IStÜ5

(2) $X_1, X_2, \dots$ i.i.d with distribution $U(0,1)$ $\quad X_{(n)} = \max_{1 \le i \le n} X_i$

$Y_n = n(1 - X_{(n)})$, $n \in \mathbb{N}$ $\quad$ show $Y_n \xrightarrow{n\to\infty} \exp(1)$

F_n... cdf of Y_n is calculated by

$$F_n(x) = \mathbb{P}(Y_n \le x) = \mathbb{P}(n(1-X_{(n)}) \le x) = \mathbb{P}\left(1 - X_{(n)} \le \frac{x}{n}\right) =$$

$$= \mathbb{P}\left(-X_{(n)} \le \frac{x}{n} - 1\right) = \mathbb{P}\left(X_{(n)} \ge 1 - \frac{x}{n}\right) = 1 - \mathbb{P}\left(X_{(n)} \le 1 - \frac{x}{n}\right)$$

We note that $\mathbb{P}(X_{(n)} \le x)$ iff $\mathbb{P}(X_1 \le x \wedge X_2 \le x \wedge \dots \wedge X_n \le x)$ is given.

$$\mathbb{P}(X_1 \le x \wedge \dots \wedge X_n \le x) = \mathbb{P}(X_1 \le x) \cdot \ldots \cdot \mathbb{P}(X_n \le x) = x \cdot \ldots \cdot x = x^n$$

Using this we have

$$F_n(x) = 1 - \mathbb{P}\left(X_{(n)} \le 1 - \frac{x}{n}\right) = 1 - \left(1 - \frac{x}{n}\right)^n \xrightarrow{n\to\infty} 1 - e^{-x}$$

which is the cdf of $\exp(1)$. Therefore $Y_n \xrightarrow{n\to\infty} \exp(1)$