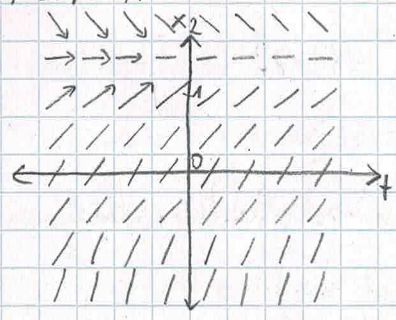
# DGL Ü3

1) a) $x' = -2x+3$




$a(t) = -2 \quad A(t) = \int a(t)\,dt = -2t \quad b(t) = 3$

$x(t) = e^{A(t)}\left(c + \int e^{-A(t)} b(t)\,dt\right) = e^{-2t}\left(c + \int e^{2t}\, 3\,dt\right)$

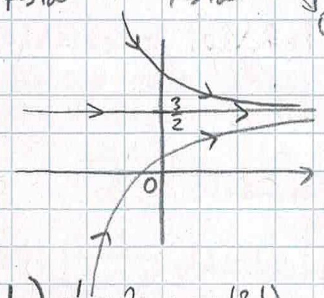
$= e^{-2t}\left(c + \frac{3}{2} e^{2t}\right) = ce^{-2t} + \frac{3}{2}$

AWP $x(0) = x_0 \qquad x(0) = ce^{-2\cdot 0} + \frac{3}{2} = c + \frac{3}{2} = x_0$

$\Rightarrow c = x_0 - \frac{3}{2} \qquad \Rightarrow x(t) = \left(x_0 - \frac{3}{2}\right) e^{-2t} + \frac{3}{2}$

$\lim\limits_{t\to+\infty} x(t) = \lim\limits_{t\to+\infty} c\underbrace{e^{-2t}}_{\to 0} + \frac{3}{2} = \frac{3}{2} \qquad \lim\limits_{t\to-\infty} x(t) = \lim\limits_{t\to-\infty} c\underbrace{e^{-2t}}_{\to\infty} + \frac{3}{2} = \operatorname{sgn}(c)\,\infty$ für $x_0 \neq \frac{3}{2}$

$\lim\limits_{t\to-\infty} x(t) = \lim\limits_{t\to-\infty} 0\cdot e^{-2t} + \frac{3}{2} = \frac{3}{2}$ für $x_0 = \frac{3}{2}$




(i) Lösungen $\exists$ für $t \in \mathbb{R}$

(ii) Lösung ist nur für $x_0 = \frac{3}{2}$ beschränkt

(iii) Lösung ist nur für $x_0 = \frac{3}{2}$ periodisch, da konstant

b) $x' = 2x + \cos(3t)$


$a(t) = 2 \quad A(t) = \int 2\,dt = 2t \qquad b(t) = \cos(3t)$

$x(t) = e^{2t}\left(c + \int e^{-2t} \cos(3t)\,dt\right)$

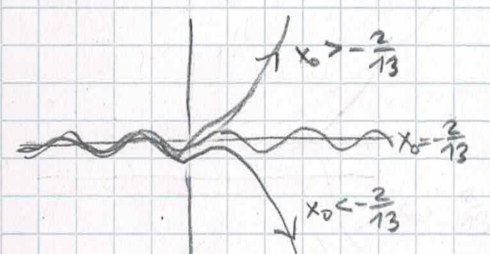
$= ce^{2t} + e^{2t} \frac{1}{13} e^{-2t}(3\sin(3t) - 2\cos(3t))$

$= ce^{2t} + \frac{3}{13}\sin(3t) - \frac{2}{13}\cos(3t)$

AWP $x_0 = x(0) = c - \frac{2}{13} \qquad \Rightarrow c = x_0 + \frac{2}{13}$

$\lim\limits_{t\to+\infty} x(t) = \lim\limits_{t\to+\infty} \left(x_0 + \frac{2}{13}\right)\underbrace{e^{2t}}_{\to\infty} + \frac{3}{13}\sin(3t) - \frac{2}{13}\cos(3t) = \infty$ für $x_0 \neq -\frac{2}{13}$ undefiniert für $x_0 = -\frac{2}{13}$

$\lim\limits_{t\to-\infty} x(t) = \lim\limits_{t\to-\infty} \left(x_0 + \frac{2}{13}\right)\underbrace{e^{2t}}_{\to 0} + \frac{3}{13}\sin(3t) - \frac{2}{13}\cos(3t) = 0$ für $x_0 \neq -\frac{2}{3}$ undefiniert für $x_0 = \frac{2}{13}$




(i) Lösungen $\exists$ für $t \in \mathbb{R}$

(ii) Lösung nur für $x_0 = -\frac{2}{13}$ beschränkt

(iii) Lösung nur für $x_0 = -\frac{2}{13}$ beschränkt

...

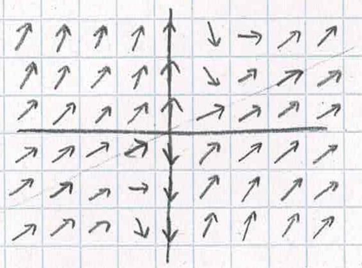
DGL Ü3

1)... c) $x' = -\frac{1}{t}x + 1$

$a(t) = -\frac{1}{t}$ $A(t) = \int -\frac{1}{t}\,dt = -\ln(t)$ $b(t) = 1$

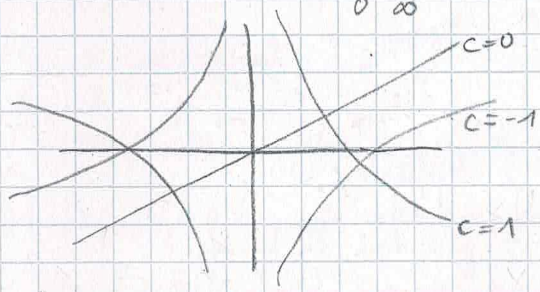
$x(t) = e^{-\ln(t)}\left(c + \int e^{\ln(t)}\,dt\right) = \frac{1}{t}\left(c + \int t\,dt\right)$

$= \frac{c}{t} + \frac{1}{t}\,\frac{1}{2}t^2 = \frac{c}{t} + \frac{t}{2}$



AWP $x_0 = x(0) = \mathrm{sgn}(c)\infty$ $\Rightarrow$ AWP nicht wohldefiniert

$\lim\limits_{t\to+\infty} x(t) = \lim\limits_{t\to+\infty} \underbrace{\frac{c}{t}}_{\to 0} + \underbrace{\frac{t}{2}}_{\to\infty} = +\infty$ $\lim\limits_{t\to-\infty} x(t) = \lim\limits_{t\to-\infty} \underbrace{\frac{c}{t}}_{\to 0} + \underbrace{\frac{t}{2}}_{\to -\infty} = -\infty$




(i) Lösung existiert für $t \in \mathbb{R}\setminus\{0\}$ für $c \in \mathbb{R}\setminus\{0\}$ und für $t \in \mathbb{R}$ für $c = 0$

(ii) keine Lösung ist beschränkt

(iii) keine Lösung ist periodisch

DGL Ü3

2) $x' = ax - x^2 \qquad x \in \mathbb{R},\ a > 0$

a) (i) Bernoulli DGL $\quad p(t) = a \quad q(t) = -1 \quad \alpha = 2 \quad x' = p(t)x + q(t)x^\alpha$

$$\Rightarrow y' = (1-\alpha)p(t)y + (1-\alpha)q(t) = -ay + 1$$

$$a(t) = -a \qquad b(t) = 1 \qquad A(t) = \int -a\, dt = -at$$

$$\Rightarrow y = e^{-at}\left(c + \int e^{at} dt\right) = e^{-at}\left(c + \frac{1}{a}e^{at}\right) = ce^{-at} + \frac{1}{a}$$

$$\Rightarrow x(t) = \left(ce^{-at} + \frac{1}{a}\right)^{-1} = \frac{1}{ce^{-at} + \frac{1}{a}}$$

(ii) separable DGL $\quad f(x) = ax - x^2 \quad g(t) = 1 \quad x' = f(x)g(t)$

$$\frac{dx}{dt} = ax - x^2 \iff \frac{1}{ax - x^2}dx = dt \iff \int \frac{1}{ax-x^2}dx = \int 1\, dt$$

$$\iff \int \frac{1}{ax-x^2}dx = \frac{1}{a}\left(\log(x) - \log(x-a)\right) = t + c$$

$$\iff \log\left(\frac{x}{x-a}\right) = at + ac \iff \frac{x}{x-a} = e^{a(t+c)}$$

$$\iff \frac{x-a}{x} = e^{-a(t+c)} \iff 1 - \frac{a}{x} = e^{-a(t+c)} \iff \frac{1}{x} = \frac{1}{a}\left(-e^{-a(t+c)} + 1\right)$$

$$\iff x = a\left(-e^{-at} \cdot e^{-ac} + 1\right)^{-1} = a\frac{1}{\hat{c}e^{-at} + 1} = \frac{1}{\frac{\hat{c}e^{-at} + 1}{a}} = \frac{1}{\hat{c}e^{-at} + \frac{1}{a}}$$

b) AWP $\quad x_0 = x(0)$

$$x_0 = x(0) = \frac{1}{c + \frac{1}{a}} = \frac{1}{\frac{ac+1}{a}} = \frac{a}{ac+1} \iff \frac{1}{x_0} = \frac{ac+1}{a} = c + \frac{1}{a}$$

$$\iff c = \frac{1}{x_0} - \frac{1}{a} \quad \text{für } x_0 \neq 0 \qquad \text{für } x_0 = 0 \Rightarrow x(t) = 0$$

$$\Rightarrow x(t) = \frac{1}{\left(\frac{1}{x_0} - \frac{1}{a}\right)e^{-at} + \frac{1}{a}} \quad \text{für } x_0 \neq 0$$

c)

$$\frac{1}{x_0}e^{-at} - \frac{1}{a}e^{-at} + \frac{1}{a} = 0$$

DGL Ü3

3) $t x' = x + t\cos^2(\frac{x}{t}) \iff x' = \frac{x}{t} + \cos^2(\frac{x}{t}) = f(\frac{x}{t})$ $\qquad x = ty$

$\Rightarrow y' = \frac{1}{t}(y + \cos^2(y) - y) = \frac{1}{t}\cos^2(y)$

$\Rightarrow \frac{dy}{dt} = \frac{1}{t}\cos^2(y) \iff \frac{1}{\cos^2(y)}\,dy = \frac{1}{t}\,dt \iff \int \frac{1}{\cos^2(y)}\,dy = \int \frac{1}{t}\,dt$

$\iff \tan(y) = \ln(t) + c \qquad \iff y = \tan^{-1}(c + \ln(t))$

$\Rightarrow x(t) = t \cdot \tan^{-1}(c + \ln(t))$

AWP $x(1) = 0$:

$x(1) = 1 \cdot \tan^{-1}(c + \ln(1)) = \tan^{-1}(c) = 0 \iff c = \tan(0) = 0$

$\Rightarrow x(t) = t \cdot \tan^{-1}(\ln(t))$ ist Lösung des AWP

DGL Ü3

4) $x' = \frac{x^2+2tx}{t^2}$

a) $x' = \frac{x^2}{t^2} + 2\frac{tx}{t^2} = \left(\frac{x}{t}\right)^2 + 2\frac{x}{t}$  $x = ty$  homogene Variablen

$\Rightarrow y' = \frac{y^2+2y-y}{t} = \frac{y^2+y}{t}$

$\Leftrightarrow \frac{dy}{dt} = \frac{y^2+y}{t} \Leftrightarrow \frac{1}{y^2+y}\,dy = \frac{1}{t}\,dt \Leftrightarrow \int\frac{1}{y^2+y}\,dy = \int\frac{1}{t}\,dt$

$\Leftrightarrow \int \frac{1}{y(y+1)}\,dy = \ln(t)+c \Leftrightarrow \ln(y) - \ln(y+1) = \ln(t)+c$

$\Leftrightarrow \exp\left(\ln\left(\frac{y}{y+1}\right)\right) = \exp(\ln(t)+c) \Leftrightarrow \frac{y}{y+1} = \hat{c}t \Leftrightarrow \frac{y+1}{y} = \frac{1}{\hat{c}t}$

$\Leftrightarrow 1+\frac{1}{y} = \frac{1}{\hat{c}t} \Leftrightarrow \frac{1}{y} = \frac{1}{\hat{c}t} - 1 \Leftrightarrow y = \frac{1}{\frac{1-\hat{c}t}{\hat{c}t}} = \frac{\hat{c}t}{1-\hat{c}t}$

$\Rightarrow x(t) = \frac{\hat{c}t^2}{1-\hat{c}t} = \frac{t^2}{\frac{1}{\hat{c}} - t} = \frac{t^2}{\hat{c} - t}$  $t\in\mathbb{R}\setminus\{\hat{c}\}$  $\hat{c} \in \mathbb{R}$

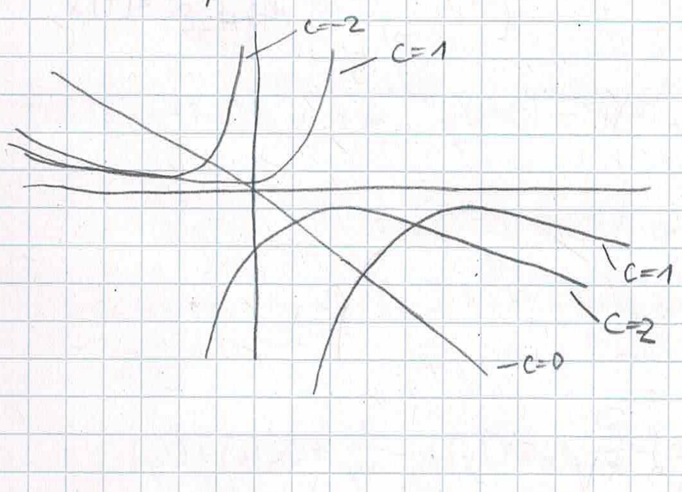
b) $x' = \frac{1}{t^2}x^2 + \frac{2}{t} x$  $p(t)=\frac{2}{t}$  $q(t)=\frac{1}{t^2}$  $\alpha=2$  Bernoulli

$y = x^{1-\alpha} = x^{-1}$

$y' = -\frac{2}{t} y - \frac{1}{t^2}$  $a(t) = -\frac{2}{t}$  $A(t) = -2\log(t)$  $b(t) = -\frac{1}{t^2}$

$y = e^{-2\log(t)}\left(c - \int e^{2\log(t)}\frac{1}{t^2}\,dt\right) = \frac{1}{t^2}\left(c-\int\frac{t^2}{t^2}\,dt\right) = \frac{1}{t^2}(c-t) = \frac{c-t}{t^2}$

$\Rightarrow x(t) = \frac{1}{y(t)} = \frac{t^2}{c-t}$



DGL Ü3

6) $(2y-3x+2)\,dx + x\,dy = 0$

integrierender Faktor $m(x)$

$\frac{\partial}{\partial y}(2y-3x+2)m(x) = 2m(x) \qquad \frac{\partial}{\partial x} x m(x) = m(x) + x\, m_x(x)$

für $m(x)=x$ gilt $2x \qquad x + x\cdot 1 = 2x$

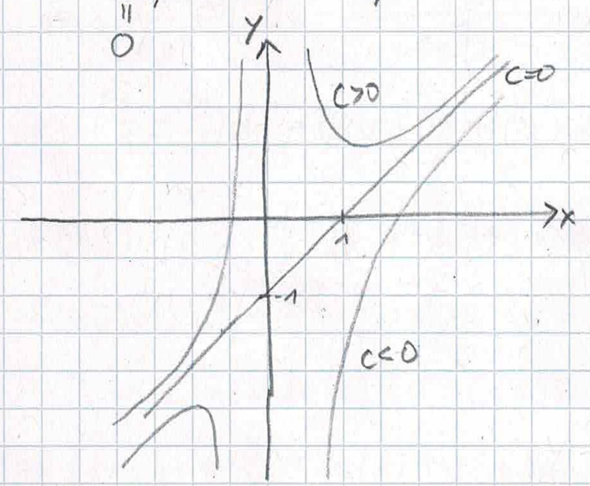
$(2xy - 3x^2 + 2x)\,dx + x^2\,dy = 0$

$F(x,y) = \int 2xy - 3x^2 + 2x \, dx + C(y) = x^2 y - x^3 + x^2 + C(y)$

$F(x,y) = \int x^2 dy + C(x) = x^2 y + C(x)$

$\Rightarrow C(x) = -x^3 + x^2 \qquad C(y) = 0$

$\Rightarrow F(x,y) = -x^3 + x^2 y + x^2 - c \overset{!}{=} 0$




$x' = -x$

$y' = 2y - 3x + 2$

da $g(x,y)\,dx - f(x,y)\,dy = 0$

äquivalent zu $x' = f(x,y)$

$y' = g(x,y)$ ist.

<u>DGL Ü3</u>

7) $2xy\,dx + (x^2+3y^2-1)\,dy = 0$

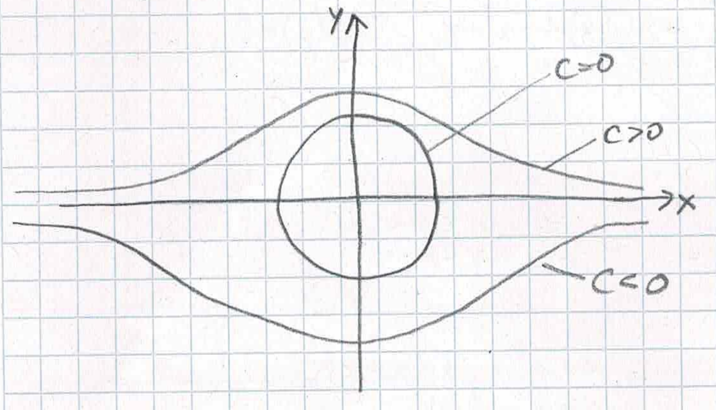
$\frac{\partial}{\partial y} 2xy = 2x \qquad \frac{\partial}{\partial x}(x^2+3y^2-1) = 2x \quad \Rightarrow$ exakt!

$F(x,y) = \int 2xy\,dx = x^2y + C(y)$

$F(x,y) = \int x^2+3y^2-1\,dy = x^2y + y^3 - y + C(x)$

$\Rightarrow C(x) = 0 \qquad C(y) = y^3 - y$

$\Rightarrow \underset{\overset{\shortparallel}{0}}{F(x,y)} = x^2y + y^3 - y - c$



DGL Ü3

8) $y'(x) = -\dfrac{y^2 - xy}{2xy^3 + xy + x^2}$

$$\frac{dy}{dx} = -\frac{y^2 - xy}{2xy^3 + xy + x^2} \Leftrightarrow (y^2 - xy)\,dx - (2xy^3 + xy + x^2)\,dy = 0$$

integrierender Faktor $m(x,y) = x^\alpha y^\beta$

$$\frac{\partial}{\partial y}(y^2 - xy)\,m(x,y) = \frac{\partial}{\partial y}\, x^\alpha y^{\beta+2} - x^{\alpha+1} y^{\beta+1} = (\beta+2)x^\alpha y^{\beta+1} - (\beta+1)x^{\alpha+1}y^\beta$$

$$\frac{\partial}{\partial x}(2xy^3 + xy + x^2)\,m(x,y) = \frac{\partial}{\partial x}\, 2x^{\alpha+1}y^{\beta+3} + x^{\alpha+1}y^{\beta+1} + x^{\alpha+2}y^\beta = 2(\alpha+1)x^\alpha y^{\beta+3} + (\alpha+1)x^\alpha y^{\beta+1} + (\alpha+2)x^{\alpha+1}y^\beta$$

$$\Rightarrow x^\alpha y^\beta((\beta+2)y - (\beta+1)x) = x^\alpha y^\beta((2\alpha+2)y^3 + (\alpha+1)y + (\alpha+2)x)$$

$$\Leftrightarrow -\beta - 1 = \alpha + 2, \quad \beta + 2 = \alpha + 1, \quad 2\alpha + 2 = 0 \quad \Rightarrow \alpha = -1, \ \beta = -2$$

$$\Rightarrow \frac{y^2 - xy}{xy^2}\,dx - \frac{2xy^3 + xy + x^2}{xy^2}\,dy = 0$$

$$\Leftrightarrow \left(\frac{1}{x} - \frac{1}{y}\right)dx - \left(2y + \frac{1}{y} + \frac{x}{y^2}\right)dy = 0$$

$$F(x,y) = \int \frac{1}{x} - \frac{1}{y}\,dx = \ln(x) - \frac{x}{y} + C(y)$$

$$F(x,y) = \int -2y + \frac{1}{y} + \frac{x}{y^2}\,dy = -y^2 + \ln(y) - \frac{x}{y} + C(x)$$

$$\Rightarrow C(x) = \ln(x) \qquad C(y) = y^2 + \ln(y)$$

$$\Rightarrow F(x,y) = -\frac{x}{y} + \ln(x) + y^2 + \ln(y) + c$$

DGL Ü3

9) a) $y''(t) = -y(t) \iff 1 \cdot y''(t) + 0 \cdot y'(t) + 1\, y(t) = 0$

$\chi(\lambda) = \lambda^2 - 1 = 0 \iff \lambda_{1,2} = \pm\sqrt{-1} = \pm i$

$\Rightarrow y_1 = ce^{it} \quad y_2 = de^{-it} \quad \Rightarrow y = c\,e^{it} + d\,e^{-it} = (c+d)\cos(t) + i(c-d)\sin(t)$

$Re(y) = (c+d)\cos(t) \quad Im(y) = (c-d)\sin(t)$

$1 = y(0) = (c+d)\cos(0) + (c-d)\sin(0) = c+d \qquad 0 = y'(0) = -(c+d)\sin(0) + (c-d)\cos(0) = c-d$

$\Rightarrow c = d = \frac{1}{2} \qquad \Rightarrow y(t) = \cos(t)$

b) $y''(t) = y(t) \iff 1 \cdot y''(t) + 0\, y'(t) - 1\, y(t) = 0$

$\chi(\lambda) = \lambda^2 - 1 = 0 \iff \lambda_{1,2} = \pm\sqrt{1} = \pm 1$

$\Rightarrow y_1 = ce^{t} \quad y_2 = de^{-t} \qquad \Rightarrow y(t) = ce^{t} + de^{-t} \qquad y' = ce^{t} - de^{-t}$

$1 = y(0) = c+d \qquad 0 = y'(0) = c-d \qquad \Rightarrow c = d = \frac{1}{2}$

$\Rightarrow y(t) = \frac{1}{2}(e^{t} + e^{-t})$

c) $y''(t) = 2y'(t) - y(t) \iff y''(t) - 2y'(t) + y(t) = 0$

$\chi(\lambda) = \lambda^2 - 2\lambda + 1 = 0 \qquad \lambda_{1,2} = 1 \pm \sqrt{1-1} = 1$

$\Rightarrow y_1 = ce^{t} \qquad y_2 = dte^{t} \qquad \Rightarrow y(t) = e^{t}(c + dt) \qquad y'(t) = e^{t}(c+dt) + e^{t}d$

$1 = y(0) = c \qquad 0 = y'(0) = c + d \qquad \Rightarrow c = 1 \qquad d = -1$

$\Rightarrow y(t) = e^{t}(1-t)$

# DGL Ü3

5) $(y+xy^2) - x y' = 0$

$$\Leftrightarrow (y+xy^2) - x\frac{dy}{dx} = 0 \quad \Leftrightarrow (y+xy^2)dx - x\,dy = 0$$

$$m(x,y) = x^\alpha y^\beta$$

$$\frac{\partial}{\partial y}(y+xy^2)m(x,y) = \frac{\partial}{\partial y} x^\alpha y^{\beta+1} + x^{\alpha+1} y^{\beta+2} = (\beta+1)x^\alpha y^\beta + (\beta+2)x^{\alpha+1} y^{\beta+1}$$

$$= x^\alpha y^\beta \left((\beta+1) + xy(\beta+2)\right)$$

$$\frac{\partial}{\partial x} x\,m(x,y) = \frac{\partial}{\partial x} x^{\alpha+1} y^\beta = (\alpha+1)x^\alpha y^\beta$$

$$\Rightarrow \beta+2 = 0 \quad \Leftrightarrow \beta = -2$$

$$\alpha+1 = \beta+1 \quad \Leftrightarrow \alpha = \beta = -2$$

$$\Rightarrow m(x,y) = \frac{1}{x^2y^2}$$

$$F(x,y) = \int \frac{y+xy^2}{x^2y^2}\,dx = \frac{1}{y}\int \frac{1}{x^2}dx + \int \frac{1}{x}dx + C(y) = -\frac{1}{xy} + \ln(x) + C(y)$$

$$F(x,y) = \int \frac{x}{x^2y^2}\,dy = \frac{1}{x}\int \frac{1}{y^2}dy + C(x) = -\frac{1}{xy} + C(x)$$

$$\Rightarrow C(x) = \ln(x) \qquad C(y) = 0$$

$$\Rightarrow \underset{\overset{\shortparallel}{0}}{F(x,y)} = -\frac{1}{xy} + \ln(x) - c \qquad c \in \mathbb{R}$$

$$c = -\frac{1}{2(-2)} + \ln(2) = \frac{1}{4} + \ln(2)$$

$$\Rightarrow \underset{\overset{\shortparallel}{0}}{F(x,y)} = -\frac{1}{xy} + \ln(x) - \frac{1}{4} - \ln(2)$$