MAS Ü11

4.) Gammaverteilung $\Gamma(\alpha, \lambda)$ Dichte $f(x) = \frac{x^{\alpha-1} \lambda^{\alpha}}{\Gamma(\alpha)} e^{-\lambda x} [x>0]$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{0}^{\infty} x \cdot \frac{x^{\alpha-1} \lambda^{\alpha}}{\Gamma(\alpha)} e^{-\lambda x}\, dx$$

$$\left[\begin{array}{ll} u = \lambda x & \frac{du}{dx} = \lambda \\ x = \frac{u}{\lambda} & dx = \frac{1}{\lambda} du \end{array}\right.$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha} e^{-\lambda x}\, dx = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \frac{u^{\alpha}}{\lambda^{\alpha}} e^{-u} \frac{1}{\lambda}\, du$$

$$= \frac{\lambda^{\alpha}}{\lambda^{\alpha+1} \Gamma(\alpha)} \int_0^{\infty} \lambda^{\alpha} e^{-u}\, du = \frac{1}{\lambda \Gamma(\alpha)} \Gamma(\alpha+1) = \frac{\alpha \cdot \Gamma(\alpha)}{\lambda \cdot \Gamma(\alpha)} = \frac{\alpha}{\lambda}$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \int_0^{\infty} x^2 \cdot \frac{x^{\alpha-1} \lambda^{\alpha}}{\Gamma(\alpha)} e^{-\lambda x}\, dx$$

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$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha+1} e^{-\lambda x}\, dx = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \frac{u^{\alpha+1}}{\lambda^{\alpha+1}} e^{-u} \frac{1}{\lambda}\, du$$

$$= \frac{\lambda^{\alpha}}{\lambda^{\alpha+2} \Gamma(\alpha)} \int_0^{\infty} u^{\alpha+1} e^{-u}\, du = \frac{1}{\lambda^2} \frac{1}{\Gamma(\alpha)} \Gamma(\alpha+2) = \frac{(\alpha+1)\Gamma(\alpha+1)}{\lambda^2 \Gamma(\alpha)} = \frac{(\alpha+1)\alpha\, \Gamma(\alpha)}{\lambda^2\, \Gamma(\alpha)} = \frac{\alpha^2+\alpha}{\lambda^2}$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{\alpha^2+\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$