Ist Ü3

(1) F... continous cdf

a) $U \sim U(0,1)$ $\quad Y = F^{-1}(U)$ $\quad$ cdf of Y?

$$f_U : \mathbb{R} \to \mathbb{R}^+ \qquad \text{...pdf of } U \qquad \text{sample space } \mathbb{R}$$
$$x \mapsto \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{else} \end{cases}$$

F^{-1}... invertible, $(F^{-1})^{-1} = F$...differentiable with $F' = f_F$...pdf

$\Rightarrow f_Y(y) = f_U(F(y))\,|f_F(y)|$ ...pdf of Y

$$\Rightarrow \text{cdf of } Y \text{ is } \int_{-\infty}^{y} f_U(\underbrace{F(z)}_{\in(0,1)})\,|f_F(z)|\,dz$$

$$= \int_{-\infty}^{y} \underbrace{|f_F(z)|}_{\geq 0}\,dz = \int_{-\infty}^{y} f_F(z)\,dz = F(y)$$

b) X... random variable $\quad F$... cdf of X $\quad Z = F(X)$ $\quad$ cdf of Z?

$\Rightarrow F'$... pdf of X

$\Rightarrow$ pdf $f_Z(z) = f_X(F^{-1}(z))\,|(F^{-1})'(z)|$

$$\text{cdf} = \int_{-\infty}^{z} f_X(F^{-1}(s))\,|(F^{-1})'(s)|\,ds \qquad (g(h(x)))' = g'(h(x)) \cdot h'(x)$$

$$= F(F^{-1}(z)) = z \quad \text{for } 0 \leq z \leq 1 \quad (F^{-1} : [0,1] \to \mathbb{R})$$

since a cdf is monotonely increasing the cdf of values $z < 0$ is 0 and of values $z > 1$ is 1.

$$\Rightarrow \text{cdf } F_Z(z) = \begin{cases} 0, & z < 0 \\ z, & 0 \leq z \leq 1 \\ 1, & z > 1 \end{cases} \qquad \text{which is the uniform distribution}$$