ANA Ü3

1) $K: [0,1] \times [0,1] \to \mathbb{R}$ ... stetig $\quad T: C[0,1] \to C[0,1]$

$$f(x) \mapsto \int_0^1 K(x,y) f(y)\,dy$$

zz: T bildet nach $C[0,1]$ ab

Sei $f \in C[0,1]$ bel. $\quad K \in C[0,1]^2 \Rightarrow K(x,y)f(y) \in C[0,1]^2$

Da $[0,1]^2$ kompakt ist, ist $(x,y) \mapsto K(x,y)f(y)$ sogar gleichmäßig stetig.

$\Rightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall u,v \in [0,1] \; \forall x \in [0,1]: |u-v| < \delta \Rightarrow |K(u,y)f(y) - K(v,y)f(y)| < \varepsilon$

$\Rightarrow |Tf(u) - Tf(v)| = \left| \int_0^1 K(u,y)f(y)\,dy - \int_0^1 K(v,y)f(y)\,dy \right| = \left| \int_0^1 K(u,y)f(y) - K(v,y)f(y)\,dy \right|$

$\leq \int_0^1 |K(u,y)f(y) - K(v,y)f(y)|\,dy < \int_0^1 \varepsilon\,dy = \varepsilon \quad$ für $|u-v| < \delta$

$\Rightarrow T$ ist sogar glm. stetig auf $[0,1]$.

$M := \{ Tf : f \in C[0,1], \|f\|_\infty \leq 1 \}$ $\quad$ wobei $\|.\|_\infty$ ... Supremumsnorm

zz: M ist relativ kompakt $\quad$ also $\overline{M}$ ist kompakt

Satz von Arzelà-Ascoli: X ... kompakte Menge, Φ ... Familie stetiger Funktionen $X \to \mathbb{R}$ oder $X \to \mathbb{C}$

Φ ist totalbeschränkt in $(C(X), \|.\|_\infty)$, wenn Φ ... punktweise beschränkt, gleichgradig stetig

zz: $\{ \|Tf\|_\infty : Tf \in M \}$ ist beschränkt $\quad$ (= punktweise Beschränktheit)

Da T linear ist $\left(T(\lambda f + g)(x) = \int_0^1 K(x,y)(\lambda f(y) + g(y))\,dy = \lambda \int_0^1 K(x,y) f(y)\,dy + \int_0^1 K(x,y) g(y)\,dy \right.$

$\left. = \lambda Tf + Tg \right)$ und nach oben stetig $\Longrightarrow$ aus Satz 9.2.6 $\quad$ T ist beschränkt

$\Rightarrow \{ \|Tf\|_\infty : Tf \in M \} \leq \|T\| < \infty$ also beschränkt

zz: M ist gleichgradig stetig also $\forall x \in [0,1] \; \forall \varepsilon > 0 \; \exists V \in \mathcal{U}(x): \forall z \in V \; \forall f \in \Phi: |f(x) - f(z)| < \varepsilon$

Sei $x \in [0,1]$ bel. Sei $\varepsilon > 0$ bel. Wähle $V = U_\delta(x)$ mit $|x-y| < \delta \Rightarrow \|T1(x) - T1(y)\| < \varepsilon$

Sei $z \in V$ bel. Sei $Tf \in M$ bel.

$\|Tf(x) - Tf(z)\|_\infty = \| \int_0^1 K(x,y) \underbrace{f(y)}_{\leq 1} dy - \int_0^1 K(z,y) \underbrace{f(y)}_{\leq 1} dy \|_\infty \leq \| \int_0^1 K(x,y)\,dy - \int_0^1 K(z,y)\,dy \|_\infty$

$= \| T1(x) - T1(z) \|_\infty < \varepsilon$

$\Rightarrow M$ ist relativ kompakt

ANA Ü3

3) $$f_n(x) := \begin{cases} n, & x \in [0, \frac{1}{n}) \\ 0, & x \in [\frac{1}{n}, 1] \end{cases}$$

ges: $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$

für $n \in \mathbb{N}$: $\int_0^1 f_n(x)\,dx = \int_0^1 n\, \mathbb{1}_{[0,\frac{1}{n})}(x)\,dx = n \int_0^{\frac{1}{n}} dx = n\,\frac{1}{n} = 1$

$\Rightarrow \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$

ges: $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx$

Sei $x \in (0,1]$ bel. Wähle $N \in \mathbb{N}$, sodass $\frac{1}{N} \leq x$ (existiert, da $x > 0$)

$\Rightarrow \forall\, n \geq N$: $f_n(x) = 0$, da $0 < \frac{1}{n} \leq \frac{1}{N} \leq x \leq 1$

Für $x = 0$ gilt $f_n(x) = n$ aber $\{0\}$ ist Nullmenge.

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$ fast überall

$\Rightarrow \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0$

Warum gilt Satz 2.1.5 nicht?

Satz 2.1.5. Sei $f_n(x)$ für $n \in (a,b)$ eine Familie messbarer Funktionen auf einem Maßraum (Ω, μ).
Für μ-fast alle x sei $n \mapsto f_n(x)$ in n_0 stetig.
$\exists\, g$ ... integrierbar auf Ω $\exists\, \delta > 0\ \forall t$: $|n - n_0| < \delta$ μ-f.ü. $\Rightarrow |f_n| \leq g$
$\Rightarrow \lim_{n\to n_0} \int_\Omega f_n\,d\mu = \int_\Omega f_{n_0}\,d\mu$

$|f_n| = n\ \forall n \in \mathbb{N}$ $\Rightarrow$ da $|f_n| \leq g$ gilt $\forall n \in \mathbb{N}$: $g > n \Rightarrow g$ ist nicht beschränkt auf $[0,1]$ also gibt es kein passendes g.

ANA Ü3

4) $p,q>0$ zz: $\int_0^1 \frac{x^{p-1}}{1+x^q}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{p+nq} \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \int_0^1 \frac{x^{1-1}}{1+x^2}dx = \int_0^1 \frac{1}{1+x^2}dx = \tan^{-1}(x)\Big|_0^1 = \tan^{-1}(1)-\tan^{-1}(0) = \frac{\pi}{4}-0$$

zz: $\int_0^1 \frac{x^{p-1}}{1+x^q}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{p+nq}$ für $p,q>0$

Es gilt für $|x|^q<1$: $\sum_{n=0}^{\infty}(-1)^n x^{nq} = \frac{1}{1+x^q}$

$$\Rightarrow \int_0^a \frac{x^{p-1}}{1+x^q}dx = \int_0^a x^{p-1}\sum_{n=0}^{\infty}(-1)^n x^{nq}dx = \int_0^a \sum_{n=0}^{\infty}(-1)^n x^{nq+p-1}dx$$

$$= \int_0^a \sum_{n=0}^{\infty} x^{2nq+p-1}dx - \int_0^a \sum_{n=0}^{\infty} x^{(2n-1)q+p-1}dx = \sum_{n=0}^{\infty}\int_0^a x^{2nq+p-1}dx - \sum_{n=0}^{\infty}\int_0^a x^{(2n-1)q+p-1}dx$$

$$= \sum_{n=0}^{\infty}\frac{x^{2nq+p}}{2nq+p}\Big|_0^a - \sum_{n=0}^{\infty}\frac{x^{(2n-1)q+p}}{(2n-1)q+p}\Big|_0^a = \sum_{n=0}^{\infty}(-1)^n\frac{x^{nq+p}}{nq+p}\Big|_0^a = \sum_{n=0}^{\infty}(-1)^n\frac{a^{nq+p}}{nq+p}$$

$$\lim_{a\to 1}\int_0^a \frac{x^{p-1}}{1+x^q}dx = \lim_{a\to 1}\sum_{n=0}^{\infty}(-1)^n\frac{a^{nq+p}}{nq+p} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{nq+p} \qquad \square$$

ANA Ü3

5) $f \in L^1(\mathbb{R}^n)$ fest $\quad T_f : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, $T_f(g) := f * g$

zz: T_f ist stetig

$$T_f(g)(x) := \int_{\mathbb{R}^n} f(x-y) g(y) \, d\lambda^n(y)$$

stetig: $\forall g \in L^1(\mathbb{R}^n) \, \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall h \in L^1(\mathbb{R}^n): \|g-h\|_1 < \delta \Rightarrow \|T_f(g) - T_f(h)\|_1 < \varepsilon$

Sei $g \in L^1(\mathbb{R}^n)$ bel. Sei $\varepsilon > 0$ bel.

Wähle $\delta := \frac{\varepsilon}{\|f\|_1}$

Sei $h \in L^1(\mathbb{R}^n)$ mit $\|g-h\|_1 < \delta$ bel.

$\Rightarrow \|T_f(g) - T_f(h)\|_1 = \|f*g - f*h\|_1 = \left\| \int_{\mathbb{R}^n} f(x-y) g(y) \, d\lambda^n(y) - \int_{\mathbb{R}^n} f(x-y) h(y) \, d\lambda^n(y) \right\|_1$

$= \left\| \int_{\mathbb{R}^n} f(x-y) (g-h)(y) \, d\lambda^n(y) \right\|_1 = \|f * (g-h)\|_1 \leq \|f\|_1 \|g-h\|_1 < \|f\|_1 \delta = \varepsilon$

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ANA Ü3

6) $1 \le p, q, r \le \infty$ mit $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ bel. (wobei $\frac{1}{\infty} := 0$)

$f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$

zz: $\|f * g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$ ... Young'sche Faltungsungleichung

$$\|f\|_p = \|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f|^p \, d\lambda^n\right)^{\frac{1}{p}}$$

Hölder Ungleichung: $\sum_{i=1}^{k} \frac{1}{p_i} = 1 \Rightarrow \left\| \prod_{i=1}^{k} f_i \right\|_1 \le \prod_{i=1}^{k} \|f_i\|_{p_i}$

$$|f * g(x)| \le \int_{\mathbb{R}^n} |f(x-y)||g(y)| \, d\lambda^n(y) = \int_{\mathbb{R}^n} |f(x-y)|^{1+\frac{p}{r}-\frac{p}{r}} |g(y)|^{1+\frac{q}{r}-\frac{q}{r}} \, d\lambda^n(y)$$

$$= \int_{\mathbb{R}^n} |f(x-y)|^{\frac{p}{r}} |g(y)|^{\frac{q}{r}} |f(x-y)|^{1-\frac{p}{r}} |g(y)|^{1-\frac{q}{r}} \, d\lambda^n(y)$$

$$\left[\frac{1}{r} + \frac{r-p}{pr} + \frac{r-q}{qr} = \frac{1}{r} + \frac{1}{p} - \frac{1}{r} + \frac{1}{q} - \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1 \right.$$

$$= \int_{\mathbb{R}^n} \left(|f(x-y)|^p |g(y)|^q\right)^{\frac{1}{r}} |f(x-y)|^{\frac{r-p}{r}} |g(y)|^{\frac{r-q}{r}} \, d\lambda^n(y)$$

$$\overset{\text{Hölder}}{\le} \underbrace{\left\| \left(|f(x-y)|^p |g(y)|^q\right)^{\frac{1}{r}} \right\|_r}_{} \left\| |f(x-y)|^{\frac{r-p}{r}} \right\|_{\frac{pr}{r-p}} \cdot \underbrace{\left\| |g(y)|^{\frac{r-q}{r}} \right\|_{\frac{qr}{r-q}}}_{}$$

$$= \left(\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q \, d\lambda^n(y)\right)^{\frac{1}{r}}$$

$$= \left(\int_{\mathbb{R}^n} |f(x-y)|^{\frac{r-p}{r} \frac{pr}{r-p}} \, d\lambda^n(y)\right)^{\frac{r-p}{pr}}$$
$$= \left(\int_{\mathbb{R}^n} |f(x-y)|^p \, d\lambda^n(y)\right)^{\frac{1}{p} \frac{r-p}{r}}$$
$$= \|f\|_p^{\frac{r-p}{r}}$$

$$= \left(\int_{\mathbb{R}^n} |g(y)|^{\frac{r-q}{r} \frac{qr}{r-q}} \, d\lambda^n(y)\right)^{\frac{r-q}{qr}}$$
$$= \left(\int_{\mathbb{R}^n} |g(y)|^q \, d\lambda^n(y)\right)^{\frac{1}{q} \frac{r-q}{r}}$$
$$= \|g\|_q^{\frac{r-q}{r}}$$

$$\Rightarrow \|f * g\|_r^r = \int_{\mathbb{R}^n} |f * g(x)|^r \, d\lambda(x) \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q \, d\lambda^n(y) \, \|f\|_p^{r-p} \|g\|_q^{r-q} \, d\lambda^n(x)$$

da ≥ 0, Fubini: $= \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q \, d\lambda^n(x) \, d\lambda^n(y)$ $\quad [v = x - y \quad d\lambda^n(x) = d\lambda^n(v)$

$$= \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}^n} |g(y)|^q \int_{\mathbb{R}^n} |f(v)|^p \, d\lambda^n(v) \, d\lambda^n(y) = \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}^n} |g(y)|^q \|f\|_p^p \, d\lambda^n(y)$$

$$= \|f\|_p^r \|g\|_q^{r-q} \int_{\mathbb{R}^n} |g(y)|^q \, d\lambda^n(y) = \|f\|_p^r \|g\|_q^{r-q} \|g\|_q^q = \|f\|_p^r \|g\|_q^r$$

$$\Rightarrow \|f * g\|_r \le \|f\|_p \|g\|_q$$

□

ANA Ü3

7) $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$ $\quad Z := X + Y$

ges: Dichte von Z

$$f_Z(z) = f_X * f_Y(z) = \int_{\mathbb{R}} f_X(z-y) f_Y(y) \, d\lambda(y)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \qquad f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right)$$

$$\Rightarrow f_Z(z) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(z-y-\mu_1)^2}{2\sigma_1^2}\right) \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right) d\lambda(y)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \int_{\mathbb{R}} \exp\left(-\frac{(z-y-\mu_1)^2}{2\sigma_1^2} - \frac{(y-\mu_2)^2}{2\sigma_2^2}\right) d\lambda(y)$$

$$\frac{(z-y-\mu_1)^2}{2\sigma_1^2} + \frac{(y-\mu_2)^2}{2\sigma_2^2} = \frac{1}{2}\left(\frac{z^2 - 2yz - 2\mu_1 z + 2\mu_1 y + y^2 + \mu_1^2}{\sigma_1^2} + \frac{y^2 - 2y\mu_2 + \mu_2^2}{\sigma_2^2}\right)$$

$$= \frac{1}{2}\left(y^2\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) - 2y\left(\frac{\mu_1 - z}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2}\right) + \left(\frac{(z-\mu_1)^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2}\right)\right)$$

$$= \frac{1}{2} \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}\left(y^2 - 2y\left(\frac{(\mu_1 - z)\sigma_2^2 + \mu_2^2\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right) + \left(\frac{(z-\mu_1)^2\sigma_2^2 + \mu_2^2\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)\right)$$

$$= \frac{1}{2} \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}\left(\left(y - \left(\frac{(z-\mu_1)\sigma_2^2 + \mu_2^2\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)\right)^2 + \underbrace{\left(\frac{(z-\mu_1)^2\sigma_2^2 + \mu_2^2\sigma_1^2}{\sigma_1^2 + \sigma_2^2} - \left(\frac{(z-\mu_1)\sigma_2^2 + \mu_2^2\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right)}_{\frac{\sigma_1^2\sigma_2^2 (z-\mu_1-\mu_2)^2}{(\sigma_1^2 + \sigma_2^2)^2}}\right)$$

$$\Rightarrow f_Z(z) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2}\frac{(z-\mu_1-\mu_2)^2}{\sigma_1^2 + \sigma_2^2}\right) \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}\left(y - \frac{(z-\mu_1)\sigma_2^2 + \mu_2^2\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right) d\lambda(y)$$

substitution $s = \frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{\sqrt{2}\sigma_1\sigma_2}\left(y - \frac{(z-\mu_1)\sigma_2^2 + \mu_2^2\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right) \qquad d\lambda(s) = \frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{\sqrt{2}\sigma_1\sigma_2} d\lambda(y)$

$$\Rightarrow f_Z(z) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2}\frac{(z-\mu_1-\mu_2)^2}{\sigma_1^2 + \sigma_2^2}\right) \int_{\mathbb{R}} \exp(-s^2) \frac{\sqrt{2}\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} d\lambda(s)$$

$$= \frac{\sqrt{2}\sigma_1\sigma_2}{2\pi\sigma_1\sigma_2\sqrt{\sigma_1^2 + \sigma_2^2}} \exp\left(-\frac{1}{2}\frac{(z-\mu_1-\mu_2)^2}{\sigma_1^2 + \sigma_2^2}\right) \underbrace{\int_{\mathbb{R}} \exp(-s^2) d\lambda(s)}_{= \sqrt{\pi}} = \frac{\sqrt{2\pi}}{2\pi\sqrt{\sigma_1^2 + \sigma_2^2}} \exp\left(-\frac{1}{2}\frac{(z-\mu_1-\mu_2)^2}{\sigma_1^2 + \sigma_2^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 + \sigma_2^2}} \exp\left(-\frac{(z-\mu_1-\mu_2)^2}{2(\sigma_1^2 + \sigma_2^2)}\right)$$

Ist Dichte von Normalverteilung mit $\mu = \mu_1 + \mu_2$ und $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$.

ANA Ü3

8) $f,g \in L^\infty([-1,1])$ $\quad \langle f,g\rangle := \int_{-1}^{1} f(x)\,g(x)\frac{1}{\sqrt{1-x^2}}\,dx$

zz: $\langle .,. \rangle$ ist Skalarprodukt

(i) $\langle f+g,h\rangle = \langle f,h\rangle + \langle g,h\rangle$

$$\langle f+g,h\rangle = \int_{-1}^{1}(f(x)+g(x))h(x)\frac{1}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} f(x)h(x)\frac{1}{\sqrt{1-x^2}}\,dx + \int_{-1}^{1} g(x)h(x)\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \langle f,h\rangle + \langle g,h\rangle$$

(ii) $\langle \lambda f,g\rangle = \lambda\langle f,g\rangle$

$$\langle \lambda f,g\rangle = \int_{-1}^{1}\lambda f(x)g(x)\frac{1}{\sqrt{1-x^2}}\,dx = \lambda\int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1-x^2}}\,dx = \lambda\langle f,g\rangle$$

(iii) $\langle f,g\rangle = \overline{\langle g,f\rangle}$

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} g(x)f(x)\frac{1}{\sqrt{1-x^2}}\,dx = \langle g,f\rangle = \overline{\langle g,f\rangle}$$

(iv) $\langle f,f\rangle > 0$ für $f \neq 0$

$$\langle f,f\rangle = \int_{-1}^{1} f(x)f(x)\frac{1}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1}\underbrace{f^2(x)}_{>0}\underbrace{\frac{1}{\sqrt{1-x^2}}}_{>0}\,dx > 0$$

(wohldefiniert) $\langle f,g\rangle < \infty$

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1-x^2}}\,dx \leq \int_{-1}^{1}\|f\|_\infty\|g\|_\infty\frac{1}{\sqrt{1-x^2}}\,dx = \|f\|_\infty\|g\|_\infty\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx = \|f\|_\infty\|g\|_\infty\pi < \infty$$

$T_n(x) := \cos(n\arccos(x))$, $n\in\mathbb{N}$ $\quad$ zz: $(T_n)_{n\in\mathbb{N}}$ ist Orthogonalsystem bzgl. $\langle .,.\rangle$

Sei $n,m\in\mathbb{N}$, $n\neq m$ bel. $\langle T_n,T_m\rangle = \int_{-1}^{1}\cos(n\arccos(x))\cos(m\arccos(x))\frac{1}{\sqrt{1-x^2}}\,dx$

$$= +\int_0^\pi \cos(nu)\cos(mu)\,du = +\int_0^\pi \tfrac{1}{2}(\cos((m-n)u)+\cos((m+n)u))\,du \qquad \left[\, u=\arccos(x) \quad \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}\right.$$

$$= +\frac{1}{2}\left(\int_0^{\pi(m-n)}\cos(a)\frac{1}{m-n}\,da + \int_0^{\pi(m+n)}\cos(b)\frac{1}{m+n}\,db\right) \qquad \left[\, a=(m-n)u \quad \frac{da}{du}=m-n\right.$$

$$= +\frac{1}{2}\left(\frac{1}{m-n}\sin(a)\Big|_0^{\pi(m-n)} + \frac{1}{m+n}\sin(b)\Big|_0^{\pi(m+n)}\right) \qquad \left[\, b=(m+n)u \quad \frac{db}{du}=m+n\right.$$

$$= +\frac{1}{2}\left(\frac{1}{m-n}(0-0) + \frac{1}{m+n}(0-0)\right) = 0$$

Für $m=n$: $\langle T_n,T_n\rangle = +\frac{1}{2}\int_0^\pi \cos(0)+\cos(2nu)\,du = +\frac{1}{2}\left(\int_0^\pi 1\,du + \int_0^\pi\cos(2nu)\,du\right)$

$$= +\frac{1}{2}\left(\pi + \int_0^{2n\pi}\frac{1}{2n}\cos(a)\,da\right) = +\frac{\pi}{2} + \frac{1}{4n}\sin(a)\Big|_0^{2n\pi} = +\frac{\pi}{2} \neq 0 \qquad \left[\, a=2nu \quad \frac{da}{du}=2n\right.$$

---

ges: $\|T_n\| := \sqrt{\langle T_n,T_n\rangle}$

$$\|T_n\| = \sqrt{\langle T_n,T_n\rangle} = \sqrt{\frac{\pi}{2}}$$