IST Ü5

(1) $f, g \ldots$ probability density functions $X \ldots$ random variable with pdf f

prove $\mathbb{E}\left(\log \frac{f(X)}{g(X)}\right) \geq 0$!

We know that $\log(x)$ is a concave function, therefore $-\log(x)$ is convex.

$\mathbb{E}\left(\log \frac{f(X)}{g(X)}\right) = \mathbb{E}\left(-\log \frac{g(X)}{f(X)}\right) \geq -\log\left(\mathbb{E}\left(\frac{g(X)}{f(X)}\right)\right)$ according to Jensen's inequality.

$\mathbb{E}\left(\frac{g(X)}{f(X)}\right) = \int_{-\infty}^{+\infty} \frac{g(x)}{f(x)} f(x)\, dx = \int_{-\infty}^{+\infty} g(x)\, dx = 1$

$\Rightarrow -\log\left(\mathbb{E}\left(\frac{g(X)}{f(X)}\right)\right) = -\log(1) = 0$ which shows that $\mathbb{E}\left(\log\left(\frac{f(X)}{g(X)}\right)\right) \geq 0$

$\mathbb{E}\left(\log\left(\frac{f(X)}{g(X)}\right)\right) = \mathbb{E}\left(\log(f(X)) - \log(g(X))\right) = \mathbb{E}(\log(f(X))) - \mathbb{E}(\log(g(X)))$

$\Rightarrow \mathbb{E}(\log(f(X))) - \mathbb{E}(\log(g(X))) \geq 0$

$\Rightarrow \mathbb{E}(\log(f(X))) \geq \mathbb{E}(\log(g(X)))$ and obviously for $f = g$ there holds equality.

Therefore $\mathbb{E}(\log(g(X)))$ is maximized when $g = f$.

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(2) $X_1, X_2, \ldots$ i.i.d with distribution $U(0,1)$ $\quad X_{(n)} = \max_{1 \le i \le n} X_i$

$Y_n = n(1 - X_{(n)})$, $n \in \mathbb{N}$ $\quad$ show $Y_n \xrightarrow{n\to\infty} \exp(1)$

F_n... cdf of Y_n is calculated by

$$F_n(x) = \mathbb{P}(Y_n \le x) = \mathbb{P}(n(1-X_{(n)}) \le x) = \mathbb{P}\left(1 - X_{(n)} \le \frac{x}{n}\right) =$$

$$= \mathbb{P}\left(-X_{(n)} \le \frac{x}{n} - 1\right) = \mathbb{P}\left(X_{(n)} \ge 1 - \frac{x}{n}\right) = 1 - \mathbb{P}\left(X_{(n)} \le 1 - \frac{x}{n}\right)$$

We note that $\mathbb{P}(X_{(n)} \le x)$ iff $\mathbb{P}(X_1 \le x \wedge X_2 \le x \wedge \ldots \wedge X_n \le x)$ is given.

$$\mathbb{P}(X_1 \le x \wedge \ldots \wedge X_n \le x) = \mathbb{P}(X_1 \le x) \cdot \ldots \cdot \mathbb{P}(X_n \le x) = x \cdot \ldots \cdot x = x^n$$

Using this we have

$$F_n(x) = 1 - \mathbb{P}\left(X_{(n)} \le 1 - \frac{x}{n}\right) = 1 - \left(1 - \frac{x}{n}\right)^n \xrightarrow{n\to\infty} 1 - e^{-x}$$

which is the cdf of $\exp(1)$. Therefore $Y_n \xrightarrow{n\to\infty} \exp(1)$

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(3) $n=600$ $p=\frac{1}{4}$ 600 coin flips, probability of tails is $\frac{1}{4}$

(a) Binomial distribution to calculate X... number of heads $P(|X-450| \leq 10)$

$P(|X-450| \leq 10) = P(440 \leq X \leq 460) = P(X \leq 460) - P(X < 440)$

$= \sum_{k=0}^{460} \binom{600}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{600-k} - \sum_{k=0}^{439} \binom{600}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{600-k} = \sum_{k=440}^{460} \binom{600}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{600-k}$

$= \text{pbinom}(460, 600, \frac{3}{4}) - \text{pbinom}(439, 600, \frac{3}{4})$

$\approx 0{,}6778428$

(b) Use Normal approximation to calculate the probability

$Bin(n,p) \approx N(np, np(1-p))$ in our case we have $Y \sim N(600\frac{3}{4}, 600\frac{3}{4}\frac{1}{4})$

$P(X \leq 460) - P(X \leq 439) = \text{pnorm}(460, 600\frac{3}{4}, \sqrt{600\frac{3}{4}\frac{1}{4}}) - \text{pnorm}(439, 600\frac{3}{4}, \sqrt{600\frac{3}{4}\frac{1}{4}})$

$\approx 0{,}6772637$ without continuity correction

$P(X \leq 460{,}5) - P(X - 439{,}5) \approx 0{,}6778012$ with continuity correction

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(5) a) $X_1, \dots, X_n$ ... i.i.d. Normal with unknown μ and known σ^2

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ find $\lim_{n\to\infty} \sqrt{n}(\bar{X}^3 - c)$ for an appropriate constant c

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow \sqrt{n}(\bar{X}-\mu) \sim N(\mu-\mu, n\frac{\sigma^2}{n}) = N(0,\sigma^2)$

we know this from exercise 4

$g(x) = x^3$ is differentiable with $g'(x) = 3x^2$

Using the delta method we calculate $\left(n^\alpha(X_n - \Theta) \to Y \Rightarrow n^\alpha(g(X_n) - g(\Theta)) \to g'(\Theta)Y\right)$

$\sqrt{n}(\bar{X}^3 - \mu^3) \to 3\mu^2 Y$ with $Y \sim N(0,\sigma^2)$

with results in $N(0, 9\mu^4\sigma^2)$

b) $Y \sim \text{binom}(n,p)$ $\text{logit}(y) = \ln\left(\frac{y}{1-y}\right)$ $0<y<1$

determine the approximate distribution of $\text{logit}\left(\frac{Y}{x}\right)$

$Y = n\bar{X}_n \approx N(np, np(1-p)) \Rightarrow \frac{Y}{n} \approx N(p, \frac{1}{n}p(1-p))$

$\Rightarrow \frac{Y}{n} - p \approx N(0, \frac{1}{n}p(1-p)) \Rightarrow \sqrt{n}(\frac{Y}{n} - p) \approx N(0, p(1-p))$

$g(y) = \text{logit}(y)$ if $g'(y)$ existist for $y=p$ then

$\sqrt{n}(\bar{X}_n - p) \to N(0, p(1-p)) \Rightarrow \sqrt{n}(\text{logit}(\bar{X}_n) - \text{logit}(p)) \to \text{logit}'(p)N(0,p(1-p))$

$\text{logit}'(y) = \left(\ln\left(\frac{y}{1-y}\right)\right)' = (\log(y) - \log(1-y))' = \frac{1}{y} + \frac{1}{1-y} = \frac{1}{y(1-y)}$

$\Rightarrow \sqrt{n}\left(\text{logit}(\bar{X}_n) - \ln\left(\frac{p}{1-p}\right)\right) \to \frac{1}{p(1-p)} N(0, p(1-p)) = N\left(0, \frac{1}{p(1-p)}\right)$

$\Rightarrow \text{logit}(\bar{X}_n) - \ln\left(\frac{p}{1-p}\right) \to \frac{1}{\sqrt{n}} N\left(0, \frac{1}{p(1-p)}\right) = N\left(0, \frac{1}{np(1-p)}\right)$

$\text{logit}(\bar{X}_n) \to N\left(0, \frac{1}{np(1-p)}\right) + \ln\left(\frac{p}{1-p}\right) = N\left(\ln\left(\frac{p}{1-p}\right), \frac{1}{np(1-p)}\right)$

$\text{logit}\left(\frac{Y}{n}\right) \approx N\left(\ln\left(\frac{p}{1-p}\right), \frac{1}{np(1-p)}\right)$