LINAG Ü10

9.4.2. $Bil(V)$... VR der Bilinearformen $V\times V\to K$ $Sym(V)\subseteq Bil(V)$.. symmetrische Bilinearformen

$Alt(V)\subseteq Bil(V)$... alternierende Bilinearformen

a) zz: $Sym(V)$ und $Alt(V)$ sind UR von $Bil(V)$

- $Sym(V)\neq\emptyset$: $\sigma_0\in Sym(V)$ $\sigma_0: V\times V\to K$ $(x,y)\mapsto 0_K$ ist offensichtlich sym.
- $\forall \sigma_1,\sigma_2\in Sym(V): \sigma_1+\sigma_2\in Sym(V)$: Sei $\sigma_1,\sigma_2\in Sym(V)$ bel.

  $(\sigma_1+\sigma_2)(x,y)=\sigma_1(x,y)+\sigma_2(x,y)=\sigma_1(y,x)+\sigma_2(y,x)=(\sigma_1+\sigma_2)(y,x)$
- $\forall\sigma\in Sym(V)\ \forall c\in K: c\cdot\sigma\in Sym(V)$: Sei $\sigma\in Sym(V)$ bel. Sei $c\in K$ bel.

  $(c\cdot\sigma)(x,y)=c\cdot\sigma(x,y)=c\cdot\sigma(y,x)=(c\cdot\sigma)(y,x)$
- $Alt(V)\neq\emptyset$: $\sigma_0\in Alt(V)$, da $\forall x\in V$: $\sigma_0(x,x)=0$
- $\forall\sigma_1,\sigma_2\in Alt(V): \sigma_1+\sigma_2\in Alt(V)$: Sei $\sigma_1,\sigma_2\in Alt(V)$ bel.

  $(\sigma_1+\sigma_2)(x,x)=\sigma_1(x,x)+\sigma_2(x,x)=0$
- $\forall\sigma\in Alt(V)\ \forall c\in K: c\cdot\sigma\in Alt(V)$: Sei $\sigma\in Alt(V)$, $c\in K$ bel

  $(c\cdot\sigma)(x,x)=c\cdot\sigma(x,x)=c\cdot 0=0$

$\Rightarrow Sym(V)$ und $Alt(V)$ sind UR

b) $\operatorname{char} K\neq 2$ zz: $Bil(V)=Sym(V)\oplus Alt(V)$

Sei $\sigma\in Sym(V)\cap Alt(V)$ bel. Da $\sigma\in Alt(V)$ gilt

$0=\sigma(v+w,v+w)=\sigma(v,v+w)+\sigma(w,v+w)=\sigma(v,v)+\sigma(v,w)+\sigma(w,v)+\sigma(w,w)$

$\Leftrightarrow \sigma(v,w)=-\sigma(w,v)$ da σ aber auch aus $Sym(V)$ ist gilt auch

$\sigma(v,w)=\sigma(w,v)$ $\Rightarrow \forall v,w\in V: \sigma(v,w)=0$ $\Leftrightarrow \sigma=0_{Bil(V)}$

Sei $\sigma\in Bil(V)$ bel. Sei $B=(b_i)_{i\in I}$ eine Basis von V

$$\sigma_s(b_i,b_j)=\begin{cases}\sigma(b_i,b_j), & \text{falls } i=j\\ \frac{\sigma(b_i,b_j)+\sigma(b_j,b_i)}{2}, & \text{sonst}\end{cases}\qquad \sigma_a(b_i,b_j)=\begin{cases}0, & \text{falls } i=j\\ \frac{\sigma(b_i,b_j)-\sigma(b_j,b_i)}{2}, & \text{sonst}\end{cases}$$

(hätte man auch einfacher definieren können)

Sei $i\in I$ bel. $\sigma_s(b_i,b_i)+\sigma_a(b_i,b_i)=\sigma(b_i,b_i)$

Sei $i,j\in I$, $i\neq j$ bel. $\sigma_s(b_i,b_j)+\sigma_a(b_i,b_j)=\frac{\sigma(b_i,b_j)}{2}+\cancel{\frac{\sigma(b_j,b_i)}{2}}+\frac{\sigma(b_i,b_j)}{2}-\cancel{\frac{\sigma(b_j,b_i)}{2}}=\sigma(b_i,b_j)$

$\sigma_s(b_i,b_j)=\frac{\sigma(b_i,b_j)+\sigma(b_j,b_i)}{2}=\frac{\sigma(b_j,b_i)+\sigma(b_i,b_j)}{2}=\sigma_s(b_j,b_i)$ $\Rightarrow$ symmetrisch

$\sigma_a(b_i,b_j)=\frac{\sigma(b_i,b_j)-\sigma(b_j,b_i)}{2}=-\frac{\sigma(b_j,b_i)-\sigma(b_i,b_j)}{2}=-\sigma_a(b_j,b_i)$ $\Rightarrow$ schiefsymmetrisch und alternierend

$\Rightarrow Bil(V)=Sym(V)\oplus Alt(V)$

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9.4.2. ... c) $\operatorname{char} K \neq 2$ $\quad A \in K^{n \times n}$

zz: $\exists\, S, T \in K^{n \times n}$ S... symmetrisch, T... schiefsymmetrisch $(A = S + T)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

$$S = \begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} & \cdots & \frac{a_{1n}+a_{n1}}{2} \\ \frac{a_{21}+a_{12}}{2} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ \frac{a_{n1}+a_{1n}}{2} & \cdots & & a_{nn} \end{pmatrix}$$ ist symmetrisch $\frac{a_{ij}+a_{ji}}{2} = \frac{a_{ji}+a_{ij}}{2}$

$$T = \begin{pmatrix} 0 & \frac{a_{12}-a_{21}}{2} & \cdots & \frac{a_{1n}-a_{n1}}{2} \\ \frac{a_{21}-a_{12}}{2} & 0 & & \vdots \\ \vdots & & \ddots & \\ \frac{a_{n1}-a_{1n}}{2} & \cdots & & 0 \end{pmatrix}$$ ist schiefsymmetrisch $\frac{a_{ij}-a_{ji}}{2} = -\frac{a_{ji}-a_{ij}}{2}$

$$S + T = \begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} + \frac{a_{12}-a_{21}}{2} & \cdots & \frac{a_{1n}+a_{n1}}{2} + \frac{a_{1n}-a_{n1}}{2} \\ \frac{a_{21}+a_{12}}{2} + \frac{a_{21}-a_{12}}{2} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ \frac{a_{n1}+a_{1n}}{2} + \frac{a_{n1}-a_{1n}}{2} & \cdots & & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$