ANA Ü5

9.) $\alpha \in \mathbb{R}\setminus\{0\}$ Für welche α existieren: $\int_0^1 x^\alpha dx$, $\int_1^{+\infty} x^\alpha dx$, $\int_0^{+\infty} x^\alpha dx$

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}$$

$$\int_0^1 x^\alpha dx = \frac{1^{\alpha+1}}{\alpha+1} = \frac{1}{\alpha+1}$$ existiert für alle $\alpha \in \mathbb{R}\setminus\{0,-1\}$

$$\int_1^{+\infty} x^\alpha dx = \lim_{\beta\to+\infty} \int_1^\beta x^\alpha dx = \lim_{\beta\to+\infty} \frac{\beta^{\alpha+1}}{\alpha+1} - \frac{1^{\alpha+1}}{\alpha+1} = \lim_{\beta\to+\infty} \frac{\beta^{\alpha+1}-1}{\alpha+1}$$

1.Fall $\alpha = -1$: $\lim_{\beta\to+\infty} \frac{\beta^{-1+1}-1}{-1+1} = \lim_{\beta\to+\infty} \frac{1-1}{0}$ $\Rightarrow$ existiert nicht

2.Fall $\alpha < -1$: $\lim_{\beta\to+\infty} \frac{\beta^{\alpha+1}-1}{\alpha+1} = \frac{1}{\alpha+1}\cdot \lim_{\beta\to+\infty} \underbrace{\beta^{\alpha+1}}_{\to 0} - 1 = -\frac{1}{\alpha+1}$ $\Rightarrow$ existiert

3.Fall $\alpha > -1$: $\lim_{\beta\to+\infty} \frac{\beta^{\alpha+1}-1}{\alpha+1} = +\infty$ $\Rightarrow$ existiert nicht

$$\int_0^{+\infty} x^\alpha dx = \int_0^1 x^\alpha dx + \int_1^{+\infty} x^\alpha dx = \frac{1}{\alpha+1} + \int_1^{+\infty} x^\alpha dx \quad \text{für } \alpha \neq -1$$

$\Rightarrow$ existiert falls $\alpha < -1$

$$\int_1^{+\infty} \frac{\ln(t)}{t^2} dt$$

$$\int \frac{\ln(t)}{t^2} dt = \int \ln(t)\cdot\left(-\frac{1}{t}\right)' dt = -\frac{1}{t}\cdot\ln(t) - \int -\frac{1}{t}(\ln(t))' dt$$

$$= -\frac{\ln(t)}{t} - \int -\frac{1}{t}\cdot\frac{1}{t} dt = -\frac{\ln(t)}{t} + \int \frac{1}{t^2} dt = \frac{t^{-1}}{-1} - \frac{\ln(t)}{t}$$

$$= -\frac{1}{t} - \frac{\ln(t)}{t} = \frac{-\ln(t)-1}{t}$$

$$\int_1^{+\infty} \frac{\ln(t)}{t^2} dt = \lim_{\beta\to+\infty} \int_1^\beta \frac{\ln(t)}{t^2} dt = \lim_{\beta\to+\infty} \frac{-\ln(\beta)-1}{\beta} - \frac{-\ln(1)-1}{1}$$

$$= \lim_{\beta\to+\infty} \frac{-\ln(\beta)-1}{\beta} - (0-1) = \lim_{\beta\to+\infty} \frac{-\ln(\beta)-1+\beta}{\beta}$$

$$= \lim_{\beta\to+\infty}\left(-\frac{\ln(\beta)}{\beta}\right) - \lim_{\beta\to+\infty}\left(\frac{1}{\beta}\right) + \lim_{\beta\to+\infty}\left(\frac{\beta}{\beta}\right) = 1 - \lim_{\beta\to+\infty} \frac{\overbrace{\ln(\beta)}^{\to+\infty}}{\underbrace{\beta}_{\to+\infty}}$$

$$= 1 - \lim_{\beta\to+\infty} \frac{\frac{1}{\beta}}{1} = 1 - \lim_{\beta\to+\infty} \frac{1}{\beta} = 1$$