

ANA 08

2) i) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Quotientenkriterium mit $q = \frac{1}{2}$

$$\frac{\left| \frac{(n+1)!}{(n+1)^{(n+1)}} \right|}{\left| \frac{n!}{n^n} \right|} = \frac{\frac{n! \cdot (n+1)}{(n+1)^n \cdot (n+1)}}{\frac{n!}{n^n}} = \frac{n! \cdot n^n}{n! \cdot (n+1)^n} = \left(\frac{n}{n+1} \right)^n$$
$$= \left(\frac{\frac{n}{n+1}}{\frac{n+1}{n}} \right)^n = \left(\frac{1}{\frac{n}{n+1} + \frac{1}{n}} \right)^n = \left(\frac{1}{1 + \frac{1}{n}} \right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\left(1 + \frac{1}{n}\right)^n \geq 1 + \frac{1}{n} \cdot n = 2$$

$$\frac{1}{\left(1 + \frac{1}{n}\right)^n} \leq \frac{1}{2} = q \Rightarrow \sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ konvergiert}$$

ii) $\sum_{n=1}^{\infty} \left(\frac{n^2}{n^2+1} \right)^{n^3}$ Wurzelkriterium mit $q = \frac{1}{2}$

$$\sqrt[n]{\left(\frac{n^2}{n^2+1} \right)^{n^3}} = \left(\frac{n^2}{n^2+1} \right)^{n^2} = \left(\frac{\frac{n^2}{n^2}}{\frac{n^2+1}{n^2}} \right)^{n^2} = \left(\frac{1}{1 + \frac{1}{n^2}} \right)^{n^2} = \frac{1}{\left(1 + \frac{1}{n^2}\right)^{n^2}}$$

$$\left(1 + \frac{1}{n^2}\right)^{n^2} \geq 1 + \frac{1}{n^2} \cdot n^2 = 2$$

$$\frac{1}{\left(1 + \frac{1}{n^2}\right)^{n^2}} \leq \frac{1}{2} = q \Rightarrow \sum_{n=1}^{\infty} \left(\frac{n^2}{n^2+1} \right)^{n^3} \text{ konvergiert}$$