ANA Ü10

3.) $z \in \mathbb{C}$ $\quad f(z) = \overline{z}(z-2) - 2\cdot \operatorname{Re} z$

ges: lokale Extrema von $|f(z)|$

$z = x+iy$ $\quad f(z) = f(x+iy) = (x-iy)(x+iy-2) - 2\cdot x =$

$= x^2 + ixy - 2x - ixy + y^2 + 2iy - 2x = x^2 - 4x + y^2 + i2y$

$|f(z)| = |x^2-4x+y^2 + i2y| = \sqrt{(x^2-4x+y^2)^2 + 4y^2}$

$g(x,y) := |f(x+iy)|^2 = (x^2-4x+y^2)^2 + 4y^2 = x^4 - 8x^3 + 2x^2y^2 + 16x^2 - 8xy^2 + y^4 + 4y^2$

$\frac{\partial}{\partial x} g(x,y) = 4x^3 - 24x^2 + 4xy^2 + 32x - 8y^2$ $\qquad \frac{\partial^2}{\partial x^2} g(x,y) = 12x^2 - 48x + 4y^2 + 32$

$\frac{\partial}{\partial y} g(x,y) = 4x^2y - 16xy + 4y^3 + 8y$ $\qquad \frac{\partial^2}{\partial y^2} g(x,y) = 4x^2 - 16x + 12y^2 + 8$

$\frac{\partial}{\partial x \partial y} g(x,y) = 8xy - 16y$

$dg(x,y) = (4x^3 - 24x^2 + 4xy^2 + 32x - 8y^2 \quad 4x^2y - 16xy + 4y^3 + 8y) = (0,0)$

$\Leftrightarrow x^3 - 6x^2 + xy^2 + 8x - 2y^2 = 0 \quad \wedge \quad x^2y - 4xy + y^3 + 2y = 0$

$\Leftrightarrow (x-2)(x^2-4x+y^2) = 0 \quad \wedge \quad y(x^2-4x+y^2+2) = 0$

Damit $=0$ muss entweder $x=2$ (dann folgt $y=0$ oder $4-4+y^2+2=0 \Leftrightarrow y = \pm\sqrt{2}$)

oder $x^2-4x+y^2=0 \Rightarrow x^2-4x+y^2+2 = 2 \neq 0 \Rightarrow y=0$ also $x = 2\pm\sqrt{4} = 2\pm 2$

Also sind $\binom{2}{0}, \binom{2}{\sqrt{2}}^{*}, \binom{4}{0}, \binom{0}{0}$ mögliche Extremstellen $\quad * \binom{2}{-\sqrt{2}}$

- $\binom{2}{0}$: $\begin{pmatrix} \frac{\partial^2 g}{\partial^2 x}\binom{2}{0} & \frac{\partial^2 g}{\partial x \partial y}\binom{2}{0} \\ \frac{\partial^2 g}{\partial y \partial x}\binom{2}{0} & \frac{\partial^2 g}{\partial y^2}\binom{2}{0} \end{pmatrix} = \begin{pmatrix} 12\cdot 4 - 48\cdot 2 + 32 & 0 \\ 0 & 4\cdot 4 - 16\cdot 2 + 8 \end{pmatrix} = \begin{pmatrix} -16 & 0 \\ 0 & -8 \end{pmatrix}$ negativ definit $\Rightarrow$ lokales Maximum
- $\binom{2}{\sqrt{2}}$: $\begin{pmatrix} 12\cdot 4 - 48\cdot 2 + 4\cdot 2 + 32 & 8\cdot 2\cdot\sqrt{2} - 16\cdot\sqrt{2} \\ 8\cdot 2\sqrt{2} - 16\cdot\sqrt{2} & 4\cdot 4 - 16\cdot 2 + 12\cdot 2 + 8 \end{pmatrix} = \begin{pmatrix} -8 & 0 \\ 0 & 16 \end{pmatrix}$ indefinit
- $\binom{4}{0}$: $\begin{pmatrix} 12\cdot 16 - 48\cdot 4 + 32 & 0 \\ 0 & 4\cdot 16 - 16\cdot 4 + 8 \end{pmatrix} = \begin{pmatrix} 32 & 0 \\ 0 & 8 \end{pmatrix}$ positiv definit $\Rightarrow$ lokales Minimum
- $\binom{0}{0}$: $\begin{pmatrix} 32 & 0 \\ 0 & 8 \end{pmatrix}$ positiv definit $\Rightarrow$ lokales Minimum

Δ

Da $|f(z)| \geq 0$ ist jedes Extremum von $|f(z)|^2$ auch eines von $|f(z)|$.

Δ $\binom{2}{-\sqrt{2}}$: $\begin{pmatrix} 12\cdot 4 - 48\cdot 2 + 4\cdot 2 + 32 & -8\cdot 2\cdot\sqrt{2} + 16\sqrt{2} \\ -8\cdot 2\cdot\sqrt{2} + 16\cdot\sqrt{2} & 4\cdot 4 - 16\cdot 2 + 12\cdot 2 + 8 \end{pmatrix} = \begin{pmatrix} -8 & 0 \\ 0 & 16 \end{pmatrix}$ indefinit