ANA Ü13

2.) $w \in \mathbb{C}$ $\quad \gamma: [a,b] \to \mathbb{C}$ ... geschlossen, sssd $\quad w \notin \gamma([a,b])$

$n(\gamma, w) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - w} \, d\xi$ ... Umlaufzahl von $\gamma$ um $w$

zz: $n(\gamma, w) \in \mathbb{Z}$

$$\int_\gamma \frac{1}{\xi - w} \, d\xi = \int_a^b \gamma'(s) \frac{1}{\gamma(s) - w} \, ds \quad \text{, da } \frac{1}{\xi - w} \text{ für } \xi \neq w \text{ holomorph}$$

$$g(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s) - w} \, ds \qquad \Rightarrow g'(s) = \frac{\gamma'(s)}{\gamma(s) - w}$$

$$\frac{d}{dt} \exp(-g(t))(\gamma(t) - w) = \exp(-g(t)) \cdot (-g'(t))(\gamma(t) - w) + \exp(-g(t)) \gamma'(t)$$

$$= \exp(-g(t)) \frac{\gamma'(t)}{\gamma(t) - w} (\gamma(t) - w) + \exp(-g(t)) \gamma'(t)$$

$$- \exp(-g(t)) \gamma'(t) - \exp(-g(t)) \gamma'(t) = 0$$

$$\Rightarrow \exists c \; \forall t \in \mathbb{C} : \exp(-g(t))(\gamma(t) - w) = C \iff \gamma(t) - w = C \cdot \exp(g(t))$$

$$C \cdot \exp(g(a)) = \gamma(a) - w \overset{\text{da geschlossen}}{=} \gamma(b) - w = C \cdot \exp(g(b))$$

Da $\gamma(a) - w \neq 0$ (da $w \notin \gamma([a,b])$) $\Rightarrow C \neq 0$

$$\frac{C \cdot \exp(g(a))}{C} = \frac{C \cdot \exp(g(b))}{C} \iff \exp(g(a)) = \exp(g(b))$$

$$\iff g(a) - g(b) \in 2\pi i \mathbb{Z} \qquad g(a) = \int_a^a \frac{\gamma'(s)}{\gamma(s) - w} \, ds = 0$$

$$\Rightarrow g(b) = \int_a^b \frac{\gamma'(s)}{\gamma(s) - w} \, ds = \int_\gamma \frac{1}{\xi - w} \, d\xi \in 2\pi i \mathbb{Z}$$

$$\Rightarrow n(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - w} \, d\xi \in \mathbb{Z}$$