IST Ü5

(5) a) $X_1, \dots, X_n$ ... i.i.d. Normal with unknown μ and known σ^2

$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ find $\lim_{n\to\infty} \sqrt{n}(\bar{X}^3 - c)$ for an appropriate constant c

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow \sqrt{n}(\bar{X} - \mu) \sim N(\mu - \mu, n\frac{\sigma^2}{n}) = N(0, \sigma^2)$

we know this from exercise 4

$g(x) = x^3$ is differentiable with $g'(x) = 3x^2$

Using the delta method we calculate $(n^\alpha(X_n - \Theta) \to Y \Rightarrow n^\alpha(g(X_n) - g(\Theta)) \to g'(\Theta)Y)$

$\sqrt{n}(\bar{X}^3 - \mu^3) \to 3\mu^2 Y$ with $Y \sim N(0, \sigma^2)$

with results in $N(0, 9\mu^4\sigma^2)$

b) $Y \sim \text{binom}(n, p)$ $\quad \text{logit}(y) = \ln(\frac{y}{1-y})$ $\quad 0 < y < 1$

determine the approximate distribution of $\text{logit}(\frac{Y}{x})$

$Y = n\bar{X}_n \approx N(np, np(1-p)) \Rightarrow \frac{Y}{n} \approx N(p, \frac{1}{n}p(1-p))$

$\Rightarrow \frac{Y}{n} - p \approx N(0, \frac{1}{n}p(1-p)) \Rightarrow \sqrt{n}(\frac{Y}{n} - p) \approx N(0, p(1-p))$

$g(y) = \text{logit}(y)$ if $g'(y)$ existist for $y = p$ then

$\sqrt{n}(\bar{X}_n - p) \to N(0, p(1-p)) \Rightarrow \sqrt{n}(\text{logit}(\bar{X}_n) - \text{logit}(p)) \to \text{logit}'(p) N(0, p(1-p))$

$\text{logit}'(y) = (\ln(\frac{y}{1-y}))' = (\log(y) - \log(1-y))' = \frac{1}{y} + \frac{1}{1-y} = \frac{1}{y(1-y)}$

$\Rightarrow \sqrt{n}(\text{logit}(\bar{X}_n) - \ln(\frac{p}{1-p})) \to \frac{1}{p(1-p)} N(0, p(1-p)) = N(0, \frac{1}{p(1-p)})$

$\Rightarrow \text{logit}(\bar{X}_n) - \ln(\frac{p}{1-p}) \to \frac{1}{\sqrt{n}} N(0, \frac{1}{p(1-p)}) = N(0, \frac{1}{np(1-p)})$

$\text{logit}(\bar{X}_n) \to N(0, \frac{1}{np(1-p)}) + \ln(\frac{p}{1-p}) = N(\ln(\frac{p}{1-p}), \frac{1}{np(1-p)})$

$\text{logit}(\frac{Y}{n}) \approx N(\ln(\frac{p}{1-p}), \frac{1}{np(1-p)})$