ANA Ü5

1) $F_n(x) = \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} e^{ikx}$ $\quad V_n := 2F_{2n} - F_n$

(i) zz: $\int_{-\pi}^{\pi} V_n(x)\,dx = 2\pi$

$\int_{-\pi}^{\pi} F_n(x)dx = \int_{-\pi}^{\pi} \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} e^{ikx} dx = \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} \int_{-\pi}^{\pi} e^{ikx} dx = \frac{n-0}{n} \int_{-\pi}^{\pi} 1 dx = 2\pi$

$\Rightarrow \int_{-\pi}^{\pi} V_n(x)dx = \int_{-\pi}^{\pi} 2F_{2n} - F_n(x)dx = 2\int_{-\pi}^{\pi} F_{2n}(x)dx - \int_{-\pi}^{\pi} F_n(x)dx = 4\pi - 2\pi = 2\pi$

(ii) zz: $\int_{-\pi}^{\pi} |V_n(x)|\,dx \leq C$ für ein C unabhängig von n

Da $F_n(x) = \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} e^{ikx} = \frac{1}{n}\sum_{k=0}^{n-1} D_k(x) = \frac{1}{n}\frac{1}{\sin(\frac{x}{2})} \sum_{k=0}^{n-1} \sin\left(\frac{k+1}{2}x\right) = \frac{1}{n}\frac{1}{\sin(\frac{x}{2})} \frac{\sin^2(\frac{nx}{2})}{\sin(\frac{x}{2})}$

$= \frac{\sin^2(\frac{nx}{2})}{n\sin^2(\frac{x}{2})} \geq 0$

$\Rightarrow \int_{-\pi}^{\pi} |V_n(x)|\,dx \leq \int_{-\pi}^{\pi} 2|F_{2n}(x)| + |F_n(x)|\,dx = 2\int_{-\pi}^{\pi} F_{2n}(x)dx + \int_{-\pi}^{\pi} F_n(x)dx = 4\pi + 2\pi = 6\pi =: C$

(iii) $\forall A \subseteq [-\pi,\pi]$ (A…abgeschlossen): $0 \notin A \Rightarrow V_n(x) \to 0$ glm. auf A

zz: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall x \in A\ \forall n \geq N: |V_n(x) - 0| < \varepsilon$

oder auch $\|V_n\|_\infty \xrightarrow{n\to\infty} 0$

$\|V_n\|_\infty = \|2F_{2n} - F_n\|_\infty \leq 2\|F_{2n}\|_\infty + \|F_n\|_\infty$

Wir zeigen $\forall A \subseteq [-\pi,\pi]$ abgeschlossen, $0 \notin A$: $\forall x \in A: F_n(x) \to 0$ glm.

A ist beschränkt und abgeschlossen $\Rightarrow$ kompakt also reicht $F_n(x) \to 0$ zu zeigen.

$F_n(x) = \frac{\sin^2(\frac{nx}{2})}{n\sin^2(\frac{x}{2})} \leq \underbrace{\frac{1}{n}}_{\searrow 0} \frac{1}{\underbrace{\sin^2(\frac{x}{2})}_{\in[0,1]}} \longrightarrow 0$

Es folgt $\|V_n\|_\infty \leq 2\|F_{2n}\|_\infty + \|F_n\|_\infty \xrightarrow{n\to\infty} 0$

<u>ANA Ü5</u>

2) (iv) zz: $\frac{1}{2\pi} D_n * V_n = D_n$

$$\frac{1}{2\pi} D_n * V_n(x) = \int_{-\pi}^{\pi} \frac{1}{2\pi} D_n(x-y) V_n(y)\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x-y)\,(2F_{2n}-F_n)(y)\,dy$$

$$D_n(x-y)(2F_{2n}(y)-F_n(y)) = \Big(\sum_{k=-n}^{n} e^{ik(x-y)}\Big)\Big(\sum_{k=-(n-1)}^{n-1} \frac{2n-|k|}{n} e^{iky} - \sum_{k=-(n-1)}^{n-1}\frac{n-|k|}{n} e^{iky}\Big) = \Big(\sum_{k=-n}^{n} e^{ik(x-y)}\Big)\Big(\sum_{k=-(n-1)}^{n-1} e^{iky}\Big(\frac{2\cancel{n}-\cancel{|k|}-n+\cancel{|k|}}{\cancel{n}}\Big)\Big)$$

$$= \Big(\sum_{k=-n}^{n} e^{ik(x-y)}\Big)\Big(\sum_{k=-(n-1)}^{n-1} e^{iky}\Big) = \Big(\sum_{k=-n}^{n} e^{ik(x-y)}\Big)\Big(\sum_{k=-n}^{n} e^{iky} - e^{-i(n-1)y} - e^{i(n-1)y}\Big)$$

$$= D_n(x-y) D_n(y) - D_n(x-y)\big(e^{-i(n-1)y} + e^{i(n-1)y}\big)$$

$$\Rightarrow \frac{1}{2\pi} D_n * V_n(x) = \frac{1}{2\pi}\Big(\int_{-\pi}^{\pi} D_n(x-y) D_n(y)\,dy - \int_{-\pi}^{\pi} D_n(x-y)\big(e^{-i(n-1)y} + e^{i(n-1)y}\big)\,dy$$

$$= \frac{1}{2\pi}\Big(D_n * D_n - \int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{ik(x-y)}\big(e^{-i(n-1)y} + e^{i(n-1)y}\big)\,dy$$

$$= \frac{1}{2\pi}\Big(2\pi D_n - \sum_{k=-n}^{n}\int_{-\pi}^{\pi} e^{ik(x-y)-i(n-1)y} + e^{ik(x-y)+i(n-1)y}\,dy\Big)$$

$$= D_n - \frac{1}{2\pi}\sum_{k=-n}^{n} \frac{i\exp(iy(-k-n+1)+ikx)}{k+n-1} - \frac{i\exp(iy(-k+n-1)+ikx)}{-k+n-1}\Bigg|_{-\pi}^{\pi}$$

$$= D_n - \frac{1}{2\pi}\sum_{k=-n}^{n} 0 = D_n$$

(v) $f \in C^0([-\pi,\pi])$ stetig : $\frac{1}{2\pi} V_n * f \to f$ glm.

Nach 3.1.14 gilt $\lim_{n\to\infty} \frac{1}{2\pi} F_n * f = f$ glm.

$$\frac{1}{2\pi} V_n * f = \frac{1}{2\pi}(2F_{2n}-F_n) * f = \frac{2F_{2n}}{2\pi} * f - \frac{F_n}{2\pi} * f \longrightarrow 2f - f = f \text{ glm.}$$

(vi) $f \in L^1([-\pi,\pi])$ zz: $\forall |k| < n : \frac{1}{2\pi}\widehat{V_n * f}(k) = \hat{f}(k)$

Da $f, V_n \in L^1(\mathbb{T}) \Rightarrow \widehat{V_n * f}(k) = \sqrt{2\pi}\,\hat{V}_n(k)\hat{f}(k)$

$$\frac{1}{2\pi}\sqrt{2\pi}\,\hat{V}_n(k)\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\hat{V}_n(k)\hat{f}(k)$$

$$\hat{V}_n(k) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} V_n(x) e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}} 2\int_{-\pi}^{\pi} F_{2n}(x) e^{-ikx}\,dx - \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} F_n(x) e^{-ikx}\,dx$$

$$\overset{\Delta}{=} 2\,\widehat{F_{2n}}(k) - \hat{F}_n(k) = 2\,\frac{2n-|k|}{2n}\sqrt{2\pi} - \frac{n-|k|}{n}\sqrt{2\pi} = \sqrt{2\pi}\,\frac{2n-|k|-n+|k|}{n} = \sqrt{2\pi}\,\frac{n}{n} = \sqrt{2\pi}$$

$$\hat{V}_n(0) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} V_n(x)\,dx = \frac{1}{\sqrt{2\pi}}\, 2\pi = \sqrt{2\pi}$$

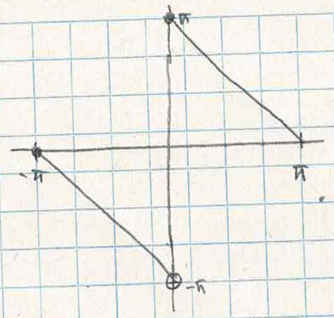
$$\Rightarrow \frac{1}{\sqrt{2\pi}}\widehat{V_n * f}(k) = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi}\,\hat{f}(k) = \hat{f}(k)$$

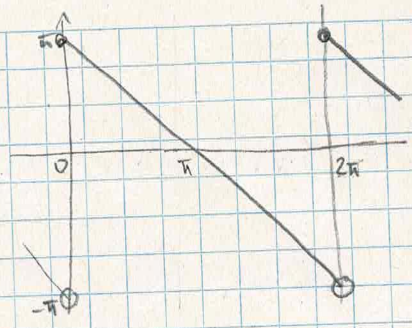
$$\Delta \int_{-\pi}^{\pi} F_n(x) e^{-ikx}\,dx = \int_{-\pi}^{\pi}\sum_{j=-(n-1)}^{n-1}\frac{n-|j|}{n} e^{ijx-ikx}\,dx = \sum_{j=-(n-1)}^{n-1}\frac{n-|j|}{n}\int_{-\pi}^{\pi} e^{ix(j-k)}\,dx = \sum_{j=-(n-1)}^{n-1}\frac{n-|j|}{n}\,\frac{e^{ix(j-k)}}{i(j-k)}\Bigg|_{-\pi}^{\pi}$$

$$= \sum_{j=-(n-1)}^{n-1}\frac{n-|j|}{n}\, 2\pi\, \mathbb{1}_j\{k\} = \frac{n-|k|}{n}\, 2\pi$$

ANA Ü5

3) $f(x) := \begin{cases} -\pi - x, & -\pi \le x < 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$





f ist stückweise stetig diffbar

$\Rightarrow$ auf $[-\pi, 0)$ und $(0, \pi]$ konvergiert die Fourierreihe gegen f und bei 0 gegen $0 = \frac{\pi + (-\pi)}{2}$

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\pi}^{0} (-\pi - x) e^{-inx} dx + \int_{0}^{\pi} (\pi - x) e^{-inx} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(-\pi \int_{-\pi}^{0} e^{-inx} dx - \int_{-\pi}^{0} x e^{-inx} dx + \pi \int_{0}^{\pi} e^{-inx} dx - \int_{0}^{\pi} x e^{-inx} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{i\pi}{n} (-1 + (-1)^n) - \frac{1 + (-1)^n (-1 + i\pi n)}{n^2} - \frac{i\pi}{n} (1 - (-1)^n) - \frac{-1 + (-1)^n (1 + i\pi n)}{n^2} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(-\frac{2i\pi}{n} + \frac{2i\pi}{n} (-1)^n - \left(\frac{\cancel{1 - (-1)^n} + (-1)^n i\pi n \cancel{- 1 + (-1)^n} + (-1)^n i\pi n}{n^2} \right) \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{2i\pi}{n} (-1 + (-1)^n) - \frac{2i\pi n (-1)^n}{n^2} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2i\pi}{n} (-1 + (-1)^n - (-1)^n) \right) = -\frac{2i\pi}{\sqrt{2\pi}\, n} = \frac{\sqrt{2\pi}}{in}$$

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\pi}^{0} -\pi - x\, dx + \int_{0}^{\pi} \pi - x\, dx \right) = \frac{1}{\sqrt{2\pi}} \left(-\pi x - \frac{x^2}{2} \Big|_{-\pi}^{0} + \pi x - \frac{x^2}{2} \Big|_{0}^{\pi} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(-\frac{\pi^2}{2} + \frac{\pi^2}{2} \right) = 0$$

$$S_N f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = \frac{1}{\sqrt{2\pi}} \hat{f}(0) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(\hat{f}(n) e^{inx} + \hat{f}(-n) e^{-inx} \right)$$

$$S_N f(0) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(\frac{\sqrt{2\pi}}{in} - \frac{\sqrt{2\pi}}{in} \right) = 0 \quad \text{wie erwartet. :-)}$$

ANA Ü5

4) zz: $\forall x \in [-\pi,\pi] : x\sin(x) = 1 - \frac{1}{2}\cos(x) - 2\sum_{n=2}^{\infty} \frac{(-1)^n \cos(nx)}{n^2-1}$

$(-x)\sin(-x) = x\sin(x) \Rightarrow$ Funktion ist gerade $\overset{5)}{\Rightarrow}$ Sinusterme verschwinden

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\frac{1}{2}(\sin(x+nx)+\sin(x-nx))\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(x(n+1))\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(x(1-n))\,dx$$

$$= \frac{1}{2\pi}\,\frac{\sin((n+1)x)-(n+1)x\cos((n+1)x)}{(n+1)^2}\Big|_{-\pi}^{\pi} + \frac{1}{2\pi}\,\frac{(n-1)x\cos((n-1)x)-\sin((n-1)x)}{(n-1)^2}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{-(n+1)\pi\cos((n+1)\pi)}{(n+1)^2} - \frac{+(n+1)\pi\cos(-(n+1)\pi)}{(n+1)^2}\right) + \frac{1}{2\pi}\left(\frac{(n-1)\pi\cos((n-1)\pi)}{(n-1)^2} - \frac{-(n-1)\pi\cos((n-1)\pi)}{(n-1)^2}\right)$$

$$= \frac{1}{2\pi}\left(\frac{-\pi(-1)^{n+1}}{n+1} - \frac{\pi(-1)^{n+1}}{n+1} + \frac{\pi(-1)^{n-1}}{n-1} + \frac{\pi(-1)^{n-1}}{n-1}\right) = \frac{1}{2\pi}\left(\frac{2\pi(-1)^n}{n+1} + \frac{2\pi(-1)^{n-1}}{n-1}\right)$$

$$= \frac{(-1)^n}{n+1} + \frac{(-1)^{n-1}}{n-1} = \frac{(-1)^n(n-1)-(-1)^n(n+1)}{n^2-1} = \frac{-(-1)^n-(-1)^n}{n^2-1} = -2\,\frac{(-1)^n}{n^2-1}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)\cos(0)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)\,dx = \frac{1}{\pi}(\sin(x)-x\cos(x))\Big|_{-\pi}^{\pi} = \frac{1}{\pi}(0+\pi-0-\pi(-1)) = \frac{2\pi}{\pi} = 2$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)\cos(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\frac{1}{2}(\sin(2x)+\sin(0))\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(2x)\,dx = \frac{1}{2\pi}\left(\frac{1}{4}\sin(2x)-\frac{1}{2}x\cos(2x)\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(-\frac{1}{2}\pi+\frac{1}{2}(-\pi)\right) = -\frac{\pi}{2\pi} = -\frac{1}{2}$$

Δ

$x\sin(x)$ ist stetig diffbar auf $(-\pi,\pi)$ und stückweise stetig diffbar auf $\mathbb{T}$

$\Rightarrow$ Korollar 3.1.9. an allen Punkten konvergiert die Fourierreihe gegen $\frac{f(x_0+0)+f(x_0-0)}{2}$

also in $(-\pi,\pi)$ gegen den Funktionswert und an $-\pi$ und π gegen

$\lim_{x\to-\pi^+} f(x) = -\pi\sin(-\pi) = 0$ $\qquad$ $\lim_{x\to\pi^-} f(x) = \pi\sin(\pi) = 0$ $\qquad$ $\frac{0+0}{2} = 0$ also den Funktionswert

$\square$

$$\Delta\quad x\sin(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx) = \frac{2}{2} + \left(-\frac{1}{2}\right)\cos(x) + \sum_{n=2}^{\infty}\left(-2\,\frac{(-1)^n}{n^2-1}\cos(nx)\right)$$

$$= 1 - \frac{1}{2}\cos(x) - 2\sum_{n=2}^{\infty}\frac{(-1)^n\cos(nx)}{n^2-1}$$

ANA Ü5

5) f...gerade $\Leftrightarrow f(-x)=f(x)$ $\quad f$...ungerade $\Leftrightarrow f(-x)=-f(x)$ für (fast) alle x

$f\in L^1([-\pi,\pi])$

i) zz: f...gerade $\Leftrightarrow \forall n=1,\dots : b_n=0$

$\Rightarrow$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)\sin(nx)\,dx+\int_{0}^{\pi}f(x)\sin(nx)\,dx\right)$$
$$=\frac{1}{\pi}\left(-\int_{\pi}^{0}f(-y)\sin(-ny)\,dy+\int_{0}^{\pi}f(x)\sin(nx)\,dx\right) \qquad \left[y=-x \quad \frac{dy}{dx}=-1\right.$$
$$=\frac{1}{\pi}\left(-\int_{0}^{\pi}f(y)\sin(ny)\,dy+\int_{0}^{\pi}f(x)\sin(nx)\,dx\right)=0$$
Nullmengen können wir ignorieren

$\Leftarrow$ Es gilt $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx)$ fast überall
$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)$$
$$f(-x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(-nx)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)=f(x) \Rightarrow \text{gerade}$$

ii) zz: f...ungerade $\Leftrightarrow \forall n=0,\dots : a_n=0$

$\Rightarrow$
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)\,dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)\cos(nx)\,dx+\int_{0}^{\pi}f(x)\cos(nx)\,dx\right)$$
$$=\frac{1}{\pi}\left(-\int_{\pi}^{0}f(-y)\cos(-ny)\,dy+\int_{0}^{\pi}f(x)\cos(nx)\,dx\right) \qquad \left[y=-x \quad \frac{dy}{dx}=-1\right.$$
$$=\frac{1}{\pi}\left(\int_{0}^{\pi}-f(y)\cos(ny)\,dy+\int_{0}^{\pi}f(x)\cos(nx)\,dx\right)=0$$

$\Leftarrow$
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx)=\sum_{n=1}^{\infty}b_n\sin(nx) \quad \text{fast überall}$$
$$f(-x)=\sum_{n=1}^{\infty}b_n\sin(-nx)=-\sum_{n=1}^{\infty}b_n\sin(nx)=-f(x) \Rightarrow \text{ungerade}$$

ANA Ü5

6) i) $f(x) := \frac{\sin(x)}{x}$ ges: $S_N f(x)$

$f(-x) = \frac{\sin(-x)}{-x} = \frac{\sin(x)}{x} = f(x) \Rightarrow$ gerade $\overset{5)}{\Rightarrow} b_n = 0 \ \forall n \in \{0, \dots\}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin(x)\cos(nx)}{x} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(x+nx) + \sin(x-nx)}{x} dx$

$= \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} \frac{\sin(x(n+1))}{x} dx + \int_{-\pi}^{\pi} \frac{\sin(x(1-n))}{x} dx \right) = \frac{1}{2\pi} \left(2 \int_0^{\pi} \frac{\sin(x(n+1))}{x} dx + 2 \int_0^{\pi} \frac{\sin(x(1-n))}{x} dx \right)$

$= \frac{1}{\pi} \left(\int_0^{\pi} \frac{\sin(x(n+1))}{x} dx + \int_0^{\pi} \frac{-\sin(x(n-1))}{x} dx \right)$ $\quad u = x(n+1) \quad \frac{du}{dx} = n+1 \quad v = (n-1)x \quad \frac{dv}{dx} = n-1$

$= \frac{1}{\pi} \left(\int_0^{(n+1)\pi} \frac{\sin(u)}{u} \frac{\cancel{n+1}}{\cancel{n+1}} du - \int_0^{(n-1)\pi} \frac{\sin(v)}{v} \frac{\cancel{n-1}}{\cancel{n-1}} dv \right) = \frac{1}{\pi} \left(\int_{(n-1)\pi}^{0} \frac{\sin(x)}{x} dx + \int_0^{(n+1)\pi} \frac{\sin(x)}{x} dx \right)$

$= \frac{1}{\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin(x)}{x} dx$

$\Rightarrow S_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx)$

ii) Darstellung von $\int_0^{\infty} \frac{\sin(x)}{x} dx$ durch a_n

$a_n = \frac{1}{\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin(x)}{x} dx \Leftrightarrow \pi a_n = \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin(x)}{x} dx \Rightarrow \sum_{k=1}^{\infty} \pi a_{2k-1} = \sum_{k=1}^{\infty} \int_{2k-1-1}^{2k-1+1} \frac{\sin(x)}{x} dx$

$= \sum_{k=1}^{\infty} \int_{2k-2}^{2k} \frac{\sin(x)}{x} dx = \int_0^{\infty} \frac{\sin(x)}{x}$ $\quad$ also $\pi \sum_{k=1}^{\infty} a_{2k-1} = \int_0^{\infty} \frac{\sin(x)}{x}$

auch $\pi \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_{2k} \right) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin(x)}{x} dx + \sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{(2k+1)\pi} \frac{\sin(x)}{x} dx = \int_0^{\pi} \frac{\sin(x)}{x} dx + \int_{\pi}^{\infty} \frac{\sin(x)}{x} dx$

$= \int_0^{\infty} \frac{\sin(x)}{x} dx$

iii) f ist stetig $\quad f'(x) = \frac{x\cos(x) - \sin(x)}{x^2}$ ist stetig

$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 \qquad \lim_{x \to 0} \frac{x\cos(x) - \sin(x)}{x^2} = \lim_{x \to 0} \frac{-x \sin(x)}{2x} = -\frac{1}{2}\sin(0) = 0$

$\Rightarrow$ stetig diffbar $\Rightarrow S_N f(x) \overset{N\to\infty}{\to} f(x)$ punktweise

iv) Berechnung von $\int_0^{\infty} \frac{\sin(x)}{x} dx$ und $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx$

$1 = \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n} + \sum_{n=1}^{\infty} a_{2n-1}$

$\overset{ii)}{=} \frac{1}{\pi} \left(\int_0^{\infty} \frac{\sin(x)}{x} dx + \int_0^{\infty} \frac{\sin(x)}{x} dx \right) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin(x)}{x} dx \quad \Rightarrow \int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2}$

$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \int_{-\infty}^{0} \frac{\sin(x)}{x} dx + \int_0^{\infty} \frac{\sin(x)}{x} dx = 2 \int_0^{\infty} \frac{\sin(x)}{x} dx = 2 \frac{\pi}{2} = \pi$

ANA Ü5

7) $f \in L^1(\mathbb{R}^n)$ $\quad \tau_z f(x) = f(x-z)$ $\quad Mod_z f(x) = e^{izx} f(x)$ $\quad Dil_\lambda f(x) = \frac{1}{\lambda^{n/2}} f\left(\frac{x}{\lambda}\right)$

zz: $\widehat{\tau_z f}(\xi) = (Mod_{-z} \hat{f})(\xi)$

$$\widehat{\tau_z f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(x-z) e^{-i\xi x} dx = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(u) e^{-i\xi(u+z)} du \qquad \left[u = x-z \quad \frac{du}{dx} = 1 \right.$$

$$= e^{-i\xi z} \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-i\xi u} f(u) du = e^{-i\xi z} \hat{f}(\xi) = (Mod_{-z} \hat{f})(\xi)$$

zz: $\widehat{Mod_z f}(\xi) = (\tau_z \hat{f})(\xi)$

$$\widehat{Mod_z f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{izx} f(x) e^{-i\xi x} dx = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-ix(\xi - z)} f(x) dx = \hat{f}(\xi - z) = (\tau_z \hat{f})(\xi)$$

zz: $\widehat{Dil_\lambda f}(\xi) = Dil_{\frac{1}{\lambda}} \hat{f}(\xi)$

$$\widehat{Dil_\lambda f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \frac{1}{\lambda^{n/2}} f\left(\frac{x}{\lambda}\right) e^{-i\xi x} dx \qquad \left[u = \frac{x}{\lambda} \quad \frac{du}{dx} = \frac{1}{\lambda} \right.$$

$$= \frac{1}{\sqrt{2\pi}^n} \frac{1}{\lambda^{n/2}} \int_{\mathbb{R}^n} f(u) e^{-i\xi\lambda u} \lambda \, du = \frac{\lambda^n}{\lambda^{n/2}} \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} f(u) e^{-i\xi\lambda u} du$$

$$= \lambda^{n/2} \hat{f}(\lambda\xi) = (Dil_{\frac{1}{\lambda}} \hat{f})(\xi)$$

ANA Ü5

8) $f(x) := 1\!\!1_{[-L,L]}$ für $L>0$

ges: Fouriertransformierte

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{-in} e^{-inx}\Big|_{-L}^{L}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{i}{n} e^{-inL} - \frac{1}{\sqrt{2\pi}} \frac{i}{n} e^{inL} = \frac{i}{n\sqrt{2\pi}} (e^{-inL} - e^{inL}) = \frac{i}{n\sqrt{2\pi}} (-2i \sin(nL))$$

$$= \frac{2}{n\sqrt{2\pi}} \sin(nL) \quad \text{und} \quad \hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} 1\, dx = \frac{2L}{\sqrt{2\pi}}$$

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx} = \frac{2L}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \frac{2}{n\sqrt{2\pi}} \sin(nL) - \frac{2}{n\sqrt{2\pi}} \sin(-nL) = \frac{2}{\sqrt{2\pi}} \left(L + \sum_{n=1}^{\infty} 2 \frac{\sin(nL)}{n}\right)$$

$\operatorname{supp} f = [-L, L]$ ist kompakt

$$\hat{f}(n) = 0 \iff \frac{2}{n\sqrt{2\pi}} \sin(nL) = 0 \iff \sin(nL) = 0 \iff nL \in \pi\mathbb{Z} \iff n \in \frac{\pi}{L}\mathbb{Z}$$

$\Rightarrow \operatorname{supp} \hat{f} = \mathbb{Z} \setminus \frac{\pi}{L}\mathbb{Z} = \mathbb{Z}$ ist nicht kompakt

Ist $\hat{f} \in L^1(\mathbb{R})$?

$$\int_{\mathbb{R}} \frac{2}{x\sqrt{2\pi}} \sin(xL)\, dx = \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\sin(xL)}{x} dx = \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\sin(u)}{u} L \frac{1}{L} du \qquad u = xL \quad \frac{du}{dx} = L$$

$$= \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\sin(u)}{u} du = \frac{2\pi}{\sqrt{2\pi}} = \sqrt{2\pi} < \infty$$

Ist $\hat{f} \in L^2(\mathbb{R})$?

$$\int_{\mathbb{R}} \left(\frac{2}{x\sqrt{2\pi}} \sin(xL)\right)^2 dx = \frac{4}{2\pi} \int_{\mathbb{R}} \frac{\sin^2(xL)}{x^2} dx = \frac{2}{\pi} \int_{\mathbb{R}} \frac{\sin^2(u)}{u^2} L^2 \frac{1}{L} du \qquad \left[u = xL \quad \frac{du}{dx} = L \right.$$

$$= \frac{2L}{\pi} \int_{\mathbb{R}} \frac{\sin^2(u)}{u^2} du = 2L < \infty$$