ANA Ü12

1.) $\gamma: [0,1] \to \mathbb{R}^2$ ... stetiger Weg $\quad \Phi: (1,+\infty)\times\mathbb{R} \to L(\mathbb{R}^2,\mathbb{R}) \cong \mathbb{R}^{1\times 2}$

$t \mapsto \begin{pmatrix} 1+t^2 \\ 1+t^3 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto (\sin(\sqrt{x-1}), xy)$

ges: $l(\gamma)$ und $\int_\gamma \Phi(x)\,dx$

$\gamma'(t) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$ ... stetig $\Rightarrow \gamma \in C^1[0,1]$ Nach Satz 11.1.8. gilt nun

$$l(\gamma) = \int_0^1 \|\gamma'(x)\|_2\,dx = \int_0^1 \left\|\begin{pmatrix} 2x \\ 3x^2 \end{pmatrix}\right\|_2 dx = \int_0^1 \sqrt{4x^2+9x^4}\,dx = \int_0^1 x\cdot\sqrt{4+9x^2}\,dx$$

$\int x\sqrt{4+9x^2}\,dx = \int x\sqrt{u}\,\frac{1}{18x}\,du \qquad \left[u = 4+9x^2 \quad \frac{du}{dx} = 18x \quad dx = \frac{1}{18x}\,du\right.$

$= \frac{1}{18}\int\sqrt{u}\,du = \frac{1}{18}\cdot\frac{2}{3}u^{\frac{3}{2}} = \frac{1}{27}(4+9x^2)^{\frac{3}{2}}$

$= \frac{1}{27}(4+9x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{1}{27}\cdot 13^{\frac{3}{2}} - \frac{1}{27}\cdot 4^{\frac{3}{2}} \approx 1,4397$

Φ ist auf $(1,+\infty)\times\mathbb{R}$ stetig. Nach Satz 11.2.5 gilt nun

$$\int_\gamma \Phi(x)\,dx = \int_0^1 \Phi(\gamma(t))\cdot\gamma'(t)\,dt = \int_0^1 \left(\sin(\sqrt{1+t^2-1}), (1+t^2)(1+t^3)\right)\cdot\begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} dt$$

$$= \int_0^1 2\sin(t)\cdot t + 3t^2(1+t^3+t^2+t^5)\,dt = \int_0^1 2\sin(t)t + 3t^2 + 3t^5 + 3t^4 + 3t^7\,dt$$

$$= 2\int_0^1 \sin(t)t\,dt + 3\left(\int_0^1 t^2\,dt + \int_0^1 t^5\,dt + \int_0^1 t^4\,dt + \int_0^1 t^7\,dt\right)$$

$\int \sin(t)\cdot t\,dt = -t\cdot\cos(t) + \int\cos(t)\,dt = -\cos(t)\cdot t + \sin(t)$

$$\Rightarrow \int_\gamma \Phi(x)\,dx = 2\cdot(-\cos(1)+\sin(1)-0) + 3\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{5}+\frac{1}{8}\right) = \frac{99}{40} + 2\sin(1) - 2\cos(1) \approx 3,07734$$

ANA Ü12

2.) $\Phi: \mathbb{R}^3 \to L(\mathbb{R}^3, \mathbb{R}) \cong \mathbb{R}^{1\times 3}$ Ist Φ ein Gradientenfeld? Falls ja suche eine Stammfunktion.

i) $\Phi\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = (1\ 1\ 1)$ $\quad \frac{\partial\Phi}{\partial\xi} = \frac{\partial\Phi}{\partial\eta} = \frac{\partial\Phi}{\partial\zeta} = (0\ 0\ 0) \Rightarrow$ Gradientenfeld

$\frac{\partial f}{\partial \xi} = 1 \quad \Rightarrow f\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = \int 1\, d\xi = \xi + c(\eta, \zeta)$

$\frac{\partial f}{\partial \eta} = c'(\eta,\zeta) = 1 \quad \Rightarrow c(\eta,\zeta) = \int 1\, d\eta = \eta + c(\zeta) \quad \Rightarrow f\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = \xi + \eta + c(\zeta)$

$\frac{\partial f}{\partial \zeta} = c'(\zeta) = 1 \quad \Rightarrow c(\zeta) = \int 1\, d\zeta = \zeta + c \quad \Rightarrow f\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = \xi + \eta + \zeta + c$

ii) $\Phi\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = (-\xi, -\eta, -\zeta)$

$\frac{\partial\Phi}{\partial\xi} = (-1,0,0) \quad \frac{\partial\Phi}{\partial\eta} = (0,-1,0) \quad \frac{\partial\Phi}{\partial\zeta} = (0,0,-1) \quad \Rightarrow$ Gradientenfeld

$\frac{\partial f}{\partial \xi} = -\xi \quad \Rightarrow f\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = \int -\xi\, d\xi = -\frac{1}{2}\xi^2 + c(\eta,\zeta)$

$\frac{\partial f}{\partial \eta} = c'(\eta,\zeta) = -\eta \quad \Rightarrow c(\eta,\zeta) = \int -\eta\, d\eta = -\frac{1}{2}\eta^2 + c(\zeta) \quad \Rightarrow f\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = -\frac{1}{2}\xi^2 - \frac{1}{2}\eta^2 + c(\zeta)$

$\frac{\partial f}{\partial \zeta} = c'(\zeta) = -\zeta \quad \Rightarrow c(\zeta) = \int -\zeta\, d\zeta = -\frac{1}{2}\zeta^2 \quad \Rightarrow f\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = -\frac{1}{2}(\xi^2 + \eta^2 + \zeta^2)$

iii) $\Phi\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = (\eta^3, 2, \xi^2)$

$\frac{\partial\Phi}{\partial\xi} = (0,0,2\xi) \quad \frac{\partial\Phi}{\partial\eta} = (3\eta^2, 0, 0) \quad \frac{\partial\Phi}{\partial\zeta} = (0,0,0)$

$\frac{\partial\Phi}{\partial\xi} e_3 = 2\xi \quad \frac{\partial\Phi}{\partial\zeta} e_1 = 0 \quad \Rightarrow$ kein Gradientenfeld

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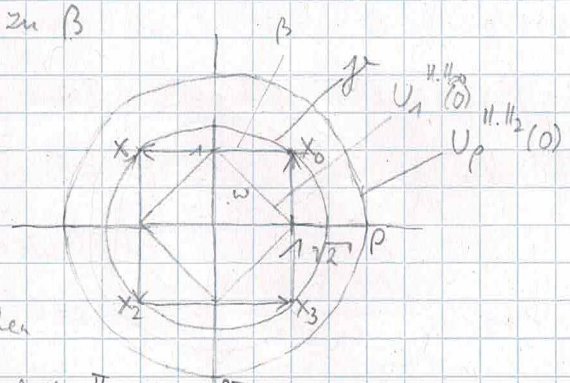
3.) $x_0=\begin{pmatrix}1\\1\end{pmatrix}$ $x_1=\begin{pmatrix}-1\\1\end{pmatrix}$ $x_2=\begin{pmatrix}-1\\-1\end{pmatrix}$ $x_3=\begin{pmatrix}1\\-1\end{pmatrix}$ $\gamma:[\frac{\pi}{4}, 2\pi+\frac{\pi}{4}] \longrightarrow \mathbb{R}^2$

$t \longmapsto \begin{pmatrix}\sqrt{2}\cos(t)\\ \sqrt{2}\sin(t)\end{pmatrix}$

$\rho > \sqrt{2}$ $\quad w \in U_1^{\|.\|_\infty}(0) \subseteq \mathbb{R}^2$

$\beta:[\frac{\pi}{4}, 2\pi+\frac{\pi}{4}] \to \mathbb{R}^2$ $\quad \alpha \ldots$ linear $\quad \beta = \overrightarrow{x_0x_1}, \overrightarrow{x_1x_2}, \overrightarrow{x_2x_3}, \overrightarrow{x_3x_0} \circ \alpha$

zz: γ ist in $U_\rho^{\|.\|_2}(0)\setminus\{w\}$ homotop zu β



$\gamma(\frac{\pi}{4}) = \begin{pmatrix}\sqrt{2}\cos(\frac{\pi}{4})\\ \sqrt{2}\sin(\frac{\pi}{4})\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = x_0$

$\gamma(2\pi+\frac{\pi}{4}) = \begin{pmatrix}\sqrt{2}\cos(2\pi+\frac{\pi}{4})\\ \sqrt{2}\sin(2\pi+\frac{\pi}{4})\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = x_0$

$\Rightarrow \gamma$ und $\overrightarrow{x_0x_1}, \overrightarrow{x_1x_2}, \overrightarrow{x_2x_3}, \overrightarrow{x_3x_0}$ haben gleichen Anfangs- und Endpunkt

$$\beta:[\tfrac{\pi}{4}, 2\pi+\tfrac{\pi}{4}] \to \mathbb{R}^2 \qquad t \mapsto \begin{cases} \begin{pmatrix}\tan(t-\frac{\pi}{2})\\ 1\end{pmatrix} & \text{falls } \frac{\pi}{4} \le t < \frac{3\pi}{4} \\ \begin{pmatrix}-1\\ \tan(t)\end{pmatrix} & \text{falls } \frac{3\pi}{4} < t \le \pi+\frac{\pi}{4} \\ \begin{pmatrix}\tan(t-\frac{\pi}{2})\\ -1\end{pmatrix} & \text{falls } \pi+\frac{\pi}{4} < t \le \pi+\frac{3\pi}{4} \\ \begin{pmatrix}1\\ \tan(t)\end{pmatrix} & \text{falls } \pi+\frac{3\pi}{4} < t \le 2\pi+\frac{\pi}{4} \end{cases}$$

$\Gamma:[\frac{\pi}{4}, 2\pi+\frac{\pi}{4}] \times [0,1] \longrightarrow \mathbb{R}^2$

$$(s,t) \longmapsto \beta(s) + t\cdot(\gamma(s)-\beta(s)) = t\cdot\gamma(s) + (1-t)\beta(s)$$

$\Gamma(s,0) = \beta(s) + 0\cdot(\gamma(s)-\beta(s)) = \beta(s)$ $\qquad \Gamma(s,1) = \beta(s) + 1(\gamma(s)-\beta(s)) = \gamma(s)$

Γ ist als Zusammensetzung stetigen Funktionen stetig.

zz: $\forall s \in [\frac{\pi}{4}, 2\pi+\frac{\pi}{4}]$ $\forall t \in [0,1]$: $\|\Gamma(s,t)\|_\infty \geq 1$

$\|\Gamma(s,t)\|_\infty \geq \|\beta(s)\|_\infty$, da $\Gamma(s,t) = \beta(s) + t(\gamma(s)-\beta(s))$

$\|\beta(s)\|_\infty = \max(|\tan(t)|, |1|) \geq 1$ oder

$\|\beta(s)\|_\infty = \max(|\tan(t-\frac{\pi}{2})|, |1|) \geq 1$

$\Rightarrow \gamma$ ist zu β homotop

ANA Ü12

4.) Kettenregel

$G \subseteq \mathbb{C}$ ... offen $\quad Y$... komplexer Banachraum $\quad \phi: D \to \mathbb{C} \quad g: G \to Y$

ϕ, g ... holomorphe Funktionen

$\Rightarrow g \circ \phi : \phi^{-1}(G) \to Y$ ist holomorph mit $(g \circ \phi)'(z) = \phi'(z) \cdot g'(\phi(z)) \ \forall z \in \phi^{-1}(G)$

ges: Beweis wie im Kapitel 7 und Beweis mit mehrdimensionale Kettenregel

ϕ, g ... holomorph $\Rightarrow \forall z \in D \ \ \phi$ ist bei z komplex diffbar und $z \mapsto \phi'(z)$ .. stetig

$\forall h \in G \ \ g$ ist bei h komplex diffbar und $h \mapsto g'(h)$ .. stetig

$$\psi(s) = \begin{cases} \dfrac{g(s) - g(\phi(z))}{s - \phi(z)} & \text{falls } s \neq \phi(z) \\ g'(\phi(z)) & \text{falls } s = \phi(z) \end{cases}$$

ist bei $\phi(z)$ stetig

$\forall t \in D \setminus \{z\}$

$$\psi(\phi(t)) \cdot \frac{\phi(t) - \phi(z)}{t - z} = \frac{g(\phi(t)) - g(\phi(z))}{\phi(t) - \phi(z)} \cdot \frac{\phi(t) - \phi(z)}{t - z} = \frac{g(\phi(t)) - g(\phi(z))}{t - z}$$

$$(g \circ \phi)'(z) = \lim_{t \to z} \psi(\phi(t)) \cdot \frac{\phi(t) - \phi(z)}{t - z} = \psi(\lim_{t \to z} \phi(t)) \cdot \phi'(z) = g'(\phi(z)) \cdot \phi'(z)$$

$\phi: D \to \mathbb{R}^2 \cong \mathbb{C} \qquad g: G_{\subseteq \mathbb{C} \cong \mathbb{R}^2} \to Y \qquad$ Beide stetig diffbar, da holomorph

$\Rightarrow \forall x \in D: \ d(g \circ \phi)(x) = dg(\phi(x))\, d\phi(x)$

$d(g \circ \phi)(x) = (g \circ \phi)'(x)$

$dg(\phi(x))\, d\phi(x) = g'(\phi(x)) \cdot \phi'(x)$

ANA Ü12

5.) Y ... Komplexer Banachraum $\quad D, G \subseteq \mathbb{C}$ ... offen $\quad \Phi: D \to \mathbb{C}$ ... holomorph

$\Phi(D) \subseteq G \quad f: G \to Y$ ... stetig $\quad \gamma: [a,b] \to D$ ... sssd

zz: $\displaystyle\int_{\Phi\circ\gamma} f(z)\,dz = \int_{\gamma} \Phi'(z)\cdot(f\circ\Phi)(z)\,dz$

Da Φ diffbar ist folgt Φ ist stetig $\Rightarrow$ $\Phi\circ\gamma$ ist sssd. Nach Satz 11.2.5 folgt

$$\int_{\Phi\circ\gamma} f(z)\,dz = \int_a^b f(\Phi(\gamma(t)))\,(\Phi\circ\gamma)'(t)\,dt = \int_a^b (f\circ\Phi)(\gamma(t))\,\Phi'(\gamma(t))\cdot\gamma'(t)\,dt$$

$$= \int_a^b ((f\circ\Phi)\cdot\Phi')(\gamma(t))\cdot\gamma'(t)\,dt = \int_\gamma ((f\circ\Phi)\cdot\Phi')(x)\,dx = \int_\gamma \Phi'(x)\cdot(f\circ\Phi)(x)\,dx$$

ANA Ü12:

6.) $k \in \mathbb{Z}$ $\quad \gamma : [0, 2\pi] \to \mathbb{C}$ $\quad r > 0$ $\quad a \in \mathbb{C}$

$t \mapsto a + r \cdot \exp(it)$

ges: $\int_\gamma (z-a)^k \, dz$

$\gamma'(t) = r \cdot i \cdot \exp(it)$ ist stetig $\qquad (z-a)^k$ ist für alle $k \in \mathbb{Z}$, $a \in \mathbb{C}$ stetig

Nach Satz 11.2.5 folgt

$$\int_\gamma (z-a)^k \, dz = \int_0^{2\pi} (\gamma(t) - a)^k \cdot \gamma'(t) \, dt = \int_0^{2\pi} (a + r \cdot \exp(it) - a)^k \cdot r \cdot i \cdot \exp(it) \, dt$$

$$= \int_0^{2\pi} r^k \cdot \exp(it)^k \cdot r \cdot i \cdot \exp(it) \, dt = r^{k+1} \cdot i \cdot \int_0^{2\pi} \exp(it)^{k+1} \, dt$$

$$= r^{k+1} \cdot i \cdot \left(-\frac{1}{k+1} \cdot i \cdot \exp(it)^{k+1} \right) \Big|_0^{2\pi} = r^{k+1} \cdot i \left(-\frac{i}{k+1} \exp(2\pi i)^{k+1} + \frac{1}{k+1} \cdot i \cdot \exp(0)^{k+1} \right)$$

$$= r^{k+1} \cdot i \left(-\frac{i}{k+1} + \frac{i}{k+1} \right) = 0$$

ANA Ü12

7.) $f: D \to \mathbb{C}$ $\quad z \in D \quad$ f ... ist bei z komplex differenzierbar

zz: $\det df(z) = |f'(z)|^2$

$$\det df = \det \begin{pmatrix} \frac{\partial Re(f)}{\partial x} & \frac{\partial Im(f)}{\partial x} \\ \frac{\partial Re(f)}{\partial y} & \frac{\partial Im(f)}{\partial y} \end{pmatrix} = \frac{\partial Re(f)}{\partial x} \cdot \frac{\partial Im(f)}{\partial y} - \frac{\partial Re(f)}{\partial y} \cdot \frac{\partial Im(f)}{\partial x}$$

$$\overset{\uparrow}{=} \frac{\partial Re(f)}{\partial x} \cdot \frac{\partial Re(f)}{\partial x} + \frac{\partial Im(f)}{\partial x} \cdot \frac{\partial Im(f)}{\partial x} = \left(\frac{\partial Re(f)}{\partial x}\right)^2 + \left(\frac{\partial Im(f)}{\partial x}\right)^2$$

Cauchy-Riemannsche Differenzialgleichungen

$$|f'(z)|^2 = \left|\frac{\partial f}{\partial x}(z)\right|^2 = \left|\frac{\partial Re(f)}{\partial x} + i\,\frac{\partial Im(f)}{\partial x}\right|^2 = \sqrt{\left(\frac{\partial Re(f)}{\partial x}\right)^2 + \left(\frac{\partial Im(f)}{\partial x}\right)^2}^{\,2}$$

$$= \left(\frac{\partial Re(f)}{\partial x}\right)^2 + \left(\frac{\partial Im(f)}{\partial x}\right)^2$$

ANA Ü12

8.) $D \subseteq \mathbb{C}$ ... Gebiet $f: D \to \mathbb{C}$ ... holomorph

zz: $f(D) \subseteq \mathbb{R} \vee f(D) \subseteq i\mathbb{R} \implies f$ .. konstant

$f(z) = u(z) + i\,v(z)$ o.B.d.A. $f(D) \subseteq \mathbb{R}$

$$df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \\ -\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Cauchy-Riemannsche Differentialgleichungen

$v(z) = 0 \;\forall z \in D \implies v'(z) = 0 \;\forall z \in D$

Da die erste Ableitung gleich 0 ist f konstant.