

Computer Aided Geometric Design Compendium WS2023

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October 13, 2023

Orginazation

Lecture each Thursday 12:00 to 14:00 (full 2 hours).

Oral exam. Write email to fix date and time.

Problem session each Thursday 14:00 to 16:00. Mandatory attendance!

Kreuzerübung.

1 Bezier curves

Example 1. Linear combination $\lambda a + \mu b$

TODO image

Affine combination $\lambda a + \mu b$ and $\lambda + \mu = 1$

TODO image

What is μ so that $\lambda a + \mu b$ is on the line?

$$\lambda a + \mu b = a + t(b - a) \implies a(\underbrace{\lambda - 1 + t}_{=0}) + b(\underbrace{\mu - t}_{=0}) = 0$$

If a, b are linearly independent $\implies \mu = t \wedge \lambda + \mu = 1$

Convex combination $\lambda a + \mu b$ and $\lambda + \mu = 1$ and $\lambda, \mu \geq 0$

TODO image

Line is $a + t(b - a)$ with $t \in [0, 1] \implies \mu, \lambda \in [0, 1]$

Definition 1 (combinations). *linear combination* $\sum_{i=1}^n \lambda_i v_i$ with $v_1, \dots, v_n \in \mathbb{R}^d, \lambda_1, \dots, \lambda_n \in \mathbb{R}$
affine combination $\sum_{i=1}^n \lambda_i v_i$ with $\sum_{i=1}^n \lambda_i = 1$
convex combination $\sum_{i=1}^n \lambda_i v_i$ with $\sum_{i=1}^n \lambda_i = 1$ and $\forall i : \lambda_i \geq 0$

Algorithm 1 (of de Casteljau, Bezier curve). Given: $b_0, \dots, b_n \in \mathbb{R}^d$ (called control points / Kontrollpunkte), $t \in \mathbb{R}$

Recursion: $b_i^0(t) := b_i$

$b_i^j(t) := (1 - t)b_i^{j-1}(t) + tb_{i+1}^{j-1}(t)$ for $j = 1, \dots, n$ and $i = 0, \dots, n - j$

Result: $b(t) := b_0^n(t)$ (called Bezier curve)

Remark 1. In the algorithm above often we choose $t \in [0, 1]$.

Example 2. *TODO images*

Remark 2. In this course $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

Recap 1. $0! := 1, n! := n(n-1)(n-2) \cdots 1$ for $n \geq 1$.

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!} & , n \geq k \geq 0 \\ 0 & , k > n \end{cases} \text{ for } n, k \in \mathbb{N}_0$$

Definition 2 (Bernstein polynomials). For $n, i \in \mathbb{N}_0$ we define $B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i} \in \mathbb{R}[t]$

Remark 3. Special cases of Bernstein polynomials

$$\begin{aligned} i > n &\implies B_i^n(t) = 0 & B_i^n(0) &= \begin{cases} 0, i \neq 0 \\ 1, i = 0 \end{cases} \\ B_i^n(1) &= \begin{cases} 0, i \neq n \\ 1, i = n \end{cases} & B_0^n(t) &= 1 \end{aligned}$$

Theorem 1. $b_i^j(t) = \sum_{l=0}^j B_l^j(t) b_{i+l}$

Proof. Induction over j : $j = 0$:

$$j = 0 : \quad b_i^0(t) := b_i = 1 \cdot b_i = B_0^0(t) \cdot b_i \quad \checkmark$$

$$j - 1 \rightarrow j : \quad b_i^j(t) := (1-t)b_i^{j-1}(t) + t b_{i+1}^{j-1}(t) \stackrel{IA}{=} (1-t) \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+l} + t \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+1+l} =$$

$$(1-t) \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+l} + t \sum_{l=1}^j B_{l-1}^{j-1}(t) b_{i+l} = \sum_{l=0}^j \underbrace{\left((1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t) \right)}_{= B_l^j(t) \text{ using the following lemma}} b_{i+l} = \sum_{l=0}^j B_l^j(t) b_{i+l} \quad \checkmark$$

□

Corollary 1. The Bezier curve equals $b(t) = b_0^n(t) = \sum_{l=0}^n B_l^n(t) b_{i+l}$, which is called the Bernstein representation of the Bezier curve.

Remark 4. As $b(t) = \sum_{l=0}^n B_l^n(t) b_l \in C^\infty$ it is a polynomial curve of degree n , which is in C^∞ and therefore "very smooth".

Lemma 1. $B_l^j(t) = (1-t)B_l^{j-1}(t) + tB_{l-1}^{j-1}(t)$

Proof.

$$(1-t)B_l^{j-1}(t) + tB_{l-1}^{j-1}(t) = (1-t) \binom{j-1}{l} t^l (1-t)^{j-1-l} + t \binom{j-1}{l-1} t^{l-1} (1-t)^{j-1-l+1} =$$

$$\binom{j-1}{l} t^l (1-t)^{j-l} + \binom{j-1}{l-1} t^l (1-t)^{j-l} = \left(\binom{j-1}{l} + \binom{j-1}{l-1} \right) t^l (1-t)^{j-l} = \binom{j}{l} t^l (1-t)^{j-l} = B_l^j(t)$$

□

Remark 5. What is $b(0)$? $b(0) = \sum_{i=0}^n B_i^n(0) b_i = b_0 + 0 + 0 + \dots + 0 = b_0$
What is $b(1)$? $b(1) = \sum_{i=0}^n B_i^n(1) b_i = 0 + \dots + 0 + b_n = b_n$

Definition 3 (end-point-interpolating). Curves which pass through the first and last point are called end-point-interpolating (Endpunktinterpolierend).

Remark 6. Bezier curves are end-point-interpolating.

Remark 7. How many intersection points are there between a planar (i.e. in \mathbb{R}^2) Bezier curve and a straight line?

$$\text{Straight line: } p + t(q - p) \quad \text{Bezier curve: } b(t) = \sum_{i=0}^n B_i^n(t) \underbrace{b_i}_{\in \mathbb{R}^2}$$

Solving $p + t(q - p) = \sum_{i=0}^n B_i^n(t) b_i$ results in at most n solutions.

Lemma 2. $\frac{d}{dt} B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$

Proof.

$$\frac{d}{dt} B_i^n(t) = \frac{d}{dt} \binom{n}{i} t^i (1-t)^{n-i} = \binom{n}{i} i t^{i-1} (1-t)^{n-i} - \binom{n}{i} t^i (n-i) (1-t)^{n-i-1} =$$

$$\frac{n!}{(i-1)! (n-i)!} t^{i-1} (1-t)^{n-i} - \frac{n!}{i! (n-i-1)!} t^i (1-t)^{n-i-1} =$$

$$n \left(\frac{(n-1)!}{(i-1)! (n-i)!} t^{i-1} (1-t)^{n-i} - \frac{(n-1)!}{i! (n-i-1)!} t^i (1-t)^{n-i-1} \right) =$$

$$n \left(\binom{n-1}{i-1} t^{i-1} (1-t)^{n-i} - \binom{n-1}{i} t^i (1-t)^{n-i-1} \right) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$$

□

Theorem 2. $\dot{b}(t) := \frac{d}{dt}b(t) = n \sum_{i=0}^{n-1} B_i^{n-1}(t)(b_{i+1} - b_i) = n(b_1^{n-1}(t) - b_0^{n-1}(t))$

Proof.

$$\begin{aligned} \dot{b}(t) &= \frac{d}{dt} \left(\sum_{i=0}^n B_i^n(t) b_i \right) = \sum_{i=0}^n \frac{d}{dt} B_i^n(t) b_i = \sum_{i=0}^n n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)) b_i = n \left(\sum_{i=0}^n B_{i-1}^{n-1}(t) b_i - \sum_{i=0}^n B_i^{n-1}(t) b_i \right) = \\ &= n \left(\sum_{i=1}^n B_{i-1}^{n-1}(t) b_i - \sum_{i=0}^n B_i^{n-1}(t) b_i \right) = n \left(\underbrace{\sum_{i=0}^{n-1} B_i^{n-1}(t) b_{i+1}}_{=b_1^{n-1}(t)} - \underbrace{\sum_{i=0}^{n-1} B_i^{n-1}(t) b_i}_{=b_0^{n-1}(t)} \right) = n \left(\sum_{i=0}^{n-1} B_i^{n-1}(t) (b_{i+1} - b_i) \right) \end{aligned}$$

□

Corollary 2. • $\dot{b}(0) = n(b_1 - b_0)$

- $\dot{b}(1) = n(b_n - b_{n-1})$
- The last segment in the algorithm of de Casteljou is the tangent of the Bezier curve in $b(t)$.
- The derivative of a bezier curve of degree n is a bezier curve of degree $n-1$ with control points $(b_1, b_0), (b_2 - b_1), \dots, (b_n - b_{n-1})$.

Corollary 3. $\ddot{b}(t) = n(n-1) \sum_{i=0}^{n-2} B_i^{n-2}(t)(b_{i+2} - 2b_{i+1} + b_i)$
 $\ddot{b}(0) = n(n-1)(b_2 - 2b_1 + b_0)$, $\ddot{b}(1) = n(n-1)(b_n - 2b_{n-1} + b_{n-2})$

Corollary 4. The curvature of a bezier curve in the point $b(0)$ depends only on b_0, b_1, b_2 .
The curvature of a bezier curve in the point $b(1)$ depends only on b_{n-2}, b_{n-1}, b_n .

Example 3. Quadratic Bezier curve

$$b(t) = \sum_{i=0}^2 B_i^2(t) b_i = \binom{2}{0} t^0 (1-t)^2 b_0 + \binom{2}{1} t^1 (1-t)^1 b_1 + \binom{2}{2} t^2 (1-t)^0 b_2 = t^2 (b_2 - 2b_1 + b_0) + t(2b_1 - 2b_0) + b_0$$

which is an affine transformation of a parabel and therefore a parabel.

Quadratic bezier curves are parabolas.

Remark 8. Line at infinity (Ferngerade) is the collection of points where parallel lines intersect.
TODO fig

Remark 9. Different applications using these curves are Rhino, OpenSCAD, Autocad, Geogebra, ...

Definition 4. $c : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^3$ is called a parameterized curve.

TODO figure

$\dot{c}(t) := \frac{d}{dt}c(t)$ is called the tangential vector. For \mathbb{R}^3 we have $\dot{c}(t) = (\dot{c}_1(t), \dot{c}_2(t), \dot{c}_3(t))$.

The velocity is defined as $\|\dot{c}(t)\|$.

A point $c(t)$ is called regular, if $\dot{c}(t) \neq 0$ and is called singular, if $\dot{c}(t) = 0$.

Example 4. A helix (Schraublinie) is defined by $c(t) = (\cos(t), \sin(t), t)^T$.

TODO figure

$$\dot{c}(t) = (-\sin(t), \cos(t), 1)^T \quad \|\dot{c}(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

We see that the helix is passed through with constant velocity. Furthermore all points are regular.

Example 5. $c : \mathbb{R} \rightarrow \mathbb{R}^3, t \mapsto (t^2, t^3, t^4)$, $\dot{c}(t) = (2t, 3t^2, 4t^3)$. We see that 0 is singular as $\dot{c}(0) = (0, 0, 0)$. Everywhere else the curve is regular.

Remark 10. A point being regular or singular depends on the parameterization of the curve.

For example $c(t) = (t, t)$ produces a regular curve, while $c(t) = (t^3, t^3)$ produces a curve where 0 is singular.

TODO figure

There are curves and points where no parameterization exists such that the point is regular.

Definition 5. $c : I \rightarrow \mathbb{R}^2 \in C^2(I, \mathbb{R}^2)$

The curvature of the curve in the point $c(t)$ is defined as $\kappa(t) = \frac{\det(\dot{c}(t), \ddot{c}(t))}{\|\dot{c}(t)\|^3}$

Example 6. TODO figure

The circle of curvature has a radius of $\frac{1}{\kappa(t)}$. $m(t)$ is called the center of curvature.

$m(t) = c(t) + \frac{1}{\kappa(t)}n(t)$ where $n(t) = \frac{(-\dot{c}_2(t), \dot{c}_1(t))}{\|\dot{c}(t)\|}$.

Remark 11. Exercise: compare this definition of curvature with the school version concerning graphs.

Example 7. For a circle we have $c(t) = (r \cos(t), r \sin(t))^T$, $\dot{c}(t) = (-r \sin(t), r \cos(t))^T$, $\ddot{c}(t) = (-r \cos(t), -r \sin(t))^T$

$$\kappa(t) = \frac{\det \begin{pmatrix} -r \sin(t) & -r \cos(t) \\ r \cos(t) & -r \sin(t) \end{pmatrix}}{r^3} = \frac{r^2 \sin^2(t) + r^2 \cos^2(t)}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$n(t) = \frac{(-r \cos(t), -r \sin(t))}{r} = (-\cos(t), -\sin(t))$$

TODO fig

Definition 6. A point $c(t)$ with $\kappa(t) = 0$ is called a vertex.

Example 8. An ellipse has four vertices. TODO fig

$(t, \exp t)$ has no vertex.

Klothoids are curves with $\kappa(t) = t$. They are used in road construction and have no vertex.

Definition 7. $c : I \rightarrow \mathbb{R}^3$

$\kappa(t) = \frac{\|\dot{c}(t) \times \ddot{c}(t)\|}{\|\dot{c}(t)\|^3}$ is called the curvature of a space curve.

$\tau(t) = \frac{\det(\dot{c}(t), \ddot{c}(t), \ddot{c}'(t))}{\|\dot{c}(t) \times \ddot{c}(t)\|^2}$ is called torsion of a space curve.

Example 9. For the helix $t \mapsto (\cos(t), \sin(t), pt)$ the torsion depends on p .