3.) $z, w \in \mathbb{C}$

zz: $k \in \mathbb{Z}$ $\quad \cos(z + k\pi) = (-1)^k \cdot \cos(z)$

$$\cos(z+k\pi) = \frac{\exp(iz+ik\pi) + \exp(-iz-ik\pi)}{2} = \frac{\exp(iz)\cdot\exp(ik\pi) + \exp(iz)\exp(-ik\pi)}{2}$$

$$= (\exp(iz)\cdot\exp(i\pi)^k + \exp(iz)\cdot\exp(-i\pi)^k)\cdot\frac{1}{2} = \frac{\exp(iz)\cdot(-1)^k + \exp(-iz)\cdot(-1)^k}{2}$$

$$= (-1)^k \cdot \frac{\exp(iz)+\exp(-iz)}{2} = (-1)^k \cdot \cos(z)$$

zz: $k \in \mathbb{Z}$ $\quad \sin(z + k\pi) = (-1)^k \cdot \sin(z)$

$$\sin(z+k\pi) = \frac{\exp(iz+ik\pi) - \exp(-iz-ik\pi)}{2i} = \frac{\exp(iz)\cdot\exp(ik\pi) - \exp(-iz)\cdot\exp(-ik\pi)}{2i}$$

$$= \frac{\exp(iz)\cdot(-1)^k - \exp(-iz)\cdot(-1)^k}{2i} = (-1)^k \cdot \sin(z)$$

zz: $\cos(\frac{\pi}{2} - z) = \sin(z)$

$$\cos(\tfrac{\pi}{2} - z) = \frac{\exp(i\frac{\pi}{2} - iz) + \exp(-i\frac{\pi}{2} + iz)}{2} = \frac{\exp(i\frac{\pi}{2})\exp(-iz) + \exp(-i\frac{\pi}{2})\exp(iz)}{2}$$

$$= \frac{i\cdot\exp(-iz) + (-i)\cdot\exp(iz)}{2} = \frac{-\exp(-iz)+\exp(iz)}{2i} = \sin(z)$$

...

ANA 01

3.) ... zz: $\sin(\frac{\pi}{2}-z)=\cos(z)$

$$\sin\left(\frac{\pi}{2}-z\right)=\frac{\exp(i\frac{\pi}{2}-iz)-\exp(-i\frac{\pi}{2}+iz)}{2i}$$

$$=\frac{\exp(i\frac{\pi}{2})\cdot\exp(-iz)-\exp(-i\frac{\pi}{2})\cdot\exp(iz)}{2i}$$

$$=\frac{i\cdot\exp(-iz)-(-i)\cdot\exp(iz)}{2i}=\frac{i\cdot(\exp(-iz)+\exp(iz))}{i\cdot 2}$$

$$=\frac{\exp(iz)+\exp(-iz)}{2}=\cos(z)$$

zz: $\cos(2z)=(\cos(z))^2-(\sin(z))^2$

$$(\cos(z))^2-(\sin(z))^2=\left(\frac{\exp(iz)+\exp(-iz)}{2}\right)^2-\left(\frac{\exp(iz)-\exp(-iz)}{2i}\right)^2$$

$$=\frac{(\exp(iz))^2+2\exp(iz)\exp(-iz)+(\exp(-iz))^2}{4}-\frac{(\exp(iz)-\exp(-iz))^2}{-4}$$

$$=\frac{\exp(i2z)+2\exp(iz-iz)+\exp(-i2z)}{4}+\frac{(\exp(iz))^2-2\exp(iz)\exp(-iz)+(\exp(-iz))^2}{4}$$

$$=\frac{\exp(i2z)+2+\exp(-i2z)+\exp(i2z)-2\exp(iz-iz)+\exp(-i2z)}{4}$$

$$=\frac{2\exp(i2z)+2\exp(-i2z)+2-2}{4}=\frac{\exp(i2z)+\exp(-i2z)}{2}$$

$$=\cos(2z)$$

zz: $\sin(2z)=2\sin(z)\cdot\cos(z)$

$$2\sin(z)\cdot\cos(z)=2\,\frac{\exp(iz)-\exp(-iz)}{2i}\cdot\frac{\exp(iz)+\exp(-iz)}{2}=\frac{(\exp(iz)-\exp(-iz))(\exp(iz)+\exp(-iz))}{2i}$$

$$=\frac{(\exp(iz))^2+\exp(iz)\exp(-iz)-\exp(-iz)\cdot\exp(iz)-(\exp(-iz))^2}{2i}$$

$$=\frac{\exp(i2z)+1-1-\exp(-i2z)}{2i}=\sin(2z)$$