NUM Ü7

25) zz: $x_0,\dots,x_n$ ... Nullstellen des $(n+1)$-ten Orthogonalpolynoms p_{n+1} sind EW zu den EV $v^{(0)},\dots,v^{(n)}$

$v^{(j)} = \begin{pmatrix} J_0 p_0(x_j) \\ \vdots \\ J_n p_n(x_j)\end{pmatrix}$ Zu zeigen $\forall j: Av^{(j)} = x_j v^{(j)}$

Sei $j\in\{0,\dots,n\}$ bel. Wir schauen uns nun $(Av^{(j)})_l$ und $(x_j v^{(j)})_l$ an.

$l=0$: $(Av^{(j)})_0 = \beta_0 J_0 p_0(x_j) - \gamma_1 J_1 p_1(x_j) = \beta_0 p_0(x_j) + \cancel{\gamma_1}\cdot\cancel{\gamma_1^{-1}}\, p_1(x_j) = \cancel{\beta_0} + x_j - \cancel{\beta_0} = x_j J_0 p_0(x_j) = (x_j v^{(j)})_0$

$J_{l+1} = (-1)^{l+1}\prod_{k=1}^{l+1}\gamma_k^{-1} = (-1)\gamma_{l+1}^{-1}\cdot(-1)^l\prod_{k=1}^{l}\gamma_k^{-1} = -\gamma_{l+1}^{-1} J_l$

$l=n$: $(Av^{(j)})_n = -\gamma_n J_{n-1}p_{n-1}(x_j) + \beta_n J_n p_n(x_j) = -\gamma_n(-J_n\gamma_n)p_{n-1}(x_j) + \beta_n J_n p_n(x_j)$

$= J_n(\gamma_n^2 p_{n-1}(x_j) + \beta_n p_n(x_j)) = -J_n(-\gamma_n^2 p_{n-1}(x_j) - \beta_n p_n(x_j) + x_j p_n(x_j) - x_j p_n(x_j))$

$= -J_n((x_j-\beta)p_n(x_j) - \gamma_n^2 p_{n-1}(x_j) - x_j p_n(x_j)) = -J_n(\underbrace{p_{n+1}(x_j)}_{=0} - x_j p_n(x_j))$

$= x_j J_n p_n(x_j) = (x_j v^{(j)})_n$

$l\in\{1,\dots,n-1\}$: $(Av^{(j)})_l = -\gamma_l J_{l-1}p_{l-1}(x_j) + \beta_l J_l p_l(x_j) - \gamma_{l+1}J_{l+1}p_{l+1}(x_j)$

$= \gamma_l^2 J_l p_{l-1}(x_j) + \beta_l J_l p_l(x_j) + J_l p_{l+1}(x_j)$

$= J_l(\cancel{\gamma_l^2 p_{l-1}(x_j)} + \cancel{\beta_l p_l(x_j)} + (x_j - \cancel{\beta_j})p_l(x_j) - \cancel{\gamma_l^2 p_{l-1}(x_j)})$

$= x_j J_l p_l(x_j) = (x_j v^{(j)})_l$

zz: Gewichte der Gauß-Quadratur w_j erfüllen $w_j = \left(\int_a^b w(x)\,dx\right)\left(\sum_{k=0}^n J_k^2(p_k(x_j))^2\right)^{-1}$ $\forall j=0,\dots,n$

Sei $j\in\{0,\dots,n\}$ bel.

$\sum_{k=0}^n w_k p_j(x_k) = Q_n p_j = Q p_j = \int_a^b p_j(x)w(x)\,dx = \int_a^b p_j(x)\,p_0(x)\,w(x)\,dx = \langle p_j,p_0\rangle = \delta_{j0}\int_a^b w(x)\,dx$

Gauß-Quadratur Exaktheitsgrad $2n+1$ — nach Def — $p_0(x)=1$ — nach Def — $=0$ für $j\neq 0$ da Orthogonal, $\neq 0$ für $j=0$ klar

Daraus folgt: $\int_a^b w(x)\,dx = \sum_{k=0}^n w_k p_0(x_k) \overset{*}{=} \sum_{l=0}^n J_l^2 p_l(x_j)\sum_{k=0}^n w_k p_l(x_k)$

$= \sum_{l=0}^n\sum_{k=0}^n J_l^2 p_l(x_j)\, w_k p_l(x_k) = \sum_{k=0}^n\sum_{l=0}^n J_l^2 p_l(x_j)\, w_k p_l(x_k) = \sum_{k=0}^n w_k \sum_{l=0}^n J_l p_l(x_j)\, J_l p_l(x_k)$

$= \sum_{k=0}^n w_k \left(v^{(j)T}\cdot v^{(k)}\right) = w_j \left(v^{(j)T}\cdot v^{(j)}\right)$

sonst orthogonal

$\Rightarrow \int_a^b w(x)\,dx = w_j\sum_{k=0}^n J_k^2(p_k(x_j))^2 \Leftrightarrow w_j = \int_a^b w(x)\,dx\left(\sum_{k=0}^n J_k^2(p_k(x_j))^2\right)^{-1}$

$*$ $J_l^2 p_l(x_j)\sum_{k=0}^n w_k p_l(x_k) = J_l p_l(x_j)\sum_{k=0}^n w_k J_l p_l(x_k) = (v^{(j)})_l\sum_{k=0}^n w_k (v^{(k)})_l$

$J_0^2 p_0(x_j) = 1$

□

NUM Ü7

26)

$$p_0(x) = 1 \qquad p_1(x) = -\frac{1}{2}\left(1-x^2\right)' = -\frac{1}{2}(-2x) = x$$

$$p_2(x) = \frac{2}{24}\left(1-2x^2+x^4\right)'' = \frac{1}{12}\left(-4x+4x^3\right)' = \frac{1}{12}\left(-4+12x^2\right) = x^2 - \frac{1}{3}$$

$$p_3(x) = -\frac{6}{720}\left(1-3x^2+3x^4-x^6\right)''' = -\frac{1}{120}\left(72x-120x^3\right) = x^3 - \frac{3}{5}x$$

$n=0$ $\quad p_1(x) = x$ hat Nullstellen $\{0\}$ $\Rightarrow x_0 = 0$

$$w_0 = \int_a^b L_0 w \, dx = \int_{-1}^{1} \prod_{\substack{k=0\\k\neq 0}}^{0} \frac{x-x_k}{x_0-x_k} dx = \int_{-1}^{1} 1 \, dx = 2$$

$n=1$ $\quad p_2(x) = x^2 - \frac{1}{3}$ hat Nullstellen $\left\{\sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}\right\}$ $\Rightarrow x_0 = \sqrt{\frac{1}{3}} \quad x_1 = -\sqrt{\frac{1}{3}}$

$$w_0 = \int_a^b L_0 w \, dx = \int_{-1}^{1} \prod_{\substack{k=0\\k\neq 0}}^{1} \frac{x-x_k}{x_0-x_k} dx = \int_{-1}^{1} \frac{x-x_1}{x_0-x_1} dx = \frac{1}{x_0-x_1}\left(\frac{x^2}{2} - x_1 x\right)\Big|_{-1}^{1} =$$

$$= \frac{1}{2\frac{1}{\sqrt{3}}}\left(\cancel{\frac{1}{2}} - x_1 - \cancel{\frac{1}{2}} - x_1\right) = -\frac{\sqrt{3}}{\cancel{2}}\,\cancel{2}x_1 = -\sqrt{3}\left(-\frac{1}{\sqrt{3}}\right) = 1$$

$$w_1 = \int_{-1}^{1} \prod_{\substack{k=0\\k\neq 1}}^{1} \frac{x-x_k}{x_1-x_k} dx = \int_{-1}^{1} \frac{x-x_0}{x_1-x_0} dx = \frac{1}{x_1-x_0}\left(\frac{x^2}{2} - x_0 x\right)\Big|_{-1}^{1} = \frac{1}{-\frac{2}{\sqrt{3}}}\left(\cancel{\frac{1}{2}} - x_0 - \cancel{\frac{1}{2}} - x_0\right)$$

$$= + \frac{\sqrt{3}}{2}\, 2 \,\frac{1}{\sqrt{3}} = 1$$

$n=2$ $\quad p_3(x) = x\left(x^2 - \frac{3}{5}\right)$ hat Nullstellen $\left\{0, \sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right\}$ $\Rightarrow x_0 = -\sqrt{\frac{3}{5}} \quad x_1 = 0 \quad x_2 = \sqrt{\frac{3}{5}}$

$$w_0 = \int_{-1}^{1} \prod_{\substack{k=0\\k\neq 0}}^{2} \frac{x-x_k}{x_0-x_k} dx = \int_{-1}^{1} \frac{x-x_1}{x_0-x_1}\,\frac{x-x_2}{x_0-x_2} dx = \frac{1}{x_0-x_1}\,\frac{1}{x_0-x_2}\int_{-1}^{1} x^2 - x_2 x - x_1 x + x_1 x_2 \, dx$$

$$= \frac{1}{x_0-x_1}\,\frac{1}{x_0-x_2}\left(\frac{x^3}{3} - x_2\frac{x^2}{2} - x_1\frac{x^2}{2} + x_1 x_2 x\right)\Big|_{-1}^{1} = \frac{1}{-\sqrt{\frac{3}{5}}}\,\frac{1}{-2\sqrt{\frac{3}{5}}}\left(\frac{1}{3} - \frac{1}{2}x_2 - \frac{1}{2}x_1 + x_1 x_2 + \frac{1}{3} + \frac{1}{2}x_2 + \frac{1}{2}x_1 + x_1 x_2\right)$$

$$= \frac{1}{2\left(\frac{3}{5}\right)}\left(\frac{1}{3} - \cancel{\frac{1}{2}\sqrt{\frac{3}{5}}} + \frac{1}{3} + \cancel{\frac{1}{2}\sqrt{\frac{3}{5}}}\right) = \frac{5}{6}\,\frac{2}{3} = \frac{5}{9}$$

$$w_1 = \int_{-1}^{1} \frac{x-x_0}{x_1-x_0}\,\frac{x-x_2}{x_1-x_2} dx = \frac{1}{\sqrt{\frac{3}{5}}}\,\frac{1}{-\sqrt{\frac{3}{5}}}\int_{-1}^{1} x^2 - x_2 x - x_0 x + x_0 x_2 \, dx = -\frac{5}{3}\left(\frac{x^3}{3} - x_2\frac{x^2}{2} - x_0\frac{x^2}{2} + x_0 x_2 x\right)\Big|_{-1}^{1}$$

$$= -\frac{5}{3}\left(\frac{1}{3} - \cancel{x_2\frac{1}{2}} - \cancel{x_0\frac{1}{2}} + x_0 x_2 + \frac{1}{3} + \cancel{x_2\frac{1}{2}} + \cancel{x_0\frac{1}{2}} + x_0 x_2\right) = -\frac{5}{3}\left(\frac{2}{3} - 2\,\frac{3}{5}\right) = \frac{8}{9}$$

$$w_2 = \int_{-1}^{1} \frac{x-x_0}{x_2-x_0}\,\frac{x-x_1}{x_2-x_1} dx = \frac{1}{2\sqrt{\frac{3}{5}}}\,\frac{1}{\sqrt{\frac{3}{5}}}\int_{-1}^{1} x^2 - x x_0 \, dx = \frac{1}{2\frac{3}{5}}\left(\frac{x^3}{3} - x_0\frac{x^2}{2}\right)\Big|_{-1}^{1}$$

$$= \frac{5}{6}\left(\frac{1}{3} - \cancel{x_0\frac{1}{2}} + \frac{1}{3} + \cancel{x_0\frac{1}{2}}\right) = \frac{5}{6}\,\frac{2}{3} = \frac{5}{9}$$

Einfacher: Wir wissen die Stützpunkt und Gewichte sind symmetrisch

$\Rightarrow$ aus $w_0 = \frac{5}{9}$ ergibt sich schon $w_2 = \frac{5}{9}$ und so weiter

NUM Ü7

28) (i) $n=1 \quad a=0 \quad b=1 \quad w(x)=\sqrt{x}$

$$w_0=\int_a^b L_0 w\,dx=\int_0^1 \prod_{\substack{k=0\\k\neq 0}}^1 \frac{x-x_k}{x_0-x_k}\sqrt{x}\,dx=\int_0^1 \frac{x-x_1}{x_0-x_1}\sqrt{x}\,dx=\frac{1}{x_0-x_1}\left(\int_0^1 x\sqrt{x}\,dx-x_1\int_0^1\sqrt{x}\,dx\right)$$

$$=\frac{1}{x_0-x_1}\left(\frac{2}{5}x^{\frac{5}{2}}-x_1\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_0^1=\frac{1}{x_0-x_1}\left(\frac{2}{5}-\frac{2}{3}x_1\right)$$

$$w_1=\int_a^b L_1 w\,dx=\int_0^1 \prod_{\substack{k=0\\k\neq 1}}^1 \frac{x-x_k}{x_1-x_k}\sqrt{x}\,dx=\int_0^1 \frac{x-x_0}{x_1-x_0}\sqrt{x}\,dx=\frac{1}{x_1-x_0}\left(\int_0^1 x\sqrt{x}\,dx-x_0\int_0^1\sqrt{x}\,dx\right)$$

$$=\frac{1}{x_1-x_0}\left(\frac{2}{5}x^{\frac{5}{2}}-x_0\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_0^1=\frac{1}{x_1-x_0}\left(\frac{2}{5}-\frac{2}{3}x_0\right)$$

$$p_0(x)=1 \quad \beta_0=\frac{\langle xp_0,p_0\rangle}{\|p_0\|^2}=\frac{\int_0^1 x\sqrt{x}\,dx}{\int_0^1\sqrt{x}\,dx}=\frac{\frac{2}{5}}{\frac{2}{3}}=\frac{3}{5} \quad p_1(x)=x-\frac{3}{5}$$

$$\beta_1=\frac{\langle x^2-\frac{3}{5}x,x-\frac{3}{5}\rangle}{\|x-\frac{3}{5}\|^2}=\frac{\int_0^1(x^2-\frac{3}{5}x)(x-\frac{3}{5})\sqrt{x}\,dx}{\int_0^1(x-\frac{3}{5})^2\sqrt{x}\,dx}=\frac{\frac{184}{7875}}{\frac{8}{175}}=\frac{23}{45} \quad \gamma_1=\frac{\|p_1\|}{\|p_0\|}=\frac{\sqrt{\frac{8}{175}}}{\sqrt{\frac{2}{3}}}=\frac{2\sqrt{21}}{35}$$

$p_2(x)=(x-\frac{23}{45})(x-\frac{3}{5})-\frac{12}{175}$ hat Nullstellen $x_0=\frac{5}{9}-\frac{2\sqrt{\frac{10}{7}}}{9}$ und $x_1=\frac{5}{9}+\frac{2\sqrt{\frac{10}{7}}}{9}$

$$\Rightarrow w_0=-\frac{63}{4\sqrt{70}}\left(\frac{4}{945}(7-5\sqrt{70})\right)=\frac{712+175\sqrt{70}}{30}\approx 72{,}54 \qquad w_1=-\frac{712+175\sqrt{70}}{30}\approx -72{,}54$$

(ii) $n=0 \quad a=0 \quad b=1 \quad w(x)=2x^2+1$

$$w_0=\int_a^b L_0 w\,dx=\int_0^1 \prod_{\substack{k=0\\k\neq 0}}^0 \frac{x-x_k}{x_0-x_k}\,2x^2+1\,dx=\int_0^1 2x^2+1\,dx=\left(\frac{2}{3}x^3+x\right)\Big|_0^1=\frac{2}{3}+1=\frac{5}{3}$$

$$p_0(x)=1 \quad \beta_0=\frac{\langle xp_0,p_0\rangle}{\|p_0\|^2}=\frac{\int_0^1 x(2x^2+1)\,dx}{\int_0^1 2x^2+1\,dx}=\frac{1}{\frac{5}{3}}=\frac{3}{5} \quad p_1(x)=x-\frac{3}{5}$$

hat Nullstellen $x_0=\frac{3}{5}$.

(iii) $n=1 \quad a=-1 \quad b=1 \quad w(x)=\begin{cases}2 & \text{für } x\in(0,1]\\ 1 & \text{für } x\in[-1,0]\end{cases}$

$$w_0=\int_{-1}^1 \prod_{\substack{k=0\\k\neq 0}}^1 \frac{x-x_k}{x_0-x_k}w(x)\,dx=\frac{1}{x_0-x_1}\left(\int_{-1}^0 x-x_1\,dx+2\int_0^1 x-x_1\,dx\right)=\frac{1}{x_0-x_1}\left(\left(\frac{x^2}{2}-x_1x\right)\Big|_{-1}^0+2\left(\frac{x^2}{2}-x_1x\right)\Big|_0^1\right)$$

$$=\frac{1}{x_1-x_0}\left(-\frac{1}{2}-x_1+1-2x_1\right)=\frac{1}{x_1-x_0}\left(\frac{1}{2}-3x_1\right)$$

$$w_1=\int_{-1}^1 \prod_{\substack{k=0\\k\neq 1}}^1 \frac{x-x_k}{x_1-x_k}w(x)\,dx=\frac{1}{x_1-x_0}\left(\int_{-1}^0 x-x_0\,dx+2\int_0^1 x-x_0\,dx\right)=\frac{1}{x_1-x_0}\left(\left(\frac{x^2}{2}-x_0x\right)\Big|_{-1}^0+2\left(\frac{x^2}{2}-x_0x\right)\Big|_0^1\right)$$

$$=\frac{1}{x_1-x_0}\left(-\frac{1}{2}-x_0+1-2x_0\right)=\frac{1}{x_1-x_0}\left(\frac{1}{2}-3x_0\right)$$

$$p_0(x)=1 \quad \beta_0=\frac{\langle xp_0,p_0\rangle}{\|p_0\|^2}=\frac{\int_{-1}^1 x\,w(x)\,dx}{\int_{-1}^1 w(x)\,dx}=\frac{\int_{-1}^0 x\,dx+2\int_0^1 x\,dx}{\int_{-1}^0 1\,dx+2\int_0^1 1\,dx}=\frac{\left(\frac{x^2}{2}\right)\Big|_{-1}^0+2\left(\frac{x^2}{2}\right)\Big|_0^1}{x\big|_{-1}^0+2\,x\big|_0^1}$$

$$=\frac{-\frac{1}{2}+2\frac{1}{2}}{1+2}=\frac{\frac{1}{2}}{3}=\frac{1}{6} \qquad p_1(x)=x-\frac{1}{6} \qquad \gamma_1=\frac{\|p_1\|}{\|p_0\|}=\frac{\sqrt{\frac{33}{36}}}{\sqrt{3}}=\frac{\sqrt{11}}{6}$$

$$\beta_1=\frac{\langle xp_1,p_1\rangle}{\|p_1\|^2}=\frac{\int_{-1}^1(x^2-\frac{1}{6}x)(x-\frac{1}{6})w(x)\,dx}{\int_{-1}^1(x-\frac{1}{6})^2w(x)\,dx}=\frac{\int_{-1}^0 x^3-\frac{1}{6}x^2-\frac{1}{6}x^2+\frac{1}{36}\,dx+2\int_0^1 x^3-\frac{1}{3}x^2+\frac{1}{36}\,dx}{\int_{-1}^0 x^2-\frac{1}{3}x+\frac{1}{36}\,dx+2\int_0^1 x^2-\frac{1}{3}x+\frac{1}{36}\,dx}$$

$$=\frac{-\frac{3}{8}+2\frac{11}{72}}{\frac{19}{36}+2\frac{7}{36}}=\frac{-\frac{5}{72}}{\frac{11}{12}}=-\frac{5}{66} \qquad p_2(x)=(x+\frac{5}{66})(x-\frac{1}{6})-\frac{11}{36}$$ hat Nullstellen

$$x_0=\frac{1}{22}-\frac{\sqrt{155}}{22} \qquad x_1=\frac{1}{22}+\frac{\sqrt{155}}{22}$$

$$\Rightarrow w_0=\frac{11}{\sqrt{155}}\left(\frac{1}{2}-\frac{3}{22}-\frac{3-\sqrt{155}}{22}\right)=-\frac{3}{2}+\frac{4}{\sqrt{155}}\approx -1{,}179 \qquad w_1=\frac{11}{\sqrt{155}}\left(\frac{1}{2}-\frac{3}{22}+\frac{3-\sqrt{155}}{22}\right)=\frac{3}{2}+\frac{4}{\sqrt{155}}\approx 1{,}82$$