many intersection points are there between a planar (i.e. in $\mathbb{R}^2$) Bezier curve and a straight line?*

Straight line: $p + t(q-p)$ $\qquad$ *Bezier curve:* $b(t) = \sum_{i=0}^{n} B_i^n(t) \underbrace{b_i}_{\in\mathbb{R}^2}$

Solving $p + t(q-p) = \sum_{i=0}^{n} B_i^n(t)b_i$ results in at most n solutions.

Lemma 2. $\frac{d}{dt}B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$

Proof.

$$\begin{aligned}
\frac{d}{dt}B_i^n(t) = \frac{d}{dt}\binom{n}{i}t^i(1-t)^{n-i} = \binom{n}{i}it^{i-1}(1-t)^{n-i} - \binom{n}{i}t^i(n-i)(1-t)^{n-i-1} = \\
\frac{n!}{\underbrace{\cancel{i!}}_{(i-1)!}(n-i)!}\cancel{i}t^{i-1}(1-t)^{n-i} - \frac{n!}{i!\underbrace{\cancel{(n-i)!}}_{(n-i-1)!}}\cancel{(n-i)}t^i(1-t)^{n-i-1} = \\
n\left(\frac{(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{(n-1)!}{i!(n-i-1)!}t^i(1-t)^{n-i-1}\right) = \\
n\left(\binom{n-1}{i-1}t^{i-1}(1-t)^{n-i} - \binom{n-1}{i}t^i(1-t)^{n-i-1}\right) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))
\end{aligned}$$

□

Theorem 2. $\dot{b}(t) := \frac{d}{dt}b(t) = n\sum_{i=0}^{n-1} B_i^{n-1}(t)(b_{i+1} - b_i) = n(b_1^{n-1}(t) - b_0^{n-1}(t))$

Proof.

$$\dot{b}(t) = \frac{d}{dt}\left(\sum_{i=0}^{n} B_i^n(t)b_i\right) = \sum_{i=0}^{n}\frac{d}{dt}B_i^n(t)b_i = \sum_{i=0}^{n} n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))b_i = n\left(\sum_{i=0}^{n} B_{i-1}^{n-1}(t)b_i - \sum_{i=0}^{n} B_i^{n-1}(t)b_i\right) =$$

$$n\left(\sum_{i=1}^{n} B_{i-1}^{n-1}(t)b_i - \sum_{i=0}^{n} B_i^{n-1}(t)b_i\right) = n\left(\underbrace{\sum_{i=0}^{n-1} B_i^{n-1}(t)b_{i+1}}_{=b_1^{n-1}(t)} - \underbrace{\sum_{i=0}^{n-1} B_i^{n-1}(t)b_i}_{=b_0^{n-1}(t)}\right) = n\left(\sum_{i=0}^{n-1} B_i^{n-1}(t)(b_{i+1} - b_i)\right)$$

□

Corollary 2.

- $\dot{b}(0) = n(b_1 - b_0)$
- $\dot{b}(1) = n(b_n - b_{n-1})$
- *The last segment in the algorithm of de Casteljou is the tangent of the Bezier curve in $b(t)$.*
- *The derivative of a bezier curve of degree n is a bezier curve of degree $n-1$ with control points $(b_1, b_0), (b_2 - b_1), \cdots, (b_n - b_{n-1})$.*

Corollary 3. $\ddot{b}(t) = n(n-1)\sum_{i=0}^{n-2} B_i^{n-2}(t)(b_{i+2} - 2b_{i+1} + b_i)$
$\ddot{b}(0) = n(n-1)(b_2 - 2b_1 + b_0)$, $\ddot{b}(1) = n(n-1)(b_n - 2b_{n-1} + b_{n-2})$

Corollary 4. *The curvature of a bezier curve in the point $b(0)$ depends only on b_0, b_1, b_2.*
The curvature of a bezier curve in the point $b(1)$ depends only on b_{n-2}, b_{n-1}, b_n.

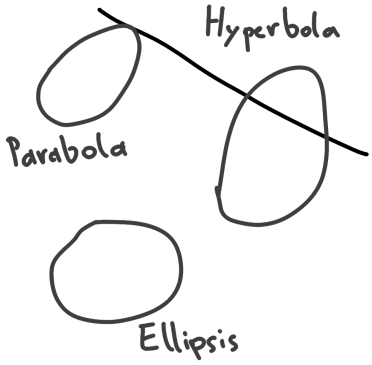

Figure 3: Categorization of Parabolas, Hyperbolas and Ellipsis as intersection points with the line at infinity.

Example 3. *Quadratic Bezier curve*

$$b(t) = \sum_{i=0}^{2} B_i^2(t) b_i = \binom{2}{0} t^0 (1-t)^2 b_0 + \binom{2}{1} t^1 (1-t)^1 b_1 + \binom{2}{2} t^2 (1-t)^0 b_2 = t^2(b_2 - 2b_1 + b_0) + t(2b_1 - 2b_0) + b_0$$

which is an affine transformation of a parabola and therefore a parabola.
Quadratic bezier curves are parabolas.

Remark 8. *Line at infinity (Ferngerade) is the collection of points where parallel lines intersect.*

Remark 9. *Different applications using these curves are Rhino, OpenSCAD, Autocad, Geogebra, ...*

2 Parameterized curves

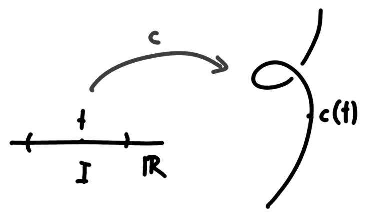

Figure 4: parameterized curve $c(t)$

Definition 4. *$c : I \subseteq \mathbb{R} \to \mathbb{R}^3$ is called a parameterized curve.*
$\dot{c}(t) := \frac{d}{dt} c(t)$ is called the tangential vector. For $\mathbb{R}^3$ we have $\dot{c}(t) = (\dot{c_1}(t), \dot{c_2}(t), \dot{c_3}(t))$.
The velocity is defined as $||\dot{c}(t)||$.
A point $c(t)$ is called regular, if $\dot{c}(t) \neq 0$ and is called singular, if $\dot{c}(t) = 0$.

Example 4. *A helix (Schraublinie) is defined by $c(t) = (\cos(t), \sin(t), t)^T$.*

$$\dot{c}(t) = (-\sin(t), \cos(t), 1)^T \qquad ||\dot{c}(t)|| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

We see that the helix is passed through with constant velocity. Furthermore all points are regular.

Example 5. *$c : \mathbb{R} \to \mathbb{R}^3, t \mapsto (t^2, t^3, t^4)$, $\dot{c}(t) = (2t, 3t^2, 4t^3)$. We see that 0 is singular as $\dot{c}(0) = (0, 0, 0)$. Everywhere else the curve is regular.*

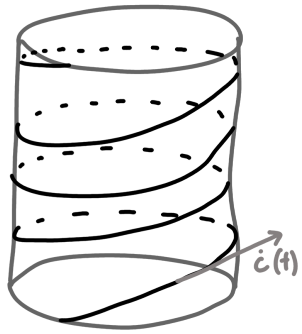

Figure 5: Helix with tangential vector

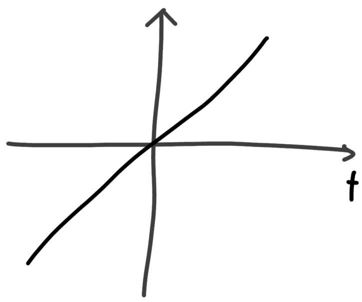

Figure 6: identity line can be parameterized such that $(0, 0)$ is singular.

Remark 10. *A point being regular or singular depends on the parameterisation of the curve.*
For example $c(t) = (t, t)$ produces a regular curve, while $c(t) = (t^3, t^3)$ produces a curve where 0 is singular.
There are curves and points where no parameterisation exists such that the point is regular.

Definition 5. $c : I \to \mathbb{R}^2 \in C^2(I, \mathbb{R}^2)$
The curvature of the curve in the point $c(t)$ is defined as $\kappa(t) = \frac{\det(\dot{c}(t), \ddot{c}(t))}{||\dot{c}(t)||^3}$

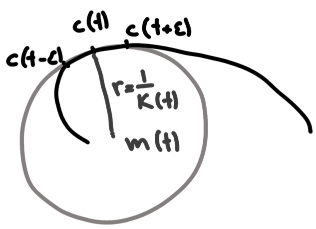

Figure 7: Circle of curvature

Example 6. *The circle of curvature has a radius of $\frac{1}{\kappa(t)}$. $m(t)$ is called the center of curvature.*
$m(t) = c(t) + \frac{1}{\kappa(t)} n(t)$ *where* $n(t) = \frac{(-\dot{c}_2(t), \dot{c}_1(t))}{||\dot{c}(t)||}$.

Remark 11. *Exercise: compare this definition of curvature with the school version concerning graphs.*

Example 7. *For a circle we have* $c(t) = (r\cos(t), r\sin(t))^T$, $\dot{c}(t) = (-r\sin(t), r\cos(t))^T$, $\ddot{c}(t) = (-r\cos(t), -r\sin(t))^T$

$$\kappa(t) = \frac{\det\begin{pmatrix} -r\sin(t) & -r\cos(t) \\ r\cos(t) & -r\sin(t)\end{pmatrix}}{r^3} = \frac{r^2\sin^2(t) + r^2\cos^2(t)}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$
$$n(t) = \frac{(-r\cos(t), -r\sin(t))}{r} = (-\cos(t), -\sin(t))$$

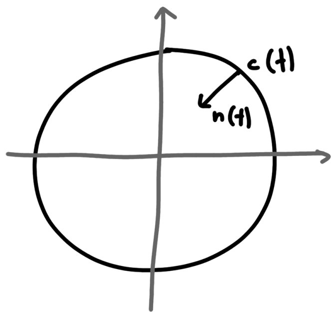

Figure 8: Circle with normal vector

Definition 6. *A point $c(t)$ with $\dot\kappa(t) = 0$ is called a vertex.*

Example 8. *An ellipse has four vertices.*

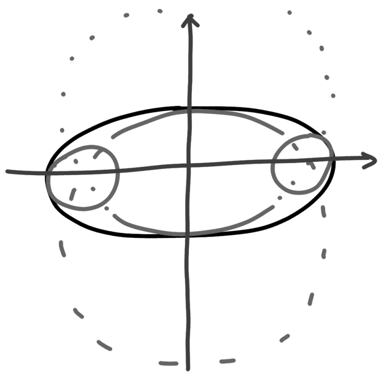

Figure 9: Ellipse and the four vertices.

$(t, \exp t)$ *has no vertex.*
Klothoids are curves with $\kappa(t) = t$. They are used in road construction and have no vertex.

Definition 7. $c : I \to \mathbb{R}^3$
$\kappa(t) = \frac{||\dot c(t)\times \ddot c(t)||}{||\dot c(t)||^3}$ *is called the curvature of a space curve.*
$\tau(t) = \frac{\det(\dot c(t), \ddot c(t), \dddot c(t))}{||\dot c(t)\times\ddot c(t)||^2}$ *is called torsion of a space curve.*

Example 9. *For the helix $t \mapsto (\cos(t), \sin(t), pt)$ the torsion depends on p.*

3 Properties of Bezier curves

Definition 8. $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ *is called* ***affine*** *if $\exists l : \mathbb{R}^n \to \mathbb{R}^m$...linear $\exists v \in \mathbb{R}^m : \alpha(x) = l(x) + v$.*
α is called ***affinity*** *if α is affine and bijective.*

Example 10. *An example of an linear function is shear (Scherung).*

$$l\begin{pmatrix} x \\ y\end{pmatrix} := \begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}$$

Area is preserved.

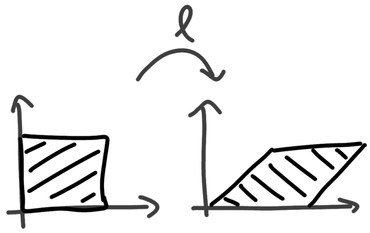

Figure 10: Shear preserves area

Theorem 3. *Bezier curves are invariant under affine transformations.*

Proof. Let $b(t) = \sum_{i=0}^{n} B_i^n(t) b_i$ be a bezier curve and $\alpha(x) = l(x) + v$ where $b_i \in \mathbb{R}^d, l : \mathbb{R}^d \to \mathbb{R}^m, v \in \mathbb{R}^m$

$$\alpha(b(t)) = l(b(t)) + v = l\left(\sum_{i=0}^{n} B_i^n(t) b_i\right) + v = \sum_{i=0}^{n} B_i^n(t) l(b_i) + v =$$

$$\sum_{i=0}^{n} B_i^n(t) l(b_i) + \sum_{i=0}^{n} B_i^n(t) v = \sum_{i=0}^{n} B_i^n(t)(l(b_i) + v) = \sum_{i=0}^{n} B_i^n(t) \alpha(b_i)$$

$\square$

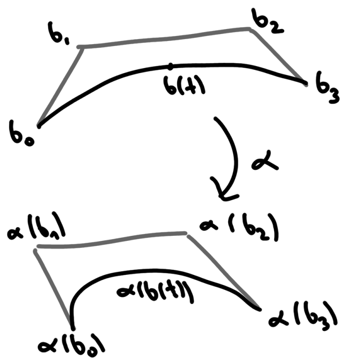

Figure 11: Bezier curve is invariant under affine transformation

Lemma 3. $\sum_{i=0}^{n} B_i^n(t) = 1$

Proof. Binomial theorem: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ for $x = t, y = 1 - t$ we get

$$1 = (t + (1-t))^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} = \sum_{i=0}^{n} B_i^n(t)$$

$\square$

Lemma 4. *For $t \in [0,1]$ it holds that $B_i^n(t) \geq 0$*

Proof.

$$B_i^n(t) = \underbrace{\binom{n}{i}}_{\geq 0} \underbrace{t^i}_{\geq 0} \underbrace{(1-t)^{n-i}}_{\geq 0} \geq 0$$

$\square$

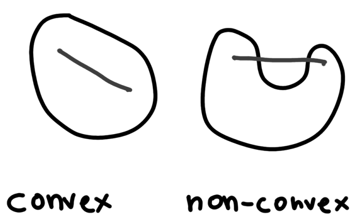

Figure 12: example of convex and non-convex set

Definition 9. *$M \subseteq \mathbb{R}^d$ then $conv(M) := \{\sum_{i=1}^k \lambda_i v_i : v_1, ..., v_k \in M, \lambda_1, ..., \lambda_k \geq 0, \sum_{i=1}^k \lambda_i = 1, k \in \mathbb{N}\}$ is called the convex hull of M.*
M is called convex set, iff $\forall x, y \in M \forall t \in [0,1] : tx + (1-t)y \in M$.

Example 11.

Remark 12 (Pick theorem)**.** *For any polygon formed of points in $\mathbb{Z} \times \mathbb{Z}$ it holds that the area can be calculated by $I + R/2 - 1$ where I is the number of points inside the polygon and R is the number of points on the edges of the polygon.*

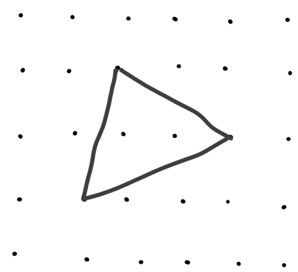

Figure 13: example picks theorem where $I = 2, R = 3$ and therefore $A = \frac{5}{2}$

Theorem 4. $\forall t \in [0,1] : b(t) \in conv(\{b_0, b_1, ..., b_n\})$

Proof. $b(t) = \sum_{i=0}^n \underbrace{B_i^n(t)}_{\geq 0} b_i \forall t \in [0,1]$ and $\sum_{i=0}^n B_i^n(t) = 1 \forall t \in [0,1]$. Therefore $b(t) \in conv(\{b_0, ..., b_n\})$ □

Remark 13 (Montecarlo method for calculating area)**.** *random points, count how many fall within the area from the total amount of points, estimate volume from that.*

Theorem 5. *Bezier curves are symmetric, meaning that if $b(t)$ is a bezier curves of $b_0, ..., b_n$ and $\tilde{b}(t)$ is a bezier curve of $b_n, ..., b_0$ then $b(t) = \tilde{b}(1-t)$*

Proof.

$$\tilde{b}(t) = \sum_{i=0}^n B_i^n(t) b_{n-i} = \sum_{i=0}^n \underbrace{B_i^n(1-t)(1-t)}_{B_i^n(t)} b_{n-i} = \sum_{i=0}^n B_{n-i}^n(1-t) b_i = \sum_{i=0}^n B_i^n(1-t) b_i = b(1-t)$$

□

Lemma 5. $\alpha \in \mathbb{R} \implies B_i^n(\alpha t) = \sum_{j=0}^n B_i^j(\alpha) B_j^n(t)$

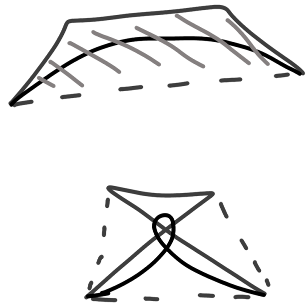

Figure 14: example of a bezier curve and its convex hull

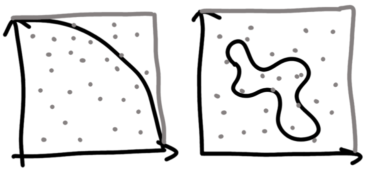

Figure 15: montecarlo method for a quarter circle and some random shape

Proof. can be done by induction, but is tedious ☐

Theorem 6 (Sub-division-property (Unterteilungseigenschaft)). *Let $b(t)$ be a bezier curve to $b_0, ..., b_n$ and $\alpha \in \mathbb{R}$.*

$$\tilde{b}(t) := b(\alpha t)\hat{b}(t) := b((1-\alpha)t + \alpha)$$

Then $\tilde{b}$ is a bezier curve to $b_0^0(\alpha), b_0^1(\alpha), ..., b_0^n(\alpha)$; $\hat{b}$ is a bezier curve to $b_0^n(\alpha), b_1^{n-i}(\alpha), ..., b_n^0(\alpha)$ and "glued together" they result in the original bezier curve.

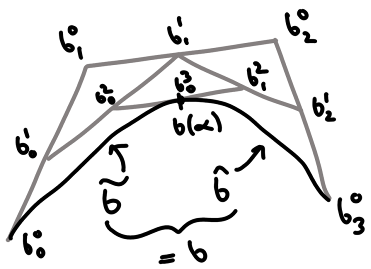

Figure 16: bezier curve b is divided at point $b(\alpha)$ into two bezier curves $\tilde{b}$ and $\hat{b}$

Proof.

$$\tilde{b}(t) = b(\alpha t) = \sum_{i=0}^{n} B_i^n(\alpha t) b_i = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} B_i^j(\alpha) B_j^n(t) \right) b_i = \sum_{j=0}^{n} \sum_{i=0}^{n} B_i^j(\alpha) B_j^n(t) b_i =$$

$$\sum_{j=0}^{n} B_j^n(t) \underbrace{\sum_{i=0}^{\cancel{n}\,j} B_i^j(\alpha) b_i}_{b_0^j(\alpha)} = \sum_{j=0}^{n} B_j^n(t) b_0^j(\alpha)$$

Where we used $b_i^j(t) = \sum_{l=0}^{j} B_l^j(t) b_{i+l}$, which we already showed.

$\hat{b}$ is analogous. □

Definition 10. *The polynomial ring is defined as* $\Pi_n := \{f \in \mathbb{R}[t] : deg(f) \leq n\}$*, which is a vector space.*

Theorem 7. $\{B_0^n(t), ..., B_n^n(t)\}$ *is a basis of* Π_n*.*

Proof. $\forall i : B_i^n(t) \in \Pi_n$ therefore $span(B_0^n(t), ..., B_n^n(t)) \subseteq \Pi_n$.

We know that $dim(\Pi_n) = n + 1$ as $\{1, t, t^2, ..., t^n\}$ is a basis.

Let $k \leq n$.

$$1 = \sum_{i=0}^{n-k} B_i^{n-k}(t) = \sum_{i=k}^{n} B_{i-k}^{n-k}(t) = \sum_{i=k}^{n} \binom{n-k}{i-k} t^{i-k}(1-t)^{n-i}$$

$$\implies \binom{n}{k} t^k = \sum_{i=k}^{n} \binom{n}{k} \binom{n-k}{i-k} t^i (1-t)^{n-i} = \sum_{i=k}^{n} \binom{n}{i} \binom{i}{k} t^i (1-t)^{n-i} = \sum_{i=k}^{n} \binom{i}{k} B_i^n(t)$$

$$\implies t^k \in span(B_0^n(t), ..., B_n^n(t)) \implies \Pi_n \subseteq span(B_0^n(t), ..., B_n^n(t))$$

□

Corollary 5. *Every polynomial curve is a bezier curve.*

Theorem 8 (corner-cutting (Eckenabschneiden))**.** *Define the sequence* P_n *by this (recursive) definition* $P_0 := (b_0^0, b_0^1, ..., b_0^n)$*,* $P_1 := (b_0^0, b_0^1, ..., b_0^n, b_1^{n-1}, b_2^{n-2}, ..., b_n^0)$*, ...,* P_n *consists of the points we get by sub-dividing the bezier curve (with* $t = \frac{1}{2}$*).*

Then this sequence "converges" to the bezier curve

Proof. $d := \max_{0 \leq i \leq n} ||b_i - b_{i+1}$ ... maximum length of edges of control polygon

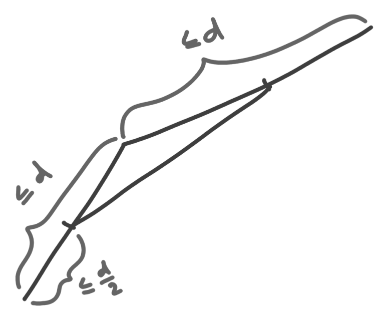

Figure 17: the new edges are at most $\frac{d}{2}$ long

From the image we can deduce that the maximum length of the edges of the new control polygon is at most $\frac{d}{2}$. With induction we get that the control polygon of P_n has the maximum edge length of $\frac{d}{2^n}$.

How far are the points of the control polygon from the bezier curve b? $||P_k(i) - b|| \leq (n-1)\frac{d}{2^k} \to 0$ for $k \to \infty$. □

Remark 14. *Not only does the sequence from the theorem converge pointwise, but it holds that* $P_n \to b$*.*

Remark 15. *Disadvantages of bezier curves are: unintuitive, local changes are not possible as every change in a control point changes the entire curve, points can be far away from the curve.*

4 B-Spline-Curves

Remark 16. *B-Spline curves are compositions of bezier curves.*

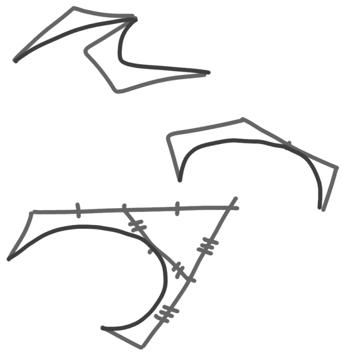

Figure 18: continuous, tangential continuous and curvature continuous composition

Example 12.

Definition 11 (B-Spline-Curves)**.** *$m \in \mathbb{N}$ is the number of control points, $n \in \mathbb{N}$ is the degree of the b spline curve, $T = (t_0, t_1, ..., t_{m+n+1})$ where $t_i \in \mathbb{R}$, $t_i \leq t_{i+1}$ and $t_i < t_{i+n+1}$ is called knot vector (Knotenvektor).*

$$\alpha_i^r(t) := \begin{cases} \frac{t-t_i}{t_{i+r}-t_i} & t_{i+r} - t_i \neq 0 \\ 0 & else \end{cases}$$

$$N_i^0(t) := \begin{cases} 1 & t \in [t_i, t_{i+1}) \\ 0 & else \end{cases} \qquad N_i^r(t) := \alpha_i^r(t) N_i^{r-1}(t) + (1 - \alpha_{i+1}^r(t)) N_{i+1}^{r-1}(t)$$

$N_i^r : \mathbb{R} \to \mathbb{R}$ are called b-spline basis functions (B-Spline-Basisfunktionen) and are piece-wise polynomial curves.

Theorem 9. *Every b-spline curve $s(t)$ can be expressed in the form $s(t) = \sum_{i=0}^m N_i^n(t) c_i$ where $c_0, ..., c_m \in \mathbb{R}^d$ are the control points and n is the degree of the b-spline curve.*

Proof. without proof ☐

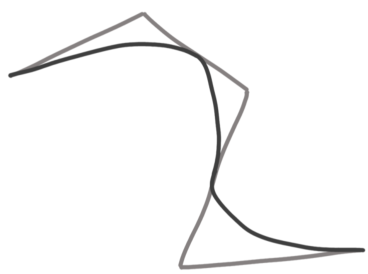

Figure 19: from this set of control points there are different possible quadratic b-spline curves (depending on the chosen knot vector).

Example 13.

Remark 17. *Typically a knot vector of a similar form as $T = (0, 0, 0, 0, 1, 2, 3, 4, 5, 5, 5, 5)$ is chosen. 0 often repeated, then counting up, then as many numbers of the same value as 0 in the start.*

Example 14. *Every line segment in the control polygon is cut into thirds.*

Example 15. *closed b-spline curves*

Remark 18. *B-spline curves of degree n with $n + 1$ control points ($\implies m = n$) are bezier curves.*

Figure 20: cubic b-spline curve

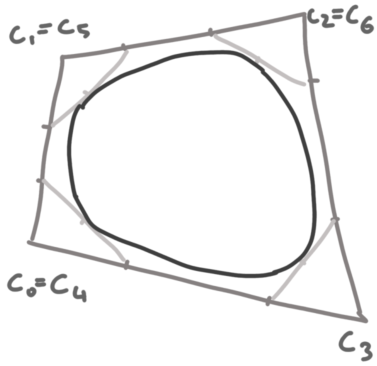

Figure 21: curvature continuous, closed b-spline curve.

5 NURBS-Curve

Remark 19. *Non-Uniform-Rational-B-Spline-Curve*

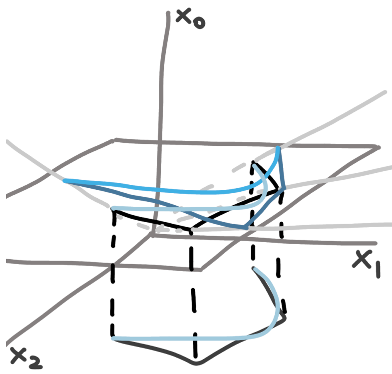

Figure 22: construction of a nurbs curve

Lemma 6 (Construction of a NURBS curve). *1. given are control points $c_0, ..., c_m \in \mathbb{R}^d$*

2. $d_i := (1, c_i) \in \mathbb{R}^{d+1}$ (this is called transforming into homogeneous coordinates (homogenisieren))

3. choose weights $w_i > 0$. $e_i := w_i d_i \in \mathbb{R}^{d+1}$

4. calculate b-spline curve with control points e_i

$$s(t) := \sum_{i=0}^{m} N_i^n(t) e_i = \sum_{i=0}^{m} N_i^n(t) w_i (1, c_i) = \left(\sum_{i=0}^{m} N_i^n(t) w_i, \sum_{i=0}^{m} N_i^n(t) w_i c_i \right)$$

5. project $s(t)$ back into the $(1 \times \mathbb{R}^d)$ plane

$$\tilde{x}(t) = pr(s(t)) = \left(1, \frac{\sum_{i=0}^{m} N_i^n(t) w_i c_i}{\sum_{i=0}^{m} N_i^n(t) w_i} \right)$$

6. dehomogenate

$$x(t) := pr_{\mathbb{R}^d}(\tilde{x}) = \frac{\sum_{i=0}^{m} N_i^n(t) w_i c_i}{\sum_{i=0}^{m} N_i^n(t) w_i}$$

$x(t)$ is called the NURBS curve with weights $w_0, .., w_m$

Remark 20. *In the special case $w_i = w \in \mathbb{R}^+$ the NURBS curve is a B-spline curve.*

Remark 21.

$$\sum_{i=0}^{m} N_i^n(t) = 1 \qquad\qquad N_i^n(t) \geq 0$$

Remark 22. *The bigger w_i is the more the curve is attracted towards c_i.*

Remark 23. *NURBS and B-splines are invariant under affine transformations.*

Example 16. *NURBS-curve over a quadratic bezier curve.*

$$b(t) = (1-t)^2 b_0 + 2(1-t) t b_1 + t^2 b_2$$
$$x(t) = \frac{(1-t)^2 w_0 b_0 + 2(1-t) t w_1 b_1 + t^2 w_2 b_2}{(1-t)^2 w_0 + 2(1-t) t w_1 + t^2 w_2}$$

$x(t)$ is a central projection of a parabola, and therefore a conic section (Kegelschnitt).

Example 17. *$w_0 = w_1 = w_2$ then $x(t) = b(t)$ which is a parabola.*

Example 18. *$w_0 = w_2 = 1$, $w := w_1$ How do we have to choose w in order for $x(t)$ to be a different conic section from a parabola?*

$$x(t) \text{ is a } \begin{cases} \text{ellipse} \\ \text{parabola} \\ \text{hyperbola} \end{cases} \text{if } x(t) \text{ has } \begin{cases} 0 \\ 1 \\ 2 \end{cases} \text{ points at infinity (Fernpunkt).}$$

Therefore we investigate how often the denominator is 0.

$$(1-t)^2 + 2(1-t)tw + t^2 = 1 - 2t + t^2 + 2tw - 2t^2 w + t^2 = (2-2w)t^2 + (-2+2w)t + 1$$
$$t_{1,2} = \frac{2 - 2w \pm \sqrt{(2-2w)^2 - 4(2-2w)}}{2(2-2w)}$$
$$(2-2w)^2 - 4(2-2w) = 4 - 8w + 4w^2 - 8 + 8w = 4w^2 - 4 = 4(w^2 - 1)$$

If $w = 1$ we have already seen that the result is a parabola. Otherwise we do not divide by 0. We are interested in what is under the square root.

$$\text{As } 4(w^2-1) = 0 \iff w = 1 \text{ we have } \begin{cases} 0 \\ 1 \\ 2 \end{cases} \text{ real roots if } 4(w^2-1) \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} .$$

Theorem 10. *$x(t)$ is a* $\begin{cases} \textit{ellipse} \\ \textit{parabola} \\ \textit{hyperbola} \end{cases}$ *if* $\begin{cases} w < 1 \\ w = 1 \\ w > 1 \end{cases}$.

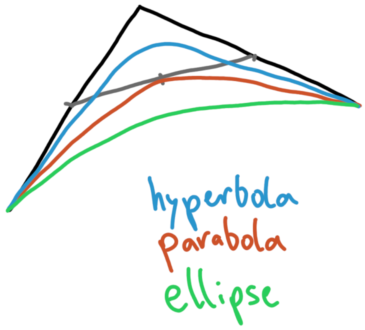

Figure 23: NURBS curve for the different values of w

Example 19. *Is it possible to make a circle?*

Obviously the distance between b_1 and the two points b_0 and b_2 must be equal.

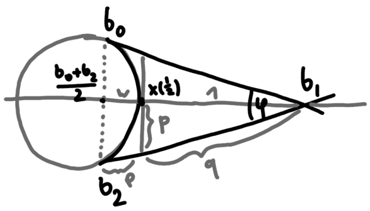

Figure 24: depection of conditions in order for $x(t)$ to be a circle

A necessary condition is therefore that b_0, b_1, b_2 form an isosceles (gleichschenkelig) triangle.

$$x(\frac{1}{2}) = \frac{(1-\frac{1}{2})^2 b_0 + 2(1-\frac{1}{2})\frac{1}{2} w b_1 + (\frac{1}{2})^2 b_2}{(1-\frac{1}{2})^2 + 2(1-\frac{1}{2})\frac{1}{2} w + (\frac{1}{2})^2} = \frac{\frac{1}{4}b_0 + \frac{1}{2}b_1 + \frac{1}{4}b_2}{\frac{1}{4} + \frac{1}{2}w + \frac{1}{4}} = \frac{b_0 + 2wb_1 + b_2}{2 + 2w} =$$
$$\frac{1}{2+2w}(b_0 + b_2 + 2wb_1) = \frac{1}{1+w}(\frac{b_0+b_2}{2} + wb_1) = \frac{1}{1+w}\frac{b_0+b_2}{2} + \frac{w}{1+w}b_1$$

which is a affine combination of $\frac{b_0+b_2}{2}$ and b_1.

$$c = (1-\lambda)a + \lambda b$$
$$\frac{x}{y} = \frac{c-1}{b-c} = \frac{(1-\lambda)a + \lambda b - a}{b - (1-\lambda)a - \lambda b} = \frac{\lambda(b-a)}{(1-\lambda)(b-a)} = \frac{\lambda}{1-\lambda}$$

For $(1-\lambda)a + \lambda b = (\frac{1}{1-w})\frac{b_0+b_2}{2} + \frac{w}{1+w}b_1$ we get $\frac{x}{y} = \frac{\lambda}{1-\lambda} = \frac{\frac{w}{1+w}}{\frac{1}{1+w}} = w$.

With the intercept theorem (Strahlensatz) and law of sines (trigonometrischer Satz) we get $\sin(\frac{\phi}{2}) = \frac{p}{q} = w$.

$\implies w = \sin(\frac{\phi}{2})$.

Theorem 11. *$x(t)$ is a circle, if b_0, b_1, b_2 is an isosceles triangle and $w_0 = w_2 = 1$, $w_1 = \sin(\frac{\phi}{2})$ where $\phi = \angle(b_0 - b_1, b_2, b_1)$.*

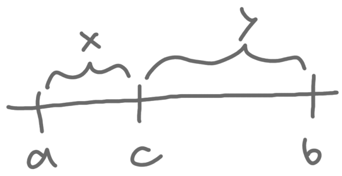

Figure 25: ratio between distances between points on a line

Remark 24. *How can we calculate the angle?*

$$\cos(\phi) = \frac{< b_0 - b_1, b_2 - b_1 >}{||b_0 - b_1|| \cdot ||b_2 - b_1||}$$

With further calculations we get

$$w = \sin(\frac{\phi}{2}) = \frac{||(b_0 - b_1) \times (\frac{b_0 - b_2}{2} - b_1)||}{||b_0 - b_1|| \cdot ||\frac{b_0 - b_2}{2} - b_1||}$$

Remark 25. *Area of a triangle*

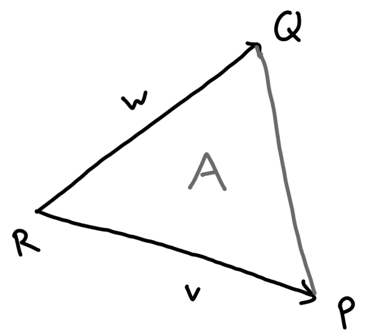

Figure 26: we want to calculate the area of the triangle between two vectors

$$A = \frac{|\det(v, w)|}{2} = \frac{|\det(P - R, Q - R)|}{2} = \frac{\left|\det\begin{pmatrix}1 & R \\ 0 & P - R \\ 0 & Q - R\end{pmatrix}\right|}{2} = \frac{\left|\det\begin{pmatrix}1 & R \\ 1 & P \\ 1 & Q\end{pmatrix}\right|}{2}$$

6 Free form surfaces

Definition 12. *given a control mesh $b_{00}, b_{01}, ..., b_{mn} \in \mathbb{R}^d$, $m, n \in \mathbb{N}$ and two curve schemes $g(s) = \sum_{i=0}^{m} D_i(s)p_i$ and $h(t) = \sum_{j=0}^{n} E_j(t)q_j$.*
$f(s,t) := \sum_{i=0}^{m}\sum_{j=0}^{n} D_i(s)E_j(t)b_{ij}$ is called free form surface (also called tensor product surface) (Freiformfläche, Tensorproduktfläche).

Example 20. *$D_i(s) = B_i^m(s)$ and $E_j(t) = B_j^n(t)$ ... bernstein polynomials then $f(s,t)$ is a bezier surface from degree (m, n).*

Example 21. *$D_i(s) = N_i^m(s)$ and $E_j(t) = N_j^n(t)$ ... b-spline base functions then $f(s,t)$ is a b-spline surface from degree (m, n).*

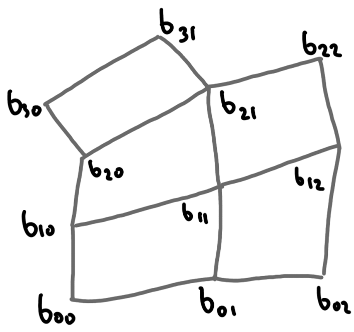

Figure 27: control mesh

Lemma 7.

$$f(s,t) = \sum_i \sum_j D_i(s)E_j(t)b_{ij} = \sum_i D_i(s) \underbrace{\sum_i E_j(t)b_{ij}}_{=:h_i(t)} = \sum_i D_i(s)h_i(t)$$

$$f(s,t) = ... = \sum_j E_j(t) \underbrace{\sum_i D_i(s)b_{ij}}_{=:g_j(s)} = \sum_j E_j(t)g_j(s)$$

For some t_0 $f(s,t_0)$ is called a s-parameter line.
$f(s,t_0) = \sum_i D_i(s) \underbrace{h_i(t_0)}_{p_i}$. A s-parameter line is a curve of the type $g(s)$. The same is true for t-parameter line of type $h(t)$.

Example 22. *bezier mesh of grade* $(2,3)$

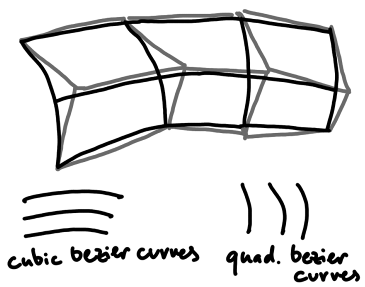

Figure 28: bezier mesh

Algorithm 2 (de casteljou for meshes)**.**

All lemmas and theorems are very similar to free form curves. Therefore we only show selected theorems.

Theorem 12. *If D_i, E_i are affine invariant, then f is affine invariant.*

Proof. Let α be affine invariant. We have to show $\alpha(f(s,t)) = \sum_i \sum_j D_i(s)E_j(t)\alpha(b_{i,j})$.

$$\alpha(f(s,t)) = \alpha(\sum_{i=0}^{m}\sum_{j=0}^{n} D_i(s)E_j(t)b_{ij}) = \alpha(\sum_{i=0}^{m} D_i(s) \overbrace{\sum_{j=0}^{n} E_j(t)b_{ij}}^{h_{ij}}) = \alpha(\sum_{i=0}^{m} D_i(s)\alpha(h_{ij})) =$$

$$\alpha(\sum_{i=0}^{m} D_i(s)\alpha(\sum_{j=0}^{n} E_j(t)b_{ij})) = \sum_{i=0}^{m}\sum_{j=0}^{n} D_i(s)E_j(t)\alpha(b_{i,j})$$

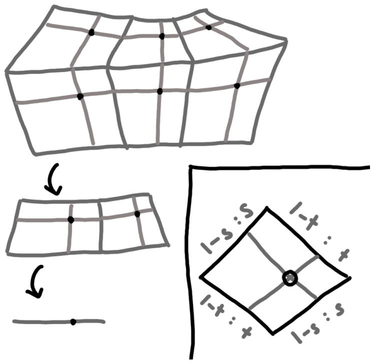

Figure 29: cateljou algorithm for meshes

□

Other theorems such as convex hull, end-point-interpolating are analogous.

7 Subdivision algorithms

7.1 Subdivision for curves

Algorithm 3 (Chaikin)**.** *given some polygon* $p_i, p_{i+1}, ... \in \mathbb{R}^d$

1. *copy every vertex* $p_i \rightarrow p_i^1$
2. *calculate average of two neighboring* $p_i^1 s$ $m_i^1 = \frac{1}{2}(p_i^1 + p_{i+1}^1)$
3. *average the* $m_i^1 s$ $p_i^2 = \frac{1}{2}(m_i^1 + m_{i+1}^1)$

The resulting polygon looks similar to the original one, but the corners are rounded. Continue by repeating the procedure with the new polygon as input.

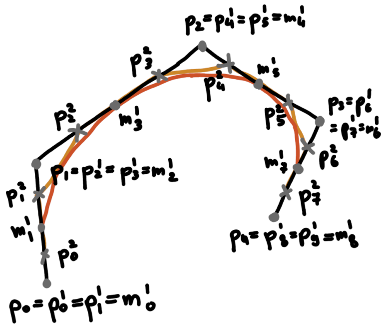

Figure 30: Chaikin algorithm for five given points.

Theorem 13. *The algorithm of Chaikin gives polygons that converge to a quadratic B-Spline curve.*

Remark 26. *Remember Chaikin as: copy once, average twice*

Algorithm 4 (Lane-Riesenfeld)**.** *same as Chaikin, but copy once, average* n *times*

Theorem 14. *The algorithm of Lane-Riesenfeld gives polygons that converge to a B-Spline curve of degree* n*. Lane-Riesenfeld is:*

- *a approximating scheme*
- *is affine invariant.*

Remark 27. *Chaikin is the* $n = 2$ *special case of Lane-Riesenfeld.*

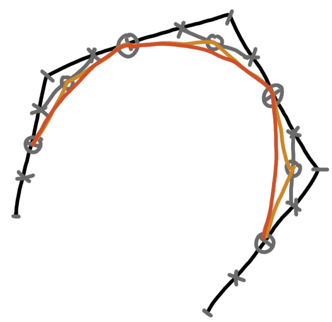

Figure 31: Lane Riesenfeld algorithm with $n = 3$ for five given points.

Algorithm 5 (Four-Point-Scheme)**.**

$$p_i^1 := -\frac{1}{16}p_{i-1} + \frac{9}{16}p_i + \frac{9}{16}p_{i+1} - \frac{1}{16}p_{i+2}$$

this is an affine combination. Inductive $p_i^2, \ldots$.

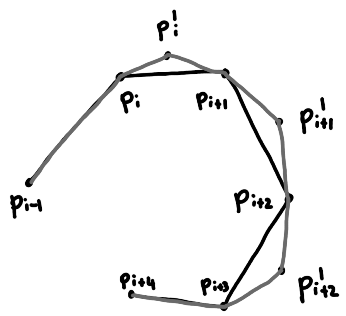

Figure 32: Four Point Scheme for six given points.

Theorem 15. *The curve resulting from the four point scheme converge to a* C^1 *curve.*
Four Point Scheme is:

- *a interpolating*
- *is affine invariant.*

Remark 28. *The weights of the four point scheme can be chosen differently as well:*

$$p_i^1 := -wp_{i-1} + (\frac{1}{2} + w)p_i + (\frac{1}{2}w)p_{i+1} - wp_{i+2}$$

Above we choose $w = \frac{1}{16}$. *This is always interpolating, but only for* $w \in (0, \frac{\sqrt{5}-1}{8})$ *the resulting curve will be* C^1.

Remark 29. *All of the above schemes preserve sub spaces and therefore have the property of linear precision (lineare Präzesion).*

They do not have "circular precision", meaning that if all points lie on a sphere then the curve can leave the sphere.

7.2 Subdivision for meshes

Definition 13. *The degree (Valenz) of a vertex is the number of edges connected to the vertex.*

Algorithm 6 (Loopwe (triangle meshes))**.**

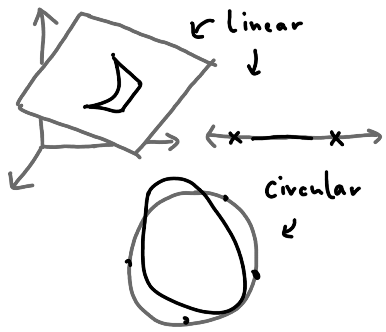

Figure 33: Example for linear precision, but not circular precision.

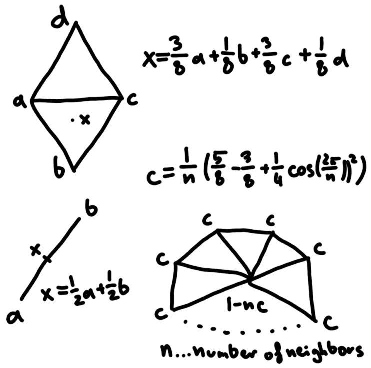

Figure 34: Loop subdivision.

Theorem 16. *Loop subdivision*

- *is approximating*
- *converges to a C^2 surface for generic $n = 6$ vertices and to a C^1 surface otherwise*
- *is face splitting (faces are subdivided)*

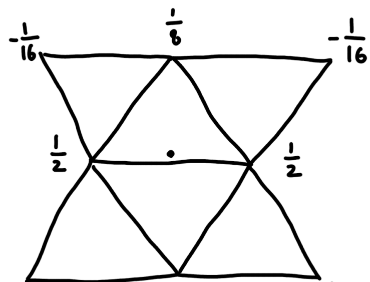

Figure 35: Modified Butterfly subdivision.

Algorithm 7 (Modified-Butterfly-Scheme (triangle meshes))**.** *As in the image we generate new vertices from edges. For non-generic $k \neq 6$ we specify*

$$s_j = \frac{1}{k}\left(\frac{1}{4} + \cos\left(\frac{2j\pi}{k}\right) + \frac{1}{2}\cos\left(\frac{4j\pi}{k}\right)\right)$$

Theorem 17. *The Modified Butterfly Scheme:*

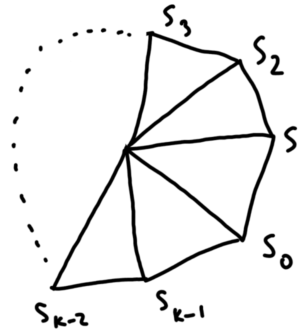

Figure 36: Special case in Modified Butterfly subdivision.

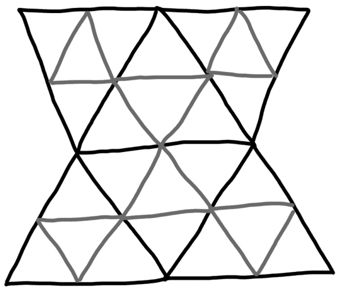

Figure 37: Result of Modified Butterfly subdivision.

- *is interpolating*
- *converges to a C^1 surface*
- *is face-splitting*

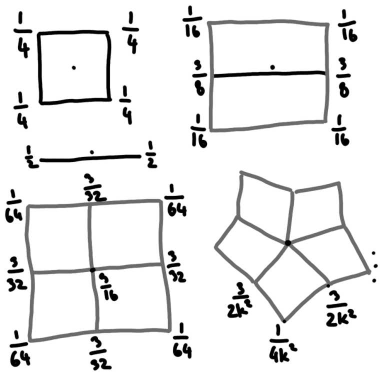

Figure 38: Catmul Clark subdivision.

Algorithm 8 (Catmull-Clark (square meshes))**.**

Theorem 18. *Catmull Clark is:*

- *approximating*
- *converges to a C^2 surface for generic vertices*
- *is face-splitting*

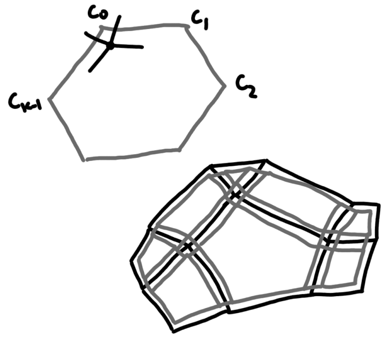

Figure 39: Doo Sabin subdivision.

Algorithm 9 (Doo-Sabin)**.**

$$c_j = \frac{\left(3 + 2\cos\left(\frac{2\pi j}{k}\right)\right)}{4k}, j > 0 \qquad\qquad c_0 = \frac{1}{4} + \frac{5}{4k}$$

Theorem 19. *Doo-Sabin is:*

- *approximating*
- *converges to C^1 surface*
- *is vertex-splitting.*

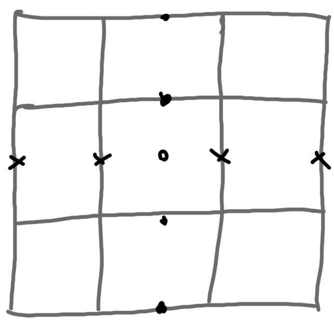

Figure 40: Kobbelt subdivision.

Algorithm 10 (Kobbelt-Scheme (square meshes))**.** *Apply four-point-scheme for vertical and horizontal vertexes. Then apply four point scheme on resulting points. The "horizontal" result and the "vertical" result are the same.*

Theorem 20. *The Kobbelt scheme is:*

- *interpolating*
- *converges to a C^1 surface*
- *is face-splitting.*

Algorithm 11 (Half-Edge-Data-Structure)**.** *given: mesh, that represents a orientable surface.*

A mesh is a collection of lists. These lists include vertices v_i, edges e_i, faces f_i and half-edges h_i.

Every vertex v is assigned a half-edge `v->h`. Every face, every edge is assigned a half-edge. Every half-edge is assigned the opposing (`flip`), next (`next`), previous (`prev`) half-edge, the corresponding face (`f`), vertex (`v`) and edge (`e`).

Remark 30. *In python use the package `geopy`. Find it via tiss/Lehrunterlagen. The documentation is at `www.geometrie.tuwien.ac.at/kilian/docs/geopy`.*

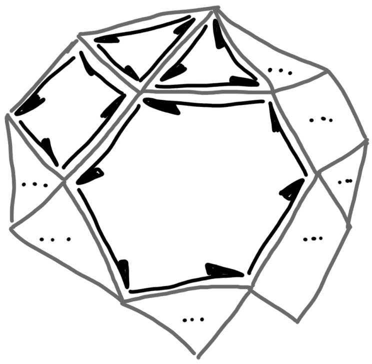

Figure 41: Halfedge datastructure.

8 Conform Pattern Matching

(Musterübertragung)

8.1 Euclidean Geometry

Transformations: Translation, Rotation, Reflection

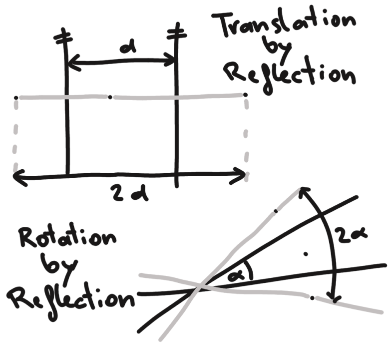

Figure 42: Representation of euclidean transformations as reflections.

From the figure it follows that euclidean transformations can be represented as reflections along hyper-planes.

8.2 Möbius Geometry

Remark 31. *This section is closely following the paper Conformal equivalence of triangle meshes by Boris Springborn, Peter Schröder and Ulrich Pinkall from the year 2008.*

Definition 14. *Möbius transformations are compositions of reflections along hyper-spheres.*

Definition 15 (reflection along sphere)**.**

$$\tilde{\omega} := \frac{r^2}{||\omega||^2}\omega$$

Remark 32. *Reflecting twice yields the identity.*

$$\tilde{(\tilde{\omega})} = \frac{r^2}{||\tilde{\omega}||^2}\tilde{\omega} = \frac{r^2}{||\frac{r^2}{||\omega||^2}\omega||^2}\frac{r^2}{||\omega||^2}\omega = \frac{r^2}{r^4||\omega||^2}||\omega||^2 r^2\omega = \omega$$

Reflecting ω on a sphere with center c and radius r.

$$\tilde{\omega} = \frac{r^2(\omega - c)}{||\omega - c||^2} + c$$

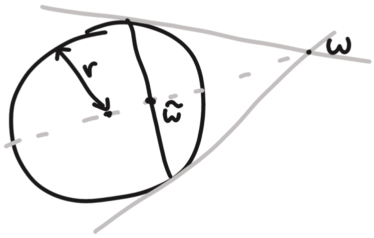

Figure 43: Reflection of point along a circle.

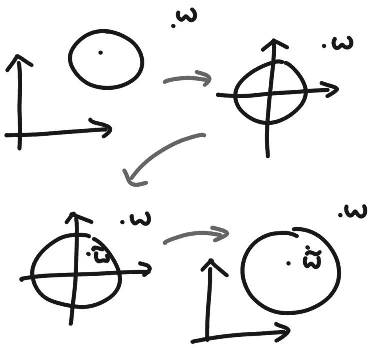

Figure 44: Reflection of point along a circle with different origin.

Remark 33. *Special case: reflection of point c:* $\tilde{c} := \infty$ *and* $\tilde{\infty} := c$

Remark 34. *We can identify* $\mathbb{R}^2$ *with* $\mathbb{C}$*. Then we get*

$$\tilde{\omega} = \frac{r^2(\omega - c)}{(\omega - c)(\bar{\omega} - \bar{c})} + c = \frac{r^2}{\bar{\omega} - \bar{c}} + c$$

In complex analysis möbius transformations are of the form $m(z) = (az+b)/(cz+d)$ *with* $a, b, c, d \in \mathbb{C}, ad - bc \neq 0$*. Anti-möbius transformations are* $m(z) = (a\bar{z}+b)/(c\bar{z}+d)$*.*

$$m(z) = \frac{az+b}{cz+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$$

Embedding $\mathbb{C} \to \mathbb{P}(\mathbb{C}) := \{u \subseteq \mathbb{C}^2 | \dim u = 1\}$ *with* $z \in \mathbb{C} \mapsto [(z, 1)]$ *and* $[(z_0, z_1)] \mapsto \frac{z_0}{z_1}$*,* $z_1 \neq 0$*.*
Then möbius transformations in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ *can be represented as a composition of*

- *Rotation:* $z \mapsto pz$*, with* $|p| = 1$
- *Scaling:* $z \mapsto Az$*, with* $A \in \mathbb{R}$
- *Inversion:* $z \mapsto \frac{1}{z}$
- *Translation:* $z \mapsto z + B$*, with* $B \in \mathbb{C}$

Proof on Wikipedia.

Definition 16 (Cross-Ratio (Doppelverhältnis))**.** *For* $a, b, c, d \in \mathbb{C}$ *... pairwise different we define* $DV(a, b, c, d) := \frac{(a-b)(c-d)}{(b-c)(d-a)} \in \mathbb{C}$ *as the cross-ratio.*

Example 23. $a, b, c, d \in \mathbb{C}$

$$DV(a,b,c,d) = \frac{(a-b)}{(c-b)}\frac{(c-d)}{(a-d)} = \frac{re^{i\phi}}{se^{i\psi}}\frac{te^{i\xi}}{ue^{i\nu}} = \frac{r}{s}e^{i(\phi-\psi)}\frac{t}{u}e^{i(\xi-\nu)} = \underbrace{\frac{rt}{su}}_{\in\mathbb{R}}\ e^{i(\alpha+\beta)}$$

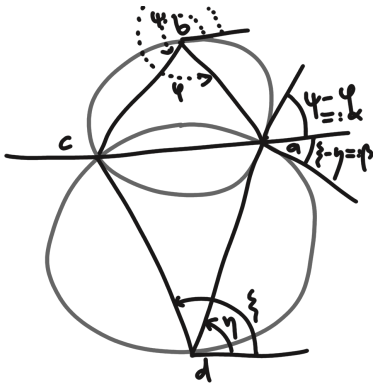

Figure 45: DV as the angle.

The argument of DV is the angle of circumscribed circle of abc and cda. As seen in the image.

Remark 35. *$DV(a,b,c,d) \in \mathbb{R}$, if and only if they share the circumscribed circle.*

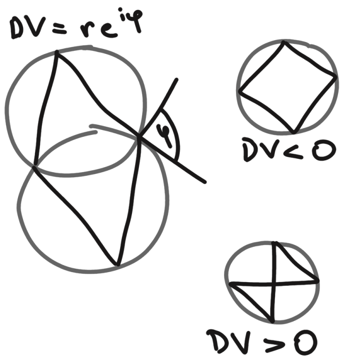

Figure 46: DV and the corresponding angle, as well as examples where the circumscribed circle is shared.

Theorem 21. *The DV is invariant under all möbius transformations.*

Proof.

- Scaling:

$$DV(Aa, Ab, Ac, Ad) = \frac{(Aa-Ab)(Ac-Ad)}{(Ac-Ab)(Aa-Ad)} = \frac{A(a-b)(c-d)}{A(c-b)(a-d)} = DV(a,b,c,d)$$

- Translation:

$$DV(a+B, b+B, c+B, d+B) = \frac{((a+B)-(b+B))((c+B)-(d+B))}{((c+B)-(b+B))((a+B)-(d+B))} = DV(a,b,c,d)$$

- Rotation: $p = e^{i\phi}, z = re^{i\psi}$ then $pz = re^{i(\phi+\psi)}$
- Inversion:

$$DV\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}\right) = \frac{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{c}-\frac{1}{d}\right)abcd}{\left(\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{d}-\frac{1}{a}\right)abcd} = \frac{(b-a)(d-c)}{(c-b)(a-d)} = DV(a,b,c,d)$$

$\square$

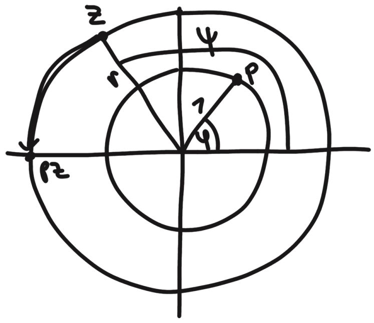

Figure 47: DV is rotation invariant.

Theorem 22. *Möbius transformations are circle-preserving (kreistreu).*

Proof. $a, b, c, z \in K$ where K is any circle. Then $DV(a, b, c, z) \in \mathbb{R}$. As $DV(m(a), m(b), m(c), m(z)) = DV(a, b, c, z) \in \mathbb{R}$ it follows that $m(a), m(b), m(c), m(z)$ lie on a circle. $\square$

Theorem 23. *Möbius transformations are conformal, which means preserving angles (konform, d.h. winkeltreu).*

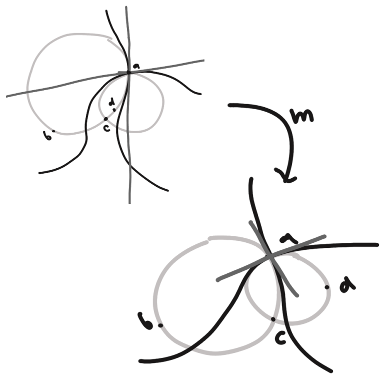

Figure 48: Corresponding circles are transformed into intersecting circles.

Proof. Consider tangential circles as in the figure. They will be transformed into intersecting circles, as they already are intersecting.

$DV(a, b, c, d) = re^{i\phi} \implies \phi$ is the angle. $\implies DV(m(a), m(b), m(c), m(d)) = re^{i\phi}$. Therefore the angle is preserved.

$\square$

Remark 36. *Holomorphic (holomorphe = komplex diffbare) functions preserve angles, but do not preserve circles.*

Example 24. $z \mapsto sin(z^2)$ *is holomorphic, but does not preserve circles.*

Remark 37. *Möbius transformations are the only conformal, bijective functions from open subsets of* $\mathbb{R}^n$*, where* $n \geq 3$*. (Theorem of Liouville).*

Remark 38. *Let* $a, b, c, d \in Im\mathbb{H}$ *(Quaternions)* $\cong \mathbb{R}^3$.

$DV(a, b, c, d) = (a - b)(b - c)^{-1}(c - d)(d - a)^{-1}$ *as* $\mathbb{H}$ *is not commutative (Schiefkörper).*

It still holds that $DV(a, b, c, d) \in \mathbb{R} \iff a, b, c, d$ *lie on a common circle.*

Definition 17. $f, \tilde{f}$... *bijective,* $\phi : f(U) \to \tilde{f}(U), x \mapsto \tilde{f} \circ f^{-1}(x)$.
ϕ *is described over* $f, \tilde{f}$ *by the same parameter (durch den gleichen Parameter beschrieben).*

We ask our-self when ϕ is isometric (isometrisch = längentreu).

Definition 18 (preserving length)**.** ϕ *is preserving length* $\iff \forall$ *curves* c *in* $f(U) : \phi(c)$ *is as long as* c
$c : I \to \mathbb{R}^3$ ... *curve, then the length of* c *is defined by* $L(c) := \int_I ||\dot{c}(t)||dt$

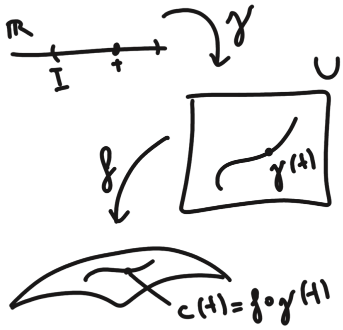

Figure 49: Definition preserving length.

Lemma 8. *ϕ is isometric* $\iff \forall\gamma : I \to U : L(f\circ\gamma) = L(\phi(f\circ\gamma))$.

Proof.

$$||\dot c(t)||^2 = ||\frac{dc(t)}{dt}||^2 = ||fu\dot\gamma_1 + fv\dot\gamma_2||^2 =< \dot\gamma_1 fu + \dot\gamma_2 fv + \dot\gamma_1 fu + \dot\gamma_2 fv >=$$

$$\dot\gamma_1{}^2 \underbrace{< fu, fu >}_{=E} + 2\dot\gamma_1\dot\gamma_2 \underbrace{< fu, fv >}_{=F} + \dot\gamma_2{}^2 \underbrace{< fv, fv >}_{=G} = (\dot\gamma_1, \dot\gamma_2)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \end{pmatrix} = \dot\gamma^T \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{=I} \dot\gamma$$

I is matrix of first fundamental form (1. Fundamentalform).
$\implies ||\dot c(t)||^2 = \dot\gamma^T I \dot\gamma$. Therefore $L(c) = \int_a^b \sqrt{\dot\gamma^T I \dot\gamma} dt$
ϕ is preserving length $\iff \forall\gamma : L(f\circ\gamma) = L(\phi(f\circ\gamma))$
$\gamma : (a, b) \to U, t \mapsto q + (t_0 + t, 0)^T$.

$$L(f\circ\gamma) = \int_a^b \sqrt{(1,0)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} dt = \int_a^b \sqrt{E} dt$$
$$L(\phi(f\circ\gamma)) = \int_a^b \sqrt{\tilde E} dt$$
$$\tilde E =< \tilde f u, \tilde f u >, \tilde F =< \tilde f u, \tilde f v >, \tilde G =< \tilde f v, \tilde f v >$$

ϕ is preserving length $\iff \forall a, b : \int_a^b \sqrt{E} dt = \int_a^b \sqrt{\tilde E} dt \iff \sqrt{E} = \sqrt{\tilde E} \iff E = \tilde E(\iff < fu, fu >=< \tilde f u, \tilde f u >)$.
Analogous $\gamma(t) = q + (0, t_0 + t)^T \implies G = \tilde G$.

$$\gamma(t) = q + \begin{pmatrix} t_0 + t \\ t_1 + t \end{pmatrix} \implies \sqrt{E + 2F + G} = \sqrt{\tilde E + 2\tilde F + \tilde G}$$
$$E + 2F + G = ||fu||^2 + 2 < fu, fv > + ||fv||^2 \geq ||fu||^2 - 2||fu|| \cdot ||fv|| + ||fv||^2 = (||fu|| - ||fv||)^2$$
$$\implies F = \tilde F$$

$\implies \phi$ is preserving length $\implies I = \tilde I$ and $I = \tilde I \implies \phi$ is preserving length. □

Theorem 24. *ϕ is preserving area* $\iff \det I = \det \tilde I$.

Proof. without proof. □

Corollary 6. *preserving length* $\implies I = \tilde I \implies \det I = \det \tilde I \implies$ *preserving area.*

Theorem 25. *ϕ is conformal (preserving angles)* $\iff \exists\lambda : \mathbb{R} \to \mathbb{R} \setminus \{0\} : I = \lambda^2 \tilde I$.

Remark 39. *preserving length* $\iff$ *preserving angles and preserving area.*
Stereographic projection are preserving angles.

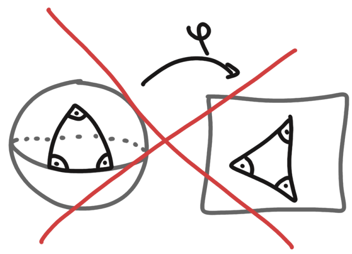

Figure 50: There exists no such length preserving ϕ.

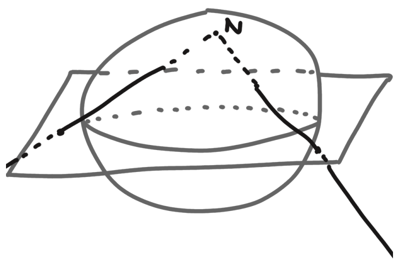

Figure 51: Stereographic projections are preserving angles.

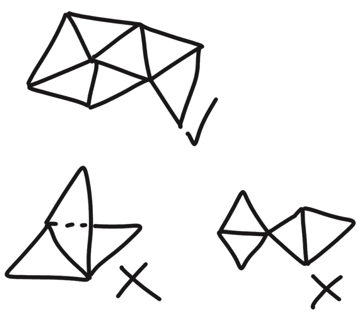

Figure 52: Triangles mesh (top) and ones consisting of triangles that are not triangle meshes (bottom).

Definition 19 (Triangle mesh)**.** *Triangle mesh = polyhedral area (local like open subset of* $\mathbb{R}^2$*) and all areas are triangles.*
$T = (V, E, F)$ *where* V *... vertices,* E *... edges,* F *... faces*

Remark 40. $\forall$ *polyhedron (Polyeder) that are homeomorph (homöomorph):* $F - E + V = 2$.

Definition 20. *A function $L : E \to \mathbb{R}_{\geq 0}$ for which $\forall f \in F : f$ satisfies the triangle equations, is called discrete metric (diskrete Metrik) auf T.*
The triangle equations are given by:

$$l_{ij} = l(v_i, v_j) \ldots \text{ length of the edge between } v_i and v_j$$
$$l_{ij} \leq l_{jk} + l_{ki}$$
$$l_{jk} \leq l_{ki} + l_{ij}$$
$$l_{ki} \leq l_{ij} + l_{jk}$$
$$l_{ij} = l_{ji} \forall i, j, k$$

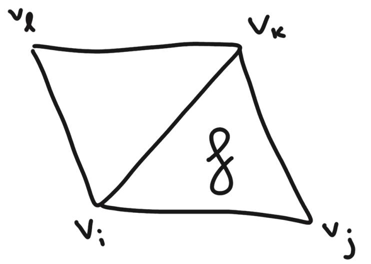

Figure 53: Vertices associated to face f in definition of discrete metric.

Example 25. *The "true" lengths $l_{ij} = ||v_i - v_j||$ of the edges form a discrete metric.*

Definition 21. *Two combinatorial equivalent triangle meshes $T, \tilde{T}$ are called discrete conform equivalent (diskret konform äquivalent), if $\exists \lambda : E \to \mathbb{R}_{\geq 0}$ with $\tilde{l}_{ij} = \lambda_i \lambda_j l_{ij} \forall (i,j) \in E$.*

Lemma 9. *Let m be a möbius tranformation $m : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$.*
$\implies \exists \rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0} : ||m(x) - m(y)|| = \rho(x)\rho(y)||x - y||$.

Proof. without proof □

Definition 22. *length cross-ration (Längendoppelverhältnis) $LDV(v_i, v_j, v_k, v_m) := \frac{l_{ij} l_{km}}{l_{jk} l_{mi}}$*

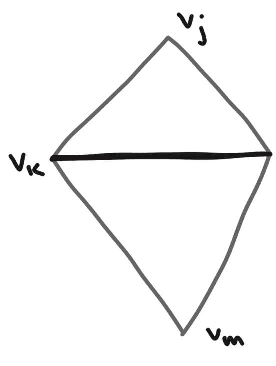

Figure 54: Definition of length cross ratio.

Remark 41. *$a, b, c, d \in \mathbb{C}$ then $LDV(a, b, c, d) = |DV(a, b, c, d)|$*

Theorem 26. *(T, l) and $(\tilde{T}, \tilde{l})$ are discrete conform equivalent $\iff \forall ik \in E : LDV(v_i, v_j, v_k, v_m) = LDV(\tilde{v_i}, \tilde{v_j}, \tilde{v_k}, \tilde{v_m})$.*

Theorem 27. *Let T be a triangle mesh with metric $l_{ij} := ||v_i - v_j||$. Let m be a möbius transformation. Then T and $\tilde{T}$ are conform equivalent.*

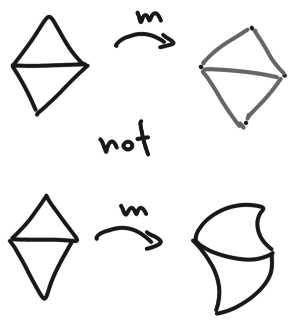

Figure 55: Möbius transformation of the triangle mesh.

Remark 42. *In the above theorem we means that when the edges are möbius transformed and then connected with straight edges to get a triangle mesh.*

Proof. use the above lemma. □

Remark 43. *In a plane möbius transformations are uniquely given by three points.*

Definition 23 ((Milnov's) Lobachevsky-Function)**.**

$$\Pi(x) := \int_0^x \log|2\sin(t)|dt$$

Cyrillic L looks like Π.

Lemma 10. *Properties of* $\Pi(x)$*:*

- $\Pi(0) = 0$
- $\Pi(x + \pi) = \Pi(x)$ ... *pi-periodic*
- $\Pi(k\pi) = 0$
- $\Pi(-x) = -\Pi(x)$

Definition 24.

$$M := \{(x, y, z) \in \mathbb{R}^3 | e^x, e^y, e^z \text{ are the lengths of sides of a trianlge}\} = \\ \{(x, y, z) \in \mathbb{R}^3 | e^x \leq e^y + e^z, e^y \leq e^x + e^z, e^z \leq e^x + e^y\}$$

$$f(x, y, z) := \alpha x + \beta y + \gamma z + \Pi(\alpha) + \Pi(\beta) + \Pi(\gamma)$$

where $\alpha(x, y, z), \beta(x, y, z), \gamma(x, y, z)$ *are the angles of the triangle.* $x, y, z \in M$.

Lemma 11.

$$\frac{\delta f}{\delta x} = \alpha \qquad \frac{\delta f}{\delta y} = \beta \qquad \frac{\delta f}{\delta z} = \gamma$$

Proof.

$$\frac{\delta f}{\delta x} = \frac{\delta \alpha}{\delta x}x + \alpha + \frac{\delta \beta}{\delta x}y + \frac{\delta \gamma}{\delta x}z + (-\log|2\sin(\alpha)|)\frac{\delta \alpha}{\delta x} + (-\log|2\sin(\beta)|)\frac{\delta \beta}{\delta x} + (-\log(|2\sin\gamma)|)\frac{\delta \gamma}{\delta x} = \\ \alpha + \frac{\delta \alpha}{\delta x}(x - \log(|2\sin\alpha|)) + \frac{\delta \beta}{\delta x}(y - \log(|2\sin\beta|)) + \frac{\delta \gamma}{\delta x}(z - \log(|2\sin\gamma|)) = ...$$

We do a short calculation on the side:

$$\sin(\alpha) = \frac{\frac{e^x}{2}}{R} \implies 2R\sin(\alpha) = e^x \implies R = \frac{e^x}{|2\sin(\alpha)|} \implies \log R = x - \log|2\sin(\alpha)|$$

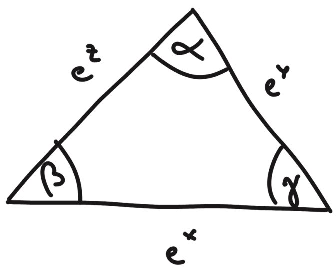

Figure 56: Definition of M and corresponding angles.

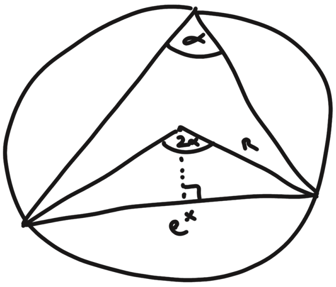

Figure 57: Figure for side calculation.

Which gives us

$$... = \alpha + \frac{\delta\alpha}{\delta x}\log R + \frac{\delta\beta}{\delta x}\log R + \frac{\delta\gamma}{\delta x}\log R = \alpha + \log R(\frac{\delta(\alpha+\beta+\gamma)}{\delta x}) = \alpha + \log R\underbrace{(\frac{\delta\Pi}{\delta x})}_{=0} = \alpha$$

□

Remark 44. *Discrete conform equivalent $\tilde{l}_{ij} = \lambda_i\lambda_j l_{ij}$ in different wording:*
$\exists\lambda_i > 0 \implies \exists u_i : \lambda_i = e^{\frac{u_i}{2}}$ *with* $u_i \in \mathbb{R}$.
$\gamma_{ij} := \log l_{ij}$, $\tilde{\gamma}_{ij} := \log \tilde{l}_{ij}$ *gives us*
$\log \tilde{l}_{ij} = \log\lambda_i + \log\lambda_j + \log l_{ij} \iff \tilde{\lambda}_{ij} = \frac{u_i}{2} + \frac{u_j}{2} + \gamma_{ij}$
which is an additive condition for conformal equivalency.

Definition 25 (Interior angle (Innenwinkel) at the position i). α^i_{jk}
Sum of interior angles $\sum_{jk\in star(i)} \alpha^i_{jk}$ where $star(i)$ is the vertex star around the edge i.

Goal: Given a triangle mesh $T, \Phi_i \in \mathbb{R}{>}\, 0 \forall i \in V$. We want a triangle mesh $\tilde{T}$, which is conform equivalent to T with sum of interior angles equal to $\Phi_i \forall i \in V$.

Remark 45. *If such a $\tilde{T}$ exists and $\Phi_i = 2\pi\forall i$ ... interior vertex, then $\tilde{T}$ can be thought of as in a plane.*
Let $u \in \mathbb{R}^{\#V}$ with $\tilde{l}_{ij} = e^{u_i/2}e^{u_j/2}l_{ij} = e^{(u_i+u_j)/2}l_{ij}$ are lengths of a triangle mesh.

$$E(u) := \sum_{i,j,k\in F}\left[2f\left(\frac{\tilde{\gamma}_{ij}}{2}, \frac{\tilde{\gamma}_{jk}}{2}, \frac{\tilde{\gamma}_{ki}}{2}\right) - \frac{\pi}{2}(\tilde{\gamma}_{ij} + \tilde{\gamma}_{jk} + \tilde{\gamma}_{ki})\right] + \sum_{i\in V} u_i\Phi_i$$

$$\tilde{\gamma}_{ij} = \frac{u_i+u_j}{2} + \gamma_{ij} = \frac{u_i}{2} + \frac{u_j}{2} + \log l_{ij}$$

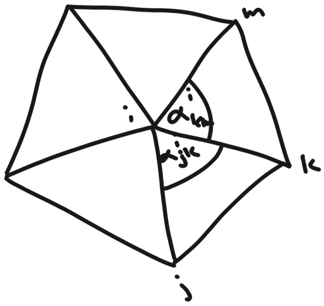

Figure 58: Interior angle definition.

Lemma 12. *Properties of $E(u)$:*

- $$\frac{\delta E(u)}{\delta u_i} = \Phi_i - \sum_{jk \in star(i)} \alpha^i_{jk}$$

  Therefore the critical points are those interesting us.

- *E can be extended as a convex function onto $\mathbb{R}^{\#V}$*

Theorem 28. *The critical points of E describe the triangle mesh $\tilde{T}$. This means $\tilde{T}$ is conform equivalent to T and $\forall i \in V : \Phi_i = \sum_{jk \in star(i)} \alpha^i_{jk}$.*

As E is convex there exists such a critical point.

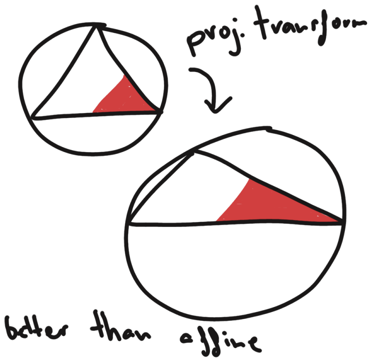

Figure 59: Using projective transformations provides better results than affine transformations.