ISt Ü7

3) $W_1, \dots, W_k$ ... unbiased estimators of θ with $\mathbb{V}ar = \sigma_i^2$ and $\mathbb{C}or(W_i, W_j) = 0$ if $i \neq j$

show a_i ... constant and $\mathbb{E}_\theta(\Sigma a_i W_i) = \theta$

of all estimators $\Sigma a_i W_i$ $\quad W^* = \dfrac{\Sigma W_i / \sigma_i^2}{\Sigma(1-\sigma_i^2)}$ has minimum variance

$$\theta = \mathbb{E}(\Sigma a_i W_i) = \Sigma a_i \, \mathbb{E}(W_i) = \Sigma a_i \theta = \theta(\Sigma a_i) \quad \Rightarrow \Sigma a_i = 1$$

$$\mathbb{V}ar(\Sigma a_i W_i) = \Sigma a_i^2 \mathbb{V}ar(W_i) = \Sigma a_i^2 \sigma_i^2$$

our goal is to minimize $\Sigma a_i^2 \sigma_i^2$ with the condition $\Sigma a_i - 1 = 0$

$$L(a, \lambda) = \Sigma a_i^2 \sigma_i^2 - \lambda(\Sigma a_i - 1)$$

$$\frac{\partial L}{\partial a_i} = 2 a_i \sigma_i^2 - \lambda = 0 \qquad \frac{\partial L}{\partial \lambda} = -(\Sigma a_i - 1) = 0$$

$$\Rightarrow a_i = \frac{\lambda}{2\sigma_i^2} \qquad \Rightarrow \Sigma a_i = \Sigma \cdot \frac{\lambda}{2\sigma_i^2} = \frac{\lambda}{2} \Sigma \sigma_i^{-2}$$

$$\Rightarrow \lambda = 2(\Sigma \sigma_i^{-2})^{-1} \qquad \Rightarrow a_i = \frac{\frac{1}{\sigma_i^2}}{\sum_j \sigma_j^{-2}}$$

$$\Rightarrow \Sigma a_i W_i = \frac{\Sigma \frac{W_i}{\sigma_i^2}}{\Sigma \frac{1}{\sigma_i^2}} = W^*$$

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show $\mathbb{V}ar\, W^* = \dfrac{1}{\Sigma \frac{1}{\sigma_i^2}}$

$$\mathbb{V}ar(W^*) = \Sigma a_i^2 \sigma_i^2 = \left(\sum_i \left(\frac{\frac{1}{\sigma_i^2}}{\sum_j \sigma_j^{-2}}\right)^2\right) \sigma_i^2 = \sum_i \frac{\frac{1}{\sigma_i^2}}{\sum_j \sigma_j^{-2}} = \frac{1}{(\sum_j \sigma_j^{-2})^2} \sum_i \frac{1}{\sigma_i^2}$$

$$= \frac{1}{\Sigma \sigma_j^{-2}}$$