ANA Ü10

6.) $\gamma: [0,1] \to \mathbb{R}^2$ ges: $l(\gamma)$, $\beta \sim \gamma$ mit Bogenlänge als Parameter

$t \mapsto \begin{pmatrix} t \\ t^{\frac{3}{2}} \end{pmatrix}$

$\gamma \in C^1[0,1]$, da $t, t^{\frac{3}{2}} \in C^1[0,1]$

Satz 11.1.8. besagt nun, dass $l(\gamma) = \int_0^1 \|\gamma'(x)\|_2 \, dx$

$\frac{d}{dt}\gamma(t) = \begin{pmatrix} 1 \\ \frac{3}{2}\cdot\sqrt{t} \end{pmatrix}$ $\qquad \|\gamma'(t)\|_2 = \sqrt{1^2 + (\frac{3}{2}\sqrt{t})^2} = \sqrt{1+\frac{9}{4}t}$

$\int \|\gamma'(x)\|_2 \, dx = \int \sqrt{1+\frac{9}{4}x} \, dx$ $\qquad \left[u = 1+\frac{9}{4}x \quad \frac{du}{dx} = \frac{9}{4} \quad dx = \frac{4}{9} du \right.$

$= \int \sqrt{u} \, \frac{4}{9} du = \frac{4}{9} \int \sqrt{u} \, du = \frac{4}{9} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} = \frac{8}{27}(1+\frac{9}{4}x)^{\frac{3}{2}}$

$\int_0^1 \|\gamma'(x)\|_2 \, dx = \frac{8}{27}(1+\frac{9}{4})^{\frac{3}{2}} - \frac{8}{27}(1)^{\frac{3}{2}} = \frac{13\sqrt{13}}{27} - \frac{8}{27} \approx 1{,}4397$

$l(\beta|_{[0,s]}) = l(\gamma|_{[0,s]})$

$\int_0^t \|\beta'(x)\|_2 \, dx = \int_0^t \|\gamma'(x)\|_2 \, dx = \frac{8}{27}(1+\frac{9}{4}t)^{\frac{3}{2}} - \frac{8}{27}$

$y = \frac{8}{27}\left((1+\frac{9}{4}t)^{\frac{3}{2}} - 1\right) \iff \frac{27}{8}y = (1+\frac{9}{4}t)^{\frac{3}{2}} - 1 \iff \frac{27}{8}y + 1 = (1+\frac{9}{4}t)^{\frac{3}{2}}$

$\iff (\frac{27}{8}y+1)^{\frac{2}{3}} = 1 + \frac{9}{4}t \iff (\frac{27}{8}y+1)^{\frac{2}{3}} - 1 = \frac{9}{4}t \iff \frac{4}{9}(\frac{27}{8}y+1)^{\frac{2}{3}} - \frac{4}{9} = t$

Nun soll $\beta(y) = \gamma\left(\frac{4}{9}(\frac{27}{8}y+1)^{\frac{2}{3}} - \frac{4}{9}\right) = \begin{pmatrix} \frac{4}{9}(\frac{27}{8}y+1)^{\frac{2}{3}} - \frac{4}{9} \\ \left(\frac{4}{9}(\frac{27}{8}y+1)^{\frac{2}{3}} - \frac{4}{9}\right)^{\frac{3}{2}} \end{pmatrix}$ sein.

$\beta'(y) = \begin{pmatrix} \frac{4}{9}\cdot\frac{2}{3}(\frac{27}{8}y+1)^{-\frac{1}{3}}\cdot\frac{27}{8} \\ \frac{3}{2}\left(\frac{4}{9}(\frac{27}{8}y+1)^{\frac{2}{3}} - \frac{4}{9}\right)^{\frac{1}{2}} \cdot \frac{4}{9}\cdot\frac{2}{3}\cdot(\frac{27}{8}y+1)^{-\frac{1}{3}}\cdot\frac{27}{8} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt[3]{\frac{27}{8}y+1}} \\ \frac{3}{2}\,\frac{\sqrt{\frac{4}{9}(\frac{27}{8}y+1)^{\frac{2}{3}} - \frac{4}{9}}}{\sqrt[3]{\frac{27}{8}y+1}} \end{pmatrix}$

$\|\beta'(y)\|_2 = \sqrt{(\frac{27}{8}y+1)^{-\frac{2}{3}} + \frac{9}{4}\,\frac{\frac{4}{9}(\frac{27}{8}y+1)^{\frac{2}{3}} - \frac{4}{9}}{(\frac{27}{8}y+1)^{\frac{2}{3}}}}$

$= \sqrt{\frac{1 + (\frac{27}{8}y+1)^{\frac{2}{3}} - 1}{(\frac{27}{8}y+1)^{\frac{2}{3}}}} = 1$

$\Rightarrow \int_0^s \|\beta'(x)\|_2 \, dx = \int_0^s 1 \, dx = s - 0 = s$ $\qquad \Rightarrow l(\beta|_{[0,t]}) = l(\gamma|_{[0,t]})$

$\beta = \gamma \circ \alpha$ mit $\alpha(t) = \frac{4}{9}(\frac{27}{8}t+1)^{\frac{2}{3}} - \frac{4}{9}$ ist bijektiv (Umkehrabbildung oben) und in $[0,1]$ streng monoton wachsend

$\Rightarrow \beta \sim \gamma$

...

ANA Ü10

6.) ... $\gamma(t) = \begin{pmatrix} t \\ \cosh(t) \end{pmatrix}$ $\qquad \gamma \in C^1([0,1])$

$l(\gamma) = \int_0^1 \|\gamma'(t)\|_2 \, dt \qquad \gamma'(t) = \begin{pmatrix} 1 \\ \sinh(t) \end{pmatrix}$

$\|\gamma'(t)\|_2 = \sqrt{1^2 + \sinh^2(t)} = \sqrt{1+\sinh^2(t)} = \cosh(t)$

$l(\gamma) = \int_0^1 \cosh(t) \, dt = \sinh(1) - \sinh(0) = \sinh(1) \approx 1{,}1752$

$l(\beta|_{[0,s]}) = l(\gamma|_{[0,s]})$

$\int_0^t \|\beta'(x)\|_2 \, dx = \int_0^t \|\gamma'(x)\|_2 \, dx = \sinh(t)$

$y = \sinh(t) \iff \operatorname{areasinh}(y) = t$

$\beta(y) = \gamma(\operatorname{areasinh}(y)) = \gamma\begin{pmatrix} \operatorname{areasinh}(y) \\ \sqrt{y^2-1} \end{pmatrix} \qquad \beta'(y) = \begin{pmatrix} \frac{1}{\sqrt{y^2+1}} \\ \frac{y}{\sqrt{y^2+1}} \end{pmatrix}$

$\|\beta'(y)\|_2 = \sqrt{\frac{1}{y^2+1} + \frac{y^2}{y^2+1}} = \sqrt{\frac{y^2+1}{y^2+1}} = 1$

$\Rightarrow \int_0^t \|\beta'(x)\|_2 \, dx = \int_0^t 1 \, dx = t \qquad \Rightarrow l(\beta|_{[0,s]}) = l(\gamma|_{[0,s]})$

$\beta = \gamma \circ \operatorname{areasinh}$ (wobei areasinh bijektiv und monoton ↗ ist)

$\Rightarrow \beta \sim \gamma$