IST Üb

2) Box of candles

red, blue candles probability of randomly choosing a blue candle is $\frac{1}{1+2a}$

where $a>0$; sample size n ; find MLE $\hat{a}$ of parameter a

drawing n candles k of which are blue has probability:

$$\left(\frac{1}{1+2a}\right)^k \left(1-\frac{1}{1+2a}\right)^{n-k} = \frac{1}{(1+2a)^k}\left(\frac{1+2a-1}{1+2a}\right)^{n-k} = \frac{1}{(1+2a)^k}\frac{(2a)^{n-k}}{(1+2a)^{n-k}}$$

$$= \frac{(2a)^{n-k}}{(1+2a)^n} =: f_{n,k}(a)$$

$1 \leq k \leq n$:

$$\frac{d}{da} f_{n,k}(a) = \frac{d}{da}\frac{(2a)^{n-k}}{(1+2a)^n} = 2^{n-k}\left(\frac{(n-k)\,a^{n-k-1}(1+2a)^n - a^{n-k}\,n(1+2a)^{n-1}2}{(1+2a)^{2n}}\right)$$

$$= 2^{n-k}\,a^{n-k-1}(1+2a)^{n-1}\frac{(n-k)(1+2a)-2an}{(1+2a)^{2n}}$$

$$= 2^{n-k}\,a^{n-k-1}(1+2a)^{-n-1}(n+2an-k-2ak-2an)$$

$$= 2^{n-k}\,a^{n-k-1}(1+2a)^{-n-1}(n-k-2ak)$$

$$\frac{d}{da} f_{n,k}(a) = 0 \iff a=0 \lor n-k-2ak=0$$

$$\iff a=0 \lor a=\frac{n-k}{2k}$$

$a=0$ gives the probability 1 and likelihood $L(a=0\,|\,x) = \begin{cases} 0 & \text{if } x<n \\ 1 & \text{if } x=n \end{cases}$

$a=\frac{n-k}{2k}$ gives probability $\frac{1}{1+2\frac{n-k}{2k}} = \frac{k}{n}$ and likelihood

$$L\left(a=\frac{n-k}{2k}\,\middle|\,x\right) = \mathbb{P}\left(x\,\middle|\,p=\frac{k}{n}\right) > 0 \quad \forall x$$

$$\Rightarrow \hat{a} = \begin{cases} 0 & \text{, if } k=n \\ \frac{n-k}{2k} & \text{, if } k<n \end{cases}$$

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3) $X_1, \dots, X_n \dots$ iid uniform $(0,\theta)$ $\qquad \theta > 0$

a) MME of θ

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \mu(\hat{\theta}) = \int_{\mathbb{R}} x\, f_{\hat{\theta}}(x)\, dx = \int_0^{\hat{\theta}} x \frac{1}{\hat{\theta}}\, dx = \frac{1}{\hat{\theta}} \left(\frac{x^2}{2}\right)\Big|_0^{\hat{\theta}} = \frac{1}{\hat{\theta}} \frac{\hat{\theta}^2}{2} = \frac{\hat{\theta}}{2}$$

$$\Rightarrow \hat{\theta} = 2\bar{X}$$

MLE of θ

$$L(\hat{\theta}) = \begin{cases} 0 & \text{if } \exists i \in \{1,\dots,n\}: X_i > \hat{\theta} \\ \mathbb{P}(X_1,\dots,X_n \mid \hat{\theta}) = \left(\frac{1}{\hat{\theta}}\right)^n & \text{otherwise} \end{cases}$$

$\frac{1}{\hat{\theta}^n}$ gets bigger if $\hat{\theta}$ is smaller $\Rightarrow$ we want the smallest $\hat{\theta}$ with

$\hat{\theta} \leq X_i \ \forall i \quad \Rightarrow \hat{\theta} = \max\{X_i : i \in \{1,\dots,n\}\}$

b) MSE of both estimators

$$MSE = \mathbb{V} + b^2$$

MME: $b(\hat{\theta}) = \mathbb{E}(2\bar{X}) - \theta = 2\,\mathbb{E}(\bar{X}) - \theta = \theta - \theta = 0$

$$\mathbb{V}(\hat{\theta}) = \mathbb{V}(2\bar{X}) = 4\,\mathbb{V}(\bar{X}) = 4\frac{\theta^2}{12n} = \frac{\theta^2}{3n}$$

$$\Rightarrow MSE = \frac{\theta^2}{3n} + 0 = \frac{\theta^2}{3n}$$

MLE: $b(\hat{\theta}) = \mathbb{E}(\max_{1\leq i\leq n} X_i) - \theta = \frac{n}{n+1}\theta - \theta = -\theta\frac{1}{n+1}$

$$\mathbb{V}(\hat{\theta}) = \mathbb{V}(\max_{1\leq i\leq n} X_i) = \frac{n\theta^2}{(n+1)^2(n+2)}$$

$$\Rightarrow MSE = \frac{n\theta^2}{(n+1)^2(n+2)} + \frac{\theta^2}{(n+1)^2} = \frac{n\theta^2 + \theta^2(n+2)}{(n+1)^2} = \frac{2n\theta^2 + 2\theta^2}{(n+1)^2} = 2\theta^2\frac{n+1}{(n+1)^2}$$

$$= 2\theta^2\frac{1}{n+1}$$

$\theta^2\frac{1}{3n} = \theta^2\frac{2}{n+1} \iff n+1 = 6n \iff n = \frac{1}{5} \quad \Rightarrow \forall n > 1: \frac{2\theta^2}{n+1} > \frac{\theta^2}{3n}$

therefore MME has a better MSE than MLE.

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4) Unbiased estimators

$\hat{a}, \hat{b}$ ... unbiased estimators of unknown parameters a, b

a) $\alpha, \beta \in \mathbb{R}$ $\quad \alpha\hat{a} + \beta\hat{b}$ ... unbiased estimator of $\alpha a + \beta b$?

we know $\mathbb{E}(\hat{a}) = a$ and $\mathbb{E}(\hat{b}) = b$

$\Rightarrow \mathbb{E}(\alpha\hat{a} + \beta\hat{b}) = \alpha\,\mathbb{E}(\hat{a}) + \beta\,\mathbb{E}(\hat{b}) = \alpha\hat{a} + \beta\hat{b}$ $\quad \Rightarrow$ unbiased estimator

b) $\hat{a}^2$ ... unbiased estimator of a^2?

$b(\hat{a}^2) = \mathbb{E}(\hat{a}^2) - a^2 = \mathbb{E}(\hat{a}^2) - (\mathbb{E}(\hat{a}))^2 = \mathbb{V}ar(\hat{a})$

which is only $= 0$ if $\hat{a}$ is fixed at a. Otherwise $\hat{a}^2$ is not an unbiased estimator.

c) $\mathbb{E}(a) = \mathbb{E}(\{15^2, 17^2, 16^2, 16^2, 17^2, 14^2\}) = \dfrac{15^2 + 17^2 + 16^2 + 16^2 + 17^2 + 14^2}{6}$

$= \dfrac{1511}{6}$

$a = \dfrac{1511}{6}$ is an unbiased estimator of the area.