Homework 1

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Problem 1

$$p(t) = \det(A - t \cdot \mathrm{Id}) = \begin{vmatrix} a_{11} - t & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - t & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} - t & a_{n2} & \cdots & a_{nn} - t \end{vmatrix}$$

Problem 2

The Gamma function is defined as

$$\Gamma(x) := \lim_{n \to \infty} \frac{n! n^x}{x(x+1) \cdots (x+n)}$$

There holds the Weierstraß product representation

$$\frac{1}{\Gamma(x)} = x \cdot e^{Cx} \cdot \prod_{k=1}^{\infty} (1 + \frac{x}{k}) e^{-k/k} \qquad \text{with} \qquad C := \lim_{n \to \infty} (\sum_{k=1}^{n} \frac{1}{k} - \ln n)$$

Problem 3

Let $f, g : \mathbb{R} \to \mathbb{R}$ be given functions given by

$$f(x) := \begin{cases} -1 & \text{if } x < -\frac{\pi}{2}, \\ \sin(x) & \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \\ 1 & \text{if } x > \frac{\pi}{2}. \end{cases} \quad \text{and} \qquad g(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Problem 4

For $q \in \mathbb{R}$, it holds that

$$\lim_{n\to\infty} q^n = \begin{cases} +\infty & \text{if } q > 1, \\ 1 & \text{if } q = 1, \\ 0 & \text{if } -1 < q < 1, \\ \nexists & \text{if } q \leq -1. \end{cases}$$

Problem 5

$$A := \begin{pmatrix} \alpha & 2\alpha & 3\alpha & \cdots & n\alpha \\ 0 & \alpha & 2\alpha & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 3\alpha \\ \vdots & \ddots & \ddots & \ddots & 2\alpha \\ 0 & \cdots & 0 & 0 & \alpha \end{pmatrix} \in \mathbb{R}^{n\times n}_{\text{tria}}$$

Problem 6

$$\begin{aligned} \det(V) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} \\ = a^3b^2c - a^3b^2d - a^3bc^2 + a^3bd^2 \\ +a^3c^2d - a^3cd^2 - a^2b^3c + a^2b^3d \\ +a^2bc^3 - a^2bd^3 - a^2c^3d + a^2cd^3 \\ +ab^3c^2 - ab^3d^2 - ab^2c^3 + ab^2d^3 \\ +ac^3d^2 - ac^2d^3 - b^3c^2d + b^3cd^2 \\ +b^2c^3d - b^2cd^3 - bc^3d^2 + bc^2d^3 \\ = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \end{aligned}$$

Problem 7

Theorem 1 *For $a, b \in \mathbb{R}$ and a continuous function $f : (a, b) \to \mathbb{R}$, the following two assertions are equivalent:*

(i) f is uniformly continuous.

(ii) f has a continuous extension onto the compact interval $[a, b]$, i.e., there exists a function $\hat{f} : [a, b] \to \mathbb{R}$ with $\hat{f} = f(x)$ for all $x \in (a, b)$.

In this case the continuous extension $\hat{f}$ is even unique.

Proof *(i)* $\implies$ *(ii)* Uniform continuity of f is defined as

$$\exists \delta \; \forall \epsilon \; \forall x, y \in (a,b) : |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in (a,b) which converges to a. Since every convergent sequence is a Cauchy-sequence it holds that

$$\forall \delta \; \exists N \in \mathbb{N} \, \forall n, m > N : |x_n - x_m| < \delta$$

The uniform continuity now gives us

$$\forall n, m > N : |f(x_n) - f(x_m)| < \epsilon$$

which indicates, that $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy-Series in $\mathbb{R}$. Now we define $\hat{f}(a) := \lim_{n\to\infty} f(x_n)$.

The same can be done for $(x_n)_{n\in\mathbb{N}}$ converging towards b.

(ii) $\implies$ *(i)* Proof by contradiction. Suppose $\hat{f}$ and therefore f is not uniformly continuous on (a,b). Then there exist sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in (a,b) with

$$\lim_{n\to\infty} |x_n - y_n| = 0 \quad \text{and} \quad |\hat{f}(x_n) - \hat{f}(y_n)| > \epsilon \quad \forall n \in \mathbb{N} \tag{1}$$

Since $[a,b]$ is compact a subsequence $(x_{n(k)})_{k\in\mathbb{N}}$ with

$$\lim_{k\to\infty} x_{n(k)} = x$$

exists. The subsequence $(y_{n(k)})$ also converges to x, since $(y_n - x_n) \to 0$ and

$$\lim_{k\to\infty} y_{n(k)} = \lim_{k\to\infty} (y_{n(k)} - (y_{n(k)} - x_{n(k)})) = \lim_{k\to\infty} x_{n(k)} = x$$

Since $\hat{f}$ is continuous

$$\lim_{k\to\infty} |\hat{f}(x_{n(k)}) - \hat{f}(y_{n(k)})| = |\lim_{k\to\infty} \hat{f}(x_{n(k)}) - \lim_{k\to\infty} \hat{f}(y_{n(k)})| = |\hat{f}(x) - \hat{f}(x)| = 0$$

which contradicts (1).

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Problem 8

Theorem 2 *For real numbers $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Proof Let $x, y \in \mathbb{R}$ be arbitrary. Proof by induction:
$n = 0:$

$$(x+y)^0 = 1 = \binom{0}{0} x^0 y^0$$

$n+1:$

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)^n \cdot (x+y) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \right) (x+y) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \right) x + \left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \right) y \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k \right) + \left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1} \right) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k \right) + \left(\sum_{k=1}^{n+1} \binom{n}{k-1} x^{n-(k-1)} y^k \right) \\
&= \left(\sum_{k=1}^{n} \binom{n}{k} x^{n-k+1} y^k \right) + \left(\sum_{k=1}^{n} \binom{n}{k-1} x^{n-(k-1)} y^k \right) \\
&\quad + \binom{n}{0} x^{n-0+1} y^0 - \binom{n}{(n+1)-1} x^{n-(n+1)+1)} y^{n+1} \\
&= \left(\sum_{k=1}^{n} \binom{n}{k} x^{n-k+1} y^k \right) + \left(\sum_{k=1}^{n} \binom{n}{k-1} x^{n-k+1} y^k \right) \\
&\quad + \binom{n}{0} x^{n+1} y^0 - \binom{n}{n} x^0 y^{n+1} \\
&= \left(\sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) x^{n-k+1} y^k \right) + \binom{n}{0} x^{n+1} y^0 - \binom{n}{n} x^0 y^{n+1} \\
&= \left(\sum_{k=1}^{n} \binom{n+1}{k} x^{n-k+1} y^k \right) + \binom{n+1}{0} x^{n+1} y^0 - \binom{n+1}{n+1} x^0 y^{n+1} \\
&= \left(\sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^k \right)
\end{aligned}$$

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