LINAG Ü13

11.5.2. $\mathbb{K}^{n\times 1}$ mit kanonischem euklidischem oder unitärem Skalarprodukt

$B = (b_1, b_2, \dots, b_n) \dots$ Basis

a) $n=3 \quad \mathbb{K}=\mathbb{R} \quad b_1 = \begin{pmatrix}2\\3\\1\end{pmatrix} \quad b_2 = \begin{pmatrix}1\\-2\\2\end{pmatrix} \quad b_3 = \begin{pmatrix}2\\2\\1\end{pmatrix}$ ges: Orthogonal/Orthonormalbasis

$a_1 = b_1 = \begin{pmatrix}2\\3\\1\end{pmatrix}$

$$a_2 = b_2 - \sum_{j=1}^{1} \frac{a_j \cdot b_2}{a_j \cdot a_j} a_j = b_2 - \frac{a_1 \cdot b_2}{a_1 \cdot a_1} a_1 = \begin{pmatrix}1\\-2\\2\end{pmatrix} - \frac{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}}{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\1\end{pmatrix}} \begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix} - \frac{-2}{14}\begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix} + \frac{1}{7}\begin{pmatrix}2\\3\\1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{9}{7}\\-\frac{11}{7}\\\frac{15}{7}\end{pmatrix}$$

$$a_3 = b_3 - \sum_{j=1}^{2} \frac{a_j \cdot b_3}{a_j \cdot a_j} a_j = b_3 - \left(\frac{a_1 \cdot b_3}{a_1 \cdot a_1} a_1 + \frac{a_2 b_3}{a_2 a_2} a_2\right) = b_3 - \left(\frac{11}{14} a_1 + \frac{\frac{11}{7}}{\frac{61}{7}} a_2\right)$$

$$= \begin{pmatrix} 2 - \left(\frac{11}{14}\cdot 2 + \frac{11}{61}\cdot\frac{9}{7}\right) \\ 2 - \left(\frac{11}{14}\cdot 3 + \frac{11}{61}\cdot\left(-\frac{11}{7}\right)\right) \\ 1 - \left(\frac{11}{14}\cdot 1 + \frac{11}{61}\cdot\frac{15}{7}\right) \end{pmatrix} = \begin{pmatrix}\frac{12}{61}\\-\frac{9}{122}\\-\frac{21}{122}\end{pmatrix}$$

$\|a_1\| = \sqrt{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\1\end{pmatrix}} = \sqrt{14}$ $\qquad c_1 = \left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)^T$

$\|a_2\| = \sqrt{\frac{61}{7}}$ $\qquad c_2 = \left(\frac{9}{\sqrt{427}}, -\frac{11}{\sqrt{427}}, \frac{15}{\sqrt{427}}\right)^T$

$\|a_3\| = \sqrt{\frac{9}{122}} = \frac{3}{\sqrt{122}}$ $\qquad c_3 = \left(\frac{4\sqrt{122}}{61}, -\frac{3}{\sqrt{122}}, -\frac{7}{\sqrt{122}}\right)^T$

γ) $n=3 \quad K=\mathbb{C} \quad b_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad b_2 = \begin{pmatrix}0\\i\\0\end{pmatrix} \quad b_3 = \begin{pmatrix}i\\0\\0\end{pmatrix}$

$a_1 = b_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$$a_2 = b_2 - \frac{a_1 \cdot b_2}{a_1 \cdot a_1} \cdot a_1 = b_2 - \frac{i}{3} a_1 = \begin{pmatrix}0 - \frac{i}{3}\\ i - \frac{i}{3}\\ 0 - \frac{i}{3}\end{pmatrix} = \begin{pmatrix}-\frac{i}{3}\\ \frac{2i}{3}\\ -\frac{i}{3}\end{pmatrix}$$

$$a_3 = b_3 - \left(\frac{a_1 b_3}{a_1 a_1} a_1 + \frac{a_2 b_3}{a_2 a_2} a_2\right) = b_3 - \left(\frac{i}{3} a_1 + \frac{\frac{i}{3}}{-\frac{2}{3}} a_2\right) = b_3 - \frac{i}{3} a_1 + \frac{1}{2} a_2$$

$$= \begin{pmatrix} i - \frac{i}{3} + \frac{1}{2}\left(-\frac{i}{3}\right) \\ 0 - \frac{i}{3} + \frac{1}{2}\,\frac{2i}{3} \\ 0 - \frac{i}{3} + \frac{1}{2}\left(-\frac{i}{3}\right) \end{pmatrix} = \begin{pmatrix}\frac{i}{2}\\0\\-\frac{i}{2}\end{pmatrix}$$

$\|a_1\| = \sqrt{3}$ $\qquad c_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T$

$\|a_2\| = \sqrt{-\frac{1}{9} - \frac{4}{9} - \frac{1}{9}} = i \cdot \sqrt{\frac{2}{3}}$ $\qquad c_2 = \left(-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}\right)^T$

$\|a_3\| = \sqrt{-\frac{1}{4} - \frac{1}{4}} = \frac{i}{\sqrt{2}}$ $\qquad c_3 = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$

Dann ist (a_1, a_2, a_3) immer eine Orthogonalbasis und (c_1, c_2, c_3) sogar eine Orthonormalbasis. (Proberechnungen $a_1 \cdot a_2 = 0 \quad a_1 \cdot a_3 = 0 \quad a_2 \cdot a_3 = 0$

$\|c_1\| = 1 \quad \|c_2\| = 1 \quad \|c_3\| = 1$ mit Taschenrechner)