1st Ü2

(1) $\mathbb{P}(X=x) = a(x+1)(6-x) \qquad x \in \{0, 1, \dots, 5\}$

$$1 = \sum_{x=0}^{5} \mathbb{P}(X=x) = a(6+10+12+12+10+6) = 56a \quad \Rightarrow a = \frac{1}{56}$$

a) pmf?

$f(0) = \frac{6}{56} \quad f(1) = \frac{10}{56} \quad f(2) = \frac{12}{56} \quad f(3) = \frac{12}{56} \quad f(4) = \frac{10}{56} \quad f(5) = \frac{6}{56}$

b) $\mathbb{P}(X \geq 4)$?

$$\mathbb{P}(X \geq 4) = \mathbb{P}(X=4) + \mathbb{P}(X=5) = \frac{10}{56} + \frac{6}{56} = \frac{16}{56} = 0{,}2857$$

c) expectation $\mathbb{E}(X)$, standard deviation $\sqrt{\mathbb{V}(x)}$?

$$\mathbb{E}(X) = \sum_{x=0}^{5} \mathbb{P}(X=x) \cdot x = \frac{6}{56} \cdot 0 + \frac{10}{56} \cdot 1 + \frac{12}{56} \cdot 2 + \frac{12}{56} \cdot 3 + \frac{10}{56} \cdot 4 + \frac{6}{56} \cdot 5$$

$$= \frac{1}{56}(10+24+36+40+30) = \frac{140}{56} = 2{,}5$$

$$\sqrt{\mathbb{V}(X)} = \sqrt{\mathbb{E}((X-\mathbb{E}(X))^2)} = \sqrt{\mathbb{E}(X^2) - (\mathbb{E}(X))^2} = \sqrt{\mathbb{E}(X^2) - 6{,}25}$$

$$\mathbb{E}(X^2) = \sum_{x=0}^{5} \mathbb{P}(X=x) \cdot x^2 = \frac{6}{56} \cdot 0 + \frac{10}{56} 1 + \frac{12}{56} \cdot 4 + \frac{12}{56} 9 + \frac{10}{56} 16 + \frac{6}{56} 25$$

$$= \frac{1}{56}(10+48+108+160+150) = \frac{476}{56} = 8{,}5$$

$$\sqrt{\mathbb{V}(X)} = \sqrt{8{,}5 - 6{,}25} = \sqrt{2{,}25} = 1{,}5$$

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2) Basketball 10 throws each $p_T = 0{,}8$ $p_J = 0{,}85$ ... probability of success for Tom / John

a) $\mathbb{P}(X_T = 7) = \binom{10}{7} p_T^{\,7} (1-p_T)^3 =$

$= 120 \cdot 0{,}8^7 \cdot 0{,}2^3 = 0{,}2013$

b) $\mathbb{P}(X_J \geq 8) = \sum_{k=8}^{10} \binom{10}{k} p_J^{\,k} (1-p_J)^{10-k} =$

$= \binom{10}{8} \cdot 0{,}85^8 \cdot 0{,}15^2 + \binom{10}{9} \cdot 0{,}85^9 \cdot 0{,}15^1 + \binom{10}{10} \cdot 0{,}85^{10} \cdot 0{,}15^0$

$= 0{,}275897 + 0{,}347425 + 0{,}196874$

$= 0{,}8201960$

c) $\mathbb{P}(X_T > X_J) = \sum_{k=1}^{10} \mathbb{P}(X_T = k) \cdot \mathbb{P}(X_J < k-1)$

$= \sum_{k=1}^{10} \binom{10}{k} p_T^{\,k} (1-p_T)^{10-k} \left(1 - \sum_{l=k}^{10} \binom{10}{l} p_J^{\,l} (1-p_J)^{10-l}\right)$

$= 0{,}2738$ calculated with WolframAlpha

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3) $f(x) = \begin{cases} a x^2 e^{-bx^2} & , x \geq 0 \\ 0 & , x < 0 \end{cases} \qquad b = \frac{m}{2kT}$

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} a x^2 e^{-bx^2}\,dx = a \int_0^{\infty} \underbrace{x}_{u} \cdot \underbrace{(x e^{-bx^2})}_{v'}\,dx \qquad \Rightarrow v = -\frac{e^{-bx^2}}{2b}$$

$$= a\left(x \cdot \left(-\frac{e^{-bx^2}}{2b}\right)\Big|_0^{\infty} - \int_0^{\infty} 1 \cdot \left(-\frac{e^{-bx^2}}{2b}\right) dx\right)$$

$$= a\left(\left(-\frac{1}{2b} x e^{-bx^2}\right)\Big|_0^{\infty} + \frac{1}{2b}\int_0^{\infty} e^{-bx^2}\,dx\right)$$

$$= \frac{a}{2b}\left(\left(\lim_{x\to+\infty}(-x e^{-bx^2})\right) + \int_0^{\infty} \frac{1}{\sqrt{b}} e^{-y^2}\,dy\right) \qquad \left[\begin{array}{l} y = \sqrt{b}\,x \\ dx = \frac{1}{\sqrt{b}}\,dy \end{array}\right.$$

$$= \frac{a}{2b}\left(0 + \frac{1}{\sqrt{b}} \frac{\sqrt{\pi}}{2}\right)$$

$$= \frac{a\sqrt{\pi}}{4b\sqrt{b}}$$

$$\Rightarrow 4b\sqrt{b} = a\sqrt{\pi} \iff a = \frac{4b\sqrt{b}}{\sqrt{\pi}} = \frac{4b\sqrt{b\pi}}{\pi}$$

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4) a) $Y \sim \exp(\lambda)$ $\qquad f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & , y \geq 0 \\ 0 & , \text{else} \end{cases}$

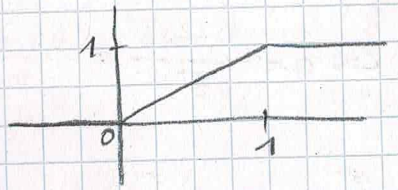
$$P(Y > y) = 1 - P(Y < y) = 1 - \int_{-\infty}^{y} f_Y(x)\,dx = 1 - \int_0^y \lambda e^{-\lambda x}\,dx$$

$$= 1 - \lambda\left(-\frac{1}{\lambda} e^{-\lambda x}\Big|_0^y\right) = 1 - \lambda\left(-\frac{1}{\lambda} e^{-\lambda y} + \frac{1}{\lambda} e^{-\lambda \cdot 0}\right)$$

$$= 1 + e^{-\lambda y} - 1 = e^{-\lambda y}$$

b) $X \sim$ uniform $(0,1)$

cummulative distribution function of X?



$$cdf_X(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \leq x \leq 1 \\ 1 & , x > 1 \end{cases}$$

distribution $Z = -\ln(X)$?

i) $P(Z \leq z) = P(-\ln(X) \leq z) = P(\ln(X) \geq -z) = P(X \geq \exp(-z)) =$

$$= 1 - P(X < \exp(-z)) = \begin{cases} 1 - e^{-z} & , \text{if } 0 \leq e^{-z} \leq 1 \iff z \geq 0 \\ 1 - 1 = 0 & , \text{if } e^{-z} > 1 \iff z < 0 \end{cases}$$

ii) $$f_Z(z) = f_X(h(z))\,|h'(z)| = f_X(e^{-z})\,e^{-z} = \begin{cases} e^{-z} & , \text{if } 0 \leq e^{-z} \leq 1 \iff z \geq 0 \\ 0 & , \text{else} \end{cases}$$

iii) $$\int_{-\infty}^{z} f_Z(x)\,dx = \int_0^z e^{-x}\,dx = -e^{-x}\Big|_0^z = -e^{-z} + e^0 = 1 - e^{-z} \quad \text{for } z \geq 0$$

$$\int_{-\infty}^{z} f_Z(x)\,dx = 0 \quad \text{for } z < 0$$

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5) X ... annual rainfall in Cleveland $\quad X \sim N(40.2, 8.4)$

a) $\mathbb{P}(X > 44)$?

$\mathbb{P}(X > 44) = 1 - \mathbb{P}(X \leq 44)$

transform X into standard $N(0,1)$

$$Y = \frac{X - \mu}{\sigma} = \frac{X - 40{,}2}{8{,}4}$$

$\mathbb{P}(X \leq 44) = \mathbb{P}(Y \leq 0{,}45) \doteq 0{,}6736 \quad \Rightarrow \mathbb{P}(X > 44) = 0{,}3264$

b) rainfalls exceeds 44 inches exactly 3 out of 7 years?

$$\binom{7}{3} (\mathbb{P}(X > 44))^3 (\mathbb{P}(X \leq 44))^4 = 35 \cdot 0{,}3264^3 \cdot 0{,}6736^4$$

$= 0{,}25067$