Problem_2

Ida Hönigmann

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Real roots

Let A, B and C be independent random variables, uniformly distributed on (0, 1).

Calculating the probability

What is the probability that the quadratic equation $Ax^2 + Bx + C = 0$ has real roots?

Using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we get that a, b, c must satisfy

$$b^2 - 4ac > 0 \text{ or equivalently } b^2 > 4ac.$$

If we have that $a < \frac{b^2}{4}$ it follows that for all $c \in (0,1)$ it holds that $ac < \frac{b^2}{4} \iff b^2 > 4ac$.

If $a \geq \frac{b^2}{4}$ we have to have $c < \frac{b^2}{4a}$ in order to satisfy $b^2 > 4ac$.

Using this we can use these limits of integration to get

$$\begin{aligned}
\mathbb{P}(B^2 > 4AC) &= \int_0^1 (\mathbb{P}(b^2 > 4AC))db = \int_0^1 (\int_0^{\frac{b^2}{4}} (\mathbb{P}(b^2 > 4aC))da + \int_{\frac{b^2}{4}}^1 (\mathbb{P}(b^2 > 4aC))da)db \\
&= \int_0^1 (\int_0^{\frac{b^2}{4}} (1)da + \int_{\frac{b^2}{4}}^1 (\int_0^{\frac{b^2}{4a}} (\mathbb{P}(b^2 > 4ac))dc)da)db = \int_0^1 (\frac{b^2}{4} + \int_{\frac{b^2}{4}}^1 (\int_0^{\frac{b^2}{4a}} (1)dc)da)db \\
&= \int_0^1 (\frac{b^2}{4} + \int_{\frac{b^2}{4}}^1 (\frac{b^2}{4a})da)db = \int_0^1 (\frac{b^2}{4} + \frac{b^2}{4}\log(\frac{4}{b^2}))db = \frac{1}{36}(3\log(4) + 5) \approx 0.2544134
\end{aligned}$$

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1/36 * (3*log(4) + 5)
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## [1] 0.2544134
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Estimating the probability

We generate 10000 instances of this problem and check what percentage satisfies the inequality. This gives us an estimate of the probabilty that $ax^2 + bc + c = 0$ has real roots.

```
n=10000
a=runif(n)
b=runif(n)
c=runif(n)
sum(b^2>4*a*c)/n
```

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## [1] 0.2537
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