ANA Ü4

1) $(H, \langle.,.\rangle)$... Hilbertraum $\quad n \in \mathbb{N} \quad y_i \in H \setminus \{0\}$ für $i = 1, \dots, n$

zz: $\displaystyle \|x\|^2 \geq \sum_{i=1}^n \frac{|\langle x, y_i\rangle|^2}{\sum_{j=1}^n |\langle y_i, y_j\rangle|}$

Sei $\displaystyle \alpha_i := \frac{\langle x, y_i\rangle}{\sum_{j=1}^n |\langle y_i, y_j\rangle|}$ für $i = 1, \dots, n$

$$\|x\|^2 - \sum_{i=1}^n \alpha_i \langle y_i, x\rangle - \sum_{i=1}^n \overline{\alpha_i}\langle x, y_i\rangle + \sum_{i,j=1}^n \alpha_i \overline{\alpha_j}\langle y_i, y_j\rangle$$
$$= \|x\|^2 + \sum_{i=1}^n \langle -\alpha_i y_i, x\rangle + \sum_{i=1}^n \langle x, -\alpha_i y_i\rangle + \sum_{i,j=1}^n \langle -\alpha_i y_i, -\alpha_j y_j\rangle$$
$$= \langle x, x\rangle + \langle \sum_{i=1}^n (-\alpha_i y_i), x\rangle + \langle x, \sum_{j=1}^n (-\alpha_j y_j)\rangle + \langle \sum_{i=1}^n (-\alpha_i y_i), \sum_{j=1}^n (-\alpha_j y_j)\rangle$$
$$= \langle x, x + \sum_{j=1}^n (-\alpha_j y_j)\rangle + \langle \sum_{i=1}^n (-\alpha_i y_i), x + \sum_{j=1}^n (-\alpha_j y_j)\rangle$$
$$= \langle x + \sum_{i=1}^n (-\alpha_i y_i), x + \sum_{j=1}^n (-\alpha_j y_j)\rangle$$
$$= \langle x - \sum_{i=1}^n \alpha_i y_i, x - \sum_{j=1}^n \alpha_j y_j\rangle = \|x - \sum_{i=1}^n \alpha_i y_i\|^2 \geq 0$$

$$0 \leq (|\alpha_i| - |\alpha_j|)^2 = |\alpha_i|^2 - 2|\alpha_i \alpha_j| + |\alpha_j|^2 = |\alpha_i|^2 - 2|\alpha_i \overline{\alpha_j}| + |\alpha_j|^2$$
$$\Leftrightarrow 2|\alpha_i \overline{\alpha_j}| \leq |\alpha_i|^2 + |\alpha_j|^2$$

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von oben folgt $\displaystyle \|x\|^2 \geq \sum_{i=1}^n \alpha_i \langle y_i, x\rangle + \sum_{i=1}^n \overline{\alpha_i}\langle x, y_i\rangle - \sum_{i,j=1}^n \alpha_i \overline{\alpha_j}\langle y_i, y_j\rangle$

$$\sum_{i=1}^n \frac{|\langle x, y_i\rangle|^2}{\sum_{j=1}^n |\langle y_i, y_j\rangle|} = \sum_{i=1}^n \frac{\langle x, y_i\rangle \overline{\langle x, y_i\rangle}}{\sum_{j=1}^n |\langle y_i, y_j\rangle|} = \sum_{i=1}^n \alpha_i \overline{\langle x, y_i\rangle} = \sum_{i=1}^n \alpha_i \langle y_i, x\rangle$$

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zz: $\displaystyle \sum_{i=1}^n \overline{\alpha_i}\langle x, y_i\rangle - \sum_{i,j=1}^n \alpha_i \overline{\alpha_j}\langle y_i, y_j\rangle \geq 0$

$$\left|\sum_{i,j=1}^n \alpha_i \overline{\alpha_j}\langle y_i, y_j\rangle\right| \leq \sum_{i,j=1}^n |\alpha_i \overline{\alpha_j}||\langle y_i, y_j\rangle| \leq \sum_{i,j=1}^n \frac{1}{2}(|\alpha_i|^2 + |\alpha_j|^2)|\langle y_i, y_j\rangle|$$
$$= \frac{1}{2}\sum_{i,j=1}^n |\alpha_i|^2 |\langle y_i, y_j\rangle| + \frac{1}{2}\sum_{i,j=1}^n |\alpha_j|^2 |\langle y_i, y_j\rangle| = \frac{1}{2}\sum_{i,j=1}^n \frac{|\langle x, y_i\rangle|^2}{(\sum_{k=1}^n |\langle y_i, y_k\rangle|)^2}|\langle y_i, y_j\rangle| + \frac{1}{2}\sum_{i,j=1}^n \frac{|\langle x, y_j\rangle|^2}{(\sum_{k=1}^n |\langle y_j, y_k\rangle|)^2}|\langle y_i, y_j\rangle|$$
$$= \frac{1}{2}\sum_{i=1}^n |\langle x, y_i\rangle|^2 \frac{\sum_{j=1}^n |\langle y_i, y_j\rangle|}{(\sum_{k=1}^n |\langle y_i, y_k\rangle|)^2} + \frac{1}{2}\sum_{j=1}^n |\langle x, y_j\rangle|^2 \frac{\sum_{i=1}^n |\langle y_i, y_j\rangle|}{(\sum_{k=1}^n |\langle y_j, y_k\rangle|)^2} = \frac{1}{2}\sum_{i=1}^n \frac{|\langle x, y_i\rangle|^2}{\sum_{k=1}^n |\langle y_i, y_k\rangle|} + \frac{1}{2}\sum_{j=1}^n \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^n |\langle y_j, y_k\rangle|}$$
$$= \sum_{i=1}^n \frac{|\langle x, y_i\rangle|^2}{\sum_{k=1}^n |\langle y_i, y_k\rangle|} = \sum_{i=1}^n \overline{\alpha_i}\langle x, y_i\rangle$$

$$\Rightarrow \|x\|^2 \geq \sum_{i=1}^n \alpha_i \langle y_i, x\rangle = \sum_{i=1}^n \frac{|\langle x, y_i\rangle|^2}{\sum_{j=1}^n |\langle y_i, y_j\rangle|} \qquad \square$$

ANA Ü4

2) $\Big[$ X ... reeller, normierter Raum heißt strikt konvex

$:\Leftrightarrow \forall x \neq y \in X$ mit $\|x\| = 1 = \|y\|$ $\forall \lambda \in (0,1): \|\lambda x + (1-\lambda) y\| < \lambda \|x\| + (1-\lambda)\|y\|$

zz: X... reeller Hilbertraum $\Rightarrow$ X... strikt konvex

Sei $x, y \in X$ mit $x \neq y$ und $\|x\| = \sqrt{\langle x, x\rangle} = 1$ sowie $\|y\| = \sqrt{\langle y, y\rangle} = 1$

$\Rightarrow \langle x, x\rangle = 1 = \langle y, y\rangle$

Sei $\lambda \in (0,1)$ bel.

$\|\lambda x + (1-\lambda) y\|^2 = \sqrt{\langle \lambda x + (1-\lambda) y, \lambda x + (1-\lambda) y\rangle}^2 = |\langle \lambda x + (1-\lambda) y, \lambda x + (1-\lambda) y\rangle|$

$= |\langle \lambda x, \lambda x\rangle + \langle \lambda x, (1-\lambda) y\rangle + \langle (1-\lambda) y, \lambda x\rangle + \langle (1-\lambda) y, (1-\lambda) y\rangle|$

$= |\lambda^2 \langle x, x\rangle + \lambda(1-\lambda)\langle x, y\rangle + \lambda(1-\lambda)\langle x, y\rangle + (1-\lambda)^2 \langle y, y\rangle|$

$= |\lambda^2 + 2\lambda(1-\lambda)\langle x, y\rangle + (1-\lambda)^2| \leq |\lambda^2| + |2\lambda(1-\lambda)\langle x, y\rangle| + |(1-\lambda)^2|$

$= \lambda^2 + 2\lambda(1-\lambda)|\langle x, y\rangle| + (1-\lambda)^2 \overset{*}{<} \lambda^2 + 2\lambda(1-\lambda)\|x\| \cdot \|y\| + (1-\lambda)^2$

$= \lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2 = (\lambda + (1-\lambda))^2$

$\Rightarrow \|\lambda x + (1-\lambda) y\| < \lambda + (1-\lambda) = \lambda\|x\| + (1-\lambda)\|y\|$

* Cauchy-Schwarz-Ungleichung $\forall x, y \in X: |\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

wobei Gleichheit $\Leftrightarrow$ x, y linear abhängig

Angenommen x und y sind linear abhängig, d.h. $\exists \alpha \in \mathbb{R}: x = \alpha y$

$\Rightarrow \|x\|^2 = \langle x, x\rangle = \langle \alpha y, \alpha y\rangle = \alpha^2 \langle y, y\rangle = \alpha^2 \|y\|^2$

Da $\|x\| = 1 = \|y\|$ folgt $1 = \alpha \Rightarrow x = y$ $\lightning$ zu $x \neq y$

oder $-1 = \alpha \Rightarrow y = -x \Rightarrow \|\lambda x + (1-\lambda) y\| = \|\lambda x + y - \lambda y\| = \|\lambda x - x + \lambda x\|$

$= \|(2\lambda - 1)x\| = |2\lambda - 1| \, \|x\| = |2\lambda - 1| \in [0, 1)$ also < 1 für die linke Seite

und $\lambda\|x\| + (1-\lambda)\|y\| = \lambda + 1 - \lambda = 1$ für die rechte Seite, womit die Behauptung gilt.

$\square$

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ANA Ü4

2)... $l^1(\mathbb{N})$ .. Banachraum aller absolut konvergenten reellen / komplexen Folgen

$x \neq y \in l^1(\mathbb{N})$ bel. mit $\|x\| = \sum_{i=1}^{\infty} |x_i| = 1$ und $\|y\| = \sum_{i=1}^{\infty} |y_i| = 1$ $\lambda \in (0,1)$ bel.

$$\|\lambda x + (1-\lambda) y\|_1 = \sum_{i=1}^{\infty} |\lambda x_i + (1-\lambda) y_i| \leq \sum_{i=1}^{\infty} \lambda |x_i| + (1-\lambda)|y_i| = \lambda \sum_{i=1}^{\infty} |x_i| + (1-\lambda) \sum_{i=1}^{\infty} |y_i|$$

$$= \lambda \|x\|_1 + (1-\lambda) \|y\|_1$$ also konvex

Sei $x = (1,0,0,\dots)$ und $y = (0,1,0,0,\dots)$. Dann ist $x \neq y$ und $\|x\|_1 = 1 = \|y\|_1$.

$\Rightarrow \|\lambda x + (1-\lambda) y\|_1 = \sum_{i=1}^{\infty} |\lambda x_i + (1-\lambda) y_i| = \lambda + (1-\lambda) = \lambda \|x\|_1 + (1-\lambda) \|y\|_1$ also nicht strikt

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$l^\infty(\mathbb{N})$ .. Banachraum aller beschränkten reellen / komplexen Folgen

$x \neq y \in l^\infty(\mathbb{N})$ bel. mit $\|x\| = \sup_{i \in \mathbb{N}} |x_i| = 1$ und $\|y\| = \sup_{i \in \mathbb{N}} |y_i| = 1$ $\lambda \in (0,1)$ bel.

$$\|\lambda x + (1-\lambda) y\|_\infty = \sup_{i \in \mathbb{N}} |\lambda x_i + (1-\lambda) y_i| \leq \sup_{i \in \mathbb{N}} \lambda |x_i| + (1-\lambda) |y_i| = \lambda \sup_{i \in \mathbb{N}} |x_i| + (1-\lambda) \sup_{i \in \mathbb{N}} |y_i|$$

$$= \lambda \|x\|_\infty + (1-\lambda) \|y\|_\infty$$ also konvex

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$x = (1,1,0,\dots)$, $y = (1,0,1,0,\dots)$ $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i| = 1 = \sup_{i \in \mathbb{N}} |y_i| = \|y\|_\infty$, $x \neq y$

$\Rightarrow \|\lambda x + (1-\lambda) y\|_\infty = \sup_{i \in \mathbb{N}} |\lambda x_i + (1-\lambda) y_i| = \sup_{i \in \mathbb{N}} \{\lambda + (1-\lambda), \lambda, (1-\lambda), 0, \dots\} = \max\{\lambda, (1-\lambda), 1\} = 1$

$\lambda \|x\|_\infty + (1-\lambda) \|y\|_\infty = \lambda \sup_{i \in \mathbb{N}} |x_i| + (1-\lambda) \sup_{i \in \mathbb{N}} |y_i| = \lambda + (1-\lambda) = 1$ also wieder nicht konvex

Nur für $p=2$ ist l^p ein Hilbertraum, daher kein Widerspruch zu oben.

ANA Ü4

3) $1 \leq p \leq \infty$ $\quad f \in L^p(\mathbb{R}^d)$ fest $\quad \tau : \mathbb{R}^d \to L^p(\mathbb{R}^d)$
$z \mapsto \tau_z f := f(\cdot - z)$

Für welche Werte von p ist τ stetig?

Sei $1 \leq p < \infty$. Sei $\varepsilon > 0$ bel. $\exists g \in C_c^\infty : \|f - g\|_p \leq \frac{\varepsilon}{3}$ nach Satz 2.5.1.

g ist als stetige Funktion mit kompakten Träger nach Satz 1.4.12. glm. stetig.

$\Rightarrow \exists \delta > 0 \ \forall |z| < \delta : \|g - \tau_z g\|_p < \frac{\varepsilon}{3}$

Für $|z| < \delta$ folgt mit Dreiecksungleichung und Translationsinvarianz der L^p-Norm:

$$\|f - \tau_z f\|_p \leq \|f - g\|_p + \|g - \tau_z g\|_p + \|\tau_z g - \tau_z f\|_p = 2\|f - g\|_p + \|g - \tau_z g\|_p \leq \varepsilon.$$

Für $p = \infty$:

$$f(x) = \begin{cases} 1, & 0 \leq x \\ 0, & x < 0 \end{cases}$$

$\|f - \tau_z f\|_\infty \leq |f - \tau_z f(0)| = |f(0) - f(0 - z)| = |1 - 0| = 1$ für alle $z \in (0, \infty)$

$\Rightarrow$ nicht stetig

ANA Ü4

4) $a \in \mathbb{C}, |a| < 1$ zz: $\int_0^1 \frac{1-t}{1-at^3}\,dt = \sum_{n=0}^{\infty} \frac{a^n}{(3n+1)(3n+2)}$

zz: für $|a|<1$, $t \in [0,1)$ gilt $\sum_{n=0}^{k} a^n t^{3n} = \frac{a^{k+1}t^{3k+3}-1}{at^3-1}$

$k=0$: $a^0 t^{3\cdot 0} = 1$ $\quad \frac{a^1t^3-1}{at^3-1} = 1$ ✓

$k+1$: $\sum_{n=0}^{k+1} a^n t^{3n} = \sum_{n=0}^{k} a^n t^{3n} + a^{k+1} t^{3(k+1)} = \frac{a^{k+1}t^{3k+3}-1}{at^3-1} + a^{k+1}t^{3k+3}$

$= \frac{a^{k+1}t^{3k+3} - 1 + a^{k+2}t^{3k+6} - a^{k+1}t^{3k+3}}{at^3-1} = \frac{a^{k+2}t^{3(k+2)}-1}{at^3-1}$ ✓

zz: $|a|<1$, $t\in[0,1)$ gilt $\sum_{n=0}^{\infty} a^n t^{3n} = \frac{1}{1-at^3}$

$\lim_{k\to\infty} \frac{a^{k+1}t^{3k+3}-1}{at^3-1} = \frac{\lim_{k\to\infty} a^{k+1}t^{3k+3} - 1}{at^3-1} = \frac{-1}{at^3-1} = \frac{1}{1-at^3}$

zz: $|a|<1$ gilt $\int_0^1 \frac{1-t}{1-at^3}\,dt = \sum_{n=0}^{\infty} \frac{a^n}{(3n+1)(3n+2)}$

$\int_0^1 \frac{1-t}{1-at^3}\,dt = \int_0^1 (1-t) \sum_{n=0}^{\infty} a^n t^{3n}\,dt = \int_0^1 \sum_{n=0}^{\infty} (1-t) a^n t^{3n}\,dt \overset{*}{=} \sum_{n=0}^{\infty} \int_0^1 (1-t) a^n t^{3n}\,dt$

$= \sum_{n=0}^{\infty} a^n \int_0^1 t^{3n} - t^{3n+1}\,dt = \sum_{n=0}^{\infty} a^n \left(\frac{t^{3n+1}}{3n+1}\Big|_0^1 - \frac{t^{3n+2}}{3n+2}\Big|_0^1 \right) = \sum_{n=0}^{\infty} a^n \left(\frac{1}{3n+1} - \frac{1}{3n+2}\right)$

$= \sum_{n=0}^{\infty} a^n \frac{3n+2-(3n+1)}{(3n+1)(3n+2)} = \sum_{n=0}^{\infty} a^n \frac{3n+2-3n-1}{(3n+1)(3n+2)} = \sum_{n=0}^{\infty} \frac{a^n}{(3n+1)(3n+2)}$

$* \sum_{n=0}^{\infty} |(1-t)a^n t^{3n}| = |1-t| \sum_{n=0}^{\infty} \underbrace{|at^3|}_{<1}{}^n = |1-t| \frac{1}{1-|at^3|} < \frac{1}{1-|at^3|} < \infty$

zz: $\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)} = \frac{\pi}{3\sqrt{3}}$

$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)} \overset{\Delta}{=} \int_0^1 \frac{1-t}{1-t^3}\,dt = \int_0^1 \frac{1}{t^2+t+1}\,dt = \int_0^1 \frac{1}{(t+\frac{1}{2})^2 + \frac{3}{4}}\,dt$ $\quad [u = t + \frac{1}{2} \quad \frac{du}{dt} = 1$

$= \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{u^2 + \frac{3}{4}}\,du = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{4}{3(\frac{4u^2}{3}+1)}\,du = \frac{4}{3}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\frac{4}{3}u^2+1}\,du$ $\quad [v = \frac{2u}{\sqrt{3}} \quad \frac{dv}{du} = \frac{2}{\sqrt{3}}$

$= \frac{4}{3}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{v^2+1} \frac{\sqrt{3}}{2}\,dv = \frac{2\sqrt{3}}{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{v^2+1}\,dv = \frac{2\sqrt{3}}{3}\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$

$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}\left(\frac{2\pi-\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$

Δ $p(a) = \sum_{n=0}^{\infty} \frac{a^n}{(3n+1)(3n+2)}$ hat Konvergenzradius 1. Nach Abelschem Grenzwertsatz $p(1) = \sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)}$ $\quad \frac{1-t}{1-at^3} \to \frac{1-t}{1-t^3}$ punktweise auf $[0,1)$ und monoton ↘ $\Rightarrow \lim_{a\to 1-} \int_0^1 \frac{1-t}{1-at^3}\,dt = \int_0^1 \lim_{a\to 1-} \frac{1-t}{1-at^3} = \int_0^1 \frac{1-t}{1-t^3}$ $\quad \Rightarrow \sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)} = \int_0^1 \frac{1-t}{1-t^3}\,dt$

ANA Ü4

5) $D_n(x) := \sum_{k=-n}^{n} e^{ikx}$ zz: $D_n * D_n = 2\pi D_n$

$$D_n * D_n = \int_{-\pi}^{\pi} D_n(x-y) D_n(y)\,dy = \int_{-\pi}^{\pi} \Big(\sum_{k=-n}^{n} e^{ik(x-y)}\Big)\Big(\sum_{k=-n}^{n} e^{iky}\Big)dy$$

$$\overset{*}{=} \int_{-\pi}^{\pi} \sum_{k=-n}^{n} \sum_{j=-n}^{n} e^{ik(x-y)} e^{ijy}\,dy$$

$$= \int_{-\pi}^{\pi} \sum_{k=-n}^{n} e^{ikx} \sum_{j=-n}^{n} e^{iy(j-k)}\,dy = \sum_{k=-n}^{n} e^{ikx} \sum_{j=-n}^{n} \int_{-\pi}^{\pi} e^{iy(j-k)}\,dy$$

$$= \sum_{k=-n}^{n} e^{ikx}\Big(\sum_{\substack{j=-n\\ j\neq k}}^{n} \int_{-\pi}^{\pi} e^{iy(j-k)}\,dy + \int_{-\pi}^{\pi} \underbrace{e^{iy0}}_{=1}\,dy\Big) = \sum_{k=-n}^{n} e^{ikx}\Big(\sum \frac{e^{iy(j-k)}}{i(j-k)}\Big|_{-\pi}^{\pi} + y\Big|_{-\pi}^{\pi}\Big)$$

$$= \sum_{k=-n}^{n} e^{ikx}\Big(\sum_{\substack{j=-n\\ j\neq k}}^{n} \frac{1}{i(j-k)} \underbrace{\big(e^{i\pi(j-k)} - e^{-i\pi(j-k)}\big)}_{=0} + (\pi+\pi)\Big)$$

$$= \sum_{k=-n}^{n} e^{ikx}\, 2\pi = 2\pi \sum_{k=-n}^{n} e^{ikx} = 2\pi D_n(x)$$

$*$ $n=0$: $\Big(\sum_{k=0}^{0} a_k\Big)\Big(\sum_{k=0}^{0} b_k\Big) = a_0 b_0 = \sum_{k=0}^{0}\sum_{l=0}^{0} a_k b_l$

$n+1$: $$\Big(\sum_{k=-(n+1)}^{n+1} a_k\Big)\Big(\sum_{k=-(n+1)}^{n+1} b_k\Big) = \Big(\sum_{k=-n}^{n} a_k + a_{n+1} + a_{-(n+1)}\Big)\Big(\sum_{k=-n}^{n} b_k + b_{n+1} + b_{-(n+1)}\Big)$$

$$= \Big(\sum_{k=-n}^{n} a_k\Big)\Big(\sum_{k=-n}^{n} b_k\Big) + \sum_{k=-n}^{n} a_k(b_{n+1}+b_{-(n+1)}) + \sum_{k=-n}^{n} b_k(a_{n+1}+a_{-(n+1)}) + (a_{n+1}+a_{-(n+1)})(b_{n+1}+b_{-(n+1)})$$

$$= \underbrace{\sum_{k=-n}^{n}\sum_{l=-n}^{n} a_k b_l + \sum_{k=-n}^{n} a_k(b_{n+1}+b_{-(n+1)})} + \underbrace{\sum_{l=-n}^{n} b_k(a_{n+1}+a_{-(n+1)}) + (a_{n+1}+a_{-(n+1)})(b_{n+1}+b_{-(n+1)})}$$

$$= \sum_{k=-n}^{n} a_k \sum_{l=-(n+1)}^{n+1} b_l + (a_{n+1}+a_{-(n+1)}) \sum_{l=-(n+1)}^{n+1} b_k = \sum_{k=-(n+1)}^{n+1} a_k \sum_{l=-(n+1)}^{n+1} b_l$$

ANA Ü4

6) Basel-Problem $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(i) Fourierreihe von $f(x) := x$

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left(\frac{e^{-inx}}{-in}\right)' x \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{i}{n} e^{-inx} x \Big|_{-\pi}^{\pi} - \frac{i}{n} \int_{-\pi}^{\pi} e^{-inx} dx \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{i}{n} \left(e^{-in\pi} \pi + e^{in\pi} \pi \right) - \frac{i}{n} \frac{e^{-inx}}{-in} \Big|_{-\pi}^{\pi} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2i\pi}{n} (-1)^n + \frac{1}{n^2} \left(e^{-in\pi} - e^{in\pi} \right) \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{2i(-1)^n \pi}{n} + \frac{2}{n^2} \left((-1)^n - (-1)^n \right) \right)$$

$$= \frac{\sqrt{2\pi}}{n} i (-1)^n \qquad \text{für } n > 0$$

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-ix\cdot 0} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x \, dx = \frac{1}{\sqrt{2\pi}} \frac{x^2}{2} \Big|_{-\pi}^{\pi} = \frac{1}{\sqrt{2\pi}} \left(\frac{\pi^2}{2} - \frac{\pi^2}{2} \right) = 0$$

(ii) ges: $\|f\|_2$

$$\|f\|_2^2 = \int_{-\pi}^{\pi} f^2(x) \, dx = \int_{-\pi}^{\pi} x^2 dx = \frac{x^3}{3} \Big|_{-\pi}^{\pi} = \frac{\pi^3}{3} - \frac{-\pi^3}{3} = \frac{2}{3} \pi^3 \qquad \Rightarrow \|f\|_2 = \sqrt{\frac{2}{3}\pi^3}$$

(iii) Parsevalsche Gleichung um $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ zu zeigen

Parsevalsche Gleichung: $\|x\|^2 = \sum_{i \in I} |\langle x, \varepsilon_i \rangle|^2$

Für $\varepsilon_i = D_n$ ist $\langle f, D_n \rangle = \hat{f}(n)$

$$\Rightarrow \frac{2}{3}\pi^3 = \|f\|^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 = 2 \sum_{n=1}^{\infty} \left| \frac{\sqrt{2\pi}}{n} i (-1)^n \right|^2 = 2 \sum_{n=1}^{\infty} \frac{2\pi}{n^2} = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\frac{2}{3}\pi^3}{4\pi} = \frac{2}{3 \cdot 4} \pi^2 = \frac{1}{6} \pi^2$$

ANA Ü4

7) ges: $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}$

(i) Fourierreihe von $f(x)=|x|$

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |x| e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\pi}^{0} -x e^{-inx} dx + \int_{0}^{\pi} x e^{-inx} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{0}^{\pi} x e^{-inx} dx - \int_{-\pi}^{0} x e^{-inx} dx \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-inx}(1+inx)}{n^2} \Big|_0^{\pi} - \frac{e^{-inx}(1+inx)}{n^2} \Big|_{-\pi}^{0} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-in\pi}(1+in\pi)}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{e^{in\pi}(1-in\pi)}{n^2} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{(-1)^n (1+in\pi+1-in\pi)}{n^2} - \frac{2}{n^2} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{n^2} ((-1)^n - 1)$$

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\sqrt{2\pi}} \left(\int_0^{\pi} x dx - \int_{-\pi}^{0} x dx \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{\pi^2}{2} + \frac{\pi^2}{2} \right) = \frac{\pi^2}{\sqrt{2\pi}}$$

(ii) $\|f\|_2$

$$\|f\|_2^2 = \int_{-\pi}^{\pi} f^2(x) dx = \int_{-\pi}^{\pi} x^2 dx = \frac{2}{3}\pi^3 \qquad \Rightarrow \|f\|_2 = \sqrt{\frac{2}{3}\pi^3}$$

(iii) Parsevalsche Gleichung um $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}$ zu berechnen

$$\frac{2}{3}\pi^3 = \|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\langle f, D_n \rangle|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \left|\frac{\pi^2}{\sqrt{2\pi}}\right|^2 + 2 \sum_{n=1}^{\infty} \left| \frac{2}{\sqrt{2\pi} n^2} ((-1)^n - 1) \right|^2$$

$$= \left|\frac{\pi^2}{\sqrt{2\pi}}\right|^2 + 2 \sum_{n=1}^{\infty} \frac{4}{2\pi n^4} |1-(-1)^n|^2 \qquad |1-(-1)^n| = \begin{cases} 0, & n \text{ gerade} \\ 2, & n \text{ ungerade} \end{cases}$$

$$= \left|\frac{\pi^2}{\sqrt{2\pi}}\right|^2 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^4} 4 \cdot 1\!\!1_{2\mathbb{N}-1}(n) = \frac{\pi^4}{2\pi} + \frac{16}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^3}{2} + \frac{16}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}$$

$$\Rightarrow \frac{16}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{2}{3}\pi^3 - \frac{\pi^3}{2} = \pi^3 \left(\frac{4}{6} - \frac{3}{6} \right) = \pi^3 \frac{1}{6}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^3}{6} \frac{\pi}{16} = \frac{\pi^4}{96}$$

(iv) Berechnung von $\zeta(4)$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{(2n)^4} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{1}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=0}^{\infty} \frac{1}{(2(n+1)-1)^4} = \frac{1}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}$$

$$= \frac{1}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} + \frac{\pi^4}{96}$$

$$\Rightarrow \frac{\pi^4}{96} = \sum_{n=1}^{\infty} \frac{1}{n^4} \left(1 - \frac{1}{16}\right) \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96} \frac{16}{15} = \frac{\pi^4}{90}$$

$\square$

ANA Ü4

8) f heißt Hölder-stetig mit Exponent $\alpha \in (0,1]$ $:\Longleftrightarrow \exists C\ \forall x,y: |f(x)-f(y)| \leq C|x-y|^{\alpha}$

$f \in L^1(\mathbb{T})$ ... auf (a,b) Hölder-stetig mit Exponent $\alpha \in (0,1]$ mit α bel. aber fest

zz: Fourierreihe von f konvergiert auf (a,b) punktweise.

Sei $x_0 \in [-\pi, \pi]$ bel. $\quad c = f(x_0) \quad$ f ist auf $\mathbb{T}$ integrierbar

Klingt nach Dini-Test, also Frage ist $t \mapsto \frac{1}{t}\left|\frac{f(x_0-t)+f(x_0+t)}{2} - c\right|$ auf $[0,\pi]$ integrierbar?

$$\int_0^\pi \frac{1}{t}\left|\frac{f(x_0-t)+f(x_0+t)}{2} - c\right| dt = \int_0^\pi \left|\frac{f(x_0+t)-2f(x_0)+f(x_0-t)}{2t}\right| dt$$

$$\leq \frac{1}{2}\left(\int_0^\pi \left|\frac{f(x_0+t)-f(x_0)}{t}\right| dt + \int_0^\pi \left|\frac{-f(x_0)+f(x_0-t)}{t}\right| dt\right) \leq \int_0^\pi \left|\frac{f(x_0+t)-f(x_0)}{t}\right| dt + \int_0^\pi \left|\frac{f(x_0)-f(x_0-t)}{t}\right| dt$$

$$= \int_0^\pi \left|\frac{f(x_0+t)-f(x_0)}{t}\right| dt - \int_0^{-\pi} \left|\frac{f(x_0)-f(x_0+u)}{-u}\right| du \qquad \left[\begin{array}{l} u = -t \\ \frac{du}{dt} = -1 \end{array}\right.$$

$$= \int_0^\pi \left|\frac{f(x_0+t)-f(x_0)}{t}\right| dt + \int_{-\pi}^0 \left|\frac{f(x_0+u)-f(x_0)}{t}\right| du = \int_{-\pi}^\pi \left|\frac{f(x_0+t)-f(x_0)}{t}\right| dt$$

$$\leq \int_{-\pi}^\pi \left|\frac{1}{t}\right| C |x_0+t-x_0|^{\alpha} dt = C\int_{-\pi}^\pi \left|\frac{t^{\alpha}}{t}\right| dt = C\int_{-\pi}^\pi |t^{\alpha-1}| dt = C\left(\int_{-\pi}^0 (-t)^{\alpha-1} dt + \int_0^\pi t^{\alpha-1} dt\right)$$

$$= C\left(-\int_\pi^0 u^{\alpha-1} du + \int_0^\pi t^{\alpha-1} dt\right) = C\left(\int_0^\pi u^{\alpha-1} du + \int_0^\pi t^{\alpha-1} dt\right) = 2C \left.\frac{t^{\alpha}}{\alpha}\right|_0^\pi \qquad \left[u = -t \quad \frac{du}{dt} = -1 \right.$$

$$= 2C\left(\frac{\pi^{\alpha}}{\alpha} - 0\right) = \frac{2C}{\alpha}\pi^{\alpha} < \infty$$

$\Rightarrow$ Nach Dini Test gilt $\lim_{n\to\infty} S_n f(x_0) = c = f(x_0)$

$\square$