IST Ü8

2) $X_1, \dots, X_n$ ... random sample from population with pdf

$$f_\theta(x) = \begin{cases} \frac{\theta}{x^2}, & \theta \le x \\ 0, & \text{otherwise} \end{cases} \qquad \theta > 0 \text{ ... unknown}$$

Factorization Theorem $\quad T(X)$ ... sufficient statistic for θ iff

$$\exists g(t,\theta)\ \exists h(x): f_\theta(x) = g(T(x),\theta) \cdot h(x)$$

pdf of $X = (X_1, \dots, X_n)$ is $\prod_{i=1}^{n} f_\theta(x_i)$ as they are iid

$$f_\theta(x) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} \frac{\theta}{x_i^2} 1\!\!1_{[\theta,\infty)}(x_i) = 1\!\!1_{[\theta,\infty)}(\min_i x_i)\, \theta^n \prod_{i=1}^{n} \frac{1}{x_i{}^2}$$

$f_\theta(x) = g(T(x),\theta) \cdot h(x)$ for $\quad h(x) = \prod_{i=1}^{n} \frac{1}{x_i{}^2}$

$T(x) = \min_i x_i$

$g(T(x),\theta) = \theta^n 1\!\!1_{[\theta,\infty)}(\min_i x_i)$

$\Rightarrow T(x)$ is a sufficient statistic for θ

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3) $X_1, \dots, X_n$ ... random sample from population with $\mathcal{N}(\mu,\mu)$ distribution $\mu > 0$ .. unknown

a) show $\sum X_i^2$ ... minimal sufficient in $\mathcal{N}(\mu,\mu)$ family

$f_\mu(x) = \frac{1}{\sqrt{2\pi\mu}} e^{-\frac{(x-\mu)^2}{2\mu}}$ for every X_i therefore

$\mathbf{f}_\mu(x) = \prod_{i=1}^n f_\mu(x) = \frac{1}{(\sqrt{2\pi\mu})^n} e^{-\frac{1}{2\mu}(\sum(x_i-\mu)^2)}$

Theorem Suppose there exists a function $T(X)$ such, that for every sample points x and y the ratio $\frac{f_\theta(x)}{f_\theta(y)}$ is constant as a function of θ iff $T(x) = T(y)$ then $T(X)$ is a minimal sufficient statistic for θ.

$$\frac{\mathbf{f}_\mu(x)}{\mathbf{f}_\mu(y)} = \frac{\frac{1}{(\sqrt{2\pi\mu})^n} e^{-\frac{1}{2\mu}(\sum(x_i-\mu)^2)}}{\frac{1}{(\sqrt{2\pi\mu})^n} e^{-\frac{1}{2\mu}(\sum(y_i-\mu)^2)}} = e^{\frac{1}{2\mu}(-\sum(x_i-\mu)^2 + \sum(y_i-\mu)^2)}$$

$$= e^{\frac{1}{2\mu}(\sum(y_i-\mu)^2 - (x_i-\mu)^2)} = e^{\frac{1}{2\mu}(\sum y_i^2 - \sum x_i^2 - 2\mu(\sum x_i - y_i)}$$

is constant as a function of μ iff $\sum x_i^2 = \sum y_i^2$

(in this case we get $e^{-\frac{1}{2\mu}2\mu(\sum x_i - y_i)} = e^{\sum y_i - x_i}$)

Therefore $T(x) = \sum x_i^2$ is a minimal sufficient statistic for μ.

b) show $(\sum X_i, \sum X_i^2)$ is sufficient but not minimal sufficient in the $\mathcal{N}(\mu,\mu)$ family

not minimal as there exists no function r such that

$(\sum X_i, \sum X_i^2) = r(\sum X_i^2)$ ($\sum X_i^2$ is a sufficient statistic)

$(\sum X_i, \sum X_i^2)$ is a sufficient statistic as it contains $\sum X_i^2$ which we have shown to be a sufficient statistic.

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4) $X_1, \dots, X_n$ ... random sample from population with pdf $f_\theta(x) = \begin{cases} \frac{2x}{\theta^2}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$

$\theta > 0$ ... unknown find minimal sufficient statistic

$$f_\theta(x) = \frac{2x}{\theta^2} 1\!\!1_{(0,\theta)}(x)$$

$$f_\theta(x) = \prod_{i=1}^{n} f_\theta(x) = \frac{2^n}{\theta^{2n}} 1\!\!1_{(0,\infty)}(\min_i x_i)\, 1\!\!1_{(-\infty,\theta)}(\max_i x_i) \prod_{i=1}^{n} x_i$$

$$\frac{f_\theta(x)}{f_\theta(y)} = \frac{1\!\!1_{(0,\infty)}(\min_i x_i)\, 1\!\!1_{(-\infty,\theta)}(\max_i x_i) \prod_{i=1}^{n} x_i}{1\!\!1_{(0,\infty)}(\min_i y_i)\, 1\!\!1_{(-\infty,\theta)}(\max_i y_i) \prod_{i=1}^{n} y_i}$$ is constant as a function of θ iff

$1\!\!1_{(-\infty,\theta)}(\max_i x_i) = 1\!\!1_{(-\infty,\theta)}(\max_i y_i)$ $\rightarrow T(x) = \max_i x_i$ ... minimal sufficient statistic