NUM Ü6

21) $n \in \mathbb{N}_0$ $\quad b_{k,n}(x) := \binom{n}{k} x^k (1-x)^{n-k}$ $\quad \forall k=0,\dots,n$ und $x \in [0,1]$

(i) zz: $\sum_{k=0}^{n} b_{k,n}(x) = 1$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\sum_{k=0}^{n} b_{k,n}(x) = \sum_{k=0}^{n} \binom{n}{k} (1-x)^{n-k} x^k \overset{\vdots}{=} (1-x+x)^n = 1^n = 1$$

(ii) zz: $\forall n \geq 1: \sum_{k=0}^{n} \frac{k}{n} b_{k,n}(x) = x$

Sei $n \geq 1$ bel.

$$\sum_{k=0}^{n} \frac{k}{n} b_{k,n}(x) = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=1}^{n} \frac{k}{n} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x \cdot x^{k-1} (1-x)^{n-k} = x \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-(k+1))!} x^k (1-x)^{n-(k+1)}$$

$$= x \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} x^k (1-x)^{n-1-k} = x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k}$$

$$= x \sum_{k=0}^{n-1} b_{k,n-1}(x) = x \cdot 1 = x$$

(iii) zz: $\forall n \geq 2: \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 b_{k,n}(x) = \frac{x(1-x)}{n}$

Sei $n \geq 2$ bel.

$$\sum_{k=0}^{n} \left(x-\frac{k}{n}\right)^2 b_{k,n}(x) = \sum_{k=0}^{n} \left(x^2 - 2x\frac{k}{n} + \frac{k^2}{n^2}\right) b_{k,n}(x) = x^2 \sum_{k=0}^{n} b_{k,n}(x) - 2x \sum_{k=0}^{n} \frac{k}{n} b_{k,n}(x) + \sum_{k=0}^{n} \frac{k^2}{n^2} b_{k,n}(x)$$

$$= x^2 - 2x^2 + \frac{x((n-1)x+1)}{n} = \frac{-nx^2 + nx^2 - x^2 + x}{n} = \frac{x(1-x)}{n}$$

$$\sum_{k=0}^{n} \frac{k^2}{n^2} b_{k,n}(x) \overset{l=n-2}{\overset{\vdots}{=}} \sum_{k=0}^{l+2} \frac{k^2}{(l+2)^2} \binom{l+2}{k} x^k (1-x)^{l+2-k} = \sum_{k=0}^{l+2} \frac{k^2}{(l+2)^2} \frac{(l+2)!}{k!(l+2-k)!} x^k (1-x)^{l+2-k}$$

$$= \sum_{k=1}^{l+2} \frac{k(l+1)!}{(l+2)(k-1)!(l+2-k)!} x^k (1-x)^{l+2-k} = \frac{1}{l+2} \sum_{k=0}^{l+1} \frac{(k+1)(l+1)!}{k!(l+1-k)!} x^k (1-x)^{l+1-k} x$$

$$= \frac{x}{l+2} \sum_{k=0}^{l+1} (k+1) b_{k,l+1}(x) = \frac{x}{l+2} \left((l+1) \sum_{k=0}^{l+1} \frac{k}{l+1} b_{k,l+1}(x) + \sum_{k=0}^{l+1} b_{k,l+1}(x)\right)$$

$$= \frac{x}{l+2}((l+1)x+1) \overset{n=l+2}{\overset{\vdots}{=}} \frac{x((n-1)x+1)}{n}$$

$\square$

NUM Ü6

24) $T(f) := \frac{b-a}{2}(f(a)+f(b))$ $\quad E(f) = T(f) - \int_a^b f(x)\,dx$ $\quad h := b-a$

$$E(f) = \frac{h}{2}(f(a)+f(a+h)) - \int_a^{a+h} f(x)\,dx$$

$$E'(f) = \frac{1}{2}f(a) - \frac{1}{2}f(a+h) + \frac{1}{2}h f'(a+h)$$

$$E''(f) = \frac{1}{2}h f''(a+h) \quad , \text{da } |f''(x)| \leq f''(\xi) \text{ für ein } \xi \in [a,b]$$

$$\Rightarrow -\frac{f''(\xi)h}{2} \leq \frac{1}{2}h f''(a+h) = E''(f) \leq \frac{f''(\xi)h}{2}$$

$$\int_0^h E''(x)\,dx = E'(h) - E'(0) = E'(h)$$

$$\int_0^h E'(x)\,dx = E(h) - E(0) = E(h)$$

$$\Rightarrow -\frac{h^2}{4}f''(\xi) \leq E'(h) \leq \frac{h^2}{4}f''(\xi) \quad \Rightarrow -\frac{h^3}{12}f''(\xi) \leq E(h) \leq \frac{h^3}{12}f''(\xi)$$

$$\Rightarrow |E| \leq \frac{(b-a)^3}{12}\|f''\|_\infty$$

$$\Rightarrow \int_a^b f(x)\,dx = T(f) + E(f) \approx \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^3}{12}\|f''\|_\infty$$

$$= T(f) + \frac{(b-a)^2}{12}(b-a)\max_{a\leq x\leq b}|f''(x)|$$

$$\leq T(f) + \frac{(b-a)^2}{12}\int_a^b f''(x)\,dx = T(f) + \frac{(b-a)^2}{12}(f'(b)-f'(a))$$

zz: $|Qf - Q_1^H f| \leq C\,(b-a)^5 \|f^{(4)}\|_\infty$

Sei $p(x)$ das eindeutige kubische, hermitsche Interpolationspolynom.

Dann gilt $|p(x) - f(x)| \leq C\frac{\|f^{(4)}\|_\infty}{4!}|x-a|^2|x-b|^2$

$$|Qf - Q_1^H f| = |Qf - Qp| = \left|\int_a^b f(x)\,dx - \int_a^b p(x)\,dx\right| = \left|\int_a^b f(x)-p(x)\,dx\right|$$

$$\leq \int_a^b |f(x)-p(x)|\,dx \leq \int_a^b C\frac{\|f^{(4)}\|_\infty}{4!}|x-a|^2|x-b|^2\,dx$$

$$= C\frac{\|f^{(4)}\|_\infty}{4!}\int_a^b (x-a)^2(x-b)^2\,dx = -C\frac{\|f^{(4)}\|_\infty}{4!}\frac{1}{30}(a-b)^5 = \underbrace{\frac{C}{4!\cdot 30}\|f^{(4)}\|_\infty (b-a)^5}_{F(f)}$$

$$\int_a^b p(x)\,dx = \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{12}(f'(a)-f'(b)) + F(p)$$

Da $F(p)$ von $\|p^{(4)}\|_\infty$ abhängt und $p^{(4)}(x) = 0$ werden diese Polynome exakt integriert.

$\square$

NUM Ü6

22) a) zz: $\forall \delta>0, x\in[0,1], n\geq 2: \sum\limits_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|\geq\delta}} b_{k,n}(x) \leq \frac{x(1-x)}{\delta^2 n}$

Offensichtlich gilt $b_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \geq 0 \;\forall x\in[0,1]$.

Weiters gilt $\frac{(x-\frac{k}{n})^2}{\delta^2} \geq 1$ für alle Summanden für die $|x-\frac{k}{n}|\geq\delta$ gilt.

$$\Rightarrow \sum_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|\geq\delta}} b_{k,n}(x) \leq \sum_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|\geq\delta}} \frac{(x-\frac{k}{n})^2}{\delta^2} b_{k,n}(x) \leq \frac{1}{\delta^2}\sum_{k=0}^{n}(x-\tfrac{k}{n})^2 b_{k,n}(x) = \frac{x(1-x)}{\delta^2 n}$$

b) $f\in C[0,1]$ $\quad \forall n\in\mathbb{N}: f_n := B_n(f) = \sum\limits_{k=0}^{n} f(\frac{k}{n})\, b_{k,n}$

zz: $\lim\limits_{n\to\infty} \|f-f_n\|_{L^\infty(0,1)} = 0$

$$f(x)-f_n(x) = f(x)\underbrace{\sum_{k=0}^{n} b_{k,n}(x)}_{=1} - \sum_{k=0}^{n} f(\tfrac{k}{n}) b_{k,n}(x) = \sum_{k=0}^{n}(f(x)-f(\tfrac{k}{n}))b_{k,n}(x)$$

Da f auf einem Kompaktum ($[0,1]$) stetig ist gilt f ist gleichmäßig stetig, also

$\forall \varepsilon>0\ \exists\delta>0\ \forall x,y\in[0,1]: |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$$\left|\sum_{k=0}^{n}(f(x)-f(\tfrac{k}{n}))b_{k,n}(x)\right| = \left|\sum_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|<\delta}}(f(x)-f(\tfrac{k}{n}))b_{k,n}(x) + \sum_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|\geq\delta}}(f(x)-f(\tfrac{k}{n}))b_{k,n}(x)\right|$$

$$\leq \sum_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|<\delta}} |f(x)-f(\tfrac{k}{n})|\, b_{k,n}(x) + \sum_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|\geq\delta}} |f(x)-f(\tfrac{k}{n})|\, b_{k,n}(x)$$

$$\leq \sum_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|<\delta}} \varepsilon\, b_{k,n}(x) + \sum_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|\geq\delta}} 2\|f\|_\infty\, b_{k,n}(x) \leq \varepsilon\sum_{k=0}^{n} b_{k,n}(x) + 2\|f\|_\infty \sum_{\substack{k\in\{0,\dots,n\}\\ |x-\frac{k}{n}|\geq\delta}} b_{k,n}(x)$$

$$\leq \varepsilon + 2\|f\|_\infty \frac{x(1-x)}{\delta^2 n} = \varepsilon + 2\|f\|_\infty\frac{x-x^2}{\delta^2 n} \leq \varepsilon + 2\frac{\|f\|_\infty}{\delta^2 n}$$

$\Rightarrow \forall n\in\mathbb{N}: |f(x)-f_n(x)| \leq \varepsilon + 2\frac{\|f\|_\infty}{\delta^2 n}$

$\exists N\in\mathbb{N}: N > \frac{\|f\|_\infty}{\delta^2\varepsilon} \quad \Rightarrow \forall n\geq N: \varepsilon + 2\frac{\|f\|_\infty}{\delta^2 n} \leq \varepsilon + 2\frac{\|f\|_\infty}{\delta^2 N}$

$$= \varepsilon + 2\frac{\|f\|_\infty}{\delta^2 \frac{\|f\|_\infty}{\delta^2\varepsilon}} = \varepsilon + 2\varepsilon = 3\varepsilon$$

$\Rightarrow \forall\varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\geq N: \|f-f_n\|_\infty < \varepsilon$ $\square$