Computer Aided Geometric Design Compendium WS2023

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Organization

Lecture each Thursday 12:00 to 14:00 (full 2 hours).
Oral exam. Write email to fix date and time.
Problem session each Thursday 14:00 to 16:00. Mandatory attendance!
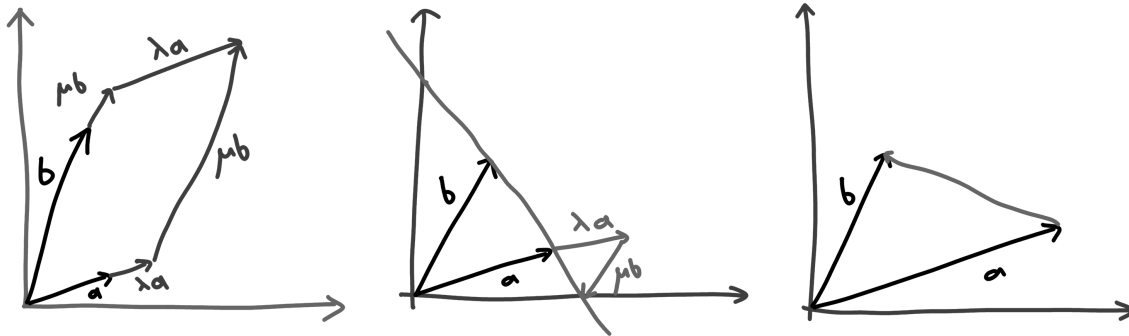Kreuzerlübung.

1 Bezier curves

Figure 1: Linear combination, affine combination and convex combination

Example 1. ***Linear combination*** $\lambda a + \mu b$
    Affine combination $\lambda a + \mu b$ *and* $\lambda + \mu = 1$
    What is μ *so that* $\lambda a + \mu B$ *is on the line?*

$$\lambda a + \mu b = a + t(b - a) \implies a(\underbrace{\lambda - 1 + t}_{=0}) + b(\underbrace{\mu - t}_{=0}) = 0$$

If a, b *are linearly independent* $\implies \mu = t \land \lambda + \mu = 1$
    Convex combination $\lambda a + \mu b$ *and* $\lambda + \mu = 1$ *and* $\lambda, \mu \geq 0$
    Line is $a + t(b - a)$ *with* $t \in [0, 1] \implies \mu, \lambda \in [0, 1]$

Definition 1 (combinations). *linear combination* $\sum_{i=1}^{n} \lambda_i v_i$ *with* $v_1, ..., v_n \in \mathbb{R}^d, \lambda_1, ..., \lambda_n \in \mathbb{R}$
affine combination $\sum_{i=1}^{n} \lambda_i v_i$ *with* $\sum_{i=1}^{n} \lambda_i = 1$
convex combination $\sum_{i=1}^{n} \lambda_i v_i$ *with* $\sum_{i=1}^{n} \lambda_i = 1$ *and* $\forall i : \lambda_i \geq 0$

Algorithm 1 (of de Casteljou, Bezier curve). *Given:* $b_0, ..., b_n \in \mathbb{R}^d$ *(called control points / Kontrollpunkte),*
$t \in \mathbb{R}$
Recursion: $b_i^0(t) := b_i$
$b_i^j(t) := (1 - t)b_i^{j-1}(t) + tb_{i+1}^{j-1}(t)$ *for* $j = 1, ..., n$ *and* $i = 0, ..., n - j$
Result: $b(t) := b_0^n(t)$ *(called Bezier curve)*

Remark 1. *In the algorithm above often we choose* $t \in [0, 1]$.

Example 2.

Remark 2. *In this course* $\mathbb{N} = \{1, 2, 3, ...\}$ *and* $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$
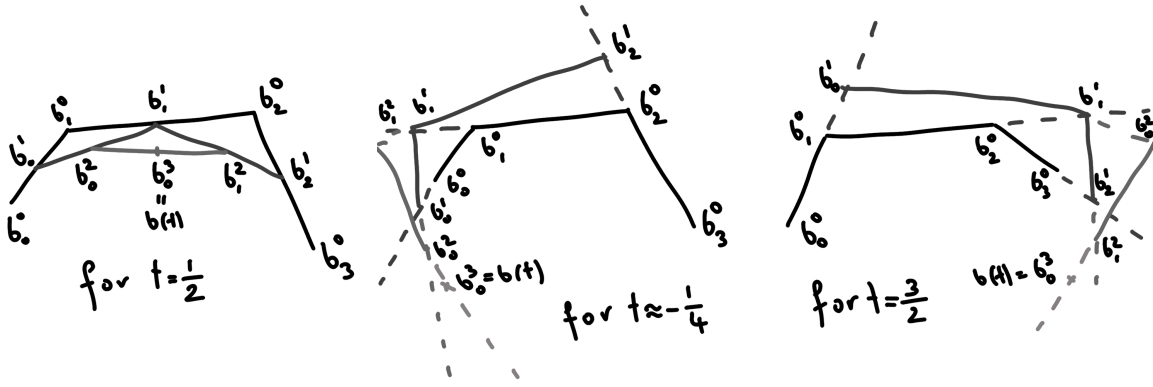
Figure 2: Examples of the de casteljou algorithm

Recap 1. $0! := 1, n! := n(n-1)(n-2)\cdots 1$ *for* $n \geq 1$.

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!} & , n \geq k \geq 0 \\ 0 & , k > n \end{cases} \text{ for } n, k \in \mathbb{N}_0$$

Definition 2 (Bernstein polynomials). *For* $n, i \in \mathbb{N}_0$ *we define* $B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i} \in \mathbb{R}[t]$

Remark 3. *Special cases of Bernstein polynomials*

$$i > n \implies B_i^n(t) = 0 \qquad\qquad\qquad B_i^n(0) = \begin{cases} 0, i \neq 0 \\ 1, i = 0 \end{cases}$$

$$B_i^n(1) = \begin{cases} 0, i \neq n \\ 1, i = n \end{cases} \qquad\qquad\qquad B_0^0(t) = 1$$

Theorem 1. $b_i^j(t) = \sum_{l=0}^{j} B_l^j(t) b_{i+l}$

Proof. Induction over j: $j = 0$:

$$j = 0: \qquad\qquad\qquad b_i^0(t) := b_i = 1 \cdot b_i = B_0^0(t) \cdot b_i \qquad \checkmark$$

$$j - 1 \to j: \qquad b_i^j(t) := (1-t) b_i^{j-1}(t) + t b_{i+1}^{j-1}(t) \overset{\text{IA}}{=} (1-t) \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+l} + t \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+1+l} =$$

$$(1-t) \sum_{l=0}^{j-1} B_l^{j-1}(t) b_{i+l} + t \sum_{l=1}^{j} B_{l-1}^{j-1}(t) b_{i+l} = \sum_{l=0}^{j} (\underbrace{(1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t)}_{= B_l^j(t) \text{ using the following lemma}}) b_{i+l} =$$

$$\sum_{l=0}^{j} B_l^j(t) b_{i+l} \qquad \checkmark$$

$\square$

Corollary 1. *The Bezier curve equals* $b(t) = b_0^n(t) = \sum_{l=0}^{n} B_l^j(t) b_{i+l}$, *which is called the Bernstein representation of the Bezier curve.*

Remark 4. *As* $b(t) = \sum_{l=0}^{n} B_l^n(t) b_l \in C^\infty$ *it is a polynomial curve of degree* n, *which is in* C^∞ *and therefore "very smooth".*

Lemma 1. $B_l^j(t) = (1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t)$

Proof.

$$(1-t) B_l^{j-1}(t) + t B_{l-1}^{j-1}(t) = (1-t) \binom{j-1}{l} t^l (1-t)^{j-1-l} + t \binom{j-1}{l-1} t^{l-1} (1-t)^{j-1-l+1} =$$

$$\binom{j-1}{l} t^l (1-t)^{j-l} + \binom{j-1}{l-1} t^l (1-t)^{j-l} = \left(\binom{j-1}{l} + \binom{j-1}{l-1} \right) t^l (1-t)^{j-l} = \binom{j}{l} t^l (1-t)^{j-l} = B_l^j(t)$$

$\square$

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Remark 5. *What is $b(0)$?* $b(0) = \sum_{i=0}^{n} B_i^n(0)b_i = b_0 + 0 + 0 + \cdots + 0 = b_0$
What is $b(1)$? $b(1) = \sum_{i=0}^{n} B_i^n(1)b_i = 0 + \cdots + 0 + b_n = b_n$

Definition 3 (end-point-interpolating)**.** *Curves which pass through the first and last point are called end-point-interpolating (Endpunktinterpolierend).*

Remark 6. *Bezier curves are end-point-interpolating.*

Remark 7. *How many intersection points are there between a planar (i.e. in $\mathbb{R}^2$) Bezier curve and a straight line?*

$$\text{Straight line: } p + t(q - p) \qquad\qquad \text{Bezier curve: } b(t) = \sum_{i=0}^{n} B_i^n(t) \underbrace{b_i}_{\in \mathbb{R}^2}$$

Solving $p + t(q - p) = \sum_{i=0}^{n} B_i^n(t)b_i$ results in at most n solutions.

Lemma 2. $\frac{d}{dt}B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$

Proof.

$$\frac{d}{dt}B_i^n(t) = \frac{d}{dt}\binom{n}{i}t^i(1-t)^{n-i} = \binom{n}{i}it^{i-1}(1-t)^{n-i} - \binom{n}{i}t^i(n-i)(1-t)^{n-i-1} =$$

$$\frac{n!}{\underset{(i-1)!}{\cancel{i!}}(n-i)!}it^{i-1}(1-t)^{n-i} - \frac{n!}{i!\underset{(n-i-1)!}{\cancel{(n-i)!}}}\cancel{(n-i)}t^i(1-t)^{n-i-1} =$$

$$n\left(\frac{(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{(n-1)!}{i!(n-i-1)!}t^i(1-t)^{n-i-1}\right) =$$

$$n\left(\binom{n-1}{i-1}t^{i-1}(1-t)^{n-i} - \binom{n-1}{i}t^i(1-t)^{n-i-1}\right) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$$

$\square$

Theorem 2. $\dot{b}(t) := \frac{d}{dt}b(t) = n\sum_{i=0}^{n-1}B_i^{n-1}(t)(b_{i+1} - b_i) = n(b_1^{n-1}(t) - b_0^{n-1}(t))$

Proof.

$$\dot{b}(t) = \frac{d}{dt}\left(\sum_{i=0}^{n}B_i^n(t)b_i\right) = \sum_{i=0}^{n}\frac{d}{dt}B_i^n(t)b_i = \sum_{i=0}^{n}n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))b_i = n\left(\sum_{i=0}^{n}B_{i-1}^{n-1}(t)b_i - \sum_{i=0}^{n}B_i^{n-1}(t)b_i\right) =$$

$$n\left(\sum_{i=1}^{n}B_{i-1}^{n-1}(t)b_i - \sum_{i=0}^{n}B_i^{n-1}(t)b_i\right) = n\left(\underbrace{\sum_{i=0}^{n-1}B_i^{n-1}(t)b_{i+1}}_{=b_1^{n-1}(t)} - \underbrace{\sum_{i=0}^{n-1}B_i^{n-1}(t)b_i}_{=b_0^{n-1}(t)}\right) = n\left(\sum_{i=0}^{n-1}B_i^{n-1}(t)(b_{i+1} - b_i)\right)$$

$\square$

Corollary 2.   • $\dot{b}(0) = n(b_1 - b_0)$

• $\dot{b}(1) = n(b_n - b_{n-1})$

• *The last segment in the algorithm of de Casteljou is the tangent of the Bezier curve in $b(t)$.*

• *The derivative of a bezier curve of degree n is a bezier curve of degree $n-1$ with control points $(b_1, b_0), (b_2 - b_1), \cdots, (b_n - b_{n-1})$.*

Corollary 3. $\ddot{b}(t) = n(n-1)\sum_{i=0}^{n-2}B_i^{n-2}(t)(b_{i+2} - 2b_{i+1} + b_i)$
$\ddot{b}(0) = n(n-1)(b_2 - 2b_1 + b_0)$, $\ddot{b}(1) = n(n-1)(b_n - 2b_{n-1} + b_{n-2})$

Corollary 4. *The curvature of a bezier curve in the point $b(0)$ depends only on b_0, b_1, b_2.*
The curvature of a bezier curve in the point $b(1)$ depends only on b_{n-2}, b_{n-1}, b_n.
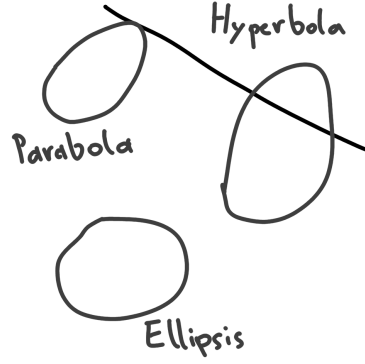
Figure 3: Categorization of Parabolas, Hyperbolas and Ellipsis as intersection points with the line at infinity.

Example 3. *Quadratic Bezier curve*

$$b(t) = \sum_{i=0}^{2} B_i^2(t)b_i = \binom{2}{0}t^0(1-t)^2b_0 + \binom{2}{1}t^1(1-t)^1b_1 + \binom{2}{2}t^2(1-t)^0b_2 = t^2(b_2 - 2b_1 + b_0) + t(2b_1 - 2b_0) + b_0$$

which is an affine transformation of a parabola and therefore a parabola.
  Quadratic bezier curves are parabolas.

Remark 8. *Line at infinity (Ferngerade) is the collection of points where parallel lines intersect.*

Remark 9. *Different applications using these curves are Rhino, OpenSCAD, Autocad, Geogebra, ...*

2 Parameterized curves
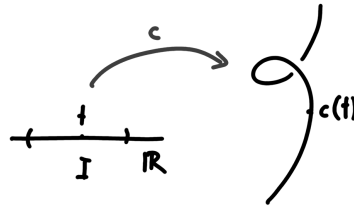
Figure 4: parameterized curve $c(t)$

Definition 4. $c : I \subseteq \mathbb{R} \to \mathbb{R}^3$ *is called a parameterized curve.*
$\dot{c}(t) := \frac{d}{dt}c(t)$ *is called the tangential vector. For $\mathbb{R}^3$ we have $\dot{c}(t) = (\dot{c}_1(t), \dot{c}_2(t), \dot{c}_3(t))$.*
The velocity is defined as $||\dot{c}(t)||$.
A point $c(t)$ is called regular, if $\dot{c}(t) \neq 0$ and is called singular, if $\dot{c}(t) = 0$.

Example 4. *A helix (Schraublinie) is defined by $c(t) = (\cos(t), \sin(t), t)^T$.*

$$\dot{c}(t) = (-\sin(t), \cos(t), 1)^T \qquad\qquad ||\dot{c}(t)|| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

  We see that the helix is passed through with constant velocity. Furthermore all points are regular.

Example 5. $c : \mathbb{R} \to \mathbb{R}^3, t \mapsto (t^2, t^3, t^4)$, $\dot{c}(t) = (2t, 3t^2, 4t^3)$. *We see that 0 is singular as $\dot{c}(0) = (0, 0, 0)$. Everywhere else the curve is regular.*
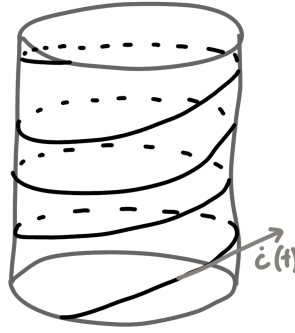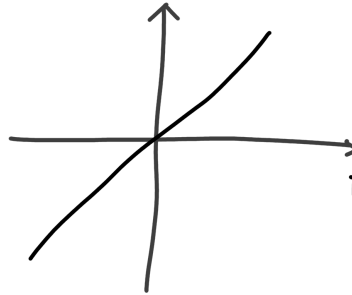
4

Figure 5: Helix with tangential vector

Figure 6: identity line can be parameterized such that $(0,0)$ is singular.

Remark 10. *A point being regular or singular depends on the parameterisation of the curve.*
For example $c(t) = (t, t)$ produces a regular curve, while $c(t) = (t^3, t^3)$ produces a curve where 0 is singular.
There are curves and points where no parameterisation exists such that the point is regular.

Definition 5. $c : I \to \mathbb{R}^2 \in C^2(I, \mathbb{R}^2)$
The curvature of the curve in the point $c(t)$ is defined as $\kappa(t) = \frac{\det(\dot{c}(t), \ddot{c}(t))}{||\dot{c}(t)||^3}$
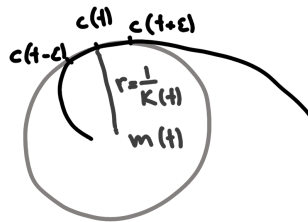

Figure 7: Circle of curvature

Example 6. *The circle of curvature has a radius of $\frac{1}{\kappa(t)}$. $m(t)$ is called the center of curvature.*
$m(t) = c(t) + \frac{1}{\kappa(t)} n(t)$ *where* $n(t) = \frac{(-\dot{c}_2(t), \dot{c}_1(t))}{||\dot{c}(t)||}$.

Remark 11. *Exercise: compare this definition of curvature with the school version concerning graphs.*

Example 7. *For a circle we have $c(t) = (r\cos(t), r\sin(t))^T$, $\dot{c}(t) = (-r\sin(t), r\cos(t))^T$, $\ddot{c}(t) = (-r\cos(t), -r\sin(t))^T$*

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$$\kappa(t) = \frac{\det \begin{pmatrix} -r\sin(t) & -r\cos(t) \\ r\cos(t) & -r\sin(t) \end{pmatrix}}{r^3} = \frac{r^2\sin^2(t) + r^2\cos^2(t)}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

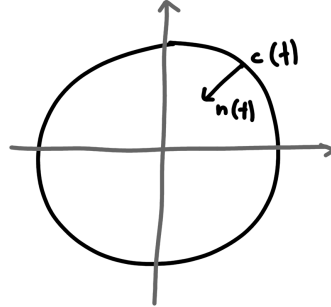$$n(t) = \frac{(-r\cos(t), -r\sin(t))}{r} = (-\cos(t), -\sin(t))$$

Figure 8: Circle with normal vector

Definition 6. *A point $c(t)$ with $\dot{\kappa}(t) = 0$ is called a vertex.*

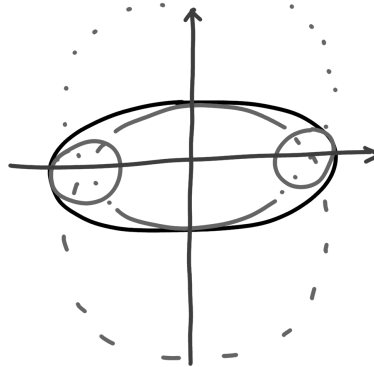**Example 8.** *An ellipse has four vertices.*

Figure 9: Ellipse and the four vertices.

$(t, \exp t)$ *has no vertex.*
Klothoids are curves with $\kappa(t) = t$. They are used in road construction and have no vertex.

Definition 7. $c : I \to \mathbb{R}^3$
$\kappa(t) = \frac{||\dot{c}(t) \times \ddot{c}(t)||}{||\dot{c}(t)||^3}$ *is called the curvature of a space curve.*
$\tau(t) = \frac{\det(\dot{c}(t), \ddot{c}(t), \dddot{c}(t))}{||\dot{c}(t) \times \ddot{c}(t)||^2}$ *is called torsion of a space curve.*

Example 9. *For the helix $t \mapsto (\cos(t), \sin(t), pt)$ the torsion depends on p.*

3 Properties of Bezier curves

Definition 8. $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ *is called **affine** if $\exists l : \mathbb{R}^n \to \mathbb{R}^m ...linear\ \exists v \in \mathbb{R}^m : \alpha(x) = l(x) + v$.*
α *is called **affinity** if α is affine and bijective.*

Example 10. *An example of an linear function is shear (Scherung).*

$$l\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
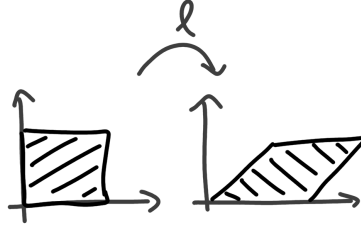
Area is preserved.

Figure 10: Shear preserves area

Theorem 3. *Bezier curves are invariant under affine transformations.*

Proof. Let $b(t) = \sum_{i=0}^{n} B_i^n(t)b_i$ be a bezier curve and $\alpha(x) = l(x) + v$ where $b_i \in \mathbb{R}^d, l : \mathbb{R}^d \to \mathbb{R}^m, v \in \mathbb{R}^m$

$$\alpha(b(t)) = l(b(t)) + v = l\left(\sum_{i=0}^{n} B_i^n(t)b_i\right) + v = \sum_{i=0}^{n} B_i^n(t)l(b_i) + v =$$

$$\sum_{i=0}^{n} B_i^n(t)l(b_i) + \sum_{i=0}^{n} B_i^n(t)v = \sum_{i=0}^{n} B_i^n(t)(l(b_i) + v) = \sum_{i=0}^{n} B_i^n(t)\alpha(b_i)$$
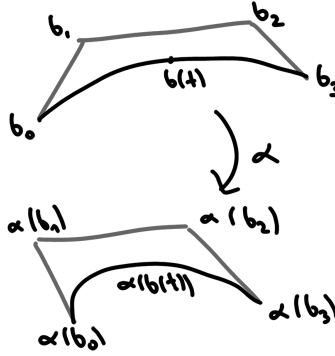
$\square$

Figure 11: Bezier curve is invariant under affine transformation

Lemma 3. $\sum_{i=0}^{n} B_i^n(t) = 1$

Proof. Binomial theorem: $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ for $x = t, y = 1 - t$ we get

$$1 = (t + (1 - t))^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1 - t)^{n-i} = \sum_{i=0}^{n} B_i^n(t)$$

$\square$

Lemma 4. *For $t \in [0, 1]$ it holds that $B_i^n(t) \geq 0$*

Proof.

$$B_i^n(t) = \underbrace{\binom{n}{i}}_{\geq 0} \underbrace{t^i}_{\geq 0} \underbrace{(1 - t)^{n-i}}_{\geq 0} \geq 0$$
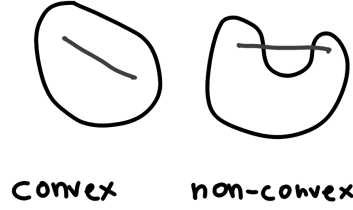
$\square$

Figure 12: example of convex and non-convex set

Definition 9. $M \subseteq \mathbb{R}^d$ then $conv(M) := \{\sum_{i=1}^{k} \lambda_i v_i : v_1, ..., v_k \in M, \lambda_1, ..., \lambda_k \geq 0, \sum_{i=1}^{k} \lambda_i = 1, k \in \mathbb{N}\}$ is called the convex hull of M.
M is called convex set, iff $\forall x, y \in M \forall t \in [0,1] : tx + (1-t)y \in M$.

Example 11.

Remark 12 (Pick theorem). *For any polygon formed of points in $\mathbb{Z} \times \mathbb{Z}$ it holds that the area can be calculated by $I + R/2 - 1$ where I is the number of points inside the polygon and R is the number of points on the edges of the polygon.*
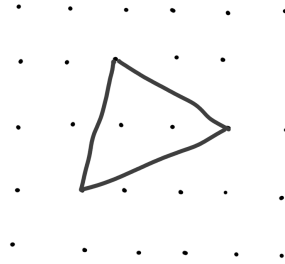

Figure 13: example picks theorem where $I = 2, R = 3$ and therefore $A = \frac{5}{2}$

Theorem 4. $\forall t \in [0,1] : b(t) \in conv(\{b_0, b_1, ..., b_n\})$

Proof. $b(t) = \sum_{i=0}^{n} \underbrace{B_i^n(t)}_{\geq 0} b_i \forall t \in [0,1]$ and $\sum_{i=0}^{n} B_i^n(t) = 1 \forall t \in [0,1]$. Therefore $b(t) \in conv(\{b_0, ..., b_n\})$ $\square$

Remark 13 (Montecarlo method for calculating area). *random points, count how many fall within the area from the total amount of points, estimate volume from that.*

Theorem 5. *Bezier curves are symmetric, meaning that if $b(t)$ is a bezier curves of $b_0, ..., b_n$ and $\tilde{b}(t)$ is a bezier curve of $b_n, ..., b_0$ then $b(t) = \tilde{b}(1-t)$*

Proof.

$$\tilde{b}(t) = \sum_{i=0}^{n} B_i^n(t) b_{n-i} = \sum_{i=0}^{n} \underbrace{B_i^n(1-t)(1-t)}_{B_i^n(t)} b_{n-i} = \sum_{i=0}^{n} B_{n-i}^n(1-t) b_i = \sum_{i=0}^{n} B_i^n(1-t) b_i = b(1-t)$$

$\square$

Lemma 5. $\alpha \in \mathbb{R} \implies B_i^n(\alpha t) = \sum_{j=0}^{n} B_i^j(\alpha) B_j^n(t)$
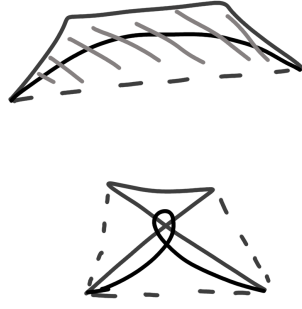
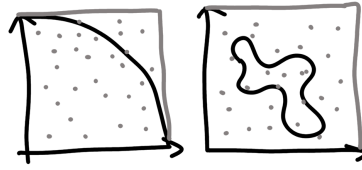Figure 14: example of a bezier curve and its convex hull

Figure 15: montecarlo method for a quarter circle and some random shape

Proof. can be done by induction, but is tedious $\qquad\square$

Theorem 6 (Sub-division-property (Unterteilungseigenschaft))**.** *Let $b(t)$ be a bezier curve to $b_0, ..., b_n$ and $\alpha \in \mathbb{R}$.*

$$\tilde{b}(t) := b(\alpha t) \quad \hat{b}(t) := b((1-\alpha)t + \alpha)$$

Then $\tilde{b}$ is a bezier curve to $b_0^0(\alpha), b_0^1(\alpha), ..., b_0^n(\alpha)$; $\hat{b}$ is a bezier curve to $b_0^n(\alpha), b_1^{n-i}(\alpha), ..., b_n^0(\alpha)$ and "glued together" they result in the original bezier curve.
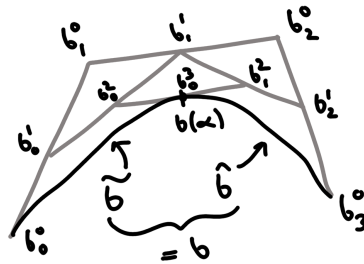

Figure 16: bezier curve b is divided at point $b(\alpha)$ into two bezier curves $\tilde{b}$ and $\hat{b}$

Proof.

$$\tilde{b}(t) = b(\alpha t) = \sum_{i=0}^{n} B_i^n(\alpha t) b_i = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} B_i^j(\alpha) B_j^n(t) \right) b_i = \sum_{j=0}^{n} \sum_{i=0}^{n} B_i^j(\alpha) B_j^n(t) b_i =$$

$$\sum_{j=0}^{n} B_j^n(t) \underbrace{\sum_{i=0}^{n^{j}} B_i^j(\alpha) b_i}_{b_0^j(\alpha)} = \sum_{j=0}^{n} B_j^n(t) b_0^j(\alpha)$$

Where we used $b_i^j(t) = \sum_{l=0}^{j} B_l^j(t) b_{i+l}$, which we already showed.
$\hat{b}$ is analogous. $\qquad \square$

Definition 10. *The polynomial ring is defined as* $\Pi_n := \{ f \in \mathbb{R}[t] : deg(f) \leq n \}$, *which is a vector space.*

Theorem 7. $\{ B_0^n(t), ..., B_n^n(t) \}$ *is a basis of* Π_n.

Proof. $\forall i : B_i^n(t) \in \Pi_n$ therefore $span(B_0^n(t), ..., B_n^n(t)) \subseteq \Pi_n$.
We know that $dim(\Pi_n) = n + 1$ as $\{1, t, t^2, ..., t^n\}$ is a basis.
Let $k \leq n$.

$$1 = \sum_{i=0}^{n-k} B_i^{n-k}(t) = \sum_{i=k}^{n} B_{i-k}^{n-k}(t) = \sum_{i=k}^{n} \binom{n-k}{i-k} t^{i-k} (1-t)^{n-i}$$

$$\implies \binom{n}{k} t^k = \sum_{i=k}^{n} \binom{n}{k} \binom{n-k}{i-k} t^i (1-t)^{n-i} = \sum_{i=k}^{n} \binom{n}{i} \binom{i}{k} t^i (1-t)^{n-i} = \sum_{i=k}^{n} \binom{i}{k} B_i^n(t)$$

$$\implies t^k \in span(B_0^n(t), ..., B_n^n(t)) \implies \Pi_n \subseteq span(B_0^n(t), ..., B_n^n(t))$$

$\square$

Corollary 5. *Every polynomial curve is a bezier curve.*

Remark 14. *Disadvantages of bezier curves are: unintuitive, local changes are not possible as every change in a control point changes the entire curve, points can be far away from the curve.*