3.6.1 $\quad f \in L(V,V) \qquad f \circ f = f$ (idempotent)

a) zz $g := id - f$ ist idempotent

$$g(x) = (id - f)(x) = id(x) - f(x) = x - f(x)$$

$$g \circ g(x) = (id-f)(id-f)(x) = (id-f)(id(x) - f(x)) = id(id(x) - f(x))$$
$$- f(id(x) - f(x)) = id(x - f(x)) - f(x - f(x)) = x - f(x) - f(x - f(x))$$
$$= x - f(x) - (f(x) - f(f(x))) = x - f(x) - f(x) + f(f(x))$$
$$= x - f(x) - f(x) + f(x) = x - f(x) = g(x)$$

$$\implies g \circ g = g$$

b) $g \circ f = g(f(x)) = id(f(x)) - f(f(x)) = f(x) - f(x) = 0$

$ f \circ g = f(id(x) - f(x)) = f(id(x)) - f(f(x)) = f(x) - f(x) = 0$

c) Sei $x \in V$ mit $g(x) = 0$ bel.

$$0 = g(x) = id(x) - f(x) = x - f(x) \iff x = f(x)$$

$$\text{d.h. } x \in f(V)$$

Sei $x \in V$ mit $x \in f(V)$ bel.

$\exists z \in V: f(z) = x \qquad g(x) = id(x) - f(x) = x - f(x) = f(z) - f(f(z)) =$
$$= f(z) - f(z) = 0$$

$$\implies ker(g) = f(V)$$

Sei $x \in V$ mit $f(x) = 0$ bel.

$$0 = f(x) \iff g(x) = f(x) + g(x) = f(x) + id(x) - f(x) = id(x) = x$$

$$\text{d.h. } x \in g(V)$$

Sei $x \in V$ mit $x \in g(V)$ bel.

$\exists z \in V: g(z) = x \qquad f(x) = f(g(z)) = f(id(z) - f(z)) = f(z) - f(f(z))$
$$= f(z) - f(z) = 0$$

$$\implies ker(f) = g(V)$$