ANA Ü1

8.) $z \in \mathbb{C}$ $\quad E = \mathbb{N} \cup \{0\}$ $\quad f_n : E \to \mathbb{C}$ $\quad$ für $n \in E$

$$f_n(K) = \frac{1}{K^n} \binom{K}{n} z^n$$

zz: $\sum_{n=0}^{\infty} f_n$ konvergiert als Funktionenreihe absolut

$$\|f_n\|_\infty = \sup_{K \in E} \left| \frac{1}{K^n} \binom{K}{n} z^n \right| = \sup_{K \in E} \left| \frac{1}{K^n} \frac{K!}{n!(K-n)!} \cdot z^n \right|$$

$$= \frac{|z|^n}{n!} \cdot \sup_{K \in E} \frac{K!}{(K-n)!} \cdot \frac{1}{K^n} = \frac{|z|^n}{n!} \cdot \sup_{K \in E} \frac{K \cdot (K-1) \cdot \ldots \cdot 2 \cdot 1}{(K-n)(K-n-1) \cdots 2 \cdot 1} \cdot \underbrace{\frac{1}{K^n}}_{n\text{-Mal}}$$

$$= \frac{|z|^n}{n!} \cdot \sup_{K \in E} \underbrace{\frac{K \cdot (K-1) \cdot \ldots \cdot (K-n+1)}{K \cdot K \cdot \ldots \cdot K}}_{n\text{-Mal}} \leq \frac{|z|^n}{n!} \cdot \sup_{K \in E} \frac{K^n}{K^n} = \frac{|z|^n}{n!}$$

$$\sum_{n=0}^{\infty} \|f_n\|_\infty \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = \exp(|z|) \quad \Rightarrow \text{absolut konvergent}$$

ges: Grenzfunktion von $\sum_{n=0}^{\infty} f_n$

$$\sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} \frac{1}{K^n} \binom{K}{n} z^n = \sum_{n=0}^{K} \left(\frac{z}{K}\right)^n \cdot \binom{K}{n} \cdot 1^{K-n} = \left(\frac{z}{K} + 1\right)^K$$

$S_N = \sum_{n=0}^{N} f_n$ $\quad N \in \mathbb{N}$ $\qquad x_K = k$ $\quad k \in \mathbb{N}$

$$\lim_{K \to \infty} \lim_{N \to \infty} S_N(x_K) = \lim_{N \to \infty} \lim_{K \to \infty} S_N(x_K)$$

$$\lim_{K \to \infty} \overset{\shortparallel}{\sum_{n=0}^{\infty}} \frac{1}{K^n} \binom{K}{n} z^n \qquad \overset{\shortparallel}{\lim_{N \to \infty} \lim_{K \to \infty} \sum_{n=0}^{N}} \frac{1}{K^n} \binom{K}{n} z^n$$

$$\overset{\shortparallel}{\lim_{K \to \infty}} \left(\frac{z}{K} + 1\right)^K \overset{1.\text{Fall } z=0}{=} \lim_{K \to \infty} \left(\frac{0}{K} + 1\right)^K = 1 = \exp(0) = \exp(z)$$

$\shortparallel$ 2. Fall $z \neq 0$

$$\lim_{K \to \infty} \left(\frac{1}{\frac{K}{z}} + 1\right)^K = \lim_{K \to \infty} \left(\left(1 + \frac{1}{\frac{K}{z}}\right)^{\frac{K}{z}}\right)^z = e^z = \exp(z)$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z) = \lim_{N \to \infty} \lim_{K \to \infty} \sum_{n=0}^{N} \frac{1}{K^n} \binom{K}{n} z^n$$

$$= \lim_{N \to \infty} \sum_{n=0}^{N} \lim_{K \to \infty} \frac{1}{K^n} \binom{K}{n} \cdot z^n = \sum_{n=0}^{\infty} z^n \cdot \lim_{K \to \infty} \frac{1}{K^n} \binom{K}{n}$$

$$\Rightarrow \lim_{K \to \infty} \frac{1}{K^n} \binom{K}{n} = \frac{1}{n!}$$