ISt Ü6

3) $X_1, \dots, X_n$ ... iid uniform $(0,\theta)$ $\theta > 0$

a) MME of $\theta$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \mu(\hat{\theta}) = \int_{\mathbb{R}} x\, f_{\hat{\theta}}(x)\,dx = \int_0^{\hat{\theta}} x \frac{1}{\hat{\theta}}\,dx = \frac{1}{\hat{\theta}}\left(\frac{x^2}{2}\right)\Big|_0^{\hat{\theta}} = \frac{1}{\hat{\theta}}\frac{\hat{\theta}^2}{2} = \frac{\hat{\theta}}{2}$$

$\Rightarrow \hat{\theta} = 2\bar{X}$

MLE of $\theta$

$$L(\hat{\theta}) = \begin{cases} 0 & \text{if } \exists i \in \{1,\dots,n\}: X_i > \hat{\theta} \\ \mathbb{P}(X_1,\dots,X_n \mid \hat{\theta}) = \left(\frac{1}{\hat{\theta}}\right)^n & \text{otherwise} \end{cases}$$

$\frac{1}{\hat{\theta}^n}$ gets bigger if $\hat{\theta}$ is smaller $\Rightarrow$ we want the smallest $\hat{\theta}$ with

$\hat{\theta} \leq X_i \ \forall i \quad \Rightarrow \hat{\theta} = \max\{X_i : i \in \{1,\dots,n\}\}$

b) MSE of both estimators

$MSE = \mathbb{V} + b^2$

MME: $b(\hat{\theta}) = \mathbb{E}(2\bar{X}) - \theta = 2\,\mathbb{E}(\bar{X}) - \theta = \theta - \theta = 0$

$\mathbb{V}(\hat{\theta}) = \mathbb{V}(2\bar{X}) = 4\,\mathbb{V}(\bar{X}) = 4\frac{\theta^2}{12n} = \frac{\theta^2}{3n}$

$\Rightarrow MSE = \frac{\theta^2}{3n} + 0 = \frac{\theta^2}{3n}$

MLE: $b(\hat{\theta}) = \mathbb{E}(\max_{1\leq i\leq n} X_i) - \theta = \frac{n}{n+1}\theta - \theta = -\theta\frac{1}{n+1}$

$\mathbb{V}(\hat{\theta}) = \mathbb{V}(\max_{1\leq i\leq n} X_i) = \frac{n\theta^2}{(n+1)^2(n+2)}$

$\Rightarrow MSE = \frac{n\theta^2}{(n+1)^2(n+2)} + \frac{\theta^2}{(n+1)^2} = \frac{n\theta^2 + \theta^2(n+2)}{(n+1)^2} = \frac{2n\theta^2 + 2\theta^2}{(n+1)^2} = 2\theta^2\frac{n+1}{(n+1)^2}$

$= 2\theta^2\frac{1}{n+1}$

$\theta^2\frac{1}{3n} = \theta^2\frac{2}{n+1} \iff n+1 = 6n \iff n = \frac{1}{5} \quad \Rightarrow \forall n > 1: \frac{2\theta^2}{n+1} > \frac{\theta^2}{3n}$

therefore MME has a better MSE than MLE.