

Computer Aided Geometric Design Compendium WS2023

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Organization

Lecture each Thursday 12:00 to 14:00 (full 2 hours).

Oral exam. Write email to fix date and time.

Problem session each Thursday 14:00 to 16:00. Mandatory attendance!

Kreuzerübung.

1 Bezier curves

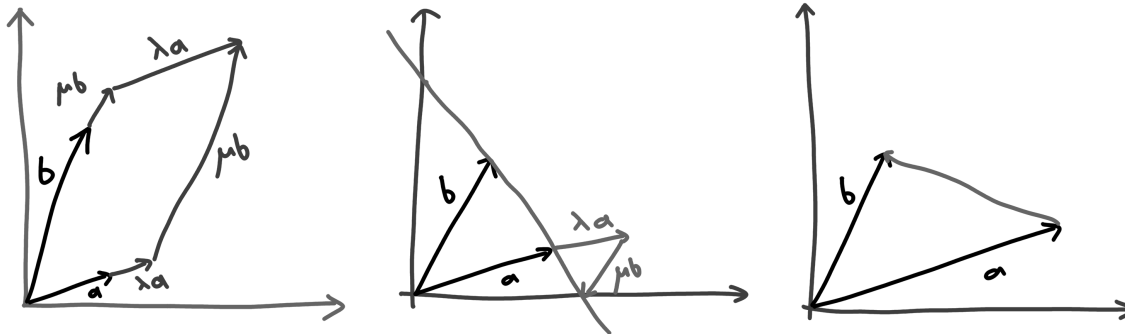


Figure 1: Linear combination, affine combination and convex combination

Example 1. Linear combination $\lambda a + \mu b$

Affine combination $\lambda a + \mu b$ and $\lambda + \mu = 1$

What is μ so that $\lambda a + \mu b$ is on the line?

$$\lambda a + \mu b = a + t(b - a) \implies \underbrace{a(\lambda - 1 + t)}_{=0} + \underbrace{b(\mu - t)}_{=0} = 0$$

If a, b are linearly independent $\implies \mu = t \wedge \lambda + \mu = 1$

Convex combination $\lambda a + \mu b$ and $\lambda + \mu = 1$ and $\lambda, \mu \geq 0$

Line is $a + t(b - a)$ with $t \in [0, 1] \implies \mu, \lambda \in [0, 1]$

Definition 1 (combinations). *linear combination* $\sum_{i=1}^n \lambda_i v_i$ with $v_1, \dots, v_n \in \mathbb{R}^d, \lambda_1, \dots, \lambda_n \in \mathbb{R}$
affine combination $\sum_{i=1}^n \lambda_i v_i$ with $\sum_{i=1}^n \lambda_i = 1$
convex combination $\sum_{i=1}^n \lambda_i v_i$ with $\sum_{i=1}^n \lambda_i = 1$ and $\forall i : \lambda_i \geq 0$

Algorithm 1 (of de Casteljau, Bezier curve). Given: $b_0, \dots, b_n \in \mathbb{R}^d$ (called control points / Kontrollpunkte), $t \in \mathbb{R}$

Recursion: $b_i^0(t) := b_i$

$b_i^j(t) := (1 - t)b_i^{j-1}(t) + tb_{i+1}^{j-1}(t)$ for $j = 1, \dots, n$ and $i = 0, \dots, n - j$

Result: $b(t) := b_0^n(t)$ (called Bezier curve)

Remark 1. In the algorithm above often we choose $t \in [0, 1]$.

Example 2.

Remark 2. In this course $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

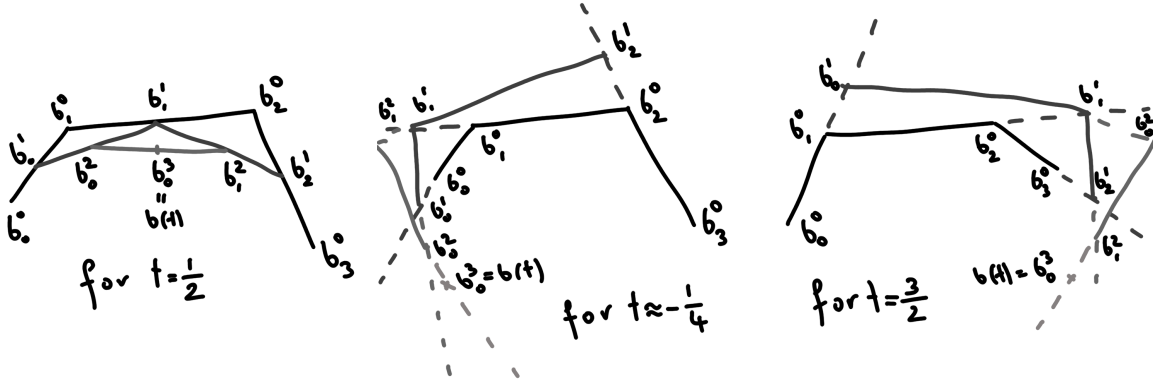


Figure 2: Examples of the de Casteljou algorithm

Recap 1. $0! := 1, n! := n(n-1)(n-2) \cdots 1$ for $n \geq 1$.

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!} & , n \geq k \geq 0 \\ 0 & , k > n \end{cases} \text{ for } n, k \in \mathbb{N}_0$$

Definition 2 (Bernstein polynomials). For $n, i \in \mathbb{N}_0$ we define $B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i} \in \mathbb{R}[t]$

Remark 3. Special cases of Bernstein polynomials

$$\begin{aligned} i > n &\implies B_i^n(t) = 0 & B_i^n(0) &= \begin{cases} 0, i \neq 0 \\ 1, i = 0 \end{cases} \\ B_i^n(1) &= \begin{cases} 0, i \neq n \\ 1, i = n \end{cases} & B_0^0(t) &= 1 \end{aligned}$$

Theorem 1. $b_i^j(t) = \sum_{l=0}^j B_l^j(t) b_{i+l}$

Proof. Induction over j : $j = 0$:

$$j = 0 : \quad b_i^0(t) := b_i = 1 \cdot b_i = B_0^0(t) \cdot b_i \quad \checkmark$$

$$j-1 \rightarrow j : \quad b_i^j(t) := (1-t)b_i^{j-1}(t) + t b_{i+1}^{j-1}(t) \stackrel{\text{IA}}{=} (1-t) \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+l} + t \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+1+l} =$$

$$(1-t) \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+l} + t \sum_{l=1}^j B_{l-1}^{j-1}(t) b_{i+l} = \sum_{l=0}^j \underbrace{((1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t))}_{= B_l^j(t) \text{ using the following lemma}} b_{i+l} =$$

$$\sum_{l=0}^j B_l^j(t) b_{i+l} \quad \checkmark$$

□

Corollary 1. The Bezier curve equals $b(t) = b_0^n(t) = \sum_{l=0}^n B_l^n(t) b_{i+l}$, which is called the Bernstein representation of the Bezier curve.

Remark 4. As $b(t) = \sum_{l=0}^n B_l^n(t) b_l \in C^\infty$ it is a polynomial curve of degree n , which is in C^∞ and therefore "very smooth".

Lemma 1. $B_l^j(t) = (1-t)B_l^{j-1}(t) + tB_{l-1}^{j-1}(t)$

Proof.

$$\begin{aligned} (1-t)B_l^{j-1}(t) + tB_{l-1}^{j-1}(t) &= (1-t) \binom{j-1}{l} t^l (1-t)^{j-1-l} + t \binom{j-1}{l-1} t^{l-1} (1-t)^{j-1-l+1} = \\ \binom{j-1}{l} t^l (1-t)^{j-l} + \binom{j-1}{l-1} t^l (1-t)^{j-l} &= \left(\binom{j-1}{l} + \binom{j-1}{l-1} \right) t^l (1-t)^{j-l} = \binom{j}{l} t^l (1-t)^{j-l} = B_l^j(t) \end{aligned}$$

□

Remark 5. What is $b(0)$? $b(0) = \sum_{i=0}^n B_i^n(0)b_i = b_0 + 0 + 0 + \dots + 0 = b_0$
What is $b(1)$? $b(1) = \sum_{i=0}^n B_i^n(1)b_i = 0 + \dots + 0 + b_n = b_n$

Definition 3 (end-point-interpolating). Curves which pass through the first and last point are called end-point-interpolating (Endpunktinterpolierend).

Remark 6. Bezier curves are end-point-interpolating.

Remark 7. How many intersection points are there between a planar (i.e. in \mathbb{R}^2) Bezier curve and a straight line?

$$\text{Straight line: } p + t(q - p) \qquad \text{Bezier curve: } b(t) = \sum_{i=0}^n B_i^n(t) \underbrace{b_i}_{\in \mathbb{R}^2}$$

Solving $p + t(q - p) = \sum_{i=0}^n B_i^n(t)b_i$ results in at most n solutions.

Lemma 2. $\frac{d}{dt} B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$

Proof.

$$\begin{aligned} \frac{d}{dt} B_i^n(t) &= \frac{d}{dt} \binom{n}{i} t^i (1-t)^{n-i} = \binom{n}{i} i t^{i-1} (1-t)^{n-i} - \binom{n}{i} t^i (n-i) (1-t)^{n-i-1} = \\ &= \frac{n!}{i!(n-i)!} i t^{i-1} (1-t)^{n-i} - \frac{n!}{i!(n-i)!} t^i (n-i) (1-t)^{n-i-1} = \\ &= n \left(\frac{(n-1)!}{(i-1)!(n-i)!} t^{i-1} (1-t)^{n-i} - \frac{(n-1)!}{i!(n-i-1)!} t^i (1-t)^{n-i-1} \right) = \\ &= n \left(\binom{n-1}{i-1} t^{i-1} (1-t)^{n-i} - \binom{n-1}{i} t^i (1-t)^{n-i-1} \right) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)) \end{aligned}$$

□

Theorem 2. $\dot{b}(t) := \frac{d}{dt} b(t) = n \sum_{i=0}^{n-1} B_i^{n-1}(t)(b_{i+1} - b_i) = n(b_1^{n-1}(t) - b_0^{n-1}(t))$

Proof.

$$\begin{aligned} \dot{b}(t) &= \frac{d}{dt} \left(\sum_{i=0}^n B_i^n(t)b_i \right) = \sum_{i=0}^n \frac{d}{dt} B_i^n(t)b_i = \sum_{i=0}^n n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))b_i = n \left(\sum_{i=0}^n B_{i-1}^{n-1}(t)b_i - \sum_{i=0}^n B_i^{n-1}(t)b_i \right) = \\ &= n \left(\sum_{i=1}^n B_{i-1}^{n-1}(t)b_i - \sum_{i=0}^n B_i^{n-1}(t)b_i \right) = n \left(\underbrace{\sum_{i=0}^{n-1} B_i^{n-1}(t)b_{i+1}}_{=b_1^{n-1}(t)} - \underbrace{\sum_{i=0}^{n-1} B_i^{n-1}(t)b_i}_{=b_0^{n-1}(t)} \right) = n \left(\sum_{i=0}^{n-1} B_i^{n-1}(t)(b_{i+1} - b_i) \right) \end{aligned}$$

□

Corollary 2. • $\dot{b}(0) = n(b_1 - b_0)$

- $\dot{b}(1) = n(b_n - b_{n-1})$
- The last segment in the algorithm of de Casteljou is the tangent of the Bezier curve in $b(t)$.
- The derivative of a bezier curve of degree n is a bezier curve of degree $n-1$ with control points $(b_1, b_0), (b_2 - b_1), \dots, (b_n - b_{n-1})$.

Corollary 3. $\ddot{b}(t) = n(n-1) \sum_{i=0}^{n-2} B_i^{n-2}(t)(b_{i+2} - 2b_{i+1} + b_i)$
 $\ddot{b}(0) = n(n-1)(b_2 - 2b_1 + b_0), \ddot{b}(1) = n(n-1)(b_n - 2b_{n-1} + b_{n-2})$

Corollary 4. The curvature of a bezier curve in the point $b(0)$ depends only on b_0, b_1, b_2 .
The curvature of a bezier curve in the point $b(1)$ depends only on b_{n-2}, b_{n-1}, b_n .

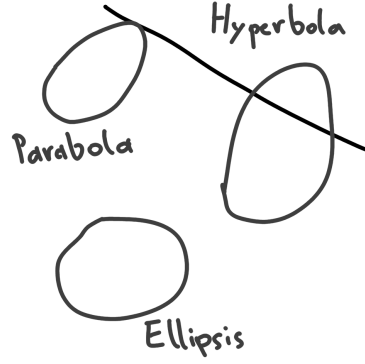


Figure 3: Categorization of Parabolas, Hyperbolas and Ellipsis as intersection points with the line at infinity.

Example 3. *Quadratic Bezier curve*

$$b(t) = \sum_{i=0}^2 B_i^2(t) b_i = \binom{2}{0} t^0 (1-t)^2 b_0 + \binom{2}{1} t^1 (1-t)^1 b_1 + \binom{2}{2} t^2 (1-t)^0 b_2 = t^2 (b_2 - 2b_1 + b_0) + t(2b_1 - 2b_0) + b_0$$

which is an affine transformation of a parabola and therefore a parabola.

Quadratic bezier curves are parabolas.

Remark 8. *Line at infinity (Ferngerade) is the collection of points where parallel lines intersect.*

Remark 9. *Different applications using these curves are Rhino, OpenSCAD, Autocad, Geogebra, ...*

2 Parameterized curves

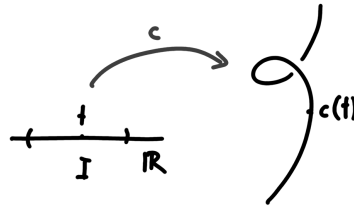


Figure 4: parameterized curve $c(t)$

Definition 4. $c : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^3$ is called a parameterized curve.

$\dot{c}(t) := \frac{d}{dt}c(t)$ is called the tangential vector. For \mathbb{R}^3 we have $\dot{c}(t) = (\dot{c}_1(t), \dot{c}_2(t), \dot{c}_3(t))$.

The velocity is defined as $\|\dot{c}(t)\|$.

A point $c(t)$ is called regular, if $\dot{c}(t) \neq 0$ and is called singular, if $\dot{c}(t) = 0$.

Example 4. A helix (Schraublinie) is defined by $c(t) = (\cos(t), \sin(t), t)^T$.

$$\dot{c}(t) = (-\sin(t), \cos(t), 1)^T$$

$$\|\dot{c}(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

We see that the helix is passed through with constant velocity. Furthermore all points are regular.

Example 5. $c : \mathbb{R} \rightarrow \mathbb{R}^3, t \mapsto (t^2, t^3, t^4)$, $\dot{c}(t) = (2t, 3t^2, 4t^3)$. We see that 0 is singular as $\dot{c}(0) = (0, 0, 0)$. Everywhere else the curve is regular.

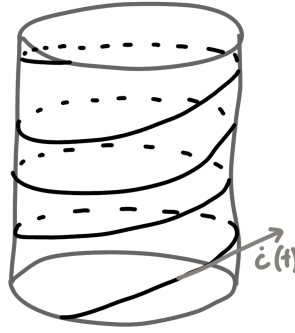


Figure 5: Helix with tangential vector

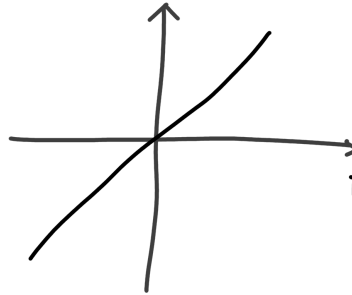


Figure 6: identity line can be parameterized such that $(0,0)$ is singular.

Remark 10. A point being regular or singular depends on the parameterisation of the curve.

For example $c(t) = (t, t)$ produces a regular curve, while $c(t) = (t^3, t^3)$ produces a curve where 0 is singular.

There are curves and points where no parameterisation exists such that the point is regular.

Definition 5. $c : I \rightarrow \mathbb{R}^2 \in C^2(I, \mathbb{R}^2)$

The curvature of the curve in the point $c(t)$ is defined as $\kappa(t) = \frac{\det(\dot{c}(t), \ddot{c}(t))}{\|\dot{c}(t)\|^3}$

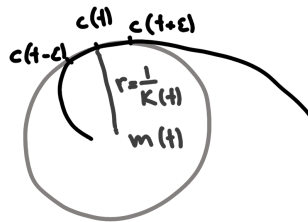


Figure 7: Circle of curvature

Example 6. The circle of curvature has a radius of $\frac{1}{\kappa(t)}$. $m(t)$ is called the center of curvature.

$$m(t) = c(t) + \frac{1}{\kappa(t)}n(t) \text{ where } n(t) = \frac{(-\dot{c}_2(t), \dot{c}_1(t))}{\|\dot{c}(t)\|}.$$

Remark 11. Exercise: compare this definition of curvature with the school version concerning graphs.

Example 7. For a circle we have $c(t) = (r \cos(t), r \sin(t))^T$, $\dot{c}(t) = (-r \sin(t), r \cos(t))^T$, $\ddot{c}(t) = (-r \cos(t), -r \sin(t))^T$

$$\kappa(t) = \frac{\det \begin{pmatrix} -r \sin(t) & -r \cos(t) \\ r \cos(t) & -r \sin(t) \end{pmatrix}}{r^3} = \frac{r^2 \sin^2(t) + r^2 \cos^2(t)}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$n(t) = \frac{(-r \cos(t), -r \sin(t))}{r} = (-\cos(t), -\sin(t))$$

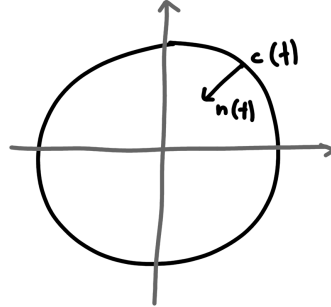


Figure 8: Circle with normal vector

Definition 6. A point $c(t)$ with $\dot{\kappa}(t) = 0$ is called a vertex.

Example 8. An ellipse has four vertices.

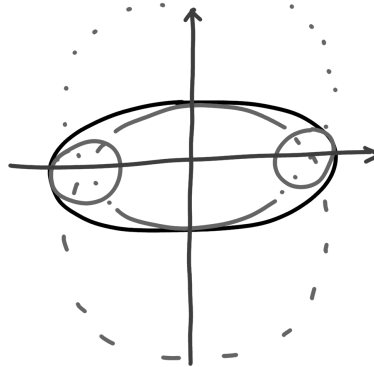


Figure 9: Ellipse and the four vertices.

$(t, \exp t)$ has no vertex.

Klothoids are curves with $\kappa(t) = t$. They are used in road construction and have no vertex.

Definition 7. $c : I \rightarrow \mathbb{R}^3$

$\kappa(t) = \frac{\|\dot{c}(t) \times \ddot{c}(t)\|}{\|\dot{c}(t)\|^3}$ is called the curvature of a space curve.

$\tau(t) = \frac{\det(\dot{c}(t), \ddot{c}(t), \ddot{\ddot{c}}(t))}{\|\dot{c}(t) \times \ddot{c}(t)\|^2}$ is called torsion of a space curve.

Example 9. For the helix $t \mapsto (\cos(t), \sin(t), pt)$ the torsion depends on p .

3 Properties of Bezier curves

Definition 8. $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called **affine** if $\exists l : \mathbb{R}^n \rightarrow \mathbb{R}^m$...linear $\exists v \in \mathbb{R}^m : \alpha(x) = l(x) + v$.
 α is called **affinity** if α is affine and bijective.

Example 10. An example of an linear function is shear (Scherung).

$$l \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Area is preserved.

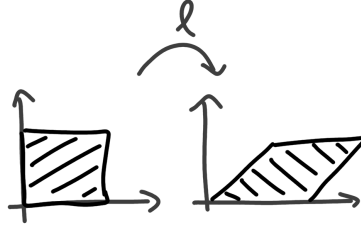


Figure 10: Shear preserves area

Theorem 3. *Bezier curves are invariant under affine transformations.*

Proof. Let $b(t) = \sum_{i=0}^n B_i^n(t)b_i$ be a bezier curve and $\alpha(x) = l(x) + v$ where $b_i \in \mathbb{R}^d, l : \mathbb{R}^d \rightarrow \mathbb{R}^m, v \in \mathbb{R}^m$

$$\begin{aligned} \alpha(b(t)) &= l(b(t)) + v = l\left(\sum_{i=0}^n B_i^n(t)b_i\right) + v = \sum_{i=0}^n B_i^n(t)l(b_i) + v = \\ &= \sum_{i=0}^n B_i^n(t)l(b_i) + \sum_{i=0}^n B_i^n(t)v = \sum_{i=0}^n B_i^n(t)(l(b_i) + v) = \sum_{i=0}^n B_i^n(t)\alpha(b_i) \end{aligned}$$

□

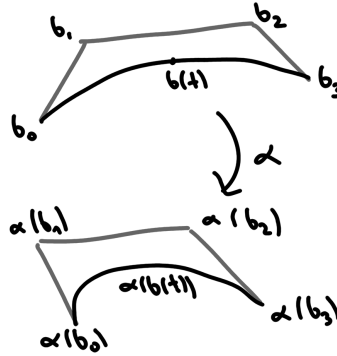


Figure 11: Bezier curve is invariant under affine transformation

Lemma 3. $\sum_{i=0}^n B_i^n(t) = 1$

Proof. Binomial theorem: $(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}$ for $x = t, y = 1 - t$ we get

$$1 = (t + (1 - t))^n = \sum_{i=0}^n \binom{n}{i} t^i (1 - t)^{n-i} = \sum_{i=0}^n B_i^n(t)$$

□

Lemma 4. *For $t \in [0, 1]$ it holds that $B_i^n(t) \geq 0$*

Proof.

$$B_i^n(t) = \underbrace{\binom{n}{i}}_{\geq 0} \underbrace{t^i}_{\geq 0} \underbrace{(1 - t)^{n-i}}_{\geq 0} \geq 0$$

□

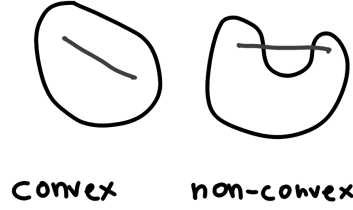


Figure 12: example of convex and non-convex set

Definition 9. $M \subseteq \mathbb{R}^d$ then $\text{conv}(M) := \{\sum_{i=1}^k \lambda_i v_i : v_1, \dots, v_k \in M, \lambda_1, \dots, \lambda_k \geq 0, \sum_{i=1}^k \lambda_i = 1, k \in \mathbb{N}\}$ is called the convex hull of M .

M is called convex set, iff $\forall x, y \in M \forall t \in [0, 1] : tx + (1 - t)y \in M$.

Example 11.

Remark 12 (Pick theorem). For any polygon formed of points in $\mathbb{Z} \times \mathbb{Z}$ it holds that the area can be calculated by $I + R/2 - 1$ where I is the number of points inside the polygon and R is the number of points on the edges of the polygon.

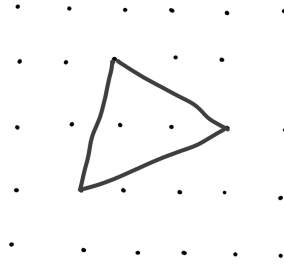


Figure 13: example picks theorem where $I = 2, R = 3$ and therefore $A = \frac{5}{2}$

Theorem 4. $\forall t \in [0, 1] : b(t) \in \text{conv}(\{b_0, b_1, \dots, b_n\})$

Proof. $b(t) = \sum_{i=0}^n \underbrace{B_i^n(t)}_{\geq 0} b_i \forall t \in [0, 1]$ and $\sum_{i=0}^n B_i^n(t) = 1 \forall t \in [0, 1]$. Therefore $b(t) \in \text{conv}(\{b_0, \dots, b_n\})$ \square

Remark 13 (Montecarlo method for calculating area). random points, count how many fall within the area from the total amount of points, estimate volume from that.

Theorem 5. Bezier curves are symmetric, meaning that if $b(t)$ is a bezier curves of b_0, \dots, b_n and $\tilde{b}(t)$ is a bezier curve of b_n, \dots, b_0 then $b(t) = \tilde{b}(1 - t)$

Proof.

$$\tilde{b}(t) = \sum_{i=0}^n B_i^n(t) b_{n-i} = \sum_{i=0}^n \underbrace{B_i^n(1-t)(1-t)}_{B_i^n(t)} b_{n-i} = \sum_{i=0}^n B_{n-i}^n(1-t) b_i = \sum_{i=0}^n B_i^n(1-t) b_i = b(1-t)$$

\square

Lemma 5. $\alpha \in \mathbb{R} \implies B_i^n(\alpha t) = \sum_{j=0}^n B_i^j(\alpha) B_j^n(t)$

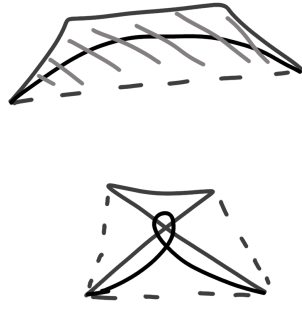


Figure 14: example of a bezier curve and its convex hull

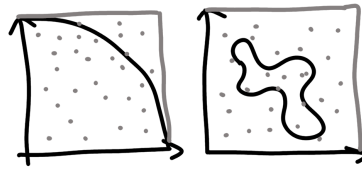


Figure 15: montecarlo method for a quarter circle and some random shape

Proof. can be done by induction, but is tedious □

Theorem 6 (Sub-division-property (Unterteilungseigenschaft)). Let $b(t)$ be a bezier curve to b_0, \dots, b_n and $\alpha \in \mathbb{R}$.

$$\tilde{b}(t) := b(\alpha t)$$

$$\hat{b}(t) := b((1 - \alpha)t + \alpha)$$

Then \tilde{b} is a bezier curve to $b_0^0(\alpha), b_0^1(\alpha), \dots, b_0^n(\alpha)$; \hat{b} is a bezier curve to $b_0^n(\alpha), b_1^{n-1}(\alpha), \dots, b_n^0(\alpha)$ and "glued together" they result in the original bezier curve.

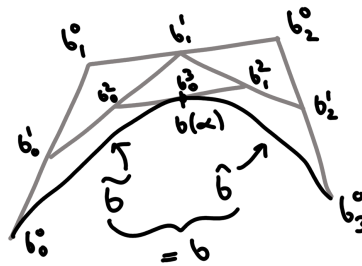


Figure 16: bezier curve b is divided at point $b(\alpha)$ into two bezier curves \tilde{b} and \hat{b}

Proof.

$$\begin{aligned}\tilde{b}(t) = b(\alpha t) &= \sum_{i=0}^n B_i^n(\alpha t) b_i = \sum_{i=0}^n \left(\sum_{j=0}^n B_i^j(\alpha) B_j^n(t) \right) b_i = \sum_{j=0}^n \sum_{i=0}^n B_i^j(\alpha) B_j^n(t) b_i = \\ &= \sum_{j=0}^n B_j^n(t) \underbrace{\sum_{i=0}^n B_i^j(\alpha) b_i}_{b_0^j(\alpha)} = \sum_{j=0}^n B_j^n(t) b_0^j(\alpha)\end{aligned}$$

Where we used $b_i^j(t) = \sum_{l=0}^j B_l^j(t) b_{i+l}$, which we already showed.
 \hat{b} is analogous. □

Definition 10. The polynomial ring is defined as $\Pi_n := \{f \in \mathbb{R}[t] : \deg(f) \leq n\}$, which is a vector space.

Theorem 7. $\{B_0^n(t), \dots, B_n^n(t)\}$ is a basis of Π_n .

Proof. $\forall i : B_i^n(t) \in \Pi_n$ therefore $\text{span}(B_0^n(t), \dots, B_n^n(t)) \subseteq \Pi_n$.

We know that $\dim(\Pi_n) = n + 1$ as $\{1, t, t^2, \dots, t^n\}$ is a basis.

Let $k \leq n$.

$$\begin{aligned}1 &= \sum_{i=0}^{n-k} B_i^{n-k}(t) = \sum_{i=k}^n B_{i-k}^{n-k}(t) = \sum_{i=k}^n \binom{n-k}{i-k} t^{i-k} (1-t)^{n-i} \\ \implies \binom{n}{k} t^k &= \sum_{i=k}^n \binom{n}{k} \binom{n-k}{i-k} t^i (1-t)^{n-i} = \sum_{i=k}^n \binom{n}{i} \binom{i}{k} t^i (1-t)^{n-i} = \sum_{i=k}^n \binom{i}{k} B_i^n(t) \\ \implies t^k &\in \text{span}(B_0^n(t), \dots, B_n^n(t)) \implies \Pi_n \subseteq \text{span}(B_0^n(t), \dots, B_n^n(t))\end{aligned}$$

□

Corollary 5. Every polynomial curve is a bezier curve.

Theorem 8 (corner-cutting (Eckenabschneiden)). Define the sequence P_n by this (recursive) definition $P_0 := (b_0^0, b_0^1, \dots, b_0^n)$, $P_1 := (b_0^0, b_0^1, \dots, b_0^n, b_1^{n-1}, b_2^{n-2}, \dots, b_n^0)$, ..., P_n consists of the points we get by sub-dividing the bezier curve (with $t = \frac{1}{2}$).

Then this sequence "converges" to the bezier curve

Proof. $d := \max_{0 \leq i \leq n} \|b_i - b_{i+1}\|$... maximum length of edges of control polygon

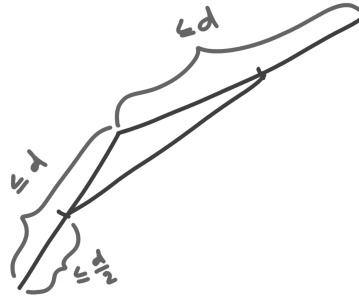


Figure 17: the new edges are at most $\frac{d}{2}$ long

From the image we can deduce that the maximum length of the edges of the new control polygon is at most $\frac{d}{2}$. With induction we get that the control polygon of P_n has the maximum edge length of $\frac{d}{2^n}$.

How far are the points of the control polygon from the bezier curve b ? $\|P_k(i) - b\| \leq (n-1) \frac{d}{2^k} \rightarrow 0$ for $k \rightarrow \infty$. □

Remark 14. Not only does the sequence from the theorem converge pointwise, but it holds that $P_n \rightarrow b$.

Remark 15. Disadvantages of bezier curves are: unintuitive, local changes are not possible as every change in a control point changes the entire curve, points can be far away from the curve.

4 B-Spline-Curves

Definition 11. *B-Spline curves are compositions of bezier curves.*



Figure 18: continuous, tangential continuous and curvature continuous composition

Example 12.

Definition 12 (B-Spline-Curves). $m \in \mathbb{N}$ is the number of control points, $n \in \mathbb{N}$ is the degree of the b spline curve, $T = (t_0, t_1, \dots, t_{m+n+1})$ where $t_i \in \mathbb{R}$, $t_i \leq t_{i+1}$ and $t_i < t_{i+n+1}$ is called knot vector (Knotenvektor).

$$\alpha_i^r(t) := \begin{cases} \frac{t-t_i}{t_{i+r}-t_i} & t_{i+r} - t_i \neq 0 \\ 0 & \text{else} \end{cases}$$

$$N_i^0(t) := \begin{cases} 1 & t \in [t_i, t_{i+1}) \\ 0 & \text{else} \end{cases} \quad N_i^r(t) := \alpha_i^r(t)N_i^{r-1}(t) + (1 - \alpha_{i+1}^r(t))N_{i+1}^{r-1}(t)$$

$N_i^r : \mathbb{R} \rightarrow \mathbb{R}$ are called b-spline basis functions (B-Spline-Basisfunktionen) and are piece-wise polynomial curves.

Theorem 9. Every b-spline curve $s(t)$ can be expressed in the form $s(t) = \sum_{i=0}^m N_i^n(t)c_i$ where $c_0, \dots, c_m \in \mathbb{R}^d$ are the control points and n is the degree of the b-spline curve.

Proof. without proof □

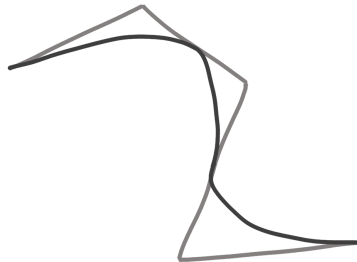


Figure 19: from this set of control points there are different possible quadratic b-spline curves (depending on the chosen knot vector).

Example 13.

Remark 16. Typically a knot vector of a similar form as $T = (0, 0, 0, 0, 1, 2, 3, 4, 5, 5, 5, 5)$ is chosen. 0 often repeated, then counting up, then as many numbers of the same value as 0 in the start.

Example 14. Every line segment in the control polygon is cut into thirds.



Figure 20: cubic b-spline curve

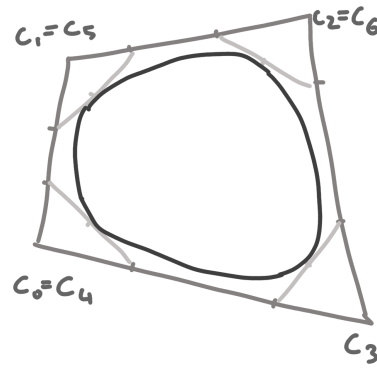


Figure 21: curvature continuous, closed b-spline curve.

Example 15. *closed b-spline curves*

Remark 17. *B-spline curves of degree n with $n + 1$ control points ($\implies m = n$) are bezier curves.*

5 NURBS-Curve

Remark 18. *Non-Uniform-Rational-B-Spline-Curve*

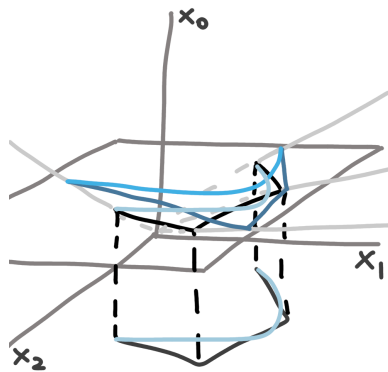


Figure 22: construction of a nurbs curve

Lemma 6 (Construction of a NURBS curve). 1. given are control points $c_0, \dots, c_m \in \mathbb{R}^d$

2. $d_i := (1, c_i) \in \mathbb{R}^{d+1}$ (this is called transforming into homogeneous coordinates (homogenisieren))

3. choose weights $w_i > 0$. $e_i := w_i d_i \in \mathbb{R}^{d+1}$

4. calculate b-spline curve with control points e_i

$$s(t) := \sum_{i=0}^m N_i^n(t) e_i = \sum_{i=0}^m N_i^n(t) w_i (1, c_i) = \left(\sum_{i=0}^m N_i^n(t) w_i, \sum_{i=0}^m N_i^n(t) w_i c_i \right)$$

5. project $s(t)$ back into the $(1 \times \mathbb{R}^d)$ plane

$$\tilde{x}(t) = pr(s(t)) = \left(1, \frac{\sum_{i=0}^m N_i^n(t) w_i c_i}{\sum_{i=0}^m N_i^n(t) w_i} \right)$$

6. dehomogenate

$$x(t) := pr_{\mathbb{R}^d}(\tilde{x}) = \frac{\sum_{i=0}^m N_i^n(t) w_i c_i}{\sum_{i=0}^m N_i^n(t) w_i}$$

$x(t)$ is called the NURBS curve with weights w_0, \dots, w_m

Remark 19. In the special case $w_i = w \in \mathbb{R}^+$ the NURBS curve is a B-spline curve.

Remark 20.

$$\sum_{i=0}^m N_i^n(t) = 1 \quad N_i^n(t) \geq 0$$

Remark 21. The bigger w_i is the more the curve is attracted towards c_i .

Remark 22. NURBS and B-splines are invariant under affine transformations.

Example 16. NURBS-curve over a quadratic bezier curve.

$$b(t) = (1-t)^2 b_0 + 2(1-t)t b_1 + t^2 b_2$$

$$x(t) = \frac{(1-t)^2 w_0 b_0 + 2(1-t)t w_1 b_1 + t^2 w_2 b_2}{(1-t)^2 w_0 + 2(1-t)t w_1 + t^2 w_2}$$

$x(t)$ is a central projection of a parabola, and therefore a conic section (Kegelschnitt).

Example 17. $w_0 = w_1 = w_2$ then $x(t) = b(t)$ which is a parabola.

Example 18. $w_0 = w_2 = 1$, $w := w_1$ How do we have to choose w in order for $x(t)$ to be a different conic section from a parabola?

$$x(t) \text{ is a } \begin{cases} \text{ellipse} \\ \text{parabola} \\ \text{hyperbola} \end{cases} \quad \text{if } x(t) \text{ has } \begin{cases} 0 \\ 1 \\ 2 \end{cases} \text{ points at infinity (Fernpunkt).}$$

Therefore we investigate how often the denominator is 0.

$$(1-t)^2 + 2(1-t)t w + t^2 = 1 - 2t + t^2 + 2tw - 2t^2 w + t^2 = (2-2w)t^2 + (-2+2w)t + 1$$

$$t_{1,2} = \frac{2-2w \pm \sqrt{(2-2w)^2 - 4(2-2w)}}{2(2-2w)}$$

$$(2-2w)^2 - 4(2-2w) = 4 - 8w + 4w^2 - 8 + 8w = 4w^2 - 4 = 4(w^2 - 1)$$

If $w = 1$ we have already seen that the result is a parabola. Otherwise we do not divide by 0. We are interested in what is under the square root.

$$\text{As } 4(w^2 - 1) = 0 \iff w = 1 \text{ we have } \begin{cases} 0 \\ 1 \\ 2 \end{cases} \text{ real roots if } 4(w^2 - 1) \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} .$$

Theorem 10. $x(t)$ is a $\begin{cases} \text{ellipse} \\ \text{parabola} \\ \text{hyperbola} \end{cases}$ if $\begin{cases} w < 1 \\ w = 1 \\ w > 1 \end{cases}$.

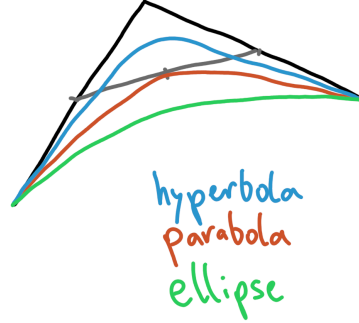


Figure 23: NURBS curve for the different values of w

Example 19. Is it possible to make a circle?

Obviously the distance between b_1 and the two points b_0 and b_2 must be equal.

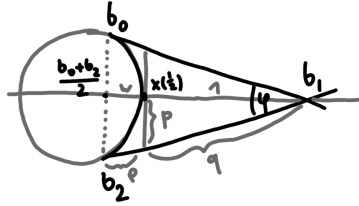


Figure 24: depiction of conditions in order for $x(t)$ to be a circle

A necessary condition is therefore that b_0, b_1, b_2 form an isosceles (gleichschenkelig) triangle.

$$\begin{aligned} x\left(\frac{1}{2}\right) &= \frac{(1 - \frac{1}{2})^2 b_0 + 2(1 - \frac{1}{2})\frac{1}{2} w b_1 + (\frac{1}{2})^2 b_2}{(1 - \frac{1}{2})^2 + 2(1 - \frac{1}{2})\frac{1}{2} w + (\frac{1}{2})^2} = \frac{\frac{1}{4} b_0 + \frac{1}{2} b_1 + \frac{1}{4} b_2}{\frac{1}{4} + \frac{1}{2} w + \frac{1}{4}} = \frac{b_0 + 2w b_1 + b_2}{2 + 2w} = \\ &= \frac{1}{2 + 2w} (b_0 + b_2 + 2w b_1) = \frac{1}{1 + w} \left(\frac{b_0 + b_2}{2} + w b_1 \right) = \frac{1}{1 + w} \frac{b_0 + b_2}{2} + \frac{w}{1 + w} b_1 \end{aligned}$$

which is a affine combination of $\frac{b_0 + b_2}{2}$ and b_1 .

$$\begin{aligned} c &= (1 - \lambda)a + \lambda b \\ \frac{x}{y} &= \frac{c - 1}{b - c} = \frac{(1 - \lambda)a + \lambda b - a}{b - (1 - \lambda)a - \lambda b} = \frac{\lambda(b - a)}{(1 - \lambda)(b - a)} = \frac{\lambda}{1 - \lambda} \end{aligned}$$

For $(1 - \lambda)a + \lambda b = (\frac{1}{1 - w})\frac{b_0 + b_2}{2} + \frac{w}{1 + w} b_1$ we get $\frac{x}{y} = \frac{\lambda}{1 - \lambda} = \frac{\frac{w}{1 + w}}{\frac{1}{1 - w}} = w$.

With the intercept theorem (Strahlensatz) and law of sines (trigonometrischer Satz) we get $\sin(\frac{\phi}{2}) = \frac{p}{q} = w$.

$\implies w = \sin(\frac{\phi}{2})$.

Theorem 11. $x(t)$ is a circle, if b_0, b_1, b_2 is an isosceles triangle and $w_0 = w_2 = 1$, $w_1 = \sin(\frac{\phi}{2})$ where $\phi = \angle(b_0 - b_1, b_2, b_1)$.

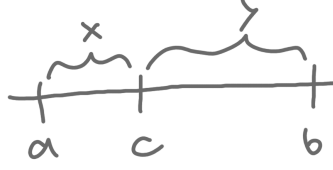


Figure 25: ratio between distances between points on a line

Remark 23. How can we calculate the angle?

$$\cos(\phi) = \frac{\langle b_0 - b_1, b_2 - b_1 \rangle}{\|b_0 - b_1\| \cdot \|b_2 - b_1\|}$$

With further calculations we get

$$w = \sin\left(\frac{\phi}{2}\right) = \frac{\|(b_0 - b_1) \times (\frac{b_0 - b_2}{2} - b_1)\|}{\|b_0 - b_1\| \cdot \|\frac{b_0 - b_2}{2} - b_1\|}$$

Remark 24. Area of a triangle

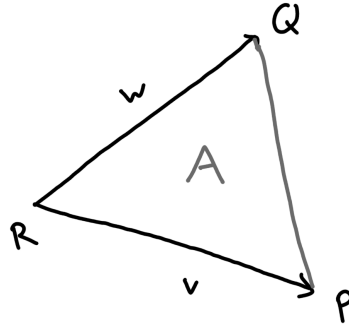


Figure 26: we want to calculate the area of the triangle between two vectors

$$A = \frac{|\det(v, w)|}{2} = \frac{|\det(P - R, Q - R)|}{2} = \frac{\left| \det \begin{pmatrix} 1 & R \\ 0 & P - R \\ 0 & Q - R \end{pmatrix} \right|}{2} = \frac{\left| \det \begin{pmatrix} 1 & R \\ 1 & P \\ 1 & Q \end{pmatrix} \right|}{2}$$

6 Free form surfaces

Definition 13. given a control mesh $b_{00}, b_{01}, \dots, b_{mn} \in \mathbb{R}^d$, $m, n \in \mathbb{N}$ and two curve schemes $g(s) = \sum_{i=0}^m D_i(s)p_i$ and $h(t) = \sum_{j=0}^n E_j(t)q_j$.
 $f(s, t) := \sum_{i=0}^m \sum_{j=0}^n D_i(s)E_j(t)b_{ij}$ is called free form surface (also called tensor product surface) (Freiformfläche, Tensorproduktfläche).

Example 20. $D_i(s) = B_i^m(s)$ and $E_j(t) = B_j^n(t)$... bernstein polynomials then $f(s, t)$ is a bezier surface from degree (m, n) .

Example 21. $D_i(s) = N_i^m(s)$ and $E_j(t) = N_j^n(t)$... b-spline base functions then $f(s, t)$ is a b-spline surface from degree (m, n) .

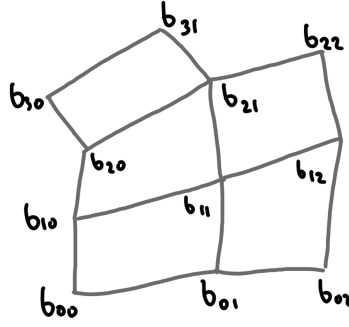


Figure 27: control mesh

Lemma 7.

$$f(s, t) = \sum_i \sum_j D_i(s) E_j(t) b_{ij} = \sum_i D_i(s) \underbrace{\sum_j E_j(t) b_{ij}}_{=: h_i(t)} = \sum_i D_i(s) h_i(t)$$

$$f(s, t) = \dots = \sum_j E_j(t) \underbrace{\sum_i D_i(s) b_{ij}}_{=: g_j(s)} = \sum_j E_j(t) g_j(s)$$

For some t_0 $f(s, t_0)$ is called a s -parameter line.

$f(s, t_0) = \sum_i D_i(s) \underbrace{h_i(t_0)}_{p_i}$. A s -parameter line is a curve of the type $g(s)$. The same is true for t -parameter line of type $h(t)$.

Example 22. bezier mesh of grade (2, 3)

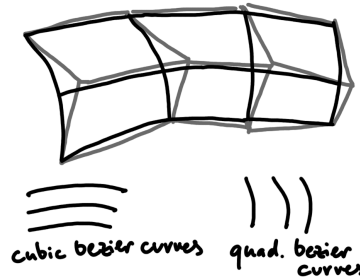


Figure 28: bezier mesh

Algorithm 2 (de casteljau for meshes).

All lemmas and theorems are very similar to free form curves. Therefore we only show selected theorems.

Theorem 12. If D_i, E_i are affine invariant, then f is affine invariant.

Proof. Let α be affine invariant. We have to show $\alpha(f(s, t)) = \sum_i \sum_j D_i(s) E_j(t) \alpha(b_{i,j})$.

$$\alpha(f(s, t)) = \alpha\left(\sum_{i=0}^m \sum_{j=0}^n D_i(s) E_j(t) b_{ij}\right) = \alpha\left(\sum_{i=0}^m D_i(s) \sum_{j=0}^n E_j(t) b_{ij}\right) = \alpha\left(\sum_{i=0}^m D_i(s) \underbrace{\sum_{j=0}^n E_j(t) b_{ij}}_{h_{ij}}\right) =$$

$$\alpha\left(\sum_{i=0}^m D_i(s) \alpha\left(\sum_{j=0}^n E_j(t) b_{ij}\right)\right) = \sum_{i=0}^m \sum_{j=0}^n D_i(s) E_j(t) \alpha(b_{i,j})$$

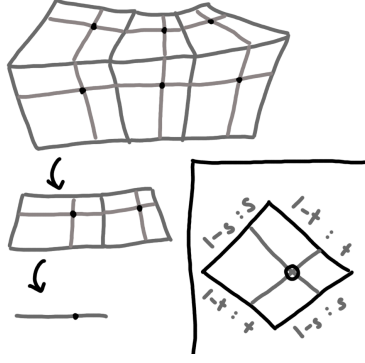


Figure 29: cateljou algorithm for meshes

□

Other theorems such as convex hull, end-point-interpolating are analogous.

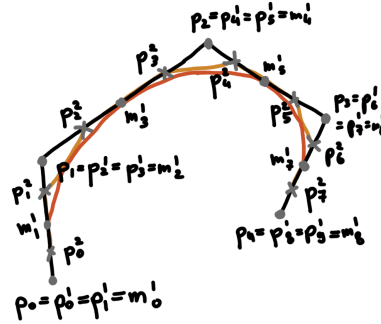
7 Subdivision algorithms

7.1 Subdivision for curves

Algorithm 3 (Chaikin). *given some polygon $p_i, p_{i+1}, \dots \in \mathbb{R}^d$*

1. *copy every vertex $p_i \rightarrow p_i^1$*
2. *calculate average of two neighboring p_i^1 s $m_i^1 = \frac{1}{2}(p_i^1 + p_{i+1}^1)$*
3. *average the m_i^1 s $p_i^2 = \frac{1}{2}(m_i^1 + m_{i+1}^1)$*

The resulting polygon looks similar to the original one, but the corners are rounded. Continue by repeating the procedure with the new polygon as input.



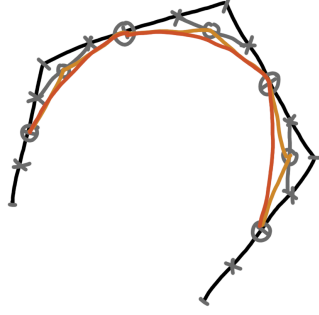


Figure 31: Lane Riesenfeld algorithm with $n = 3$ for five given points.

Algorithm 5 (Four-Point-Scheme).

$$p_i^1 := -\frac{1}{16}p_{i-1} + \frac{9}{16}p_i + \frac{9}{16}p_{i+1} - \frac{1}{16}p_{i+2}$$

this is an affine combination. Inductive p_i^2, \dots

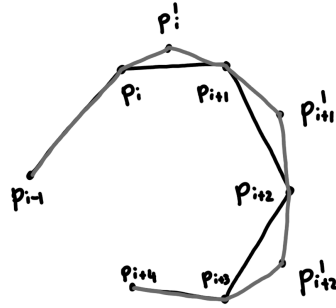


Figure 32: Four Point Scheme for six given points.

Theorem 15. The curve resulting from the four point scheme converge to a C^1 curve. Four Point Scheme is:

- a interpolating
- is affine invariant.

Remark 27. The weights of the four point scheme can be chosen differently as well:

$$p_i^1 := -wp_{i-1} + \left(\frac{1}{2} + w\right)p_i + \left(\frac{1}{2}w\right)p_{i+1} - wp_{i+2}$$

Above we choose $w = \frac{1}{16}$. This is always interpolating, but only for $w \in (0, \frac{\sqrt{5}-1}{8})$ the resulting curve will be C^1 .

Remark 28. All of the above schemes preserve sub spaces and therefore have the property of linear precision (lineare Präzision).

They do not have "circular precision", meaning that if all points lie on a sphere then the curve can leave the sphere.

7.2 Subdivision for meshes

Definition 14. The degree (Valenz) of a vertex is the number of edges connected to the vertex.

Algorithm 6 (Loop (triangle meshes)).

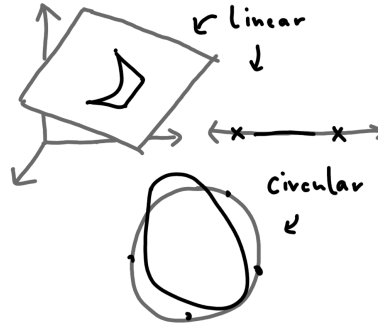


Figure 33: Example for linear precision, but not circular precision.

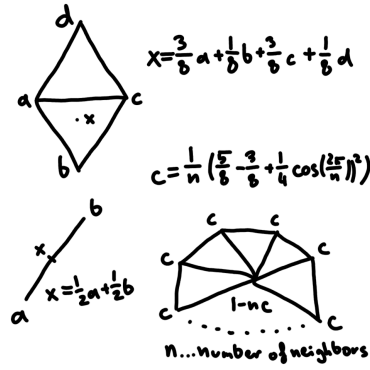


Figure 34: Loop subdivision.

Theorem 16. *Loop subdivision*

- *is approximating*
- *converges to a C^2 surface for generic $n = 6$ vertices and to a C^1 surface otherwise*
- *is face splitting (faces are subdivided)*

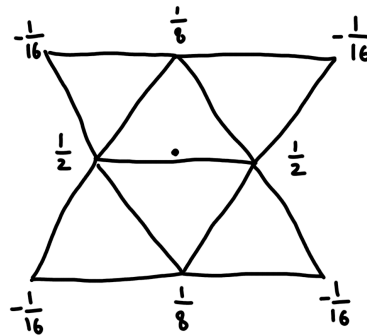


Figure 35: Modified Butterfly subdivision.

Algorithm 7 (Modified-Butterfly-Scheme (triangle meshes)). *As in the image we generate new vertices from edges. For non-generic $k \neq 6$ we specify*

$$s_j = \frac{1}{k} \left(\frac{1}{4} + \cos\left(\frac{2j\pi}{k}\right) + \frac{1}{2} \cos\left(\frac{4j\pi}{k}\right) \right)$$

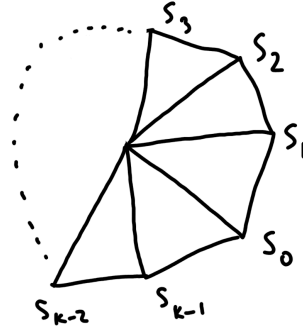


Figure 36: Special case in Modified Butterfly subdivision.

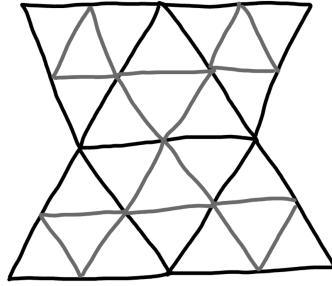


Figure 37: Result of Modified Butterfly subdivision.

Theorem 17. *The Modified Butterfly Scheme:*

- *is interpolating*
- *converges to a C^1 surface*
- *is face-splitting*

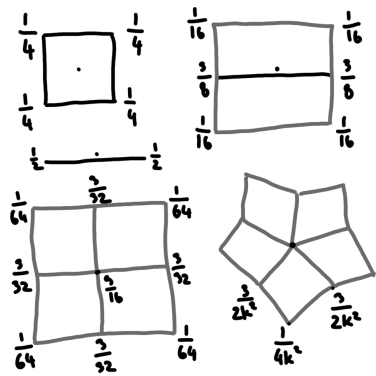


Figure 38: Catmul Clark subdivision.

Algorithm 8 (Catmull-Clark (square meshes)).

Theorem 18. *Catmull Clark is:*

- *approximating*
- *converges to a C^2 surface for generic vertices*
- *is face-splitting*

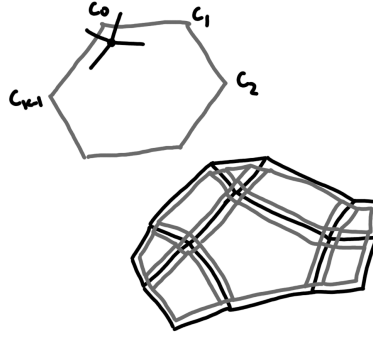


Figure 39: Doo Sabin subdivision.

Algorithm 9 (Doo-Sabin).

$$c_j = \frac{(3 + 2 \cos(\frac{2\pi j}{k}))}{4k}, j > 0 \qquad c_0 = \frac{1}{4} + \frac{5}{4k}$$

Theorem 19. *Doo-Sabin is:*

- *approximating*
- *converges to C^1 surface*
- *is vertex-splitting.*

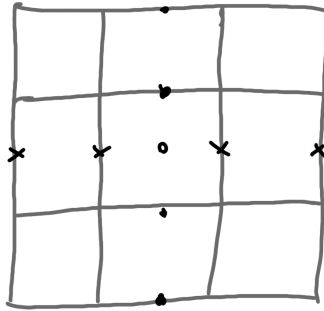


Figure 40: Kobbelt subdivision.

Algorithm 10 (Kobbelt-Scheme (square meshes)). *Apply four-point-scheme for vertical and horizontal vertices. Then apply four point scheme on resulting points. The "horizontal" result and the "vertical" result are the same.*

Theorem 20. *The Kobbelt scheme is:*

- *interpolating*
- *converges to a C^1 surface*
- *is face-splitting.*

Algorithm 11 (Half-Edge-Data-Structure). *given: mesh, that represents a orientable surface.*

A mesh is a collection of lists. These lists include vertices v_i , edges e_i , faces f_i and half-edges h_i .

*Every vertex v is assigned a half-edge $v \rightarrow h$. Every face, every edge is assigned a half-edge. Every half-edge is assigned the opposing (*flip*), next (*next*), previous (*prev*) half-edge, the corresponding face (*f*), vertex (*v*) and edge (*e*).*

Remark 29. *In python use the package `geopy`. Find it via `tiss/Lehrunterlagen`. The documentation is at `www.geometrie.tuwien.ac.at/kilian/docs/geopy`.*

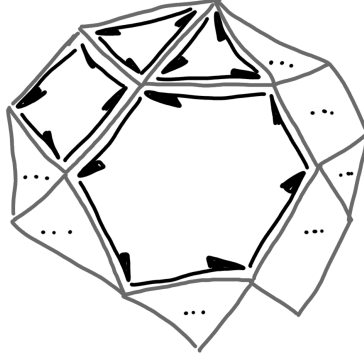


Figure 41: Halfedge datastructure.

8 Conform Pattern Matching

(Musterübertragung)

8.1 Euclidean Geometry

Transformations: Translation, Rotation, Reflection

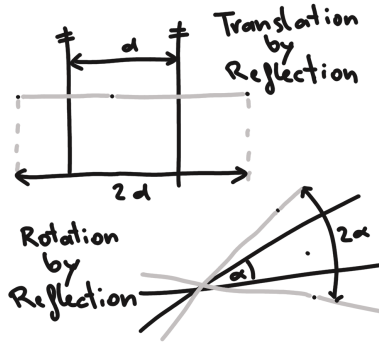


Figure 42: Representation of euclidean transformations as reflections.

From the figure it follows that euclidean transformations can be represented as reflections along hyper-planes.

8.2 Möbius Geometry

Remark 30. This section is closely following the paper *Conformal equivalence of triangle meshes* by Boris Springborn, Peter Schröder and Ulrich Pinkall from the year 2008.

Definition 15. Möbius transformations are compositions of reflections along hyper-spheres.

Definition 16 (reflection along sphere).

$$\tilde{\omega} := \frac{r^2}{\|\omega\|^2} \omega$$

Remark 31. Reflecting twice yields the identity.

$$(\tilde{\tilde{\omega}}) = \frac{r^2}{\|\tilde{\omega}\|^2} \tilde{\omega} = \frac{r^2}{\|\frac{r^2}{\|\omega\|^2} \omega\|^2} \frac{r^2}{\|\omega\|^2} \omega = \frac{r^2}{r^4 \|\omega\|^2} \|\omega\|^2 r^2 \omega = \omega$$

Reflecting ω on a sphere with center c and radius r .

$$\tilde{\omega} = \frac{r^2(\omega - c)}{\|\omega - c\|^2} + c$$

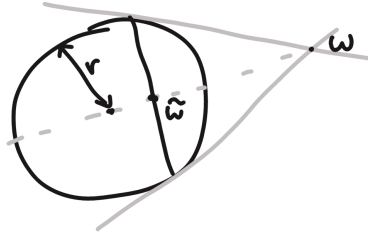


Figure 43: Reflection of point along a circle.

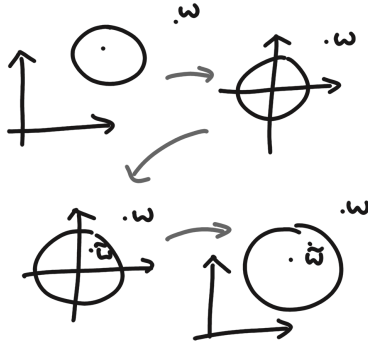


Figure 44: Reflection of point along a circle with different origin.

Remark 32. *Special case: reflection of point c : $\tilde{c} := \infty$ and $\tilde{\infty} := c$*

Remark 33. *We can identify \mathbb{R}^2 with \mathbb{C} . Then we get*

$$\tilde{\omega} = \frac{r^2(\omega - c)}{(\omega - c)(\bar{\omega} - \bar{c})} + c = \frac{r^2}{\bar{\omega} - \bar{c}} + c$$

In complex analysis möbius transformations are of the form $m(z) = (az + b)/(cz + d)$ with $a, b, c, d \in \mathbb{C}, ad - bc \neq 0$. Anti-möbius transformations are $m(z) = (a\bar{z} + b)/(c\bar{z} + d)$.

$$m(z) = \frac{az + b}{cz + d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az + b \\ cz + d \end{pmatrix}$$

Embedding $\mathbb{C} \rightarrow \mathbb{P}(\mathbb{C}) := \{u \subseteq \mathbb{C}^2 \mid \dim u = 1\}$ with $z \in \mathbb{C} \mapsto [(z, 1)]$ and $[(z_0, z_1)] \mapsto \frac{z_0}{z_1}, z_1 \neq 0$.

Then möbius transformations in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ can be represented as a composition of

- *Rotation: $z \mapsto pz$, with $|p| = 1$*
- *Scaling: $z \mapsto Az$, with $A \in \mathbb{R}$*
- *Inversion: $z \mapsto \frac{1}{z}$*
- *Translation: $z \mapsto z + B$, with $B \in \mathbb{C}$*

Proof on Wikipedia.

Definition 17 (Cross-Ratio (Doppelverhältnis)). *For $a, b, c, d \in \mathbb{C} \dots$ pairwise different we define $DV(a, b, c, d) := \frac{(a-b)(c-d)}{(b-c)(d-a)} \in \mathbb{C}$ as the cross-ratio.*

Example 23. $a, b, c, d \in \mathbb{C}$

$$DV(a, b, c, d) = \frac{(a-b)(c-d)}{(c-b)(a-d)} = \frac{re^{i\phi} te^{i\xi}}{se^{i\psi} ue^{i\nu}} = \frac{r}{s} e^{i(\phi-\psi)} \frac{t}{u} e^{i(\xi-\nu)} = \underbrace{\frac{rt}{su}}_{\in \mathbb{R}} e^{i(\alpha+\beta)}$$

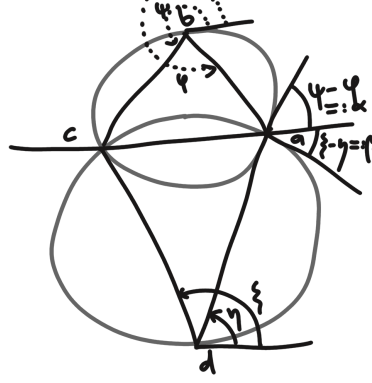


Figure 45: DV as the angle.

The argument of DV is the angle of circumscribed circle of abc and cda. As seen in the image.

Remark 34. $DV(a, b, c, d) \in \mathbb{R}$, if and only if they share the circumscribed circle.

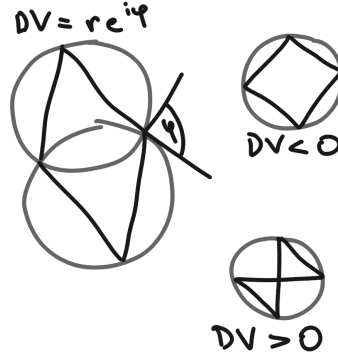


Figure 46: DV and the corresponding angle, as well as examples where the circumscribed circle is shared.

Theorem 21. The DV is invariant under all möbius transformations.

Proof. • Scaling:

$$DV(Aa, Ab, Ac, Ad) = \frac{(Aa - Ab)(Ac - Ad)}{(Ac - Ab)(Aa - Ad)} = \frac{A(a-b)(c-d)}{A(c-b)(a-d)} = DV(a, b, c, d)$$

• Translation:

$$DV(a+B, b+B, c+B, d+B) = \frac{((a+B) - (b+B))((c+B) - (d+B))}{((c+B) - (b+B))((a+B) - (d+B))} = DV(a, b, c, d)$$

• Rotation: $p = e^{i\phi}$, $z = re^{i\psi}$ then $pz = re^{i(\phi+\psi)}$

• Inversion:

$$DV\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right) = \frac{\left(\frac{1}{a} - \frac{1}{b}\right)\left(\frac{1}{c} - \frac{1}{d}\right)abcd}{\left(\frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{d} - \frac{1}{a}\right)abcd} = \frac{(b-a)(d-c)}{(c-b)(a-d)} = DV(a, b, c, d)$$

□

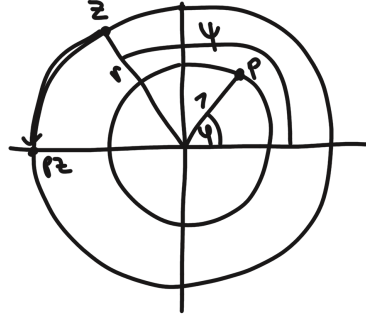


Figure 47: DV is rotation invariant.

Theorem 22. Möbius transformations are circle-preserving (kreistreu).

Proof. $a, b, c, z \in K$ where K is any circle. Then $DV(a, b, c, z) \in \mathbb{R}$. As $DV(m(a), m(b), m(c), m(z)) = DV(a, b, c, z) \in \mathbb{R}$ it follows that $m(a), m(b), m(c), m(z)$ lie on a circle. \square

Theorem 23. Möbius transformations are conformal, which means preserving angles (konform, d.h. winkeltreu).

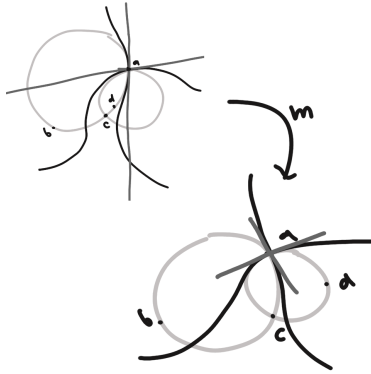


Figure 48: Corresponding circles are transformed into intersecting circles.

Proof. Consider tangential circles as in the figure. They will be transformed into intersecting circles, as they already are intersecting.

$DV(a, b, c, d) = re^{i\phi} \implies \phi$ is the angle. $\implies DV(m(a), m(b), m(c), m(d)) = re^{i\phi}$. Therefore the angle is preserved. \square

Remark 35. Holomorphic (holomorphe = komplex diffbare) functions preserve angles, but do not preserve circles.

Example 24. $z \mapsto \sin(z^2)$ is holomorphic, but does not preserve circles.

Remark 36. Möbius transformations are the only conformal, bijective functions from open subsets of \mathbb{R}^n , where $n \geq 3$. (Theorem of Liouville).

Remark 37. Let $a, b, c, d \in \text{Im}\mathbb{H}$ (Quaternions) $\cong \mathbb{R}^3$.

$DV(a, b, c, d) = (a - b)(b - c)^{-1}(c - d)(d - a)^{-1}$ as \mathbb{H} is not commutative (Schiefkörper).

It still holds that $DV(a, b, c, d) \in \mathbb{R} \iff a, b, c, d$ lie on a common circle.

Definition 18. $f, \tilde{f} \dots$ bijective, $\phi: f(U) \rightarrow \tilde{f}(U), x \mapsto \tilde{f} \circ f^{-1}(x)$.

ϕ is described over f, \tilde{f} by the same parameter (durch den gleichen Parameter beschrieben).

We ask our-self when ϕ is isometric (isometrisch = längentreu).

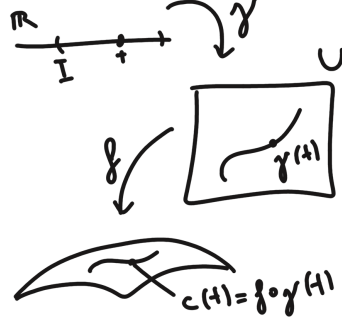


Figure 49: Definition preserving length.

Definition 19 (preserving length). ϕ is preserving length $\iff \forall$ curves c in $f(U) : \phi(c)$ is as long as c
 $c : I \rightarrow \mathbb{R}^3 \dots$ curve, then the length of c is defined by $L(c) := \int_I \|\dot{c}(t)\| dt$

Lemma 8. ϕ is isometric $\iff \forall \gamma : I \rightarrow U : L(f \circ \gamma) = L(\phi(f \circ \gamma))$.

Proof.

$$\begin{aligned} \|\dot{c}(t)\|^2 &= \left\| \frac{dc(t)}{dt} \right\|^2 = \|fu\dot{\gamma}_1 + fv\dot{\gamma}_2\|^2 = \langle \dot{\gamma}_1 fu + \dot{\gamma}_2 fv + \dot{\gamma}_1 fu + \dot{\gamma}_2 fv \rangle = \\ &= \underbrace{\dot{\gamma}_1^2 \langle fu, fu \rangle}_{=E} + 2\dot{\gamma}_1 \dot{\gamma}_2 \underbrace{\langle fu, fv \rangle}_{=F} + \dot{\gamma}_2^2 \underbrace{\langle fv, fv \rangle}_{=G} = (\dot{\gamma}_1, \dot{\gamma}_2) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} = \dot{\gamma}^T \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{=I} \dot{\gamma} \end{aligned}$$

I is matrix of first fundamental form (1. Fundamentalform).

$\implies \|\dot{c}(t)\|^2 = \dot{\gamma}^T I \dot{\gamma}$. Therefore $L(c) = \int_a^b \sqrt{\dot{\gamma}^T I \dot{\gamma}} dt$

ϕ is preserving length $\iff \forall \gamma : L(f \circ \gamma) = L(\phi(f \circ \gamma))$

$\gamma : (a, b) \rightarrow U, t \mapsto q + (t_0 + t, 0)^T$.

$$\begin{aligned} L(f \circ \gamma) &= \int_a^b \sqrt{(1, 0) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} dt = \int_a^b \sqrt{E} dt \\ L(\phi(f \circ \gamma)) &= \int_a^b \sqrt{\tilde{E}} dt \\ \tilde{E} &= \langle \tilde{f}u, \tilde{f}u \rangle, \tilde{F} = \langle \tilde{f}u, \tilde{f}v \rangle, \tilde{G} = \langle \tilde{f}v, \tilde{f}v \rangle \end{aligned}$$

ϕ is preserving length $\iff \forall a, b : \int_a^b \sqrt{E} dt = \int_a^b \sqrt{\tilde{E}} dt \iff \sqrt{E} = \sqrt{\tilde{E}} \iff E = \tilde{E} (\iff \langle fu, fu \rangle = \langle \tilde{f}u, \tilde{f}u \rangle)$.

Analogous $\gamma(t) = q + (0, t_0 + t)^T \implies G = \tilde{G}$.

$$\begin{aligned} \gamma(t) &= q + \begin{pmatrix} t_0 + t \\ t_1 + t \end{pmatrix} \implies \sqrt{E + 2F + G} = \sqrt{\tilde{E} + 2\tilde{F} + \tilde{G}} \\ E + 2F + G &= \|fu\|^2 + 2\langle fu, fv \rangle + \|fv\|^2 \geq \|fu\|^2 - 2\|fu\| \cdot \|fv\| + \|fv\|^2 = (\|fu\| - \|fv\|)^2 \\ &\implies F = \tilde{F} \end{aligned}$$

$\implies \phi$ is preserving length $\implies I = \tilde{I}$ and $I = \tilde{I} \implies \phi$ is preserving length. □

Theorem 24. ϕ is preserving area $\iff \det I = \det \tilde{I}$.

Proof. without proof. □

Corollary 6. preserving length $\implies I = \tilde{I} \implies \det I = \det \tilde{I} \implies$ preserving area.

Theorem 25. ϕ is conformal (preserving angles) $\iff \exists \lambda : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\} : I = \lambda^2 \tilde{I}$.

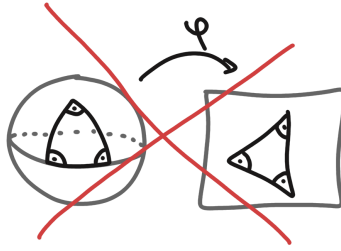


Figure 50: There exists no such length preserving ϕ .

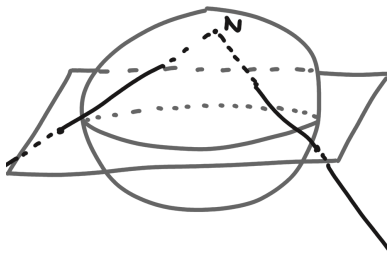


Figure 51: Stereographic projections are preserving angles.

Remark 38. *preserving length \iff preserving angles and preserving area.
Stereographic projection are preserving angles.*

Definition 20 (Triangle mesh). *Triangle mesh = polyhedral area (local like open subset of \mathbb{R}^2) and all areas are triangles.*

$T = (V, E, F)$ where $V \dots$ vertices, $E \dots$ edges, $F \dots$ faces

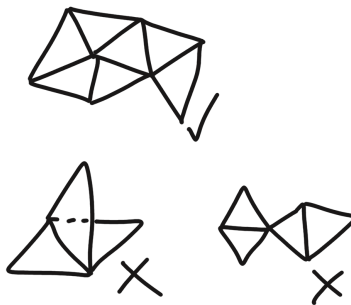


Figure 52: Triangles mesh (top) and ones consisting of triangles that are not triangle meshes (bottom).

Remark 39. \forall polyhedron (Polyeder) that are homeomorph (homöomorph): $F - E + V = 2$.

Definition 21. A function $L : E \rightarrow \mathbb{R}_{\geq 0}$ for which $\forall f \in F : f$ satisfies the triangle equations, is called discrete metric (diskrete Metrik) auf T .
The triangle equations are given by:

$$\begin{aligned} l_{ij} &= l(v_i, v_j) \dots \text{length of the edge between } v_i \text{ and } v_j \\ l_{ij} &\leq l_{jk} + l_{ki} \\ l_{jk} &\leq l_{ki} + l_{ij} \\ l_{ki} &\leq l_{ij} + l_{jk} \\ l_{ij} &= l_{ji} \forall i, j, k \end{aligned}$$

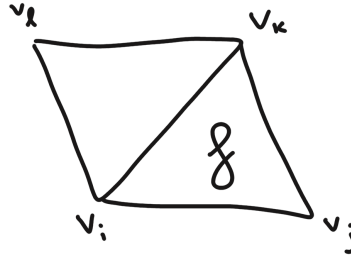


Figure 53: Vertices associated to face f in definition of discrete metric.

Example 25. The "true" lengths $l_{ij} = \|v_i - v_j\|$ of the edges form a discrete metric.

Definition 22. Two combinatorial equivalent triangle meshes T, \tilde{T} are called discrete conform equivalent (diskret konform äquivalent), if $\exists \lambda : E \rightarrow \mathbb{R}_{\geq 0}$ with $\tilde{l}_{ij} = \lambda_i \lambda_j l_{ij} \forall (i, j) \in E$.

Lemma 9. Let m be a möbius transformation $m : \mathbb{R}^n \cup \{\infty\} \rightarrow \mathbb{R}^n \cup \{\infty\}$.
 $\implies \exists \rho : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} : \|m(x) - m(y)\| = \rho(x)\rho(y)\|x - y\|$.

Proof. without proof □

Definition 23. length cross-ratio (Längendoppelverhältnis) $LDV(v_i, v_j, v_k, v_m) := \frac{l_{ij}l_{km}}{l_{jk}l_{mi}}$

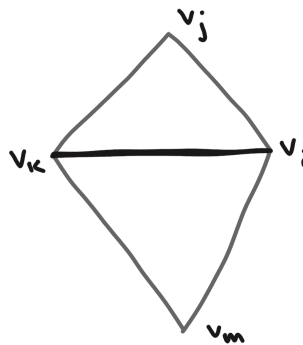


Figure 54: Definition of length cross ratio.

Remark 40. $a, b, c, d \in \mathbb{C}$ then $LDV(a, b, c, d) = |DV(a, b, c, d)|$

Theorem 26. (T, l) and (\tilde{T}, \tilde{l}) are discrete conform equivalent $\iff \forall i, k \in E : LDV(v_i, v_j, v_k, v_m) = LDV(\tilde{v}_i, \tilde{v}_j, \tilde{v}_k, \tilde{v}_m)$.

Theorem 27. Let T be a triangle mesh with metric $l_{ij} := \|v_i - v_j\|$. Let m be a möbius transformation. Then T and \tilde{T} are conform equivalent.

Remark 41. In the above theorem we means that when the edges are möbius transformed and then connected with straight edges to get a triangle mesh.

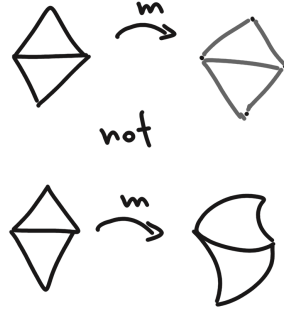


Figure 55: Möbius transformation of the triangle mesh.

Proof. use the above lemma. □

Remark 42. In a plane möbius transformations are uniquely given by three points.

Definition 24 ((Milnov's) Lobachevsky-Function).

$$\Pi(x) := \int_0^x \log |2 \sin(t)| dt$$

Cyrillic L looks like Π .

Lemma 10. Properties of $\Pi(x)$:

- $\Pi(0) = 0$
- $\Pi(x + \pi) = \Pi(x) \dots \pi$ -periodic
- $\Pi(k\pi) = 0$
- $\Pi(-x) = -\Pi(x)$

Definition 25.

$$M := \{(x, y, z) \in \mathbb{R}^3 | e^x, e^y, e^z \text{ are the lengths of sides of a triangle}\} = \\ \{(x, y, z) \in \mathbb{R}^3 | e^x \leq e^y + e^z, e^y \leq e^x + e^z, e^z \leq e^x + e^y\}$$

$$f(x, y, z) := \alpha x + \beta y + \gamma z + \Pi(\alpha) + \Pi(\beta) + \Pi(\gamma)$$

where $\alpha(x, y, z), \beta(x, y, z), \gamma(x, y, z)$ are the angles of the triangle. $x, y, z \in M$.

Lemma 11.

$$\frac{\delta f}{\delta x} = \alpha$$

$$\frac{\delta f}{\delta y} = \beta$$

$$\frac{\delta f}{\delta z} = \gamma$$

Proof.

$$\frac{\delta f}{\delta x} = \frac{\delta \alpha}{\delta x} x + \alpha + \frac{\delta \beta}{\delta x} y + \frac{\delta \gamma}{\delta x} z + (-\log |2 \sin(\alpha)|) \frac{\delta \alpha}{\delta x} + (-\log |2 \sin(\beta)|) \frac{\delta \beta}{\delta x} + (-\log(|2 \sin \gamma|)) \frac{\delta \gamma}{\delta x} = \\ \alpha + \frac{\delta \alpha}{\delta x} (x - \log(|2 \sin \alpha|)) + \frac{\delta \beta}{\delta x} (y - \log(|2 \sin \beta|)) + \frac{\delta \gamma}{\delta x} (z - \log(|2 \sin \gamma|)) = \dots$$

We do a short calculation on the side:

$$\sin(\alpha) = \frac{e^x}{2R} \implies 2R \sin(\alpha) = e^x \implies R = \frac{e^x}{|2 \sin(\alpha)|} \implies \log R = x - \log |2 \sin(\alpha)|$$

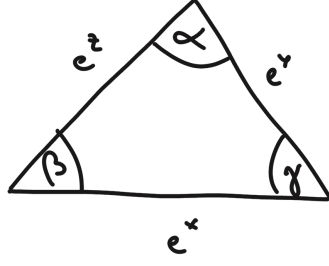


Figure 56: Definition of M and corresponding angles.

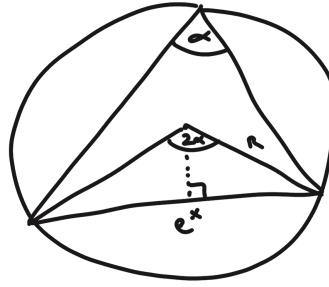


Figure 57: Figure for side calculation.

Which gives us

$$\dots = \alpha + \frac{\delta\alpha}{\delta x} \log R + \frac{\delta\beta}{\delta x} \log R + \frac{\delta\gamma}{\delta x} \log R = \alpha + \log R \left(\frac{\delta(\alpha + \beta + \gamma)}{\delta x} \right) = \alpha + \log R \underbrace{\left(\frac{\delta\Pi}{\delta x} \right)}_{=0} = \alpha$$

□

Remark 43. Discrete conform equivalent $\tilde{l}_{ij} = \lambda_i \lambda_j l_{ij}$ in different wording:

$\exists \lambda_i > 0 \implies \exists u_i : \lambda_i = e^{\frac{u_i}{2}}$ with $u_i \in \mathbb{R}$.

$\gamma_{ij} := \log l_{ij}$, $\tilde{\gamma}_{ij} := \log \tilde{l}_{ij}$ gives us

$\log \tilde{l}_{ij} = \log \lambda_i + \log \lambda_j + \log l_{ij} \iff \tilde{\lambda}_{ij} = \frac{u_i}{2} + \frac{u_j}{2} + \gamma_{ij}$

which is an additive condition for conformal equivalency.

Definition 26 (Interior angle (Innenwinkel) at the position i). α_{jk}^i

Sum of interior angles $\sum_{jk \in \text{star}(i)} \alpha_{jk}^i$ where $\text{star}(i)$ is the vertex star around the edge i .

Goal: Given a triangle mesh T , $\Phi_i \in \mathbb{R}_{>0} \forall i \in V$. We want a triangle mesh \tilde{T} , which is conform equivalent to T with sum of interior angles equal to $\Phi_i \forall i \in V$.

Remark 44. If such a \tilde{T} exists and $\Phi_i = 2\pi \forall i$... interior vertex, then \tilde{T} can be thought of as in a plane.

Let $u \in \mathbb{R}^{\#V}$ with $\tilde{l}_{ij} = e^{u_i/2} e^{u_j/2} l_{ij} = e^{(u_i+u_j)/2} l_{ij}$ are lengths of a triangle mesh.

$$E(u) := \sum_{i,j,k \in F} \left[2f \left(\frac{\tilde{\gamma}_{ij}}{2}, \frac{\tilde{\gamma}_{jk}}{2}, \frac{\tilde{\gamma}_{ki}}{2} \right) - \frac{\pi}{2} (\tilde{\gamma}_{ij} + \tilde{\gamma}_{jk} + \tilde{\gamma}_{ki}) \right] + \sum_{i \in V} u_i \Phi_i$$

$$\tilde{\gamma}_{ij} = \frac{u_i + u_j}{2} + \gamma_{ij} = \frac{u_i}{2} + \frac{u_j}{2} + \log l_{ij}$$

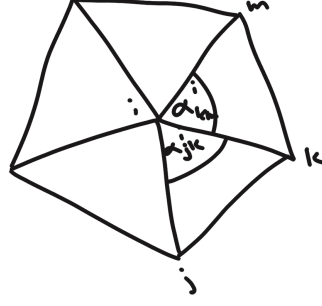


Figure 58: Interior angle definition.

Lemma 12. *Properties of $E(u)$:*

-

$$\frac{\delta E(u)}{\delta u_i} = \Phi_i - \sum_{jk \in \text{star}(i)} \alpha_{jk}^i$$

Therefore the critical points are those interesting us.

- *E can be extended as a convex function onto $\mathbb{R}^{\#V}$*

Theorem 28. *The critical points of E describe the triangle mesh \tilde{T} . This means \tilde{T} is conform equivalent to T and $\forall i \in V : \Phi_i = \sum_{jk \in \text{star}(i)} \alpha_{jk}^i$. As E is convex there exists such a critical point.*

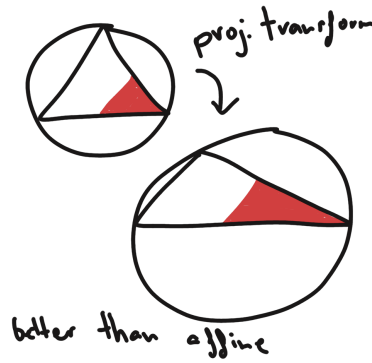


Figure 59: Using projective transformations provides better results than affine transformations.