LINAG Ü12

11.1.3 $K$... Körper $(V, \sigma)$... symplektischer VR/$K$ $\dim(V) = 4$

symplektisch: $\forall x \in V \setminus \{0\}: [x] \cap [x]^\perp \neq \{0\}$

$[x]^\perp = \{v \in V: \forall k \in K: k \cdot x \perp v\}$ $\quad \sigma(k \cdot x, v) = 0 \iff \omega(k) \cdot \sigma(x,v) = 0 \implies \sigma(x,v) = 0$

$[x]^\perp = \{v \in V: x \perp v\}$ $\quad [x] \cap [x]^\perp \neq \{0\} \implies \exists k \in K^\times: k \cdot x \perp x \iff \sigma(k \cdot x, x) = 0$

$\iff \omega(k) \cdot \sigma(x,x) = 0 \implies x \perp x$

also symplektisch: $\forall x \in V \setminus \{0\}: x \perp x$

zz: $\forall U \subseteq V$ ... 2-dim UR: $(U \oplus U^\perp = V \quad \vee \quad U = U^\perp)$

Sei $x, y \in V$ bel. mit $x, y$ l.u. $\implies U := [\{x,y\}]$ ist ein 2-dim UR

1. Fall $U$ ist nicht isotrop

D.h. $U \cap U^\perp = \{0\}$

radikalfrei

$\dim(U \cap U^\perp) + \dim(U + U^\perp) = \dim(U) + \dim(U^\perp) = 4 + \dim(U \cap V^\perp) \overset{!}{=} 4$

$\implies \dim(U + U^\perp) = 4$ und $U \cap U^\perp = \{0\}$ also $U \oplus U^\perp = V$

2. Fall $U$ ist isotrop

D.h. $U \cap U^\perp \neq \{0\}$ $\quad U \cap U^\perp$ ist ein Unterraum von $V$ ($U \cap U^\perp = U^{\perp U}$ ... Radikal)

$\exists a, b \in K \; a \neq 0$ oder $b \neq 0$ $\; \forall k, l \in K \; \sigma(ax + by, k \cdot x + l y) = 0$

$\iff \omega(a)\omega(k)\sigma(x,x) + \omega(a) \cdot l \, \sigma(x,y) + b \cdot \omega(k) \, \sigma(y,x) + b \cdot l \, \sigma(y,y) = 0$

$\iff (\omega(a) \cdot l + b \cdot \omega(k)) \, \sigma(x,y) = 0 \quad \implies \sigma(x,y) = 0$

$\forall k, l \in K: \; k \cdot \sigma(x,x) + l \cdot \sigma(x,y) = 0 \iff \sigma(x, k \cdot x + l \cdot y) = 0 \implies x \in U \cap U^\perp$

$\forall k, l \in K: \; k \cdot \sigma(y,y) + l \cdot \sigma(y,x) = 0 \iff \sigma(y, l \cdot x + k \cdot y) = 0 \implies y \in U \cap U^\perp$

Da $U \cap U^\perp$ ein UR ist folgt $\forall k, l \in K: k \cdot x + l \cdot y \in U \cap U^\perp$

$\implies U \subseteq U^\perp$ $\quad \dim(U) + \dim(U^\perp) = 4 \implies \dim(U^\perp) = 2 \quad \implies U = U^\perp$

Beispiele: $\sigma\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}\right) = x_2 y_1 - x_1 y_2 + x_4 y_3 - x_3 y_4$ $\quad K = \mathbb{R} \quad V = \mathbb{R}^4$

$x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad U = [\{x, y\}]$

$x^\perp = \left[\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}\right] \quad y^\perp = \left[\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}\right]$

$U^\perp = x^\perp \cap y^\perp = \left[\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right\}\right] = U$

$\vdots$

Satz 9.5.5 c)

$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad U = [\{x, y\}]$

$x^\perp = \left[\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}\right] \quad y^\perp = \left[\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}\right]$

$U^\perp = x^\perp \cap y^\perp = \left[\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}\right]$

$U \oplus U^\perp = V$