ANA Ü5

4.) $\int_{\frac{\pi}{8}}^{\frac{\pi}{2}} \frac{1}{\sin(x)}\, dx$

$\int \frac{1}{\sin(x)} dx = \int \frac{\sin(x)}{(\sin(x))^2} dx = \int \frac{\sin(x)}{1-(\cos(x))^2}$ $\quad$ [$u = \cos(x) \quad \frac{du}{dx} = -\sin(x) \quad dx = \frac{1}{-\sin(x)} du$

$= \int \frac{\sin(x)}{1-u^2} \frac{1}{-\sin(x)} du = -\int \frac{1}{1-u^2} du = -\int \frac{1}{(1+u)(1-u)} du$

$\frac{1}{(1+u)(1-u)} = \frac{A}{1+u} + \frac{B}{1-u} = \frac{A(1-u)+B(1+u)}{(1+u)(1-u)} = \frac{A+B+(B-A)u}{(1+u)(1-u)}$

$\Rightarrow A+B=1 \qquad B-A=0 \quad \Rightarrow A=B \quad \Rightarrow A=B=\frac{1}{2}$

$\Rightarrow \frac{1}{(1+u)(1-u)} = \frac{1}{2}\cdot\left(\frac{1}{1+u}\right) + \frac{1}{2}\cdot\left(\frac{1}{1-u}\right)$

$-\int \frac{1}{(1+u)(1-u)} du = -\int \frac{1}{2}\cdot\left(\frac{1}{1+u} + \frac{1}{1-u}\right) du = -\frac{1}{2}\cdot\left(\int \frac{1}{1+u} du - \int \frac{1}{u-1} du\right)$

$= -\frac{1}{2}\cdot\left(\ln(1+u) - \ln(u-1)\right) = -\frac{1}{2}\cdot\left(\ln(1+\cos(x)) - \ln(\cos(x)-1)\right)$

$= -\frac{1}{2}\cdot\left(\ln\left(\frac{1+\cos(x)}{\cos(x)-1}\right)\right) = \frac{1}{2}\cdot\left(\ln\left(\frac{\cos(x)-1}{\cos(x)+1}\right)\right) = \frac{1}{2}\cdot\ln\left(\frac{(\cos(x))^2-1}{(\cos(x)+1)^2}\right)$

$= \frac{1}{2}\cdot\ln\left(\frac{(\sin(x))^2}{(\cos(x)+1)^2}\right) = \ln\left(\frac{\sin(x)}{\cos(x)+1}\right)$

$\int_{\frac{\pi}{8}}^{\frac{\pi}{2}} \frac{1}{\sin(x)} dx = \ln\left(\frac{\sin(\frac{\pi}{2})}{\cos(\frac{\pi}{2})+1}\right) - \ln\left(\frac{\sin(\frac{\pi}{8})}{\cos(\frac{\pi}{8})+1}\right) \approx 1{,}615$

$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{(\sin(x))^2\cdot(\cos(x))^4} dx$

$\int \frac{1}{(\sin(x))^2\cdot(\cos(x))^4} dx = \int \frac{1}{(\sin(x))^2}\cdot\left(\frac{1}{(\cos(x))^2}\right)^2 dx = \int \frac{(\sin(x))^2+(\cos(x))^2}{(\sin(x))^2}\cdot\left(\frac{(\sin(x))^2+(\cos(x))^2}{(\cos(x))^2}\right)^2 dx$

$= \int \left(1+\frac{(\cos(x))^2}{(\sin(x))^2}\right)\cdot\left((\tan(x))^2+1\right)^2 dx = \int \left(1+\frac{1}{(\tan(x))^2}\right)\cdot\left((\tan(x))^2+1\right)^2 dx$

[$u = \tan(x) \qquad \frac{du}{dx} = \frac{1}{(\cos(x))^2} \qquad dx = (\cos(x))^2 du = \frac{1}{\frac{1}{(\cos(x))^2}} du = \frac{1}{(\tan(x))^2+1} du$

$= \int \left(1+\frac{1}{u^2}\right)\cdot\left(u^2+1\right)^{\cancel{2}}\cdot\frac{1}{\cancel{u^2+1}} du = \int u^2+1+1+\frac{1}{u^2} du = \int u^2+\frac{1}{u^2}+2\, du$

$= \int u^2 du + \int \frac{1}{u^2} du + \int 2\, du = \frac{u^3}{3} - \frac{1}{u} + 2u = \frac{(\tan(x))^3}{3} - \frac{1}{\tan(x)} + 2\tan(x)$

$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{(\sin(x))^2\cdot(\cos(x))^4} dx = \frac{(\tan(\frac{\pi}{4}))^3}{3} - \frac{1}{\tan(\frac{\pi}{4})} + 2\tan(\frac{\pi}{4}) - \left(\frac{(\tan(\frac{\pi}{8}))^3}{3} - \frac{1}{\tan(\frac{\pi}{8})} + 2\tan(\frac{\pi}{8})\right)$

$= \frac{1}{3} - 1 + 2 - \left(\frac{(\tan(\frac{\pi}{8}))^3}{3} - \frac{1}{\tan(\frac{\pi}{8})} + 2\tan(\frac{\pi}{8})\right) \approx 2{,}895$