PDGL Ü5

1) $\Omega \subseteq \mathbb{R}^n$ ... beschränktes Gebiet $\quad u \in C^2(\overline{\Omega})$

(i) u ... subharmonisch in Ω (d.h. $-\Delta u \leq 0$ in Ω) $\Rightarrow \forall R>0 \, \forall x \in \Omega$ mit $\overline{B_R(x)} \subseteq \Omega$:

$$u(x) \leq \frac{1}{S_n R^{n-1}} \int_{\partial B_R(x)} u \, ds \qquad \text{wobei } S_n \text{ ... Oberflächenmaß von } B_1(0).$$

$x_0 \in \Omega$, $r := |x - x_0|$, $\omega := \frac{x - x_0}{r}$ $\quad \Rightarrow 0 < r \leq R$. Mit Satz von Gauß folgt

$$0 \geq \int_{B_r(x_0)} -\Delta u \, dx \underset{\Delta = \mathrm{div}(\mathrm{grad})}{=} -\int_{B_r(x_0)} \mathrm{div}(\mathrm{grad}(u)) \, dx \underset{\text{Gauß}}{=} -\int_{\partial B_r(x_0)} \mathrm{grad}(u) \cdot \nu \, ds = -\int_{\partial B_r(x_0)} \frac{\partial u}{\partial \nu} ds = -\int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x_0 + r\omega) r^{n-1} d\omega$$

$$\Rightarrow 0 \cdot r^{-n+1} \geq -\int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x_0 + r\omega) \, d\omega \quad \Rightarrow 0 \geq -\int_{\partial B_1(0)} \int_0^R \frac{\partial u}{\partial r}(x_0 + r\omega) \, dr \, d\omega = -\int_{\partial B_1(0)} u(x_0 + R\omega) - u(x_0) \, d\omega$$

Mit $\omega' = R\omega$ folgt $0 \leq \frac{1}{R^{n-1}} \int_{\partial B_1(0)} u(x_0 + \omega') \, d\omega' - u(x_0) \int_{\partial B_1(0)} d\omega = \frac{1}{R^{n-1}} \int_{\partial B_R(x_0)} u \, ds - S_n u(x_0)$

$$\Rightarrow S_n u(x_0) \leq \frac{1}{R^{n-1}} \int_{\partial B_R(x_0)} u \, ds \quad \Rightarrow u(x_0) \leq \frac{1}{S_n R^{n-1}} \int_{\partial B_R(x_0)} u \, ds$$

(Beweis aus dem Skript 4.9.)

(ii) $\Phi \in C^\infty(\mathbb{R})$ ... konvex $\qquad u$ ... harmonisch $\quad \Rightarrow v := \Phi(u)$ ... subharmonisch

Φ ... konvex $:\Leftrightarrow \forall x, y \in \mathbb{R} \, \forall t \in [0,1]: \Phi(tx + (1-t)y) \leq t\Phi(x) + (1-t)\Phi(y)$

u ... harmonisch $:\Leftrightarrow u \in C^2 \wedge \Delta u = 0$

$$\Delta v = \Delta \Phi(u) = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} \Phi(u) = \sum_{k=1}^n \frac{\partial}{\partial x_k}\left(\Phi'(u) \cdot \frac{\partial}{\partial x_k} u\right) = \sum_{k=1}^n \frac{\partial}{\partial x_k}(\Phi'(u)) \cdot \frac{\partial}{\partial x_k} u + \Phi'(u) \frac{\partial^2}{\partial x_k^2} u$$

$$= \sum_{k=1}^n \Phi''(u) \cdot \left(\frac{\partial}{\partial x_k} u\right)^2 + \Phi'(u) \frac{\partial^2}{\partial x_k^2} u = \underbrace{\Phi''(u)}_{\geq 0} \underbrace{\sum_{k=1}^n \left(\frac{\partial}{\partial x_k} u\right)^2}_{\geq 0} + \Phi'(u) \underbrace{\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} u}_{= \Delta u = 0} \geq 0 \quad \Rightarrow -\Delta v \leq 0$$

$\Big($ Φ ... konvex $\Rightarrow \Phi'' \geq 0$: $\Phi''(x) = \lim_{h \to 0} \frac{\Phi(x+h) + \Phi(x-h) - 2\Phi(x)}{h^2}$

$\Phi(x) = \Phi\left(\frac{1}{2}(x+h) + \frac{1}{2}(x-h)\right) \leq \frac{1}{2}\Phi(x+h) + \frac{1}{2}\Phi(x-h) \Rightarrow 2\Phi(x) \leq \Phi(x+h) + \Phi(x-h) \Rightarrow \Phi''(x) \geq 0$ $\Big)$

(iii) $u \in C^3(\Omega)$ ... harmonisch $\Rightarrow v := |\nabla u|^2$ ... subharmonisch

$$\Delta v = \Delta(|\nabla u|^2) = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} |\nabla u|^2 = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}\left(\left(\frac{\partial}{\partial x_1} u\right)^2 + \left(\frac{\partial}{\partial x_2} u\right)^2 + \dots + \left(\frac{\partial}{\partial x_n} u\right)^2\right) = \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_k^2}\left(\frac{\partial}{\partial x_j} u\right)^2$$

$$= \sum_{k=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_k}\left(2 \frac{\partial^2}{\partial x_k \partial x_j} u \cdot \frac{\partial}{\partial x_j} u\right) = \sum_{k=1}^n \sum_{j=1}^n 2\left(\frac{\partial^3}{\partial x_k^2 \partial x_j} u \cdot \frac{\partial}{\partial x_j} u + \frac{\partial^2}{\partial x_k \partial x_j} u \cdot \frac{\partial^2}{\partial x_k \partial x_j} u\right)$$

$$= 2 \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^3 u}{\partial x_k^2 \partial x_j} \cdot \frac{\partial u}{\partial x_j} + \left(\frac{\partial^2 u}{\partial x_k \partial x_j}\right)^2 = 2\left(\sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} \cdot \frac{\partial}{\partial x_j}\underbrace{\left(\sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2}\right)}_{= \Delta u = 0}\right) + \underbrace{\sum_{k=1}^n \sum_{j=1}^n \left(\frac{\partial^2 u}{\partial x_k \partial x_j}\right)^2}_{\geq 0}\right) \geq 0$$

$\Rightarrow -\Delta v \leq 0$ also v ... subharmonisch.

$\square$

PDGL Ü5

2) $\Omega \subseteq \mathbb{R}^n$ ... beschränktes Gebiet, $\partial\Omega \in C^1$, $c \in L^\infty(\Omega)$, $c \geq c_0 > 0$, $f \in L^2(\Omega)$, $g \in H^1(\Omega)$, $\alpha > 0$

$-\Delta u + cu = f$ in Ω $\quad \nabla u \cdot \nu + \alpha u = g$ auf $\partial\Omega$

(i) ges: schwache Formulierung des Randwertproblems

$$\int_\Omega f v \, dx = \int_\Omega (-\Delta u + cu) v \, dx = -\int_\Omega \Delta u \cdot v \, dx + c\int_\Omega u \cdot v \, dx = -\int_\Omega \operatorname{div}(\operatorname{grad} u)) v \, dx + c\int_\Omega u \cdot v \, dx =$$

$$= +\int_\Omega \nabla v \, \nabla u \, dx - \int_{\partial\Omega} v \underbrace{(\nabla u \cdot \nu)}_{= g - \alpha u} ds + c\int_\Omega u \cdot v dx \quad \left[\text{Gauß: } \int_\Omega f \operatorname{div} g \, dx = -\int_\Omega \nabla f \cdot g \, dx + \int_{\partial\Omega} f (g \cdot \nu) ds\right]$$

$$= \int_\Omega \nabla v \nabla u \, dx - \int_{\partial\Omega} v(g - \alpha u) ds + c\int_\Omega u \cdot v \, dx = \int_\Omega \nabla v \nabla u + cuv \, dx - \int_{\partial\Omega} vg - \alpha uv \, ds$$

Suche $u \in H_0^1(\Omega)$ mit $\int_\Omega f v dx = \int_\Omega \nabla v \nabla u + cuv \, dx + \int_{\partial\Omega} \alpha uv - vg ds \quad \forall v \in H_0^1(\Omega)$

(ii) zz: $\exists$ schwache Lösung $\quad$ Eindeutig?

$$(u,v) := \int_\Omega \nabla v \nabla u + cuv \, dx + \int_{\partial\Omega} \alpha uv \, ds$$

linear ✓ $\quad$ symmetrisch ✓ $\quad$ positiv definit, da $c > 0, \alpha > 0 \Rightarrow (u,u) \geq 0$

$0 = (u,u) = \int_\Omega (\nabla u)^2 + cu^2 dx + \int_{\partial\Omega} \alpha u^2 ds \Rightarrow (\nabla u)^2 + cu^2 = 0 \wedge \alpha u^2 = 0 \Rightarrow u = 0$ ✓

$\forall v \in H_0^1(\Omega)$: $(u,v) = \int_\Omega f v \, dx + \int_{\partial\Omega} vg \, ds =: F(v) \in H^{-1}$

Mit dem Satz von Lax-Milgram folgt $\exists!\, u \in H_0^1 : (u,v) = F(v)$

Also existiert eine schwache Lösung und diese ist eindeutig.

$(.,..)$ stetig? $\quad |(u,v)| = \left|\alpha \int_{\partial\Omega} uv \, ds + \int_\Omega \nabla u \nabla v + cuv \, d\lambda\right| \leq \quad$ [T ... Spur auf Rand

$\alpha \|T(u)\|_{L^2} \|T(v)\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|c\|_\infty \|u\|_{L^2} \|v\|_{L^2} \leq$

$\alpha \|T\|^2 \|u\|_{H_1} \|v\|_{H_1} + \|u\|_{H_1} \|v\|_{H_1} + \|c\|_\infty \|u\|_{H_1} \|v\|_{H_1} \leq$

$(\underbrace{\alpha \|T\|^2}_{>0} + 1 + \underbrace{\|c\|_\infty}_{>0}) \|u\|_{H_1} \|v\|_{H_1}$

$\Rightarrow$ stetig ✓

$(.,..)$ koerziv? $\quad (u,u) = \underbrace{\int_{\partial\Omega} \overbrace{\alpha}^{>0} u^2 ds}_{>0} + \int_\Omega (\nabla u)^2 + \underbrace{c}_{>c_0} u^2 dx > \int_\Omega \underbrace{\frac{c_0}{1+c_0}}_{<1} (\nabla u)^2 + \underbrace{\frac{1}{1+c_0}}_{<1} c_0 u^2 d\lambda$

$= \frac{c_0}{1+c_0} \int_\Omega u^2 + (\nabla u)^2 d\lambda = \frac{c_0}{1+c_0} \sum_{|\alpha| \leq 1} \int_\Omega (D^\alpha u)^2 d\lambda = \frac{c_0}{1+c_0} \|u\|_{H_1}^2$

$\Rightarrow$ koerziv ✓

F stetig? $|F(v)| = \left|\int_\Omega f v dx + \int_{\partial\Omega} vg \, ds\right| \leq \|fv\|_{L^1(\Omega)} + \|vg\|_{L^1(\partial\Omega)} \leq \|f\|_{L^2} \|v\|_{L^2} + \|T\|^2 \|g\|_{H^1} \|v\|_{H^1} \leq$

$(\|f\|_{L^2} + \|T\|^2 \|g\|_{H^1}) \|v\|_{H_1}$

$\Rightarrow$ F stetig ✓

PDGL Ü5

3) $\Delta u = 0$ für $(x,y) \in \Omega := \mathbb{R} \times (0,\pi)$ $\quad u = 0$ auf $\partial\Omega = \mathbb{R}\times\{0\} \cup \mathbb{R}\times\{\pi\}$

ges: nicht triviale Lösung des RWP mittels Separationsansatz

$u(x,y) = X(x) \cdot Y(y)$

$0 = \Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = X''(x) \cdot Y(y) + X(x) \cdot Y''(y)$

$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$ Da linke Seite nur von x abhängt und rechte nur von y muss dieser Ausdruck "konstant" sein.

$\Rightarrow X''(x) = c X(x) \wedge Y''(y) = cY(y)$ Klingt nach sin und cos also

$X(x) = \cos(ax) \quad Y(y) = \sin(by)$

$\Rightarrow u(x,y) = \cos(ax)\sin(by) \cdot c$

$0 = u(x,0) = \cos(ax) \cdot \sin(0)c = 0$

$0 = u(x,\pi) = \cos(ax) \cdot \sin(b\pi)c = 0$ für $b \in \mathbb{Z}$

$0 = \Delta u = -\cos(ax)a^2 \sin(by)c - \cos(ax)\sin(by)b^2c = -c\cos(ax)\sin(by)(a^2+b^2) \quad \forall x,y$

$\Rightarrow a = ib$

$\Rightarrow u(x,y) = c \cdot \cos(ibx)\sin(by)$ für $c \in \mathbb{R}, b \in \mathbb{Z}$ $\quad$ z.B. $\cos(ix)\sin(y)$

Ist schwaches Maximumsprinzip anwendbar?

$\Omega \subseteq \mathbb{R}^n$ ... beschränktes Gebiet $\quad u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \quad \Delta u \geq 0$ bzw. $\Delta u \leq 0$

$\Rightarrow \sup_{x\in\Omega} u(x) = \sup_{x\in\partial\Omega} u(x)$ bzw. $\inf_{x\in\Omega} u(x) = \inf_{x\in\partial\Omega} u(x)$

Es gilt zwar $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ und $\Delta u \geq 0$ sowie $\Delta u \leq 0$, aber nicht Ω ... beschränkt!

Daher gilt auch nicht $\sup_{x\in\Omega} u(x) = \sup_{x\in\partial\Omega} u(x) = 0 = \inf_{x\in\partial\Omega} u(x) = \inf_{x\in\Omega} u(x)$.

$\square$

PDGL Ü5

4) $\beta > 0 \quad u(x) := \begin{cases} |x|^{-\beta}, & x \in B_1(0)\setminus\{0\} \\ 0, & x = 0 \end{cases} \qquad W^{1,p}(B_1(0)) = \{u \in L^p(B_1(0)) : \forall |\alpha| \leq 1 : D^\alpha u \in L^p(B_1(0))\}$

(i) ges: für welche β gilt $u \in W^{1,p}(B_1(0))$?

$$\|u\|_p^p = \int_{B_1(0)} |u(x)|^p dx = \int_{B_1(0)\setminus\{0\}} |x|^{-\beta p} dx = \int_0^1 \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} r^{-\beta p} d\varphi_{n-1} \cdots d\varphi_2\, d\varphi_1\, dr$$

$$= 2\pi^{n-1} \int_0^1 r^{-\beta p} dr = \begin{cases} 2\pi^{n-1} \frac{r^{1-\beta p}}{1-\beta p}\Big|_0^1 = \frac{2\pi^{n-1}}{1-\beta p}, & \beta p \neq 1 \\ 2\pi^{n-1} \ln(r)\Big|_0^1 = \infty, & \beta p = 1 \end{cases} \qquad \boxed{\Rightarrow \beta \neq \frac{1}{p}}$$

$$\frac{d}{dx_i} u(x) = \frac{d}{dx_i} \left(\sqrt{x_1^2 + \ldots + x_n^2}\right)^{-\beta} = \frac{d}{dx_i} \left(\sum_{j=1}^n x_j^2\right)^{-\frac{\beta}{2}} = -\frac{\beta}{2} \left(\sum_{j=1}^n x_j^2\right)^{-\frac{\beta}{2}-1} 2x_i =$$

$$= -\beta x_i \left(\sum_{j=1}^n x_j^2\right)^{-\frac{\beta+2}{2}} = -\beta x_i |x|^{-(\beta+2)} \quad \text{für } x \neq 0 \text{ und } 0 \text{ für } x = 0$$

$$\left\|\frac{d}{dx_i} u\right\|_p^p = \int_{B_1(0)\setminus\{0\}} \left|-\beta x_i |x|^{-(\beta+2)}\right|^p dx = \beta^p \int_{B_1(0)\setminus\{0\}} |x_i|^p |x|^{-(\beta+2)p} dx =$$

o.B.d.A. $i = 1$

$$= \beta^p \int_0^1 \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} |r\cos(\varphi_1)|^p r^{-(\beta+2)p} d\varphi_{n-1} \cdots d\varphi_2\, d\varphi_1\, dr =$$

$$= \beta^p \pi^{n-2} \int_0^1 \int_0^{2\pi} r^{-(\beta+2)p+p} |\cos(\varphi_1)|^p d\varphi_1\, dr = \beta^p \pi^{n-2} \int_0^{2\pi} |\cos(\varphi_1)|^p \int_0^1 r^{-(\beta+2)p+p} dr\, d\varphi_1 =$$

$$= \beta^p \pi^{n-2} \left(-r^{1-(\beta+1)p} \frac{1}{\beta p + p - 1}\right)\Big|_0^1 \left(\int_0^{\frac{\pi}{2}} \cos(\varphi_1)^p d\varphi_1 - \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \cos(\varphi_1)^p + \int_{\frac{3}{2}\pi}^{2\pi} \cos(\varphi_1)^p d\varphi_1\right) =$$

$$= -\beta^p \pi^{n-2} \frac{1}{\beta p + p - 1} (- \,||\, -) < \infty \quad \text{für } (\beta+1)p \neq 1$$

sonst

$$= \beta^p \pi^{n-2} \int_0^{2\pi} |\cos(\varphi_1)|^p d\varphi_1 \int_0^1 r^{-1} dr = \beta^p \pi^{n-2} \int_0^{2\pi} |\cos(\varphi_1)|^p d\varphi_1 (\ln(r))\Big|_0^1 = \infty$$

$$\boxed{\Rightarrow \beta \neq \frac{1}{p} - 1}$$

$n \geq 2,\ p \in [1, \infty) \Rightarrow \beta \in [0, \infty) \setminus \{\frac{1}{p}, \frac{1}{p} - 1\}$

$p = \infty$: $\|u\|_\infty = \inf\{C \geq 0 : |u(x)| \leq C$ für f.a. $x \in B_1(0)\} = \inf\{C \geq 0 : |x|^{-\beta} \leq C$ für f.a. $x \in B_1(0)\setminus\{0\}\} =$

$$x_n = \left(\frac{1}{n}, 0, \ldots, 0\right) \in B_1(0)\ \forall n \qquad |x_n|^{-\beta} = \left(\frac{1}{n}\right)^{-\beta} = n^\beta \xrightarrow{n \to \infty} \infty \quad \Rightarrow \|u\|_\infty = \infty\ \forall n$$

$n = 1,\ p \in [1, \infty)$: $\|u\|_p^p = \int_{-1}^1 |x|^{-\beta p} dx = \int_{-1}^0 (-x)^{-\beta p} dx + \int_0^1 x^{-\beta p} dx = \begin{cases} \frac{1}{1-\beta p} + \frac{1}{1-\beta p}, & \beta p \neq 1 \\ \infty, & \beta p = 1 \end{cases}$

$$\left\|\frac{d}{dx} u(x)\right\|_p^p = \int_{-1}^1 \left|-\beta |x|^{-(\beta+1)}\right|^p dx = \beta \int_{-1}^1 |x|^{-(\beta+1)p} dx = 2\beta \int_0^1 x^{-(\beta+1)p} dx = \qquad \boxed{\Rightarrow \beta \neq \frac{1}{p}}$$

$$= \begin{cases} 2\beta \frac{x^{1-(\beta+1)p}}{1-(\beta+1)p}\Big|_0^1, & (\beta+1)p \neq 1 \\ 2\beta \ln(x)\Big|_0^1, & (\beta+1)p = 1 \end{cases} = \begin{cases} \frac{2\beta}{1-(\beta+1)p}, & (\beta+1)p \neq 1 \\ \infty, & \beta + 1 = \frac{1}{p} \end{cases} \qquad \boxed{\Rightarrow \beta \neq \frac{1}{p} - 1}$$

...

PDGL Ü5

4)... (ii) zz: $\delta_0 \in H^{-1}(B_1(0)) \Leftrightarrow n=1$

$n=1$: $\delta_0 : H^1(-1,1) \to \mathbb{R}$, $\phi \mapsto \phi(0)$ linear ✓

$n \geq 2$: $\delta_0 : H^1(B_1(0)) \to \mathbb{R}$, $\phi \mapsto \phi(0,\dots,0)$

$p=2$: $W^{1,2}(B_1(0)) = \{u \in L^2(B_1(0)) : D^\alpha u \in L^2(B_1(0)) \ \forall |\alpha| \le 1\} = H^1(B_1(0))$

$$0 = u(0) = \langle \delta, u \rangle \overset{\text{Skript}}{=} \lim_{\varepsilon \to 0} \int_{B_1(0)} f_\varepsilon(x) u(x)\, dx = \lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)\setminus\{0\}} \frac{1}{2\varepsilon} |x|^{-\beta} dx = \qquad \left[f_\varepsilon(x) := \begin{cases} \frac{1}{2\varepsilon}, & |x| < \varepsilon \\ 0, & |x| > \varepsilon \end{cases} \right.$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^\varepsilon \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi r^{-\beta} d\varphi_{n-1} \cdots d\varphi_2 d\varphi_1 dr = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} 2\pi^{n-1} \int_0^\varepsilon r^{-\beta} dr =$$

$$= \lim_{\varepsilon \to 0} \frac{\pi^{n-1}}{\varepsilon} \frac{r^{1-\beta}}{1-\beta}\Big|_0^\varepsilon = \frac{\pi^{n-1}}{1-\beta} \lim_{\varepsilon \to 0} \frac{\varepsilon^{1-\beta}}{\varepsilon} = \frac{\pi^{n-1}}{1-\beta} \lim_{\varepsilon \to 0} \varepsilon^{-\beta} = \frac{\pi^{n-1}}{1-\beta} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\beta} = \infty \quad \lightning$$

$n=1$ $$0 = u(0) = \langle \delta, u \rangle = \lim_{\varepsilon \to 0} \int_{-1}^{1} f_\varepsilon(x) u(x)\, dx = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} \mathbb{1}_{(-1,1)\setminus\{0\}}(x) |x|^{-\beta} dx =$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} 2 \int_0^\varepsilon x^{-\beta} dx = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{x^{1-\beta}}{1-\beta}\Big|_0^\varepsilon = \lim_{\varepsilon \to 0} \frac{\varepsilon^{1-\beta}}{\varepsilon} = \lim_{\varepsilon \to 0} \varepsilon^{-\beta} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\beta} = \infty$$

PDGL Ü5

5) $\Omega \subseteq \mathbb{R}^n$ ... beschränktes Gebiet $\partial\Omega \in C^1$ $\begin{cases} -\operatorname{div}(A\nabla u) + cu = f & \text{für } x \in \Omega \\ u = 0 & \text{für } x \in \partial\Omega \end{cases}$

$A(x) = (a_{ij}(x))$ ... symmetrisch, glm. pos. definit

$a_{ij} \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$

(i) ges: schwache Formulierung des RWP

$$\int_\Omega (-\operatorname{div}(A\nabla u) + cu)\phi \, dx = \int_\Omega f\phi \, dx$$

$$\| $$

$$-\int_\Omega \operatorname{div}(A\nabla u)\phi\,dx + c\int_\Omega u\phi\,dx = +\int_\Omega \nabla\phi^T A\nabla u\,dx - \underbrace{\int_{\partial\Omega}\phi(A\nabla u\cdot\nu)\,ds}_{=0} \overset{+c\int_\Omega u\phi dx}{=} \int_\Omega \nabla\phi^T A\nabla u + cu\phi\,dx$$

Also schwache Formulierung $u \in C^1(\Omega)$ mit $u = 0$ auf $\partial\Omega$ mit $\forall \phi \in C^1(\Omega)$ mit $\phi = 0$ auf $\partial\Omega$:

$$F(\phi) := \int_\Omega f\phi\,dx = \int_\Omega \nabla\phi^T A\nabla u + cu\phi\,dx =: \langle u, \phi\rangle$$

(ii) ges: $\mu > 0$ für fast alle $x \in \Omega$: $c(x) > -\mu$ F ... stetig, koerziv $H = H_0^1(\Omega)$

$$|\langle u, v\rangle| = \left|\int_\Omega \nabla v^T A\nabla u + c\,u\,v\,dx\right| \leq \|\nabla v\|_{L^2} \max_{i,j}\|a_{ij}\|_\infty \|\nabla u\|_{L^2} + \|c\|_\infty \|u\|_{L^2}\|v\|_{L^2}$$

$$\leq \max_{i,j}\|a_{ij}\|_\infty \|u\|_{H^1}\|v\|_{H^1} + \|c\|_\infty\|u\|_{H^1}\|v\|_{H^1} = \underbrace{(\max_{i,j}\|a_{ij}\|_\infty + \|c\|_\infty)}_{K>0}\|u\|_{H^1}\|v\|_{H^1}$$

$$\langle u, u\rangle = \int_\Omega \nabla u^T A\nabla u + cu^2\,dx = \int_\Omega \nabla u^T B^T D B\nabla u + cu^2\,dx$$

$$= \int_\Omega \underbrace{(B\nabla u)^T}_{=:w} D\underbrace{(B\nabla u)}_{w} + cu^2\,dx = \int_\Omega \sum_{i=1}^n \underbrace{\lambda_i w_i^2}_{>0\ \geq 0} + cu^2\,dx \geq$$

$$\geq \int_\Omega \underbrace{\min_{i=1,\dots,n}\lambda_i}_{=:\lambda}\left(\sum_{i=1}^n w_i^2\right) + cu^2\,dx = \int_\Omega \lambda(w^Tw) + cu^2\,dx =$$

$$= \int_\Omega \lambda(\nabla u^T \underbrace{B^T B}_{=B^{-1}B=I_n}\nabla u) + cu^2\,dx = \int_\Omega \lambda\,\nabla u^2 + cu^2\,dx \geq$$

$$\geq \lambda\int_\Omega \nabla u^2\,dx - \mu\int_\Omega u^2\,dx = \lambda\|\nabla u\|_{L^2}^2 - \mu\|u\|_{L^2}^2 \geq$$

$$\geq \lambda\frac{1}{\xi^2}\|u\|_{L^2}^2 - \mu\|u\|_{L^2}^2 = \underbrace{\left(\frac{\lambda}{\xi^2} - \mu\right)}_{=K>0}\|u\|_{L^2}^2 \quad \text{also muss } c(x) > -\frac{\lambda}{\xi^2}$$

[$A = B^{-1}DB$ wobei $B^T = B^{-1}$ und $D = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ mit λ_i sind die EW von A.

$\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}\begin{pmatrix}\vdots\end{pmatrix}$ $\begin{pmatrix}\lambda_1 w_1\\ \vdots \\ \lambda_n w_n\end{pmatrix}$ $\lambda_i > 0\ \forall i$

$(\ \cdots\ )$ $\sum \lambda_i w_i^2$]

[$u \in H_0^1 \Rightarrow \exists\, \xi > 0: \|u\|_{L^2} \leq \xi\|\nabla u\|_{L^2}$ gilt nach Poincaré-Ungleichung]

[$(u,v): H\times H \to \mathbb{R}$ ... stetig $:\Leftrightarrow \exists K > 0\ \forall u, v \in H: |(u,v)| \leq K\|u\|_H\|v\|_H$

$(u,v): H\times H \to \mathbb{R}$ ... koerziv $:\Leftrightarrow \exists K > 0\ \forall u \in H: (u,u) \geq K\|u\|_H^2$]

(iii) Lax-Millgram Bedingungen in (ii) geprüft bzw. klar. *

$\Rightarrow \exists!\, u \in H_0^1(\Omega): \langle u, \phi\rangle = F(\phi)\ \forall\phi$ Dies ist die gesuchte schwache Lsg.

* F ... stetig: $|F(\phi)| = \left|\int_\Omega f\phi\,dx\right| \leq \|f\|_{L^2}\|\phi\|_{L^2} \leq \underbrace{\|f\|_{L^2}}_{<\infty}\|\phi\|_{H^1} < \infty$