IST Ü4

3) $X \sim N(\mu, \sigma^2)$ show that $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

We know that $M_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$

$$\int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(tx - \frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$tx - \frac{(x-\mu)^2}{2\sigma^2} \overset{?}{=} \mu t + \frac{\sigma^2 t^2}{2} - \frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}$$

$$\frac{2\sigma^2 tx - x^2 + 2\mu x - \mu^2}{2\sigma^2} \overset{?}{=} \frac{2\sigma^2 \mu t + \sigma^4 t^2 - x^2 + 2(\mu+\sigma^2 t)x - (\mu+\sigma^2 t)^2}{2\sigma^2}$$

$$2\sigma^2 tx + 2\mu x - \mu^2 \overset{?}{=} \cancel{2\sigma^2\mu t} + \cancel{\sigma^4 t^2} + 2\mu x + 2\sigma^2 tx - \mu^2 - \cancel{2\sigma^2\mu t} - \cancel{\sigma^4 t^2}$$

$$2\sigma^2 tx + 2\mu x - \mu^2 = 2\sigma^2 tx + 2\mu x - \mu^2$$

$$\Rightarrow M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\mu t + \frac{\sigma^2 t^2}{2} - \frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right) dx$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}}\,dx$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

because $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}}\,dx = 1$ as it is the pdf of $N(\mu+\sigma^2 t, \sigma^2)$