

# Homework 1

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## Problem 1

$$p(t) = \det(A - t \cdot \text{Id}) = \begin{vmatrix} a_{11} - t & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - t & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} - t & a_{n2} & \cdots & a_{nn} - t \end{vmatrix}$$

## Problem 2

The Gamma function is defined as

$$\Gamma(x) := \lim_{n \rightarrow \infty} \frac{n! n^x}{x(x+1) \cdots (x+n)}$$

There holds the Weierstraß product representation

$$\frac{1}{\Gamma(x)} = x \cdot e^{Cx} \cdot \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{-x/k} \quad \text{with} \quad C := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n\right)$$

## Problem 3

Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be given functions given by

$$f(x) := \begin{cases} -1 & \text{if } x < -\frac{\pi}{2}, \\ \sin(x) & \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \\ 1 & \text{if } x > \frac{\pi}{2}. \end{cases} \quad \text{and} \quad g(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

## Problem 4

For  $q \in \mathbb{R}$ , it holds that

$$\lim_{n \rightarrow \infty} q^n = \begin{cases} +\infty & \text{if } q > 1, \\ 1 & \text{if } q = 1, \\ 0 & \text{if } -1 < q < 1, \\ \nexists & \text{if } q \leq -1. \end{cases}$$

### Problem 5

$$A := \begin{pmatrix} \alpha & 2\alpha & 3\alpha & \cdots & n\alpha \\ 0 & \alpha & 2\alpha & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 3\alpha \\ \vdots & \ddots & \ddots & \ddots & 2\alpha \\ 0 & \cdots & 0 & 0 & \alpha \end{pmatrix} \in \mathbb{R}_{\text{tria}}^{n \times n}$$

### Problem 6

$$\begin{aligned} \det(V) &= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} \\ &= a^3b^2c - a^3b^2d - a^3bc^2 + a^3bd^2 \\ &\quad + a^3c^2d - a^3cd^2 - a^2b^3c + a^2b^3d \\ &\quad + a^2bc^3 - a^2bd^3 - a^2c^3d + a^2cd^3 \\ &\quad + ab^3c^2 - ab^3d^2 - ab^2c^3 + ab^2d^3 \\ &\quad + ac^3d^2 - ac^2d^3 - b^3c^2d + b^3cd^2 \\ &\quad + b^2c^3d - b^2cd^3 - bc^3d^2 + bc^2d^3 \\ &= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \end{aligned}$$

### Problem 7

**Theorem 1** For  $a, b \in \mathbb{R}$  and a continuous function  $f : (a, b) \rightarrow \mathbb{R}$ , the following two assertions are equivalent:

- (i)  $f$  is uniformly continuous.
- (ii)  $f$  has a continuous extension onto the compact interval  $[a, b]$ , i.e., there exists a function  $\hat{f} : [a, b] \rightarrow \mathbb{R}$  with  $\hat{f} = f(x)$  for all  $x \in (a, b)$ .

In this case the continuous extension  $\hat{f}$  is even unique.

**Proof** (i)  $\implies$  (ii) Uniform continuity of  $f$  is defined as

$$\exists \delta \forall \epsilon \forall x, y \in (a, b) : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $(a, b)$  which converges to  $a$ . Since every convergent sequence is a Cauchy-sequence it holds that

$$\forall \delta \exists N \in \mathbb{N} \forall n, m > N : |x_n - x_m| < \delta$$

The uniform continuity now gives us

$$\forall n, m > N : |f(x_n) - f(x_m)| < \epsilon$$

which indicates, that  $(f(x_n))_{n \in \mathbb{N}}$  is a Cauchy-Series in  $\mathbb{R}$ . Now we define  $\hat{f}(a) := \lim_{n \rightarrow \infty} f(x_n)$ .

The same can be done for  $(x_n)_{n \in \mathbb{N}}$  converging towards  $b$ .

(ii)  $\implies$  (i) Proof by contradiction. Suppose  $\hat{f}$  and therefore  $f$  is not uniformly continuous on  $(a, b)$ . Then there exist sequences  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$  in  $(a, b)$  with

$$\lim_{n \rightarrow \infty} |x_n - y_n| = 0 \quad \text{and} \quad |\hat{f}(x_n) - \hat{f}(y_n)| > \epsilon \quad \forall n \in \mathbb{N} \quad (1)$$

Since  $[a, b]$  is compact a subsequence  $(x_{n(k)})_{k \in \mathbb{N}}$  with

$$\lim_{k \rightarrow \infty} x_{n(k)} = x$$

exists. The subsequence  $(y_{n(k)})$  also converges to  $x$ , since  $(y_n - x_n) \rightarrow 0$  and

$$\lim_{k \rightarrow \infty} y_{n(k)} = \lim_{k \rightarrow \infty} (y_{n(k)} - (y_{n(k)} - x_{n(k)})) = \lim_{k \rightarrow \infty} x_{n(k)} = x$$

Since  $\hat{f}$  is continuous

$$\lim_{k \rightarrow \infty} |\hat{f}(x_{n(k)}) - \hat{f}(y_{n(k)})| = \left| \lim_{k \rightarrow \infty} \hat{f}(x_{n(k)}) - \lim_{k \rightarrow \infty} \hat{f}(y_{n(k)}) \right| = |\hat{f}(x) - \hat{f}(x)| = 0$$

which contradicts (1). ■

## Problem 8

**Theorem 2** For real numbers  $x, y \in \mathbb{R}$  and  $n \in \mathbb{N}$  we have

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

**Proof** Let  $x, y \in \mathbb{R}$  be arbitrary. Proof by induction:  
 $n = 0$  :

$$(x + y)^0 = 1 = \binom{0}{0} x^0 y^0$$

$n + 1$  :

$$\begin{aligned}
(x + y)^{n+1} &= (x + y)^n \cdot (x + y) \\
&= \left( \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \right) (x + y) \\
&= \left( \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \right) x + \left( \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \right) y \\
&= \left( \sum_{k=0}^n \binom{n}{k} x^{n-k+1} y^k \right) + \left( \sum_{k=0}^n \binom{n}{k} x^{n-k} y^{k+1} \right) \\
&= \left( \sum_{k=0}^n \binom{n}{k} x^{n-k+1} y^k \right) + \left( \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n-(k-1)} y^k \right) \\
&= \left( \sum_{k=1}^n \binom{n}{k} x^{n-k+1} y^k \right) + \left( \sum_{k=1}^n \binom{n}{k-1} x^{n-(k-1)} y^k \right) \\
&\quad + \binom{n}{0} x^{n-0+1} y^0 - \binom{n}{(n+1)-1} x^{n-(n+1)+1} y^{n+1} \\
&= \left( \sum_{k=1}^n \binom{n}{k} x^{n-k+1} y^k \right) + \left( \sum_{k=1}^n \binom{n}{k-1} x^{n-k+1} y^k \right) \\
&\quad + \binom{n}{0} x^{n+1} y^0 - \binom{n}{n} x^0 y^{n+1} \\
&= \left( \sum_{k=1}^n \left( \binom{n}{k} + \binom{n}{k-1} \right) x^{n-k+1} y^k \right) + \binom{n}{0} x^{n+1} y^0 - \binom{n}{n} x^0 y^{n+1} \\
&= \left( \sum_{k=1}^n \binom{n+1}{k} x^{n-k+1} y^k \right) + \binom{n+1}{0} x^{n+1} y^0 - \binom{n+1}{n+1} x^0 y^{n+1} \\
&= \left( \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^k \right)
\end{aligned}$$

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