373) ges: Grad von $\mathbb{Q}(\sqrt{6}, \sqrt{10}, \sqrt{15})$ über $\mathbb{Q}$

Da $\dfrac{\sqrt{6}\,\sqrt{10}}{2} = \dfrac{\sqrt{2}\sqrt{3}\,\sqrt{2}\sqrt{5}}{2} = \sqrt{3}\cdot\sqrt{5} = \sqrt{15}$ gilt

$\mathbb{Q}(\sqrt{6}, \sqrt{10}, \sqrt{15}) = \left\{ \dfrac{a + b\sqrt{6} + c\sqrt{10} + d\sqrt{15} + e\sqrt{6}\sqrt{10} + f\sqrt{6}\sqrt{15} + g\sqrt{10}\sqrt{15} + h\sqrt{6}\sqrt{10}\sqrt{15}}{a' + b'\sqrt{6} + c'\sqrt{10} + d'\sqrt{15} + e'\sqrt{6}\sqrt{10} + f'\sqrt{6}\sqrt{15} + g'\sqrt{10}\sqrt{15} + h'\sqrt{6}\sqrt{10}\sqrt{15}} \right.$,

$a, a', b, b', c, c', d, d', e, e', f, f', g, g', h, h' \in \mathbb{Q}$,

$\left. (a', b', c', d', e', f', g', h') \neq (0,0,0,0,0,0,0) \right\}$

$= \left\{ \dfrac{a + b\sqrt{6} + c\sqrt{10} + e\sqrt{6}\sqrt{10}}{a' + b'\sqrt{6} + c'\sqrt{10} + e'\sqrt{6}\sqrt{10}} \,,\; a, a', b, b', c, c', e, e' \in \mathbb{Q}, (a', b', c', e') \neq (0,0,0,0) \right\}$

$= \mathbb{Q}(\sqrt{6}, \sqrt{10})$ ⌐ Da $\mathbb{Q} \leq \mathbb{Q}(\sqrt{6})$, $\sqrt{6} \in \mathbb{Q}(\sqrt{6})$ ... algebraisch über $\mathbb{Q}$

(nämlich $x^2 - 6$) $\xrightarrow{\text{Satz } 6.1.3.4.\,(2)}$ $\mathbb{Q}(\sqrt{6}) = \mathbb{Q}[\sqrt{6}]$

$B_1 := \{1, \sqrt{6}\}$ ist l.u. da $\sqrt{6} \notin \mathbb{Q}$ und erzeugt $\mathbb{Q}[\sqrt{6}] = \mathbb{Q}(\sqrt{6})$.

$\Rightarrow B_1$ ist Basis von $\mathbb{Q}(\sqrt{6})$ über $\mathbb{Q}$

$\mathbb{Q}(\sqrt{6}) \leq \mathbb{Q}(\sqrt{6}, \sqrt{10}) = \left\{ \dfrac{a + b\sqrt{6} + c\sqrt{10} + d\sqrt{60}}{a' + b'\sqrt{6} + c'\sqrt{10} + d'\sqrt{60}} \;\middle|\; \begin{array}{l} a, a', b, b', c, c', d, d' \in \mathbb{Q}, \\ (a', b', c', d') \neq (0,0,0,0) \end{array} \right\}$

$= \left\{ \dfrac{x + y\sqrt{10}}{x' + y'\sqrt{10}} \;\middle|\; x, y, x', y' \in \mathbb{Q}(\sqrt{6}), (x', y') \neq (0,0) \right\} = (\mathbb{Q}(\sqrt{6}))(\sqrt{10})$

$\sqrt{10} \in \mathbb{Q}(\sqrt{6})(\sqrt{10})$ ... algebraisch über $\mathbb{Q}(\sqrt{6})$ (nämlich $x^2 - 10$)

$\xrightarrow{6.1.3.4.\,(2)} \mathbb{Q}(\sqrt{6})(\sqrt{10}) = \mathbb{Q}(\sqrt{6})[\sqrt{10}]$

$B_2 := \{1, \sqrt{10}\}$ erzeugt $\mathbb{Q}(\sqrt{6})[\sqrt{10}] = \mathbb{Q}(\sqrt{6}, \sqrt{10})$

Sei $x = a + b\sqrt{6} \in \mathbb{Q}[\sqrt{6}] = \mathbb{Q}(\sqrt{6})$ mit $1 \cdot x = \sqrt{10}$ bel.

$\Rightarrow \sqrt{2}\sqrt{5} = \sqrt{10} = a + b\sqrt{6} = a + b\sqrt{2}\sqrt{3}$

$\Leftrightarrow \sqrt{5} = \dfrac{1}{\sqrt{2}}a + b\sqrt{3} = \dfrac{a}{2}\dfrac{2}{\sqrt{2}} + b\sqrt{3} = \dfrac{a}{2}\sqrt{\dfrac{4}{2}} + b\sqrt{3} = \dfrac{a}{2}\sqrt{2} + b\sqrt{3}$ ↯ , da

$\sqrt{5} \notin \sqrt{2}\,\mathbb{Q} + \sqrt{3}\,\mathbb{Q}$ $\Rightarrow B_2$ ... l.u. $\rightarrow B_2$ ... Basis von $\mathbb{Q}(\sqrt{6}, \sqrt{10})$ über $\mathbb{Q}(\sqrt{6})$

$\Rightarrow [\mathbb{Q}(\sqrt{6}, \sqrt{10}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{6}, \sqrt{10}) : \mathbb{Q}(\sqrt{6})] \cdot [\mathbb{Q}(\sqrt{6}) : \mathbb{Q}] = 2 \cdot 2 = 4$

‖

$[\mathbb{Q}(\sqrt{6}, \sqrt{10}, \sqrt{15}) : \mathbb{Q}]$ nach Gradsatz und

$B := B_1 \cdot B_2 = \{1, \sqrt{6}, \sqrt{10}, \sqrt{60}\}$ ist Basis von $\mathbb{Q}(\sqrt{6}, \sqrt{10}, \sqrt{15})$ über $\mathbb{Q}$

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$(\sqrt{2}\sqrt{2}\cdot\sqrt{3}\sqrt{5} = 2\sqrt{15})$