DGL Ü2

2) $\varphi''(t) + w^2 \sin(\varphi(t)) = 0$

$$\frac{1}{2}(\varphi'(t))^2 - w^2 \cos(\varphi(t)) = c \quad \Big| \frac{d}{dt}$$

$$\frac{1}{2} 2 \varphi'(t) \cdot \varphi''(t) + w^2 \sin(\varphi(t)) \varphi'(t) = 0$$

$$\varphi'(t) \left(\varphi''(t) + w^2 \sin(\varphi(t))\right) = 0$$

$\Rightarrow$ entweder $\varphi'(t) \equiv 0$ oder $\varphi''(t) + w^2 \sin(\varphi(t)) \equiv 0$

Angenommen $\varphi'(t) = 0$: $\Rightarrow \varphi''(t) = 0 \quad \varphi(t) = c$

$$\varphi''(t) + w^2 \sin(\varphi(t)) = 0 + w^2 \sin(c) = w^2 \sin(c)$$

---

gegeben: $\varphi''(t) + w^2 \sin(\varphi(t)) = 0$

$$\Rightarrow \varphi'(t) \left(\varphi''(t) + w^2 \sin(\varphi(t))\right) = 0$$

$$\Leftrightarrow \varphi'(t) \varphi''(t) + \varphi'(t) w^2 \sin(\varphi(t)) = 0 \quad \Big| \int$$

$$\int \varphi'(t) \varphi''(t)\, dt + \int w^2 \varphi'(t) \sin(\varphi(t))\, dt = \int 0\, dt$$

$$\int u\, du + w^2 \int \sin(v)\, dv = c$$

$$\frac{1}{2} u^2 - w^2 \cos(v) = c$$

$$\frac{1}{2}(\varphi'(t))^2 - w^2 \cos(\varphi(t)) = c$$

$$\Rightarrow \frac{1}{2}(\varphi'(t))^2 - w^2 \cos(\varphi(t)) = c$$

$\left[\, u = \varphi' \quad \frac{du}{dt} = \varphi'' \quad dt = \frac{du}{\varphi''} \right.$

$\left[\, v = \varphi \quad \frac{dv}{dt} = \varphi' \quad dt = \frac{dv}{\varphi'} \right.$

DGL Ü2

3) $\phi''(t) + w^2 \sin(\phi(t)) = 0$ $\quad w^2 = \frac{g}{l}$ $\quad$ g.. Erdbeschleunigung

$\phi: t \to \phi(t)$ .. Auslenkung des Pendels $\quad$ l.. Faden Länge des Pendel

a) ges: System 1. Ordnung

$$\phi_1 = \phi \qquad \phi_2 = \phi'$$

$$\phi_1' = \phi_2 \qquad \phi_2' = -w^2 \sin(\phi_1) = -\frac{g}{l}\sin(\phi_1)$$

ϕ_2 ... Winkelgeschwindigkeit also wie schnell ändert sich der Winkel mit der Zeit?

b) ges: Erhaltungsgröße, d.h. $h: \mathbb{R} \to \mathbb{R}$ mit $\nabla h(t) \cdot f(t) = 0 \ \forall t \in \mathbb{R}$

$$\phi_1' = \phi_2 \iff w^2 \sin(\phi_1)\,\phi_1' = w^2 \sin(\phi_1)\,\phi_2$$

$$\phi_2' = -w^2 \sin(\phi_1) \iff \phi_2'\,\phi_2 = -w^2 \sin(\phi_1)\,\phi_2$$

$$\Rightarrow \quad w^2 \sin(\phi_1)\,\phi_1' + \phi_2'\,\phi_2 = 0$$

$$\left(-w^2 \cos(\phi_1) + \tfrac{1}{2}(\phi_2)^2\right)' = 0$$

$\Rightarrow h = -w^2 \cos(\phi) + \frac{1}{2}(\phi')^2$ ist eine Erhaltungsgröße, da $\nabla h(\phi) = 0$ $\forall \phi$ die die Differentialgleichung lösen.

DGL Ü2

4) $x' = -x + 2y \qquad y' = -2x - y$

a) zz: Integralkurven der DGL sind logarithmische Spiralen

Gleichungen der logarithmischen Spirale sind:

$$x(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$$

$$y(t) = -c_1 e^{\alpha t}\sin(\beta t) + c_2 e^{\alpha t}\cos(\beta t)$$

$$x' = c_1(\alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t)) + c_2(\alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\cos(\beta t))$$

$$= e^{\alpha t}(c_1\alpha\cos(\beta t) - c_1\beta\sin(\beta t) + c_2\alpha\sin(\beta t) + c_2\beta\cos(\beta t))$$

$$= \alpha\cdot x + \beta y$$

$$y' = -c_1(\alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\cos(\beta t)) + c_2(\alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t))$$

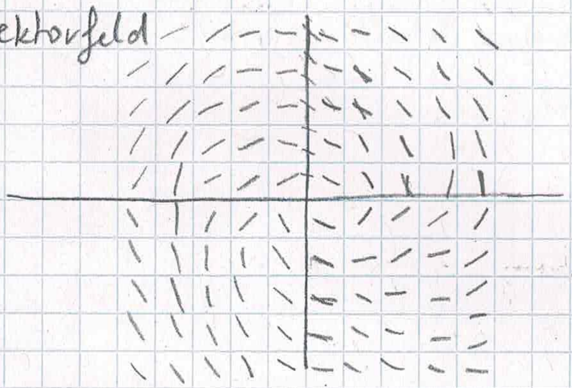
$$= e^{\alpha t}(-c_1\beta\cos(\beta t) - c_2\beta\sin(\beta t) - \alpha c_1\sin(\beta t) + \alpha c_2\cos(\beta t))$$

$$= -\beta x + \alpha y$$

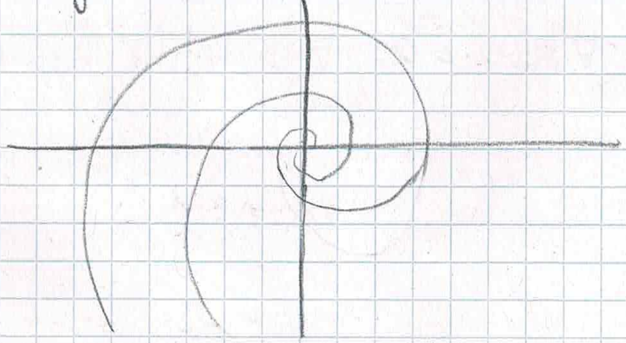
$$\Rightarrow \alpha = -1 \qquad \beta = 2$$

$$\Rightarrow x = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t), \quad y = -c_1 e^{-t}\sin(2t) + c_2 e^{-t}\cos(2t)$$

Vektorfeld



Integralkurve



b) $x'' = -x' + 2y' = -x' + 2(-2x-y) = -x' - 4x - 2y = -x' - 4x - x' - x = -2x' - 5x$

Ansatz $x = e^{\lambda t} \qquad x' = \lambda e^{\lambda t} \qquad x'' = \lambda^2 e^{\lambda t}$

$\Rightarrow \lambda^2 e^{\lambda t} = -2\lambda e^{\lambda t} - 5e^{\lambda t} \iff \lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = -1 \pm \sqrt{1-5} = -1 \pm 2i$

c) Eigenwerte der Matrix?

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \left|\begin{pmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{pmatrix}\right| = (-1-\lambda)^2 + 4 = 1 + 2\lambda + \lambda^2 + 4 = \lambda^2 + 2\lambda + 5 = 0$$

$$\iff \lambda_{1,2} = -1 \pm 2i$$

DGL Ü2

5) a) $y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Vektorfeld $y_1(x)$

Vektorfeld $y_2(x)$

Lösung $y_1' = 2 \quad y_2' = 3$

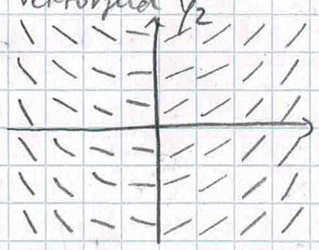
$\Rightarrow y_1 = 2x + c \quad y_2 = 3x + d$

$\Rightarrow y = \begin{pmatrix} 2 \\ 3 \end{pmatrix} x + \begin{pmatrix} c \\ d \end{pmatrix} \quad c, d \in \mathbb{R}$

b) $y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 \\ y_1 \end{pmatrix}$

Vektorfeld y_1

Vektorfeld y_2



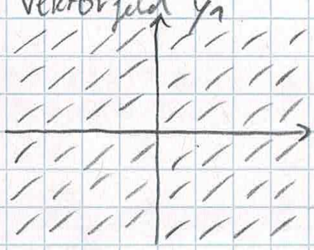
Lösung $y_1' = 1 \quad y_2' = y_1$

$\Rightarrow y_1 = x + c \quad y_2 = \frac{1}{2} x^2 + cx + d$

$\Rightarrow y = \begin{pmatrix} 0 \\ 0{,}5 \end{pmatrix} x^2 + \begin{pmatrix} 1 \\ c \end{pmatrix} x + \begin{pmatrix} c \\ d \end{pmatrix} \quad c, d \in \mathbb{R}$

c) $y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 \\ y_2 \end{pmatrix}$

Vektorfeld y_1



Vektorfeld y_2

Lösung $y_1' = 1 \quad y_2' = y_2$

$\Rightarrow y_1 = x + c \quad y_2 = de^x$

$\Rightarrow y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ d \end{pmatrix} e^x$

d) $y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Vektorfeld y_1

Vektorfeld y_2

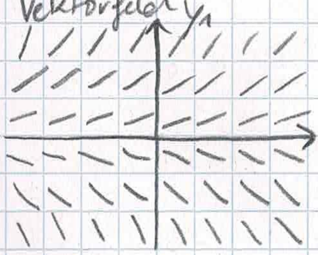
Lösung $y_1' = y_1 \quad y_2' = y_2$

$\Rightarrow y_1 = ce^x \quad y_2 = de^x$

$\Rightarrow y = \begin{pmatrix} c \\ d \end{pmatrix} e^x$

e) $y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 \\ -y_2 \end{pmatrix}$

Vektorfeld y_1



Vektorfeld y_2

Lösung $y_1' = y_1 \quad y_2' = -y_2$

$\Rightarrow y_1 = ce^x \quad y_2 = de^{-x}$

$\Rightarrow y = \begin{pmatrix} c \\ 0 \end{pmatrix} e^x + \begin{pmatrix} 0 \\ d \end{pmatrix} e^{-x}$

DGL Ü2

6) $x' = x(1-x) \qquad x \in \mathbb{R}$

a) Richtungsfeld

Für $x = 0 \lor x = 1$

gilt $x' = 0(1-0) = 0 = 0'$
$x' = 1(1-1) = 0 = 1'$

Ruhelagen

b) $x' = x(1-x)$

$$\frac{dx}{dt} = x(1-x)$$

$$\frac{1}{x(1-x)}\,dx = 1\,dt$$

$$\int \frac{1}{x(1-x)}\,dx = \int 1\,dt$$

$$\log(x) - \log(1-x) = t + c$$

$$\log\left(\frac{x}{1-x}\right) = t + c$$

$$\frac{x}{1-x} = e^c e^t = \tilde{c} e^t$$

$$\frac{1}{x} - 1 = \frac{1-x}{x} = \frac{1}{\tilde{c} e^t}$$

$$\frac{1}{x} = \frac{1}{\tilde{c} e^t} + 1 = \frac{\tilde{c} e^t + 1}{\tilde{c} e^t}$$

$$x = \frac{\tilde{c} e^t}{\tilde{c} e^t + 1} \qquad \tilde{c} \in \mathbb{R}^+$$

c) AWP $\quad x(0) = x_0 \quad x_0 \in \mathbb{R} \quad t \to \pm\infty$?

$$x(0) = \frac{\tilde{c} e^0}{\tilde{c} e^0 + 1} = \frac{\tilde{c}}{\tilde{c}+1} = x_0$$

Für $x_0 = 0 \Rightarrow \tilde{c} = 0 \Leftrightarrow x = 0$ $\qquad$ Für $x_0 = 1 \Rightarrow x = 1$

$$\Leftrightarrow \frac{\tilde{c}+1}{\tilde{c}} = \frac{1}{x_0} \Leftrightarrow 1 + \frac{1}{\tilde{c}} = \frac{1}{x_0} \Leftrightarrow \frac{1}{\tilde{c}} = \frac{1}{x_0} - 1 = \frac{1-x_0}{x_0} \Leftrightarrow \tilde{c} = \frac{x_0}{1-x_0}$$

1. Fall $x_0 \neq 1 \land x_0 \neq 0$:

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{\tilde{c} e^t}{\tilde{c} e^t + 1} = \tilde{c} \lim_{t\to\infty} \frac{1}{\tilde{c} + \underbrace{\frac{1}{e^t}}_{\to 0}} = \tilde{c}\,\frac{1}{\tilde{c}} = 1$$

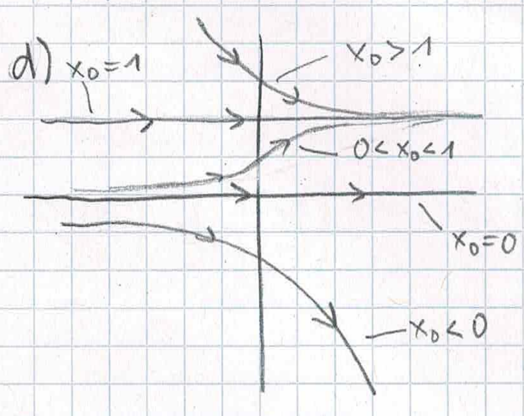
$$\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} \frac{\tilde{c} e^t}{\tilde{c} e^t + 1} = \tilde{c} \lim_{t\to-\infty} \frac{\overbrace{e^t}^{\to 0}}{\tilde{c}\underbrace{e^t}_{\to 0} + 1} = \tilde{c} \cdot 0 = 0$$

?

2. Fall $x_0 = 1 \lor x_0 = 0$:

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x_0 = x_0 = \lim_{t\to-\infty} x_0 = \lim_{t\to-\infty} x(t)$$

d)



DGL Ü2

7) $x'(t) + \cos(t)\,x(t) = \frac{1}{2}\sin(2t)$

ges: allgemeine Lösung

1. Ordnung, linear, explizit, skalar

$$x'(t) = \underbrace{-\cos(t)}_{a(t)}\,x(t) + \underbrace{\frac{1}{2}\sin(2t)}_{b(t)}$$

a, b ... stetig

$A = \int a\,dt$

$$\Rightarrow x(t) = e^{A(t)}\left(c + \int e^{-A(s)}\,b(s)\,ds\right)$$

$$A(t) = \int -\cos(t)\,dt = -\sin(t)$$

$$\int e^{-A(s)}\,b(s)\,ds = \int e^{\sin(s)}\,\frac{1}{2}\sin(2s)\,ds = \frac{1}{2}\int 2e^{\sin(s)}\sin(s)\cos(s)\,ds$$

$$= \int e^{u}\,u\,du = e^{u}(u-1) = e^{\sin(s)}(\sin(s)-1) \quad \left[\begin{array}{l} u = \sin(s) \\ ds = \frac{1}{\cos(s)}\,du \end{array}\right.$$

$$\Rightarrow x(t) = e^{-\sin(t)}\left(c + e^{\sin(t)}(\sin(t)-1)\right) = c\,e^{-\sin(t)} + \sin(t) - 1$$

DGL Ü2

8) $x'(t) = \cos(t)\, x(t) + \sin(t)\, (x(t))^2$

mit $a(t) = \cos(t)$ $\quad b(t) = \sin(t)$ $\quad \alpha = 2$

$x'(t) = a(t)\, x(t) + b(t)\, (x(t))^\alpha$ Bernoulli DGL

$y(t) = (x(t))^{1-\alpha} = x(t)^{-1} = \frac{1}{x(t)}$ $\qquad x(t) = \frac{1}{y(t)}$

$x'(t) = \left(\frac{1}{y(t)}\right)' = (y^{-1}(t))' = -\frac{1}{y(t)^2}\, y'(t)$

$x'(t) = \cos(t)\, x(t) + \sin(t)\, (x(t))^2$

$-\frac{y'}{y^2} = \cos(t)\, \frac{1}{y} + \sin(t)\, \frac{1}{y^2}$

$y' = -y\cos(t) - \sin(t)$ lineare, skalare DGL 1. Ordnung

$y' = y\, a(t) + b(t)$ mit $a(t) = -\cos(t)$ $\quad b(t) = -\sin(t)$

$$A(t) = \int a(t)\, dt = \int -\cos(t)\, dt = -\sin(t)$$

$y(t) = e^{A(t)} \left(c + \int e^{-A(t)}\, b(t)\, dt\right)$

$= e^{-\sin(t)} \left(c + \int e^{\sin(t)} (-\sin(t))\, dt\right)$

$= \frac{1}{e^{\sin(t)}} \left(c - \int e^{\sin(t)} \sin(t)\, dt\right)$

$\Rightarrow x(t) = e^{\sin(t)} \frac{1}{c - \int e^{\sin(t)} \sin(t)\, dt}$

DGL Ü2

g) $x' = x^2 + \frac{x}{t} + \frac{1}{t^2}$ Riccati-DGL

$x = \frac{a}{t}$ $x' = -\frac{a}{t^2}$

$x' = x^2 + \frac{x}{t} + \frac{1}{t^2}$

$-\frac{a}{t^2} = \frac{a^2}{t^2} + \frac{a}{t^2} + \frac{1}{t^2}$

$\varphi = -\frac{1}{t}$ löst die DGL $0 = a^2 + 2a + 1 \Rightarrow a = -1$

$x' - \frac{1}{t}x - x^2 = \frac{1}{t^2}$ $g(t) = -\frac{1}{t}$ $h(t) = -1$ $r(t) = \frac{1}{t^2}$

$x' + g(t)x + h(t)x^2 = r(t)$

$x = -\frac{1}{t} + \frac{1}{z(t)}$ $z' = (g(t) + 2h(t)\varphi(t))z + h(t)$

$= (-\frac{1}{t} + 2\frac{1}{t})z - 1$

$= \frac{z}{t} - 1$

$z' = \frac{1}{t}z - 1$ $a(t) = \frac{1}{t}$ $b(t) = -1$ $A(t) = \int \frac{1}{t} dt = \log(t)$

nach Formel für lineare, skalare DGL 1. Ordnung.

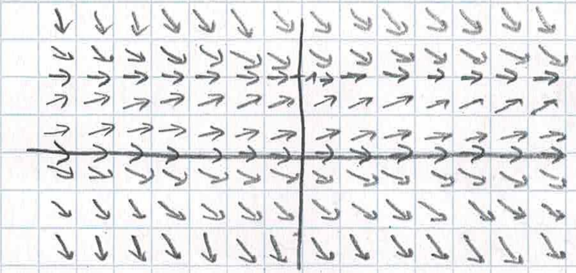
$z(t) = e^{\log(t)}(c + \int e^{-\log(t)}(-1)\,dt) = t\,(c - \int \frac{1}{t}\,dt) = t(c - \log(t))$

$\Rightarrow x = -\frac{1}{t} + \frac{1}{t(c-\log(t))}$

DGL Ü2

6) $x' = x(1-x)$

a) Richtungsfeld



b) $x = \frac{1}{y}$ $\quad x' = \left(\frac{1}{y}\right)' = -\frac{1}{y^2}y'$

$-\frac{y'}{y^2} = \frac{1}{y}\left(1-\frac{1}{y}\right) \iff -\frac{y'}{y^2} = \frac{y-1}{y^2} \iff -y' = y-1$

$\iff y' = -y + 1 = a(t)y + b(t)$ $\qquad a(t) = -1 \quad b(t) = 1$

$\Rightarrow y = e^{A(t)}\left(c + \int e^{-A(t)}\, b(t)\, dt\right)$ $\qquad A(t) = \int -1\, dt = -t$

$= e^{-t}\left(c + \int e^{t}\, dt\right)$

$= e^{-t}\left(c + e^{t}\right) = \frac{c}{e^t} + 1 \qquad c \in \mathbb{R}$

$x \equiv 0 \ \wedge \ x \equiv 1$ sind auch Lösungen nämlich Ruhelagen

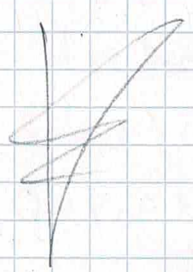
$x = \frac{1}{y} = \frac{1}{\frac{c}{e^t}+1} = \frac{e^t}{c+e^t}$ ist die allgemeine Lösung

c) AWP $x(0) = x_0 \in \mathbb{R}$

$x(0) = \frac{1}{c+1} = x_0$ $\qquad$ Für $x_0 = 0$ ist $x \equiv 0$ die Lösung, für $x_0 = 1$, $x \equiv 1$.

$\iff c+1 = \frac{1}{x_0} \quad \Rightarrow c = \frac{1}{x_0} - 1 = \frac{1-x_0}{x_0}$

Für $x_0 > 0$



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