ANA Ü9

2) ges: $I := \int_{\mathbb{R}} e^{-x^2}\,dx$ indem I^2 als Integral über $\mathbb{R}^2$ dargestellt wird und Polarkoordinaten verwenden

Lemma $\left(\int_{\mathbb{R}} f(x)\,dx\right)\left(\int_{\mathbb{R}} g(y)\,dy\right) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\,g(y)\,dy\,dx$ falls existent

Bew $\int_{\mathbb{R}} f(x)\,dx =: a \in \mathbb{R} \Rightarrow a \cdot \int_{\mathbb{R}} g(y)\,dy = \int_{\mathbb{R}} a\,g(y)\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\,dx\,g(y)\,dy$

$\forall y \in \mathbb{R}: g(y) = b \in \mathbb{R} \Rightarrow \int_{\mathbb{R}} f(x)\,dx\; b = \int_{\mathbb{R}} f(x)\,b\,dx = \int_{\mathbb{R}} f(x)\,g(y)\,dx$

$\Rightarrow \left(\int_{\mathbb{R}} f(x)\,dx\right)\left(\int_{\mathbb{R}} g(y)\,dy\right) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\,g(y)\,dx\,dy$ $\square$

$$\Rightarrow I^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2} e^{-y^2}\,dx\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)}\,dx\,dy$$

$$\varphi: (0,\infty)\times[0,2\pi) \to \mathbb{R}^2 \qquad d\varphi = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \qquad \det d\varphi = r\cos^2\theta + r\sin^2\theta = r$$

$$(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

Transformationsformel $\int_{\mathbb{R}^2 = \varphi((0,\infty)\times[0,2\pi))} e^{-(x^2+y^2)}\,d\lambda^2(x,y) = \int_{(0,\infty)\times[0,2\pi)} f(\varphi(r,\theta))\cdot|\det d\varphi|\,d\lambda^2(r,\theta)$

$$= \int_0^\infty\int_0^{2\pi} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}\, r\,d\lambda\theta\,d\lambda r = \int_0^\infty\int_0^{2\pi} e^{-r^2}\, r\,d\lambda\theta\,dr = 2\pi\int_0^\infty e^{-r^2}\,r\,dr$$

$\left[u = r^2 \quad \frac{du}{dr} = 2r \right.$

$$= 2\pi\int_0^\infty e^{-u}\,\frac{1}{2}\,du = \pi \; -e^{-u}\Big|_0^\infty = \pi(-0+1) = \pi$$

$$\Rightarrow I^2 = \pi \quad \Rightarrow I = \sqrt{\pi}$$

ANA Ü9

3) $A \in \mathbb{R}^{n\times n}$ ... symmetrische, pos. definite Matrix $\quad Q(x) = x^T A x \quad ; x \in \mathbb{R}^n$

ges: $\int_{\mathbb{R}^n} e^{-Q(x)} dx$

A ... symmetrisch, pos. definit $\Rightarrow \exists O$ ... orthogonal $\exists D = \text{diag}(d_1, \dots, d_n) : A = O^T D O \quad \forall i : d_i \geq 0$

$$\Rightarrow Q(x) = x^T A x = x^T O^T D O x = (Ox)^T D (Ox) = y^T D y \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := Ox$$

$$y^T D y = \sum_{i=1}^n d_i y_i^2 = \sum_{i=1}^n (\sqrt{d_i}\, y_i)^2$$

$$\int_{\mathbb{R}^n = \varphi(\mathbb{R}^n)} e^{-Q(y)} dy = \int_{\mathbb{R}^n} e^{-Q(Ox)} |\det O| dx \qquad \varphi : \mathbb{R}^n \to \mathbb{R}^n \quad x \mapsto Ox$$

$$\det d\varphi = \det O$$

$$= |\det O| \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n (\sqrt{d_i}\, y_i)^2} dy = |\det O| \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-(\sqrt{d_i}\, y_i)^2} dy = |\det O| \prod_{i=1}^n \int_{\mathbb{R}} e^{-(\sqrt{d_i}\, y_i)^2} dy_i$$

$$= |\det O| \prod_{i=1}^n \int_{\mathbb{R}} e^{-u^2} \frac{1}{\sqrt{d_i}} du = |\det O| \prod_{i=1}^n \frac{1}{\sqrt{d_i}} \int_{\mathbb{R}} e^{-u^2} du \qquad \left[u = \sqrt{d_i}\, y_i \quad \frac{du}{dy_i} = \sqrt{d_i} \right.$$

$$= |\det O| \prod_{i=1}^n \frac{1}{\sqrt{d_i}} \sqrt{\pi} = |\det O| \sqrt{\pi}^n \sqrt{\frac{1}{\prod_{i=1}^n d_i}}$$

$$= |\det O| \sqrt{\pi}^n \sqrt{\det D^{-1}} = \sqrt{\det O^T \det O \det D^{-1}}\, \sqrt{\pi}^n = \sqrt{\det(O^T D^{-1} O)}\, \sqrt{\pi}^n$$

$$= \sqrt{\det(A^{-1})\, \pi^n} = \sqrt{\frac{\pi^n}{\det(A)}}$$

---

oder auch

$$\varphi : \mathbb{R}^n \to \mathbb{R}^n \quad x \mapsto Ox \qquad \mathbb{R}^n = \varphi(\mathbb{R}^n) \qquad d\varphi(x) = O \quad \Rightarrow \sqrt{\det d\varphi^T d\varphi} = \sqrt{\det O^T O}$$

$$= \sqrt{\det O^{-1} O} = \sqrt{\det E_n} = 1$$

$$\int_{\mathbb{R}^n} e^{-Q(x)} dx = \int_{\varphi(\mathbb{R}^n)} e^{-x^T A x} dx = \int_{\mathbb{R}^n} e^{-\varphi(x)^T A \varphi(x)} \cdot 1\, dx \qquad A = O^T D O \Rightarrow D = O^T A O$$

$$= \int_{\mathbb{R}^n} e^{-(Ox)^T A (Ox)} dx = \int_{\mathbb{R}^n} e^{-x^T (O^T A O) x} dx = \int_{\mathbb{R}^n} e^{-x^T D x} dx$$

$$= \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n (\sqrt{d_i}\, x_i)^2} d\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \prod_{i=1}^n \int_{\mathbb{R}} e^{-(\sqrt{d_i}\, x_i)^2} dx_i \qquad u = \sqrt{d_i}\, x_i \quad \frac{du}{dx} = \sqrt{d_i}$$

$$= \prod_{i=1}^n \frac{1}{\sqrt{d_i}} \int_{\mathbb{R}} e^{-u^2} du \overset{2)}{=} \prod_{i=1}^n \frac{\sqrt{\pi}}{\sqrt{d_i}} = \sqrt{\frac{\pi^n}{\prod_{i=1}^n d_i}}$$

ANA Ü9

$0<R<r$

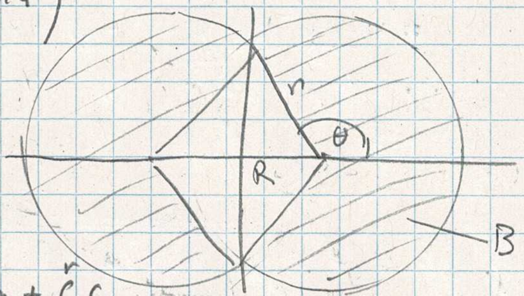
4) Spindeltorus $S:=\{((R+r\cos\theta)\cos\phi,\ (R+r\cos\theta)\sin\phi,\ r\sin\theta) : \theta\in(-\pi,\pi), \phi\in[0,2\pi), R+r\cos\theta\geq 0\}$

$R+r\cos\theta\geq 0 \Leftrightarrow |\theta|\leq|\arccos(-\frac{R}{r})| \Rightarrow \theta\in(-\arccos(-\frac{R}{r}), \arccos(-\frac{R}{r}))=: D_2$

$D:=(0,r)\times D_2\times[0,2\pi)$ ...offen

$\varphi: D\to\mathbb{R}^3 \quad \varphi(a,\theta,\phi):=((R+a\cos\theta)\cos\phi, (R+a\cos\theta)\sin\phi, a\sin\theta)$ ...injektiv

$$d\varphi=\begin{pmatrix}\cos\theta\cos\phi & -(R+a\cos\theta)\sin\phi & -a\sin\theta\cos\phi\\ \cos\theta\sin\phi & (R+a\cos\theta)\cos\phi & -a\sin\theta\sin\phi\\ \sin\theta & 0 & a\cos\theta\end{pmatrix} \quad \det d\varphi=(R+a\cos\theta)a\overset{*}{>}0$$



$B=\varphi(D)$

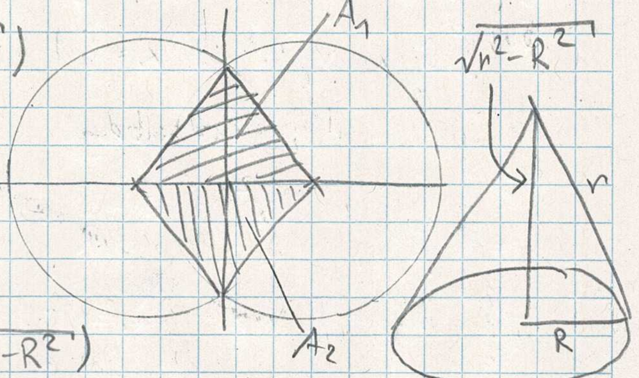
$$\mathcal{H}^3(B)=\int_D|\det d\varphi|\,d\lambda^3=\int_D Ra+a^2\cos\theta\,d\lambda^3$$

$$=\int_0^r\int_{D_2}\int_0^{2\pi}Ra+a^2\cos\theta\,d\phi\,d\theta\,da=2\pi\Big(R\int_0^r\int_{D_2}a\,d\theta\,da+\int_0^r\int_{D_2}a^2\cos\theta\,d\theta\,da\Big)$$

$$\int_0^r\int_{D_2}a\,d\theta\,da=2\arccos(-\tfrac{R}{r})\int_0^r a\,da=2\arccos(-\tfrac{R}{r})\frac{r^2}{2}=r^2\arccos(-\tfrac{R}{r})$$

$$\int_0^r\int_{D_2}a^2\cos\theta\,d\theta\,da=\int_0^r a^2\,2\underbrace{\sin(\arccos(-\tfrac{R}{r}))}_{=\sqrt{1-\frac{R^2}{r^2}}}\,da=2\sqrt{\frac{r^2-R^2}{r^2}}\int_0^r a^2\,da=\frac{2}{r}\sqrt{r^2-R^2}\,\frac{r^3}{3}=\frac{2}{3}r^2\sqrt{r^2-R^2}$$

$$\Rightarrow\mathcal{H}^3(B)=2\pi\big(Rr^2\arccos(-\tfrac{R}{r})+\tfrac{2}{3}r^2\sqrt{r^2-R^2}\big)$$



$$\mathcal{H}^3(A_1\cup A_2)=2\,\mathcal{H}^3(A_1)=2\underbrace{\tfrac{1}{3}R^2\pi\sqrt{r^2-R^2}}_{\text{Kegelvolumen}}$$

$$\Rightarrow\mathcal{H}^3(S)=\mathcal{H}^3(B)+\mathcal{H}^3(A_1\cup A_2)$$

$$=2\pi\big(Rr^2\arccos(-\tfrac{R}{r})+\tfrac{2}{3}r^2\sqrt{r^2-R^2}+\tfrac{1}{3}R^2\sqrt{r^2-R^2}\big)$$

$*$ Ist $R+a\cos\theta>0$? $\quad 0<R<r,\ 0<a<r,\ \theta\in(-\arccos(-\frac{R}{r}),\arccos(-\frac{R}{r}))\subset(-\pi,\pi)$

Minimum, wenn $\cos\theta$ Minimal und das ist bei $\pm\arccos(-\frac{R}{r})$

$$R+a\cos(\pm\arccos(-\tfrac{R}{r}))=R+a(-\tfrac{R}{r})=R(1-\tfrac{a}{r})>0$$

$$\Rightarrow\forall\theta\in D_2: R+a\cos\theta>0$$

ANA Ü9

5) ges: Flächenmaß von S ... Spindeltorus

$$f(x,y,z) := \sqrt{\left(\sqrt{x^2+y^2}-R\right)^2+z^2}$$

$$\nabla f(x,y,z) = \left(\frac{x\left(\sqrt{x^2+y^2}-R\right)}{\sqrt{x^2+y^2}\sqrt{\left(\sqrt{x^2+y^2}-R\right)^2+z^2}}, \frac{y\left(\sqrt{x^2+y^2}-R\right)}{\sqrt{x^2+y^2}\sqrt{\left(\sqrt{x^2+y^2}-R\right)^2+z^2}}, \frac{z}{\sqrt{\left(\sqrt{x^2+y^2}-R\right)^2+z^2}}\right)$$

$$|\nabla f|^2 = \frac{x^2\left(\sqrt{x^2+y^2}-R\right)^2+y^2\left(\sqrt{x^2+y^2}-R\right)^2+z^2\left(x^2+y^2\right)}{\left(x^2+y^2\right)\left(\left(\sqrt{x^2+y^2}-R\right)^2+z^2\right)}$$

$$= \frac{\left(x^2+y^2\right)\left(\left(\sqrt{x^2+y^2}-R\right)^2+z^2\right)}{\left(x^2+y^2\right)\left(\left(\sqrt{x^2+y^2}-R\right)^2+z^2\right)} = 1 \quad \Rightarrow |\nabla f| = 1$$

$$g(x) := 1\!\!1_{[R,r]}(f(x))$$

$$f(\varphi(a,\theta,\phi)) = \sqrt{\left(\sqrt{(R+a\cos\theta)^2\cos^2\phi+(R+a\cos\theta)\sin^2\phi}-R\right)^2+a^2\sin^2\theta}$$

$$= \sqrt{\left(\sqrt{(R+a\cos\theta)^2}-R\right)^2+a^2\sin^2\theta} = \sqrt{(R+a\cos\theta-R)^2+a^2\sin^2\theta}$$

$$= \sqrt{a^2\cos^2\theta+a^2\sin^2\theta} = \sqrt{a^2} = a$$

$\tilde{f} := f\circ\varphi(y) = a \qquad \tilde{g} := 1\!\!1_{[R,r]}(f\circ\varphi(y)) = 1\!\!1_{[R,r]}(a) \qquad \nabla\tilde{f} = \sqrt{1^2+0^2+0^2} = 1$

$$\int_S \tilde{g}(x)\,|\nabla\tilde{f}(x)|\,d\lambda^3(x) = \int_S 1\!\!1_{[R,r]}(a)\,d\lambda^3(x) = V(r)-V(R)$$

$\Vert$ Koflächenformel

$$\int_{\mathbb{R}}\int_{\tilde{f}^{-1}(t)} \tilde{g}(y)\,d\mathcal{H}^2(y)\,dt = \int_{\mathbb{R}}\int_{\tilde{f}^{-1}(t)} 1\!\!1_{[R,r]}(a)\,d\mathcal{H}^2(y)\,dt = \int_0^r \mathcal{H}^2(\partial S)\,dt$$

$$\mathcal{H}^2(\partial S) = \frac{d}{dr}V(r) \overset{*}{=} 4\pi r\left(\sqrt{r^2-R^2}+R\arccos\left(-\frac{R}{r}\right)\right)$$

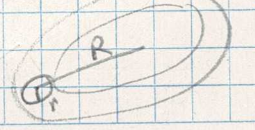
Tests: Spezialfall $R=0$ sollte Kugel sein

$4\pi r\left(\sqrt{r^2}\right) = 4\pi r^2$ ist tatsächlich Kugeloberfläche.

Spezialfall $R=r$ sollte Torus mit Lochgröße $=0$ sein

$4\pi r(r\pi) = 4\pi^2 r^2$

Torusoberfläche ist $4\pi^2 rR$ also mit $R=r$ gleich $4\pi^2r^2$ ✓



$*$ bei Bsp 6)

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6) $f \in L^2([-\pi, \pi))$ 2π-periodisch fortgesetzt auf $\mathbb{R}$ $\alpha \in [-\pi, \pi)$, $\beta \in [\alpha, \alpha + 2\pi)$

$S := \{(r\cos\phi, r\sin\phi) \in \mathbb{R}^2 : \alpha < \phi < \beta,\ 0 < r < f(\phi)\}$

ZZ: $\mathcal{H}^2(S) = \frac{1}{2}\int_\alpha^\beta f^2(\phi)\,d\phi$

$\varphi : (0,\infty) \times (\alpha, \beta) \longrightarrow \mathbb{R}^2$ ...inj. $\quad d\varphi = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}$ $\quad \det d\varphi = r(\cos^2\phi + \sin^2\phi) = r$

$(r, \phi) \mapsto (r\cos\phi, r\sin\phi)$ $\qquad \Rightarrow C_1$

$D = \{(r, \phi) \in (0,\infty) \times (\alpha, \beta) : r < f(\phi)\}$

$$\mathcal{H}^2(S) = \int_{\varphi(D)} d\mathcal{H}^2(y) = \int_D |\det d\varphi|\, d\lambda^2 = \int_\alpha^\beta \int_0^{f(\phi)} r\,dr\,d\phi = \int_\alpha^\beta \frac{1}{2} r^2 \Big|_0^{f(\phi)} d\phi$$

$$= \frac{1}{2}\int_\alpha^\beta f^2(\phi)\,d\phi$$

$\square$

Bsp 5)

$$* \frac{d}{dr}V(r) = 2\pi\Big(R\,2r\arccos\left(-\tfrac{R}{r}\right) - Rr^2 \frac{1}{\sqrt{1-\frac{R^2}{r^2}}} R \frac{1}{r^2} + \frac{2}{3}2r\sqrt{r^2-R^2} - \frac{2}{3}r^2 \frac{1}{2}\frac{1}{\sqrt{r^2-R^2}}2r$$

$$+ \frac{1}{3}R^2 \frac{1}{2}\sqrt{r^2-R^2}\,2r\Big)$$

$$= 2\pi\left(2rR\arccos\left(-\tfrac{R}{r}\right) - \frac{R^2}{\sqrt{r^2-R^2}} + \frac{2}{3}\frac{(3r^3-2rR^2)}{\sqrt{r^2-R^2}} + \frac{rR^2}{3\sqrt{r^2-R^2}}\right)$$

$$= 2\pi\left(2rR\arccos\left(-\tfrac{R}{r}\right) + \frac{r\,1}{\sqrt{r^2-R^2}}\left(-R^2r + 2r^3 - \frac{4}{3}rR^2 + \frac{1}{3}rR^2\right)\right)$$

$$= 2\pi\left(2Rr\arccos\left(-\tfrac{R}{r}\right) + \frac{1}{\sqrt{r^2-R^2}}(2r^3 - R^2r\,2)\right)$$

$$= 4\pi r\left(R\arccos\left(-\tfrac{R}{r}\right) + \frac{1}{\sqrt{r^2-R^2}}(r^2-R^2)\right) = 4\pi r\left(R\arccos\left(-\tfrac{R}{r}\right) + \sqrt{r^2-R^2}\right)$$

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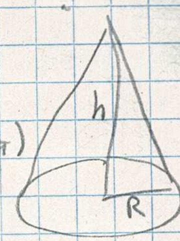
7) $R, h > 0$ $M := \{x \in \mathbb{R}^3 : 0 < z < h,\ x^2+y^2 = \frac{R^2}{h^2}(h-z)^2\}$



$\varphi : (0,R) \times [0, 2\pi) \longrightarrow \mathbb{R}^3$ ... inj. $D = (0,R) \times [0, 2\pi)$

$(r, \theta) \mapsto (r\cos\theta, r\sin\theta, h(1-\frac{r}{R}))$, da

$x^2+y^2 = \frac{R^2}{h^2}(h-z)^2 \Leftrightarrow r^2\cos^2\theta + r^2\sin^2\theta = \frac{R^2}{h^2}(h-z)^2 \Leftrightarrow r^2h^2 = R^2(h-z)^2$

$\Leftrightarrow \frac{rh}{R} = h-z \Leftrightarrow z = h - \frac{rh}{R}$

$$d\varphi = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ -\frac{h}{R} & 0 \end{pmatrix}$$

$$d\varphi^T d\varphi = \begin{pmatrix} \sin^2\theta + \cos^2\theta + \frac{h^2}{R^2} & -r\sin\theta\cos\theta + r\sin\theta\cos\theta \\ -r\sin\theta\cos\theta + r\sin\theta\cos\theta & r^2\sin^2\theta + r^2\cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{h^2}{R^2} & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\sqrt{\det d\varphi^T d\varphi} = \sqrt{r^2(1+\frac{h^2}{R^2})} = r\frac{\sqrt{R^2+h^2}}{R}$$

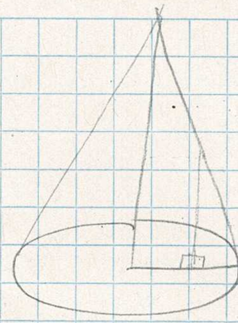
$$\mathcal{H}^2(M) = \int_{\varphi(D)} d\mathcal{H}^2 = \int_D \sqrt{\det d\varphi^T d\varphi}\, d\lambda^2$$

$$= \int_0^R \int_0^{2\pi} \frac{r}{R}\sqrt{R^2+h^2}\, d\theta\, dr$$

$$= \sqrt{R^2+h^2}\,\frac{1}{R}\int_0^{2\pi} d\theta \int_0^R r\, dr$$

$$= \sqrt{R^2+h^2}\,\frac{1}{R}\, 2\pi\, \frac{R^2}{2} = R\pi\sqrt{R^2+h^2}$$

ANA Ü9



8) ges: Flächenmaß des Kegelmantel M mittels Koflächenformel

wobei $V(R,h) = \frac{\pi R^2 h}{3}$

Idee: Kegel mit s in Größe verändern, dann darüber rausintegrieren.

$$M := \{(x,y,z) \in \mathbb{R}^3 : 0 < z < sh \;,\; x^2+y^2 = \frac{(sR)^2}{(sh)^2}(sh-z)^2\}$$

$$x^2+y^2 = \frac{(sR)^2}{(sh)^2}(sh-z)^2 \iff \sqrt{x^2+y^2} = \frac{R}{h}(sh-z) \iff \frac{h}{R}\sqrt{x^2+y^2} = sh-z$$

$$\iff \frac{h}{R}\sqrt{x^2+y^2}+z = sh \iff s = \frac{1}{R}\sqrt{x^2+y^2}+\frac{z}{h}$$

$$f: \mathbb{R}^3 \to \mathbb{R} \quad (x,y,z) \mapsto \frac{1}{R}\sqrt{x^2+y^2}+\frac{z}{h}$$

$$\nabla f = \left(\frac{1}{R}\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}2x \;,\; \frac{1}{R}\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}2y \;,\; \frac{1}{h}\right) = \left(\frac{x}{R\sqrt{x^2+y^2}}, \frac{y}{R\sqrt{x^2+y^2}}, \frac{1}{h}\right)$$

$$|\nabla f| = \sqrt{\frac{x^2+y^2}{R^2(x^2+y^2)}+\frac{1}{h^2}} = \sqrt{\frac{1}{R^2}+\frac{1}{h^2}} = \sqrt{\frac{h^2+R^2}{R^2h^2}} = \frac{1}{Rh}\sqrt{h^2+R^2}$$

$$g(x) = 1\!\!1_{(0,s)}(f(x))$$

$$\frac{\pi s^2R^2sh}{3} = s^3\frac{\pi R^2 h}{3} = \mathcal{H}^3(K(sR,sh)) = \int_{K(\dots)} d\lambda^3 = \frac{Rh}{\sqrt{h^2+R^2}}\int_{K(\dots)} g(x)\,|\nabla f|\,d\lambda^3(x)$$

$$= \frac{Rh}{\sqrt{h^2+R^2}}\int_{\mathbb{R}}\int_{f^{-1}(t)} g \, d\mathcal{H}^2\,dt = \frac{Rh}{\sqrt{h^2+R^2}}\int_0^s \mathcal{H}^2(\partial K(tR,th))\,dt$$

$$\Rightarrow \mathcal{H}^2(\partial K(sR,sh)) = \frac{\sqrt{h^2+R^2}}{Rh}\frac{\partial}{\partial s}s^3\frac{\pi R^2h}{3} = \frac{\sqrt{h^2+R^2}}{\cancel{Rh}}\frac{\pi R^{\cancel{2}}\cancel{h}}{\cancel{3}}\cancel{3}\cdot s^2$$

$$= \pi\sqrt{h^2+R^2}\,R\,s^2$$

$$\Rightarrow \mathcal{H}^2(\partial K(R,h)) = \pi\sqrt{h^2+R^2}\,R$$

$$\mathcal{H}^2(O) = \mathcal{H}^2(\partial K) + \mathcal{H}^2(B_R(0)) = R\pi\sqrt{h^2+R^2} + R^2\pi = R\pi(R+\sqrt{h^2+R^2})$$

Notiz ∂K ist nur die Mantelfläche zu der noch die Grundfläche fehlt. O ist die ganze Oberfläche.