Short CAGD

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I. General

Definition 1 (Combinations).

Linear Combination	$\sum_{i=1}^{n} \lambda_i v_i$	$v_i \in \mathbb{R}^d, \lambda_i \in \mathbb{R}$
Affine Combination	$\sum_{i=1}^{n} \lambda_i v_i$	above and $\sum_{i=1}^{n} \lambda_i = 1$
Convex Combination	$\sum_{i=1}^{n} \lambda_i v_i$	above and $\forall i : \lambda_i \geq 0$

Definition 2 (Affine Transformation).

$$\alpha : \mathbb{R}^n \to \mathbb{R}^m \textit{ with}$$
$$\exists l : \mathbb{R}^n \to \mathbb{R}^m \textit{ ... linear } \exists v \in \mathbb{R}^m : \alpha(x) = l(x) + v$$

Definition 3 (Convex Set, Convex Hull).

$$M ... convex \iff \forall x, y \in M \forall t \in [0,1] : tx + (1-t)y \in M$$
$$conv(M) := \{\sum_{i=0}^{n} \lambda_i v_i : v_i \in M, \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1, n \in \mathbb{N}\}$$

Definition 4 (Described by Same Parameter). *ϕ is described over f, $\tilde{f}$ by the same parameter, iff $f, \tilde{f}$ ... bij.*

$$\phi : f(U) \to \tilde{f}(U), x \mapsto \tilde{f} \circ f^{-1}(x)$$

II. Parameterized Curves

Definition 5 (Tangential Vector, Velocity).

$$tangent := \dot{c}(t) \qquad vel := ||\dot{c}(t)||$$

Definition 6 (Regular, Singular).

$$regular \iff \dot{c}(t) \neq 0 \qquad singular \iff \dot{c}(t) = 0$$

Definition 7 (Curvature, Torsion).

$$\kappa(t) := \frac{\det(\dot{c}(t), \ddot{c}(t))}{||\dot{c}(t)||^3} \qquad \textit{for } c : I \to \mathbb{R}^2$$
$$\kappa(t) := \frac{||\dot{c}(t) \times \ddot{c}(t)||}{||\dot{c}(t)||^3} \qquad \textit{for } c : I \to \mathbb{R}^3$$
$$\tau(t) := \frac{\det(\dot{c}(t), \ddot{c}(t), \dddot{c}(t))}{||\dot{c}(t) \times \ddot{c}(t)||^2} \qquad \textit{for } c : I \to \mathbb{R}^3$$

Definition 8 (Vertex). *$c(t)$ with $\dot{\kappa}(t) = 0$*

Definition 9 (Length).

$$L(c) := \int_I ||\dot{c}(t)|| dt$$

Definition 10 (Isometric = Preserving Length).

$$\forall c : I \to M : L(\phi(c)) = L(c)$$

III. Bezier Curves

Algorithm 1 (of de Casteljou).

$$b_i^0(t) := b_i \quad b_i^j(t) := (1-t)b_i^{j-1}(t) + tb_{i+1}^{j-1}(t) \quad b(t) := b_0^n(t)$$

Definition 11 (Bernstein Polynomials).

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i} \in \mathbb{R}[t]$$

Theorem 1 (Bernstein Representation of Bezier Curve).

$$b_i^j(t) = \sum_{l=0}^{j} B_l^j(t) b_{i+l}$$

Proof. Induction over j using the fact that $B_l^j(t) = (1-t)B_l^{j-1}(t) + tB_{l-1}^{j-1}(t)$. □

Lemma 1 (Properties of Bernstein Polynomials).

$$\sum_{i=1}^{n} B_i^n(t) = 1$$
$$t \in [0,1] \implies B_i^n(t) \geq 0$$
$$B_i^n(\alpha t) = \sum_{j=0}^{n} B_i^j(\alpha) B_j^n(t)$$
$$\{B_0^n(t), ..., B_n^n(t)\} \textit{ forms a basis of } \Pi_n$$

Lemma 2 (Endpoint Interpolating).

$$b(0) = b_0 \qquad b(1) = b_n$$

Lemma 3 (Tangent of Bezier Curves).

$$\dot{b}(t) = n(b_1^{n-1}(t) - b_0^{n-1}(t))$$

Proof. Calculation using the fact that $\frac{d}{dt} B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$. □

Theorem 2 (Invariant under Affine Transformations).

$$\alpha(b(t)) = \sum_{i=0}^{n} B_i^n(t) \alpha(b_i)$$

Theorem 3 (Bezier Curve is in Convex Hull).

$$t \in [0,1] \implies b(t) \in conv(\{b_0, ..., b_n\})$$

Theorem 4 (Bezier Curve is Symmetric).

$$b(t) \textit{ from points } b_0, ..., b_n \quad \tilde{b}(t) \textit{ from points } b_n, ..., b_0$$
$$\implies b(t) = \tilde{b}(1-t)$$

Theorem 5 (Sub-division property)**.**

$$\tilde{b}(t) := b(\alpha t) \qquad \hat{b}(t) := b((1-\alpha)t + \alpha)$$

Then

$$\tilde{b} \textit{ from points } b_0^0(\alpha), ..., b_0^n(\alpha)$$
$$\hat{b} \textit{ from points } b_0^n(\alpha), b_1^{n-1}(\alpha), ..., b_n^0(\alpha)$$

and "gluing" them together results in $b(t)$*.*

Theorem 6. *Every polynomial curve is a Bezier curve.*

Theorem 7 (Corner-cutting)**.**

$$P_0 := (b_0, ..., b_n) \quad P_n := \textit{ sub-divide } n \textit{ times with } n = 1/2$$
$$\implies P_n \to b(t)$$

IV. B-Spline and NURBS

Definition 12 (B-Spline)**.** *Composition of Bezier Curves.*

Definition 13 (B-Spline Base Functions)**.**

$$\alpha_i^r(t) := \begin{cases} \frac{t-t_i}{t_{i+r}-t_i} & t_{i+r} - t_i \neq 0, \\ 0, & \textit{else} \end{cases}$$
$$N_i^0(t) := \begin{cases} 1, & t \in [t_i, t_{i+1}) \\ 0, & \textit{else} \end{cases}$$
$$N_i^r(t) := \alpha_i^r(t) N_i^{r-1}(t) + (1 - \alpha_i^r(t)) N_{i+1}^{r-1}(t)$$

Lemma 4 (Properties of B-Spline Base Functions)**.**

$$\sum_{i=0}^{m} N_i^n(t) = 1 \qquad N_i^n(t) \geq 0$$

Theorem 8 (B-Spline Representation)**.** m *... number control points,* n *... degree,* $T = (t_0, ..., t_{m+n+1})$ *... knot vector with* $t_i \leq t_{i+1}$, $t_i < t_{i+n+1}$

$$s(t) = \sum_{i=0}^{m} N_i^n(t) c_i$$

Definition 14 (NURBS)**.** *Construction by:*

1) *homogenize control points*
2) *scale by weight*
3) *construct B-Spline*
4) *project back onto* $1 \times \mathbb{R}^d$
5) *dehomogenize*

Lemma 5. *B-Spline curve with* $m = n$ *is a Bezier curve. NURBS with* $w_i = w$ *is a B-Spline.*

Remark 1. w_i *bigger* $\implies$ *curve is attracted towards* c_i*.*

Remark 2. *NURBS can construct conic sections (such as circles).*

Lemma 6. *B-Spline and NURBS are both affine invariant.*

V. Free Form Surfaces

Definition 15 (Free Form Surface)**.** $b_{00}, b_{01}, ..., b_{mn} \in \mathbb{R}^d$, $g(s) = \sum_{i=0}^{m} D_i(s) p_i$, $h(t) = \sum_{j=0}^{n} E_j(t) q_j$

$$f(s,t) := \sum_{i=0}^{m} \sum_{j=0}^{n} D_i(s) E_j(t) b_{ij}$$

Remark 3. *Many same theorems as in Bezier, B-Spline and NURBS hold.*

VI. Subdivision of Curves

Algorithm 2 (Chaikin)**.** *copy once, average twice*

- $\to$ *quadratic B-Spline*
- *approximating*
- *affine invariant*
- *linear precision*

Algorithm 3 (Lane-Riesenfeld)**.** *copy once, average* n *times*

- $\to$ *B-Spline of degree* n
- *approximating*
- *affine invariant*
- *linear precision*

Algorithm 4 (Four Point Scheme)**.** $p_i^0 := p_i$
$p_i^l := -\frac{1}{16} p_{i-1}^{l-1} + \frac{9}{16} p_i^{l-1} + \frac{9}{16} p_{i+1}^{l-1} - \frac{1}{16} p_{i+2}^{l-1}$

- $\to C^1$ *curve*
- *interpolating*
- *affine invariant*
- *linear precision*

VII. Subdivision of Meshes

Algorithm 5 (Loop)**.** *triangle mesh*

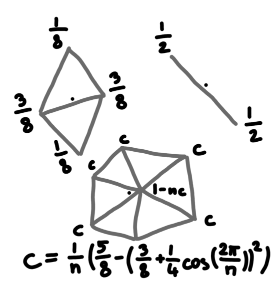

Fig. 1. Loop

- $\to C^2$ *in generic* ($n = 6$) *case*
- $\to C^1$ *otherwise*
- *approximating*
- *face splitting*

Algorithm 6 (Modified Butterfly)**.** *triangle mesh*

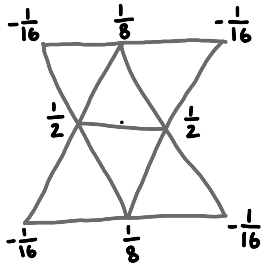

Fig. 2. Modified Butterfly

- $\rightarrow C^1$ *surface*
- *interpolating*
- *face splitting*

Algorithm 7 (Catmull Clark)**.** *square mesh*

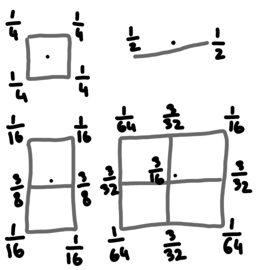

Fig. 3. Catmull Clark

- $\rightarrow C^2$ *in generic* ($n = 4$) *case*
- *approximating*
- *face splitting*

Algorithm 8 (Doo Sabin)**.** *any mesh*

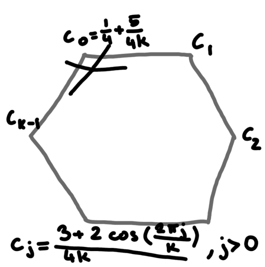

Fig. 4. Doo Sabin

- $\rightarrow C^1$ *surface*
- *approximating*
- *vertex splitting*

Algorithm 9 (Kobbelt)**.** *square mesh*

four point scheme horizontal or vertical, then four point scheme on results

- $\rightarrow C^1$ *surface*
- *interpolating*
- *face splitting*

VIII. Conformal Pattern Matching

Definition 16 (Möbius Transformation)**.** *Composition of reflections on hyper-sphere.*

Definition 17 (Reflection on Sphere)**.**

$$\tilde{\omega} := \frac{r^2}{||\omega||^2}\omega \qquad \tilde{0} := \infty \qquad \tilde{\infty} := 0$$

Definition 18 (Cross Ratio)**.**

$$DV(a, b, c, d) := \frac{(a-b)(c-d)}{(b-c)(d-a)} \in \mathbb{C}$$

Theorem 9. *Cross Ratio is invariant under Möbius transformations.*

Theorem 10. *Möbius transformations preserve angles (=conformal) and circles.*

Definition 19 (Discrete Metric)**.**

$$L : E \rightarrow \mathbb{R}_{\geq 0}$$

$$\forall f \in F : l_{ij} \leq l_{jk} + l_{ki}, l_{jk} \leq l_{ki} + l_{ij}, l_{ki} \leq l_{ij} + l_{jk}, l_{ij} = l_{ji}$$

Definition 20 (Discrete Conform Equivalent)**.** $T, \tilde{T}$ *.. combinatorial equivalent triangle meshes with*

$$\exists \lambda : E \rightarrow \mathbb{R}_{\geq 0} \forall (i, j) \in E : \tilde{l}_{ij} = \lambda_i \lambda_j l_{ij}$$

Definition 21 (Length Cross Ratio)**.**

$$LDV(v_i, v_j, v_k, v_m) := \frac{l_{ij} l_{km}}{l_{jk} l_{mi}}$$

Lemma 7. $(T, l), (\tilde{T}, \tilde{l})$ *are discrete conformal equivalent, iff* $LDV(v_i, v_j, v_k, v_m) = LDV(\tilde{v}_i, \tilde{v}_j, \tilde{v}_k, \tilde{v}_m)$

Lemma 8. T *... triangle mesh,* $l_{ij} := ||v_i - v_j||$, m *... Möbius transformation, then* $T, m(T)$ *are discrete conformal equivalent.*

Theorem 11. *Let* T *be a triangle mesh and* $\Phi_i \geq 0 \forall i \in V$.

There exists a discrete conformal equivalent triangle mesh $\tilde{T}$, *such that the sum of interior angles is* $\Phi_i \forall i \in V$.

Proof. There is a function $E(u)$ with $\frac{dE(u)}{du_i} = \Phi_i - \sum_{jk \in star(i)} \alpha^i_{jk}$, that can be extended to be a convex function on $\mathbb{R}^{\#V}$. Finding the critical points, that exist as $E(u)$ is convex, gives us the equivalent mesh $\tilde{T}$. □