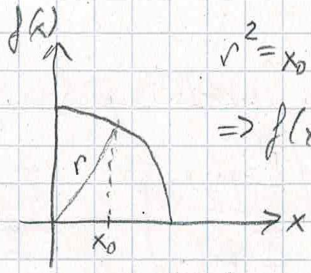
ANA Ü5

8.) $f(x)=\sqrt{r^2-x^2}$ $\qquad r>0$



$r^2=x_0^2+f(x_0)^2$

$\Rightarrow f(x_0)=\sqrt{r^2-x_0^2}$

$\int f(x)dx=\int\sqrt{r^2-x^2}\,dx=$

$\Big[\, x=r\cdot\sin(u) \quad u=\arcsin\left(\frac{x}{r}\right) \quad \frac{du}{dx}=\frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}\cdot\frac{1}{r}=\frac{1}{r\cdot\sqrt{1-\frac{x^2}{r^2}}}$

$\Big[\, dx=r\cdot\sqrt{1-\frac{x^2}{r^2}}\,du=r\cdot\cos(u)du$, da $\cos(\arcsin(x))=\sqrt{1-x^2}$

$\int\sqrt{r^2-x^2}\,dx=\int\sqrt{r^2-(r\cdot\sin(u))^2}\cdot\cos(u)\cdot r\,du=\int\sqrt{r^2\cdot(1-(\sin(u))^2)}\cdot r\cdot\cos(u)du$

$=\int r^2\cdot\sqrt{(\cos(u))^2}\cdot\cos(u)\,du=r^2\cdot\int(\cos(u))^2du=r^2\cdot\int\frac{1}{2}(\cos(u))^2+\frac{1}{2}(\cos(u))^2du$

$=r^2\cdot\int\frac{1}{2}(\cos(u))^2+\frac{1}{2}(1-(\sin(u))^2)du=r^2\cdot\int\frac{1}{2}((\cos(u))^2-(\sin(u))^2+1)du$

$=r^2\cdot\int\frac{1}{2}(\cos(u)\cos(u)-\sin(u)\sin(u)+1)\,du=r^2\cdot\int\frac{1}{2}(\cos(2\cdot)+1)\,du$

$=\frac{r^2}{2}\cdot(\int\cos(2u)du+\int 1\,du)=\frac{r^2}{2}\cdot(\int\cos(2u)du+u)$

$\Big[\, m=2u \qquad u=\frac{m}{2} \quad \frac{dm}{du}=2 \quad du=\frac{1}{2}dm$

$=\frac{r^2}{2}\cdot(\int\cos(m)\cdot\frac{1}{2}dm+u)=\frac{r^2}{2}\cdot\left(\frac{1}{2}(-\sin(m))+u\right)=\frac{r^2}{2}\cdot\left(-\frac{\sin(2u)}{2}+u\right)$

$=\frac{r^2}{2}\cdot\left(\arcsin\left(\frac{x}{r}\right)-\frac{\sin(2\arcsin(\frac{x}{r}))}{2}\right)$

$\int_0^r f(x)=\frac{r^2}{2}\cdot\left(\arcsin\left(\frac{r}{r}\right)-\frac{\sin(2\cdot\arcsin(\frac{r}{r}))}{2}\right)-\frac{r^2}{2}\cdot\left(\arcsin\left(\frac{0}{r}\right)-\frac{\sin(2\arcsin(\frac{0}{r}))}{2}\right)$

$=\frac{r^2}{2}\cdot\left(\frac{\pi}{2}-\frac{\sin(2\cdot\frac{\pi}{2})}{2}\right)-\frac{r^2}{2}\cdot\left(0-\frac{\sin(2\cdot0)}{2}\right)=\frac{r^2}{2}\cdot\left(\frac{\pi}{2}-0\right)-\frac{r^2}{2}\cdot(-0)$

$=\frac{r^2\cdot\pi}{4}$

$\Rightarrow$ Fläche eines Kreises mit Radius $r = 4\cdot\frac{r^2\cdot\pi}{4}=r^2\cdot\pi$

$\{(x,y)\in\mathbb{R}^2: x^2+y^2\leq 1\ ,\ y\geq x^2\}$ $\qquad x^2+y^2\leq1 \Leftrightarrow y^2\leq1-x^2\Leftrightarrow y\leq\sqrt{1-x^2}$

Fläche eines Kreises mit Radius 1 $\cap$ $y\geq x^2$

$y=x^2 \wedge y=\sqrt{1-x^2} \Rightarrow y=\sqrt{1-y} \Leftrightarrow y^2=1-y \Leftrightarrow y^2+y-1=0$

$\Leftrightarrow y_{1,2}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+1}=-\frac{1}{2}\pm\sqrt{\frac{5}{4}}=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}=\frac{\sqrt{5}}{2}-\frac{1}{2}$, da $y>0$ $(y\geq x^2)$

$x=\sqrt{y}=\pm\sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}}$

$\int_{-\sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}}}^{\sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}}}\sqrt{1-x^2}dx=2\cdot\int_0^{\sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}}}\sqrt{1-x^2}\,dx\approx2\cdot0{,}6952$ (da symmetrisch bzgl. y-Achse)

$\int x^2dx=\frac{x^3}{3} \qquad 2\cdot\int_0^{\sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}}}x^2=2\cdot\left(\frac{\left(\sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}}\right)^3}{3}-\frac{0^3}{3}\right)\approx2\cdot0{,}162$

$\Rightarrow$ Fläche $=2\cdot0{,}6952-2\cdot0{,}162\approx1{,}0664$