IST Ü4

4) $X_1, X_2 \ldots$ independent random variables $\mathbb{E}(X_1) = \mathbb{E}(X_2) = \mu$

$\mathbb{V}ar(X_1) = \sigma_1^2$ $\mathbb{V}ar(X_2) = \sigma_2^2$ μ...unknown

estimate μ by $\lambda X_1 + (1-\lambda) X_2$ for some appropriate value of λ

Which value of λ yields the estimate having the lowest possible variance?

$$\mathbb{V}ar(\lambda X_1 + (1-\lambda) X_2)$$
$$= \lambda^2 \mathbb{V}ar(X_1) + (1-\lambda)^2 \mathbb{V}ar(X_2) \quad \text{(as } X_1 \text{ and } X_2 \text{ are independent)}$$
$$= \lambda^2 \sigma_1^2 + (1-\lambda)^2 \sigma_2^2$$

We want to minimize $\lambda^2 \sigma_1^2 + (1-\lambda)^2 \sigma_2^2 =: f(\lambda)$

$$f'(\lambda) = 2\lambda\sigma_1^2 + 2(1-\lambda)(-1)\sigma_2^2 = 2\lambda(\sigma_1^2+\sigma_2^2) - 2\sigma_2^2 = 0$$
$$\Leftrightarrow \lambda(\sigma_1^2+\sigma_2^2) = \sigma_2^2 \quad \Leftrightarrow \lambda = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

where $\sigma_1^2+\sigma_2^2$ can only be 0 if $\mathbb{V}ar(X_1) = \mathbb{V}ar(X_2) = 0$, which is a trivial case.

To ensure $\lambda = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$ is a minimum we calculate

$$f''(\lambda) = 2(\sigma_1^2+\sigma_2^2) > 0$$

which gives us that

$$\lambda X_1 + (1-\lambda) X_2$$
$$= \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} X_1 + \left(1 - \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right) X_2$$
$$= \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} X_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} X_2$$

is the estimate of μ with the lowest possible variance.