1st Ü7

2) $X_1, \dots, X_n$ ... random sample with pdf $f_\theta(x) = \theta\, x^{\theta-1}$ $\quad 0 < x < 1$, $\theta > 0$

Is there a function $g(\theta)$ for which there exists an unbiased estimator whose variance attains the Cramér-Rao lower bound?

We search for a function $g(\theta)$ and unbiased estimator $\widehat{g(\theta)}$ with

$$\mathbb{V}ar_{g(\theta)}(\widehat{g(\theta)}_n) = \frac{1}{n I_1(g(\theta))} = I_n(g(\theta))^{-1}$$

$$I_1(\theta) = \mathbb{V}ar(z(X,\theta)) = \mathbb{V}ar\left(\frac{\partial}{\partial\theta} \log f_\theta(X)\right) = -\mathbb{E}(z'(X,\theta)) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2} \log f_\theta(X)\right)$$

$$\log f_\theta(x) = \log(\theta\, x^{\theta-1}) = \log(\theta) + (\theta-1)\log(x)$$

$$\frac{\partial}{\partial\theta} \log f_\theta(x) = \frac{\partial}{\partial\theta} \log(\theta) + (\theta-1)\log(x) = \frac{1}{\theta} + \log(x)$$

$$\frac{\partial^2}{\partial\theta^2} \log f_\theta(x) = \frac{\partial}{\partial\theta}\, \frac{1}{\theta} + \log(x) = \frac{-1}{\theta^2}$$

$$-\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2} \log f_\theta(x)\right) = -\mathbb{E}\left(-\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}$$

$$\Rightarrow \mathbb{V}ar(\widehat{g(\theta)}_n) = \frac{1}{n}\, g(\theta)^2$$

Mean of $\widehat{g(\theta)}_n = \mathbb{E}(g(\theta))$ and variance of $\widehat{g(\theta)}_n = \frac{1}{n} g(\theta)^2$

$n=1$ $\quad g(\theta) = \frac{1}{\theta} \qquad T(x) = -\ln(x)$ ... estimator of $g(\theta)$

$$\mathbb{E}(T(x)) = \int_0^1 -\ln(x)\,\theta\, x^{\theta-1}\, dx = \frac{1}{\theta}$$

$$\mathbb{E}(T(x)^2) = \int_0^1 \ln(x)^2\,\theta\, x^{\theta-1}\, dx = \frac{2}{\theta^2}$$

$$\mathbb{V}ar(T(x)) = \mathbb{E}(T(x)^2) - \mathbb{E}(T(x))^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2} = g(\theta)^2$$

$n>1$ $\quad g(\theta) = \frac{1}{\theta} \qquad T(x) = -\frac{\sum_{i=1}^n \ln(x_i)}{\sqrt{n}}$

$$\Rightarrow \mathbb{V}ar(T(x)) = \mathbb{E}(T(x)^2) - \mathbb{E}(T(x))^2 = \frac{1}{n}\,\frac{2}{\theta^2} - \frac{1}{n}\,\frac{1}{\theta^2} = \frac{1}{n}\,\frac{1}{\theta^2} = \frac{1}{n}\, g(\theta)^2$$

ISt Ü7

3) $W_1, \dots, W_k$ ... unbiased estimators of θ with $\mathbb{V}ar = \sigma_i^2$ and $\mathbb{C}or(W_i, W_j) = 0$ if $i \neq j$

show a_i ... constant and $\mathbb{E}_\theta(\sum a_i W_i) = \theta$

of all estimators $\sum a_i W_i$ $\quad W^* = \dfrac{\sum W_i / \sigma_i^2}{\sum (1-\sigma_i^2)}$ has minimum variance

$$\theta = \mathbb{E}(\sum a_i W_i) = \sum a_i \mathbb{E}(W_i) = \sum a_i \theta = \theta(\sum a_i) \quad \Rightarrow \sum a_i = 1$$

$$\mathbb{V}ar(\sum a_i W_i) = \sum a_i^2 \mathbb{V}ar(W_i) = \sum a_i^2 \sigma_i^2$$

our goal is to minimize $\sum a_i^2 \sigma_i^2$ with the condition $\sum a_i - 1 = 0$

$$L(a, \lambda) = \sum a_i^2 \sigma_i^2 - \lambda(\sum a_i - 1)$$

$$\frac{\partial L}{\partial a_i} = 2 a_i \sigma_i^2 - \lambda = 0 \qquad \frac{\partial L}{\partial \lambda} = -(\sum a_i - 1) = 0$$

$$\Rightarrow a_i = \frac{\lambda}{2\sigma_i^2} \qquad \Rightarrow \sum a_i = \sum \frac{\lambda}{2\sigma_i^2} = \frac{\lambda}{2} \sum \sigma_i^{-2}$$

$$\Rightarrow \lambda = 2(\sum \sigma_i^{-2})^{-1} \qquad \Rightarrow a_i = \frac{\frac{1}{\sigma_i^2}}{\sum_j \sigma_j^{-2}}$$

$$\Rightarrow \sum a_i W_i = \frac{\sum \frac{W_i}{\sigma_i^2}}{\sum \frac{1}{\sigma_i^2}} = W^*$$

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show $\mathbb{V}ar\, W^* = \dfrac{1}{\sum \frac{1}{\sigma_i^2}}$

$$\mathbb{V}ar(W^*) = \sum a_i^2 \sigma_i^2 = \left(\sum_i \left(\frac{\frac{1}{\sigma_i^2}}{\sum_j \sigma_j^{-2}}\right)\right)^2 \sigma_i^2 = \sum_i \frac{\frac{1}{\sigma_i^2}}{\sum_j \sigma_j^{-2}} = \frac{1}{(\sum_j \sigma_j^{-2})^2} \sum_i \frac{1}{\sigma_i^2}$$

$$= \frac{1}{\sum \sigma_j^{-2}}$$

ISt Ü7

5) show $Poi(\lambda)$ with unknown $\lambda > 0$ belongs to the exponential family

$$f_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

in order to belong to the exponential family we need the following form

$$f_\lambda(k) = h(k)\, c(\lambda)\, e^{w(\lambda) t(k)} \qquad h(k), c(\lambda) \geq 0$$

The following transformations accieve this form

$$f_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda} = \frac{1}{k!} e^{\ln(\lambda^k)} e^{-\lambda} = \frac{1}{k!} e^{-\lambda} e^{k \ln(\lambda)}$$

$$\Rightarrow h(k) = \frac{1}{k!} \geq 0; \quad c(\lambda) = e^{-\lambda} \geq 0; \quad w(\lambda) = \ln(\lambda); \quad t(k) = k$$

$\Rightarrow Poi(\lambda)$ belongs to the exponential family