LINAG Ü14

12.2.2. $A := \frac{1}{3}\begin{pmatrix} a & 2 & 2 \\ -2 & 1 & c \\ -2 & b & 1 \end{pmatrix}$ $\quad B := \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & w \\ 1 & -1 & -1 & x \\ 1 & 1 & -1 & y \\ 1 & 1 & 1 & z \end{pmatrix}$

ges: alle $a, b, c \in \mathbb{R}$ und alle $w, x, y, z \in \mathbb{C}$ sodass $A \in O_3$ und $B \in U_4$ ist

Nach Beobachtung 12.2.11 ist A genau dann orthogonal, wenn die Spalten eine Orthonormalbasis von $\mathbb{R}^{3\times 1}$ bilden.

$$\mathrm{I}\cdot\mathrm{I} = \begin{pmatrix} \frac{a}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{a}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} = \frac{a^2}{9} + \frac{4}{9} + \frac{4}{9} = \frac{a^2+8}{9} \overset{\text{ONB}}{=} 1 \Rightarrow a^2+8 = 9 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$$

$$\mathrm{I}\cdot\mathrm{II} = \begin{pmatrix} \frac{a}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{b}{3} \end{pmatrix} = \frac{2a}{9} - \frac{2}{9} - \frac{2b}{9} = \frac{2\cdot(a-1-b)}{9} \overset{\text{ONB}}{=} 0 \Rightarrow a-1-b = 0 \Rightarrow a = b+1 \Rightarrow b = 0 \vee b = -2$$

$$\mathrm{II}\cdot\mathrm{II} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{b}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{b}{3} \end{pmatrix} = \frac{4}{9} + \frac{1}{9} + \frac{b^2}{9} = \frac{b^2+5}{9} = 1 \Rightarrow b^2+5 = 9 \Rightarrow b^2 = 4 \Rightarrow b = -2$$
$$\Rightarrow a = -1$$

$$\mathrm{I}\cdot\mathrm{III} = \begin{pmatrix} \frac{a}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ \frac{c}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{2a}{9} - \frac{2c}{9} - \frac{2}{9} = \frac{2\cdot(a-c-1)}{9} = 0 \Rightarrow a-c-1 = 0 \Rightarrow c = a-1 \Rightarrow c = 0 \vee c = -2$$

$$\mathrm{III}\cdot\mathrm{III} = \begin{pmatrix} \frac{2}{3} \\ \frac{c}{3} \\ \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ \frac{c}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{4}{9} + \frac{c^2}{9} + \frac{1}{9} = \frac{c^2+5}{9} = 1 \Rightarrow c^2+5 = 9 \Rightarrow c^2 = 4 \Rightarrow c = -2$$

$$\mathrm{II}\cdot\mathrm{III} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{b}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ \frac{c}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{4}{9} + \frac{c}{9} + \frac{b}{9} = \frac{b+c+4}{9} = 0 \Rightarrow b+c+4 = 0 \Rightarrow b+c = -4 \checkmark$$

Probe: $A^{-1} = \left(\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}\right)^{-1} = \frac{1}{3}\cdot\begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} = A^T$ (mit Wolfram Alpha gerechnet)

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12.2.2.

$$\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}\cdot\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}=\frac14\cdot4=1\ \checkmark\qquad \begin{pmatrix}-\frac12\\-\frac12\\\frac12\\\frac12\end{pmatrix}\cdot\begin{pmatrix}-\frac12\\-\frac12\\\frac12\\\frac12\end{pmatrix}=\frac14\cdot4=1\ \checkmark\qquad \begin{pmatrix}\frac12\\-\frac12\\-\frac12\\\frac12\end{pmatrix}\cdot\begin{pmatrix}\frac12\\-\frac12\\-\frac12\\\frac12\end{pmatrix}=1\ \checkmark\qquad \begin{pmatrix}\frac w2\\\frac x2\\\frac y2\\\frac z2\end{pmatrix}\cdot\begin{pmatrix}\frac w2\\\frac x2\\\frac y2\\\frac z2\end{pmatrix}=(\bar w w+\bar x x+\bar y y+\bar z z)\cdot\frac14$$

$$=|w|^2+|x|^2+|y|^2+|z|^2=4$$

$$\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}\cdot\begin{pmatrix}-\frac12\\-\frac12\\\frac12\\\frac12\end{pmatrix}=2\cdot\left(-\frac12\right)\frac12+2\cdot\frac12\cdot\frac12=0\ \checkmark\qquad \begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}\cdot\begin{pmatrix}\frac12\\-\frac12\\-\frac12\\\frac12\end{pmatrix}=2\cdot\frac12\cdot\frac12-2\cdot\frac12\cdot\frac12=0\ \checkmark\qquad \begin{pmatrix}-\frac12\\-\frac12\\\frac12\\\frac12\end{pmatrix}\cdot\begin{pmatrix}\frac12\\-\frac12\\-\frac12\\\frac12\end{pmatrix}=2\cdot\frac14-2\cdot\frac14=0\ \checkmark$$

$$\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}\cdot\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=\frac12(w+x+y+z)=0 \Rightarrow w+x+y+z=0$$

$$\begin{pmatrix}-\frac12\\-\frac12\\\frac12\\\frac12\end{pmatrix}\cdot\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=\frac12(y+z-w-x)=0 \Rightarrow y+z=w+x$$

$$\begin{pmatrix}\frac12\\-\frac12\\-\frac12\\\frac12\end{pmatrix}\cdot\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=\frac12\cdot(w+z-x-y)=0 \Rightarrow w+z=x+y$$

$$\begin{pmatrix}w\\x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}=\frac12\cdot(\bar w+\bar x+\bar y+\bar z)=0 \Rightarrow \overline{w+x+y+z}=0\ \checkmark$$

$$\begin{pmatrix}w\\x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-\frac12\\-\frac12\\\frac12\\\frac12\end{pmatrix}=\frac12(\bar y+\bar z-\bar w-\bar x)=0 \quad \overline{y+z}=\overline{x+w}\ \checkmark$$

$$\begin{pmatrix}w\\x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}\frac12\\-\frac12\\-\frac12\\\frac12\end{pmatrix}=\frac12(\bar w+\bar z-\bar x-\bar y)=0 \Rightarrow \overline{w+z}=\overline{x+y}\ \checkmark$$

$$\Rightarrow |w|^2+|x|^2+|y|^2+|z|^2=4$$

$$w+x+y+z=0 \qquad y+z=w+x \qquad w+z=x+y$$

$$y+z=w+x \iff x=y+z-w \qquad w+z=x+y=y+z-w+y \iff 2w=2y \iff w=y$$

$$y+z=\underline{w}+x=y+x \iff x=z \qquad w+x+\underline{y}+\underline{z}=2w+2x=0 \iff x=-w$$

$$|\underline{w}|^2+|x|^2+|\underline{y}|^2+|\underline{z}|^2=4 \iff 4|x|^2=4 \iff |x|^2=1 \iff |x|=1$$

$$\Rightarrow w=-x=y=-z \quad \wedge\ |w|=1 \Rightarrow w=(\cos(v)+i\sin(v))$$

Sei $v\in\mathbb{R}$ bel. $\quad w:=(\cos(v)+i\sin(v))$

$$B^T=\frac12\cdot\begin{pmatrix}1&1&1&1\\-1&-1&1&1\\1&-1&-1&1\\w&-w&w&-w\end{pmatrix}\qquad B^{-1}=\frac12\cdot\begin{pmatrix}1&1&1&1\\-1&-1&1&1\\1&-1&-1&1\\\bar w&-\bar w&\bar w&-\bar w\end{pmatrix}$$