1st Ü7

2) $X_1, \dots, X_n$ … random sample with pdf $f_\theta(x) = \theta x^{\theta-1}$ $0 < x < 1$, $\theta > 0$

Is there a function $g(\theta)$ for which there exists an unbiased estimator whose variance attains the Cramér-Rao lower bound?

We search for a function $g(\theta)$ and unbiased estimator $\widehat{g(\theta)}$ with

$$\mathbb{V}ar_{g(\theta)}(\widehat{g(\theta)}_n) = \frac{1}{n I_1(g(\theta))} = I_n(g(\theta))^{-1}$$

$$I_1(\theta) = \mathbb{V}ar(z(X,\theta)) = \mathbb{V}ar\left(\frac{\partial}{\partial\theta} \log f_\theta(x)\right) = -\mathbb{E}(z'(X,\theta)) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2} \log f_\theta(x)\right)$$

$$\log f_\theta(x) = \log(\theta x^{\theta-1}) = \log(\theta) + (\theta-1)\log(x)$$

$$\frac{\partial}{\partial\theta} \log f_\theta(x) = \frac{\partial}{\partial\theta} \log(\theta) + (\theta-1)\log(x) = \frac{1}{\theta} + \log(x)$$

$$\frac{\partial^2}{\partial\theta^2} \log f_\theta(x) = \frac{\partial}{\partial\theta} \frac{1}{\theta} + \log(x) = \frac{-1}{\theta^2}$$

$$-\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2} \log f_\theta(x)\right) = -\mathbb{E}\left(-\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}$$

$$\Rightarrow \mathbb{V}ar(\widehat{g(\theta)}_n) = \frac{1}{n} g(\theta)^2$$

Mean of $\widehat{g(\theta)}_n = \mathbb{E}(g(\theta))$ and variance of $\widehat{g(\theta)}_n = \frac{1}{n} g(\theta)^2$

$n=1$ $g(\theta) = \frac{1}{\theta}$ $T(x) = -\ln(x)$ … estimator of $g(\theta)$

$$\mathbb{E}(T(x)) = \int_0^1 -\ln(x)\,\theta x^{\theta-1}\,dx = \frac{1}{\theta}$$

$$\mathbb{E}(T(x)^2) = \int_0^1 \ln(x)^2\,\theta x^{\theta-1}\,dx = \frac{2}{\theta^2}$$

$$\mathbb{V}ar(T(x)) = \mathbb{E}(T(x)^2) - \mathbb{E}(T(x))^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2} = g(\theta)^2$$

$n>1$ $g(\theta) = \frac{1}{\theta}$ $T(x) = -\frac{\sum_{i=1}^n \ln(x_i)}{\sqrt{n}}$

$$\Rightarrow \mathbb{V}ar(T(x)) = \mathbb{E}(T(x)^2) - \mathbb{E}(T(x))^2 = \frac{1}{n}\frac{2}{\theta^2} - \frac{1}{n}\frac{1}{\theta^2} = \frac{1}{n}\frac{1}{\theta^2} = \frac{1}{n} g(\theta)^2$$