<u>ANA</u> Ü8

1.) $A \in \mathbb{C}^{n\times n}$ $\quad A = T^{-1} J T$ (Jordansche Normalform) $\quad$ ges: $\exp(A)$

$\exp(A) = \exp(T^{-1} J T) = T^{-1} \cdot \exp(J) \cdot T$ laut Ü7 Nr 8

J hat Form $\begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_n \end{pmatrix}$ mit $J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & & \lambda_i \end{pmatrix}$ $\forall i \in \{1, \dots, n\}$

$K_i := \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & 0 & & \vdots \\ & & J_i \; 0 & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \leftarrow$ i-te Stelle $\qquad \Rightarrow J = \sum_{i=1}^{n} K_i$

$\uparrow$ i-te Stelle

$\exp(J) = \exp\left(\sum_{i=1}^{n} K_i\right) = \prod_{i=1}^{n} \exp(K_i)$ $\quad$ laut Ü7 Nr 9

$$K_i = \begin{array}{c} \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \begin{matrix} \lambda_i & 1 & & 0 \\ & \ddots & & 1 \\ 0 & & & \lambda_i \end{matrix} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \begin{matrix} \}a \\ \}b \\ \}c \end{matrix} \\ \quad a \qquad b \qquad c \end{array}$$

$$\exp(K_i) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot K_i^{\,n} = E_n + K_i + \frac{1}{2} K_i^{\,2} + \dots$$

$$K_i^{\,m} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_i^{\,m} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \exp(K_i) = \begin{pmatrix} E_a & 0 & 0 \\ 0 & \exp(J_i) & 0 \\ 0 & 0 & E_c \end{pmatrix}$$

$\prod_{i=1}^{n} \exp(K_i) = \begin{pmatrix} \exp(J_1) & & 0 \\ & \ddots & \\ 0 & & \exp(J_n) \end{pmatrix}$ da sich die Spalten und Zeilen der einzelnen Blöcke nicht überschneiden, siehe Beispiel:

$$\overbrace{\begin{pmatrix} E_a & 0 \\ 0 & A \end{pmatrix}}^{n=a+b} \begin{pmatrix} B & 0 \\ 0 & E_b \end{pmatrix} \Big\} n = a+b$$

$$\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$$

$$\Rightarrow \exp(A) = T^{-1} \begin{pmatrix} \exp(J_1) & & 0 \\ & \ddots & \\ 0 & & \exp(J_n) \end{pmatrix} \cdot T$$

mit $\exp(J_i) = \begin{pmatrix} \frac{\exp(\lambda_i)}{0!} & \frac{\exp(\lambda_i)}{1!} & \cdots & \frac{\exp(\lambda_i)}{(k-1)!} \\ 0 & \frac{\exp(\lambda_i)}{0!} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\exp(\lambda_i)}{0!} \end{pmatrix}$ laut Ü7 Nr 10

ANA Ü8

2.) $a, b \in \mathbb{R}$, $a < b$ $\quad f : [a; b) \to \mathbb{R}\,(\mathbb{C})$

zz: $f'(a)^+$ existiert $\iff \exists y \in \mathbb{R}(\mathbb{C})\ \exists \delta > 0\ \exists \varepsilon : [0, \delta) \to \mathbb{R}(\mathbb{C})$ ... bei 0 stetig und verschwindend:

$$f(x) = f(a) + (x-a)y + (x-a) \cdot \varepsilon(x-a) \quad \forall x \in [a, b) \cap [a, a+\delta)$$

$\Rightarrow$

$y := \lim_{t \to a^+} \frac{f(t) - f(a)}{t - a}$ existiert $\Rightarrow \exists \delta > 0\ \forall t \in [a, b) \cap [a, a+\delta) : \frac{f(t)-f(a)}{t-a}\ \exists$

$$\varepsilon(t-a) := \frac{f(t) - f(a)}{t-a} - y \qquad \lim_{x \to 0} \varepsilon(x) = \lim_{t \to a^+} (t-a) = 0$$

$$\varepsilon(t-a) = \frac{f(t) - f(a) - (t-a)y}{t-a} \iff f(t) = \varepsilon(t-a) \cdot (t-a) + f(a) + (t-a)y$$

$\Leftarrow$

$$f(t) = f(a) + (t-a)y + (t-a) \cdot \varepsilon(t-a)$$

$$\iff \varepsilon(t-a) = \frac{f(t) - f(a) - (t-a)y}{t-a} = \frac{f(t)-f(a)}{t-a} - y$$

$$\iff \frac{f(t)-f(a)}{t-a} - \varepsilon(t-a) = y \qquad \text{da } \varepsilon(t-a) \xrightarrow{t \to a^+} 0 \text{ folgt}$$

$$f'(a)^+ = \lim_{t \to a^+} \frac{f(t)-f(a)}{t-a} = y \quad \text{existiert}$$

Für die linksseitige Ableitung ist $\varepsilon : (-\delta, 0] \to \mathbb{R}(\mathbb{C})$, da $t - b \leq 0$ und $\forall t \in [a, b) \cap (b-\delta, b]$, sonst gleich.

Für die Ableitung ist $\varepsilon : (-\delta, \delta) \to \mathbb{R}(\mathbb{C})$, $\forall t \in [a, b) \cap (x-\delta, x+\delta)$ sonst auch gleich.

ANA Ü8

6. $A \in \mathbb{R}^{n \times n}$ $\quad A^T = A$ $\quad f: \mathbb{R}^n \to \mathbb{R}$, $x \mapsto x^T A x$

zz: $df(x) = 2(Ax)^T$

$$df(x) = \left(\frac{\partial f}{\partial x_j}(x)\right)_{j=1,\dots,n} \qquad \frac{\partial f}{\partial x_i}(x) = \lim_{s\to 0} \frac{1}{s}\left(f(x+se_i) - f(x)\right)$$

laut 10.1.9, 5.

$$= \lim_{s\to 0} \frac{1}{s}\left((x+se_i)^T A (x+se_i) - x^T A x\right)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad x^T A \cdot x = (x_1\, x_2 \dots x_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1\, x_2 \dots x_n) \begin{pmatrix} \sum_{i=1}^n a_{1i} \cdot x_i \\ \vdots \\ \sum_{i=1}^n a_{ni} \cdot x_i \end{pmatrix}$$

$$= \left(x_1 \cdot \sum_{i=1}^n a_{1i} x_i + \dots + x_n \cdot \sum_{i=1}^n a_{ni} x_i\right)$$

bei $i=j$ steht $(x_j + s)$

$$\lim_{s\to 0} \frac{1}{s}\left((x+se_j)^T A (x+se_j) - x^T A x\right) = \lim_{s\to 0} \frac{1}{s}\Big(x_1 \cdot \sum_{i=1}^n a_{1i} x_i + \dots + (x_j+s) \sum_{i=1}^n a_{ji} x_i + \dots + x_n \sum_{i=1}^n a_{ni} x_i$$

$$- \left(x_1 \sum_{i=1}^n a_{1i} x_i + \dots + x_n \sum_{i=1}^n a_{ni} x_i\right)\Big) =$$

$$x_1 \cdot \Big(\sum_{\substack{i=1 \\ i \neq j}}^n a_{1i} x_i + a_{1j}(x_j + s)\Big) - x_1 \Big(\sum_{\substack{i=1 \\ i \neq j}}^n a_{1i} x_i + a_{1j} x_j\Big) = x_1 a_{1j} (x_j + s) - x_1 a_{1j} x_j$$

$$= x_1 a_{1j} x_j + s x_1 a_{1j} - x_1 a_{1j} x_j = s \cdot x_1 \cdot a_{1j}$$

$$= \lim_{s\to 0} \frac{1}{s}\left(s x_1 a_{1j} + \dots + s x_n a_{nj} + s \sum_{i=1}^n a_{ji} x_i\right) = x_1 a_{1j} + \dots + x_n a_{nj} + \sum_{i=1}^n a_{ji} x_i$$

$$= \sum_{i=1}^n a_{ij} x_i + \sum_{i=1}^n a_{ji} x_i = 2 \sum_{i=1}^n a_{ij} x_i \quad , \text{da } a_{ij} = a_{ji} \text{ wegen } A^T = A$$

stetig

$$df(x) = \left(2 \sum_{i=1}^n a_{ji} x_i\right)_{j=1,\dots,n} = 2 \cdot \left(\sum_{i=1}^n a_{1i} x_i \;\cdots\; \sum_{i=1}^n a_{ni} x_i\right) = 2 (Ax)^T$$

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ges: $\frac{\partial f}{\partial x}(x)$

$$\frac{\partial f}{\partial x}(x) = \sum_{j=1}^n x_j \frac{\partial f}{\partial x_j}(x) = df(x)\, x = 2(Ax)^T x = 2\, x^T A^T x \underset{A^T = A}{=} 2\, x^T A x = 2 f(x)$$

ANA Ü8

3.) $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \xi^2 \eta \sin(\xi\eta) \\ \frac{\xi}{\xi^2+\eta^2+1} \end{pmatrix}$$

ges: partielle Ableitungen

$$\frac{\partial}{\partial x_1} f\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lim_{s\to 0} \frac{1}{s}\left(f\left(\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix}\right) - f\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = \lim_{s\to 0} \frac{1}{s}\left(\begin{pmatrix} (\xi+s)^2\eta \sin((\xi+s)\eta) \\ \frac{\xi+s}{(\xi+s)^2+\eta^2+1} \end{pmatrix} - \begin{pmatrix} \xi^2\eta\sin(\xi\eta) \\ \frac{\xi}{\xi^2+\eta^2+1} \end{pmatrix}\right)$$

$$\lim_{s\to 0} \frac{(\xi+s)^2\eta\sin((\xi+s)\eta) - \xi^2\eta\sin(\xi\eta)}{s} = \lim_{s\to 0} \eta(2(\xi+s)\cdot\sin((\xi+s)\eta) + (\xi+s)^2\cos((\xi+s)\eta)\cdot\eta)$$

$$= \eta(2\xi\sin(\xi\eta) + \xi^2\eta\cos(\xi\eta)) = \xi\eta(2\sin(\xi\eta) + \xi\eta\cos(\xi\eta))$$

$$\lim_{s\to 0}\left(\frac{(\xi+s)(\xi^2+\eta^2+1) - \xi((\xi+s)^2+\eta^2+1)}{s((\xi+s)^2+\eta^2+1)(\xi^2+\eta^2+1)}\right) = \lim_{s\to 0}\frac{(\xi^2+\eta^2+1) - \xi(2(\xi+s))}{(\xi^2+\eta^2+1)((\xi+s)^2+\eta^2+1) + s(2(\xi+s)))}$$

$$= \frac{\xi^2+\eta^2+1-2\xi^2}{(\xi^2+\eta^2+1)(\xi^2+\eta^2+1)} = \frac{-\xi^2+\eta^2+1}{(\xi^2+\eta^2+1)^2}$$

$$\Rightarrow \frac{\partial}{\partial x_1} f\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi\eta(2\sin(\xi\eta) + \xi\eta\cos(\xi\eta)) \\ \frac{-\xi^2+\eta^2+1}{(\xi^2+\eta^2+1)^2} \end{pmatrix}$$

$$\frac{\partial}{\partial x_2} f\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lim_{s\to 0}\frac{1}{s}\left(f\left(\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} 0 \\ s \end{pmatrix}\right) - f\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = \lim_{s\to 0}\frac{1}{s}\left(\begin{pmatrix} \xi^2(\eta+s)\sin(\xi(\eta+s)) \\ \frac{\xi}{\xi^2+(\eta+s)^2+1} \end{pmatrix} - \begin{pmatrix} \xi^2\eta\sin(\xi\eta) \\ \frac{\xi}{\xi^2+\eta^2+1} \end{pmatrix}\right)$$

$$\lim_{s\to 0}\frac{\xi^2(\eta+s)\sin(\xi(\eta+s)) - \xi^2\eta(\sin(\xi\eta))}{s} = \lim_{s\to 0}\xi^2(\sin(\xi(\eta+s)) + (\eta+s)\cos(\xi(\eta+s))\cdot\xi)$$

$$= \xi^2(\sin(\xi\eta) + \xi\eta\cos(\xi\eta))$$

$$\lim_{s\to 0}\frac{1}{s}\left(\frac{\xi(\xi^2+\eta^2+1) - \xi(\xi^2+(\eta+s)^2+1)}{(\xi^2+(\eta+s)^2+1)(\xi^2+\eta^2+1)}\right) = \lim_{s\to 0}\frac{-\xi(2(\eta+s))}{(\xi^2+\eta^2+1)(2(\eta+s))} = \frac{-2\xi\eta}{(\xi^2+\eta^2+1)2\eta} = \frac{\xi}{\xi^2+\eta^2+1}$$

$$\Rightarrow \frac{\partial}{\partial x_2} f\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi^2(\sin(\xi\eta) + \xi\eta\cos(\xi\eta)) \\ \frac{\xi}{\xi^2+\eta^2+1} \end{pmatrix}$$

ges: Matrixdarstellung von $df(x)$

$$\begin{pmatrix} \xi\eta(2\sin(\xi\eta) + \xi\eta\cos(\xi\eta)) & \xi^2(\sin(\xi\eta) + \xi\eta\cos(\xi\eta)) \\ -\frac{\xi^2+\eta^2+1}{(\xi^2+\eta^2+1)^2} & \frac{\xi}{\xi^2+\eta^2+1} \end{pmatrix}$$

ges: $\frac{\partial f}{\partial v}(x)$ für $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\frac{\partial f}{\partial v}(x) = \begin{pmatrix} \xi\eta(2\sin(\xi\eta) + \xi\eta\cos(\xi\eta)) + \xi^2(\sin(\xi\eta) + \xi\eta\cos(\xi\eta)) \\ -\frac{\xi^2+\eta^2+1}{(\xi^2+\eta^2+1)^2} + \frac{\xi}{\xi^2+\eta^2+1} \end{pmatrix}$$

für $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} \xi\eta(2\sin(\xi\eta) + \xi\eta\cos(\xi\eta)) - \xi^2(\sin(\xi\eta) + \xi\eta\cos(\xi\eta)) \\ -\frac{\xi^2+\eta^2+1}{(\xi^3+\eta^2+1)^2} - \frac{\xi^2}{\xi^2+\eta^2+1} \end{pmatrix}$$

ANA Ü8

10. $h: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ $\quad h(t) = (\cos(t))^{\sin(t)}$

ges: h'

$(\cos(t))^{\sin(t)} = \exp(\sin(t) \cdot \ln(\cos(t)))$

$(\exp(\sin(t) \cdot \ln(\cos(t))))' = \exp(\sin(t) \cdot \ln(\cos(t))) \cdot (\cos(t) \cdot \ln(\cos(t)) + \sin(t) \cdot \frac{1}{\cos(t)} \cdot (-\sin(t)))$

$= (\cos(t))^{\sin(t)} \cdot (\cos(t) \cdot \ln(\cos(t)) - \tan(t)\sin(t))$

$f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to (0,1)\times(-1,1)$ $\quad f(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

$g: (0,1)\times(-1,1) \to \mathbb{R}$ $\quad g\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi^{\eta}$ $\quad \Rightarrow h = g \circ f$

ges: $(g \circ f)'$

$df(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$

$dg\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{dg}{d\xi}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \\ \frac{dg}{d\eta}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \eta \cdot \xi^{\eta-1} \\ \xi^{\eta} \cdot \ln(\xi) \end{pmatrix}$

$d(g \circ f)(t) = dg(f(t)) \cdot df(t) = \begin{pmatrix} \sin(t) \cdot \cos(t)^{\sin(t)-1} \\ \cos(t)^{\sin(t)} \ln(\cos(t)) \end{pmatrix}^T \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$

$= -(\sin(t))^2 \cos(t)^{\sin(t)-1} + \cos(t)^{\sin(t)+1} \ln(\cos(t))$

$= (\cos(t))^{\sin(t)} \cdot (\cos(t) \cdot \ln(\cos(t)) - \sin(t) \cdot \tan(t))$

ANA Ü8

8.) $I: \mathbb{R} \to \mathbb{R}$ ges: $I'(\alpha)$

$$\alpha \mapsto \int_{-\exp(\alpha)}^{\alpha^2} \cos(\alpha t^2)\,dt$$

$f: \mathbb{R} \to \mathbb{R}^3$ $g: \mathbb{R}^3 \to \mathbb{R}$

$$\alpha \mapsto \begin{pmatrix} -\exp(\alpha) \\ \alpha^2 \\ \alpha \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \int_x^y \cos(z t^2)\,dt$$

$\Rightarrow I(\alpha) = (g \circ f)(\alpha) \quad \forall \alpha \in \mathbb{R}$

$$df(\alpha) = \begin{pmatrix} -\exp(\alpha) \\ 2\alpha \\ 1 \end{pmatrix} \qquad dg\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{dg}{dx}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \frac{dg}{dy}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \frac{dg}{dz}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -\cos(z x^2) \\ \cos(z y^2) \\ -\int_x^y t^2 \sin(z t^2)\,dt \end{pmatrix}$$

$$\frac{d}{dx} \int_x^y \cos(zt^2)\,dt = \frac{d}{dx} \int_x^y h(t)\,dt = \frac{d}{dx}\left(H(y) - H(x)\right) = \frac{d}{dx} H(x) = -h(x)$$

$$\frac{d}{dy} \int_x^y \cos(zt^2)\,dt = \frac{d}{dy} \int_x^y h(t)\,dt = \frac{d}{dy}\left(H(y) - H(x)\right) = h(y)$$

$$\frac{d}{dz} \int_x^y \cos(zt^2)\,dt = \int_x^y \frac{d}{dz} \cos(zt^2)\,dt = \int_x^y -\sin(zt^2) \cdot t^2 dt\,dt = -\int_x^y t^2 \sin(zt^2)\,dt$$

$$I'(\alpha) = (g \circ f)'(\alpha) = dg(f(\alpha))\,df(\alpha)$$

$$= \begin{pmatrix} -\cos(\alpha(-\exp(\alpha))^2) \\ \cos(\alpha(\alpha^2)^2) \\ -\int_{-\exp(\alpha)}^{\alpha^2} t^2 \sin(z t^2)\,dt \end{pmatrix}^T \cdot \begin{pmatrix} -\exp(\alpha) \\ 2\alpha \\ 1 \end{pmatrix} = \begin{pmatrix} -\cos(\alpha \cdot \exp(2\alpha)) \\ \cos(\alpha^5) \\ -\int_{-\exp(\alpha)}^{\alpha^2} t^2 \sin(z t^2)\,dt \end{pmatrix}^T \cdot \begin{pmatrix} -\exp(\alpha) \\ 2\alpha \\ 1 \end{pmatrix}$$

$$= \exp(\alpha)\cos(\alpha \cdot \exp(2\alpha)) + 2\alpha\cos(\alpha^5) - \int_{-\exp(\alpha)}^{\alpha^2} t^2 \sin(z t^2)\,dt$$

ANA Ü8

7.) $p \in \mathbb{N}, p \geq 2 \qquad f: \mathbb{R}^p \setminus \{0\} \to \mathbb{R}$

Für $p=2$ $\quad f(x) = \ln \|x\|_2 \quad$ sonst $f(x) = \frac{1}{(2-p)\|x\|_2^{p-2}}$

zz: $\operatorname{grad} f(x) (:= (df(x))^T) = \frac{1}{\|x\|_2^p} x$

$p=2$: $\quad df(x) = \left(\frac{\partial f}{\partial x_j}(x)\right)_{j=1,2} = \left(\frac{\partial f}{\partial x_1}(x) \quad \frac{\partial f}{\partial x_2}(x)\right)$

$$\frac{\partial f}{\partial x_2}\binom{x}{y} = \frac{d}{dy} \ln\left(\sqrt{x^2+y^2}\right) = \frac{d}{dy} \frac{1}{2} \ln(x^2+y^2) = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial x_1}\binom{x}{y} = \frac{d}{dx} \ln\left(\sqrt{x^2+y^2}\right) = \frac{d}{dx} \frac{1}{2} \ln(x^2+y^2) = \frac{1}{2} \frac{1}{x^2+y^2} \cdot 2x = \frac{x}{x^2+y^2}$$

$$\Rightarrow df\binom{x}{y} = \left(\frac{x}{x^2+y^2} \quad \frac{y}{x^2+y^2}\right)$$

$$\frac{1}{\|\binom{x}{y}\|_2^2} \binom{x}{y} = \frac{1}{\sqrt{x^2+y^2}^2} \binom{x}{y} = \frac{1}{x^2+y^2} \binom{x}{y} = \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix} = (df(x))^T$$

$p>2$: $\quad df(x) = \left(\frac{\partial f}{\partial x_1}(x) \quad \dots \quad \frac{\partial f}{\partial x_p}(x)\right)$

$$\frac{\partial f}{\partial x_i}(x) = \frac{d}{dx_i} \frac{1}{(2-p)\|x\|_2^{p-2}} = \frac{d}{dx_i} \frac{1}{(2-p)\sqrt{x_1^2+\dots+x_p^2}^{(p-2)}} = \frac{1}{2-p} \frac{d}{dx_i} \frac{1}{(x_1^2+\dots+x_p^2)^{\frac{p}{2}-1}}$$

$$= \frac{1}{2-p} \cdot -\left(\frac{p-2}{2}\right) \cdot \frac{1}{(x_1^2+\dots+x_p^2)^{\frac{p}{2}}} \cdot 2x_i = x_i \frac{1}{\sqrt{x_1^2+\dots+x_p^2}^p} = x_i \cdot \frac{1}{\|x\|_2^p}$$

$$df(x) = \left(x_1 \cdot \frac{1}{\|x\|_2^p} \quad \dots \quad x_p \cdot \frac{1}{\|x\|_2^p}\right)$$

$$\frac{1}{\|x\|_2^p} x = \begin{pmatrix} \frac{x_1}{\|x\|_2^p} \\ \vdots \\ \frac{x_p}{\|x\|_2^p} \end{pmatrix} = df(x)^T$$

ANA Ü8.

9.) $g: \mathbb{R}^2 \to \mathbb{R}$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{cases} \cos\left(\frac{\xi}{\eta}\right) \frac{\eta^3}{\xi^2+\eta^2} & \text{, falls } \eta \neq 0 \\ 0 & \text{, sonst} \end{cases}$$

ges: partielle Ableitungen an $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\frac{\partial g}{\partial x_1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lim_{s\to 0} \frac{1}{s}\left(g\begin{pmatrix} s \\ 0 \end{pmatrix} - g\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \lim_{s\to 0} \frac{1}{s} \cdot 0 = 0$$

$$\frac{\partial g}{\partial x_2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lim_{s\to 0} \frac{1}{s}\left(g\begin{pmatrix} 0 \\ s \end{pmatrix} - g\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \lim_{s\to 0} \frac{1}{s} \cos\left(\frac{0}{s}\right) \frac{s^3}{0+s^2} = \lim_{s\to 0} \cos(0) = 1$$

ges: Richtungsableitung bei $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in Richtung $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\frac{\partial g}{\partial v}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lim_{s\to 0} \frac{1}{s}\left(g\begin{pmatrix} 2s \\ s \end{pmatrix} - g\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \lim_{s\to 0} \frac{1}{s} \cos\left(\frac{2s}{s}\right) \frac{s^3}{(2s)^2+s^2}$$

$$= \lim_{s\to 0} \cos(2) \frac{s^2}{s^2(4+1)} = \frac{\cos(2)}{5}$$

Ist g auf $\mathbb{R}^2$ stetig partiell differenzierbar? $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\sum_{i=1}^{2} v_i \frac{\partial g}{\partial x_i} = 2 \cdot 0 + 1 \cdot 1 = 1 \neq \frac{\cos(2)}{5}$$ aus der Kontraposition von

Lemma 10.1.6 folgt g ist nicht stetig partiell differenzierbar.

ANA Ü8

5.) $f\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{cases} \frac{\xi\eta}{\xi^2+\eta^2} & \text{falls } \begin{pmatrix}\xi\\ \eta\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} \\ 0 & \text{sonst} \end{cases}$

ges: partielle Ableitungen bei $\begin{pmatrix}0\\0\end{pmatrix}$

$$\frac{\partial}{\partial x_1} f\begin{pmatrix}0\\0\end{pmatrix} = \lim_{s\to 0} \frac{1}{s}\left(f\begin{pmatrix}s\\0\end{pmatrix} - f\begin{pmatrix}0\\0\end{pmatrix}\right) = \lim_{s\to 0} \frac{1}{s}\,\frac{s\cdot 0}{s^2+0^2} = \lim_{s\to 0} 0 = 0$$

$$\frac{\partial}{\partial x_2} f\begin{pmatrix}0\\0\end{pmatrix} = \lim_{s\to 0} \frac{1}{s}\left(f\begin{pmatrix}0\\s\end{pmatrix} - f\begin{pmatrix}0\\0\end{pmatrix}\right) = \lim_{s\to 0} \frac{1}{s}\,\frac{0\cdot s}{0^2+s^2} = 0$$

zz: f ist bei $\begin{pmatrix}0\\0\end{pmatrix}$ nicht differenzierbar

$$f\begin{pmatrix}\frac{1}{n}\\ \frac{1}{n}\end{pmatrix} = \frac{\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{\frac{1}{n^2}}{2\frac{1}{n^2}} = \frac{1}{2} \qquad \lim_{n\to\infty} f\begin{pmatrix}\frac{1}{n}\\ \frac{1}{n}\end{pmatrix} = \frac{1}{2} \text{ aber } f\begin{pmatrix}0\\0\end{pmatrix} = 0$$

$\Rightarrow f$ ist nicht stetig bei $\begin{pmatrix}0\\0\end{pmatrix}$ $\Rightarrow f$ ist nicht differenzierbar bei $\begin{pmatrix}0\\0\end{pmatrix}$

ges: partielle Ableitungen zweiter Ordnung an $\begin{pmatrix}x\\y\end{pmatrix}$ mit $\begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$

- $\frac{\partial}{\partial x_1} f\begin{pmatrix}x\\y\end{pmatrix} = \frac{d}{d\xi} f\begin{pmatrix}x\\y\end{pmatrix} = \frac{\eta\cdot(\xi^2+\eta^2) - \xi\eta 2\xi}{(\xi^2+\eta^2)^2}\begin{pmatrix}x\\y\end{pmatrix} = \frac{\xi^2\eta+\eta^3-2\xi^2\eta}{(\xi^2+\eta^2)^2}\begin{pmatrix}x\\y\end{pmatrix} = \frac{y^3-x^2y}{(x^2+y^2)^2}$
- $\frac{\partial}{\partial x_2} f\begin{pmatrix}x\\y\end{pmatrix} = \frac{d}{d\eta} f\begin{pmatrix}x\\y\end{pmatrix} = \frac{\xi\cdot(\xi^2+\eta^2) - \xi\eta\cdot 2\eta}{(\xi^2+\eta^2)^2}\begin{pmatrix}x\\y\end{pmatrix} = \frac{\xi^3+\xi\eta^2-2\xi\eta^2}{(\xi^2+\eta^2)^2}\begin{pmatrix}x\\y\end{pmatrix} = \frac{x^3-xy^2}{(x^2+y^2)^2}$

- $\frac{\partial}{\partial x_1} \frac{y^3-x^2y}{(x^2+y^2)^2} = \frac{d}{dx} \frac{y^3-x^2y}{(x^2+y^2)^2} = \frac{-2yx(x^2+y^2)^2 - (y^3-x^2y)2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$

$= \frac{-2xy(x^2+y^2)^2 - 4x(y^3-x^2y)(x^2+y^2)}{(x^2+y^2)^4}$

- $\frac{\partial}{\partial x_2} \frac{y^3-x^2y}{(x^2+y^2)^2} = \frac{d}{dy} \frac{y^3-x^2y}{(x^2+y^2)^2} = \frac{(3y^2-x^2)(x^2+y^2)^2 - (y^3-x^2y)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$

$= \frac{(3y^2-x^2)(x^2+y^2)^2 - 4y(y^3-x^2y)(x^2+y^2)}{(x^2+y^2)^4} = \frac{3x^2y^2+3y^4-x^4-x^2y^2-4y^3+4x^2y^2}{(x^2+y^2)^3}$ $*$

- $\frac{\partial}{\partial x_2} \frac{x^3-xy^2}{(x^2+y^2)^2} = \frac{2xy(x^2+y^2)^2 - (x^3-xy^2)(2(x^2+y^2)\cdot 2y)}{(x^2+y^2)^2} = \frac{2xy(x^2+y^2)^2 - 4y(x^3-xy^2)(x^2+y^2)}{(x^2+y^2)^2}$
- $\frac{\partial}{\partial x_1} \frac{x^3-xy^2}{(x^2+y^2)^2} = \frac{(3x^2-y^2)(x^2+y^2)^2 - (x^3-xy^2)(2(x^2+y^2)2x)}{(x^2+y^2)^4} = \frac{(3x^2-y^2)(x^2+y^2) - 4x(x^3-xy^2)}{(x^2+y^2)^3}$

$= \frac{3x^4+3x^2y^2-x^2y^2-y^4-4x^4+4x^2y^2}{(x^2+y^2)^3} = \frac{-x^4-y^4+6x^2y^2}{(x^2+y^2)^3}$

$* = \frac{-x^4-y^4+6x^2y^2}{(x^2+y^2)^3}$ wie zu erwarten: $\frac{\partial^2}{\partial x_1 \partial x_2} f = \frac{\partial^2}{\partial x_2 \partial x_1} f$