ISt Ü8

3) $X_1, \dots, X_n$ ... random sample from population with $\mathcal{N}(\mu,\mu)$ distribution $\mu > 0$ .. unknown

a) show $\sum X_i^2$ ... minimal sufficient in $\mathcal{N}(\mu,\mu)$ family

$$f_\mu(x) = \frac{1}{\sqrt{2\pi\mu}} e^{-\frac{(x-\mu)^2}{2\mu}} \quad \text{for every } x_i \qquad \text{therefore}$$

$$f_\mu(x) = \prod_{i=1}^{n} f_\mu(x) = \frac{1}{(\sqrt{2\pi\mu})^n} e^{-\frac{1}{2\mu}(\sum (x_i-\mu)^2)}$$

Theorem: Suppose there exists a function $T(X)$ such, that for every sample points x and y the ratio $\frac{f_\theta(x)}{f_\theta(y)}$ is constant as a function of θ iff $T(x) = T(y)$ then $T(X)$ is a minimal sufficient statistic for θ.

$$\frac{f_\mu(x)}{f_\mu(y)} = \frac{\frac{1}{(\sqrt{2\pi\mu})^n} e^{-\frac{1}{2\mu}(\sum(x_i-\mu)^2)}}{\frac{1}{(\sqrt{2\pi\mu})^n} e^{-\frac{1}{2\mu}(\sum(y_i-\mu)^2)}} = e^{\frac{1}{2\mu}(-\sum(x_i-\mu)^2 + \sum(y_i-\mu)^2)}$$

$$= e^{\frac{1}{2\mu}(\sum(y_i-\mu)^2 - (x_i-\mu)^2)} = e^{\frac{1}{2\mu}(\sum y_i^2 - \sum x_i^2 - 2\mu(\sum x_i - y_i)}$$

is constant as a function of μ iff $\sum x_i^2 = \sum y_i^2$

(in this case we get $e^{-\frac{1}{2\mu}2\mu(\sum x_i - y_i)} = e^{\sum y_i - x_i}$)

Therefore $T(x) = \sum x_i^2$ is a minimal sufficient statistic for μ.

b) show $(\sum X_i, \sum X_i^2)$ is sufficient but not minimal sufficient in the $\mathcal{N}(\mu,\mu)$ family

not minimal as there exists no function r such that

$(\sum X_i, \sum X_i^2) = r(\sum X_i^2)$ $\qquad$ ($\sum X_i^2$ is a sufficient statistic)

$(\sum X_i, \sum X_i^2)$ is a sufficient statistic as it contains $\sum X_i^2$ which we have shown to be a sufficient statistic.