ISt Ü4

1) $P_t \sim \mathcal{P}(\lambda t)$ ... passengers having arrived until time t

$X \sim U(0,T)$ ... time at which first train arrives

What are the expectation and variance of number of passengers who enter the first train?

$$\mathbb{E}(P_X \mid X=t) = \mathbb{E}(P_t) = \lambda t \qquad \Rightarrow \mathbb{E}(P_X \mid X) = \lambda X$$

We know that

$$\mathbb{E}(P_X) = \mathbb{E}(\mathbb{E}(P_X \mid X)) = \mathbb{E}(\lambda X) = \lambda\, \mathbb{E}(X) = \lambda \frac{T-0}{2} = \frac{\lambda}{2} T$$

$$\mathbb{V}(P_X) = \mathbb{E}(\mathbb{V}(P_X \mid X)) + \mathbb{V}(\mathbb{E}(P_X \mid X))$$

$$\mathbb{V}(P_X \mid X=t) = \mathbb{V}(P_t) = \lambda t \qquad \Rightarrow \mathbb{V}(P_X \mid X) = \lambda X$$

$$\mathbb{E}(\mathbb{V}(P_X \mid X)) = \mathbb{E}(\lambda X) = \frac{\lambda}{2} T$$

$$\mathbb{V}(\mathbb{E}(P_X \mid X)) = \mathbb{V}(\lambda X) = \lambda^2\, \mathbb{V}(X) = \lambda^2 \frac{(T-0)^2}{12} = \frac{\lambda^2}{12} T^2$$

$$\mathbb{V}(P_X) = \mathbb{E}(\mathbb{V}(P_X \mid X)) + \mathbb{V}(\mathbb{E}(P_X \mid X))$$

$$= \frac{\lambda}{2} T + \frac{\lambda^2}{12} T^2$$

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2) Real roots

$A, B, C \sim U(0,1)$ ... independent

a) probability that $Ax^2 + Bx + C = 0$ has real roots?

the solution of the equation satisfies $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

to get real roots we must have $b^2 - 4ac > 0$

or $b^2 > 4ac$

$$\mathbb{P}(B^2 > 4AC) = \int_0^1 \mathbb{P}(b^2 > 4AC)\, db$$

$b^2 > 4AC \Leftrightarrow AC < \frac{b^2}{4}$

$\Rightarrow$ if $A < \frac{b^2}{4}$ $\mathbb{P}(b^2 > 4AC) = 1$

else $C < \frac{b^2}{4A}$ has to be satisfied.

$$= \int_0^1 \Big(\int_0^{\frac{b^2}{4}} \mathbb{P}(b^2 > 4aC)\, da + \int_{\frac{b^2}{4}}^1 \int_0^{\frac{b^2}{4a}} \mathbb{P}(b^2 > 4ac)\, dc\, da \Big)\, db$$

$$= \int_0^1 \Big(\int_0^{\frac{b^2}{4}} 1\, da + \int_{\frac{b^2}{4}}^1 \int_0^{\frac{b^2}{4a}} 1\, dc\, da \Big)\, db$$

$$= \int_0^1 \Big(\frac{b^2}{4} + \int_{\frac{b^2}{4}}^1 \frac{b^2}{4a}\, da \Big)\, db = \int_0^1 \frac{b^2}{4} + \frac{b^2}{4} \log\left(\frac{4}{b^2}\right) db =$$

$$= \frac{1}{36} b^3 \left(3 \log\left(\frac{4}{b^2}\right) + 5\right) \Big|_0^1 = \frac{1}{36} (3 \log(4) + 5) \approx 0{,}25441$$

b)

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n=10000
a=runif(n)
b=runif(n)
c=runif(n)
sum(b^2 > 4·a·c)/n
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generates 10 000 instances of this problem and checks what percentage satisfies $b^2 > 4ac$ or $ax^2 + bx + c = 0$ has real roots.

experimental "calculation" of the true percentage calculated in a).

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3) $X \sim N(\mu, \sigma^2)$ show that $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

We know that $M_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$

$$\int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(tx - \frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$tx - \frac{(x-\mu)^2}{2\sigma^2} \overset{?}{=} \mu t + \frac{\sigma^2 t^2}{2} - \frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}$$

$$\frac{2\sigma^2 tx - x^2 + 2\mu x - \mu^2}{2\sigma^2} \overset{?}{=} \frac{2\sigma^2 \mu t + \sigma^4 t^2 - x^2 + 2(\mu + \sigma^2 t)x - (\mu + \sigma^2 t)^2}{2\sigma^2}$$

$$2\sigma^2 tx + 2\mu x - \mu^2 \overset{?}{=} \cancel{2\sigma^2 \mu t} + \cancel{\sigma^4 t^2} + 2\mu x + 2\sigma^2 tx - \mu^2 - \cancel{2\sigma^2 \mu t} - \cancel{\sigma^4 t^2}$$

$$2\sigma^2 tx + 2\mu x - \mu^2 = 2\sigma^2 tx + 2\mu x - \mu^2$$

$$\Rightarrow M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\mu t + \frac{\sigma^2 t^2}{2} - \frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right) dx$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}} dx$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

because $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}} dx = 1$ as it is the pdf of $N(\mu+\sigma^2 t, \sigma^2)$

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4) $X_1, X_2 \ldots$ independent random variables $\mathbb{E}(X_1) = \mathbb{E}(X_2) = \mu$

$\mathbb{V}ar(X_1) = \sigma_1^2$ $\mathbb{V}ar(X_2) = \sigma_2^2$ μ...unknown

estimate μ by $\lambda X_1 + (1-\lambda) X_2$ for some appropriate value of λ

Which value of λ yields the estimate having the lowest possible variance?

$$\mathbb{V}ar(\lambda X_1 + (1-\lambda) X_2)$$
$$= \lambda^2 \mathbb{V}ar(X_1) + (1-\lambda)^2 \mathbb{V}ar(X_2) \quad \text{(as } X_1 \text{ and } X_2 \text{ are independent)}$$
$$= \lambda^2 \sigma_1^2 + (1-\lambda)^2 \sigma_2^2$$

We want to minimize $\lambda^2 \sigma_1^2 + (1-\lambda)^2 \sigma_2^2 =: f(\lambda)$

$$f'(\lambda) = 2\lambda \sigma_1^2 + 2(1-\lambda)(-1)\sigma_2^2 = 2\lambda(\sigma_1^2 + \sigma_2^2) - 2\sigma_2^2 = 0$$
$$\Longleftrightarrow \lambda(\sigma_1^2 + \sigma_2^2) = \sigma_2^2 \quad \Longleftrightarrow \lambda = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

where $\sigma_1^2 + \sigma_2^2$ can only be 0 if $\mathbb{V}ar(X_1) = \mathbb{V}ar(X_2) = 0$, which is a trivial case.

To ensure $\lambda = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$ is a minimum we calculate

$$f''(\lambda) = 2(\sigma_1^2 + \sigma_2^2) > 0$$

which gives us that

$$\lambda X_1 + (1-\lambda) X_2$$
$$= \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} X_1 + \left(1 - \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right) X_2$$
$$= \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} X_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} X_2$$

is the estimate of μ with the lowest possible variance.