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9.10.3 $A=\begin{pmatrix} 0 & i & 2i \\ -i & 0 & -i \\ -2i & i & 0 \end{pmatrix} \in \mathbb{C}^{3\times 3}$

ges: $P \in GL_3(\mathbb{C})$ mit $B := P^T A P$ ist eine zu A kongruente Matrix in Normalform

$$\left(\begin{array}{ccc} 1&0&0\\0&1&0\\0&0&1\\ \hline 0&i&2i\\-i&0&-i\\-2i&i&0 \end{array}\right) \overset{S}{\rightsquigarrow} \overset{-2\cdot II}{\left(\begin{array}{ccc} 1&0&0\\0&1&-2\\0&0&1\\ \hline 0&i&0\\-i&0&-i\\-2i&i&-2i \end{array}\right)} \overset{Z}{\rightsquigarrow} \left(\begin{array}{ccc} 1&0&0\\0&1&-2\\0&0&1\\ \hline 0&i&0\\-i&0&-i\\0&i&0 \end{array}\right) \overset{S}{\rightsquigarrow} \overset{-I}{\left(\begin{array}{ccc} 1&0&-1\\0&1&-2\\0&0&1\\ \hline 0&i&0\\-i&0&0\\0&i&0 \end{array}\right)} \overset{Z}{\rightsquigarrow} \left(\begin{array}{ccc} 1&0&-1\\0&1&-2\\0&0&1\\ \hline 0&i&0\\-i&0&0\\0&0&0 \end{array}\right) \overset{S}{\rightsquigarrow} \overset{\cdot i}{\left(\begin{array}{ccc} 1&0&-1\\0&i&-2\\0&0&1\\ \hline 0&-1&0\\-i&0&0\\0&0&0 \end{array}\right)} \overset{Z}{\rightsquigarrow} \left(\begin{array}{ccc} 1&0&-1\\0&i&-2\\0&0&1\\ \hline 0&-1&0\\1&0&0\\0&0&0 \end{array}\right)$$

$P=\begin{pmatrix} 1&0&-1\\0&i&-2\\0&0&1 \end{pmatrix}$ $\quad P^T=\begin{pmatrix} 1&0&0\\0&i&0\\-1&-2&1 \end{pmatrix}$ $\quad B=\begin{pmatrix} 0&-1&0\\1&0&0\\0&0&0 \end{pmatrix}=\mathrm{diag}(\square,0)$

ges: $Q \in GL_3(\mathbb{C})$ mit $C := \overline{Q}^T A Q$ ist eine zu A hermitesch kongruente Matrix in Normalform

$$\left(\begin{array}{ccc} 1&0&0\\0&1&0\\0&0&1\\ \hline 0&i&2i\\-i&0&-i\\-2i&i&0 \end{array}\right) \overset{S}{\rightsquigarrow} \overset{-2II}{\left(\begin{array}{ccc} 1&0&0\\0&1&-2\\0&0&1\\ \hline 0&i&0\\-i&0&-i\\-2i&i&-2i \end{array}\right)} \overset{Z}{\rightsquigarrow} \left(\begin{array}{ccc} 1&0&0\\0&1&-2\\0&0&1\\ \hline 0&i&0\\-i&0&-i\\0&i&0 \end{array}\right) \overset{S}{\rightsquigarrow} \overset{-I}{\left(\begin{array}{ccc} 1&0&-1\\0&1&-2\\0&0&1\\ \hline 0&i&0\\-i&0&0\\0&i&0 \end{array}\right)} \overset{Z}{\rightsquigarrow} \left(\begin{array}{ccc} 1&0&-1\\0&1&-2\\0&0&1\\ \hline 0&i&0\\-i&0&0\\0&0&0 \end{array}\right) \overset{S}{\rightsquigarrow} \overset{+(-i)II}{\left(\begin{array}{ccc} 1&0&-1\\-i&1&-2\\0&0&1\\ \hline 1&i&0\\-i&0&0\\0&0&0 \end{array}\right)} \overset{Z}{\rightsquigarrow} \left(\begin{array}{ccc} 1&0&-1\\-i&1&-2\\0&0&1\\ \hline 2&i&0\\-i&0&0\\0&0&0 \end{array}\right)$$

$$\overset{S}{\rightsquigarrow} \overset{-\frac{1}{2}iI}{\left(\begin{array}{ccc} 1&-\frac{1}{2}i&-1\\-i&\frac{1}{2}&-2\\0&0&1\\ \hline 2&0&0\\-i&-\frac{1}{2}&0\\0&0&0 \end{array}\right)} \overset{Z}{\rightsquigarrow} \left(\begin{array}{ccc} 1&-\frac{1}{2}i&-1\\-i&\frac{1}{2}&-2\\0&0&1\\ \hline 2&0&0\\0&-\frac{1}{2}&0\\0&0&0 \end{array}\right) \overset{S}{\rightsquigarrow} \overset{\cdot\frac{1}{\sqrt{2}}}{\left(\begin{array}{ccc} \frac{1}{\sqrt{2}}&-\frac{1}{2}i&-1\\-\frac{i}{\sqrt{2}}&\frac{1}{2}&-2\\0&0&1\\ \hline \frac{2}{\sqrt{2}}&0&0\\0&-\frac{1}{2}&0\\0&0&0 \end{array}\right)} \overset{Z}{\rightsquigarrow} \left(\begin{array}{ccc} \frac{1}{\sqrt{2}}&-\frac{1}{2}i&-1\\-\frac{i}{\sqrt{2}}&\frac{1}{2}&-2\\0&0&1\\ \hline 1&0&0\\0&-\frac{1}{2}&0\\0&0&0 \end{array}\right) \overset{S}{\rightsquigarrow} \overset{\cdot i\sqrt{2}}{\left(\begin{array}{ccc} \frac{1}{\sqrt{2}}&\frac{\sqrt{2}}{2}&-1\\-\frac{i}{\sqrt{2}}&i\frac{\sqrt{2}}{2}&-2\\0&0&1\\ \hline 1&0&0\\0&-\frac{1}{2}i\sqrt{2}&0\\0&0&0 \end{array}\right)} \overset{Z}{\rightsquigarrow} \left(\begin{array}{ccc} \frac{1}{\sqrt{2}}&\frac{\sqrt{2}}{2}&-1\\-\frac{i}{\sqrt{2}}&i\frac{\sqrt{2}}{2}&-2\\0&0&1\\ \hline 1&0&0\\0&-1&0\\0&0&0 \end{array}\right)$$

$Q=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} & -1 \\ -i\frac{1}{\sqrt{2}} & i\frac{\sqrt{2}}{2} & -2 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \overline{Q}^T=\begin{pmatrix} \frac{1}{\sqrt{2}} & +i\frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{2}}{2} & -i\frac{\sqrt{2}}{2} & 0 \\ -1 & -2 & 1 \end{pmatrix}$ $\quad C=\mathrm{diag}(1,-1,0)$

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9.10.7 Für welche $t \in \mathbb{R}$ ist q positiv definit?

a) $q: \mathbb{R}^{3\times 1} \to \mathbb{R} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto t x_1^2 + 2t x_1 x_2 + 2(t+1) x_2^2 + 2 x_2 x_3 + x_3^2$

$$(x_1 \ x_2 \ x_3) \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (a x_1 + b x_2 + d x_3 \quad b x_1 + c x_2 + e x_3 \quad d x_1 + e x_2 + f x_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$= a x_1^2 + b x_1 x_2 + d x_1 x_3 + b x_1 x_2 + c x_2^2 + e x_2 x_3 + d x_1 x_3 + e x_2 x_3 + f x_3^2$$
$$= a x_1^2 + 2b x_1 x_2 + 2d x_1 x_3 + c x_2^2 + 2e x_2 x_3 + f x_3^2 =: g\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$g\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = q\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow a = t \quad b = t \quad d = 0 \quad c = 2t+2 \quad e = 1 \quad f = 1$

$\Rightarrow q$ als symmetrische Matrix ist $\begin{pmatrix} t & t & 0 \\ t & 2t+2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Hauptminoren:

$|t| = t \qquad \begin{vmatrix} t & t \\ t & 2t+2 \end{vmatrix} = 2t^2 + 2t - t^2 = t^2 + 2t \qquad \begin{vmatrix} t & t & 0 \\ t & 2t+2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 2t^2 + 2t - t - t^2 = t^2 + t$

$\Rightarrow$ Damit q positiv definit muss $t > 0$, $t^2 + 2t > 0$ und $t^2 + t > 0$

$t^2 + 2t = 0 \Leftrightarrow t_{1,2} = -1 \pm \sqrt{1} = \begin{cases} 0 \\ -2 \end{cases} \Rightarrow t > 0$ oder $t < -2$

$t^2 + t = 0 \Leftrightarrow t_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4}} = \begin{cases} 0 \\ -1 \end{cases} \Rightarrow t > 0$ oder $t < -1$

$\Rightarrow$ Für $t > 0$ ist q positiv definit. Für $t = 0$ semidefinit (positiv und negativ?)

Damit q negativ definit $t < 0 \quad t^2 + 2t > 0 \quad t^2 + t < 0 \Rightarrow t < -2 \wedge t > -1$ $\lightning$

$\Rightarrow$ Für $t < 0$ indefinit

b) $q\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (1-t) x_1^2 - 2t x_1 x_2 + 2 x_1 x_3 - x_2^2 + x_3^2$

$\Rightarrow a = 1-t \quad b = -t \quad c = -1 \quad d = 1 \quad e = 0 \quad f = 1 \qquad \begin{pmatrix} 1-t & -t & 1 \\ -t & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$|1-t| = 1-t \qquad \begin{vmatrix} 1-t & -t \\ -t & -1 \end{vmatrix} = t - 1 - t^2 = -t^2 + t - 1$

$\begin{vmatrix} 1-t & -t & 1 \\ -t & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = t - 1 - 1 + t^2 = t^2 + t - 2 \quad \Rightarrow$ damit q positiv definit $1 - t > 0$, $-t^2 + t - 1 > 0$ und $t^2 + t - 2 > 0$

$\nexists t \in \mathbb{R}$ mit $-t^2 + t - 1 > 0 \Rightarrow \nexists \in \mathbb{R}$ mit q positiv definit (auch nicht $\geq$ also auch nicht semi pos. def)

Damit q negativ (semi) definit: $1 - t < 0 \quad \underbrace{-t^2 + t - 1 > 0}_{\lightning} \quad t^2 + t - 2 < 0$ (bzw. $\leq$, $\geq$)

$\Rightarrow q$ ist indefinit

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11.1.1. K… Körper $K^{2\times 1}$ mit kanonischem Skalarprodukt

Gibt es isotrope Vektoren? Anzahl der zu sich selbst orthogonalen 1-dim UR + Basis

i) $K = \mathbb{Z}_2$

- $\binom{0}{0}$ $[\binom{0}{0}] \cap [\binom{0}{0}]^\perp = \{\binom{0}{0}\}$, da $[\binom{0}{0}]^\perp = \mathbb{Z}_2$
- $\binom{0}{1}$ $\binom{0}{1} \cdot \binom{a}{b} = b \Rightarrow [\binom{0}{1}]^\perp = \{\binom{0}{1}\}^\perp = \{\binom{0}{0}, \binom{1}{0}\} \Rightarrow [\binom{0}{1}] \cap [\binom{0}{1}]^\perp = \{\binom{0}{0}\}$
- $\binom{1}{0}$ $\binom{1}{0} \cdot \binom{a}{b} = a \Rightarrow [\binom{1}{0}]^\perp = \{\binom{0}{0}, \binom{0}{1}\} \Rightarrow [\binom{1}{0}] \cap [\binom{1}{0}]^\perp = \{\binom{0}{0}\}$
- $\binom{1}{1}$ $\binom{1}{1} \cdot \binom{a}{b} = a+b \Rightarrow [\binom{1}{1}]^\perp = \{\binom{0}{0}, \binom{1}{1}\} \Rightarrow [\binom{1}{1}] \cap [\binom{1}{1}]^\perp = \{\binom{0}{0}, \binom{1}{1}\}$

$\Rightarrow \binom{1}{1}$ ist isotrop. Es gibt 1 zu sich selbst orthogonalen 1-dim UR, nämlich $[\binom{1}{1}]$.

ii) $K = \mathbb{Z}_3$

- $\binom{0}{0}$ $[\binom{0}{0}] \cap [\binom{0}{0}]^\perp = \{\binom{0}{0}\}$
- $\binom{0}{1} / \binom{0}{2}$ $\binom{0}{1} \cdot \binom{a}{b} = b$ $[\binom{0}{1}]^\perp = \{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}\} \Rightarrow [\binom{0}{1}] \cap [\binom{0}{1}]^\perp = \{\binom{0}{0}\}$ für $\binom{0}{2}$ genauso
- $\binom{1}{0} / \binom{2}{0}$ $\binom{1}{0} \cdot \binom{a}{b} = a$ $[\binom{1}{0}]^\perp = \{\binom{0}{0}, \binom{0}{1}, \binom{0}{2}\} \Rightarrow [\binom{1}{0}] \cap [\binom{1}{0}]^\perp = \{\binom{0}{0}\}$
- $\binom{1}{1}$ $\binom{1}{1} \cdot \binom{a}{b} = a+b$ $[\binom{1}{1}]^\perp = \{\binom{0}{0}, \binom{1}{2}, \binom{2}{1}\} \Rightarrow [\binom{1}{1}] \cap [\binom{1}{1}]^\perp = \{\binom{0}{0}\}$
- $\binom{2}{2}$ $\binom{2}{2} \cdot \binom{a}{b} = 2a+2b$ $[\binom{2}{2}]^\perp = \{\binom{0}{0}, \binom{1}{2}, \binom{2}{1}\} \Rightarrow [\binom{2}{2}] \cap [\binom{2}{2}]^\perp = \{\binom{0}{0}\}$
- $\binom{1}{2} / \binom{2}{1}$ $\binom{1}{2} \cdot \binom{a}{b} = a+2b$ $[\binom{1}{2}]^\perp = \{\binom{0}{0}, \binom{1}{1}, \binom{2}{2}\} \Rightarrow [\binom{1}{2}] \cap [\binom{1}{2}]^\perp = \{\binom{0}{0}\}$

$\Rightarrow \nexists$ isotroper Vektor. Es gibt keine zu sich selbst orthogonalen 1-dim UR.

iii) $K = GF(4)$

- $\binom{0}{0}$ $[\binom{0}{0}] \cap [\binom{0}{0}]^\perp = \{\binom{0}{0}\}$
- $\binom{0}{1} / \binom{0}{a} / \binom{0}{b}$ $[\binom{0}{1}]^\perp = \{\binom{0}{0}, \binom{1}{0}, \binom{a}{0}, \binom{b}{0}\} \Rightarrow [\binom{0}{1}] \cap [\binom{0}{1}]^\perp = \{\binom{0}{0}\}$
- $\binom{1}{0} / \binom{a}{0} / \binom{b}{0}$ $[\binom{1}{0}]^\perp = \{\binom{0}{0}, \binom{0}{1}, \binom{0}{a}, \binom{0}{b}\} \Rightarrow [\binom{1}{0}] \cap [\binom{1}{0}]^\perp = \{\binom{0}{0}\}$
- $\binom{1}{1} / \binom{a}{a} / \binom{b}{b}$ $\binom{1}{1}\binom{x}{y} = x+y = 1\cdot(x+y)$ $[\binom{1}{1}]^\perp = \{\binom{0}{0}, \binom{1}{1}, \binom{a}{a}, \binom{b}{b}\} \Rightarrow [\binom{1}{1}] \cap [\binom{1}{1}]^\perp = \{\binom{0}{0}, \binom{1}{1}, \binom{a}{a}, \binom{b}{b}\}$
- $\binom{1}{a} / \binom{a}{1}$ $\binom{1}{a}\binom{x}{y} = x+ay$ $[\binom{1}{a}]^\perp = \{\binom{0}{0}, \binom{1}{b}, \binom{a}{1}, \binom{b}{a}\} \Rightarrow [\binom{1}{a}] \cap [\binom{1}{a}]^\perp = \{\binom{0}{0}\}$
- $\binom{1}{b} / \binom{b}{1}$ $\binom{1}{b}\binom{x}{y} = x+by$ $[\binom{1}{b}]^\perp = \{\binom{0}{0}, \binom{1}{a}, \binom{a}{b}, \binom{b}{1}\} \Rightarrow [\binom{1}{b}] \cap [\binom{1}{b}]^\perp = \{\binom{0}{0}\}$
- $\binom{a}{b} / \binom{b}{a}$ $\binom{a}{b}\binom{x}{y} = ax+by$ $[\binom{a}{b}]^\perp = \{\binom{0}{0}, \binom{1}{b}, \binom{a}{1}, \binom{b}{a}\} \Rightarrow [\binom{a}{b}] \cap [\binom{a}{b}]^\perp = \{\binom{0}{0}\}$

$\Rightarrow \binom{1}{1}, \binom{a}{a}, \binom{b}{b}$ sind isotrop. Es gibt 1 zu sich selbst orthogonalen 1-dim UR, nämlich $[\binom{1}{1}]$

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11.11 ...

iv) $K = \mathbb{Z}_5$

- $\binom{0}{0}$ $\quad [\binom{0}{0}] \cap [\binom{0}{0}]^{\perp} = \{\binom{0}{0}\}$

- $\binom{0}{1} / \binom{0}{2} / \binom{0}{3} / \binom{0}{4}$ $\quad \binom{0}{1} \cdot \binom{x}{y} = y \Rightarrow [\binom{0}{1}]^{\perp} = \{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{4}{0}\}$ $\quad [\binom{0}{1}] \cap [\binom{0}{1}]^{\perp} = \{\binom{0}{0}\}$
  $\binom{1}{0} / \binom{2}{0} / \binom{3}{0} / \binom{4}{0}$

- $\binom{1}{1}, \binom{2}{2}, \binom{3}{3}, \binom{4}{4}$ $\quad \binom{1}{1} \cdot \binom{x}{y} = x + y = 1 \cdot (x+y)$ $\quad [\binom{1}{1}]^{\perp} = \{\binom{0}{0}, \binom{1}{4}, \binom{2}{3}, \binom{3}{2}, \binom{4}{1}\}$ $\quad [\binom{1}{1}] \cap [\binom{1}{1}]^{\perp} = \{\binom{0}{0}\}$

- $\binom{1}{2}, \binom{3}{1}$ $\quad \binom{1}{2} \cdot \binom{x}{y} = x + 2y$ $\quad [\binom{1}{2}]^{\perp} = \{\binom{0}{0}, \binom{1}{2}, \binom{2}{4}, \binom{3}{1}, \binom{4}{3}\}$ $\quad [\binom{1}{2}] \cap [\binom{1}{2}]^{\perp} = \{\binom{0}{0}, \binom{1}{2}, \binom{2}{4}, \binom{3}{1}, \binom{4}{3}\}$

- $\binom{2}{4}, \binom{4}{2}$ $\quad \binom{2}{4} \cdot \binom{x}{y} = 2x + 4y = 2 \cdot (x + 2y)$ wie oben

- $\binom{1}{3}, \binom{3}{1}$ — " — $\qquad$ • $\binom{3}{4}, \binom{4}{3}$ — " —

- $\binom{1}{4}, \binom{4}{1}$ $\quad \binom{1}{4} \cdot \binom{x}{y} = x + 4y$ $\quad [\binom{1}{4}]^{\perp} = \{\binom{0}{0}, \binom{1}{1}, \binom{2}{2}, \binom{3}{3}, \binom{4}{4}\} \Rightarrow [\binom{1}{4}] \cap [\binom{1}{4}]^{\perp} = \{\binom{0}{0}\}$

- $\binom{2}{3}, \binom{3}{2}$ $\quad \binom{2}{3} \cdot \binom{x}{y} = 2x + 3y$ $\quad [\binom{2}{3}]^{\perp} = \{\binom{0}{0}, \binom{1}{1}, \binom{2}{2}, \binom{3}{3}, \binom{4}{4}\} \Rightarrow [\binom{2}{3}] \cap [\binom{2}{3}]^{\perp} = \{\binom{0}{0}\}$

$\Rightarrow \binom{1}{2}, \binom{2}{4}, \binom{3}{1}, \binom{4}{3}, \binom{2}{1}, \binom{4}{2}, \binom{1}{3}, \binom{3}{4}$ sind isotrop. Es gibt 2 zu sich selbst isomorphe 1-dim UR, nämlich $[\binom{1}{2}]$ und $[\binom{2}{1}]$.

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11.1.3 K... Körper (V, σ)... symplektischer VR/K $\dim(V)=4$

symplektisch: $\forall x \in V\setminus\{0\}: [x] \cap [x]^{\perp} \neq \{0\}$

$[x]^{\perp} = \{v \in V: \forall k \in K: k\cdot x \perp v\}$ $\quad \sigma(k\cdot x, v)=0 \iff \omega(k)\cdot\sigma(x,v)=0 \Rightarrow \sigma(x,v)=0$

$[x]^{\perp} = \{v\in V: x\perp v\}$ $\quad [x]\cap[x]^{\perp}\neq\{0\} \Rightarrow \exists k\in K^{\times}: k\cdot x\perp x \iff \sigma(k\cdot x, x)=0$

$\iff \omega(k)\cdot\sigma(x,x)=0 \Rightarrow x\perp x$

also symplektisch: $\forall x\in V\setminus\{0\}: x\perp x$

zz: $\forall U\subseteq V$ ... 2-dim UR: $(U\oplus U^{\perp}=V \quad \vee \quad U=U^{\perp})$

Sei $x,y\in V$ bel. mit x,y l.u. $\Rightarrow U:=[\{x,y\}]$ ist ein 2-dim UR

1. Fall U ist nicht isotrop

D.h. $U\cap U^{\perp}=\{0\}$

radikalfrei

$\dim(U\cap U^{\perp})+\dim(U+U^{\perp})=\dim(U)+\dim(U^{\perp})=4+\dim(U\cap V^{\perp}) \stackrel{!}{=} 4$

$\Rightarrow \dim(U+U^{\perp})=4$ und $U\cap U^{\perp}=\{0\}$ also $U\oplus U^{\perp}=V$

2. Fall U ist isotrop

D.h. $U\cap U^{\perp}\neq\{0\}$ $\quad U\cap U^{\perp}$ ist ein Unterraum von V ($U\cap U^{\perp}=U^{\perp U}$ ... Radikal)

$\exists a,b\in K$ $a\neq 0$ oder $b\neq 0$ $\forall k,l\in K$ $\sigma(ax+by, k\cdot x+ly)=0$

$\iff \omega(a)\omega(k)\sigma(x,x)+\omega(a)\cdot l\,\sigma(x,y)+b\cdot\omega(k)\,\sigma(y,x)+b\cdot l\,\sigma(y,y)=0$

$\iff (\omega(a)\cdot l+b\cdot\omega(k))\,\sigma(x,y)=0 \quad \Rightarrow \sigma(x,y)=0$

$\forall k,l\in K: k\cdot\sigma(x,x)+l\cdot\sigma(x,y)=0 \iff \sigma(x, k\cdot x+l\cdot y)=0 \Rightarrow x\in U\cap U^{\perp}$

$\forall k,l\in K: k\cdot\sigma(y,y)+l\cdot\sigma(y,x)=0 \iff \sigma(y, l\cdot x+k\cdot y)=0 \Rightarrow y\in U\cap U^{\perp}$

Da $U\cap U^{\perp}$ ein UR ist folgt $\forall k,l\in K: k\cdot x+l\cdot y\in U\cap U^{\perp}$

$\Rightarrow U\subseteq U^{\perp}$ $\quad \dim(U)+\dim(U^{\perp})=4 \Rightarrow \dim(U^{\perp})=2 \quad \Rightarrow U=U^{\perp}$

Beispiele: $\sigma\left(\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}\right)=x_2y_1-x_1y_2+x_4y_3-x_3y_4$ $\quad K=\mathbb{R} \quad V=\mathbb{R}^4$

$x=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, y=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$ $\quad U=[\{x,y\}]$

$x^{\perp}=[\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\}]$ $\quad y^{\perp}=[\{\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\}]$

$U^{\perp}=x^{\perp}\cap y^{\perp}=[\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\}]=U$

⋮

Satz 9.5.5 c)

$x=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\quad y=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ $\quad U=[\{x,y\}]$

$x^{\perp}=[\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\}]$ $\quad y^{\perp}=[\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\}]$

$U^{\perp}=x^{\perp}\cap y^{\perp}=[\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\}]$

$U\oplus U^{\perp}=V$

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11.1.8. $V \ldots VR/K$ $\quad V\times V^* \ldots VR/K$ mit Addition und Multiplikation mit Skalaren komponentenweise

$$\iota : (V\times V^*)\times(V\times V^*) \to K$$

$$((x,a^*),(y,b^*)) \mapsto b^*(x) - a^*(y)$$

zz: ι ist ein symplektisches Skalarprodukt auf $V\times V^*$

Sei $(x,a^*),(y,b^*),(z,c^*) \in V\times V^*$ bel. Sei $k\in K$ bel.

- $\iota((x,a^*)+(y,b^*),(z,c^*)) = \iota((x+y, a^*+b^*),(z,c^*)) = c^*(x+y)-(a^*+b^*)(z)$
  $= c^*(x)+c^*(y) - a^*(z) - b^*(z) = \iota((x,a^*),(z,c^*)) + \iota((y,b^*),(z,c^*))$
- $\iota(k\cdot(x,a^*),(y,b^*)) = \iota((k\cdot x, k\cdot a^*),(y,b^*)) = b^*(k\cdot x) - (k\cdot a^*)(y)$
  $= k\cdot b^*(x) - k\cdot a^*(y) = k\cdot \iota((x,a^*),(y,b^*))$
- Umgekehrt genauso $\quad\Rightarrow \iota \ldots$ Bilinearform
- $\iota((x,a^*),(x,a^*)) = a^*(x) - a^*(x) = 0 \quad \Rightarrow$ alternierend / symplektisch
- Sei $(x,a^*)\in V\times V^*$ bel. $\iota((x,a^*),(0,x^*)) = x^*(x) - a^*(0) = 1$
  $\Rightarrow$ radikalfrei

Also ist ι ein symplektisches Skalarprodukt auf $V\times V^*$

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G9 $V = \mathbb{R}^{2\times 1}$ $E = (e_1, e_2)$ ... kanonische Basis $\sigma: V \times V \to V$

$\sigma(E,E) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: A$

1.) ges: alle Basen mit $\sigma(B,B) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Transformationsmatrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $T^T \cdot E \cdot T = B$ $T^T \cdot A \cdot T = A$

$T^T \cdot A \cdot T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Falls $b = 0$: $ab - cd = 0 \Leftrightarrow -cd = 0 \Rightarrow c = 0 \vee d = 0$ wenn $d = 0$: $b^2 - d^2 = b^2 \neq -1$ ↯

Falls $d = 0$: $ab - cd = 0 \Leftrightarrow ab = 0 \Rightarrow a = 0 \vee b = 0$ wenn $a = 0$: $a^2 - c^2 = -c^2 \neq 1$ ↯

Also ist beides der gleiche Fall. $b^2 - d^2 = 0 \neq -1$ ↯

Falls $b \neq 0 \wedge d \neq 0$: $ab - cd = 0 \Leftrightarrow a = \frac{cd}{b}$ $\Leftrightarrow c = \frac{ab}{d}$

$a^2 - c^2 = \frac{c^2 d^2}{b^2} - c^2 = \frac{c^2(d^2 - b^2)}{b^2} = 1$ $b^2 - d^2 = -1 \Rightarrow d^2 - b^2 = 1$

$\frac{c^2}{b^2} = 1 \Leftrightarrow b^2 = c^2 \Leftrightarrow |b| = |c|$

$a^2 - c^2 = a^2 - \frac{a^2 b^2}{d^2} = \frac{a^2 \cdot (d^2 - b^2)}{d^2} = \frac{a^2}{d^2} = 1 \Leftrightarrow a^2 = d^2 \Leftrightarrow |d| = |a|$

Also $T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ $T = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ $T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ oder $T = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$.

1. Fall $a^2 - b^2 = 1$ $a \cdot b - b \cdot a = 0$ ✓ $b^2 - a^2 = -1 \Leftrightarrow a^2 - b^2 = 1$

$\Rightarrow b^2 = a^2 - 1 \Leftrightarrow b = \pm\sqrt{a^2 - 1}$

2. Fall $a^2 - b^2 = 1$ $a \cdot b + b \cdot a = 0$ $\Rightarrow a = 0 \vee (b = 0$ ↯$)$ $\Rightarrow T$ ... nicht regulär

3. Fall $a^2 - b^2 = 1$ $a \cdot b + b \cdot a = 0$ — " —

4. Fall $a^2 - b^2 = 1$ $a \cdot b - b \cdot a = 0$ ✓ $b^2 = a^2 - 1$ $b = \pm\sqrt{a^2 - 1}$

$\Rightarrow T = \begin{pmatrix} a & \sqrt{a^2-1} \\ \sqrt{a^2-1} & a \end{pmatrix}$ $T = \begin{pmatrix} a & -\sqrt{a^2-1} \\ -\sqrt{a^2-1} & a \end{pmatrix}$ $T = \begin{pmatrix} a & \sqrt{a^2-1} \\ -\sqrt{a^2-1} & -a \end{pmatrix}$ $T = \begin{pmatrix} a & -\sqrt{a^2-1} \\ \sqrt{a^2-1} & -a \end{pmatrix}$

$T^T \cdot E \cdot T = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} a^2 + a^2 - 1 & \pm a\sqrt{a^2-1} \pm \sqrt{a^2-1} \cdot a \\ \pm a\sqrt{a^2-1} \pm \sqrt{a^2-1} \cdot a & a^2 - 1 + a^2 \end{pmatrix}$

2.) zz: $\exists\, U^+, U^-$ wie in 9.10.3. $\exists\, \tilde{U}^+, \tilde{U}^-$ wie in 9.10.3 verschieden

$U^+ = [\begin{pmatrix} 1 \\ 0 \end{pmatrix}]$ $U^- = [\begin{pmatrix} 0 \\ 1 \end{pmatrix}]$ $\tilde{U}^+ = [\begin{pmatrix} 7 \\ 4\sqrt{3} \end{pmatrix}]$ $\tilde{U}^- = [\begin{pmatrix} 4\sqrt{3} \\ 7 \end{pmatrix}]$ $(a = 2)$

3.) ges: $U_1, U_2 \subseteq V$ $V = U_1 \oplus U_2$ $\forall v \in U_1^{\times}: \sigma(v,v) > 0$ $\forall v \in U_2^{\times}: \sigma(v,v) > 0$

$U_1 \overset{a=2}{=} [\begin{pmatrix} 7 \\ 4\sqrt{3} \end{pmatrix}]$ $U_2 \overset{a=3}{=} [\begin{pmatrix} 17 \\ 12\sqrt{2} \end{pmatrix}]$ $k \cdot \begin{pmatrix} 7 \\ 4\sqrt{3} \end{pmatrix} = \begin{pmatrix} 17 \\ 12\sqrt{2} \end{pmatrix} \Rightarrow 7k = 17 \Leftrightarrow k = \frac{17}{7}$

$\frac{17}{7} \cdot 4\sqrt{3} = \frac{68}{7}\sqrt{3} \neq 12\sqrt{2} \Rightarrow U_1 \cap U_2 = \{0\}$ $\dim U_1 + U_2 = 2$ $\Rightarrow U_1 \oplus U_2 = V$