LINAG Ü11

4.6.2.

a) Sind folgende Matrizen aus $GF(4)^{3\times 3}$ äquivalent?

$$A=\begin{pmatrix}1&1&1\\1&a&0\\1&b&0\end{pmatrix} \qquad B=\begin{pmatrix}1&a&b\\0&b&b\\1&1&0\end{pmatrix}$$

$$A=\begin{pmatrix}1&1&1\\1&a&0\\1&b&0\end{pmatrix} \rightsquigarrow \begin{array}{c}\begin{matrix}-\mathrm{III}&-\mathrm{III}&\end{matrix}\\ \begin{pmatrix}0&0&1\\1&a&0\\1&b&0\end{pmatrix}\end{array} \rightsquigarrow \begin{array}{c}\begin{matrix}&\cdot b&\end{matrix}\\ \begin{pmatrix}0&0&1\\1&1&0\\1&a&0\end{pmatrix}\end{array} \rightsquigarrow \begin{array}{c}\begin{matrix}&-\mathrm{I}&\end{matrix}\\ \begin{pmatrix}0&0&1\\1&0&0\\1&b&0\end{pmatrix}\end{array} \rightsquigarrow \begin{array}{c}\begin{matrix}&\cdot a&\end{matrix}\\ \begin{pmatrix}0&0&1\\1&0&0\\1&1&0\end{pmatrix}\end{array} \rightsquigarrow \begin{array}{c}\begin{matrix}-\mathrm{II}&&\end{matrix}\\ \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\end{array}$$

$$B=\begin{pmatrix}1&a&b\\0&b&b\\1&1&0\end{pmatrix} \rightsquigarrow \begin{array}{c}\begin{matrix}&-\mathrm{I}&\end{matrix}\\ \begin{pmatrix}1&b&b\\0&b&b\\1&0&0\end{pmatrix}\end{array} \rightsquigarrow \begin{array}{c}\begin{matrix}&\cdot a&-\mathrm{II}\end{matrix}\\ \begin{pmatrix}1&1&0\\0&1&0\\1&0&0\end{pmatrix}\end{array}$$

$$\Rightarrow \operatorname{rg}(A)=3 \qquad \operatorname{rg}(B)=2$$

Da $A\sim B \Leftrightarrow \operatorname{rg}(A)=\operatorname{rg}(B)$ sind die Matrizen nicht äquivalent.

b) ges: Äquivalenzklassen von $\mathbb{Z}_2^{2\times 2}$

Da $A\sim B \Leftrightarrow \operatorname{rg}(A)=\operatorname{rg}(B)$ und der Rang max. 2 sein kann (und ≥ 0) gibt es drei Äquivalenzklassen:

$$\underset{\operatorname{rg}=0}{\left[\begin{pmatrix}0&0\\0&0\end{pmatrix}\right]_\sim} \quad \underset{\operatorname{rg}=1}{\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]_\sim} \quad \underset{\operatorname{rg}=2}{\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]_\sim}$$

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4.7.1. β)

$$\left(\begin{array}{cccc|cc} 1 & -1 & 7 & 7 & 14 & 0 \\ 0 & 1 & 2 & 2 & 5 & 1 \\ 2 & 0 & 20 & 20 & 42 & 2 \\ 1 & -2 & 5 & 5 & 10 & -1 \end{array}\right) \rightsquigarrow \begin{array}{c} +\mathrm{II} \\ \\ \\ \\ \end{array} \left(\begin{array}{cccc|cc} 1 & 0 & 9 & 9 & 19 & 1 \\ 0 & 1 & 2 & 2 & 5 & 1 \\ 2 & 0 & 20 & 20 & 42 & 2 \\ 1 & -2 & 5 & 5 & 10 & -1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \\ \\ -2\cdot\mathrm{I} \\ -\mathrm{I} \end{array} \left(\begin{array}{cccc|cc} 1 & 0 & 9 & 9 & 19 & 1 \\ 0 & 1 & 2 & 2 & 5 & 1 \\ 0 & 0 & 2 & 2 & 4 & 0 \\ 0 & -2 & -4 & -4 & -9 & -2 \end{array}\right) \rightsquigarrow \begin{array}{c} \\ -\mathrm{III} \\ \cdot(2^{-1}) \\ +2\cdot\mathrm{II} \end{array} \left(\begin{array}{cccc|cc} 1 & 0 & 9 & 9 & 19 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} -9\cdot\mathrm{III} \\ \\ \\ \\ \end{array} \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Das linke System hat keine Lösung, da $\mathrm{rg}\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&1 \\ 0&0&0&0 \end{pmatrix} \neq \mathrm{rg}\begin{pmatrix} 1&0&0&0&1 \\ 0&1&0&0&1 \\ 0&0&1&1&2 \\ 0&0&0&0&1 \end{pmatrix}$.

Das rechte System hat unendlich viele Lösungen, da

$$\dim(U) = n - r = 4 - 3 = 1$$

$$U = \left[\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}\right] \qquad \text{partikulär Lösung} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{Menge der Lösungen} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v : v \in U \right\}$$

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4.7.7. $a_i^*: \mathbb{R}^{6\times 1} \to \mathbb{R} \quad i \in \{1,2,3\}$

$\langle a_1^*, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{pmatrix} \rangle = x_1 + 2x_2 \quad - x_4 + 3x_5 + 6x_6$

$\langle a_2^*, \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \rangle = 2x_1 + x_2 + 4x_3 + 3x_4 \quad + x_6$

$\langle a_3^*, \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \rangle = \quad x_2 \quad - x_4 - x_5$ ges: Basis von $\ker a_1^* \cap \ker a_2^* \cap \ker a_3^*$

$$\left(\begin{array}{cccccc|c} 1 & 2 & 0 & -1 & 3 & 6 & 0 \\ 2 & 1 & 4 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array}\right) \begin{array}{c} -2\cdot\mathrm{III} \\ \rightsquigarrow -\mathrm{III} \\ \end{array} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 5 & 6 & 0 \\ 2 & 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} \\ \rightsquigarrow -2\cdot\mathrm{I} \\ \end{array} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 4 & 2 & -9 & -11 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array}\right) \rightsquigarrow (\cdot 4) \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{9}{4} & -\frac{11}{4} & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array}\right)$$

$r = 3 \quad n = 6 \quad \dim = n - r = 3$

$$\begin{pmatrix} -1 & -5 & -6 \\ -\frac{1}{2} & \frac{9}{4} & \frac{11}{4} \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ \frac{9}{4} \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ \frac{11}{4} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

bilden eine Basis von $\ker a_1^* \cap \ker a_2^* \cap \ker a_3^*$

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4.6.5 β $\qquad V \ldots VR/\mathbb{R} \qquad f \in L(V,V)$

$B=(b_1,b_2) \qquad \tilde{B}=(\tilde{b}_1,\tilde{b}_2) \qquad B^* \circ f \circ (B^*)^{-1} : \mathbb{R}^{2\times 1} \to \mathbb{R}^{2\times 1}$

$\langle B^*, \tilde{b}_1 \rangle = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad \langle B^*, \tilde{b}_2 \rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1+3x_2 \\ 3x_1+x_2 \end{pmatrix}$

a) ges: $\langle B^*, f(B) \rangle$, $\langle \tilde{B}^*, f(B) \rangle$, $\langle \tilde{B}^*, f(\tilde{B}) \rangle$

$$\langle B^*, \tilde{B} \rangle = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\langle \tilde{B}^*, B \rangle = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\begin{tikzcd}
\mathbb{R}^{2\times 1} \arrow[r, "{\langle \tilde{B}^*, f(\tilde{B}) \rangle}"] & \mathbb{R}^{2\times 1} \\
V \arrow[u, "\tilde{B}^*"] \arrow[r, "f"] \arrow[d, "B^*"'] & V \arrow[u, "\tilde{B}^*"'] \arrow[d, "B^*"] \\
\mathbb{R}^{2\times 1} \arrow[r, "{\langle B^*, f(B) \rangle}"'] & \mathbb{R}^{2\times 1}
\end{tikzcd}$$

$$\langle B^*, f(B) \rangle = \begin{pmatrix} 1+3\cdot 0 & 0+3\cdot 1 \\ 3\cdot 1+0 & 3\cdot 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$\langle \tilde{B}^*, f(B) \rangle = \tilde{B}^* \circ (B^*)^{-1} \circ B^* \circ f \circ (B^*)^{-1} = \langle \tilde{B}^*, B \rangle \cdot \langle B^*, f(B) \rangle$$
$$= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -1 & 1 \end{pmatrix}$$

$$\langle \tilde{B}^*, f(\tilde{B}) \rangle = \langle \tilde{B}^*, B \rangle \cdot \langle B^*, f(B) \rangle \cdot \langle B^*, \tilde{B} \rangle$$
$$= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

b)