ANA Ü9

2.) $f: \mathbb{C} \to \mathbb{C}$ $\quad z = x+iy \quad u\binom{x}{y} = \operatorname{Re} f(x+iy)$

$z \mapsto \exp(z) \quad v\binom{x}{y} = \operatorname{Im} f(x+iy)$

f...stetig differenzierbar bereits bekannt

zz: $\frac{\partial u}{\partial x}\binom{x}{y} = \frac{\partial v}{\partial y}\binom{x}{y}$ und $\frac{\partial u}{\partial y}\binom{x}{y} = -\frac{\partial v}{\partial x}\binom{x}{y}$ ...Cauchy-Riemannschen Differentialgleichungen

- $\frac{\partial u}{\partial x}\binom{x}{y} = \frac{d}{dx} \operatorname{Re}(\exp(x+iy)) = \frac{d}{dx} \operatorname{Re}(\exp(x)(\cos(y)+i\sin(y)))$

$= \frac{d}{dx} \exp(x)\cos(y) = \cos(y)\cdot\exp(x)$

- $\frac{\partial v}{\partial y}\binom{x}{y} = \frac{d}{dy} \operatorname{Im}(\exp(x+iy)) = \frac{d}{dy} \operatorname{Im}(\exp(x)(\cos(y)+i\sin(y)))$

$= \frac{d}{dy} \exp(x)\sin(y) = \cos(y)\cdot\exp(x)$

- $\frac{\partial u}{\partial y}\binom{x}{y} = \frac{d}{dy} \operatorname{Re}(\exp(x+iy)) = \frac{d}{dy} \exp(x)\cdot\cos(y) = -\sin(y)\cdot\exp(x)$

- $-\frac{\partial v}{\partial x}\binom{x}{y} = -\frac{d}{dx} \operatorname{Im}(\exp(x+iy)) = -\frac{d}{dx} \exp(x)\sin(y) = -\sin(y)\cdot\exp(x)$

$f: \mathbb{C}\setminus\{0\} \to \mathbb{C} \quad z \mapsto \frac{1}{z}$

| f...stetig diffbar bereits bekannt

$\frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

- $\frac{\partial u}{\partial x}\binom{x}{y} = \frac{d}{dx} \operatorname{Re}\left(\frac{1}{x+iy}\right) = \frac{d}{dx} \frac{x}{x^2+y^2}$

$= \frac{x^2+y^2-x2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{x^4+2x^2y^2+y^4}$

- $\frac{\partial v}{\partial y}\binom{x}{y} = \frac{d}{dy} \operatorname{Im}\left(\frac{1}{x+iy}\right) = \frac{d}{dy} -\frac{y}{x^2+y^2} = -\frac{x^2+y^2-y2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{x^4+2x^2y^2+y^4}$

- $\frac{\partial u}{\partial y}\binom{x}{y} = \frac{d}{dy} \operatorname{Re}\left(\frac{1}{x+iy}\right) = \frac{d}{dy} \frac{x}{x^2+y^2} = \frac{-x2y}{(x^2+y^2)^2} = \frac{-2xy}{x^4+2x^2y^2+y^4}$

- $-\frac{\partial v}{\partial x}\binom{x}{y} = -\frac{d}{dx} \operatorname{Im}\left(\frac{1}{x+iy}\right) = -\frac{d}{dx} -\frac{y}{x^2+y^2} = \frac{-y2x}{x^4+2x^2y^2+y^4}$

$\Rightarrow \exp(z), \frac{1}{z}$ sind holomorph

$f: \mathbb{C} \to \mathbb{C} \quad z \mapsto z$

- $\frac{\partial u}{\partial x}\binom{x}{y} = \frac{d}{dx} \operatorname{Re}(x+iy) = \frac{d}{dx} x = 1 \qquad \frac{\partial v}{\partial y}\binom{x}{y} = \frac{d}{dy} \operatorname{Im}(x+iy) = \frac{d}{dy} y = 1$

- $\frac{\partial u}{\partial y}\binom{x}{y} = \frac{d}{dy} \operatorname{Re}(x+iy) = \frac{d}{dy} x = 0 \qquad -\frac{\partial v}{\partial x}\binom{x}{y} = -\frac{d}{dx} \operatorname{Im}(x+iy) = -\frac{d}{dx} y = 0$ $\Rightarrow z \mapsto z$ ist holomorph

ANA Ü9

3.) $f\binom{x}{y} = x^3 - 2x^2y^2 + 4xy^3 + y^4 + 10$

ges: $\frac{\partial^{k+l}}{\partial x^k \partial y^l}$ für $k,l = 0,1,2$

$$\frac{\partial}{\partial x} f\binom{x}{y} = \frac{d}{dx}\, x^3 - 2x^2y^2 + 4xy^3 + y^4 + 10 = 3x^2 - 4xy^2 + 4y^3 =: f_{x_1}\binom{x}{y}$$

$$\frac{\partial}{\partial x} f_{x_1}\binom{x}{y} = \frac{d}{dx}\, 3x^2 - 4xy^2 + 4y^3 = 6x - 4y^2 =: f_{x_2}\binom{x}{y}$$

$$\frac{\partial}{\partial y} f\binom{x}{y} = \frac{d}{dy}\, x^3 - 2x^2y^2 + 4xy^3 + y^4 + 10 = -4x^2y + 12xy^2 + 4y^3 =: f_{y_1}\binom{x}{y}$$

$$\frac{\partial}{\partial y} f_{y_1}\binom{x}{y} = \frac{d}{dy}\, -4x^2y + 12xy^2 + 4y^3 = -4x^2 + 24xy + 12y^2 =: f_{y_2}\binom{x}{y}$$

$$\frac{\partial}{\partial y} f_{x_1}\binom{x}{y} = \frac{d}{dy}\, 3x^2 - 4xy^2 + 4y^3 = -8xy + 12y^2 =: f_{x_1y_1}\binom{x}{y}$$

$$\frac{\partial}{\partial y} f_{x_2}\binom{x}{y} = \frac{d}{dy}\, 6x - 4y^2 = -8y =: f_{x_2y_1}\binom{x}{y}$$

$$\frac{\partial}{\partial x} f_{y_2}\binom{x}{y} = \frac{d}{dx}\, -4x^2 + 24xy + 12y^2 = -8x + 24y =: f_{x_1y_2}\binom{x}{y}$$

$$\frac{\partial}{\partial x} f_{x_1y_2}\binom{x}{y} = \frac{d}{dx}\, -8x + 24y = -8 =: f_{x_2y_2}\binom{x}{y}$$

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} = f_{x_k y_l}\binom{x}{y}$$ f, f_{x_1}, f_{y_1} stetig partiell diffbar

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ges: $d\, f\binom{x}{y}\, v_1$, $d^2 f\binom{x}{y}(v_1, v_2)$

$$d f\binom{x}{y} = \left(\frac{\partial f}{\partial x_j}\left(\binom{x}{y}\right)\right)_{j=1,2} = \left(f_{x_1}\binom{x}{y},\, f_{y_1}\binom{x}{y}\right)^T$$

$$\left(f_{x_1}\binom{x}{y},\, f_{y_1}\binom{x}{y}\right)^T \cdot \binom{v_a}{v_b} = v_a \cdot (3x^2 - 4xy^2 + 4y^3) + v_b(-4x^2y + 12xy^2 + 4y^3)$$

$$d^2 f\binom{x}{y}(v_1, v_2) = \frac{\partial}{\partial v_1}\frac{\partial}{\partial v_2} f\binom{x}{y} = \sum_{l_1=1}^{2} v_{1,l_1} \sum_{l_2=1}^{2} v_{2,l_2} \frac{\partial^2 f}{\partial x_{l_1} x_{l_2}} \binom{x}{y}$$

$$= \sum_{l_1=1}^{2} v_{1,l_1}\left(v_{2,1}\, f_{x_{l_1},y_1}\binom{x}{y} + v_{2,2}\, f_{x_{l_1},y_2}\binom{x}{y}\right) = v_{1,1}\left(v_{2,1} f_{x_1y_1}\binom{x}{y} + v_{22} f_{x_1y_2}\binom{x}{y}\right)$$

$$+ v_{1,2}\left(v_{2,1}\, f_{x_2y_1}\binom{x}{y} + v_{22}\, f_{x_2,y_2}\binom{x}{y}\right) =$$

$$= v_{11} \cdot v_{21} \cdot (-8xy + 12y^2) + v_{11} v_{22} \cdot (-8x + 24y) + v_{12} v_{21}(-8y) + v_{12} v_{22}(-8)$$

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ANA Ü9

3.) ... ges: Taylorpolynom in $\binom{x}{y}$ mit $q=2$ und Anschlussstelle $\binom{0}{0}$

$$T_2(x) = f\binom{0}{0} + \sum_{l=1}^{2} \frac{1}{l!}\, d^l f\binom{0}{0} \underbrace{\left(x-\binom{0}{0}, \dots, x-\binom{0}{0}\right)}_{l\text{-Mal}} \qquad x=\binom{x_1}{x_2}$$

$$= 10 + df\binom{0}{0}x + \frac{1}{2}\, d^2 f\binom{0}{0}(x,x)$$

$$= 10 + x_1 \cdot 0 + x_2 \cdot 0 + \frac{1}{2}\left(0 + x_2 \cdot x_2 (-8)\right) = 10 - 4x_2^2$$

ANA Üg

4.) $f\begin{pmatrix}x\\y\\z\end{pmatrix} = \sin(3x+yz)$ ges: partielle Ableitungen bis 3. Grad

z_0	x_0	x_1	x_2	x_3
y_0	$\sin(3x+yz)$	$\cos(3x+yz)\cdot 3$	$-\sin(3x+yz)\cdot 9$	$-\cos(3x+yz)\cdot 27$
y_1	$\cos(3x+yz)\cdot z$	$-\sin(3x+yz)\cdot 3z$	$-\cos(3x+yz)9z$	
y_2	$-\sin(3x+yz)\cdot z^2$	$-\cos(3x+yz)\cdot 3z^2$		
y_3	$-\cos(3x+yz)\cdot z^3$			

z_1	x_0	x_1	x_2
y_0	$\cos(3x+yz)\cdot y$	$-\sin(3x+yz)\cdot 3y$	$-\cos(3x+yz)\cdot 9y$
y_1	$-\sin(3x+yz)\cdot zy+\cos(3x+yz)$	$-3(\cos(3x+yz)\cdot zy+\sin(3x+yz))$	
y_2	$-z(\cos(3x+yz)\cdot zy+\sin(3x+yz))-\sin(3x+yz)\cdot z$ $= -z^2y\cos(3x+yz)-2z\sin(3x+yz)$		

z_2	x_0	x_1
y_0	$-\sin(3x+yz)\cdot y^2$	$-\cos(3x+yz)\cdot 3y^2$
y_1	$-(\cos(3x+yz)\cdot z\cdot y^2+\sin(3x+yz)\cdot 2y)$	

z_3	x_0
y_0	$-\cos(3x+yz)\cdot y^3$

ges: $df\begin{pmatrix}x\\y\\z\end{pmatrix}(v)$ und $d^2f\begin{pmatrix}x\\y\\z\end{pmatrix}(v,w)$ $\quad v=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ $\quad w=\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$

$$df\begin{pmatrix}x\\y\\z\end{pmatrix}(v) = \frac{\partial}{\partial v} f\begin{pmatrix}x\\y\\z\end{pmatrix} = v_1\cdot\cos(3x+yz)\cdot 3 + v_2\cdot\cos(3x+yz)\cdot z + v_3\cdot\cos(3x+yz)\cdot y$$

$$d^2f\begin{pmatrix}x\\y\\z\end{pmatrix}(v,w) = \sum_{l_1,l_2=1}^{3} v_{l_1}\cdot w_{l_2}\cdot\frac{\partial^2}{\partial x_{l_1}\partial x_{l_2}} f\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$= v_1\cdot w_1\frac{\partial^2}{\partial x\partial x}f\begin{pmatrix}x\\y\\z\end{pmatrix} + v_1\cdot w_2\frac{\partial^2}{\partial x\partial y}f\begin{pmatrix}x\\y\\z\end{pmatrix} + v_1\cdot w_3\frac{\partial^2}{\partial x\partial z}f\begin{pmatrix}x\\y\\z\end{pmatrix} + v_2\cdot w_1\frac{\partial^2}{\partial y\partial x}f\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$+ v_2\cdot w_2\frac{\partial^2}{\partial y\partial y}f\begin{pmatrix}x\\y\\z\end{pmatrix} + v_2\cdot w_3\frac{\partial^2}{\partial y\partial z}f\begin{pmatrix}x\\y\\z\end{pmatrix} + v_3 w_1\frac{\partial^2}{\partial z\partial x}f\begin{pmatrix}x\\y\\z\end{pmatrix} + v_3 w_2\frac{\partial^2}{\partial z\partial y}f\begin{pmatrix}x\\y\\z\end{pmatrix} + v_3 w_3\frac{\partial^2}{\partial z\partial z}f\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$= v_1\cdot w_1\cdot(-\sin(3x+yz)\cdot 9) + v_1 w_2(-\sin(3x+yz)\cdot 3z) + v_1\cdot w_3(-\sin(3x+yz)\cdot 3y) + v_2\cdot w_1(-\sin(3x+yz)\cdot 3z)$$

$$+ v_2\cdot w_2(-\sin(3x+yz)\cdot z^2) + v_2\cdot w_3(-\sin(3x+yz)\cdot zy+\cos(3x+yz)) + v_3\cdot w_1(-\sin(3x+yz)\cdot 3y)$$

$$+ v_3 w_2(-\sin(3x+yz)zy+\cos(3x+yz)) + v_3 w_3(-\sin(3x+yz)\cdot y^2)$$

...

ANA Üg

4.) ... ges: Taylorpolynom in $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ mit $q=3$ und Anschlussstelle $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$T_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = f\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \sum_{l=1}^{3} \frac{1}{l!} d^l f\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \underbrace{\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)}_{l\text{-mal}}$$

$$= 0 + df\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{2} d^2 f\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) + \frac{1}{6} d^3 f\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$$

$$= 3x \cdot \cos(0) + \frac{1}{2}\left(-9x^2 \cdot \sin(0) + y \cdot z \cdot \cos(0) + z \cdot y \cdot \cos(0)\right) + \frac{1}{6}\left(-27x^3 \cdot \cos(0) - 3xy \cdot z \sin(0)\right)$$

$$= 3x + yz - 4{,}5\, x^3$$

ANA Üg

8.) $f: [0,1]^2 \to \mathbb{R}^2$ $\quad f\binom{x}{y} = 3x^2 - 2(y+1)x + 3y - 1$

$= 3x^2 - 2xy - 2x + 3y - 1$

$\frac{\partial}{\partial x_1} f\binom{x}{y} = 6x - 2y - 2$ $\quad \frac{\partial}{\partial y} f\binom{x}{y} = -2x + 3$ $\quad \frac{\partial^2}{\partial x^2} f\binom{x}{y} = 6$ $\quad \frac{\partial^2}{\partial x \partial y} = -2$ $\quad \frac{\partial^2}{\partial y^2} = 0$

$df\binom{x}{y} = (6x - 2y - 2 \quad -2x + 3) = (0 \quad 0) \Leftrightarrow x = \frac{3}{2} \wedge y = \frac{7}{2}$ ... liegt nicht in $[0,1]^2$

Wegen des Definitionsbereiches gibt es sicher ein Maximum und Minimum, das an einem "Rand" liegen muss (da $df\binom{x}{y} \neq \binom{0}{0} \ \forall x,y \in [0,1]$).

- Rand $(0,1) \times \{0\}$

  $\frac{\partial}{\partial x} f\binom{x}{0} = 6x - 2 = 0 \Leftrightarrow x = \frac{1}{3}$

  $f\binom{\frac{1}{3}}{0} = -\frac{4}{3}$

- Rand $\{0\} \times (0,1)$

  $\frac{\partial}{\partial y} f\binom{0}{y} = 3 \neq 0$

- Rand $(0,1) \times \{1\}$

  $\frac{\partial}{\partial x} f\binom{x}{1} = 6x - 2 - 2 = 0 \Leftrightarrow x = \frac{2}{3}$

  $f\binom{\frac{2}{3}}{1} = \frac{2}{3}$

- Rand $\{1\} \times (0,1)$

  $\frac{\partial}{\partial y} f\binom{1}{y} = -2 + 3 = 1 \neq 0$

- Ecke $(0,0)$: $f\binom{0}{0} = -1$
- Ecke $(0,1)$: $f\binom{0}{1} = 2$
- Ecke $(1,0)$: $f\binom{1}{0} = 0$
- Ecke $(1,1)$: $f\binom{1}{1} = 1$

$\Rightarrow f$ hat ein globales Minimum bei $\binom{\frac{1}{3}}{0}$ und ein globales Maximum bei $\binom{0}{1}$

ANA Ü9

9.) $f: \{ \binom{x}{y} \in \mathbb{R}^2 : x^2+y^2 \leq 1 \} \to \mathbb{R}$

$$\binom{x}{y} \mapsto 8x^2 - 2xy + 3y - 1$$

$\frac{\partial}{\partial x} f\binom{x}{y} = 16x - 2y$ $\qquad$ $\frac{\partial}{\partial y} f\binom{x}{y} = -2x + 3$

$\frac{\partial^2}{\partial x^2} f\binom{x}{y} = 16$ $\qquad$ $\frac{\partial^2}{\partial y^2} f\binom{x}{y} = 0$ $\qquad$ $\frac{\partial^2}{\partial x \partial y} f\binom{x}{y} = -2$

$df\binom{x}{y} = (16x - 2y \quad -2x+3) = (0 \quad 0) \iff x = \frac{3}{2} \wedge y = 12$ liegt nicht im Einheitskreis

Da der Definitionsbereich beschränkt ist existiert ein globales Maximum und Minimum.

Da im Inneren kein Kandidat für Extrema gibt müssen die Extrema am "Rand" liegen.

$x^2+y^2 = 1 \iff y^2 = 1-x^2 \iff y = \sqrt{1-x^2}$

$\frac{\partial}{\partial x} f\binom{x}{\sqrt{1-x^2}} = 16x - 2\sqrt{1-x^2} = 0 \iff 2\sqrt{1-x^2} = 16x$

$\iff \sqrt{1-x^2} = 8x \iff 1-x^2 = 64x^2 \iff 65x^2 - 1 = 0$

$\iff x^2 - \frac{1}{65} = 0 \iff x_{1,2} = \pm\sqrt{+\frac{1}{65}} \iff x \pm \frac{1}{\sqrt{65}} \approx \pm \frac{1}{8{,}062}$

oder $-(1-x^2) = 64x^2 \iff -1+x^2 = 64x^2 \iff 63x^2+1 = 0$

$\iff x^2 + \frac{1}{63} = 0 \iff x_{1,2} = \pm\sqrt{-\frac{1}{63}}$ nicht in $\mathbb{R}$

$$f\begin{pmatrix} \frac{1}{\sqrt{65}} \\ \sqrt{\frac{64}{65}} \end{pmatrix} = 8\frac{1}{65} - 2\frac{1}{\sqrt{65}} \cdot \frac{\sqrt{64}}{\sqrt{65}} + 3\sqrt{\frac{64}{65}} - 1 = \frac{8}{65} - \frac{2\sqrt{64}}{65} + 3\frac{\sqrt{64}}{\sqrt{65}} = -\frac{8}{65} + \frac{24}{\sqrt{65}}$$

$$f\begin{pmatrix} -\frac{1}{\sqrt{65}} \\ \sqrt{\frac{64}{65}} \end{pmatrix} = 8\frac{1}{65} + 2\frac{1}{\sqrt{65}} \cdot \frac{\sqrt{64}}{\sqrt{65}} + 3\sqrt{\frac{64}{65}} - 1 = \frac{8}{65} + \frac{2\sqrt{64}}{65} + 3\frac{\sqrt{64}}{\sqrt{65}} = \frac{24}{65} + \frac{24}{\sqrt{65}}$$

$\Rightarrow f$ hat ein globales Minimum bei $\begin{pmatrix} \frac{1}{\sqrt{65}} \\ \frac{8}{\sqrt{65}} \end{pmatrix}$ und ein

globales Maximum bei $\begin{pmatrix} -\frac{1}{\sqrt{65}} \\ \frac{8}{\sqrt{65}} \end{pmatrix}$

ANA Ü9

7.) $f: \mathbb{R}^2 \to \mathbb{R}$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x^3 e^{x-y}$

	x_0	x_1	x_2	x_3
y_0	$x^3 e^{x-y}$	$3x^2 e^{x-y} + x^3 e^{x-y}$	$6xe^{x-y} + 3x^2 e^{x-y}$	$6e^{x-y} + 6xe^{x-y} + 6xe^{x-y} + 3x^2 e^{x-y}$
y_1	$-x^3 e^{x-y}$	$-3x^2 e^{x-y} - x^3 e^{x-y}$	$-6xe^{x-y} - 3x^2 e^{x-y}$	
y_2	$x^3 e^{x-y}$	$3x^2 e^{x-y} + x^3 e^{x-y}$		
y_3	$-x^3 e^{x-y}$			

$df\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff x = 0 \qquad d^2 f\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff x = 0$

$d^3 f\begin{pmatrix} 0 \\ y \end{pmatrix} \neq 0$ da $6e^{0-y} + 6 \cdot 0 \cdot e^{0-y} + 6 \cdot 0 \cdot e^{0-y} + 3 \cdot 0^2 \cdot e^{0-y} = 6 \cdot e^{-y} \neq 0 \ \forall y \in \mathbb{R}$

q ... ungerade $\Rightarrow$ bei $\begin{pmatrix} 0 \\ y \end{pmatrix}$ kein lokales Extremum $\forall y \in \mathbb{R}$

$\left[\lim_{x \to +\infty} \lim_{y \to -\infty} f\begin{pmatrix} x \\ y \end{pmatrix} = \lim_{x \to +\infty} \lim_{y \to -\infty} x^3 \exp(x-y) = \lim_{x \to +\infty} \lim_{y \to +\infty} x^3 \cdot \exp(x+y) = +\infty \right]$

$\Rightarrow f$ hat keine globalen Extremwerte