ANA Ü12

4 a) $f_n(x)=\frac{n}{n^2+x^2}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{n}{n^2+x^2} = \lim_{n\to\infty}\frac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{x^2}{n^2}} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{x^2}{n^2}} = \frac{0}{1+0} = 0 = f(x)$$

Gleichmäßig konvergent?

$\forall \varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\geq N: d_\infty(f_n,f)\leq\varepsilon$

Sei $\varepsilon>0$ bel.

$$d_\infty(f_n,f)=\sup\{d(f_n(x),f(x)) : x\in\mathbb{R}\} = \sup\{\left|\frac{n}{n^2+x^2}\right| : x\in\mathbb{R}\}$$

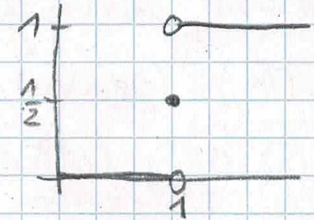
$\left|\frac{n}{n^2+x^2}\right| \xrightarrow{n\to\infty} 0$ und ist $\searrow$ $\Rightarrow \sup\{\dots\} = \frac{N}{N^2+x^2}$

Da $\lim_{N\to\infty}\frac{N}{N^2+x^2}=0$ $\exists N\in\mathbb{N}: \frac{N}{N^2+x^2}<\varepsilon$

b) $f_n(x)=\frac{x^n}{1+x^n}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{x^n}{1+x^n} = \lim_{n\to\infty}\frac{\frac{x^n}{x^n}}{\frac{1}{x^n}+\frac{x^n}{x^n}} = \lim_{n\to\infty}\frac{1}{\frac{1}{x^n}+1} = \begin{cases} 1, & \text{falls } x>1 \\ \frac{1}{2}, & \text{falls } x=1 \\ 0, & \text{falls } x<1 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 0, & \text{falls } 0\leq x<1 \\ \frac{1}{2}, & \text{falls } x=1 \\ 1, & \text{falls } x>1 \end{cases}$$



Gleichmäßig konvergent?

Nein.

Für $(x_n)_{n\in\mathbb{N}} = (\sqrt[n]{n})_{n\in\mathbb{N}}$ gilt

$$\left|\frac{(x_n)^n}{1+(x_n)^n} - 1\right| = \frac{(\sqrt[n]{n})^n}{1+(\sqrt[n]{n})^n} = \frac{n}{1+n} = \frac{\frac{n}{n}}{\frac{1}{n}+\frac{n}{n}} = \frac{1}{1+\frac{1}{n}}$$

$\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$, d.h. für $\varepsilon>1$ $\nexists N\in\mathbb{N}\ \forall n\geq N: d_\infty(f_n,f)<\varepsilon$