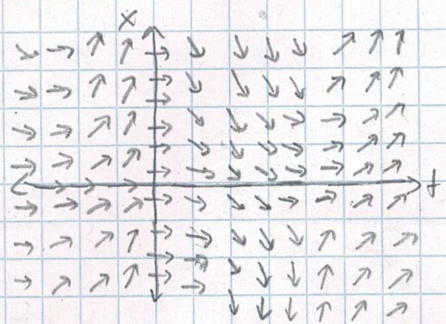
DGL Üg

1) $x' = -x^2 \sin(t)$

a)



b) AWP $x(0)=a$, $a \in \mathbb{R}$

$\frac{dx}{dt} = -x^2 \sin(t) \iff -\frac{1}{x^2}dx = \sin(t)dt \iff \int -\frac{1}{x^2}dx = \frac{1}{x} = -\cos(t) = \int \sin(t)dt$

$\Rightarrow \frac{1}{x(t)} = -\cos(t) + c \iff x(t) = \frac{1}{c-\cos(t)} \qquad \Rightarrow x(0) = \frac{1}{c-1} = a$

$\Rightarrow c = \frac{1}{a} + 1 \qquad \Rightarrow x(t) = \frac{1}{\frac{1}{a}+1-\cos(t)}$ für $a \neq 0$

$x(t) = 0$ für $a=0$

c) für $a=0$ $\exists$ Lsg für alle $t \geq 0$ $\quad x(t)=0$

für $a \neq 0$ $\exists$ Lsg für $t \geq 0 \iff \frac{1}{a} + 1 - \cos(t) \neq 0 \iff \cos(t) \neq 1 + \frac{1}{a}$

$\iff t \neq \arccos(1+\frac{1}{a})$

d) $x(t)=0$ stabil bzw asymptotisch stabil?

$\varepsilon > 0$ bel. $\quad \delta := \frac{\varepsilon}{2} \qquad |a-0| \leq \delta \Rightarrow |a| \leq \delta$

$\|x(t,a) - x(t,0)\| = \|\frac{1}{1+\frac{1}{a}-\cos(t)}\| \leq \frac{1}{\frac{1}{|a|}} = |a| \leq \delta = \frac{\varepsilon}{2} < \varepsilon$ $\Rightarrow$ stabil

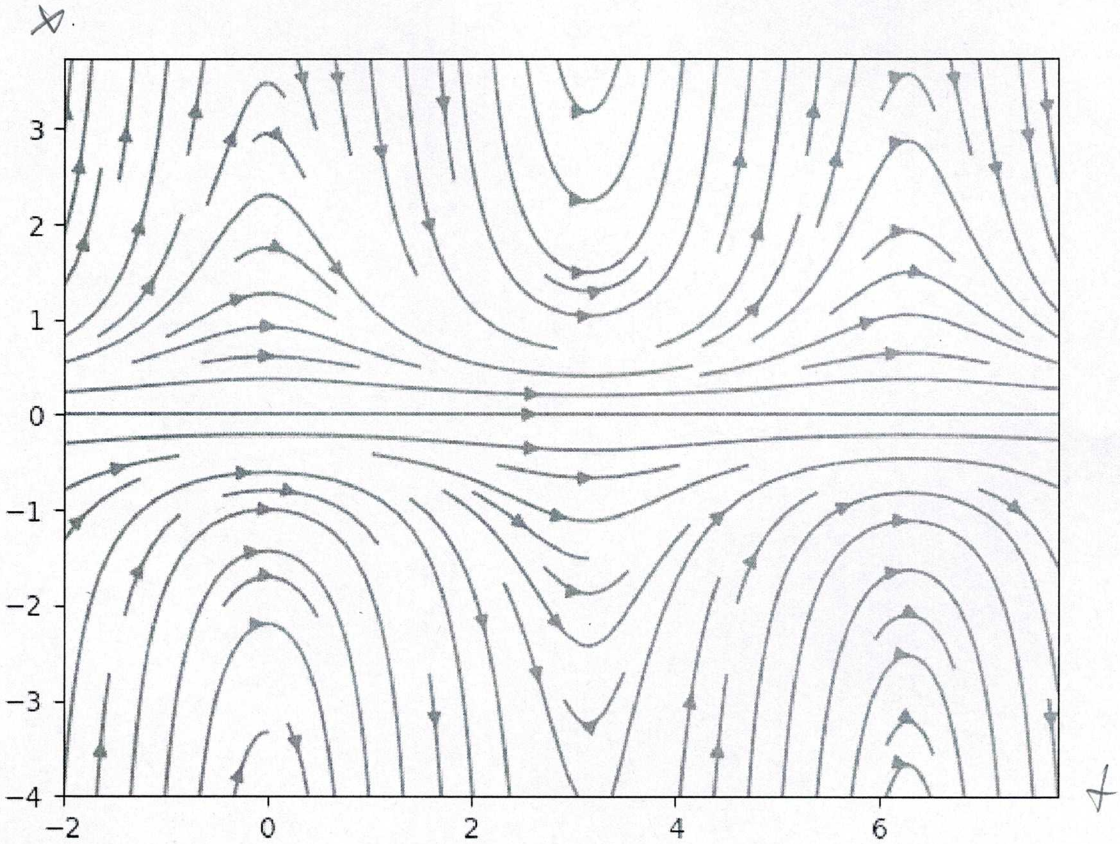
$\delta > 0$ bel. $\quad |a-0| = |a| \leq \delta \qquad a := \delta$

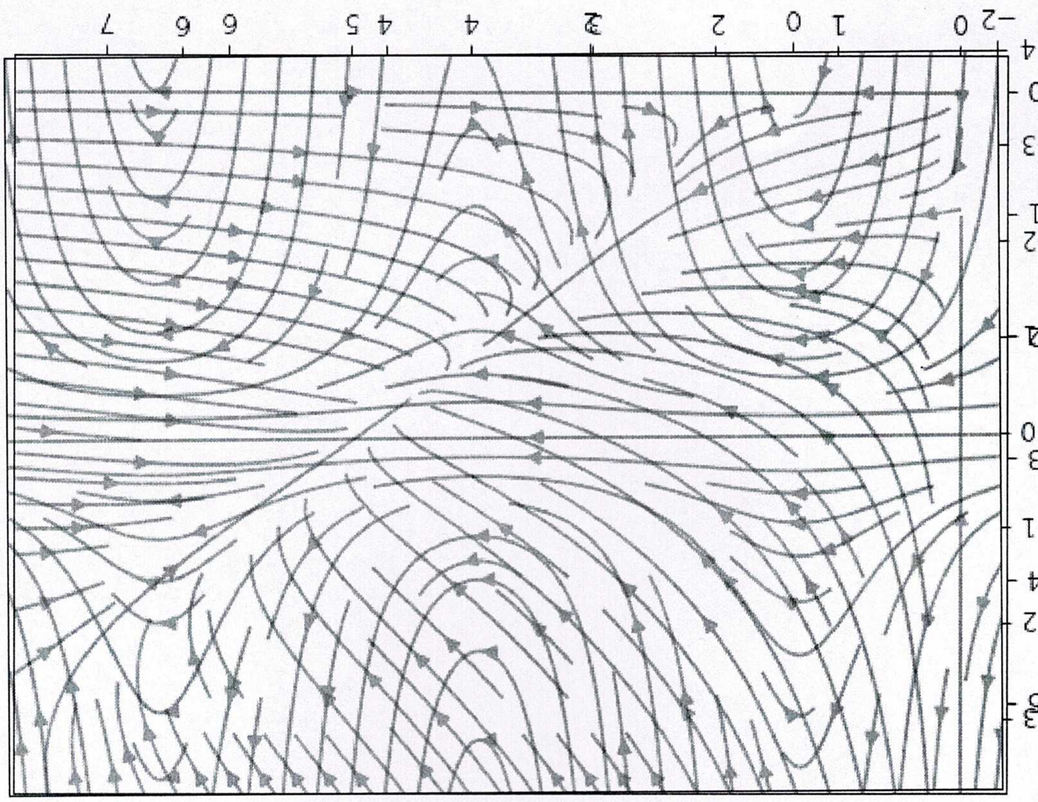
$\lim_{t\to\infty} \|x(t,a) - x(t,0)\| = \lim_{t\to\infty} \|\frac{1}{1+\frac{1}{\delta}-\cos(t)}\| \neq 0$, da $t := 2\pi n$, $n \in \mathbb{N}$

$\Rightarrow \frac{1}{1+\frac{1}{\delta}-\cos(t)} = \frac{1}{\frac{1}{\delta}} = \delta \neq 0$

also nicht asymptotisch stabil

1)





1+4)

DGL Üg

2) $x' = \frac{x^2}{1+t^2}$ $a\in\mathbb{R}$ AWP $x(0)=a$

$\frac{dx}{dt} = \frac{x^2}{1+t^2} \iff \frac{1}{x^2}dx = \frac{1}{1+t^2}dt \iff \int \frac{1}{x^2}dx = -\frac{1}{x} = \arctan(t)+c = \int\frac{1}{1+t^2}dt$

$\Rightarrow x(t) = -\frac{1}{\arctan(t)+c}$ $x(0) = -\frac{1}{c} = a \iff c = -\frac{1}{a}$

$\Rightarrow x(t) = -\frac{1}{\arctan(t)-\frac{1}{a}}$ für $a\neq 0$ $x(t)=0$ für $a=0$

$= \frac{1}{\frac{1}{a}-\arctan(t)}$

$x(0) = \frac{1}{\frac{1}{a}-\arctan(0)} = a$ $\lim\limits_{t\to\infty} \frac{1}{\frac{1}{a}-\arctan(t)} = \frac{1}{\frac{1}{a}-\frac{\pi}{2}} = \frac{1}{\frac{2-a\pi}{2a}} = \frac{2a}{2-a\pi}$

$\lim\limits_{t\to t_0} x(t) = \pm\infty \iff \lim\limits_{t\to t_0} \frac{1}{a}-\arctan(t) = 0$

$\frac{1}{a} = \arctan(t)$ für ein t_0 $\iff \frac{1}{a}\in(0,\frac{\pi}{2}) = \arctan(\mathbb{R}^+) \iff a\in(\frac{2}{\pi},\infty)$

$\Rightarrow$ Lösung ist beschränkt für $a\in(-\infty,\frac{2}{\pi}]$

Sei $\varepsilon>0$ bel. $\delta := \min(\frac{2}{\pi},\varepsilon)$ Sei $a\in\mathbb{R}$ mit $|a-0|=|a|\leq\delta$

$\Rightarrow |x(t,a)-x(t,0)| = \left|\frac{1}{\frac{1}{a}-\arctan(t)} - 0\right| = \frac{1}{|\frac{1}{a}-\arctan(t)|}$

$\Big|\underbrace{\frac{1}{a}}_{\in[-\frac{\pi}{2},\frac{\pi}{2}]} - \underbrace{\arctan(t)}_{\in[0,\frac{\pi}{2})}\Big| \in [\frac{1}{|a|}, |\frac{1}{a}-\frac{\pi}{2}|)$

$\Rightarrow \frac{1}{|\frac{1}{a}-\arctan(t)|} \leq \frac{1}{\frac{1}{|a|}} = |a| \leq \delta \leq \varepsilon$ $\Rightarrow$ stabil

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$\delta>0$ bel. $|a-0|=|a|\leq\delta$

$\lim\limits_{t\to\infty} |x(t,a)-x(t,0)| = \lim\limits_{t\to\infty}\left|\frac{1}{\frac{1}{a}-\arctan(t)} - 0\right| = \left|\frac{2a}{2-a\pi}\right| \neq 0$ für $a\neq 0$

$\Rightarrow$ nicht asymptotisch stabil

DGL Üg

3) $x' = A(t)x \qquad A(t) = \begin{pmatrix} -1+\frac{3}{2}\cos^2(t) & 1-\frac{3}{2}\sin(t)\cos(t) \\ -1-\frac{3}{2}\sin(t)\cos(t) & -1+\frac{3}{2}\sin^2(t) \end{pmatrix}$

a)
$$\det(A-\lambda I) = \left(-1+\tfrac{3}{2}\cos^2(t)-\lambda\right)\left(-1+\tfrac{3}{2}\sin^2(t)-\lambda\right) - \left(\tfrac{9}{4}\sin^2(t)\cos^2(t) - 1\right)$$
$$= \cancel{1} - \tfrac{3}{2}\sin^2(t) + \cancel{\lambda} - \tfrac{3}{2}\cos^2(t) + \cancel{\tfrac{9}{4}\sin^2(t)\cos^2(t)} - \tfrac{3}{2}\lambda\cos^2(t) + \cancel{\lambda} - \tfrac{3}{2}\lambda\sin^2(t) + \cancel{\lambda^2}$$
$$- \cancel{\tfrac{9}{4}\sin^2(t)\cos^2(t)} + 1$$
$$= \lambda^2 + 2\lambda - \tfrac{3}{2}\lambda\left(\sin^2(t)+\cos^2(t)\right) - \tfrac{3}{2}\left(\sin^2(t)+\cos^2(t)\right) + 2$$
$$= \lambda^2 + 2\lambda - \tfrac{3}{2}\lambda + 2 - \tfrac{3}{2} = \lambda^2 + \tfrac{1}{2}\lambda + \tfrac{1}{2} = 0$$
$$\Leftrightarrow \lambda_{1,2} = -\tfrac{1}{4} \pm \sqrt{\tfrac{1}{8}-\tfrac{1}{2}} = -\tfrac{1}{4} \pm \sqrt{-\tfrac{3}{8}} \qquad \mathcal{R}(\lambda_{1,2}) < 0$$

b)
$$x(t) = e^{\frac{t}{2}}\begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix} \qquad x'(t) = \tfrac{1}{2}e^{\frac{t}{2}}\begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix} + e^{\frac{t}{2}}\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$
$$\begin{pmatrix} -e^{\frac{t}{2}}\cos(t) \\ e^{\frac{t}{2}}\sin(t) \end{pmatrix}$$
$$\begin{pmatrix} -1+\frac{3}{2}\cos^2(t) & 1-\frac{3}{2}\sin(t)\cos(t) \\ -1-\frac{3}{2}\sin(t)\cos(t) & -1+\frac{3}{2}\sin^2(t) \end{pmatrix}\begin{pmatrix} e^{\frac{t}{2}}\cos(t) - \frac{3}{2}e^{\frac{t}{2}}\cos^3(t) + e^{\frac{t}{2}}\sin(t) - \frac{3}{2}e^{\frac{t}{2}}\sin^2(t)\cos(t) \\ e^{\frac{t}{2}}\cos(t) + \frac{3}{2}e^{\frac{t}{2}}\sin(t)\cos^2(t) - e^{\frac{t}{2}}\sin(t) + \frac{3}{2}e^{\frac{t}{2}}\sin^3(t) \end{pmatrix}$$
$$= e^{\frac{t}{2}}\begin{pmatrix} \cos(t) - \frac{3}{2}\cos^3(t) + \sin(t) - \frac{3}{2}\sin^2(t)\cos(t) \\ \cos(t) + \frac{3}{2}\sin(t)\cos^2(t) - \sin(t) + \frac{3}{2}\sin^3(t) \end{pmatrix}$$
$$= e^{\frac{t}{2}}\begin{pmatrix} \sin(t) + \cos(t)\left(1-\frac{3}{2}(\cos^2(t)+\sin^2(t))\right) \\ \cos(t) + \sin(t)\left(-1+\frac{3}{2}(\sin^2(t)+\cos^2(t))\right) \end{pmatrix} = e^{\frac{t}{2}}\begin{pmatrix} \sin(t) - \frac{1}{2}\cos(t) \\ \cos(t) + \frac{1}{2}\sin(t) \end{pmatrix} = x'(t)$$

c) $\delta > 0$ bel. $\quad \left\|\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right\| \leq \delta \qquad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \end{pmatrix}$

$$\|x(t,a) - x(t,0)\| = \left\| a e^{\frac{t}{2}}\begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix}\right\| = \left\|\begin{pmatrix} -\delta e^{\frac{t}{2}}\cos(t) \\ 0 \end{pmatrix}\right\|$$

$$\left|-\delta e^{\frac{t}{2}}\cos(t)\right| \xrightarrow{t\to\infty} \infty \qquad \Rightarrow \left\|\begin{pmatrix} -\delta e^{\frac{t}{2}}\cos(t) \\ 0 \end{pmatrix}\right\| \geq K \quad \forall K \in \mathbb{N}$$

also ist $\bar{x} = 0$ instabil

DGL Üg

4) $x' = 3x(2-x+y) \qquad y' = y(3+x-3y)$

a) Gleichgewichtspunkte

$3x(2-x+y)=0 \ \wedge\ y(3+x-3y)=0 \iff \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ oder

$2-x+y=0 \iff x=2+y \qquad \Rightarrow 3+x-3y = 3+2+y-3y = 5-2y=0 \Rightarrow y=\frac{5}{2}$

$\Rightarrow \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{9}{2}\\ \frac{5}{2}\end{pmatrix}$

b) Stabilität mittels Linearisierung

$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix} \qquad \begin{pmatrix}a\\b\end{pmatrix}' = df\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}6-6\bar{x}+3\bar{y} & 3\bar{x}\\ \bar{y} & 3+\bar{x}-6\bar{y}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$

(i) $\bar{x}=0=\bar{y} \qquad \Rightarrow \begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix} \qquad \begin{pmatrix}a\\b\end{pmatrix}'=\begin{pmatrix}6&0\\0&3\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$

EW 6,3 $\Rightarrow$ instabil, da Re >0

(ii) $\bar{x}=\frac{9}{2} \quad \bar{y}=\frac{5}{2} \qquad \begin{pmatrix}a\\b\end{pmatrix}' = \begin{pmatrix}6-27+\frac{15}{2} & \frac{27}{2}\\ \frac{5}{2} & 3+\frac{9}{2}-15\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}-13{,}5 & 13{,}5\\ 2{,}5 & -7{,}5\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$

$\chi(\lambda) = \det\begin{pmatrix}-13{,}5-\lambda & 13{,}5\\ 2{,}5 & -7{,}5-\lambda\end{pmatrix} = (-13{,}5-\lambda)(-7{,}5-\lambda) - 2{,}5\cdot 13{,}5$

$= 101{,}75 + 21\lambda + \lambda^2 - 33{,}75 = \lambda^2 + 21\lambda + 68 = 0$

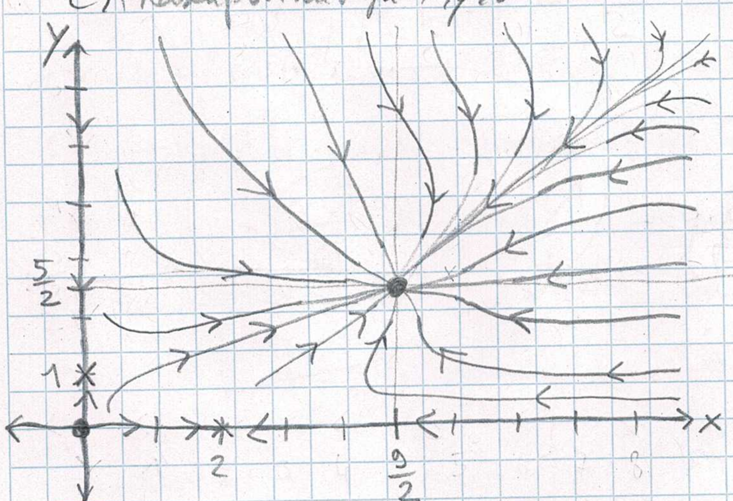
$\iff \lambda_{1,2} = -\frac{21}{2} \pm \sqrt{\frac{21^2}{4} - 68} = -\frac{21}{2} \pm \sqrt{\frac{441-272}{4}} = -\frac{21}{2} \pm \frac{\sqrt{169}}{2}$

$= \frac{-21\pm 13}{2} = \begin{cases} -\frac{8}{2} = -4 \\ -\frac{34}{2} = -17 \end{cases}$

$R(\lambda_{1,2})<0$

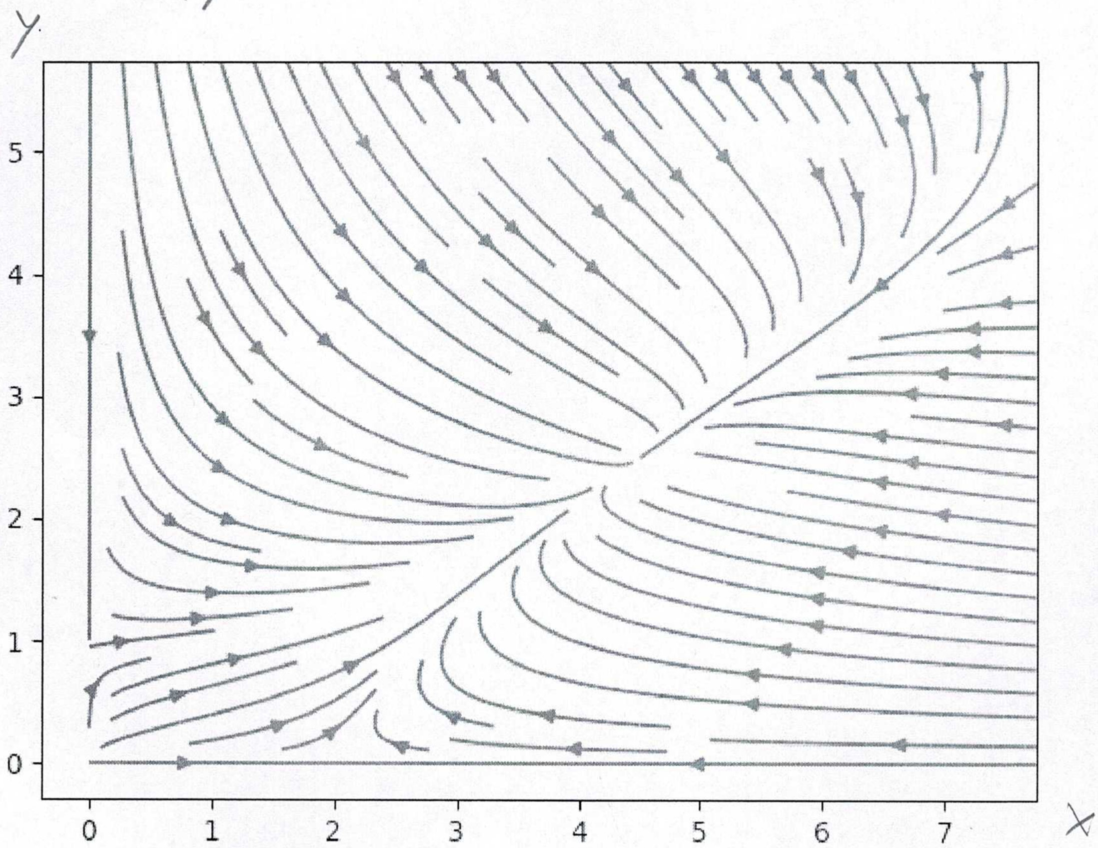
$\Rightarrow$ asymptotisch stabil

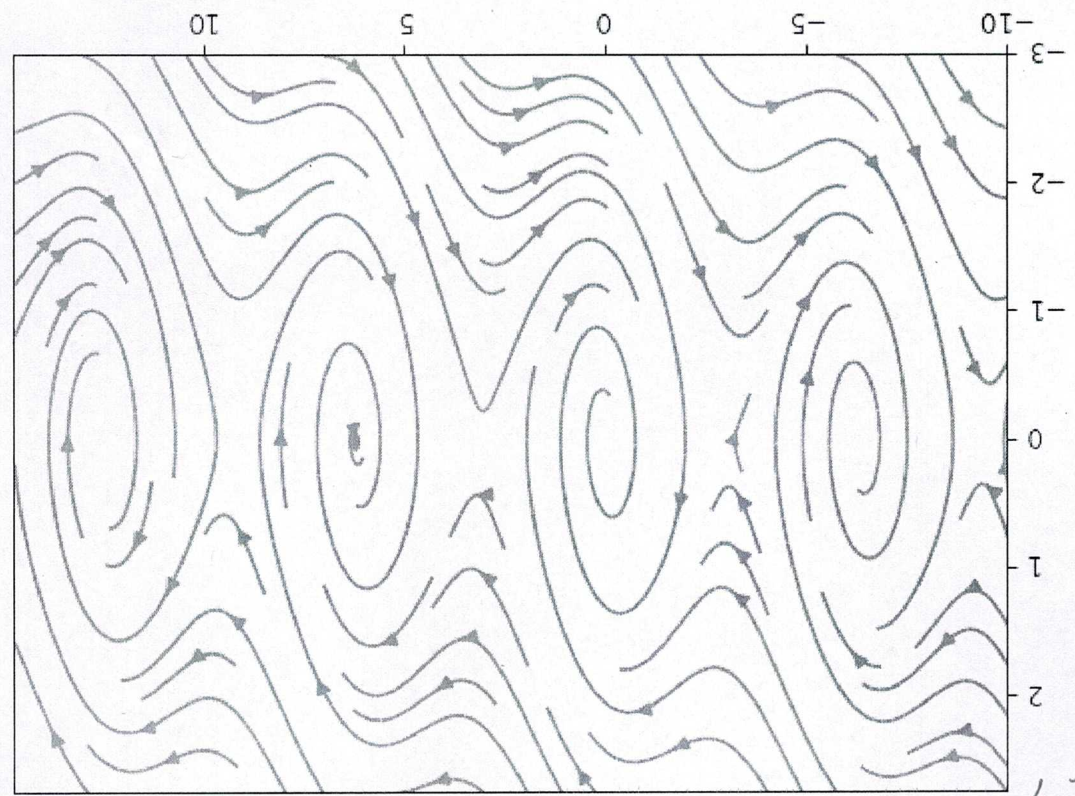
c) Phasenportrait für $x,y\geq 0$



d) tatsächlich asymptotisch stabil

(4)





(6)

DGL Üg

6) $y'' + ry' + \sin(y) = 0 \qquad r \in \mathbb{R}, r \geq 0$

a) $x_1 = y \qquad x_1' = x_2$

$x_2 = y' \qquad x_2' = -r x_2 - \sin(x_1)$

b) $x_1' = x_2 = 0 \qquad x_2' = -r x_2 - \sin(x_1) = -\sin(x_1) = 0 \Rightarrow x_1 \in \pi \mathbb{Z}$

$\Rightarrow \left\{ \begin{pmatrix} \pi k \\ 0 \end{pmatrix} \middle| k \in \mathbb{Z} \right\}$ sind Gleichgewichtspunkte

c)

$x = \bar{x} + y \qquad y' = df(\bar{x})\, y$

$df(x) = \begin{pmatrix} 0 & 1 \\ -\cos(x_1) & -r \end{pmatrix} \Rightarrow y' = \begin{pmatrix} 0 & 1 \\ -\cos(\pi k) & -r \end{pmatrix} y \qquad \begin{matrix} y_1' = y_2 \\ y_2' = -\cos(\pi k) y_1 - r y_2 \end{matrix}$

1. Fall k ist gerade

$\Rightarrow y' = \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix} y$

$\chi(\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -r-\lambda \end{pmatrix} = \lambda^2 + r\lambda + 1 = 0 \Leftrightarrow \lambda_{1,2} = -\frac{r}{2} \pm \frac{\sqrt{r^2-4}}{2}$

? • $r = 0 \Rightarrow \lambda_{1,2} = \pm i \Rightarrow \mathrm{Re}(\lambda_{1,2}) = 0$ keine Aussage

a.s. • $r \in (0,2] \Rightarrow \mathrm{Re}(\lambda_{1,2}) = -\frac{r}{2}$ asymptotisch stabil

a.s. • $r > 2 \Rightarrow \mathrm{Re}(\lambda_{1,2}) = \frac{-r \pm \sqrt{r^2-4}}{2}$

$-r + \sqrt{r^2-4} < 0 \Leftrightarrow r^2 - 4 < r^2 \Leftrightarrow -4 < 0$ ✓

$-r - \sqrt{r^2-4} < 0 \Leftrightarrow \underbrace{-r}_{<0} < \underbrace{\sqrt{r^2-4}}_{>0}$ ✓ asymptotisch stabil

2. Fall k ist ungerade

$\Rightarrow y' = \begin{pmatrix} 0 & 1 \\ 1 & -r \end{pmatrix} y \qquad \chi(\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -r-\lambda \end{pmatrix} = \lambda^2 + r\lambda - 1 = 0 \Leftrightarrow \lambda_{1,2} = -\frac{r}{2} \pm \frac{\sqrt{r^2+4}}{2}$

i. • $r = 0 \Rightarrow \lambda_{1,2} = \pm 1 \Rightarrow \mathrm{Re}(\lambda_1) < 0, \mathrm{Re}(\lambda_2) > 0$ instabil

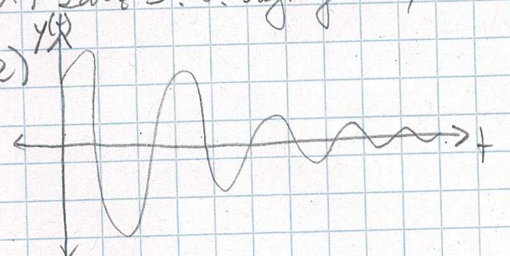
i. • $r > 0 \Rightarrow \mathrm{Re}(\lambda_1) = \frac{1}{2}(-r + \sqrt{r^2+4}) \qquad \mathrm{Re}(\lambda_2) = -\frac{1}{2}(r + \sqrt{r^2+4}) < 0$

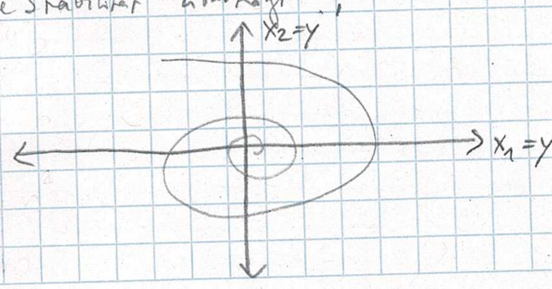
$-r + \sqrt{r^2+4} > 0 \Leftrightarrow \sqrt{r^2+4} > r \Leftrightarrow r^2 + 4 > r^2 \Leftrightarrow 4 > 0 \Rightarrow \forall r > 0$

$\Rightarrow \mathrm{Re}(\lambda_1) > 0 \quad \mathrm{Re}(\lambda_2) < 0$ instabil

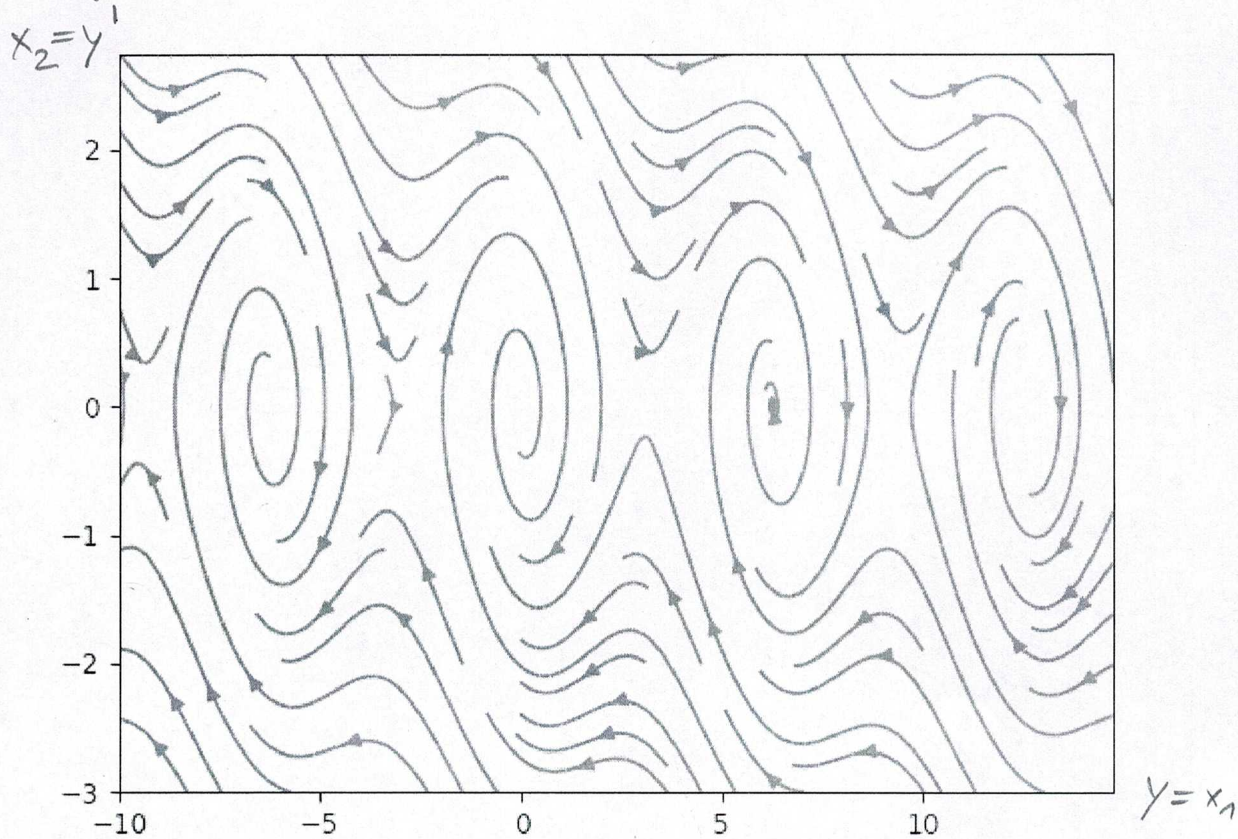
d) Satz 5.6. sagt genau, dass sich die Stabilität "überträgt".

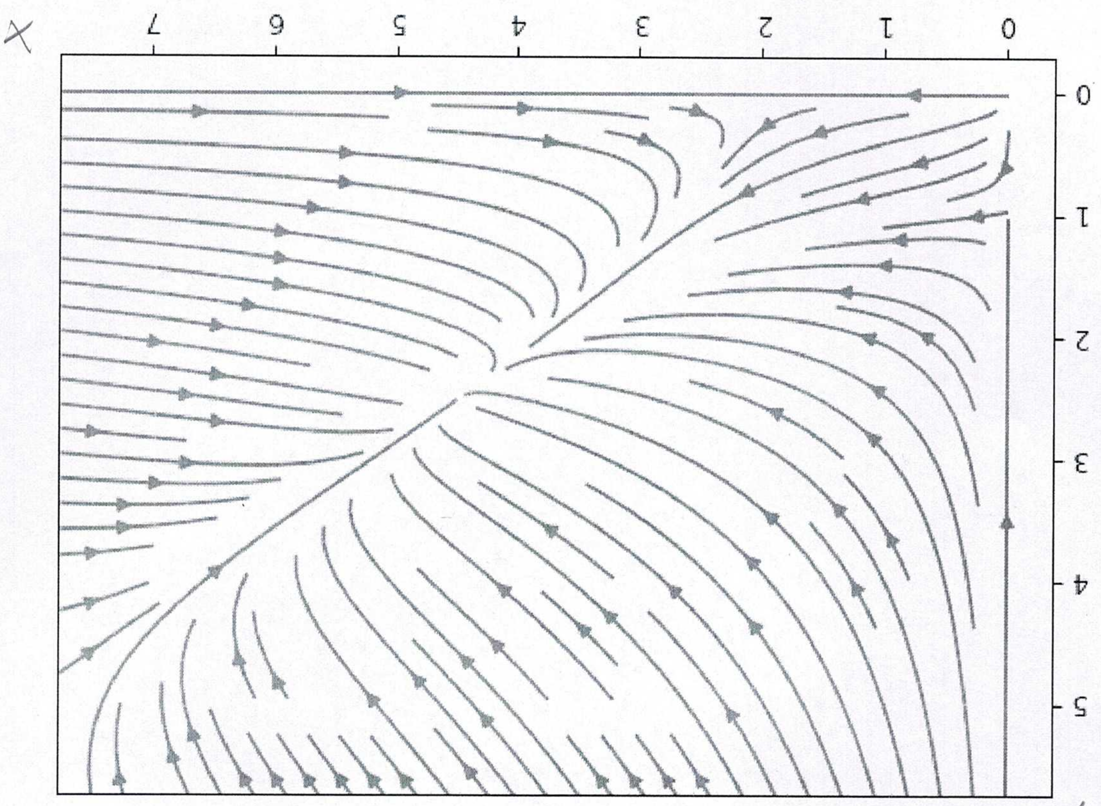
e)





6)





7)

DGL Ü9

7) $x_1' = -x_1 + x_2^2 \quad x_2' = -x_2^3 - x_1 x_2 \quad V(x) = \frac{1}{2}(x_1^2 + x_2^2) \quad \bar{x} = \binom{0}{0}$

V ist diffbar

1) $V(\bar{x}) = 0 \qquad V(\bar{x}) = \frac{1}{2}(0^2 + 0^2) = 0$ ✓

2) $\forall x \in \mathbb{R}^2 \setminus \{\bar{x}\} : V(x) > 0 \qquad V(x) = \frac{1}{2}(x_1^2 + x_2^2) = 0 \Leftrightarrow x_1 = 0 = x_2$ ✓

3) $\forall x \in \mathbb{R}^2 \setminus \{\bar{x}\} : \frac{d}{dt} V(x) \leq 0$

$$\frac{d}{dt} V(x) = \frac{d}{dt} \frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{2}(2 x_1'(t) x_1(t) + 2 x_2'(t) x_2(t))$$

$$= (-x_1 + x_2^2) x_1 + (-x_2^3 - x_1 x_2) x_2$$

$$= -x_1^2 + \cancel{x_1 x_2^2} - x_2^4 - \cancel{x_1 x_2^2} = -x_1^2 - x_2^4 < 0$$

$\Rightarrow$ V ist Ljapunovfunktion (sogar strikt)

Satz 5.13 sagt $\exists$ für $\bar{x}$ eine Ljapunovfunktion so ist $\bar{x}$ stabil

$\exists$ eine strikte Ljapunovfunktion dann sogar asymptotisch stabil

$\Rightarrow \bar{x}$ ist asymptotisch stabil

Linearisierung $\qquad x = \bar{x} + y \qquad y' = df(\bar{x}) y$

$$df(x) = \begin{pmatrix} -1 & 2x_2 \\ -x_2 & -3x_2^2 - x_1 \end{pmatrix} \Rightarrow y' = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} y$$

EW

$$\det \begin{pmatrix} -1-\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = (-1-\lambda)(-\lambda) - 0 = \lambda^2 + \lambda = 0 \Leftrightarrow \lambda_1 = 0 \lor \lambda_2 = -1$$

Da $Re(\lambda_1) = 0$ und $Re(\lambda_2) < 0$ kann man nicht auf das Verhalten der DGL schließen.