NUM Ü3

3.) $\Delta = \{x_0, \dots, x_n\}$ ... Zerlegung von $[a,b]$ $\quad h_j := x_j - x_{j-1} \; \forall j = 1, \dots, n$

$s \in S^3(\Delta)$ ... interpolierender kubischer Spline von $f \in C^0[a,b]$

mit natürlichen Randbedingungen ($s \in C^2([a,b])$, $s(x_j) = f(x_j) = y_j$, $s''(a) = 0 = s''(b)$)

$\forall j = 1, \dots, n \; \exists a_0^{(j)}, \dots, a_3^{(j)}: s(x) = a_0^{(j)} + a_1^{(j)}(x - x_j) + a_2^{(j)}(x-x_j)^2 + a_3^{(j)}(x-x_j)^3$

für $x \in [x_{j-1}, x_j]$ ... lokale Darstellung von s

i) $a_2^{(0)} := 0$ zz: $a_0^{(j)} = y_j$, $a_1^{(j)} = \frac{y_j - y_{j-1}}{h_j} + \frac{h_j}{3}(2a_2^{(j)} + a_2^{(j-1)})$, $a_3^{(j)} = \frac{a_2^{(j)} - a_2^{(j-1)}}{3h_j}$

für alle $j = 1, \dots, n$

Sei $j \in \{1, \dots, n\}$ bel.

$$y_j = s(x_j) = a_0^{(j)} + a_1^{(j)}(x_j - x_j) + a_2^{(j)}(x_j - x_j)^2 + a_3^{(j)}(x_j - x_j)^3 = a_0^{(j)} \quad \checkmark$$

$$y_{j-1} = s(x_{j-1}) = a_0^{(j)} + a_1^{(j)}(x_{j-1} - x_j) + a_2^{(j)}(x_{j-1} - x_j)^2 + a_3^{(j)}(x_{j-1} - x_j)^3$$

$$y_{j-1} = a_0^{(j)} - a_1^{(j)} h_j + a_2^{(j)} h_j^2 - a_3^{(j)} h_j^3$$

$$y_{j-1} = y_j - a_1^{(j)} h_j + a_2^{(j)} h_j^2 - \frac{a_2^{(j)} - a_2^{(j-1)}}{3h_j} h_j^3$$

$$a_1^{(j)} h_j = y_j - y_{j-1} + h_j^2 \left(a_2^{(j)} - \tfrac{1}{3}(a_2^{(j)} - a_2^{(j-1)})\right)$$

$$a_1^{(j)} = \frac{y_j - y_{j-1}}{h_j} + h_j \left(\frac{3a_2^{(j)} - a_2^{(j)} + a_2^{(j-1)}}{3}\right)$$

$$a_1^{(j)} = \frac{y_j - y_{j-1}}{h_j} + \frac{h_j}{3}(2a_2^{(j)} + a_2^{(j-1)}) \quad \checkmark$$

$$s'(x) = a_1^{(j)} + a_2^{(j)}(2x - 2x_j) + a_3^{(j)}(3x^2 - 6xx_j + 3x_j^2) \text{ in } [x_{j-1}, x_j]$$

$$s''(x) = a_2^{(j)} \cdot 2 + a_3^{(j)}(6x - 6x_j) = 2a_2^{(j)} + 6a_3^{(j)}(x - x_j) \text{ in } [x_{j-1}, x_j]$$

$$2a_2^{(j-1)} = s''(x_{j-1}) = 2a_2^{(j)} + 6a_3^{(j)}(x_{j-1} - x_j) = 2a_2^{(j)} - 6a_3^{(j)} h_j$$

$$\Leftrightarrow 6a_3^{(j)} h_j = 2a_2^{(j)} - 2a_2^{(j-1)}$$

$$a_3^{(j)} = \frac{2(a_2^{(j)} - a_2^{(j-1)})}{6h_j} = \frac{a_2^{(j)} - a_2^{(j-1)}}{3h_j} \quad \checkmark$$

für $j = 1$ gilt: $0 = s''(x_{j-1}) = s''(a) = 2a_2^{(j)} + 6a_3^{(j)}(x_{j-1} - x_j)$

$$\Leftrightarrow 6a_3^{(j)} h_j = 2a_2^{(j)} \Leftrightarrow a_3^{(j)} = \frac{a_2^{(j)}}{3h_j} = \frac{a_2^{(j)} - \overbrace{a_2^{(j-1)}}^{=0}}{3h_j} \quad \checkmark$$

...

NUM Ü3

3.) ... ii) zz: $a_2^{(n)} = 0$

$$0 = s''(b) = 2\, a_2^{(n)} + 6\, a_3^{(n)}(b - x_n) = 2a_2^{(n)} + 6\, a_3^{(n)}(b-b) = 2a_2^{(n)}$$

$$\Rightarrow a_2^{(n)} = 0$$

iii) zz: $a_2^{(1)}, \dots, a_2^{(n-1)}$ lösen

$$\begin{pmatrix} 2(h_1+h_2) & h_2 & & \\ h_2 & 2(h_2+h_3) & \ddots & \\ & \ddots & \ddots & h_{n-1} \\ & & h_{n-1} & 2(h_{n-1}-h_n) \end{pmatrix} \begin{pmatrix} a_2^{(1)} \\ \vdots \\ a_2^{(n-1)} \end{pmatrix} = 3 \begin{pmatrix} \frac{y_2-y_1}{h_2} - \frac{y_1-y_0}{h_1} \\ \vdots \\ \frac{y_n-y_{n-1}}{h_n} - \frac{y_{n-1}-y_{n-2}}{h_{n-1}} \end{pmatrix}$$

$$s'(x) = a_1^{(j)} + 2a_2^{(j)}(x-x_j) + 3a_3^{(j)}(x-x_j)^2 \quad \text{in } [x_{j-1}, x_j]$$

$$\Rightarrow a_1^{(j-1)} = s'(x_{j-1}) = a_1^{(j)} + 2a_2^{(j)}(x_{j-1}-x_j) + 3a_3^{(j)}(x_{j-1}-x_j)^2$$

$$\Leftrightarrow a_1^{(j)} - a_1^{(j-1)} = +2a_2^{(j)} h_j - 3a_3^{(j)} h_j^2$$

$$a_1^{(j)} - a_1^{(j-1)} = \frac{y_j - y_{j-1}}{h_j} + \frac{h_j}{3}(2a_2^{(j)} + a_2^{(j-1)}) - \frac{y_{j-1}-y_{j-2}}{h_{j-1}} - \frac{h_{j-1}}{3}(2a_2^{(j-1)} + a_2^{(j-2)})$$

$$\Rightarrow \frac{y_j-y_{j-1}}{h_j} - \frac{y_{j-1}-y_{j-2}}{h_{j-1}} = 2a_2^{(j)}h_j - 3a_3^{(j)}h_j^2 - \frac{h_j}{3}(2a_2^{(j)} + a_2^{(j-1)}) + \frac{h_{j-1}}{3}(2a_2^{(j-1)} + a_2^{(j-2)})$$

$$= 2a_2^{(j)}h_j - \cancel{3}\frac{a_2^{(j)} - a_2^{(j-1)}}{\cancel{3h_j}} h_j^{\cancel{2}} - \frac{2}{3}h_j a_2^{(j)} - \frac{1}{3}h_j a_2^{(j-1)} + \frac{2}{3}h_{j-1}a_2^{(j-1)} + \frac{1}{3}h_{j-1}a_2^{(j-2)}$$

$$= a_2^{(j)}\left(2h_j - h_j - \frac{2}{3}h_j\right) + a_2^{(j-1)}\left(h_j - \frac{1}{3}h_j + \frac{2}{3}h_{j-1}\right) + a_2^{(j-2)}\left(\frac{1}{3}h_{j-1}\right)$$

$$= \frac{1}{3}h_j a_2^{(j)} + \frac{2}{3}a_2^{(j-1)}(h_j + h_{j-1}) + \frac{1}{3}h_{j-1}a_2^{(j-2)}$$

$$= \frac{1}{3}\left(h_j \cdot a_2^{(j)} + 2(h_j + h_{j-1})a_2^{(j-1)} + h_{j-1}a_2^{(j-2)}\right)$$

$$\Rightarrow h_j \cdot a_2^{(j)} + 2(h_j + h_{j-1})a_2^{(j-1)} + h_{j-1}a_2^{(j-2)} = 3\left(\frac{y_j-y_{j-1}}{h_j} - \frac{y_{j-1}-y_{j-2}}{h_{j-1}}\right)$$

für $j=2$: $a_2^{(j-2)} = a_2^{(0)} = 0$

für $j=n$: $a_2^{(n)} = 0$ $\quad \Rightarrow$ gilt auch für erste und letzte Zeile

$\square$