MAS Ü9

1) (f_n) ... Funktionenfolge auf $(\Omega, \mathcal{A}, \mu)$ ... endlich $\quad f_n$ ... gmI

$\nexists g$ ... integrierbar mit $g \geq |f_n|$ μ-f.ü. $\forall n \in \mathbb{N}$

Def (f_n) heißt gmI $\Leftrightarrow \forall \varepsilon > 0 \; \exists c_\varepsilon > 0 \; \exists A_\varepsilon \in \mathcal{A}, \mu(A_\varepsilon) < \infty$:

$$\sup_{n \in \mathbb{N}} \int_{|f_n| > c_\varepsilon} |f_n| \, d\mu < \varepsilon \;\wedge\; \sup_{n \in \mathbb{N}} \int_{A_\varepsilon^c} |f_n| \, d\mu < \varepsilon$$

Da $(\Omega, \mathcal{A}, \mu)$ endlich ist $\Rightarrow \mu(\Omega) < \infty$ also können wir für $A_\varepsilon = \Omega$ wählen und somit muss für die zweite Bedingung $\sup_{n \in \mathbb{N}} \int_\emptyset |f_n| \, d\mu = 0 < \varepsilon$ was immer der Fall ist, also reicht

$$(f_n) \text{...gmI} \Leftrightarrow \forall \varepsilon > 0 \; \exists c_\varepsilon > 0 : \sup_{n \in \mathbb{N}} \int_{|f_n| > c_\varepsilon} |f_n| \, d\mu < \varepsilon$$

$(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) \qquad \mu(\{k\}) = \frac{1}{2^k} \qquad \Rightarrow \mu(\mathbb{N}) = \sum_{k=1}^\infty \frac{1}{2^k} < \infty$

$$f_n(x) = \begin{cases} \frac{2^n}{n}, & x = n \\ 0, & \text{sonst} \end{cases}$$

- Sei $\varepsilon > 0$ bel. Wähle $c_\varepsilon = 2^{\lceil \frac{1}{\varepsilon} \rceil}$ Sei $n \in \mathbb{N}$ bel.

  1. Fall $n > \lceil \frac{1}{\varepsilon} \rceil$: $\int_{|f_n| > c_\varepsilon} |f_n| \, d\mu \leq \int |f_n| \, d\mu = \frac{2^n}{n} \mu(\{n\}) = \frac{2^n}{n} \frac{1}{2^n} = \frac{1}{n} < \varepsilon$

  2. Fall $n \leq \lceil \frac{1}{\varepsilon} \rceil$: $\int_{|f_n| > c_\varepsilon} |f_n| \, d\mu = 0 < \varepsilon$, da $c_\varepsilon = 2^{\lceil \frac{1}{\varepsilon} \rceil} \geq 2^n \geq \frac{2^n}{n} = f_n = |f_n|$

  also gilt nie $|f_n| > c_\varepsilon$

- Damit $g \geq |f_n|$ $\forall n \in \mathbb{N}$ (μ-f.ü. ist egal, da nur $\emptyset$ ist Nullmenge) muss

  $\forall n \in \mathbb{N}$: $g(n) \geq \frac{2^n}{n}$

  $\Rightarrow \int_{\mathbb{N}} g \, d\mu \geq \int_{\mathbb{N}} \frac{2^n}{n} \, d\mu = \sum_{n=1}^\infty \frac{2^n}{n} \frac{1}{2^n} = \sum_{n=1}^\infty \frac{1}{n} = \infty$

  $\Rightarrow \nexists g$ ... integrierbar mit $g \geq |f_n|$ μ-f.ü. $\forall n \in \mathbb{N}$

MAS Ü9

2) $\mathcal{F}, \mathcal{G}$ ... Familien messbarer Funktionen auf $(\Omega, \mathcal{A}, \mu)$

a) $\mathcal{F} \subseteq \mathcal{L}_1$, $|\mathcal{F}| < \infty \Rightarrow \mathcal{F}$ ... gmI

$\mathcal{F} \subseteq \mathcal{L}_1 \Rightarrow \forall f \in \mathcal{F}: \int_\Omega |f| \, d\mu < \infty$, da $|\mathcal{F}| < \infty \Rightarrow \sup_{f \in \mathcal{F}} \int_\Omega |f| \, d\mu < \infty$

zz: $\forall \varepsilon > 0 \; \exists g_\varepsilon \geq 0, g_\varepsilon \in L_1 \; \exists \delta > 0 \; \forall A \in \mathcal{A}: \int_A g_\varepsilon \, d\mu < \delta \Rightarrow \sup_{f \in \mathcal{F}} \int_A |f| \, d\mu < \varepsilon$

Sei $\varepsilon > 0$ bel. Wähle $g_\varepsilon(x) = \sum_{f \in \mathcal{F}} |f(x)| \geq 0 \Rightarrow \int_\Omega |g_\varepsilon(x)| \, d\mu = \sum_{f \in \mathcal{F}} \underbrace{\int_\Omega |f(x)| \, d\mu}_{< \infty} < \infty$, da $|\mathcal{F}| < \infty$

Also ist $g_\varepsilon \in L_1$. Wähle $\delta := \varepsilon$ Sei $A \in \mathcal{A}$ bel. mit $\int_A g_\varepsilon \, d\mu < \delta$.

$\delta > \int_A g_\varepsilon \, d\mu = \sum_{f \in \mathcal{F}} \underbrace{\int_A |f(x)| \, d\mu(x)}_{\geq 0} \Rightarrow \forall f \in \mathcal{F}: \int_A |f| \, d\mu < \delta = \varepsilon \Rightarrow \sup_{f \in \mathcal{F}} \int_A |f| \, d\mu < \varepsilon$

$\Rightarrow$ Alle Bedingungen von Satz 13.LP(iv) sind erfüllt, somit ist $\mathcal{F}$ .. gmI

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b) $\mathcal{F}$ ... gmI , $\forall g \in \mathcal{G} \; \exists f \in \mathcal{F}: |g| \leq |f|$ μ-f.ü. zz: $\mathcal{G}$ ... gmI

Da $\mathcal{F}$ .. gmI gilt $\forall \varepsilon > 0 \; \exists g_\varepsilon \geq 0, g_\varepsilon \in L_1: \sup_{f \in \mathcal{F}} \int_{|f| > g_\varepsilon} |f| \, d\mu < \varepsilon$

Sei $\varepsilon > 0$ bel. Wähle g_ε wie oben $\Rightarrow \sup_{g \in \mathcal{G}} \int_{|g| > g_\varepsilon} |g| \, d\mu \leq \sup_{f \in \mathcal{F}} \int_{|f| > g_\varepsilon} |f| \, d\mu < \varepsilon$

$\Rightarrow \mathcal{G}$ ist gmI

c) $\mathcal{F}, \mathcal{G}$ ... gmI zz: $\mathcal{H} := \{f \vee g, f \pm g : f \in \mathcal{F}, g \in \mathcal{G}\}$ ... gmI

$\sup_{h \in \mathcal{H}} \int |h| \, d\mu \leq \sup_{f \in \mathcal{F}, g \in \mathcal{G}} \int |f \pm g| \, d\mu \leq \sup_{f \in \mathcal{F}} \int |f| \, d\mu + \sup_{g \in \mathcal{G}} \int |g| \, d\mu < \infty$, $\mathcal{F}, \mathcal{G}$ ... gmI

Sei $\varepsilon > 0$ bel. Wähle $g_\varepsilon := g_\varepsilon^f + g_\varepsilon^g$ wobei $g_\varepsilon^f, g_\varepsilon^g \geq 0$ und aus L_1 und $\delta := \min(\delta^f, \delta^g)$ mit

$\delta^f, \delta^g > 0 : \int_A g_\varepsilon^f \, d\mu < \delta^f \Rightarrow \sup_{f \in \mathcal{F}} \int_A |f| \, d\mu < \varepsilon \wedge \int_A g_\varepsilon^g \, d\mu < \delta^g \Rightarrow \sup_{g \in \mathcal{G}} \int_A |g| \, d\mu < \varepsilon$

$\Rightarrow \int_A g_\varepsilon \, d\mu < \delta \Rightarrow \int_A g_\varepsilon^f \, d\mu + \int_A g_\varepsilon^g \, d\mu < \min(\delta^f, \delta^g) \Rightarrow \int_A g_\varepsilon^f \, d\mu < \delta^f \wedge \int_A g_\varepsilon^g \, d\mu < \delta^g$

$\Rightarrow \sup_{f \in \mathcal{F}} \int_A |f| \, d\mu < \varepsilon \wedge \sup_{g \in \mathcal{G}} \int_A |g| \, d\mu < \varepsilon$ $\sup_{h \in \mathcal{H}} \int_A |h| \, d\mu \leq \sup_{f \in \mathcal{F}} \int_A |f| \, d\mu + \sup_{g \in \mathcal{G}} \int_A |g| \, d\mu < 2\varepsilon$

$\Rightarrow \mathcal{H}$ .. gmI

$\square$

MAS ÜG

3) $g: \mathbb{R}^+ \to \mathbb{R}^+$ $\lim\limits_{x\to\infty} \frac{g(x)}{x} = \infty$ X_n ... Folge ZGen $\sup\limits_n \mathbb{E}(g(|X_n|)) < \infty$

zz: X_n ... gleichgradig integrierbar

$K_a := \inf\limits_{x \geq a} \frac{g(x)}{x} \xrightarrow{a\to\infty} \infty$ Für $a>0$ gilt

$$\sup_n \int_{[|X_n| \geq a]} |X_n| \, d\mu \overset{*}{\leq} \frac{1}{K_a} \sup_n \int_{[|X_n| \geq a]} g(|X_n|) \, d\mu \xrightarrow[a\to\infty]{\Delta} 0$$

$$* \text{ Für } |X_n| \geq a: \ K_a \leq \frac{g(|X_n|)}{|X_n|} \Rightarrow |X_n| \leq \frac{1}{K_a} g(|X_n|)$$

$$\Delta \ \sup_n \mathbb{E}(g|X_n|) = \sup_n \int g|X_n| \, d\mu < \infty$$

$$\Rightarrow \sup_n \int_{|X_n| \geq a} g|X_n| \, d\mu \to 0$$

Also $\forall \varepsilon > 0 \ \exists g_\varepsilon \equiv a \geq 0$, $g_\varepsilon \in L_1$ klar: $\sup\limits_n \int_{|X_n| > g_\varepsilon} |X_n| \, d\mu < \varepsilon$

und somit glgmI.

$\square$

MAS Ü9

4) (f_n) ... Funktionenfolge auf $(\Omega, \mathcal{A}, \mu)$ ... endlicher Maßraum

(i) $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall A : \overset{\forall n}{\mu(A)} \leq \delta \Rightarrow \int_A |f_n| \, d\mu < \varepsilon$ aber $\sup_n \int |f_n| \, d\mu = \infty$

$(\{0,1\}, \mathcal{P}(\{0,1\}), \delta_0) \quad \Rightarrow \delta_0(\{0,1\}) = 1 < \infty$

$$f_n(x) = \begin{cases} n & , x = 0 \\ 0 & , x = 1 \end{cases}$$

- Sei $\varepsilon > 0$ bel. Wähle $\delta := \frac{1}{2}$ Sei $A \in \mathcal{P}(\{0,1\})$ mit $\delta_0(A) \leq \delta = \frac{1}{2}$ bel. $\Rightarrow 0 \notin A$

  $\int_A |f_n| \, d\delta_0 = \int_A 0 \, d\delta_0 = 0 < \varepsilon$

- $\sup_{n \in \mathbb{N}} \int_\Omega |f_n(x)| \, d\delta_0 = \sup_{n \in \mathbb{N}} n = \infty$

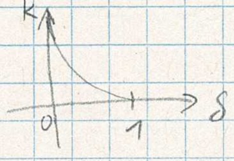
(ii) $\sup_n \int |f_n| \, d\mu < \infty$ aber $\exists \varepsilon > 0 \ \forall \delta > 0 \ \overset{\exists n}{\exists} A \in \mathcal{A} : \mu(A) \leq \delta \wedge \int_A |f_n| \, d\mu \geq \varepsilon$

$(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) \qquad \mu(\{k\}) = \frac{1}{2^k} \qquad \Rightarrow \mu(\mathbb{N}) = \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$

$$f_n(x) = \begin{cases} 2^n & x = n \\ 0 & \text{sonst} \end{cases}$$

- Wähle $\varepsilon = \frac{1}{2}$ Sei $\delta > 0$ bel. o.B.d.A. $\delta < 1$ Wähle $A := \left\{ \left\lceil -\frac{\log(\delta)}{\log(2)} \right\rceil \right\}$

  $\mu(A) = \frac{1}{2^{\left\lceil -\frac{\log(\delta)}{\log(2)} \right\rceil}} \leq \frac{1}{2^{-\frac{\log(\delta)}{\log(2)}}} = 2^{\frac{\log(\delta)}{\log(2)}} = 2^{\log_2 \delta} = \delta$



  Wähle $n = \left\lceil -\frac{\log(\delta)}{\log(2)} \right\rceil$

  $\int_A |f_n| \, d\mu = \mu(\{n\}) f_n(n) = \frac{1}{2^n} 2^n = 1 \geq \frac{1}{2}$

- $\forall n \in \mathbb{N}: \int |f_n| \, d\mu = f_n(n) \, \mu(\{n\}) = 2^n \frac{1}{2^n} = 1$

$\Rightarrow \sup_n \int |f_n| \, d\mu = 1 < \infty$

$\square$

MAS Ü9

5) $(X,Y) \sim \mathcal{N}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ ges: $\mathbb{E}(X|Y)$

zuerst $(X,Y) \sim \mathcal{N}(0,0,1,1,\rho)$ $\quad f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}$

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ und $f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ ... Randverteilung

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{\sqrt{2\pi}}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)-(-\frac{y^2}{2})} = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2-y^2(1-\rho^2))}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+\rho^2y^2)} = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x-\rho y)^2}$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right) \sim \mathcal{N}(\rho y, (1-\rho^2))$$

$\Rightarrow \mathbb{E}(X|Y) = \rho y$ $\quad$ für $(X,Y) \sim \mathcal{N}(0,0,1,1,\rho)$

Es gilt $\mathbb{E}(\sigma_x^2 X + \mu_x | Y) = \sigma^2 \mathbb{E}(X|Y) + \mu$ also sei $(X,Y) \sim \mathcal{N}(0,\mu,1,\sigma,\rho)$

$$\Rightarrow f(x,y) = \frac{1}{2\pi\sigma\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho x\left(\frac{y-\mu}{\sigma}\right) + \left(\frac{y-\mu}{\sigma}\right)^2\right)\right)$$

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\left(\frac{y-\mu}{\sigma}\right)^2 \frac{1}{2}\right)$$

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{\sqrt{2\pi}\,\sigma}{2\pi\sigma\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho x\underbrace{\left(\frac{y-\mu}{\sigma}\right)}^{z} + \left(\frac{y-\mu}{\sigma}\right)^2\right) + \frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right)$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xz + \cancel{z^2} - (\cancel{1}-\rho^2)z^2\right)\right)$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xz + \rho^2 z^2)\right) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{1}{2(1-\rho^2)}(x-\rho z)^2\right)$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{\left(x-\rho\left(\frac{y-\mu}{\sigma}\right)\right)^2}{2(1-\rho^2)}\right) \sim \mathcal{N}\left(\rho\left(\frac{y-\mu}{\sigma}\right), (1-\rho^2)\right)$$

$\Rightarrow \mathbb{E}(X|Y) = \rho\left(\frac{y-\mu}{\sigma}\right)$ $\quad$ für $(X,Y) \sim \mathcal{N}(0,\mu,1,\sigma,\rho)$

Und $\mathbb{E}(X|Y) = \sigma_x^2 \rho\left(\frac{y-\mu_y}{\sigma_y}\right) + \mu_x$ für $(X,Y) \sim \mathcal{N}(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$

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7) $X|Y \sim B_{n,Y}$ $Y \sim U_{0,1}$ ges: $Y|X$ und $\mathbb{E}(Y|X)$

$$f(X=k|Y=y) = \binom{n}{k} y^k (1-y)^{n-k}$$
$$f(Y=y) = \mathbb{1}_{[0,1]}(y)$$

$$f(X|Y) = \frac{f(X,Y)}{f(Y)} \Rightarrow f(X,Y) = f(X|Y)\, f(Y) = \binom{n}{k} y^k (1-y)^{n-k} \mathbb{1}_{[0,1]}(y)$$

$$f(Y|X) = \frac{f(X,Y)}{f(X)}$$

$$f(X=k) = \int_{\mathbb{R}} f(k,y)\,dy = \int_0^1 \binom{n}{k} y^k (1-y)^{n-k} dy = \binom{n}{k} \int_0^1 y^{(k+1)-1} (1-y)^{(n-k+1)-1} dy$$
$$= \binom{n}{k} B(k+1, n-k+1) = \frac{n!}{\cancel{k!(n-k)!}} \frac{\cancel{k!(n-k)!}}{(k+1+n-k+1-1)!} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$f(Y=y|X=k) = \frac{\binom{n}{k} y^k (1-y)^{n-k} \mathbb{1}_{[0,1]}(y)}{\binom{n}{k} B(k+1, n-k+1)} = \mathbb{1}_{[0,1]}(y) \frac{1}{B(k+1, n-k+1)} y^k (1-y)^{n-k}$$

$$\mathbb{E}(Y|X=k) = \int_{\mathbb{R}} f(Y=y|X=k)\,dy = \int_0^1 \frac{1}{B(\ldots)} y^k (1-y)^{n-k} dy$$
$$= \frac{1}{B(\ldots)} \int_0^1 y^k (1-y)^{n-k} dy = \frac{B(k+1, n-k+1)}{B(k+1, n-k+1)} = 1$$