PDGL Ü10

1) $u_{tt} - 9u_{xx} = 0 \quad , (x,t) \in (0,\pi)\times(0,\infty) \qquad u(\pi,t)=\pi \,, t>0 \qquad u(0,t)=2, \, t>0$

$u(x,0) = 5\sin(3x) + x + 2\left(1-\frac{x}{\pi}\right), x\in(0,\pi) \qquad u_t(x,0) = \sin(x), \, x\in(0,\pi)$

(i) ges: Transformation auf Problem mit homogenen Randbedingungen

$\tilde{u}(x,t) = u(x,t) - r(x) \qquad$ wobei $r(x) := \frac{(\pi - x)2 + (x-0)\pi}{\pi - 0} = \frac{2\pi - 2x + \pi x}{\pi} = 2 + \left(1-\frac{2}{\pi}\right)x$

$0 = u_{tt} - 9u_{xx} = \tilde{u}_{tt} - 9\frac{\partial^2}{\partial x^2}\tilde{u}(x,t) = \tilde{u}_{tt} - 9\tilde{u}_{xx} - \frac{\partial^2 r}{\partial x^2} = \tilde{u}_{tt} - 9\tilde{u}_{xx}$

$\tilde{u}(x,0) = u(x,0) - r(x) = 5\sin(3x) + x + 2 - \frac{2}{\pi}x - 2 - x + \frac{2}{\pi}x = 5\sin(3x)$

$\tilde{u}_t(x,0) = u_t(x,0) = \sin(x)$

Insgesamt erhalten wir $\tilde{u}_{tt} - 9\tilde{u}_{xx} = 0 \,, (x,t)\in(0,\pi)\times(0,\infty) \qquad \tilde{u}(0,t) = 0 = \tilde{u}(\pi,t) \,, \; t>0$

$\tilde{u}(x,0) = 5\sin(3x), \; x\in(0,\pi) \qquad \tilde{u}_t(x,0) = \sin(x), \; x\in(0,\pi)$

(ii) ges: Lsg bzgl. geeigneter ONB

Hinweis: $\tilde{\Phi}_k(x) = \sin(kx)$ sind EF von $-\partial_{xx}$ auf $(0,\pi)$ mit hom. Dirichletdaten $\lambda_k = k^2$ passende EW

normierte EF bilden ONB des $L^2(0,\pi)$.

$\Phi_k(x) := \tilde{\Phi}_k(x)\, \|\tilde{\Phi}_k\|_{L^2}^{-1} = \sqrt{\frac{\pi}{2}}\,\sin(kx)$

Ansatz $\tilde{u}(x,t) = \sum_{k=1}^{\infty} u_k(t)\,\Phi_k(x)$

$0 = \tilde{u}_{tt} - 9\tilde{u}_{xx} = \sum_{k=1}^{\infty} u_{k,tt}(t)\,\Phi_k(x) + 9\sum_{k=1}^{\infty} u_k(t)\overbrace{(-\partial_{xx}\Phi_k(x))}^{=k^2\Phi_k(x)} = \sum_{k=1}^{\infty}\left(u_{k,tt}(t) + 9k^2u_k(t)\right)\Phi_k(x)$

$\Rightarrow \forall l\in\mathbb{N}: \; 0 = \langle \tilde{u}_{tt} - 9\tilde{u}_{xx}, \Phi_l\rangle = u_{l,tt}(t) + 9l^2u_l(t) = u_l'' + 9l^2u_l$

$\Rightarrow u_l(t) = c_1\sin(3lt) + c_2\cos(3lt)$

$\sqrt{\frac{2}{\pi}}\,5\,\Phi_3(x) = 5\sin(3x) = \tilde{u}(x,0) = \sum_{k=1}^{\infty} u_k(0)\,\Phi_k(x) \qquad \Rightarrow u_3(0) = 5\sqrt{\frac{2}{\pi}} \;\; \forall l\in\mathbb{N}\setminus\{3\}: u_l(0)=0$

$\sqrt{\frac{2}{\pi}}\,\Phi_1(x) = \sin(x) = \tilde{u}_t(x,0) = \sum_{k=1}^{\infty} u_k'(0)\,\Phi_k(x) \qquad \Rightarrow u_1'(0) = \sqrt{\frac{2}{\pi}} \;\; \forall l\in\mathbb{N}\setminus\{1\}: u_l'(0)=0$

Wir setzen die Anfangswerte ein:

$u_1(t) = \sqrt{\frac{2}{\pi}}\,\frac{1}{3l}\sin(3lt) \qquad u_3(t) = 5\sqrt{\frac{2}{\pi}}\cos(3lt) \qquad u_l(t) = 0$

$\Rightarrow \tilde{u}(x,t) = \sum_{k=1}^{\infty} u_k(t)\,\Phi_k(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{3l}\sin(3lt)\sqrt{\frac{\pi}{2}}\sin(x) + 5\sqrt{\frac{2}{\pi}}\cos(3lt)\sqrt{\frac{\pi}{2}}\sin(3x)$

$\Rightarrow u(x,t) = \tilde{u}(x,t) + r(x) = \frac{\sin(3lt)\sin(x)}{3l} + 5\cos(3lt)\sin(3x) + 2 + x + \frac{2}{\pi}x$

$\square$

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2) $\Omega = (0,1)\times(0,2)$ $\quad u_t - 3\Delta u = 0$ in $\Omega\times(0,\infty)$ $\quad u(0,y,t) = 2y,\ y\in(0,2)$

$u(1,y,t) = \frac{7}{2}y,\ y\in(0,2)$ $\quad u(x,0,t) = 0,\ x\in(0,1)$

$u(x,2,t) = 4+3x,\ x\in(0,1)$ $\quad u_0(x,y) = \sin(4\pi x)\sin(\frac{7\pi}{2}y) + (1-x)2y + \frac{7}{2}xy$

(i) ges: Transformation auf Problem mit homogenen Randbedingungen

$\tilde{u}(x,y,t) = u(x,y,t) - r(x,y)$ wobei $r(x,y) = 2y + \frac{3}{2}xy$

$0 = u_t - 3\Delta u = \tilde{u}_t - 3\Delta\tilde{u} - 3\Delta r = \tilde{u}_t - 3\Delta\tilde{u}$

$\tilde{u}(x,y,0) = u_0(x,y) - r(x,y) = \sin(4\pi x)\sin(\frac{7\pi}{2}y) + \cancel{2y} - 2xy + \frac{7}{2}xy - \cancel{2y} - \frac{3}{2}xy = \sin(4\pi x)\sin(\frac{7\pi}{2}y)$

Wir erhalten folgendes Problem:

$\tilde{u}_t - 3\Delta\tilde{u} = 0$ $\quad \tilde{u}(0,y,t) = \tilde{u}(1,y,t) = \tilde{u}(x,0,t) = \tilde{u}(x,2,t) = 0,\ (x,y)\in\Omega$

$\tilde{u}(x,y,0) = \sin(4\pi x)\sin(\frac{7\pi}{2}y),\ (x,y)\in\Omega$

(ii) Existenz und Eindeutigkeit von Lsgen des transformierten Problems?

Das transformierte Problem hat die Form $u_t - \operatorname{div}(A\nabla u) + cu = 0$ in $\Omega\times(0,\infty)$ $\quad u(\cdot,0) = u_0$ in Ω

$u = 0$ auf $\partial\Omega\times(0,\infty)$ mit $u_0(x,y) = \sin(4\pi x)\sin(\frac{7\pi}{2}y) \overset{*}{\in} L^2(\Omega)$

Mit Theorem 6.12 folgt die Existenz einer eindeutigen Lösung.

$* \int_\Omega |u_0(x)|^2 d\binom{x}{y} = \int_0^1\int_0^2 |\sin(4\pi x)\sin(\frac{7\pi}{2}y)|^2 dy\,dx \leq \int_0^1\int_0^2 1\,dy\,dx = 2 < \infty$

(iii) ges: Lsg des ursprünglichen Problems bzgl. geeigneten ONB

Hinweis: $\tilde{\Phi}_{k,m} = \sin(k\pi x)\sin(\frac{m\pi}{2}y)$ sind EF von $-\Delta$ auf Ω mit homogenen Dirichletrandbedingungen

$\lambda_{k,m} = (k\pi)^2 + (\frac{m\pi}{2})^2$ ... zugehörigen EW $\quad$ für $k,m\in\mathbb{N}$

normierte EF bilden ONB des $L^2(\Omega)$

$\Phi_{k,m} = \tilde{\Phi}_{k,m}\,\|\tilde{\Phi}_{k,m}\|^{-1} = \tilde{\Phi}_{k,m}\sqrt{2}$

Ansatz: $\tilde{u}(x,y,t) = \sum_{k,m=1}^\infty u_{k,m}(t)\,\Phi_{k,m}(x,y)$

$0 = \tilde{u}_t - 3\Delta\tilde{u} = \sum_{k,m=1}^\infty u'_{k,m}(t)\Phi_{k,m}(x,y) + 3\sum_{k,m=1}^\infty u_{k,m}(t)\underbrace{\Delta\Phi_{k,m}(x,y)}_{=\lambda_{k,m}\Phi_{k,m}} = \sum_{k,m=1}^\infty (u'_{k,m} + 3\lambda_{k,m}u_{k,m})\,\Phi_{k,m}$

$\Rightarrow \forall k,m\in\mathbb{N}:\ 0 = \langle \tilde{u}_t - 3\Delta\tilde{u}, \Phi_{k,m}\rangle = u'_{k,m}(t) + (3k^2\pi^2 + \frac{3}{4}m^2\pi^2)\,u_{k,m}(t)$

Diese DGL lösen ergibt $u_{k,m}(t) = c_1\exp(-3\pi^2(k^2+\frac{m^2}{4})t)$

...

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2) ...

$\sqrt{2}\,\cancel{\tfrac{1}{\sqrt{2}}}\,\Phi_{4,7}(x,y) = \sin(4\pi x)\sin\left(\frac{7\pi}{2}y\right) = \tilde{v}(x,y,0) = \sum_{k,m=1}^{\infty} v_{k,m}(0)\,\Phi_{k,m}(x,y) \quad \sqrt{2}$

$\Rightarrow v_{4,7}(0) = \frac{1}{\sqrt{2}} \qquad \forall (k,m) \neq (4,7): \; v_{k,m}(0) = 0$

Einsetzen dieser Anfangsbedingungen ergibt

$v_{4,7}(t) = \exp\left(-3\pi^2\left(4^2 + \frac{7^2}{2}\right)t\right)\frac{1}{\sqrt{2}} \quad \forall (k,m) \neq (4,7): \; v_{k,m}(t) = 0$

$\Rightarrow \tilde{v}(x,y,t) = v_{4,7}(t)\,\Phi_{4,7}(x,y) = \frac{1}{\sqrt{2}} e^{-3\pi^2\left(16 + \frac{49}{2}\right)t} \sin(4\pi x)\sin\left(\frac{7\pi}{2}y\right)$

$\Rightarrow v(x,y,t) = \tilde{v} + r = \frac{1}{\sqrt{2}} e^{-\frac{243}{2}\pi^2 t} \sin(4\pi x)\sin\left(\frac{7\pi}{2}y\right) + 2y + \frac{3}{2}xy$

(iv) Ist die Lsg aus (iii) in $C^0([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$ eindeutig?

$r \in C^0([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$

$\tilde{v} \in$ —— " —— nach (ii)

$\Rightarrow v \in$ —— " ——

v_1, v_2 Lsg in —— " —— $\Rightarrow v := v_1 - v_2$ löst hom. Problem

Mit gleichem Satz wie in Punkt (ii) folgt v ist eindeutig

$v = 0$ ist Lsg $\Rightarrow v_1 - v_2 = 0$ also gleich.

$\sum_{k,m} \langle \phi_{k,m}, u_0 \rangle \phi_{k,m} = \cancel{\sum} \langle \phi_{4,7}, \phi_{4,7} \frac{1}{\sqrt{2}} \rangle u_0 \sqrt{2} =$

$= u_0$

$\iint \sin(k\pi x) \sin\left(\frac{m\pi}{2}y\right) dx\, dy$

$\langle \sum a_k \phi_k, \phi_\ell \rangle$

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3) $h \in C^2(\mathbb{R})$ $\quad G = (\mathbb{R}^3 \setminus \{0\}) \times (0,\infty)$ $\quad u(x,t) := \dfrac{h(t-|x|)}{|x|}$

zz: $u(x,t)$ löst in G die Wellengleichung

$$\frac{\partial u}{\partial x_i}(x,t) = \frac{h'(t-|x|)\left(-\frac{x_i}{|x|}\right)|x| - h(t-|x|)\frac{x_i}{|x|}}{|x|^2} = -\frac{x_i}{|x|^2}\left(h'(t-|x|) + \frac{1}{|x|}h(t-|x|)\right) =$$

$$= -\frac{x_i}{|x|^2}\left(h'(t-|x|) + u(x,t)\right)$$

$$\frac{\partial^2 u}{\partial x_i^2}(x,t) = -\frac{|x|^2 - x_i 2|x|\frac{x_i}{|x|}}{|x|^4}\left(h'(t-|x|) + u(x,t)\right) - \frac{x_i}{|x|^2}\left(h''(t-|x|)\left(-\frac{x_i}{|x|}\right) + \frac{\partial u}{\partial x_i}(x,t)\right) =$$

$$= -\frac{|x|^2 - 2x_i^2}{|x|^4}\left(h'(t-|x|) + u(x,t)\right) + \frac{x_i^2}{|x|^3}h''(t-|x|) + \frac{x_i^2}{|x|^4}\left(h'(t-|x|) + u(x,t)\right) =$$

$$= \left(-\frac{1}{|x|^2} + 2\frac{x_i^2}{|x|^4}\right)\left(h'(t-|x|) + u(x,t)\right) + \frac{x_i^2}{|x|^3}h''(t-|x|) + \frac{x_i^2}{|x|^4}h'(t-|x|) + \frac{x_i^2}{|x|^4}u(x,t) =$$

$$= \frac{x_i^2}{|x|^3}h''(t-|x|) + \left(2\frac{x_i^2}{|x|^4} + \frac{x_i^2}{|x|^4} - \frac{1}{|x|^2}\right)h'(t-|x|) + \left(2\frac{x_i^2}{|x|^4} - \frac{1}{|x|^2} + \frac{x_i^2}{|x|^4}\right)u(x,t) =$$

$$= \frac{x_i^2}{|x|^3}h''(t-|x|) + \left(3\frac{x_i^2}{|x|^4} - \frac{1}{|x|^2}\right)\left(h'(t-|x|) + u(x,t)\right)$$

$$\Delta u(x,t) = \sum_{k=1}^{3}\frac{\partial^2 u}{\partial x_i^2}(x,t) = \left(\sum_{k=1}^{3} x_i^2\right)\left(\frac{h''(t-|x|)}{|x|^3}\right) + \left(\sum_{k=1}^{3} x_i^2\right)\frac{3}{|x|^4}\left(h'(t-|x|) + u(x,t)\right) - \frac{3}{|x|^2}\left(h'(t-|x|) + u(x,t)\right) =$$

$$= |x|^2\frac{h''}{|x|^3} + |x|^2\frac{3}{|x|^4}(h' + u) - \frac{3}{|x|^2}(h' + u) = \frac{h''}{|x|}$$

$$\Rightarrow \quad u_{tt} - \Delta u = \frac{h''(t-|x|)}{|x|} - \frac{h''(t-|x|)}{|x|} = 0$$

$\square$

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4) $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$ $\quad u(x,0) = u_0(x)$ in $\mathbb{R}^n$ $\quad u_t(x,0) = u_1(x)$ in $\mathbb{R}^n$

$u_0, u_1 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ $\quad u$ ... Lsg der PDGL

ges: Darstellung von $\hat{u}$

$$0 = F[u_{tt} - \Delta u] = F[u_{tt}] - F\Big[\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u\Big] = \frac{\partial^2}{\partial t^2} F[u] - \sum_{j=1}^{n} F\Big[\frac{\partial^2}{\partial x_j^2} u\Big] =$$

$$= \frac{\partial^2}{\partial t^2} F[u] - \sum_{j=1}^{n} i^2 F[u] k_j^2 = \frac{\partial^2}{\partial t^2} F[u] + F[u]\, |k|^2$$

$\Rightarrow F[u]$ löst die DGL $w'' + |k|^2 w = 0$ und hat somit die Form

$$F[u] = c_1(k) \sin(|k| t) + c_2(k) \cos(|k| t)$$

$F[u](k,0) = F[u_0](k)$ $\quad \Rightarrow c_2(k) = F[u_0](k)$

$\frac{\partial}{\partial t} F[u](k,0) = F[u_1](k)$ $\quad \Rightarrow c_1(k) = F[u_1](k) \frac{1}{|k|}$

$$\Rightarrow F[u](k,t) = \frac{F[u_1](k)}{|k|} \sin(|k| t) + F[u_0](k) \cos(|k| t)$$

$\square$