PDGL Ü3

1) (i) $\psi \in L^1(\mathbb{R}^d)$, $d \geq 1$, $\psi \geq 0$ f.ü., $\int_{\mathbb{R}^d} \psi(x)\,dx = 1$

$\psi_\varepsilon(x) := \frac{1}{\varepsilon^d}\, \psi\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}^d$, $\varepsilon > 0$

zz: $\psi_\varepsilon \xrightarrow{\varepsilon \to 0} \delta_0$ in $D'(\mathbb{R}^d)$

Sei $\phi \in D(\mathbb{R}^d)$ bel. zz: $\langle \psi_\varepsilon, \phi \rangle \xrightarrow{\varepsilon \to 0} \langle \delta_0, \phi \rangle = \phi(0)$

$\langle \psi_\varepsilon, \phi \rangle = \int_{\mathbb{R}^d} \frac{1}{\varepsilon^d}\, \psi\left(\frac{x}{\varepsilon}\right) \phi(x)\,dx$ $\qquad \varphi(x) := \frac{x}{\varepsilon} \qquad D\varphi = \begin{pmatrix} \frac{1}{\varepsilon} & 0 & \cdots & 0 \\ 0 & \frac{1}{\varepsilon} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{1}{\varepsilon} \end{pmatrix}$

$= \int_{\mathbb{R}^d} \psi(\varphi(x))\, \phi(\varepsilon \varphi(x))\, |\det(D\varphi)|\,dx$ $\qquad |\det(D\varphi)| = \frac{1}{\varepsilon^d}$

$= \int_{\mathbb{R}^d} \psi(y)\, \phi(\varepsilon y)\,dy$ $\quad K := \operatorname{supp} \phi$ ... kompakt, $|\psi(y)\phi(\varepsilon y)| \leq \sup_{z \in \varepsilon K} \phi(z) \cdot \psi(y) \in L^1(\mathbb{R}^d)$

dominierte Konvergenz

$\lim_{\varepsilon \to 0} \langle \psi_\varepsilon, \phi \rangle = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \psi(y)\, \phi(\varepsilon y)\,dy \overset{\downarrow}{=} \int_{\mathbb{R}^d} \psi(y)\, \phi(0)\,dy = \phi(0) \cdot 1 = \phi(0)$

(ii) ges: $(\varphi_n)_{n \in \mathbb{N}} \in D(\mathbb{R}^d)$, $d \geq 1$ mit konvergent in $C^\infty(\mathbb{R}^d)$, aber nicht in $D(\mathbb{R}^d)$

$\varphi_n \xrightarrow{n \to \infty} \varphi$ in $C^\infty(\mathbb{R}^d) :\Longleftrightarrow \forall K \subseteq \mathbb{R}^d$ ... kompakt $\forall \alpha \in \mathbb{N}_0^d : \sup_{x \in K} |\partial^\alpha \varphi_n(x) - \partial^\alpha \varphi(x)| \xrightarrow{n \to \infty} 0$

$\varphi_n \xrightarrow{n \to \infty} \varphi$ in $D(\mathbb{R}^d) :\Longleftrightarrow \exists K \subseteq \mathbb{R}^d$ ... kompakt $\forall m \in \mathbb{N} : \operatorname{supp}(\varphi_m) \subseteq K \ \wedge \forall \alpha \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} |D^\alpha \varphi_n(x) - D^\alpha \varphi(x)| \xrightarrow{n \to \infty} 0$

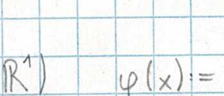
$\varphi_n(x) := \begin{cases} \exp\left(\frac{n^2}{x^2 - n^2}\right) & , |x| < n \\ 0 & , |x| \geq n \end{cases} \in C_c^\infty(\mathbb{R}^1) \qquad \varphi(x) := e^{-1} \in C^\infty(\mathbb{R}^1)$

Sei $K \subseteq \mathbb{R}^1$ ... kompakt bel. Sei $\alpha \in \mathbb{N}_0^1$ bel. 1. Fall $\alpha = 0$

$\sup_{x \in K} |\partial^\alpha \varphi_n(x) - \partial^\alpha \varphi(x)| = \sup_{x \in K} |\mathbb{1}_{[-n,n]}(x) \exp\left(\frac{n^2}{x^2 - n^2}\right) - e^{-1}| = \max\left(e^{-1} - \varphi_n(k_1),\ e^{-1} - \varphi_n(k_2)\right)$

$\xrightarrow{n \to \infty} e^{-1} - e^{-1} = 0$ $\qquad x = 0 : |e^{-1} - e^{-1}| = 0$ $\qquad$ wobei $K = [k_1, k_2]$

$x = \pm n : |0 - e^{-1}| = e^{-1}$



$\Rightarrow \underline{\underline{\varphi_n \xrightarrow{n \to \infty} \varphi \text{ in } C^\infty(\mathbb{R}^d)}}$ $\qquad \lim_{n \to \infty} \varphi_n(x) = \lim_{n \to \infty} e^{\frac{n^2}{x^2 - n^2}} = e^{\frac{2n}{-2n}} = e^{-1}$

de L'Hospital

$\operatorname{supp}(\varphi_m) = [-m, m]$ $\quad$ K müsste also so sein dass $K \supseteq \bigcup_{m \in \mathbb{N}} \operatorname{supp}(\varphi_m) = \bigcup_{m \in \mathbb{N}} [-m, m] = \mathbb{R}$

↯ zu K ist beschränkt (da kompakt)

$\Rightarrow \varphi_n$ kann in $D(\mathbb{R}^d)$ nicht konvergieren. $\square$

PDGL Ü3

2) Distribution? endliche Ordnung?

(i) $\langle u, \varphi \rangle = \sum_{k=0}^{\infty} \varphi^{(k)}(k)$

linear: klar

stetig: $\varphi_m(x) := \begin{cases} \frac{1}{m}\exp\left(\frac{1}{x^2-1}\right), & |x|<1 \\ 0, & |x|\geq 1 \end{cases} \in C_c^\infty$ und $\varphi_m \xrightarrow{m\to\infty} 0$, $\forall k\geq 1: \varphi_m^{(k)}(k)=0$

Sei $K \subseteq \mathbb{R}$ ... kompakt bel. $m=1$

$\mathrm{supp}(u(\varphi_m)) = \mathrm{supp}\left(\sum_{k=0}^{\infty} \varphi_1^{(k)}(k)\right) = \mathrm{supp}\left(\frac{1}{e}\right) = \mathbb{R} \not\subseteq K$

$\Rightarrow$ keine Distribution

(ii) $\langle u, \varphi \rangle = \int_{\mathbb{R}} \frac{\sin(x)}{x} \varphi(x)\, dx$

Theorem 3.4 $f \in L^1_{loc}(\Omega) \Rightarrow \langle f, \phi \rangle = \int_\Omega f \phi \, dx$ ... Distribution

$\int_{\mathbb{R}} \frac{\sin(x)}{x}\, dx = \pi < \infty \Rightarrow \frac{\sin(x)}{x} \in L^1_{loc}(\mathbb{R}) \Rightarrow \langle u, \varphi \rangle$ ... Distribution

Def 3.1. $\phi_m \xrightarrow{m\to\infty} 0 \iff \exists K \subseteq \Omega$ ...kompakt $\forall m \in \mathbb{N}: \mathrm{supp}(\phi_m) \subseteq K$ und

$\forall \alpha \in \mathbb{N}_0: \lim_{m\to\infty} \sup_{x\in\Omega} |D^\alpha \phi_m(x)| = 0$

Def 3.2. Distribution ist $u: D(\Omega) \to \mathbb{R}$ mit linear und stetig (also $\phi_n \to 0 \Rightarrow u(\phi_n) \to 0$)

Lemma 3.3. $u: D(\Omega) \to \mathbb{R}$ ... lin. ist Distribution $\iff \forall K \subseteq \Omega$ ... kompakt $\exists C>0\, \exists k \in \mathbb{N}_0$:

$\forall \varphi \in D(K): |\langle u, \varphi \rangle| \leq C \|\varphi\|_{C^k(K)} = C \sum_{\ell=0}^{k} \sup_{x\in K} |\varphi^{(\ell)}(x)|$

PDGL Ü3

2) ... (iii) $\langle u, \varphi \rangle = \sum_{k=0}^{m} \varphi^{(k)}(x_0)$, $x_0 \in \mathbb{R}$, $m \in \mathbb{N}_0$ fest

lin: ✓

Lemma 3.3. u ... Distribution $\iff \forall K \subseteq \mathbb{R}\ \exists C>0\ \exists n \in \mathbb{N}_0\ \forall \varphi \in D(K): |\langle u, \varphi \rangle| \leq C \sum_{k=0}^{n} \sup_{x \in K} |D^k \varphi(x)|$

In unserem Fall: $\left|\sum_{k=0}^{m} \varphi^{(k)}(x_0)\right| \leq C \sum_{k=0}^{n} \sup_{x\in K} |\varphi^{(k)}(x)|$

Sei $K \subseteq \mathbb{R}$ bel. $C := 1$, $n := m$ Sei $\varphi \in D(K)$ bel. Dann gilt

$\left|\sum_{k=0}^{m} \varphi^{(k)}(x_0)\right| \leq \sum_{k=0}^{m} |\varphi^{(k)}(x_0)| \leq \sum_{k=0}^{m} \sup_{x\in K} |\varphi^{(k)}(x)|$ also ist u eine Distribution.

von endlicher Ordnung? $(\exists \tilde{m} \in \mathbb{N}_0\ \forall K \subseteq \mathbb{R}$ ... kompakt $\exists C(K)>0\ \forall \varphi \in D(\mathbb{R}): |\langle u, \varphi\rangle| \leq C(K) \|\varphi\|_{C^{\tilde{m}}(K)})$

Sei $\tilde{m} \in \mathbb{N}_0$ bel. Wähle $K := [x_0+2, x_0+3]$. Sei $C>0$ bel. Wähle $\varphi(x) := \begin{cases} e^{\frac{1}{(x-x_0)^2-1}}, & x \in [x_0-1, x_0+1] \\ 0, & \text{sonst} \end{cases}$

$|\langle u, \varphi \rangle| = \left|\sum_{k=0}^{m} \varphi^{(k)}(x_0)\right| \underset{m=0}{=} |\varphi(x_0)| = e^{-1}$

$\|\varphi\|_{C^{\tilde{m}}(K)} = \sum_{k=0}^{\tilde{m}} \sup_{x\in K} |\varphi^{(k)}(x)| \underset{K \cap \operatorname{supp}\varphi = \emptyset}{=} 0$ $\Rightarrow |\langle u, \varphi\rangle| > C(K) \|\varphi\|_{C^{\tilde{m}}(K)}$

$\Rightarrow \neg$ endliche Ordnung.

(iv) $\langle u, \varphi \rangle = \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(-x)}{x}\, dx$

MWS

PDGL Ü3

3) (i) $u \in D'(\mathbb{R})$, $f \in C^\infty(\mathbb{R})$ zz: $(fu)' = f'u + fu'$

Sei $\phi \in D(\mathbb{R})$ bel.

$$\langle (f\cdot u)', \phi\rangle = -\langle f\cdot u, \phi'\rangle = -\langle u, f\cdot\phi'\rangle = -\langle u, (f\cdot\phi)' - f'\phi\rangle$$

$\langle D^\alpha u, \phi\rangle = (-1)^{|\alpha|}\langle u, D^\alpha\phi\rangle$

$\langle au, \phi\rangle = \langle u, a\phi\rangle$

$$\underset{\text{lin.}}{=} -\langle u, (f\cdot\phi)'\rangle + \langle u, f'\phi\rangle = \langle u', f\cdot\phi\rangle + \langle u, f'\phi\rangle = \langle fu', \phi\rangle + \langle f'u, \phi\rangle$$

$\langle D^\alpha u, \phi\rangle = (-1)^{|\alpha|}\langle u, D^\alpha\phi\rangle$ $\langle au, \phi\rangle = \langle u, a\phi\rangle$

$$\underset{\text{lin.}}{=} \langle f'u + fu', \phi\rangle \qquad \Rightarrow \quad (f\cdot u)' = f'u + fu'$$

(ii) $u(x) := \mathbb{1}_{[-1,1]}(x) \in D'(\mathbb{R})$ ges: u'

Sei $\phi \in D(\mathbb{R})$ bel.

$$\langle u', \phi\rangle = -\langle u, \phi'\rangle = -\int_{-1}^{1} \phi'(x)\,dx = -\phi(1) + \phi(-1) = \langle -\delta_1 + \delta_{-1}, \phi\rangle$$

$\Rightarrow u' = -\delta_1 + \delta_{-1}$

bzw $\mathbb{1}_{[-1,1]}(x) = \mathbb{1}_{(-\infty,1]}(x)\cdot\mathbb{1}_{[-1,\infty)}(x)$
$= -H(x-1)\,H(x+1)$
$\Rightarrow u' = -\delta_1 H(x+1) + H(x-1)\delta_{-1} = -\delta_1 + \delta_{-1}$

(iii) ges: $u \in D'(\mathbb{R})$ mit $u' = H$

1. Fall $x \le 0$: $\int_{-\infty}^{x} H(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$

2. Fall $x > 0$: $\int_{-\infty}^{x} H(t)\,dt = \int_{-\infty}^{x} \mathbb{1}_{[0,\infty)}(t)\,dt = \int_0^x 1\,dt = t\Big|_{t=0}^{x} = x$

$u(x) := \max(0, x) = \mathbb{1}_{[0,\infty)}(x)\,x$ Sei $\phi \in D(\mathbb{R})$ bel.

$$\langle u, \phi'\rangle = \int_{\mathbb{R}} u(x)\,\phi'(x)\,dx = \int_0^\infty x\,\phi'(x)\,dx = x\,\phi(x)\Big|_0^\infty - \int_0^\infty 1\cdot\phi(x)\,dx$$

$$= 0 - 0 - \int_0^\infty 1\cdot\phi(x)\,dx = -\int_{\mathbb{R}} H(x)\,\phi(x)\,dx = -\langle H, \phi\rangle = -\langle u', \phi\rangle$$

$\square$

PDGL Ü3

4) ges: Grenzwerte

(i) $u_n := \sum_{k=0}^{n} \frac{1}{n} \delta_{\frac{k}{n}}$

Sei $\phi \in D(\mathbb{R})$ bel.

$$\lim_{n\to\infty} \langle u_n, \phi \rangle = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{n} \phi\left(\frac{k}{n}\right) = \int_0^1 \phi(x)\,dx = \int_{\mathbb{R}} 1\!\!1_{[0,1]}(x)\,\phi(x)\,dx = \langle 1\!\!1_{[0,1]}, \phi \rangle$$

$$\Rightarrow \lim_{n\to\infty} u_n = 1\!\!1_{[0,1]}$$

(ii) $u_n := n(\delta_0 - \delta_{\frac{1}{n}})$

Sei $\phi \in D(\mathbb{R})$ bel. $\Rightarrow$ ϕ ... stetig diffbar

$$\lim_{n\to\infty} \langle u_n, \phi \rangle = \lim_{n\to\infty} n\left(\phi(0) - \phi\left(\tfrac{1}{n}\right)\right) = \lim_{n\to\infty} \frac{\phi(0) - \phi(\frac{1}{n})}{\frac{1}{n}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{n\to\infty} \frac{+\phi'(\frac{1}{n})\frac{1}{n^2}}{-\frac{1}{n^2}} = -\phi'(0)$$

$$\Rightarrow \lim_{n\to\infty} u_n = -\delta_0' \qquad \text{wobei } \langle \delta_x', \phi \rangle := \phi'(x)$$

(iii) $u_n := n \exp(-n|x|)$

Sei $\phi \in D(\mathbb{R})$ bel.

$$\lim_{n\to\infty} \langle u_n, \phi \rangle = \lim_{n\to\infty} \int_{\mathbb{R}} n \exp(-n|x|)\,\phi(x)\,dx = \lim_{n\to\infty} \left(\int_0^\infty n \exp(-nx)\,\phi(x)\,dx + \int_{-\infty}^0 n \exp(nx)\,\phi(x)\,dx \right)$$

$$= \lim_{n\to\infty} \left(\int_0^\infty \exp(-y)\,\phi\left(\tfrac{y}{n}\right) dy + \int_{-\infty}^0 \exp(y)\,\phi\left(\tfrac{y}{n}\right) dy \right) = \qquad \left[y = nx \quad \frac{dy}{dx} = n \quad dx = \frac{1}{n} dy \right.$$

$$= \lim_{n\to\infty} \left(\int_0^\infty \exp(-y)\,\phi\left(\tfrac{y}{n}\right) dy + \int_0^\infty \exp(-z)\,\phi\left(-\tfrac{z}{n}\right) dz \right) = \qquad \left[z = -y \quad \frac{dz}{dy} = -1 \quad dy = -dz \right.$$

$$= \lim_{n\to\infty} \left(\phi\left(\tfrac{\xi}{n}\right) \underbrace{\int_0^\infty \exp(-y)\,dy}_{=1} + \phi\left(-\tfrac{\xi'}{n}\right) \underbrace{\int_0^\infty \exp(-z)\,dz}_{=1} \right) = \qquad \text{für } \xi, \xi' \in K = \operatorname{supp} \phi$$

$$= \lim_{n\to\infty} \left(\phi\left(\tfrac{\xi}{n}\right) + \phi\left(-\tfrac{\xi'}{n}\right) \right) = 2\,\phi(0) = 2\,\delta_0(\phi)$$

$$\Rightarrow \lim_{n\to\infty} u_n = 2\,\delta_0$$

$\square$

PDGL Ü3

5) $u \in \mathcal{D}'(\Omega)$, $x_0 \in \Omega$ $\qquad x_0 \in supp(u) \Leftrightarrow \forall U \subseteq \Omega$ .. Umgebung von x_0 $\exists \varphi \in \mathcal{D}(U): \langle u, \varphi \rangle \neq 0$

ges: $supp(u)$

(i) $\langle u, \varphi \rangle = \sum_{k \in \mathbb{Z}} \varphi(k)$

Sei $x_0 \in \mathbb{Z}$ bel. Sei $U \subseteq \mathbb{R}$ .. Umgebung von x_0 bel.

Wir wissen es gibt $\varphi \in C_c^\infty = \mathcal{D}$ mit $supp(\varphi) \subseteq U \cap (x_0 - \frac{1}{2}, x_0 + \frac{1}{2})$ und $\varphi(x_0) = 1$

$\langle u, \varphi \rangle = \sum_{k \in \mathbb{Z}} \varphi(k) = \varphi(x_0) = 1 \neq 0$

Sei $x_0 \in \mathbb{R} \setminus \mathbb{Z}$ bel. Wähle $U \subseteq \mathbb{R}$, sodass U .. offen, $x_0 \in U \wedge \forall k \in \mathbb{Z}: k \notin U$.

Sei $\varphi \in \mathcal{D}(U)$ bel. $\Rightarrow \forall k \in \mathbb{Z}: \varphi(k) = 0$

$\Rightarrow \langle u, \varphi \rangle = \sum_{k \in \mathbb{Z}} \varphi(k) = 0$

$\Rightarrow supp(u) = \mathbb{Z}$

(ii) $\langle u, \varphi \rangle = \int_{-\infty}^{0} \varphi(x)\, dx$

Sei $x_0 \in (-\infty, 0]$ bel. Sei $U \subseteq \mathbb{R}$ .. Umgebung von x_0 bel.

Sei $\varphi(x) := \begin{cases} \alpha \cdot \exp\left(-\frac{1}{1-(x-x_0)^2}\right), & x \in (-1, 1) \\ 0 & \text{, sonst} \end{cases}$ wobei $\alpha > 0$, sodass $supp(\varphi) \subseteq U$

$\langle u, \varphi \rangle = \int_{-\infty}^{0} \varphi(x)\, dx \geq \alpha \int_{-1}^{0} \exp\left(-\frac{1}{1-x^2}\right) dx \approx \alpha \cdot 0{,}221997 > 0$

Sei $x_0 \in (0, \infty)$ bel. Wähle $U \subseteq \mathbb{R}$, sodass U ... offen, $x_0 \in U \wedge U \cap (-\infty, 0] = \emptyset$

Sei $\varphi \in \mathcal{D}(U)$ bel. $\Rightarrow supp(\varphi) \subseteq U \subseteq (0, \infty)$

$\langle u, \varphi \rangle = \int_{-\infty}^{0} \varphi(x)\, dx = \int_{(-\infty, 0] \cap supp(\varphi)} \varphi(x)\, dx = \int_{\emptyset} \varphi(x)\, dx = 0$

$\Rightarrow supp(u) = (-\infty, 0] = \mathbb{R}_0^-$

$\square$