8. Man zeige für $x \in \mathbb{R},\ x > 0$, und $r, s \in \mathbb{Q}$, dass $x^{-r} = \frac{1}{x^r},\ x^r x^s = x^{r+s},\ (x^r)^s = x^{rs}$.

zz: $x^{-r} = \frac{1}{x^r}$

$$\sqrt[r]{\frac{1}{x^r}} = \frac{\sqrt[r]{1}}{\sqrt[r]{x^r}} = \frac{1}{x} = x^{-1} \Rightarrow \frac{1}{x^r} = (x^{-1})^r = x^{-1\cdot r} = x^{-r}$$

zz: $x^r x^s = x^{r+s}$

$r =: \frac{a}{b}$ $\quad s := \frac{c}{d}$ $\quad r = \frac{a\cdot d}{b\cdot d}$ $\quad s = \frac{b\cdot c}{b\cdot d}$

$x^r x^s = x^{\frac{ad}{bd}} \cdot x^{\frac{bc}{bd}} = \left(\sqrt[bd]{x}\right)^{a\cdot d} \cdot \left(\sqrt[bd]{x}\right)^{b\cdot c}$ mit $\sqrt[bd]{x} \in \mathbb{R}$ und $ad, bc \in \mathbb{Z}$

$y := \sqrt[bd]{x}$ $\quad e := a\cdot d$ $\quad f := b\cdot c$ $\qquad$ zz: $y^e y^f = y^{e+f}$

vollständige Induktion nach e:

$e=0$: $y^0 \cdot y^f = y^f = y^{0+f}$ ✓

$e+1$: $y^{e+1} \cdot y^f = y \cdot y^e \cdot y^f = y \cdot y^{e+f} = y^{(e+1)+f}$ ✓

$e-1$: $y^{e-1} \cdot y^f = y^{-1} \cdot y^e \cdot y^f = y^{-1} \cdot y^{e+f} = y^{(e-1)+f}$ ✓

vollständige Induktion nach f genauso.

$$\left(\sqrt[bd]{x}\right)^{a\cdot d} \cdot \left(\sqrt[bd]{x}\right)^{b\cdot c} = \left(\sqrt[bd]{x}\right)^{ad+bc} = x^{\frac{ad+bc}{bd}} = x^{\frac{ad}{bd}+\frac{bc}{bd}} = x^{\frac{a}{b}+\frac{c}{d}} = x^{r+s}$$

zz: $(x^r)^s = x^{r\cdot s}$

$r := \frac{a}{b}$ $\quad s := \frac{c}{d}$ $\quad x^{r\cdot s} = x^{\frac{a}{b}\cdot\frac{c}{d}} = x^{\frac{ac}{bd}} = \left(\sqrt[bd]{x}\right)^{a\cdot c} = \left(x^{\frac{1}{bd}}\right)^{ac}$

$e := a\cdot c$ $\quad f := b\cdot d$ $\quad \left(x^{\frac{1}{f}}\right)^e$

$$\left(x^{\frac{1}{f}}\right)^e = \left(\sqrt[f]{x}\right)^e = x^{\frac{e}{f}}$$

$$(x^r)^s = \left(x^{\frac{1}{f}}\right)^e = x^{\frac{e}{f}} = x^{\frac{a\cdot c}{b\cdot d}} = x^{\frac{a}{b}\cdot\frac{c}{d}}$$

$\square$