LINAG Ü13

11.5.6 $\quad I=[-1,1] \quad C^0(I)$ mit Skalarprodukt $(f,g)\mapsto \int_{-1}^{1} f(x)g(x)\,dx$

a) $U_3 := [\{f_i : x\mapsto x^i \mid i\in\{0,1,2,3\}\}]$ ges: Orthonormalbasis

$a_0\ (x\mapsto 1)$, $a_1\ (x\mapsto x)$, $a_2\ (x\mapsto x^2)$, $a_3\ (x\mapsto x^3)$ bilden eine Basis von U_3

Nach Smidt: $b_0 = a_0 = 1$

$$b_1 = a_1 - \frac{b_0\cdot a_1}{b_0\cdot b_0} b_0 = a_1 - \frac{0}{2} b_0 = a_1 = x \qquad b_0\cdot a_1 = \int_{-1}^{1} 1\cdot x\,dx = 0, \quad b_0\cdot b_0 = \int_{-1}^{1} 1\,dx = 2$$

$$b_2 = a_2 - \left(\frac{b_0\cdot a_2}{b_0\cdot b_0} b_0 + \frac{b_1\cdot a_2}{b_1\cdot b_1} b_1\right) = a_2 - \left(\frac{\frac{2}{3}}{2} b_0 + \frac{0}{\frac{2}{3}} b_1\right) = a_2 - \frac{1}{3} b_0 = x^2 - \frac{1}{3}$$

$$b_0\cdot a_2 = \int_{-1}^{1} 1\cdot x^2\,dx = \frac{2}{3}, \quad b_1\cdot a_2 = \int_{-1}^{1} x\cdot x^2\,dx = 0, \quad b_1\cdot b_1 = \int_{-1}^{1} x\cdot x\,dx = \frac{2}{3}$$

$$b_3 = a_3 - \left(\frac{b_0\cdot a_3}{b_0\cdot b_0} b_0 + \frac{b_1\cdot a_3}{b_1\cdot b_1} b_1 + \frac{b_2\cdot a_3}{b_2\cdot b_2} b_2\right) = a_3 - \left(\frac{0}{2} b_0 + \frac{\frac{2}{5}}{\frac{2}{3}} b_1 + \frac{0}{\frac{8}{45}} b_2\right) = a_3 - \frac{3}{5} b_1 = x^3 - \frac{3}{5}x$$

$$b_0\cdot a_3 = \int_{-1}^{1} 1\cdot x^3\,dx = 0, \quad b_1\cdot a_3 = \int_{-1}^{1} x\cdot x^3\,dx = \frac{2}{5}, \quad b_2\cdot a_3 = \int_{-1}^{1} \left(x^2-\frac{1}{3}\right)x^3\,dx = 0, \quad b_2\cdot b_2 = \int_{-1}^{1}\left(x^2-\frac{1}{3}\right)\left(x^2-\frac{1}{3}\right)dx = \frac{8}{45}$$

$$\|b_0\| = \sqrt{1\cdot 1} = \sqrt{\int_{-1}^{1} 1\,dx} = \sqrt{2}$$

$$\|b_1\| = \sqrt{x\cdot x} = \sqrt{\int_{-1}^{1} x\cdot x\,dx} = \sqrt{\frac{2}{3}}$$

$$\|b_2\| = \sqrt{\left(x^2-\frac{1}{3}\right)\left(x^2-\frac{1}{3}\right)} = \sqrt{\frac{8}{45}}$$

$$\|b_3\| = \sqrt{\left(x^3-\frac{3}{5}x\right)\left(x^3-\frac{3}{5}x\right)} = \sqrt{\int_{-1}^{1}\left(x^3-\frac{3}{5}x\right)\left(x^3-\frac{3}{5}x\right)dx} = \sqrt{\frac{8}{175}}$$

$$c_0 = \frac{1}{\sqrt{2}} \quad c_1 = \frac{x}{\sqrt{\frac{2}{3}}} \quad c_2 = \frac{x^2-\frac{1}{3}}{\sqrt{\frac{8}{45}}} \quad c_3 = \frac{x^3-\frac{3}{5}x}{\sqrt{\frac{8}{175}}}$$

(c_0, c_1, c_2, c_3) … ONB

(mit Wolfram Alpha Probegerechnet)

b) $U_2 := [\{f_0, f_1, f_2\}] \quad p: C^0(I)\to U_2 \quad$ ges: $p(\exp|_I)$

$$p(\exp) = \frac{c_0\cdot \exp}{c_0\cdot c_0}\cdot c_0 + \frac{c_1\cdot \exp}{c_1\cdot c_1}\cdot c_1 + \frac{c_2\cdot \exp}{c_2\cdot c_2} c_2$$

$$= (c_0\cdot \exp)\cdot c_0 + (c_1\cdot \exp) c_1 + (c_2\cdot \exp)\cdot c_2$$

$$\approx 1{,}662\cdot c_0 + 0{,}90112\, c_1 + 0{,}2263\, c_2$$

$$\approx 1{,}175 + 1{,}10364x + 0{,}536718\left(x^2 - \frac{1}{3}\right)$$

$$= 0{,}536718x^2 + 1{,}10364x + 0{,}996094$$

$$c_0\cdot \exp = \int_{-1}^{1} \frac{1}{\sqrt{2}}\cdot \exp(x)\,dx \approx 1{,}6620$$

$$c_1\cdot \exp = \int_{-1}^{1} \frac{x}{\sqrt{\frac{2}{3}}} \exp(x)\,dx \approx 0{,}90112$$

$$c_2\cdot \exp = \int_{-1}^{1} \frac{x^2-\frac{1}{3}}{\sqrt{\frac{8}{45}}} \exp(x)\,dx \approx 0{,}22630$$