ANA Ü4

8.) $f(x) = x^3+1$ $\quad f:[0,1] \to \mathbb{R}$ $\quad n \in \mathbb{N}$

$R_n = ((k \cdot \frac{1}{n})_{k\in\{0,1,\dots,n\}}, (\alpha)_k)$ mit $(\alpha)_k$ bel., so dass $\forall k$ $k \cdot \frac{1}{n} < \alpha < (k+1) \cdot \frac{1}{n}$

ges: $O(R_n)$, $U(R_n)$

$$O(R_n) = \sum_{j=1}^{n(R_n)} \left(j \cdot \frac{1}{n} - (j-1) \cdot \frac{1}{n}\right) \cdot \sup_{t\in[(j-1)\cdot\frac{1}{n}, j\cdot\frac{1}{n}]} f(t)$$

Da f monoton wachsend ($f'(x) = 3x^2 \geq 0 \ \forall x \in [0,1]$) ist

$\inf\limits_{t\in[\dots]} f(t) = f((j-1)\cdot\frac{1}{n})$ und $\sup\limits_{t\in[\dots]} f(t) = f(j \cdot \frac{1}{n})$.

$$O(R_n) = \sum_{j=1}^{n} \left(\frac{j}{n} - \frac{j-1}{n}\right) \cdot f\left(\frac{j}{n}\right) = \sum_{j=1}^{n} \frac{\cancel{j} - \cancel{j} + 1}{n} \cdot \left(\left(\frac{j}{n}\right)^3 + 1\right) = \sum_{j=1}^{n} \frac{1}{n} \cdot \left(\frac{j^3}{n^3} + \frac{n^3}{n^3}\right)$$

$$= \sum_{j=1}^{n} \frac{j^3+n^3}{n^4} = \frac{1}{n^4} \cdot \sum_{j=1}^{n} j^3 + n^3 = \frac{1}{n^4} \cdot \left(n \cdot n^3 + \sum_{j=1}^{n} j^3\right) = 1 + \frac{1}{n^4} \cdot \sum_{j=1}^{n} j^3$$

$$U(R_n) = \sum_{j=1}^{n} \frac{1}{n} \cdot f\left((j-1)\cdot\frac{1}{n}\right) = \sum_{j=1}^{n} \frac{1}{n} \cdot \left(\left((j-1)\cdot\frac{1}{n}\right)^3 + 1\right) = \sum_{j=1}^{n} \frac{1}{n}\left(\frac{(j-1)^3}{n^3} + \frac{n^3}{n^3}\right)$$

$$= \sum_{j=1}^{n} \frac{1}{n} \cdot \frac{(j-1)^3 + n^3}{n^3} = \sum_{j=1}^{n} \frac{(j-1)^3 + n^3}{n^4} = \frac{1}{n^4} \cdot \left(\sum_{j=1}^{n} \left((j-1)^3 + n^3\right)\right) = \frac{1}{n^4} \sum_{j=1}^{n} \left((j-1)^3\right) + 1$$

$$\Rightarrow O(R_n) = 1 + \frac{1}{4n^4} \cdot n^2(n+1)^2 \quad \text{und} \quad U(R_n) = 1 + \frac{1}{4n^4} n^2 \cdot (n-1)^2$$

$$\lim_{n\to\infty} O(R_n) = 1 + \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4} = 1 + \lim_{n\to\infty} \frac{n^2(n^2+2n+1)}{4n^4} = 1 + \lim_{n\to\infty} \frac{n^4+2n^3+n^2}{4n^4}$$

$$= 1 + \lim_{n\to\infty} \frac{1 + 2\frac{1}{n} + \frac{1}{n^2}}{4} = 1 + \frac{1}{4} = \frac{5}{4}$$

$$\lim_{n\to\infty} U(R_n) = 1 + \lim_{n\to\infty} \frac{n^2 \cdot (n^2-2n+1)}{4n^4} = 1 + \lim_{n\to\infty} \frac{n^4 - 2n^3 + n^2}{4n^4} = 1 + \lim_{n\to\infty} \frac{1 - 2\frac{1}{n} + \frac{1}{n^2}}{4}$$

$$= 1 + \frac{1}{4} = \frac{5}{4}$$