ANA Ü11

5) $\gamma : [a,b] \to \mathbb{R}^n$ $\quad l(\gamma) < \infty$ $\quad \rho(t)$ ... Dichte an der Stelle $t$

Gesamtmasse $M = \int_a^b \rho \, dl$ mit $l(x) = l(\gamma|_{[a,x]})$

Schwerpunkt $S = \frac{1}{M} \int_a^b f \, dl$ mit $f : [a,b] \to L(\mathbb{R}, \mathbb{R}^2)$ $\quad t \mapsto \rho(t)\gamma(t)$

$\gamma : [0,1] \to \mathbb{R}^2 \quad t \mapsto \begin{pmatrix} t \\ t^2 \end{pmatrix} \quad \rho(t) = 1$

ges: Gesamtmasse und Schwerpunkt

$\gamma'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$ ... stetig $\Rightarrow \gamma \in C^1[0,1]$ nach Satz 11.1.8. gilt

$$l(x) = l(\gamma|_{[a,x]}) = \int_a^x \|\gamma'(x)\|_2 \, dx = \int_0^x \left\| \begin{pmatrix} 1 \\ 2x \end{pmatrix} \right\|_2 dx = \int_0^x \sqrt{1+4x^2} \, dx$$

$$= \int_0^x \sqrt{1+u^2} \cdot \frac{1}{2} du = \frac{1}{2} \int_0^x \sqrt{1+u^2} \, du \qquad \left[ u = 2x \quad \frac{du}{dx} = 2 \quad dx = \frac{1}{2} du \right.$$

$$= \frac{1}{2}\left( \frac{1}{2}\left(\sqrt{u^2+1} \cdot u + \sinh^{-1}(u)\right)\Big|_0^x \right) = \frac{1}{4}\left(\sqrt{4x^2+1} \cdot 2x + \sinh^{-1}(2x)\right)\Big|_0^x$$

$$= \frac{1}{4}\left(\sqrt{4x^2+1}\, 2x + \sinh^{-1}(2x)\right)$$

$\rho \in C[0,1] \Rightarrow$ laut Satz 11.2.5 gilt nun

$$\int_0^1 \rho \, dl = \int_0^1 \rho(t) \cdot l'(t) \, dt = \int_0^1 \left( \int_0^x \|\gamma'(x)\|_2 dx \right)' dt = \int_0^1 \|\gamma'(x)\|_2 \, dx$$

$$= \frac{1}{4}\left(\sqrt{4+1} \cdot 2 + \sinh^{-1}(2)\right) \approx 1{,}4789 \text{ ... Gesamtmasse}$$

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$f(t) = \gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ ... stetig nach Satz 11.1.8. gilt

$$S = \frac{1}{M} \int_0^1 f \, dl = \frac{1}{M} \int_0^1 f(t) \cdot l'(t) \, dt = \frac{1}{M} \int_0^1 f(t) \cdot \left( \int_0^x \|\gamma'(x)\|_2 dx \right)' dt = \frac{1}{M} \int_0^1 f(t) \cdot \|\gamma'(t)\|_2 \, dt$$

$$= \frac{1}{M} \int_0^1 \begin{pmatrix} t \\ t^2 \end{pmatrix} \sqrt{1+4t^2} \, dt = \frac{1}{M} \int_0^1 \begin{pmatrix} t \cdot \sqrt{1+4t^2} \\ t^2 \cdot \sqrt{1+4t^2} \end{pmatrix} dt$$

- $\int t \cdot \sqrt{1+4t^2} \, dt = \int t \cdot \sqrt{1+u} \cdot \frac{1}{8t} du \qquad \left[ u = 4t^2 \quad \frac{du}{dt} = 8t \quad dt = \frac{1}{8t} du \right.$

  $= \frac{1}{8} \int \sqrt{1+u} \, du = \frac{1}{8} \cdot \frac{2}{3} (u+1)^{\frac{3}{2}} = \frac{1}{12}(4t^2+1)^{\frac{3}{2}} \Rightarrow \int_0^1 t \sqrt{1+4t^2} \, dt = \frac{1}{12}(5\sqrt{5}-1)$

- $\int_0^1 t^2 \sqrt{1+4t^2} \, dt = \frac{1}{64}(18\sqrt{5} - \sinh^{-1}(2))$ (mit Wolfram Alpha gerechnet, da sehr mühsam)

$$\Rightarrow S = \frac{1}{\frac{1}{4}(\sqrt{5} \cdot 2 + \sinh^{-1}(2))} \begin{pmatrix} \frac{1}{12}(5\sqrt{5}-1) \\ \frac{1}{64}(18\sqrt{5} - \sinh^{-1}(2)) \end{pmatrix} = \begin{pmatrix} \frac{5\sqrt{5}-1}{3(2\sqrt{5}+\sinh^{-1}(2))} \\ \frac{18\sqrt{5}-\sinh^{-1}(2)}{16(2\sqrt{5}+\sinh^{-1}(2))} \end{pmatrix} \approx \begin{pmatrix} 0{,}57363 \\ 0{,}40998 \end{pmatrix}$$