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2.) $w \in \mathbb{C}$ $\quad \gamma : [a,b] \to \mathbb{C}$ ... geschlossen, sssd $\quad w \notin \gamma([a,b])$

$n(\gamma, w) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - w} \, d\zeta$ ... Umlaufzahl von γ um w

zz: $n(\gamma, w) \in \mathbb{Z}$

$\int_\gamma \frac{1}{\zeta - w} \, d\zeta = \int_a^b \gamma'(s) \frac{1}{\gamma(s) - w} \, ds$, da $\frac{1}{\zeta - w}$ für $\zeta \neq w$ holomorph

$g(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s) - w} \, ds \quad \Rightarrow g'(s) = \frac{\gamma'(s)}{\gamma(s) - w}$

$\frac{d}{dt} \exp(-g(t))(\gamma(t) - w) = \exp(-g(t)) \cdot (-g'(t))(\gamma(t) - w) + \exp(-g(t))\gamma'(t)$

$= -\exp(-g(t)) \frac{\gamma'(t)}{\cancel{\gamma(t) - w}} \cancel{(\gamma(t) - w)} + \exp(-g(t))\gamma'(t)$

$= \exp(-g(t))\gamma'(t) - \exp(-g(t))\gamma'(t) = 0$

$\Rightarrow \exists c \ \forall t \in \mathbb{C} : \exp(-g(t))(\gamma(t) - w) = C \quad \Leftrightarrow \gamma(t) - w = C \cdot \exp(g(t))$

$C \cdot \exp(g(a)) = \gamma(a) - w \overset{\text{da geschlossen}}{=} \gamma(b) - w = C \cdot \exp(g(b))$

Da $\gamma(a) - w \neq 0$ (da $w \notin \gamma([a,b])$) $\Rightarrow C \neq 0$

$\frac{C \cdot \exp(g(a))}{C} = \frac{C \cdot \exp(g(b))}{C} \quad \Leftrightarrow \exp(g(a)) = \exp(g(b))$

$\Leftrightarrow g(a) - g(b) \in 2\pi i \mathbb{Z} \qquad g(a) = \int_a^a \frac{\gamma'(s)}{\gamma(s) - w} ds = 0$

$\Rightarrow g(b) = \int_a^b \frac{\gamma'(s)}{\gamma(s) - w} \, ds = \int_\gamma \frac{1}{\zeta - w} \, d\zeta \in 2\pi i \mathbb{Z}$

$\Rightarrow n(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - w} \, d\zeta \in \mathbb{Z}$

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3.) zz: $\exp: \mathbb{C} \to \mathbb{C}$, $\cos: \mathbb{C} \to \mathbb{C}$, $\sin: \mathbb{C} \to \mathbb{C}$ sind holomorph

$$\exp(x+iy) = \exp(x)(\cos(y) + i\sin(y)) = \exp(x)\cos(y) + i\exp(x)\sin(y)$$

$$\frac{\partial \operatorname{Re}(\exp)}{\partial x} = \frac{\partial}{\partial x}\exp(x)\cdot\cos(y) = \exp(x)\cdot\cos(y)$$
$$\frac{\partial \operatorname{Im}(\exp)}{\partial y} = \frac{\partial}{\partial y}\exp(x)\cdot\sin(y) = \exp(x)\cdot\cos(y)$$
$$\frac{\partial \operatorname{Re}(\exp)}{\partial y} = \frac{\partial}{\partial y}\exp(x)\cos(y) = -\exp(x)\cdot\sin(y)$$
$$-\frac{\partial \operatorname{Im}(\exp)}{\partial x} = -\frac{\partial}{\partial x}\exp(x)\sin(y) = -\exp(x)\sin(y)$$

$(\exp(x+iy))' = \exp(x+iy)$ ... stetig $\Rightarrow$ holomorph Stammfunktion: exp

$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cdot\cosh(y) - i\sin(x)\sinh(y)$$
$$\frac{\partial \operatorname{Re}(\cos)}{\partial x} = \frac{\partial}{\partial x}\cos(x)\cdot\cosh(y) = -\sin(x)\cdot\cosh(y) = \frac{\partial}{\partial y} -\sin(x)\cdot\sinh(y) = \frac{\partial \operatorname{Im}(\cos)}{\partial y}$$
$$\frac{\partial \operatorname{Re}(\cos)}{\partial y} = \frac{\partial}{\partial y}\cos(x)\cdot\cosh(y) = \cos(x)\cdot\sinh(y) = -\frac{\partial}{\partial x} -\sin(x)\sinh(y) = -\frac{\partial \operatorname{Im}(\cos)}{\partial x}$$

$(\cos(x+iy))' = -\sin(x+iy)$ ... stetig $\Rightarrow$ holomorph Stammfunktion: sin

$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$
$$\frac{\partial \operatorname{Re}(\sin)}{\partial x} = \frac{\partial}{\partial x}\sin(x)\cosh(y) = \cos(x)\cosh(y) = \frac{\partial}{\partial y}\cos(x)\sinh(y) = \frac{\partial \operatorname{Im}(\sin)}{\partial y}$$
$$\frac{\partial \operatorname{Re}(\sin)}{\partial y} = \frac{\partial}{\partial y}\sin(x)\cosh(y) = -\sin(x)\sinh(y) = -\frac{\partial}{\partial x}\cos(x)\sinh(y) = -\frac{\partial \operatorname{Im}(\sin)}{\partial x}$$

$(\sin(x+iy))' = \cos(x+iy)$ ... stetig $\Rightarrow$ holomorph Stammfunktion: $-\cos$

ges: Stammfunktion von $z^2(\cos(z))^2$

$$\int z^2(\cos(z))^2\,dz = \int z^2\,\frac{1}{2}(\cos(2z)+1)\,dz = \frac{1}{2}\left(\int z^2\cos(2z)\,dz + \int z^2\,dz\right)$$
$$= \frac{1}{2}\int\left(\frac{\sin(2z)}{2}\right)'\cdot z^2\,dz + \frac{1}{2}\frac{z^3}{3} = \frac{1}{2}\left(\frac{\sin(2z)}{2}z^2 - \int\frac{\sin(2z)}{2}\cdot 2z\,dz\right) + \frac{z^3}{6}$$
$$= \frac{z^3}{6} + \frac{z^2\sin(2z)}{4} - \frac{1}{2}\int\sin(2z)\cdot z\,dz = \frac{z^3}{6} + \frac{z^2\sin(2z)}{4} - \frac{1}{2}\int\left(-\frac{\cos(2z)}{2}\right)'\cdot z\,dz$$
$$= \frac{z^3}{6} + \frac{z^2\sin(2z)}{4} + \frac{\cos(2z)}{4}z + \frac{1}{2}\int -\frac{\cos(2z)}{2}\,dz = \frac{z^3}{6} + \frac{z^2\sin(2z)}{4} + \frac{z\cos(2z)}{4} - \frac{1}{4}\int\cos(2z)\,dz$$
$$= \frac{z^3}{6} + \frac{z^2\sin(2z)}{4} + \frac{z\cos(2z)}{4} - \frac{1}{4}\int\cos(u)\frac{1}{2}\,du \qquad \left[u = 2z \quad \frac{du}{dz} = 2 \quad dz = \frac{1}{2}du\right.$$
$$= \frac{z^3}{6} + \frac{z^2\sin(2z)}{4} + \frac{z\cos(2z)}{4} - \frac{1}{8}\sin(u) = \frac{z^3}{6} + \frac{z^2\sin(2z)}{4} + \frac{z\cos(2z)}{4} - \frac{\sin(2z)}{8}$$
$$= \frac{(6z^2-3)\sin(2z) + 6z\cos(2z) + 4z^3}{24}$$

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4.) $\gamma_1 : [0, 2\pi] \to \mathbb{C}$ $\quad$ $\gamma_2 : [0, 2\pi] \to \mathbb{C}$ $\quad$ $\gamma_3 : [0, 2\pi] \to \mathbb{C}$

$t \mapsto -1 + \frac{1}{2}\exp(it)$ $\quad$ $t \mapsto 1 + \frac{1}{2}\exp(it)$ $\quad$ $t \mapsto 4\exp(it)$

$f(z) = \frac{1}{z+1} + \frac{1}{z-1}$ $\quad$ ges: $\int_{\gamma_1} f(\zeta)\,d\zeta + \int_{\gamma_2} f(\zeta)\,d\zeta$

Satz 11.6.12 (Cauchysche Integralformel)

$f : D \to Y$ $\quad$ $D \subseteq \mathbb{C}$ ... offen $\quad$ $w \in \mathbb{C}$ $\quad$ $\rho > 0$ mit $K_\rho(w) \subseteq D$

$z \in U_\rho(w)$ $\quad$ $\gamma : [0, 2\pi] \to D$ $\quad$ $t \mapsto w + \rho \cdot \exp(it)$

$\Rightarrow f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$

$$\int_{\gamma_1} f(\zeta)\,d\zeta + \int_{\gamma_2} f(\zeta)\,d\zeta = \int_{\gamma_1} \frac{1}{\zeta+1} + \frac{1}{\zeta-1}\,d\zeta + \int_{\gamma_2} \frac{1}{\zeta+1} + \frac{1}{\zeta-1}\,d\zeta$$

$$= \int_{\gamma_1} \frac{1}{\zeta+1}\,d\zeta + \int_{\gamma_1} \frac{1}{\zeta-1}\,d\zeta + \int_{\gamma_2} \frac{1}{\zeta+1}\,d\zeta + \int_{\gamma_2} \frac{1}{\zeta-1}\,d\zeta$$

$w = -1$ $\quad$ $\rho = \frac{1}{2}$ $\quad$ $z = -1$ $\quad$ $\gamma(t) = -1 + \frac{1}{2} \cdot \exp(it) = \gamma_1(t)$ $\quad$ $f \equiv 1$

$\Rightarrow 2\pi i\, f(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \iff 2\pi i = \int_{\gamma_1} \frac{1}{\zeta - (-1)}\,d\zeta$

gleich mit $w = 1$ $\quad$ $z = 1$ $\quad$ $\gamma = \gamma_2$ $\quad$ $\Rightarrow 2\pi i = \int_{\gamma_2} \frac{1}{\zeta - 1}\,d\zeta$

$$= 2\pi i + \int_{\gamma_1} \frac{1}{\zeta-1}\,d\zeta + \int_{\gamma_2} \frac{1}{\zeta+1}\,d\zeta + 2\pi i \qquad \frac{\frac{\zeta+1}{\zeta-1}}{\zeta+1} = \frac{\zeta+1}{(\zeta+1)(\zeta-1)} = \frac{1}{\zeta-1}$$

$$= 4\pi i + \int_{\gamma_1} \frac{\frac{\zeta+1}{\zeta-1}}{\zeta+1}\,d\zeta + \int_{\gamma_2} \frac{\frac{\zeta-1}{\zeta+1}}{\zeta-1}\,d\zeta$$

$w = -1$ $\quad$ $\rho = \frac{1}{2}$ $\quad$ $z = -1$ $\quad$ $\gamma = -1 + \frac{1}{2}\exp(it) = \gamma_1(t)$ $\quad$ $f(\zeta) = \frac{\zeta+1}{\zeta-1}$

$\Rightarrow 2\pi i\, f(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \iff 2\pi i \underbrace{\frac{-1+1}{-1-1}}_{=0} = \int_{\gamma_1} \frac{\frac{\zeta+1}{\zeta-1}}{\zeta - (-1)}$

gleich mit $w = 1$ $\quad$ $z = 1$ $\quad$ $\gamma = \gamma_2$ $\quad$ $f(\zeta) = \frac{\zeta-1}{\zeta+1}$

$\Rightarrow 2\pi i \underbrace{\frac{1-1}{1+1}}_{=0} = \int_{\gamma_2} \frac{\frac{\zeta-1}{\zeta+1}}{\zeta-1}\,d\zeta$

$= 4\pi i + 0 + 0 = 4\pi i$

...

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4.)... ges: $\int_{\gamma_3} f(\zeta)\,d\zeta$

$$\int_{\gamma_3} \frac{1}{\zeta+1} + \frac{1}{\zeta-1}\,d\zeta = \int_{\gamma_3} \frac{1}{\zeta+1}\,d\zeta + \int_{\gamma_3} \frac{1}{\zeta-1}\,d\zeta$$

$f(\zeta)=1$ $\quad w=0$ $\quad \rho=4$ $\quad z=-1$ $\quad \gamma(t)=4\exp(it)=\gamma_3(t)$

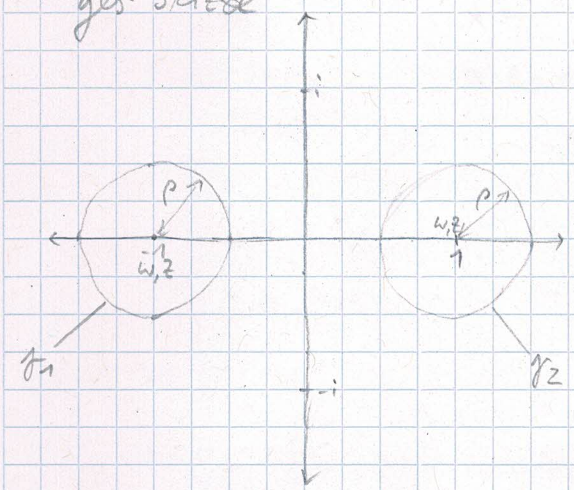
$\Rightarrow 2\pi i\, f(-1) = \int_{\gamma} \frac{1}{\zeta-(-1)}\,d\zeta \Leftrightarrow 2\pi i = \int_{\gamma_3} \frac{1}{\zeta+1}\,d\zeta$

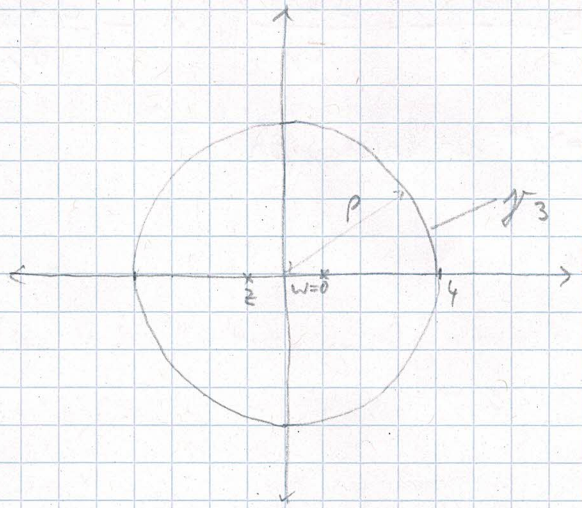
gleich mit $z=1$

$\Rightarrow 2\pi i\, f(1) = \int_{\gamma} \frac{1}{\zeta-1}\,d\zeta \Leftrightarrow 2\pi i = \int_{\gamma_3} \frac{1}{\zeta-1}\,d\zeta$

$= 2\pi i + 2\pi i = 4\pi i$

ges: Skizze





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7.) $D \subseteq \mathbb{C}$ ... offen $\quad f: D \to \mathbb{C}$ ... holomorph

$\overline{D} := \{\overline{z} : z \in D\} \quad f^*: \overline{D} \to \mathbb{C} \quad z \mapsto \overline{f(\overline{z})}$

zz: f^* ist holomorph

D... offen $\Rightarrow \forall x \in D\ \exists \varepsilon > 0\ U_\varepsilon(x) \subseteq D \Rightarrow \forall y \in \overline{D}\ \exists \varepsilon > 0\ U_\varepsilon(y) \subseteq \overline{D}$

$\Rightarrow \overline{D}$... offen

f... holomorph $\Rightarrow \dfrac{\partial Re(f)}{\partial x} = \dfrac{\partial Im(f)}{\partial y} \wedge \dfrac{\partial Re(f)}{\partial y} = -\dfrac{\partial Im(f)}{\partial x}$

$\Rightarrow \dfrac{\partial Re(f^*)}{\partial x} = \dfrac{\partial -Im(f^*)}{\partial -y} \wedge \dfrac{\partial Re(f^*)}{\partial -y} = -\dfrac{\partial -Im(f^*)}{\partial x}$

$\Rightarrow \dfrac{\partial Re(f^*)}{\partial x} = \dfrac{\partial Im(f^*)}{\partial y} \wedge -\dfrac{\partial Re(f^*)}{\partial y} = \dfrac{\partial Im(f^*)}{\partial x}$

$\Rightarrow f^*$ ist holomorph