IST Ü3

4) random points on a unit circle

X, Y ... coordinates of a point chosen uniformly on a unit circle

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi}, & \text{if } x^2+y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$R = \sqrt{X^2+Y^2}$ $\quad\quad$ $\Theta = \tan^{-1}\left(\frac{Y}{X}\right)$

$$f_{R,\Theta}\begin{pmatrix} r \\ \varphi \end{pmatrix} = f_{X,Y}\left(h\begin{pmatrix} r \\ \varphi \end{pmatrix}\right) \left| \det\left(\nabla h\begin{pmatrix} r \\ \varphi \end{pmatrix}\right)\right|$$

$g: \mathbb{R}^2 \to \mathbb{R}^+ \times [0, 2\pi)$ $\quad\quad$ $h = g^{-1}: \mathbb{R}^+ \times [0, 2\pi) \to \mathbb{R}^2$ $\quad\quad$ for $(x,y) \neq (0,0)$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{x^2+y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix}$ $\quad\quad$ $\begin{pmatrix} r \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} r \cdot \cos(\varphi) \\ r \cdot \sin(\varphi) \end{pmatrix}$

$\frac{\partial}{\partial r} h(r, \varphi) = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$ $\quad\quad$ $\frac{\partial}{\partial \varphi} h(r,\varphi) = \begin{pmatrix} -r\sin(\varphi) \\ r\cos(\varphi) \end{pmatrix}$

$$\left|\det\left(\nabla h\begin{pmatrix} r \\ \varphi \end{pmatrix}\right)\right| = \left|\det\begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix}\right| = |r\cos^2(\varphi) + r\sin^2(\varphi)| = r\,|\sin^2(\varphi) + \cos^2(\varphi)| = r$$

$$f_{R,\Theta}\begin{pmatrix} r \\ \varphi \end{pmatrix} = f_{X,Y}(r \cdot \sin(\varphi), r \cdot \cos(\varphi))\, r = \begin{cases} \frac{r}{\pi}, & \text{if } r \leq 1 \\ 0, & \text{else} \end{cases}$$

$r^2\sin^2(\varphi) + r^2\cos^2(\varphi) = r^2 \leq 1 \iff |r| \leq 1$
$\iff r \leq 1$ since $r \in \mathbb{R}^+$

$$\int_0^1 \int_0^{2\pi} \frac{r}{\pi}\, d\varphi\, dr = \int_0^1 \frac{r}{\pi} 2\pi\, dr = \int_0^1 2r\, dr = 1^2 = 1$$ "checksum"