Computer Aided Geometric Design Compendium WS2023

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Organization

Lecture each Thursday 12:00 to 14:00 (full 2 hours).
Oral exam. Write email to fix date and time.
Problem session each Thursday 14:00 to 16:00. Mandatory attendance!
Kreuzerlübung.

1 Bezier curves

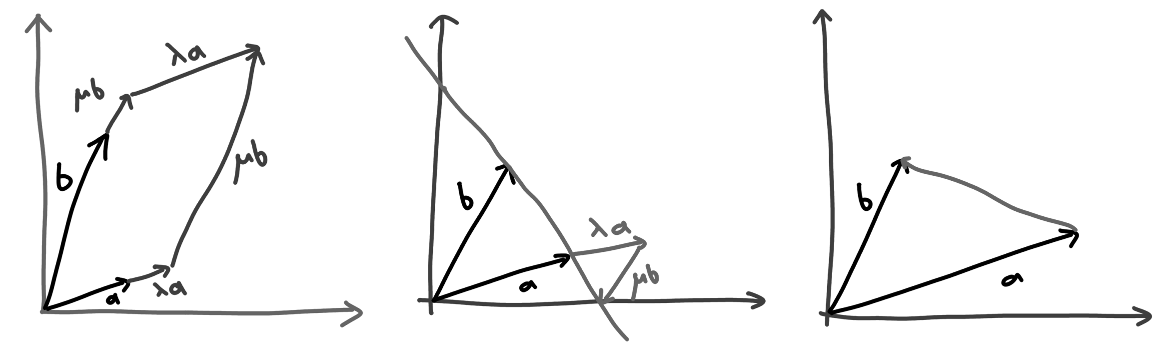

Figure 1: Linear combination, affine combination and convex combination

Example 1. ***Linear combination*** *$\lambda a + \mu b$*
Affine combination *$\lambda a + \mu b$ and $\lambda + \mu = 1$*
What is μ so that $\lambda a + \mu B$ is on the line?

$$\lambda a + \mu b = a + t(b - a) \implies a(\underbrace{\lambda - 1 + t}_{=0}) + b(\underbrace{\mu - t}_{=0}) = 0$$

If a, b are linearly independent $\implies \mu = t \wedge \lambda + \mu = 1$
Convex combination *$\lambda a + \mu b$ and $\lambda + \mu = 1$ and $\lambda, \mu \geq 0$*
Line is $a + t(b - a)$ with $t \in [0, 1] \implies \mu, \lambda \in [0, 1]$

Definition 1 (combinations). *linear combination $\sum_{i=1}^{n} \lambda_i v_i$ with $v_1, ..., v_n \in \mathbb{R}^d, \lambda_1, ..., \lambda_n \in \mathbb{R}$*
affine combination $\sum_{i=1}^{n} \lambda_i v_i$ with $\sum_{i=1}^{n} \lambda_i = 1$
convex combination $\sum_{i=1}^{n} \lambda_i v_i$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\forall i : \lambda_i \geq 0$

Algorithm 1 (of de Casteljou, Bezier curve). *Given: $b_0, ..., b_n \in \mathbb{R}^d$ (called control points / Kontrollpunkte), $t \in \mathbb{R}$*
Recursion: $b_i^0(t) := b_i$
$b_i^j(t) := (1 - t)b_i^{j-1}(t) + tb_{i+1}^{j-1}(t)$ for $j = 1, ..., n$ and $i = 0, ..., n - j$
Result: $b(t) := b_0^n(t)$ (called Bezier curve)

Remark 1. *In the algorithm above often we choose $t \in [0, 1]$.*

Example 2.

Remark 2. *In this course $\mathbb{N} = \{1, 2, 3, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$*

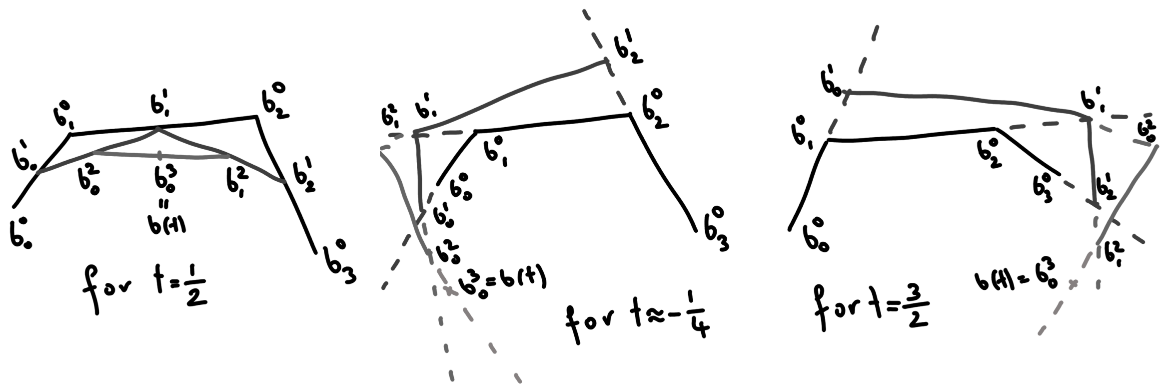

Figure 2: Examples of the de casteljou algorithm

Recap 1. $0! := 1, n! := n(n-1)(n-2)\cdots 1$ *for* $n \geq 1$.

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!} & , n \geq k \geq 0 \\ 0 & , k > n \end{cases} \quad \textit{for } n, k \in \mathbb{N}_0$$

Definition 2 (Bernstein polynomials)**.** *For* $n, i \in \mathbb{N}_0$ *we define* $B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i} \in \mathbb{R}[t]$

Remark 3. *Special cases of Bernstein polynomials*

$$i > n \implies B_i^n(t) = 0 \qquad B_i^n(0) = \begin{cases} 0, i \neq 0 \\ 1, i = 0 \end{cases}$$

$$B_i^n(1) = \begin{cases} 0, i \neq n \\ 1, i = n \end{cases} \qquad B_0^0(t) = 1$$

Theorem 1. $b_i^j(t) = \sum_{l=0}^{j} B_l^j(t) b_{i+l}$

Proof. Induction over j: $j = 0$:

$$j = 0 : \qquad b_i^0(t) := b_i = 1 \cdot b_i = B_0^0(t) \cdot b_i \qquad \checkmark$$

$$j-1 \to j : \qquad b_i^j(t) := (1-t) b_i^{j-1}(t) + t b_{i+1}^{j-1}(t) \overset{\text{IA}}{=} (1-t) \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+l} + t \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+1+l} =$$

$$(1-t) \sum_{l=0}^{\overset{j}{\cancel{j-1}}} B_l^{j-1}(t) b_{i+l} + t \sum_{l=\underset{0}{\cancel{1}}}^{j} B_{l-1}^{j-1}(t) b_{i+l} = \sum_{l=0}^{j} (\underbrace{(1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t)}_{= B_l^j(t) \text{ using the following lemma}}) b_{i+l} =$$

$$\sum_{l=0}^{j} B_l^j(t) b_{i+l} \qquad \checkmark$$

$\square$

Corollary 1. *The Bezier curve equals* $b(t) = b_0^n(t) = \sum_{l=0}^{n} B_l^j(t) b_{i+l}$*, which is called the Bernstein representation of the Bezier curve.*

Remark 4. *As* $b(t) = \sum_{l=0}^{n} B_l^n(t) b_l \in C^\infty$ *it is a polynomial curve of degree* n*, which is in* C^∞ *and therefore "very smooth".*

Lemma 1. $B_l^j(t) = (1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t)$

Proof.

$$(1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t) = (1-t) \binom{j-1}{l} t^l (1-t)^{j-1-l} + t \binom{j-1}{l-1} t^{l-1} (1-t)^{j-1-l+1} =$$

$$\binom{j-1}{l} t^l (1-t)^{j-l} + \binom{j-1}{l-1} t^l (1-t)^{j-l} = \left(\binom{j-1}{l} + \binom{j-1}{l-1} \right) t^l (1-t)^{j-l} = \binom{j}{l} t^l (1-t)^{j-l} = B_l^j(t)$$

$\square$

Remark 5. *What is $b(0)$?* $\quad b(0) = \sum_{i=0}^{n} B_i^n(0) b_i = b_0 + 0 + 0 + \cdots + 0 = b_0$
What is $b(1)$? $\quad b(1) = \sum_{i=0}^{n} B_i^n(1) b_i = 0 + \cdots + 0 + b_n = b_n$

Definition 3 (end-point-interpolating)**.** *Curves which pass through the first and last point are called end-point-interpolating (Endpunktinterpolierend).*

Remark 6. *Bezier curves are end-point-interpolating.*

Remark 7. *How many intersection points are there between a planar (i.e. in $\mathbb{R}^2$) Bezier curve and a straight line?*

$$\text{Straight line: } p + t(q-p) \qquad\qquad \text{Bezier curve: } b(t) = \sum_{i=0}^{n} B_i^n(t) \underbrace{b_i}_{\in\mathbb{R}^2}$$

Solving $p + t(q-p) = \sum_{i=0}^{n} B_i^n(t) b_i$ results in at most n solutions.

Lemma 2. $\frac{d}{dt} B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$

Proof.

$$\begin{aligned}
\frac{d}{dt} B_i^n(t) = \frac{d}{dt}\binom{n}{i} t^i (1-t)^{n-i} &= \binom{n}{i} i t^{i-1}(1-t)^{n-i} - \binom{n}{i} t^i (n-i)(1-t)^{n-i-1} = \\
&\frac{n!}{\underbrace{\cancel{i}!}_{(i-1)!}(n-i)!}\cancel{i} t^{i-1}(1-t)^{n-i} - \frac{n!}{i!\underbrace{\cancel{(n-i)!}}_{(n-i-1)!}}\cancel{(n-i)} t^i (1-t)^{n-i-1} = \\
&n\left(\frac{(n-1)!}{(i-1)!(n-i)!} t^{i-1}(1-t)^{n-i} - \frac{(n-1)!}{i!(n-i-1)!} t^i (1-t)^{n-i-1}\right) = \\
n\left(\binom{n-1}{i-1} t^{i-1}(1-t)^{n-i} - \binom{n-1}{i} t^i (1-t)^{n-i-1}\right) &= n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))
\end{aligned}$$

□

Theorem 2. $\dot{b}(t) := \frac{d}{dt} b(t) = n \sum_{i=0}^{n-1} B_i^{n-1}(t)(b_{i+1} - b_i) = n(b_1^{n-1}(t) - b_0^{n-1}(t))$

Proof.

$$\dot{b}(t) = \frac{d}{dt}\left(\sum_{i=0}^{n} B_i^n(t) b_i\right) = \sum_{i=0}^{n} \frac{d}{dt} B_i^n(t) b_i = \sum_{i=0}^{n} n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)) b_i = n\left(\sum_{i=0}^{n} B_{i-1}^{n-1}(t) b_i - \sum_{i=0}^{n} B_i^{n-1}(t) b_i\right) =$$

$$n\left(\sum_{i=1}^{n} B_{i-1}^{n-1}(t) b_i - \sum_{i=0}^{n} B_i^{n-1}(t) b_i\right) = n\left(\underbrace{\sum_{i=0}^{n-1} B_i^{n-1}(t) b_{i+1}}_{=b_1^{n-1}(t)} - \underbrace{\sum_{i=0}^{n-1} B_i^{n-1}(t) b_i}_{=b_0^{n-1}(t)}\right) = n\left(\sum_{i=0}^{n-1} B_i^{n-1}(t)(b_{i+1} - b_i)\right)$$

□

Corollary 2.

- $\dot{b}(0) = n(b_1 - b_0)$
- $\dot{b}(1) = n(b_n - b_{n-1})$
- *The last segment in the algorithm of de Casteljou is the tangent of the Bezier curve in $b(t)$.*
- *The derivative of a bezier curve of degree n is a bezier curve of degree $n-1$ with control points $(b_1, b_0), (b_2 - b_1), \cdots, (b_n - b_{n-1})$.*

Corollary 3. $\ddot{b}(t) = n(n-1)\sum_{i=0}^{n-2} B_i^{n-2}(t)(b_{i+2} - 2b_{i+1} + b_i)$
$\ddot{b}(0) = n(n-1)(b_2 - 2b_1 + b_0)$, $\ddot{b}(1) = n(n-1)(b_n - 2b_{n-1} + b_{n-2})$

Corollary 4. *The curvature of a bezier curve in the point $b(0)$ depends only on b_0, b_1, b_2.*
The curvature of a bezier curve in the point $b(1)$ depends only on b_{n-2}, b_{n-1}, b_n.

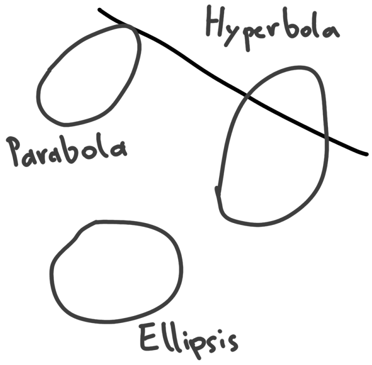

Figure 3: Categorization of Parabolas, Hyperbolas and Ellipsis as intersection points with the line at infinity.

Example 3. *Quadratic Bezier curve*

$$b(t) = \sum_{i=0}^{2} B_i^2(t)b_i = \binom{2}{0}t^0(1-t)^2b_0 + \binom{2}{1}t^1(1-t)^1b_1 + \binom{2}{2}t^2(1-t)^0b_2 = t^2(b_2 - 2b_1 + b_0) + t(2b_1 - 2b_0) + b_0$$

which is an affine transformation of a parabola and therefore a parabola.

Quadratic bezier curves are parabolas.

Remark 8. *Line at infinity (Ferngerade) is the collection of points where parallel lines intersect.*

Remark 9. *Different applications using these curves are Rhino, OpenSCAD, Autocad, Geogebra, ...*

2 Parameterized curves

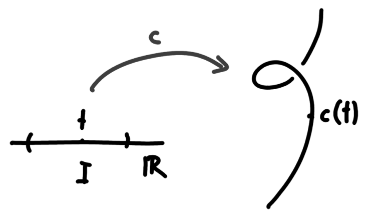

Figure 4: parameterized curve $c(t)$

Definition 4. $c : I \subseteq \mathbb{R} \to \mathbb{R}^3$ *is called a parameterized curve.*
$\dot{c}(t) := \frac{d}{dt}c(t)$ *is called the tangential vector. For* $\mathbb{R}^3$ *we have* $\dot{c}(t) = (\dot{c_1}(t), \dot{c_2}(t), \dot{c_3}(t))$.
The velocity is defined as $||\dot{c}(t)||$.
A point $c(t)$ *is called regular, if* $\dot{c}(t) \neq 0$ *and is called singular, if* $\dot{c}(t) = 0$.

Example 4. *A helix (Schraublinie) is defined by* $c(t) = (\cos(t), \sin(t), t)^T$.

$$\dot{c}(t) = (-\sin(t), \cos(t), 1)^T \qquad ||\dot{c}(t)|| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

We see that the helix is passed through with constant velocity. Furthermore all points are regular.

Example 5. $c : \mathbb{R} \to \mathbb{R}^3, t \mapsto (t^2, t^3, t^4)$, $\dot{c}(t) = (2t, 3t^2, 4t^3)$. *We see that* 0 *is singular as* $\dot{c}(0) = (0,0,0)$. *Everywhere else the curve is regular.*

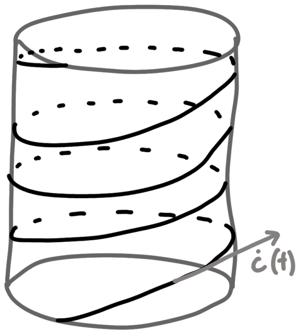

Figure 5: Helix with tangential vector

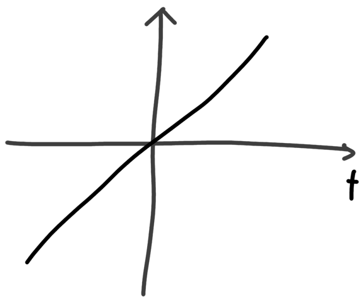

Figure 6: identity line can be parameterized such that $(0,0)$ is singular.

Remark 10. *A point being regular or singular depends on the parameterisation of the curve.*

For example $c(t) = (t,t)$ *produces a regular curve, while* $c(t) = (t^3, t^3)$ *produces a curve where* 0 *is singular.*

There are curves and points where no parameterisation exists such that the point is regular.

Definition 5. $c : I \to \mathbb{R}^2 \in C^2(I, \mathbb{R}^2)$

The curvature of the curve in the point $c(t)$ *is defined as* $\kappa(t) = \frac{\det(\dot{c}(t), \ddot{c}(t))}{||\dot{c}(t)||^3}$

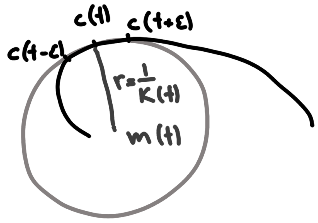

Figure 7: Circle of curvature

Example 6. *The circle of curvature has a radius of* $\frac{1}{\kappa(t)}$. $m(t)$ *is called the center of curvature.*

$m(t) = c(t) + \frac{1}{\kappa(t)} n(t)$ *where* $n(t) = \frac{(-\dot{c_2}(t), \dot{c_1}(t))}{||\dot{c}(t)||}$.

Remark 11. *Exercise: compare this definition of curvature with the school version concerning graphs.*

Example 7. *For a circle we have* $c(t) = (r\cos(t), r\sin(t))^T$, $\dot{c}(t) = (-r\sin(t), r\cos(t))^T$, $\ddot{c}(t) = (-r\cos(t), -r\sin(t))^T$

$$\kappa(t) = \frac{\det\begin{pmatrix}-r\sin(t) & -r\cos(t)\\ r\cos(t) & -r\sin(t)\end{pmatrix}}{r^3} = \frac{r^2\sin^2(t) + r^2\cos^2(t)}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$
$$n(t) = \frac{(-r\cos(t), -r\sin(t))}{r} = (-\cos(t), -\sin(t))$$

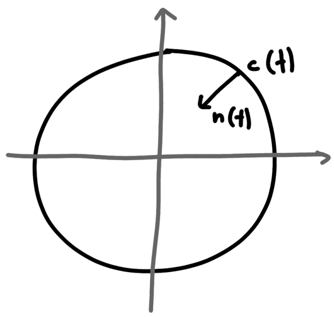

Figure 8: Circle with normal vector

Definition 6. *A point $c(t)$ with $\dot{\kappa}(t) = 0$ is called a vertex.*

Example 8. *An ellipse has four vertices.*

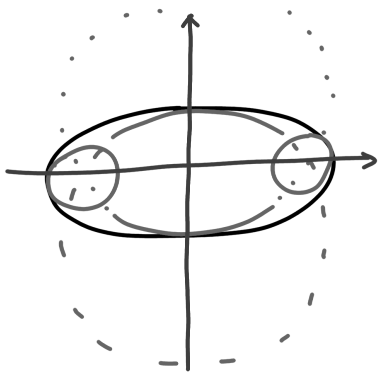

Figure 9: Ellipse and the four vertices.

$(t, \exp t)$ *has no vertex.*
Klothoids are curves with $\kappa(t) = t$. They are used in road construction and have no vertex.

Definition 7. $c : I \to \mathbb{R}^3$
$\kappa(t) = \frac{||\dot{c}(t)\times\ddot{c}(t)||}{||\dot{c}(t)||^3}$ *is called the curvature of a space curve.*
$\tau(t) = \frac{\det(\dot{c}(t), \ddot{c}(t), \dddot{c}(t))}{||\dot{c}(t)\times\ddot{c}(t)||^2}$ *is called torsion of a space curve.*

Example 9. *For the helix $t \mapsto (\cos(t), \sin(t), pt)$ the torsion depends on p.*

3 Properties of Bezier curves

Definition 8. *$\alpha : \mathbb{R}^n \to \mathbb{R}^m$ is called **affine** if $\exists l : \mathbb{R}^n \to \mathbb{R}^m$...linear $\exists v \in \mathbb{R}^m : \alpha(x) = l(x) + v$.*
*α is called **affinity** if α is affine and bijective.*

Example 10. *An example of an linear function is shear (Scherung).*

$$l\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}1 & a\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Area is preserved.