NUM Ü1

4.) a) $p \in \mathbb{R}_{\geq 1}$ $\quad F(p,x) := x^2 - 2px + 1 = 0$

$\Phi : \mathbb{R}_{\geq 1} \to \mathbb{R}^2 \quad \Phi(p) := (x_+, x_-)$ wobei $x_\pm$ die Lösungen von $F(p,x)$ sind

ges: $K_{rel}(p) = \frac{\|\Phi'(p)\| \cdot |p|}{\|\Phi(p)\|}$

$$\Phi(p) = \left(p + \sqrt{p^2-1},\ p - \sqrt{p^2-1}\right)$$

für $p > 1$: $\Phi'(p) = \left(1 + \frac{p}{\sqrt{p^2-1}},\ 1 - \frac{p}{\sqrt{p^2-1}}\right)$

$$\|\Phi(p)\| = \sqrt{(p+\sqrt{p^2-1})^2 + (p-\sqrt{p^2-1})^2} =$$

$$= \sqrt{p^2 + \cancel{2p\sqrt{p^2-1}} + p^2 - 1 + p^2 - \cancel{2p\sqrt{p^2-1}} + p^2 - 1} = \sqrt{2(2p^2-1)}$$

$$\|\Phi'(p)\| = \sqrt{\left(1+\frac{p}{\sqrt{p^2-1}}\right)^2 + \left(1 - \frac{p}{\sqrt{p^2-1}}\right)^2} =$$

$$= \sqrt{1 + \cancel{2\frac{p}{\sqrt{p^2-1}}} + \frac{p^2}{p^2-1} + 1 - \cancel{2\frac{p}{\sqrt{p^2-1}}} + \frac{p^2}{p^2-1}} = \sqrt{2\left(1 + \frac{p^2}{p^2-1}\right)}$$

$$K_{rel}(p) = \frac{\|\Phi'(p)\| \cdot |p|}{\|\Phi(p)\|} = \frac{\cancel{\sqrt{2}} \cdot \sqrt{1 + \frac{p^2}{p^2-1}}\ \sqrt{p^2}}{\cancel{\sqrt{2}} \cdot \sqrt{2p^2-1}} = \frac{\sqrt{\frac{p^2-1+p^2}{p^2-1} \cdot p^2}}{\sqrt{2p^2-1}}$$

$$= \sqrt{\frac{\frac{\cancel{(2p^2-1)} \cdot p^2}{p^2-1}}{\cancel{2p^2-1}}} = \sqrt{\frac{p^2}{p^2-1}} = p \cdot \sqrt{\frac{1}{p^2-1}}$$

$$\lim_{p \to 1+} K_{rel}(p) = \lim_{p\to 1+} \sqrt{\frac{p^2}{p^2-1}} = \sqrt{\lim_{p \to 1+} \frac{p^2}{p^2-1}} = \sqrt{\lim_{p\to 1+} p^2 \cdot \lim_{p \to 1+} \frac{1}{p^2-1}}$$

$$= \sqrt{1 \cdot \lim_{p \to 1+} \frac{1}{p^2-1}} = \infty$$

$\Rightarrow$ für p nahe an 1 ist das Problem schlecht konditioniert

b) $t \in \mathbb{R}_{\geq 1}$ $\quad G(t,x) := x^2 - \frac{1+t^2}{t} x + 1 = 0$

$$F(p,x) = G(p + \sqrt{p^2-1}, x)$$

$x^2 - 2px + 1 = 0$

$$x^2 - \frac{1 + (p+\sqrt{p^2-1})^2}{p+\sqrt{p^2-1}} x + 1 = 0$$

$$x^2 - \frac{\cancel{1} + p^2 + 2p\sqrt{p^2-1} + p^2 \cancel{-1}}{p+\sqrt{p^2-1}} x + 1 = 0$$

$$x^2 - 2p \cdot \frac{p+\sqrt{p^2-1}}{p+\sqrt{p^2-1}} x + 1 = 0$$

$$x^2 - 2px + 1 = 0 \Rightarrow \text{Umformung} \ \ldots$$

NUM Ü1

4) b) ... $\Psi : \mathbb{R}_{\geq 1} \to \mathbb{R}^2$ ... Lösungen von $G(t,x)$

$\Psi(t) = \left(t, \frac{1}{t}\right)$, da $\left(x^2 - \frac{1+t^2}{t} x + 1\right)(t) =$

$t^2 - \frac{1+t^2}{t} t + 1 = \frac{t^3}{t} - \frac{t+t^3}{t} + 1 = \frac{\cancel{t^3} - t - \cancel{t^3}}{t} + 1 = -1 + 1 = 0$ und

$\left(x^2 - \frac{1+t^2}{t} x + 1\right)\left(\frac{1}{t}\right) = \frac{1}{t^2} - \frac{1+t^2}{t} \cdot \frac{1}{t} + 1 = \frac{\cancel{1} - \cancel{1} - t^2}{t^2} + 1 = -1 + 1 = 0$

$\Psi'(t) = \left(1, -\frac{1}{t^2}\right)$ $\qquad \|\Psi'(t)\| = \sqrt{1 + \frac{1}{t^4}}$

$\|\Psi(t)\| = \sqrt{t^2 + \frac{1}{t^2}}$

$K_{rel}(t) = \frac{\|\Psi'(t)\| \cdot |t|}{\|\Psi(t)\|} = \frac{\sqrt{1 + \frac{1}{t^4}} \cdot \sqrt{t^2}}{\sqrt{t^2 + \frac{1}{t^2}}} = \sqrt{\frac{t^2 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}} = 1$

$\Rightarrow$ für alle $t (\geq 1)$ gilt, dass das Problem gut konditioniert ist