PDGL Ü2

1) Bestimmen Sie Mengen $\subseteq \mathbb{R}^2$, wo die PDGL elliptisch, parabolisch oder hyperbolisch sind:

(i) $x u_{xx} + 6 u_{xy} + (x+y) u_{yy} + x^2 y u_x = \cosh(x)$

$a = x \qquad b = 3 \qquad c = x+y \qquad f = \cosh(x) - x^2 y u_x$

$b^2 - ac = 3^2 - x(x+y) = 9 - x^2 - xy$

parabolisch: $9 - x^2 - xy = 0 \iff xy = 9 - x^2 \iff x \neq 0 \wedge y = \frac{9-x^2}{x}$

$\{(x,y) \in \mathbb{R}^2 : x \neq 0, y = \frac{9-x^2}{x}\}$

elliptisch: $9 - x^2 - xy < 0 \iff xy > 9 - x^2 \iff (x>0 \wedge y > \frac{9-x^2}{x}) \vee (x<0 \wedge y < \frac{9-x^2}{x})$

$\{(x,y) \in \mathbb{R}^2 : x>0, y > \frac{9-x^2}{x}\} \cup \{(x,y \in \mathbb{R}^2 : x<0, y < \frac{9-x^2}{x}\}$

hyperbolisch: $9 - x^2 - xy > 0 \iff xy < 9 - x^2 \iff (x>0 \wedge y < \frac{9-x^2}{x}) \vee (x<0 \wedge y > \frac{9-x^2}{x})$

$\{(x,y) \in \mathbb{R}^2 : x>0, y < \frac{9-x^2}{x}\} \cup \{(x,y) \in \mathbb{R}^2 : x<0, y > \frac{9-x^2}{x}\}$

(ii) $(7x^2 - \pi y^2) u_{xx} + 4\pi x^2 u_{xy} + y^2 u_{yy} - u^2 u_x - \sqrt{2} x = e^{2\pi x}$

$a = 7x^2 - \pi y^2 \qquad b = 2\pi x^2 \qquad c = y^2 \qquad f = e^{2\pi x} + u^2 u_x + \sqrt{2} x$

$b^2 - ac = 4\pi^2 x^4 - (7x^2 - \pi y^2) y^2 = 4\pi^2 x^4 - 7x^2 y^2 + \pi y^4$

parabolisch: $4\pi^2 x^4 - 7x^2 y^2 + \pi y^4 = 0 \iff y^2 = \frac{7x^2 \pm \sqrt{49x^4 - 4\pi \, 4\pi^2 x^4}}{2\pi} = \frac{7x^2 \pm \sqrt{49 - 16\pi^3}\, x^2}{2\pi}$

da wir in $\mathbb{R}^2$ suchen und $49 - 16\pi^3 < 0$ gibt es nur die $(0,0)$ Lösung $\{(0,0)\}$

elliptisch: $4\pi^2 x^4 - 7x^2 y^2 + \pi y^4 < 0$ hat keine Lösungen, da die parabolischen und hyperbolischen schon ganz $\mathbb{R}^2$ sind (wie wir gleich sehen werden) $\emptyset$

hyperbolisch $4\pi^2 x^4 - 7x^2 y^2 + \pi y^4 > 0 \iff \binom{x}{y} \neq \binom{0}{0}$, da stetig von x,y. Für $x = y = 0$ wie oben gesehen parabolisch und für etwa $x = y = 1$ ist der Ausdruck gleich $4\pi^2 - 7 + \pi > 4 \cdot 3^2 - 7 + 3 = 36 - 7 + 3 = 32 > 0$. Aus der Stetigkeit folgt $\{(x,y) \in \mathbb{R}^2 : \binom{x}{y} \neq \binom{0}{0}\}$

$\square$

Def $a u_{xx} + 2b u_{xy} + c u_{yy} = f$

$b^2 - ac < 0$ an (x,y) so ist die PDGL dort elliptisch

$b^2 - ac > 0$ — " — hyperbolisch

$b^2 - ac = 0$ — " — parabolisch

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2) Löse mit Charakteristiken Methode wo möglich:

(i) $x u_x + y u_y = u + 1$ mit $u(x,y) = x^2$ auf $x = y^2$

$\Gamma = \{(t^2, t) : t \in \mathbb{R}\}$

$$\left.\begin{array}{ll} \frac{\partial x}{\partial s} = x & x(0,t) = t^2 \\ \frac{\partial y}{\partial s} = y & y(0,t) = t \\ \frac{\partial u}{\partial s} = u+1 & u(0,t) = t^4 \end{array}\right\} \begin{array}{l} x(s,t) = t^2 e^s \\ y(s,t) = t e^s \\ u(s,t) = (t^4+1)e^s - 1 \end{array}$$

$$\det\begin{pmatrix} x_s(0,t) & x_t(0,t) \\ y_s(0,t) & y_t(0,t) \end{pmatrix} = \det\begin{pmatrix} t^2 e^s & 2te^s \\ te^s & e^s \end{pmatrix} = t^2 e^{2s} - 2t^2 e^{2s} = -t^2 e^{2s} = 0 \Leftrightarrow t = 0$$

$x = t^2 e^s,\ y = te^s \Rightarrow \frac{x}{t^2} = e^s = \frac{y}{t} \Rightarrow tx = t^2 y \Rightarrow x = ty \Rightarrow t = \frac{x}{y}$

$e^s = \frac{x}{t^2} \Rightarrow s = \ln\left(\frac{x}{t^2}\right) = \ln\left(\frac{x}{\frac{x^2}{y^2}}\right) = \ln\left(\frac{xy^2}{x^2}\right) = \ln\left(\frac{y^2}{x}\right)$

$\left(e^s = \frac{y}{t} \Rightarrow s = \ln\left(\frac{y}{t}\right) = \ln\left(\frac{y}{\frac{x}{y}}\right) = \ln\left(\frac{y^2}{x}\right)\right.$ sieht gut aus$\left.\right)$

$$\Rightarrow u(x,y) = u\left(\ln\left(\frac{y^2}{x}\right), \frac{x}{y}\right) = \left(\frac{x^4}{y^4}+1\right)e^{\ln\left(\frac{y^2}{x}\right)} - 1 = \left(\frac{x^4}{y^4}+1\right)\frac{y^2}{x} - 1 =$$

$$= \frac{x^3}{y^2} + \frac{y^2}{x} - 1$$

Probe: $x u_x + y u_y = x\left(3x^2\frac{1}{y^2} - y^2\frac{1}{x^2}\right) + y\left(-\frac{2x^3}{y^3} + \frac{2y}{x}\right) = \frac{3x^3}{y^2} - \frac{y^2}{x} - \frac{2x^3}{y^2} + \frac{2y^2}{x} = \frac{x^3}{y^2} + \frac{y^2}{x} = u+1$ ✓

$u(y^2, y) = \frac{y^6}{y^2} + \frac{y^2}{y^2} - 1 = y^4 + 1 - 1 = y^4 = x^2$ auf $x = y^2$ ✓

(ii) $u u_x + u_y = 3$ mit $u(x,y) = 2x$ auf $x = 4y$

$\Gamma = \{(4t, t) : t \in \mathbb{R}\}$

$$\left.\begin{array}{ll} \frac{\partial x}{\partial s} = u & x(0,t) = 4t \\ \frac{\partial y}{\partial s} = 1 & y(0,t) = t \\ \frac{\partial u}{\partial s} = 3 & u(0,t) = 8t \end{array}\right\} \begin{array}{l} \frac{\partial x}{\partial s} = 3s + 8t \Rightarrow x(s,t) = \frac{3}{2}s^2 + 8st + 4t \\ y(s,t) = s + t \\ u(s,t) = 3s + 8t \end{array}$$

$$\det\begin{pmatrix} 8t & 4 \\ 1 & 1 \end{pmatrix} = 8t - 4 = 0 \Leftrightarrow t = \frac{1}{2}$$

$y = s + t \Rightarrow t = y - s \qquad x = \frac{3}{2}s^2 + 8st + 4t = \frac{3}{2}s^2 + 8s(y-s) + 4(y-s)$

$$= \frac{3}{2}s^2 + 8sy - 8s^2 + 4y - 4s$$

$-\frac{13}{2}s^2 + (8y-4)s + 4y - x = 0$

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2)... $s = \frac{8y - 4 \mp \sqrt{64y^2 + 40y + 16 - 26x}}{13}$ $\quad t = y - \frac{8y - 4 \mp \sqrt{64y^2 + 40y + 16 - 26x}}{13}$

$$u(x,y) = 3s + 8t = \frac{3}{13}\left(8y - 4 \pm \sqrt{64y^2 + 40y + 16 - 26x}\right) + 8y - \frac{8}{13}\left(8y - 4 \mp \sqrt{64y^2 + 40y + 16 - 26x}\right)$$

$$= \frac{1}{13}\left(24y - 12 \pm 3\sqrt{64y^2 + 40y + 16 - 26x} + 104y - 64y + 32 \mp 8\sqrt{64y^2 + 40y + 16 - 26x}\right)$$

$$= \frac{1}{13}\left(64y + 20 \mp 5\sqrt{64y^2 + 40y + 16 - 26x}\right)$$

Probe: $uu_x + u_y = \ldots = 3$ mit Wolfram Alpha ✓

$$u(4y,y) = \frac{1}{13}\left(64y + 20 \mp 5\sqrt{64y^2 + 40y + 16 - 26 \cdot 4y}\right)$$

$$= \frac{1}{13}\left(64y + 20 \mp 5\sqrt{64y^2 - 64y + 16}\right) = \frac{1}{13}\left(64y + 20 \mp 5(8y - 4)\right)$$

$$= \frac{1}{13}(104y + 0) = 8y = 2x \quad \text{für } +$$

$$= \frac{1}{13}(24y + 40) \neq 2x \quad \text{für } -$$

✓

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3) $\Delta u = 0$ für $x \in \mathbb{R}^3 \setminus \{0\}$ ges: radialsymmetrische Lösung

In kartesischen Koordinaten: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

In Kugelkoordinaten: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = 0$

Wegen radialsymmetrisch ist $\frac{\partial u}{\partial \varphi} = 0 = \frac{\partial u}{\partial \theta}$ also erhalten wir

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{1}{r^2}\left(2r\frac{\partial u}{\partial r} + r^2 \frac{\partial^2 u}{\partial r^2}\right) = 2\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}$$

ist eine Euler-Cauchy DGL ($u'' + \frac{\alpha}{r}u' + \frac{\beta}{r^2}u = 0$ mit $\alpha = 2$ und $\beta = 0$)

Lösung: $u(r,\theta,\varphi) = \frac{c_1}{r} + c_2$

Probe: $u'(r,\theta,\varphi) = -\frac{c_1}{r^2}$ $\quad u''(r,\theta,\varphi) = \frac{2c_1}{r^3}$ $\quad \Rightarrow \frac{2}{r}u' + u'' = -\frac{2c_1}{r^3} + \frac{2c_1}{r^3} = 0$ ✓

Zurücktransformieren: $u(x,y,z) = \frac{c_1}{\sqrt{x^2+y^2+z^2}} + c_2$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = u_r \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = u_r \frac{x}{\sqrt{x^2+y^2+z^2}} = u_r \frac{r\sin\theta\cos\varphi}{r} = \sin\theta\cos\varphi\, u_r$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = u_r \frac{y}{\sqrt{x^2+y^2+z^2}} = u_r \sin\theta \sin\varphi$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} = u_r \frac{z}{\sqrt{x^2+y^2+z^2}} = u_r \cos\theta$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\sin\theta\cos\varphi\, u_r = \frac{\partial}{\partial r}\sin\theta\cos\varphi\, u_r \cdot \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\sin\theta\cos\varphi\, u_r \cdot \frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \varphi}\sin\theta\cos\varphi\, u_r \frac{\partial \varphi}{\partial x}$$

$$= \sin\theta\cos\varphi\, u_{rr} \cdot \frac{x}{r} + \cos\theta\cos\varphi\, u_r \cdot \frac{xz}{r^3\sqrt{1-\frac{z^2}{r^2}}} + \sin\theta\sin\varphi\, u_r \frac{y}{x^2+y^2}$$

$$= \sin^2\theta\cos^2\varphi\, u_{rr} + \cancel{r^2}\sin\theta\cos^2\theta\cos^2\varphi \frac{1}{r^3\cancel{\sqrt{1-\cos^2\theta}}} u_r + \cancel{r}\sin^2\theta\sin^2\varphi \frac{1}{r^2\sin^2\theta} u_r$$

$$= \sin^2\theta\cos^2\varphi\, u_{rr} + \frac{1}{r}(\cos^2\theta\cos^2\varphi + \sin^2\varphi)\, u_r$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} u_r \sin\theta\sin\varphi = \frac{\partial}{\partial r}\sin\theta\sin\varphi\, u_r \cdot \frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\sin\theta\sin\varphi\, u_r \cdot \frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \varphi}\sin\theta\sin\varphi\, u_r \cdot \frac{\partial \varphi}{\partial y}$$

$$= \sin\theta\sin\varphi\, u_{rr} \cdot \frac{y}{r} + \cos\theta\sin\varphi\, u_r \cdot \frac{yz}{\sqrt{x^2+y^2}\, r^2} + \sin\theta\cos\varphi\, u_r \cdot \frac{x}{x^2+y^2}$$

$$= \sin^2\theta\sin^2\varphi\, u_{rr} + \frac{r^2\sin\theta\cos^2\theta\sin^2\varphi}{r^2\sqrt{r^2\sin^2\theta(\sin^2\varphi+\cos^2\varphi)}} u_r + \frac{r\sin^2\theta\cos^2\varphi}{r^2\sin^2\theta(\sin^2\varphi+\cos^2\varphi)} u_r$$

$$= \sin^2\theta\sin^2\varphi\, u_{rr} + \left(\frac{\cos^2\theta\sin^2\varphi}{r} + \frac{\cos^2\varphi}{r}\right) u_r$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z} u_r\cos\theta = \frac{\partial}{\partial r} u_r\cos\theta \frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta} u_r\cos\theta \frac{\partial \theta}{\partial z} = u_{rr}\cos\theta\frac{z}{r} + u_r\sin\theta\frac{\sqrt{x^2+y^2}}{r^2}$$

$$= \cos^2\theta\, u_{rr} + \sin\theta\frac{\sqrt{r^2\sin^2\theta(\cos^2\varphi+\sin^2\varphi)}}{r^2} u_r = \cos^2\theta\, u_{rr} + \frac{1}{r}\sin^2\theta\, u_r$$

$r = \sqrt{x^2+y^2+z^2}$ $\qquad \theta = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ $\qquad \varphi = \arctan\left(\frac{y}{x}\right)$

$x = r\sin\theta\cos\varphi$ $\qquad y = r\cdot\sin\theta\sin\varphi$ $\qquad z = r\cos\theta$

3)... $0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta)\, u_{rr} + \frac{1}{r}(\cos^2\varphi\cos^2\theta + \sin^2\varphi + \cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta)\, u_r = (\sin^2\theta + \cos^2\theta)\, u_{rr} + \frac{1}{r}(\cos^2\theta + \sin^2\theta + 1)\, u_r = u_{rr} + \frac{2}{r} u_r$

$\Rightarrow \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = 0$

$r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} + 0\,u = 0$

$\lambda_{1,2} = \frac{a_2 - a_1}{2a_2} \pm \sqrt{\frac{(a_2-a_1)^2}{4a_2^2} - \frac{a_0}{a_2}} = \frac{(1-2)}{2} \pm \sqrt{\frac{(-1)^2}{4} - 0} = -\frac{1}{2} \pm \sqrt{\frac{1}{4}} = -\frac{1}{2} \pm \frac{1}{2}$

$\lambda_1 \neq \lambda_2$ und beide reell also:

$r^{\lambda_1}, r^{\lambda_2}$ ist Fundamentalsystem

$\Rightarrow u(r) = c_1 \cdot r^{-1} + c_2 \cdot r^{0} = \frac{c_1}{r} + c_2$

□

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4) $x\in\mathbb{R},\ t>0 \qquad u_t + u u_x = \varepsilon u_{xx} \quad , \varepsilon>0$

Wandernde-Wellen-Lösung $u(\xi)$ mit $\xi = x-ct$ für ein $c\in\mathbb{R}$ und u löst PDGL

ges: u Wandernde-Wellen-Lösung mit $\lim\limits_{\xi\to\pm\infty} u(\xi) = u_\pm$, $\lim\limits_{\xi\to\pm\infty} u'(\xi) = 0$ für $u_+, u_- \in\mathbb{R}$

(i) zz: gesuchtes u erfüllt $\varepsilon u' = \frac{1}{2}(u-c)^2 - \frac{1}{2}(u_- - c)^2$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} = u'\cdot(-c) = -cu' \qquad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = u'\cdot 1 = u'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial x} u'(x-ct) = u''(x-ct)$$

$$\Rightarrow -cu' + uu' = u'(u-c) = \varepsilon u''$$

$$\frac{1}{2}(u-c)^2 - \frac{1}{2}(u_- - c)^2 = \frac{1}{2}(u^2 - 2uc + c^2 - u_-^2 + 2u_-c - c^2) = \frac{u^2}{2} - uc - \frac{u_-^2}{2} + u_-c$$

$$= -\frac{u^2}{2} + u^2 - u_-^2 + \frac{u_-^2}{2} - cu + cu_- = -\frac{1}{2}(u^2 - u_-^2) + u(u-c) - u_-(u_- - c)$$

$$\varepsilon\cdot(u'(\xi) - 0) = \varepsilon\cdot\left(u'(\xi) - \lim_{t\to-\infty} u'(t)\right) = \varepsilon\int_{-\infty}^{\xi} u''(t)\,dt = \int_{-\infty}^{\xi} u'(u-c)\,dt = u(u-c)\Big|_{-\infty}^{\xi} - \int_{-\infty}^{\xi} uu'\,dt =$$

$$= u(\xi)(u(\xi)-c) - u_-(u_- - c) - \int_{-\infty}^{\xi} u'u\,dt \overset{*}{=} u(u-c) - u_-(u_- - c) - \frac{1}{2}(u^2 - u_-^2)$$

$$\Rightarrow \varepsilon u' = \frac{1}{2}(u-c)^2 - \frac{1}{2}(u_- - c)^2$$

(ii) zz: $u(\xi)\equiv u_-$ ist eine Lösung und die Riccati-Gleichung hat i. A. eine zweite Lösung $u\equiv u_+$.

$u(\xi)\equiv u_- \Rightarrow \varepsilon u' = 0 \qquad \frac{1}{2}(u-c)^2 - \frac{1}{2}(u_- - c)^2 = \frac{1}{2}(u_- - c)^2 - \frac{1}{2}(u_- - c)^2 = 0$ ✓

$u(\xi)\equiv u_+ \Rightarrow \varepsilon u' = 0$ also suchen wir ein $u_+\in\mathbb{R}$: $\frac{(u_+ - c)^2 - (u_- - c)^2}{2} = 0$

$$\Leftrightarrow (u_+ - c)^2 - (u_- - c)^2 = 0 \quad \Leftrightarrow (u_+ - c)^2 = (u_- - c)^2$$

Für $u(\xi)\equiv u_+ := 2c - u_- \quad \Rightarrow (2c - u_- - c)^2 = (c - u_-)^2 = (-1)^2(u_- - c)^2 = (u_- - c)^2$

(iii) ges: allgemeine Lösung mit Ansatz $u(\xi) = u_- + \frac{1}{v(\xi)}$ für Funktion v

$$\varepsilon u'(\xi) = -\frac{\varepsilon v'(\xi)}{v(\xi)^2} \qquad \frac{1}{2}(u-c)^2 - \frac{1}{2}(u_- - c)^2 = \frac{1}{2}\left(\left(u_- + \frac{1}{v(\xi)} - c\right)^2 - (u_- - c)^2\right)$$

$$= \frac{1}{2}\left(u_-^2 + 2u_-\frac{1}{v(\xi)} - 2u_-c + \frac{1}{v(\xi)^2} - 2\frac{1}{v(\xi)}c + c^2 - u_-^2 + 2u_-c - c^2\right)$$

$$= \frac{1}{2}\left(2\frac{u_-}{v(\xi)} + \frac{1}{v(\xi)^2} - 2\frac{c}{v(\xi)}\right) = \frac{1}{v(\xi)^2}\left(v(\xi)u_- + \frac{1}{2} - v(\xi)c\right)$$

$$\varepsilon u' = \frac{1}{2}(u-c)^2 - \frac{1}{2}(u_- - c)^2 \Rightarrow -\varepsilon v'(\xi) = v(\xi)(u_- - c) + \frac{1}{2} \Rightarrow v'\frac{-\varepsilon}{v(u_- - c) + \frac{1}{2}} = 1$$

$$\Rightarrow \int \frac{-\varepsilon}{v\cdot(u_- - c) + \frac{1}{2}}\,dv = \int 1\,d\xi = \xi$$

$s = v(u_- - c) + \frac{1}{2}$

$ds = (u_- - c)\,dx$

$$-\varepsilon\int \frac{1}{v(u_- - c) + \frac{1}{2}}\,dv = -\varepsilon\int\frac{1}{s}\frac{1}{(u_- - c)}\,ds = \frac{\varepsilon}{(c - u_-)}\int\frac{1}{s}\,ds =$$

$$= \frac{\varepsilon}{(c - u_-)}\ln\left(v(u_- - c) + \frac{1}{2}\right) \Rightarrow \ln\left(v(u_- - c) + \frac{1}{2}\right) = \frac{\xi(c - u_-)}{\varepsilon}$$

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4) ... (iii) $\boxed{u = u_- + \frac{1}{v}}$

$$\varepsilon \cdot u' = \varepsilon \cdot \left(u_- + \frac{1}{v}\right)' = \varepsilon\left(-\frac{1}{v^2}v'\right) = -\frac{\varepsilon v'}{v^2}$$

$$= \frac{1}{2}(u-c)^2 - \frac{1}{2}(u_- - c)^2 = \frac{1}{2}\left(u^2 - 2uc + \cancel{c^2} - u_-^2 + 2u_-c - \cancel{c^2}\right)$$

$$= \frac{1}{2}\left(u^2 + 2c(u_- - u) - u_-^2\right) = \frac{u^2}{2} + c(u_- - u) - \frac{u_-^2}{2}$$

$$= \frac{1}{2}\left(u_- + \frac{1}{v}\right)^2 + c\left(\cancel{u_-} - \left(\cancel{u_-} + \frac{1}{v}\right)\right) - \frac{1}{2}u_-^2$$

$$= \frac{1}{2}\left(\cancel{u_-^2} + 2u_-\frac{1}{v} + \frac{1}{v^2}\right) - c\frac{1}{v} - \cancel{\frac{1}{2}u_-^2} = u_-\frac{1}{v} + \frac{1}{2}\frac{1}{v^2} - c\frac{1}{v}$$

$$\Rightarrow -\frac{\varepsilon v'}{v^2} = \frac{u_-}{v} + \frac{1}{2v^2} - \frac{c}{v}$$

$$-\varepsilon v' = u_- \cdot v + \frac{1}{2} - c \cdot v = \frac{1}{2} + v(u_- - c)$$

$$v(\xi) = k_1 \cdot \exp\left(\frac{(c-u_-)\xi}{\varepsilon}\right) + \frac{1}{2(c-u_-)}$$

Wolfram Alpha

Probe mit Wolfram Alpha passt.

$$u(t,x) = u_- + \frac{1}{k_1 \cdot \exp\left(\frac{(c-u_-)(x-ct)}{\varepsilon}\right) + \frac{1}{2(c-u_-)}}$$

Erfüllt nach Wolfram Alpha die viskose Burgers-Gleichung.

$$\lim_{\xi\to\infty} u(\xi) = u_- \qquad \lim_{\xi\to-\infty} u(\xi) = 2c - u_-$$

$$\lim_{\xi\to\infty} u'(\xi) = 0 \qquad \lim_{\xi\to-\infty} u'(\xi) = 0$$

Wolfram Alpha ist mein Freund und Helfer!

$\square$