ANA Ü6

6.) $\int_0^1 \left(\int_1^2 \left(\int_1^2 z^{x+y} dx \right) dy \right) dz$

$z^{x+y} : [1,2] \times [1,2] \times [0,1] \longrightarrow \mathbb{R}$ ist stetig, da

$\lim_{z \to 0+} z^{x+y} = 0 = 0^{x+y}$ (Fortsetzung durch $f(x,y,0) = 0$)

Δ

$\int_0^1 \left(\int_1^2 \left(\int_1^2 z^{x+y} dx \right) dy \right) dz = \int_1^2 \left(\int_1^2 \left(\int_0^1 z^{x+y} dz \right) dx \right) dy$

$= \int_1^2 \left(\int_1^2 \left(\frac{1^{x+y+1}}{x+y+1} - \frac{0^{x+y+1}}{x+y+1} \right) dx \right) dy = \int_1^2 \left(\int_1^2 \frac{1}{x+y+1} dx \right) dy$

$\left[u = x+y+1 \quad \frac{du}{dx} = 1 \quad du = dx \quad \int \frac{1}{u} du = \ln(u) = \ln(x+y+1) \right)$

$= \int_1^2 \ln(2+y+1) - \ln(1+y+1) \, dy = \int_1^2 \ln(y+3) \, dy - \int_1^2 \ln(y+2) \, dy$

$\left| \begin{array}{llll} v = y+3 & \frac{dv}{dy} = 1 & dv = dy & \int \ln(v) dv = v(\ln(v) - 1) = (y+3)(\ln(y+3) - 1) \\ w = y+2 & \frac{dw}{dy} = 1 & dw = dy & \int \ln(w) dw = w(\ln(w) - 1) = (y+2)(\ln(y+2) - 1) \end{array} \right.$

$= 5(\ln(5) - 1) - 4(\ln(4) - 1) - (4(\ln(4) - 1) - 3(\ln(3) - 1))$

$= 5\ln(5) - 5 - 4\ln(4) + 4 - 4\ln(4) + 4 + 3\ln(3) - 3$

$= \underline{5\ln(5) - 8\ln(4) + 3\ln(3)} \approx 0{,}2527$

$\Delta \quad g(y,z) := \int_1^2 z^{x+y} dx \qquad u = x+y \quad \frac{du}{dx} = 1 \quad du = dx$

$\int z^{x+y} dx = \int z^u du = \frac{z^u}{\ln(z)} = \frac{z^{x+y}}{\ln(z)}$

$\int_1^2 z^{x+y} dx = \frac{z^{2+y}}{\ln(z)} - \frac{z^{1+y}}{\ln(z)} = \frac{1}{\ln(z)} \cdot (z^{y+2} - z^{y+1})$

$\Rightarrow g(y,z)$ ist stetig

Satz 8.7.10 sagt nun: $\int_0^1 \left(\int_1^2 g(y,z) \, dy \right) dz$

$= \int_1^2 \left(\int_0^1 g(y,z) \, dz \right) dy = \int_1^2 \left(\int_0^1 \left(\int_1^2 z^{x+y} dx \right) dz \right) dy$

$f(x,z) := z^{x+y}$ ist stetig

nochmals Satz 8.7.10 $\quad \int_0^1 \left(\int_1^2 f(x,z) \, dx \right) dz = \int_1^2 \left(\int_0^1 f(x,z) \, dz \right) dx$

$\Rightarrow \int_0^1 \left(\int_1^2 \left(\int_1^2 z^{x+y} dx \right) dy \right) dz = \int_1^2 \left(\int_0^1 \left(\int_1^2 z^{x+y} dx \right) dz \right) dy = \int_1^2 \left(\int_1^2 \left(\int_0^1 (z^{x+y}) dz \right) dx \right) dy$