ANA Ü6

1) $x \in \mathbb{R}$ fest $\quad f_x(t) := e^{ix\sin(t)}$ $\quad$ Fourierreihe von f_x: $f_x(t) = \sum_{n\in\mathbb{Z}} c_n(x) e^{int}$

Bessel-Funktion (erster Gattung) $J_n(x) := c_n(x)$

Berechnen Sie für alle $x \in \mathbb{R}$ $\quad \sum_{n\in\mathbb{Z}} |J_n(x)|^2$

$f_x(t) \in L^2(\mathbb{T})$, da $\int_{\mathbb{T}} |f_x(t)|^2 \, d\lambda^2(t) = \int_{-\pi}^{\pi} |e^{ix\sin(t)}|^2 \, d\lambda^2(t) = \int_{-\pi}^{\pi} 1 \, d\lambda = 2\pi = \|f_x\|_2^2$

$\sum_{n\in\mathbb{Z}} |J_n(x)|^2 = \sum_{n\in\mathbb{Z}} \left| \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f_x(t) e^{-int} \, dt \right|^2 = \sum_{n\in\mathbb{Z}} \frac{1}{2\pi} |\langle f_x, e^{-int} \rangle|^2 =$ nach Parceval'sche Gleichung

$= \frac{1}{2\pi} \|f_x\|_2^2 = \frac{2\pi}{2\pi} = 1$

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2) $f \in L^1(\mathbb{R})$. $I_1(t) := \int_{\mathbb{R}} f(x)\sin(tx)\,dx$ $\quad I_2(t) := \int_{\mathbb{R}} f(x)\cos(tx)\,dx$

Drücken Sie I_1, I_2 durch Fouriertransformierte aus!

$$I_2(t) - i\,I_1(t) = \int_{\mathbb{R}} f(x)\cos(tx)\,dx - i\int_{\mathbb{R}} f(x)\sin(tx)\,dx = \int_{\mathbb{R}} f(x)(\cos(tx) - i\sin(tx))\,dx$$

$$= \int_{\mathbb{R}} f(x)\,e^{-itx}\,dx = \sqrt{2\pi}\,\hat{f}(t)$$

$f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ $\quad$ zz: $2\pi \int_{\mathbb{R}} |f(x)|^2\,dx = \int_{\mathbb{R}} |I_1(t)|^2\,dt + \int_{\mathbb{R}} |I_2(t)|^2\,dt$

Nach Prop 3.3.3. gilt $\mathcal{F}f = \hat{f}$, da $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

$$2\pi \int_{\mathbb{R}} |f(x)|^2\,dx = 2\pi \|f\|_2^2 \overset{\text{Seite 77}}{=} 2\pi \|\hat{f}\|_2^2 = 2\pi \int_{\mathbb{R}} |\hat{f}(t)|^2\,dt = \int_{\mathbb{R}} |\sqrt{2\pi}\,\hat{f}(t)|^2\,dt$$

$$= \int_{\mathbb{R}} |\underbrace{I_2(t)}_{\in\mathbb{R}} - i\,\underbrace{I_1(t)}_{\in\mathbb{R}}|^2\,dt = \int_{\mathbb{R}} |I_2(t)|^2 + |I_1(t)|^2\,dt = \int_{\mathbb{R}} |I_2(t)|^2\,dt + \int_{\mathbb{R}} |I_1(t)|^2\,dt$$

$\square$

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3) $f(x) := x\exp(-\frac{x^2}{2})$ ges: Fouriertransformierte von f zz: $\hat{f}(\xi) = -i f(\xi)$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,d\lambda(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\, e^{-i\xi x} e^{-\frac{x^2}{2}}\,d\lambda(x)$$

$$\overset{*}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-e^{-\frac{x^2}{2}})' e^{-i\xi x}\,dx = \frac{1}{\sqrt{2\pi}}\left(-e^{-\frac{x^2}{2}} e^{-i\xi x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} i\xi e^{-i\xi x}\,dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(0 - i\xi\int_{-\infty}^{\infty} e^{-i\xi x} e^{-\frac{x^2}{2}}\,dx\right) = -\frac{i\xi}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\cos(-\xi x) + i\sin(-\xi x))e^{-\frac{x^2}{2}}\,dx$$

$$= -\frac{i\xi}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\cos(\xi x)e^{-\frac{x^2}{2}}\,dx - \underbrace{i\int_{-\infty}^{\infty}\sin(\xi x)e^{-\frac{x^2}{2}}\,dx}_{=0}\right)$$

$$= -\frac{i\xi}{\sqrt{2\pi}}\sqrt{2\pi}\,e^{-\frac{\xi^2}{2}} = \qquad \sin(-\xi x)e^{-\frac{(-x)^2}{2}} = -\sin(\xi x)e^{-\frac{x^2}{2}} \Rightarrow \text{ungerade}$$

$$= -i\xi e^{-\frac{\xi^2}{2}} = -i f(\xi)$$

$\square$

$$* \int_{-\infty}^{\infty}|x e^{-i\xi x} e^{-\frac{x^2}{2}}|\,dx = \int_{-\infty}^{\infty}|x|\,e^{-\frac{x^2}{2}}\,dx = 2\int_{0}^{\infty} x\, e^{-\frac{x^2}{2}}\,dx = -2e^{-\frac{x^2}{2}}\Big|_0^{\infty} = 0+2 = 2 < \infty$$

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4) Hermite-Polynome $P_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$

(i) zz: $P_0(x) \equiv 1$, $P_{n+1}(x) = x\, P_n(x) - P_n'(x)$

$$P_0(x) = (-1)^0 e^{\frac{x^2}{2}} \frac{d^0}{dx^0} e^{-\frac{x^2}{2}} = 1 e^{\frac{x^2}{2}} e^{-\frac{x^2}{2}} = 1 \checkmark$$

$$P_{n+1}(x) = (-1)^{n+1} e^{\frac{x^2}{2}} \frac{d^{n+1}}{dx^{n+1}} e^{-\frac{x^2}{2}} = (-1)(-1)^n e^{\frac{x^2}{2}} \frac{d}{dx} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$$

$$= (-1) e^{\frac{x^2}{2}} \frac{d}{dx} e^{-\frac{x^2}{2}} \underbrace{(-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}} = -e^{\frac{x^2}{2}} \frac{d}{dx} e^{-\frac{x^2}{2}} P_n(x)$$

$$= -e^{\frac{x^2}{2}} \left(e^{-\frac{x^2}{2}} (-x) P_n(x) + e^{-\frac{x^2}{2}} P_n'(x) \right) = x\, P_n(x) - P_n'(x) \checkmark$$

(ii) zz: $\deg P_n = n$

$\deg P_0 = \deg 1 = 0 \checkmark$

$\deg P_{n+1} = \deg x P_n(x) - P_n'(x) = \max\{\underbrace{\deg P_n}_{=n} + 1, \underbrace{\deg P_n'}_{=n-1}\} \overset{IV}{=} n+1 \checkmark$

(iii) ges: Fouriertransformierte von $P_n(x) e^{-\frac{x^2}{2}}$ für $n \in \mathbb{N}$

$$P_{n+1}(x) e^{-\frac{x^2}{2}} = x P_n(x) e^{-\frac{x^2}{2}} - P_n'(x) e^{-\frac{x^2}{2}} = -\left(\left(e^{-\frac{x^2}{2}}\right)' P_n(x) + e^{-\frac{x^2}{2}} P_n'(x) \right) = -\left(e^{-\frac{x^2}{2}} P_n(x)\right)'$$

Sei $g_n(x) := P_n(x) e^{-\frac{x^2}{2}} \Rightarrow g_{n+1}(x) = -g_n'(x)$

Prop 3.2.2. Ist $x g_n(x)$ integrierbar $\Rightarrow \widehat{g_n}(x) \in C(\mathbb{R}) \wedge \widehat{g_n'} = ix\, \widehat{g_n}$

$\int x g_n(x) dx =$

$$\Rightarrow \widehat{g_n} = \widehat{-g_{n-1}'} = -\widehat{g_{n-1}'} = -ix\, \widehat{g_{n-1}} = -ix\, (\widehat{-g_{n-2}'}) = (-ix)^2\, \widehat{g_{n-2}} = \dots = (-ix)^k\, \widehat{g_{n-k}}$$

$$\widehat{g_1} = \widehat{P_1(x) e^{-\frac{x^2}{2}}} = \widehat{-e^{\frac{x^2}{2}-\frac{x^2}{2}} \frac{d}{dx} e^{-\frac{x^2}{2}}} = \widehat{+x e^{-\frac{x^2}{2}}} = -ix e^{-\frac{x^2}{2}} = (-ix)^1 e^{-\frac{x^2}{2}}$$

$$\Rightarrow \widehat{g_n} = (-ix)^{n-1}\, \widehat{g_{n-(n-1)}} = (-ix)^{n-1}\, \widehat{g_1} = (-ix)^n e^{-\frac{x^2}{2}}$$

(iv) $e^{-\frac{x^2}{2}} \in \mathcal{S}$ zz: $\mathcal{F} e^{-\frac{x^2}{2}} \in \mathcal{S}$

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}} d\lambda(x) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} < \infty \Rightarrow e^{-\frac{x^2}{2}} \in L^1$$

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right)^2 dx = \int_{-\infty}^{\infty} \exp(-x^2) dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} < \infty \Rightarrow e^{-\frac{x^2}{2}} \in L^2 \quad \Rightarrow e^{-\frac{x^2}{2}} \in L^1 \cap L^2$$

(Nach Prop 3.3.3. gilt nun $\mathcal{F} e^{-\frac{x^2}{2}} = \widehat{e^{-\frac{x^2}{2}}} = \widehat{g_0(x)}$)

Nach Prop 3.3.1. ist die Fouriertransformation ein Automorphismus auf $\mathcal{S}$.

$\Rightarrow \mathcal{F} e^{-\frac{x^2}{2}} \in \mathcal{S}$

ANA Ü6

5) $f(x) := e^{-|x|}$ auf $\mathbb{R}$ ges: Fouriertransformierte von f

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} e^{-|x|}dx = 2\int_0^{\infty} e^{-x}dx = -2\int_0^{-\infty} e^{u}du = 2\int_{-\infty}^{0} e^{u}du \qquad \left[u=-x \quad \frac{du}{dx}=-1\right.$$

$$= 2e^u\Big|_{-\infty}^{0} = 2(1-0) = 2 < \infty \qquad \Rightarrow f \in L^1(\mathbb{R})$$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi x}e^{-|x|}d\lambda(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x-|x|}dx = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} e^{-i\xi x+x}dx + \int_0^{\infty} e^{-i\xi x-x}dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} e^{x(1-i\xi)}dx + \int_0^{\infty} e^{-x(1+i\xi)}dx\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{x(1-i\xi)}}{1-i\xi}\Big|_{-\infty}^{0} + \frac{e^{-x(1+i\xi)}}{-1-i\xi}\Big|_0^{\infty}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-i\xi} - 0 + 0 - \frac{1}{-1-i\xi}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-i\xi} + \frac{1}{1+i\xi}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{1+i\xi+1-i\xi}{1-i^2\xi^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{2}{1+\xi^2} = \frac{2}{\sqrt{2\pi}(1+\xi^2)}$$

$f(x) := e^{-|x|}$ auf $\mathbb{R}^n$ ges: Fouriertransformierte von f wobei $|.|$ die l^1-Norm auf $\mathbb{R}^n$ ist

Induktionsbehauptung: $\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\left(\frac{2}{1+\xi^2}\right)^n$

$$\hat{f}_{n+1}(\xi) = \frac{1}{(2\pi)^{\frac{n+1}{2}}}\int_{\mathbb{R}^{n+1}} e^{-|x|}e^{-i\xi x}d\lambda^{n+1}(x) = \frac{1}{(2\pi)^{n/2}}\frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{-|x_{n+1}|-\sum_{j=1}^{n}|x_j|}e^{-i\xi x_{n+1}-i\xi\sum_{j=1}^{n}x_j}d\lambda(x_{n+1})d\lambda^n(x)$$

$$= \frac{1}{(2\pi)^{n/2}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{-|x_{n+1}|}e^{-\sum_{j=1}^{n}|x_j|}e^{-i\xi x_{n+1}}e^{-i\xi\sum_{j=1}^{n}x_j}d\lambda(x_{n+1})d\lambda^n(x)$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-\sum_{j=1}^{n}|x_j|}e^{-i\xi\sum_{j=1}^{n}x_j}d\lambda^n(x)\ \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-|x_{n+1}|}e^{-i\xi x_{n+1}}d\lambda(x_{n+1}) = \frac{1}{(2\pi)^{n/2}}\left(\frac{2}{1+\xi^2}\right)^n\frac{2}{\sqrt{2\pi}(1+\xi^2)}$$

$$= \frac{2^{n+1}}{(2\pi)^{\frac{n+1}{2}}(1+\xi^2)^{n+1}}$$

$\square$

ANA Ü6

6) ges: Lösung der DGL $y(t) - y''(t) = e^{-|t|}$, $t \in \mathbb{R}$

(i) ges: formale Fouriertransformierte beider Seiten

Sei l die linke und r die rechte Seite

$$\hat{l}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t}(y(t) - y''(t))\, d\lambda(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t} y(t)\, d\lambda(t) - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t} y''(t)\, d\lambda(t)$$

$$= \hat{y}(\xi) - \widehat{y''}(\xi)$$

$$\hat{r}(\xi) = (e^{-|t|})^\wedge(\xi)$$

(ii) vereinfachen der linken Seite

$\hat{l}(\xi) = \hat{y}(\xi) - \widehat{y''}(\xi)$ nach Prop 3.2.2. gilt xy, x^2y integrierbar $\Rightarrow \left(\frac{d^2}{dx^2} y\right)^\wedge = i^2 x^2 \hat{y}$

$$\int_{-\infty}^{\infty} x y(x)\, dx < \infty \wedge \int_{-\infty}^{\infty} x^2 y(x)\, dx < \infty \Rightarrow \widehat{y''}(\xi) = -\xi^2 \hat{y}(\xi)$$

$$\Rightarrow \hat{l}(\xi) = \hat{y}(\xi) + \xi^2 \hat{y}(\xi) = (1+\xi^2)\hat{y}(\xi)$$

(iii) verwenden Sie (5) um die DGL zu lösen

Aus (5) folgt $\hat{r}(\xi) = \frac{2}{\sqrt{2\pi}(1+\xi^2)}$

$$\Rightarrow (1-\xi^2)\hat{y}(\xi) = \frac{2}{\sqrt{2\pi}(1+\xi^2)} \Leftrightarrow \hat{y}(\xi) = \frac{2}{\sqrt{2\pi}} \frac{1}{(1+\xi^2)^2} = \frac{2}{\sqrt{2\pi}} \frac{1}{(1+\xi^2)} \frac{1}{1+\xi^2}$$

$$f(t) := \frac{\sqrt{2\pi}}{2} e^{-|t|} \overset{5)}{\Longrightarrow} \hat{f}(t) = \frac{\sqrt{2\pi}}{2} (e^{-|t|})^\wedge = \frac{\sqrt{2\pi}}{2} \frac{2}{\sqrt{2\pi}} \frac{1}{(1+t^2)} = \frac{1}{1+t^2}$$

$$\Rightarrow \hat{y}(\xi) = \frac{2}{\sqrt{2\pi}} \hat{f}(\xi) \hat{f}(\xi) \overset{\text{Satz 3.2.5}}{=} \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} (f*f)^\wedge(\xi) = \frac{2}{2\pi}(f*f)^\wedge(\xi) = \frac{1}{\pi}(f*f)^\wedge(\xi)$$

$$f*f(x) = \int_{\mathbb{R}} \frac{\sqrt{2\pi}}{2} e^{-|x-y|} \frac{\sqrt{2\pi}}{2} e^{-|y|}\, dy = \frac{2\pi}{4} \int_{\mathbb{R}} e^{-|x-y|} e^{-|y|}\, dy \overset{\Delta}{=} \frac{\pi}{2} e^{-|x|}(1+|x|)$$

$$\Rightarrow \hat{y}(\xi) = \frac{1}{\pi}\left(\frac{\pi}{2} e^{-|\xi|}(1+|\xi|)\right)^\wedge = \left(\frac{1}{2} e^{-|\xi|}(1+|\xi|)\right)^\wedge \overset{\wedge \text{ inj.}}{\Longrightarrow} y(t) = \frac{1}{2} e^{-|t|}(1+|t|)$$

Δ $x \geq 0$: $\int_{\mathbb{R}} e^{-|x-y|} e^{-|y|}\, dy = \int_{-\infty}^{0} e^{-x+y} e^{y}\, dy + \int_{0}^{x} e^{-x+y} e^{-y}\, dy + \int_{x}^{\infty} e^{x-y} e^{-y}\, dy$

$$= e^{-x} \int_{-\infty}^{0} e^{2y}\, dy + e^{-x} \int_{0}^{x} e^{0}\, dy + e^{x} \int_{x}^{\infty} e^{-2y}\, dy = e^{-x} \frac{e^{2y}}{2}\Big|_{-\infty}^{0} + e^{-x} y\Big|_{0}^{x} - e^{x} \frac{e^{-2y}}{2}\Big|_{x}^{\infty}$$

$$= e^{-x}\frac{1}{2} - e^{-x} 0 + x e^{-x} - 0 e^{-x} - e^{x} 0 + e^{x} \frac{e^{-2x}}{2} = \frac{1}{2} e^{-x} + x e^{-x} + \frac{1}{2} e^{-x} = e^{-x}(1+x) = e^{-|x|}(1+|x|)$$

$x < 0$: $\int_{\mathbb{R}} e^{-|x-y|} e^{-|y|}\, dy = \int_{\mathbb{R}} e^{-|y-x|} e^{-|y|}\, dy = \int_{\mathbb{R}} e^{-|-y-x|} e^{-|-y|}\, dy = \int_{\mathbb{R}} e^{-|-x-y|} e^{-|y|}\, dy = e^{-(-x)}(1+(-x))$

nach oben, da $-x > 0$

$$= e^{-|x|}(1+|x|)$$