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(1) F... continous cdf

a) $U \sim U(0,1)$ $\quad Y = F^{-1}(U)$ $\quad$ cdf of Y?

$$f_U : \mathbb{R} \to \mathbb{R}^+ \qquad \text{...pdf of } U \quad \text{sample space } \mathbb{R}$$
$$x \mapsto \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

F^{-1} ... invertible, $(F^{-1})^{-1} = F$ ...differentiable with $F' = f_F$...pdf

$$\Rightarrow f_Y(y) = f_U(F(y))\, |f_F(y)| \quad \text{...pdf of } Y$$

$$\Rightarrow \text{cdf of } Y \text{ is } \int_{-\infty}^{y} f_U(\underbrace{F(z)}_{\in(0,1)})\, |f_F(z)|\, dz$$
$$= \int_{-\infty}^{y} \underbrace{|f_F(z)|}_{\ge 0}\, dz = \int_{-\infty}^{y} f_F(z)\, dz = F(y)$$

b) X ... random variable $\quad F$... cdf of X $\quad Z = F(X)$ $\quad$ cdf of Z?

$\Rightarrow F'$ ... pdf of X

$$\Rightarrow \text{pdf } f_Z(z) = f_X(F^{-1}(z))\, |(F^{-1})'(z)|$$

$$\text{cdf} = \int_{-\infty}^{z} f_X(F^{-1}(s))\, |(F^{-1})'(s)|\, ds \qquad (g(h(x)))' = g'(h(x)) \cdot h'(x)$$

$$= F(F^{-1}(z)) = z \quad \text{for } 0 \le z \le 1 \quad (F^{-1} : [0,1] \to \mathbb{R})$$

since a cdf is monotonely increasing the cdf of values $z < 0$ is 0 and of values $z > 1$ is 1.

$$\Rightarrow \text{cdf } F_Z(z) = \begin{cases} 0, & z < 0 \\ z, & 0 \le z \le 1 \\ 1, & z > 1 \end{cases} \qquad \text{which is the uniform distribution}$$

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2) random points on a line

X_1, X_2, X_3 independently, uniformly choosen on line of length L

probability X_2 lies between X_1 and X_3 ?

$X_1, X_2\ X_3 \sim U(0,L)$ iid

$$\mathbb{P}(X_1 < X_2 < X_3) = \int_0^L f(x_1)\, \mathbb{P}(x_1 < X_2 < X_3)\, dx_1 = \int_0^L f(x_1) \int_{x_1}^L f(x_2)\, \mathbb{P}(x_1 < x_2 < X_3)\, dx_2\, dx_1$$

$$= \int_0^L f(x_1) \int_{x_1}^L f(x_2) \int_{x_2}^L f(x_3)\, dx_3\, dx_2\, dx_1 = \int_0^L \frac{1}{L} \int_{x_1}^L \frac{1}{L} \int_{x_2}^L \frac{1}{L}\, dx_3\, dx_2\, dx_1 = \frac{1}{L^3} \int_0^L\int_{x_1}^L\int_{x_2}^L 1\, dx_3\, dx_2\, dx_1$$

$$= \frac{1}{L^3} \int_0^L \int_{x_1}^L (L - x_2)\, dx_2\, dx_1 = \frac{1}{L^3} \int_0^L L^2 - \frac{L^2}{2} - Lx_1 + \frac{x_1^2}{2}\, dx_1$$

$$= \frac{1}{L^3}\left(\frac{L^3}{2} - \frac{L^3}{2} + \frac{L^3}{6}\right) = \frac{1}{6}$$

$$\mathbb{P}(X_3 < X_2 < X_1) = \frac{1}{6}$$

$$\Rightarrow \mathbb{P}(X_2 \text{ lies between } X_1 \text{ and } X_3) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

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3) $X, Y \ldots$ random variables

$$f(x,y) = \begin{cases} c(x+2y) & , 0<y<1 \text{ and } 0<x<2 \\ 0 & , \text{otherwise} \end{cases}$$

a) find c and the marginal distribution of Y

$$1 = \int_0^1 \int_0^2 f(x,y)\,dx\,dy = \int_0^1 \int_0^2 c(x+2y)\,dx\,dy = c \int_0^1 2+4y\,dy$$

$$= c(2+2) = 4c \qquad \Rightarrow c = \frac{1}{4}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{1}{4}(x+2y)\,dx = \frac{1}{4}(2+4y) = \frac{1}{2}+y \quad \text{for } y \in (0,1)$$

and 0 elsewhere

b) joint cdf of X and Y

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds = \int_0^x \int_0^y \frac{1}{4}(s+2t)\,dt\,ds = \frac{1}{4}\int_0^x sy+y^2\,ds$$

$$= \frac{1}{4}\left(\frac{1}{2}x^2y + xy^2\right) = \frac{x^2y}{8} + \frac{xy^2}{4}$$

c) marginal distribution of X and pdf of $Z = \frac{9}{(X+1)^2}$?

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{1}{4}(x+2y)\,dy = \frac{1}{4}(x+1) = \frac{x+1}{4}$$

$X \ldots$ continuous random variable with pdf f_X

$$g: \mathbb{R} \to \mathbb{R}^+ \qquad g^{-1}: \mathbb{R}^+ \to \mathbb{R} \quad \ldots \text{differentiable} \quad (g^{-1})'(x) = -\frac{3}{2}x^{-\frac{3}{2}}$$

$$x \mapsto \frac{9}{(x+1)^2} \qquad x \mapsto \frac{3}{\sqrt{x}} - 1$$

$$Z = g(X) \quad \Rightarrow f_Z(z) = f_X(g^{-1}(z))\,|(g^{-1})'(z)| = f_X\left(\frac{3}{\sqrt{z}}-1\right)\left|-\frac{3}{2}z^{-\frac{3}{2}}\right|$$

$$= \frac{\frac{3}{\sqrt{z}} \cancel{-1} \cancel{+1}}{4}\left|\frac{3}{2}z^{-\frac{3}{2}}\right| = \frac{3}{4}\cdot\frac{1}{\sqrt{z}}\,\frac{3}{2}\,\frac{1}{\sqrt{z}^3} = \frac{9}{8}\,\frac{1}{\sqrt{z}^4} = \frac{9}{8z^2}$$

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4) random points on a unit circle

X, Y ... coordinates of a point chosen uniformly on a unit circle

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi}, & \text{if } x^2+y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$R = \sqrt{X^2+Y^2} \qquad \Theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

$$f_{R,\Theta}\begin{pmatrix} r \\ \varphi \end{pmatrix} = f_{X,Y}\left(h\begin{pmatrix} r \\ \varphi \end{pmatrix}\right) \left|\det\left(\nabla h\begin{pmatrix} r \\ \varphi \end{pmatrix}\right)\right|$$

$$g: \mathbb{R}^2 \to \mathbb{R}^+ \times [0, 2\pi) \qquad h = g^{-1}: \mathbb{R}^+ \times [0, 2\pi) \to \mathbb{R}^2 \qquad \text{for } (x,y) \neq (0,0)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{x^2+y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix} \qquad \begin{pmatrix} r \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} r \cdot \cos(\varphi) \\ r \cdot \sin(\varphi) \end{pmatrix}$$

$$\frac{\partial}{\partial r} h(r,\varphi) = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} \qquad \frac{\partial}{\partial \varphi} h(r,\varphi) = \begin{pmatrix} -r\sin(\varphi) \\ r\cos(\varphi) \end{pmatrix}$$

$$\left|\det\left(\nabla h\begin{pmatrix} r \\ \varphi \end{pmatrix}\right)\right| = \left|\det\begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix}\right| = |r\cos^2(\varphi) + r\sin^2(\varphi)| = r\,|\sin^2(\varphi)+\cos^2(\varphi)| = r$$

$$f_{R,\Theta}\begin{pmatrix} r \\ \varphi \end{pmatrix} = f_{X,Y}(r \cdot \sin(\varphi), r \cdot \cos(\varphi))\, r \qquad r^2\sin^2(\varphi) + r^2\cos^2(\varphi) = r^2 \leq 1 \iff |r| \leq 1$$

$$\iff r \leq 1 \text{ since } r \in \mathbb{R}^+$$

$$= \begin{cases} \frac{r}{\pi}, & \text{if } r \leq 1 \\ 0, & \text{else} \end{cases}$$

$$\int_0^1 \int_0^{2\pi} \frac{r}{\pi}\, d\varphi\, dr = \int_0^1 \frac{r}{\pi} 2\pi\, dr = \int_0^1 2r\, dr = 1^2 = 1 \qquad \text{"checksum"}$$

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5) $X_1, X_2 \sim Exp(\lambda)$ ...iid λ...rate $f_{X_1}(x) = f_{X_2}(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$

$Y = X_1 + X_2$ f_Y ?

according to the theorem

$$f_{X_1+X_2}(y) = \int_{-\infty}^{\infty} f_{X_1}(y - x_2)\, f_{X_2}(x_2)\, dx_2 \qquad y - x_2 > 0 \quad x_2 > 0 \Rightarrow y > x_2 > 0$$

$$= \int_0^y \lambda e^{-\lambda(y-x_2)}\, \lambda e^{-\lambda(x_2)}\, dx_2$$

$$= \lambda^2 \int_0^y e^{-\lambda(y - x_2 + x_2)}\, dx_2$$

$$= \lambda^2 \int_0^y e^{-\lambda y}\, dx_2$$

$$= \lambda^2 e^{-\lambda y} x_2 \Big|_0^y$$

$$= \lambda^2 y e^{-\lambda y}$$ (which is Gamma distribution $\Gamma(2, \lambda)$)