ANA Ü12

8.) i) $\sum_{n=0}^{\infty} \underbrace{(2+(-1)^n)^n}_{a_n} \cdot z^n$ ges: Konvergenzradius R

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} = \frac{1}{\inf\{\sup\{\sqrt[k]{|\underbrace{(2+(-1)^k)^k}_{>0}|} : k \geq n\} : n \in \mathbb{N}\}}$$

$$= \frac{1}{\inf\{\sup\{2+(-1)^k : k \geq n\} : n \in \mathbb{N}\}} = \frac{1}{\inf\{3 : n\in\mathbb{N}\}} = \frac{1}{3}$$

ii) $\sum_{n=0}^{\infty} n!\, z^n$

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} = \frac{1}{\inf\{\sup\{\sqrt[k]{|\underbrace{k!}_{>0}|} : k\geq n\} : n\in\mathbb{N}\}}$$

da $\sqrt[k]{k!} \xrightarrow{k\to\infty} \infty$ geht R gegen 0

$\Rightarrow R = 0$

iii) $\sum_{n=1}^{\infty} \frac{1}{n^2} (\sqrt{n^2+n} - \sqrt{n^2+1})^n z^n$

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} = \frac{1}{\inf\{\sup\{\sqrt[k]{|\frac{1}{k^2}(\sqrt{k^2+k} - \sqrt{k^2+1})^k|} : k\geq n\} : n \in\mathbb{N}\}}$$

$$= \frac{1}{\inf\{\sup\{\sqrt[k]{\frac{1}{k^2}}\,(\sqrt{k^2+k} - \sqrt{k^2+1}) : k\geq n\} : n\in\mathbb{N}\}}$$

$$\sqrt[k]{\frac{1}{k^2}} = \exp\left(\log\left(\sqrt[k]{\frac{1}{k^2}}\right)\right) = \exp\left(\log\left(k^{-2\cdot\frac{1}{k}}\right)\right) = \exp\left(-2\cdot\log\left(\sqrt[k]{k}\right)\right)$$

$$\lim_{k\to\infty} \exp(-2\cdot \log(\sqrt[k]{k})) = \exp(0) = 1$$

$\sqrt[k]{k} \to 1$, $\log(\sqrt[k]{k}) \to 0$

$$\sqrt{k^2+k} - \sqrt{k^2+1} = \frac{(\sqrt{k^2+k} - \sqrt{k^2+1})(\sqrt{k^2+k}+\sqrt{k^2+1})}{\sqrt{k^2+k}+\sqrt{k^2+1}} = \frac{k^2+k-k^2+1}{\sqrt{k^2+k}+\sqrt{k^2+1}}$$

$$= \frac{\frac{k}{k}+\frac{1}{k}}{\sqrt{\frac{k^2}{k^2}+\frac{k}{k^2}} + \sqrt{\frac{k^2}{k^2}+\frac{1}{k}^2}} = \frac{1+\frac{1}{k}}{\sqrt{1+\frac{1}{k}}+\sqrt{1+\frac{1}{k^2}}}$$

$$\lim_{k\to\infty} \frac{1+\frac{1}{k}}{\sqrt{1+\frac{1}{k}}+\sqrt{1+\frac{1}{k^2}}} = \frac{1+0}{\sqrt{1+0}+\sqrt{1+0}} = \frac{1}{2}$$

$$R = \frac{1}{1\cdot\frac{1}{2}} = 2$$