IST Ü4

1) $P_t \sim \mathcal{P}(\lambda t)$ ... passengers having arrived until time t

$X \sim U(0,T)$ ... time at which first train arrives

What are the expectation and variance of number of passengers who enter the first train?

$\mathbb{E}(P_X \mid X=t) = \mathbb{E}(P_t) = \lambda t \qquad \Rightarrow \mathbb{E}(P_X \mid X) = \lambda X$

We know that

$\mathbb{E}(P_X) = \mathbb{E}(\mathbb{E}(P_X \mid X)) = \mathbb{E}(\lambda X) = \lambda\, \mathbb{E}(X) = \lambda \frac{T-0}{2} = \frac{\lambda}{2} T$

$\mathbb{V}(P_X) = \mathbb{E}(\mathbb{V}(P_X \mid X)) + \mathbb{V}(\mathbb{E}(P_X \mid X))$

$\mathbb{V}(P_X \mid X=t) = \mathbb{V}(P_t) = \lambda t \qquad \Rightarrow \mathbb{V}(P_X \mid X) = \lambda X$

$\mathbb{E}(\mathbb{V}(P_X \mid X)) = \mathbb{E}(\lambda X) = \frac{\lambda}{2} T$

$\mathbb{V}(\mathbb{E}(P_X \mid X)) = \mathbb{V}(\lambda X) = \lambda^2\, \mathbb{V}(X) = \lambda^2 \frac{(T-0)^2}{12} = \frac{\lambda^2}{12} T^2$

$$\mathbb{V}(P_X) = \mathbb{E}(\mathbb{V}(P_X \mid X)) + \mathbb{V}(\mathbb{E}(P_X \mid X))$$
$$= \frac{\lambda}{2} T + \frac{\lambda^2}{12} T^2$$

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2) Real roots

$A, B, C \sim U(0,1)$ ... independent

a) probability that $Ax^2 + Bx + C = 0$ has real roots?

the solution of the equation satisfies $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

to get real roots we must have $b^2 - 4ac > 0$

or $b^2 > 4ac$

$$\mathbb{P}(B^2 > 4AC) = \int_0^1 \mathbb{P}(b^2 > 4AC)\, db$$

$b^2 > 4AC \Leftrightarrow AC < \frac{b^2}{4}$

$\Rightarrow$ if $A < \frac{b^2}{4}$ $\mathbb{P}(b^2 > 4AC) = 1$

else $C < \frac{b^2}{4A}$ has to be satisfied.

$$= \int_0^1 \left(\int_0^{\frac{b^2}{4}} \mathbb{P}(b^2 > 4aC)\, da + \int_{\frac{b^2}{4}}^1 \int_0^{\frac{b^2}{4a}} \mathbb{P}(b^2 > 4ac)\, dc\, da \right) db$$

$$= \int_0^1 \left(\int_0^{\frac{b^2}{4}} 1\, da + \int_{\frac{b^2}{4}}^1 \int_0^{\frac{b^2}{4a}} 1\, dc\, da \right) db$$

$$= \int_0^1 \left(\frac{b^2}{4} + \int_{\frac{b^2}{4}}^1 \frac{b^2}{4a}\, da \right) db = \int_0^1 \frac{b^2}{4} + \frac{b^2}{4} \log\left(\frac{4}{b^2}\right) db =$$

$$= \frac{1}{36} b^3 \left(3 \log\left(\frac{4}{b^2}\right) + 5\right) \Big|_0^1 = \frac{1}{36}(3 \log(4) + 5) \approx 0{,}25441$$

b)

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n=10000
a=runif(n)
b=runif(n)
c=runif(n)
sum(b^2 > 4·a·c)/n
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generates 10 000 instances of this problem and checks what percentage satisfies $b^2 > 4ac$ or $ax^2 + bx + c = 0$ has real roots.

experimental "calculation" of the true percentage calculated in a).