ISt Üb

2) Box of candles

red, blue candles probability of randomly choosing a blue candle is $\frac{1}{1+2a}$

where $a>0$; sample size n ; find MLE $\hat{a}$ of parameter a

drawing n candles k of which are blue has probability:

$$\left(\frac{1}{1+2a}\right)^k \left(1-\frac{1}{1+2a}\right)^{n-k} = \frac{1}{(1+2a)^k}\left(\frac{1+2a-1}{1+2a}\right)^{n-k} = \frac{1}{(1+2a)^k}\frac{(2a)^{n-k}}{(1+2a)^{n-k}}$$

$$= \frac{(2a)^{n-k}}{(1+2a)^n} =: f_{n,k}(a)$$

$1 \leq k \leq n$:

$$\frac{d}{da}f_{n,k}(a) = \frac{d}{da}\frac{(2a)^{n-k}}{(1+2a)^n} = 2^{n-k}\left(\frac{(n-k)a^{n-k-1}(1+2a)^n - a^{n-k}n(1+2a)^{n-1}2}{(1+2a)^{2n}}\right)$$

$$= 2^{n-k}a^{n-k-1}(1+2a)^{n-1}\frac{(n-k)(1+2a)-2an}{(1+2a)^{2n}}$$

$$= 2^{n-k}a^{n-k-1}(1+2a)^{-n-1}(n+2an-k-2ak-2an)$$

$$= 2^{n-k}a^{n-k-1}(1+2a)^{-n-1}(n-k-2ak)$$

$$\frac{d}{da}f_{n,k}(a) = 0 \iff a=0 \vee n-k-2ak=0$$

$$\iff a=0 \vee a=\frac{n-k}{2k}$$

$a=0$ gives the probability 1 and likelihood $L(a=0\mid x) = \begin{cases} 0 & \text{if } x<n \\ 1 & \text{if } x=n \end{cases}$

$a=\frac{n-k}{2k}$ gives probability $\frac{1}{1+2\frac{n-k}{2k}} = \frac{k}{n}$ and likelihood

$$L\left(a=\frac{n-k}{2k} \mid x\right) = \mathbb{P}\left(x \mid p=\frac{k}{n}\right) > 0 \quad \forall x$$

$$\Rightarrow \hat{a} = \begin{cases} 0 & \text{, if } k=n \\ \frac{n-k}{2k} & \text{, if } k<n \end{cases}$$