7.) Sei $c>0$ bel. zz: $\mathbb{P}(X-\mathbb{E}(X) \geq c\sqrt{\mathbb{V}(X)}) \leq \frac{1}{1+c^2}$

$\mathbb{P}(X-\mathbb{E}(X) \geq c\sqrt{\mathbb{V}(X)}) \leq \frac{1}{1+c^2}$ $\qquad Y := \frac{X-\mathbb{E}(X)}{\sqrt{\mathbb{V}(X)}}$

$\Leftrightarrow \mathbb{P}(\frac{X-\mathbb{E}(X)}{\sqrt{\mathbb{V}(X)}} \geq c) \leq \frac{1}{1+c^2} \quad \Leftrightarrow \mathbb{P}(Y \geq c) \leq \frac{1}{1+c^2}$

Sei $a \geq 0$ bel.

$$\mathbb{P}(Y+a \geq c+a) \leq \mathbb{P}((Y+a)^2 \geq (c+a)^2) \overset{\text{Markov-Ungleichung}}{\leq} \frac{\mathbb{E}((Y+a)^2)}{(c+a)^2} \overset{\text{Bsp 6}}{=} \frac{1+a^2}{(c+a)^2}$$

$$\frac{d}{da}\frac{1+a^2}{(c+a)^2} = \frac{2ac-2}{(a+c)^3} \qquad \frac{2ac-2}{(a+c)^3} = 0 \Leftrightarrow 2ac = 2 \Leftrightarrow ac = 1$$

$\Leftrightarrow a = \frac{1}{c} \quad \Rightarrow \frac{1+a^2}{(c+a)^2}$ hat Minimum bei $a = \frac{1}{c}$

$$\mathbb{P}(Y \geq c) \leq \frac{1+a^2}{(c+a)^2} \quad \forall a \in \mathbb{R}^+ \quad \Rightarrow \mathbb{P}(Y \geq c) \leq \frac{1+\frac{1}{c^2}}{(c+\frac{1}{c})^2} = \frac{\frac{c^2+1}{c^2}}{\left(\frac{c^2+1}{c}\right)^2} = \frac{c^2\cdot(c^2+1)}{c^2\cdot(c^2+1)^2}$$

$$= \frac{1}{c^2+1}$$

$\square$