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2) $x' = Ax$

a) $A = \begin{pmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ $\quad \chi(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 1 & -1 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{pmatrix}$

$= (-1-\lambda)^3 \cancel{+4} \cancel{-4} + \cancel{2(-1-\lambda)} - 4(-1-\lambda) \cancel{-2(-1-\lambda)}$

$= (-1-\lambda)\left((-1-\lambda)^2 - 4\right) = (-1-\lambda)(1 + 2\lambda + \lambda^2 - 4)$

$= (-1-\lambda)(-3-\lambda)(1-\lambda) \quad \Rightarrow \quad EW \; -3, -1, 1$

$\begin{pmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a+b-c \\ 2a-b+2c \\ 2a+2b-c \end{pmatrix} = \begin{pmatrix} -3a \\ -3b \\ -3c \end{pmatrix} = -3\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -a+b-c \\ 2a-b+2c \\ 2a+2b-c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix} = -1\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -a+b-c \\ 2a-b+2c \\ 2a+2b-c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \Rightarrow A = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$

$Y(t) = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} e^{\begin{pmatrix} -3t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & t \end{pmatrix}} = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} 2e^{-3t} & -e^{-t} & 0 \\ -3e^{-3t} & e^{-t} & t \\ e^{-3t} & e^{-t} & t \end{pmatrix}$

b) $A = \begin{pmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{pmatrix}$ $\quad EW: 2, 2, 4 \quad EV: \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow Y(t) = \begin{pmatrix} 3 & -2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} = \begin{pmatrix} 3e^{2t} & -2e^{2t} & -e^{4t} \\ e^{2t} & 0 & e^{4t} \\ 0 & e^{2t} & e^{4t} \end{pmatrix}$

c) $A = \begin{pmatrix} 6 & -17 \\ 1 & -2 \end{pmatrix}$

$EW: 2+i, 2-i \quad EV: \begin{pmatrix} 4+i \\ 1 \end{pmatrix}, \begin{pmatrix} 4-i \\ 1 \end{pmatrix}$

$Y(t) = \begin{pmatrix} 4+i & 4-i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{(2+i)t} & 0 \\ 0 & e^{(2-i)t} \end{pmatrix} = \begin{pmatrix} (4+i)e^{(2t+it)} & (4-i)e^{(2t-it)} \\ e^{(2t+it)} & e^{(2t-it)} \end{pmatrix}$

$Re(Y(t)) = \begin{pmatrix} Re\left((4+i)e^{2t}(\cos(t) + i\sin(t))\right) & Re\left((4-i)e^{2t}(\cos(-t) + i\sin(-t))\right) \\ Re\left(e^{2t}(\cos(t) + i\sin(t))\right) & Re\left(e^{2t}(\cos(-t) + i\sin(-t))\right) \end{pmatrix}$

$= \begin{pmatrix} e^{2t}(4\cos(t) - \sin(t)) & e^{2t}(4\cos(t) - \sin(t)) \\ e^{2t}\cos(t) & e^{2t}\cos(t) \end{pmatrix} \quad Im(Y(t)) = \begin{pmatrix} e^{2t}(4\sin(t) + \cos(t)) & e^{2t}(-4\sin(t) - \\ e^{2t}\sin(t) & -e^{2t}\sin(t) \end{pmatrix}$

$\Rightarrow \tilde{Y}(t) = \begin{pmatrix} e^{2t}(4\cos(t) - \sin(t)) & e^{2t}(4\sin(t) + \cos(t)) \\ e^{2t}\cos(t) & e^{2t}\sin(t) \end{pmatrix} \quad W(t) = -e^{4t} \Rightarrow \tilde{Y}(t)$ l.u.

$\mathrm{I}\quad a\,4e^{2t}\cos(t) - a e^{2t}\sin(t) + b\,4e^{2t}\sin(t) + b e^{2t}\cos(t) = 0$

$\mathrm{II}\quad a e^{2t}\cos(t) + b e^{2t}\sin(t) = 0$

$\mathrm{I}\quad e^{2t}\cos(t)(4a+b) + e^{2t}\sin(t)(4b-a) = 0$

$\mathrm{II}\quad e^{2t}(a\cos(t) + b\sin(t)) = 0$

$\mathrm{I}\quad \cos$

$$W(t) = e^{2t}(4\cos(t) - \sin(t))\, e^{2t}\sin(t) - e^{2t}\cos(t)\, e^{2t}(4\sin(t) + \cos(t))$$

$$= e^{4t}(\cancel{4\sin(t)\cos(t)} - \sin^2(t) - \cancel{4\sin(t)\cos(t)} - \cos^2(t))$$

$$= -e^{4t}(\sin^2(t) + \cos^2(t)) = -e^{4t} \neq 0 \ \forall t \in \mathbb{R}$$

$\|$

id

(1) (2) (3) (4)

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3) a) $N \in \mathbb{R}^{m\times m}$ $N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 0 & 1 & \\ & & 0 & 0 & 1 \cdots 0 \\ & & & & 1 \\ 0 & & \cdots & & 0 \end{pmatrix}$ $N_{ij} = \begin{cases} 1 & \text{falls } i+1=j \wedge i \in \{1,\dots,n-1\} \\ 0 & \text{sonst} \end{cases}$

$(N^2)_{ij} = \sum_{k=1}^{m} N_{ik} N_{kj} = \begin{cases} 1 & \text{falls } i+1 = j-1 \\ 0 & \text{sonst} \end{cases}$ $N_{ik} N_{kj} = \begin{cases} 1 & i+1=k \wedge k+1=j \\ 0 & \text{sonst} \end{cases}$

$\Rightarrow N^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & & 1 \\ & & & & & 0 \\ 0 & \cdots & & & & 0 \end{pmatrix}$

Vollständige Induktion um zu zeigen, dass $N^l = \begin{pmatrix} 0 \cdots 0 & 1 & 0 \cdots & \vdots \\ & & 1 & \vdots \\ & & \ddots & 0 \\ & & & 1 \\ 0 & & & 0 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$ also 1nur in l-ter Nebendiagonale

$l=1$ ... N^1 ... klar

$l+1$ Nach IA gilt $N^l_{ij} = \begin{cases} 1 & i+l=j \\ 0 & \text{sonst} \end{cases}$; $N^{l+1} = N^l \cdot N$

$(N^{l+1})_{ij} = \sum_{k=1}^{m} N^l_{ik} N_{kj} = \begin{cases} 1 & i+l=k \wedge k+1=j \Leftrightarrow i+(l+1)=j \\ 0 & \text{sonst} \end{cases}$

$\Rightarrow$ Für $l=m$ gilt

$(N^l)_{ij} = (N^m)_{ij} = \begin{cases} 1 & i+m=j \\ 0 & \text{sonst} \end{cases}$ ... kann nie erfüllt werden, da $i,j \in \{1,\dots,m\}$

$\Rightarrow N^m = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ also ist N nilpotent

b) ges: EW von Nilpotenten Matrizen

Sei N nilpotent bel. Sei $l \in \mathbb{N}$ minimal mit $N^l = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}$

Für alle λ ... EW mit v ... EV $\Rightarrow Nv = \lambda v$

$\Rightarrow N^l v = N^{l-1}(Nv) = N^{l-1}(\lambda v) = \dots = \lambda^l v$ $\Rightarrow \lambda^l$ ist EW von N^l mit EV v

$\Rightarrow \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} v = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda^l v$ da $v \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ $\Rightarrow \lambda^l = 0$ $\Rightarrow \lambda = 0$

c) $x' = Nx$ $m=3$

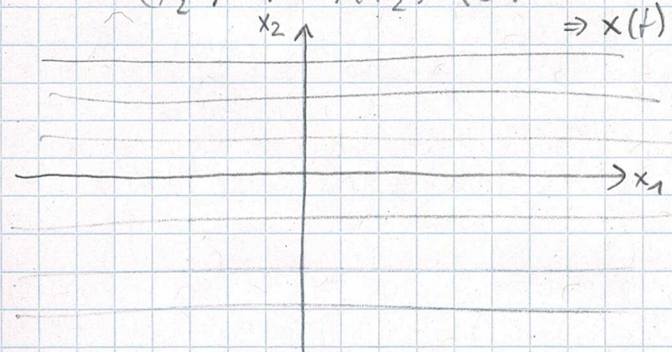
$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ 0 \end{pmatrix}$ $\Rightarrow x_3' = 0 \Rightarrow x_3 = c \in \mathbb{R}$

$\Rightarrow x_2' = c \Rightarrow x_2 = ct + d$ $d \in \mathbb{R}$

$\Rightarrow x = \left(\frac{1}{2}ct^2 + dt + e,\ ct+d,\ c\right)^T$ $\Rightarrow x_1' = ct + d \Rightarrow x_1 = \frac{1}{2}ct^2 + dt + e$ $e \in \mathbb{R}$

$m=2$ $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ $\Rightarrow x_2' = 0 \Rightarrow x_2 = c \in \mathbb{R}$ $x_1' = c \Rightarrow x_1 = ct + d$

$\Rightarrow x(t) = \begin{pmatrix} ct+d \\ c \end{pmatrix}$



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4) $A \in \mathbb{R}^{n \times n}$

a) zz: $\det e^A = e^{\text{spur} A}$

$$A = TJT^{-1} \Rightarrow e^A = e^{TJT^{-1}} = T e^J T^{-1}$$

$$\Rightarrow \det(e^A) = \det(T e^J T^{-1}) = \det(T) \cdot \det(e^J) \cdot \det(T)^{-1} = \det(e^J)$$

$$e^J = \begin{pmatrix} e^{J_1} & & 0 \\ & e^{J_2} & \\ 0 & & e^{J_m} \end{pmatrix} \text{ mit } e^{J_i} = e^{\lambda_i} \underbrace{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{k} \text{ also ist } e^J$$

eine obere Dreiecksmatrix

$$\Rightarrow \det(e^J) = \prod_{i=1}^{n} e^J_{i,i} = \prod_{i=1}^{m} \prod_{l=1}^{k} e^{\lambda_i} = e^{\sum_{i=1}^{m} \sum_{l=1}^{k} \lambda_i} = e^{\text{spur} A}$$

$\left(\text{spur} A = \sum_{i=1}^{m} \sum_{l=1}^{k} \lambda_i\right)$

$$\boxed{\begin{aligned} \det(A - \lambda I) &= (-1)^n (\lambda^n - (\text{spur} A)\lambda^{n-1} + \ldots + (-1)^n \det(A)) \\ &= (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \Rightarrow \text{spur} A = \lambda_1 + \ldots + \lambda_n \end{aligned}}$$

b) $e^{A^T} = (e^A)^T$

$$e^{A^T} = \sum_{k=1}^{\infty} \frac{1}{k!} (A^T)^k = \sum_{k=1}^{\infty} \frac{1}{k!} (A^k)^T = \left(\sum_{k=1}^{\infty} \frac{1}{k!} A^k\right)^T = (e^A)^T$$

c) $A^T = -A$ zz: e^{Ax} .. orthogonal und $\det e^{Ax} = 1$

- $e^A (e^A)^T = e^A e^{A^T} = e^A e^{-A}$, da A mit $-A$ kommutiert gilt
  $= e^{A-A} = e^0 = \text{diag}(e^0, e^0, \ldots, e^0) = \text{diag}(1, 1, \ldots, 1) = I$

- $e^{Ax} (e^{Ax})^T = e^{Ax} e^{(Ax)^T} = e^{Ax} e^{A^T x} = e^{Ax} e^{-Ax}$

$$(Ax)(-Ax) = x A(-A) x = x(-A) A x = (-Ax)(Ax)$$

$$\Rightarrow e^{Ax} e^{-Ax} = e^{Ax - Ax} = e^0 = I$$

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5) a) $A, B \in \mathbb{R}^{n\times n}$, $AB = BA$

zz: $e^{A+B} = e^A e^B$.

Korollar 5.4.11

ANA 1 Umordnen von Reihen

$$e^A e^B = \left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k\right)\left(\sum_{k=0}^{\infty} \frac{1}{k!} B^k\right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{1}{(k-j)!} B^{k-j} \frac{1}{j!} A^j\right)$$

Binomischer Lehrsatz

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (A+B)^k = e^{A+B}$$

zz: $e^{A+B} = e^B e^A$

$$e^{A+B} = e^{B+A} = e^B e^A \text{ nach oben}$$

zz: $e^{(s+t)A} = e^{sA} e^{tA}$ $\quad s, t \in \mathbb{R}$

$$e^{(s+t)A} = e^{sA + tA} = e^{sA} e^{tA}, \text{ da } sA \cdot tA = tA \cdot sA$$

b) Gegenbsp zu $e^{A+B} = e^A e^B$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$e^{A+B} = e^{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \qquad A+B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= e^{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}}$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} e^{\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & e^3 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^3 + \frac{1}{2}e & \frac{1}{2}e^3 - \frac{1}{2}e \\ \frac{1}{2}e^3 - \frac{1}{2}e & \frac{1}{2}e^3 + \frac{1}{2}e \end{pmatrix}$$

$$e^A = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix} \qquad e^B = \begin{pmatrix} e & 0 \\ e & e \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow e^A = e^1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow e^B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e & e \\ 0 & e \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e & 0 \\ e & e \end{pmatrix}$$

$$\Rightarrow e^A e^B = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix}\begin{pmatrix} e & 0 \\ e & e \end{pmatrix} = \begin{pmatrix} 2e^2 & e^2 \\ e^2 & e^2 \end{pmatrix} \neq e^{A+B}$$

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6) $x' = \frac{1}{t}\begin{pmatrix}1 & 2t^2\\ 0 & 1\end{pmatrix}x + e^{-t}\begin{pmatrix}2t\\ \frac{t+1}{t}\end{pmatrix}$ $\quad x\in\mathbb{R}^2,\ t>0$

a) ges: FM $Y(t)$

homogene DGL $x' = \begin{pmatrix}\frac{1}{t} & 2t\\ 0 & \frac{1}{t}\end{pmatrix}x$

$$\Rightarrow \begin{pmatrix}x_1'\\ x_2'\end{pmatrix} = \begin{pmatrix}\frac{1}{t} & 2t\\ 0 & \frac{1}{t}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}\frac{1}{t}x_1 + 2t x_2\\ \frac{1}{t}x_2\end{pmatrix}$$

$\Rightarrow x_2' = \frac{1}{t}x_2 \Rightarrow x_2(t) = ct \quad c\in\mathbb{R}$

$\Rightarrow x_1' = \frac{1}{t}x_1 + 2t\,ct = \frac{1}{t}x_1 + 2ct^2 \Rightarrow x_1(t) = ct^3 + dt \quad d\in\mathbb{R}$

$$\Rightarrow Y(t) = \begin{pmatrix}t^3 & t^3+t\\ t & t\end{pmatrix}$$

$at^3 + bt^3 + bt = 0 \ \wedge\ at + bt = 0 \Rightarrow (a+b)t = 0 \Rightarrow a+b=0$

$(a+b)t^3 + bt = 0 \Rightarrow bt = 0 \Rightarrow b = 0 \Rightarrow a = 0 \Rightarrow$ l.u.

b)
$W(t) = \det Y(t) = t^3\cdot t - t(t^3+t) = t^4 - t^4 - t^2 = -t^2$

$W'(t) = -2t \overset{?}{=} \operatorname{spur} A(t)\,W(t) = \frac{2}{t}(-t^2) = -2t$

c) ges: allgemeine Lösung

$$x(t) = Y(t)Y^{-1}(t_0)x_0 + Y(t)\int_{t_0}^{t} Y^{-1}(s)b(s)\,ds$$

$Y^{-1}(t) = \begin{pmatrix}-\frac{1}{t} & t+\frac{1}{t}\\ \frac{1}{t} & -t\end{pmatrix} \qquad b(s) = e^{-t}\begin{pmatrix}2t\\ \frac{t+1}{t}\end{pmatrix}$

$$\Rightarrow x(t) = \begin{pmatrix}t^3 & t^3+t\\ t & t\end{pmatrix}\begin{pmatrix}-\frac{1}{t_0} & t_0+\frac{1}{t_0}\\ \frac{1}{t_0} & -t_0\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix} + \begin{pmatrix}t^3 & t^3+t\\ t & t\end{pmatrix}\int_{t_0}^{t}\begin{pmatrix}-\frac{1}{s} & s+\frac{1}{s}\\ \frac{1}{s} & -s\end{pmatrix}\begin{pmatrix}2e^{-s}s\\ e^{-s}\frac{s+1}{s}\end{pmatrix}ds$$

$$= \begin{pmatrix}-\frac{t^3}{t_0} + \frac{t^3+t}{t_0} & t^3t_0 + \frac{t^3}{t_0} - t^3t_0 - tt_0\\ -\frac{t}{t_0} + \frac{t}{t_0} & tt_0 + \frac{t}{t_0} - tt_0\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix} + \begin{pmatrix}t^3 & t^3+t\\ t & t\end{pmatrix}\int_{t_0}^{t}\begin{pmatrix}-2e^{-s} + e^{-s}(s+\frac{1}{s})\frac{s+1}{s}\\ 2e^{-s} - e^{-s}(s+1)\end{pmatrix}ds$$

$$= \begin{pmatrix}\frac{t}{t_0} & \frac{t^3}{t_0} - tt_0\\ 0 & \frac{t}{t_0}\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix} + \begin{pmatrix}t^3 & t^3+t\\ t & t\end{pmatrix}\begin{pmatrix}-\frac{e^{-t}(t^2+1)}{t} + \frac{e^{-t_0}(t_0^2+1)}{t_0}\\ e^{-t}t - e^{-t_0}t_0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{tx_0}{t_0} + \frac{t^3y_0}{t_0} - tt_0y_0\\ \frac{ty_0}{t_0}\end{pmatrix} + \begin{pmatrix}-e^{-t}(t^2+1)t^2 + \frac{t^3}{t_0}e^{-t_0}(t_0^2+1) + e^{-t}t(t^3+t) - e^{-t_0}t_0(t^3+t)\\ -e^{-t}(t^2+1) + \frac{t}{t_0}e^{-t_0}(t_0^2+1) + e^{-t}t^2 - e^{-t_0}t_0t\end{pmatrix}$$

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6) ...

$$= \begin{pmatrix} \frac{t x_0}{t_0} + \frac{t^3 y_0}{t_0} - t t_0 y_0 + e^{-t_0}\left(\frac{t^3}{t_0}(t_0^2+1) - t_0 t (t^2+1) \right) \\ \frac{t y_0}{t_0} + e^{-t}(t^2 - t^2 - 1) + e^{-t_0}\left(\frac{t}{t_0}(t_0^2+1) - t_0 t \right) \end{pmatrix}$$

$$= \begin{pmatrix} t\left(\frac{x_0 + t^2 y_0}{t_0} + t_0 y_0 + e^{-t_0}\left(\frac{t^2}{t_0} - t_0 \right)\right) \\ \frac{t y_0}{t_0} - e^{-t} + \frac{t}{t_0} e^{-t_0} \end{pmatrix}$$

d) AWP $x(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad t_0 = 2 \quad x_0 = 1 \quad y_0 = 0$

$$x(t) = \begin{pmatrix} t\left(\frac{1}{2} + e^{-2}\left(\frac{t^2}{2} - 2 \right)\right) \\ -e^{-t} + \frac{t}{2} e^{-2} \end{pmatrix}$$

$$x(2) = \begin{pmatrix} 2\left(\frac{1}{2} + e^{-2}\left(\frac{4}{2} - 2 \right)\right) \\ -e^{-2} + \frac{2}{2} e^{-2} \end{pmatrix} = \begin{pmatrix} 1 + 2e^{-2} \cdot 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x'(t) = \begin{pmatrix} \left(\frac{1}{2} + e^{-2}\left(\frac{t^2}{2} - 2 \right)\right) + t(e^{-2} t) \\ e^{-t} + \frac{1}{2} e^{-2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + e^{-2}\frac{t^2}{2} - 2e^{-2} + t^2 e^{-2} \\ e^{-t} + \frac{1}{2} e^{-2} \end{pmatrix}$$

$$\frac{1}{t}\begin{pmatrix} 1 & 2t^2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t\left(\frac{1}{2} + e^{-2}\left(\frac{t^2}{2} - 2 \right)\right) \\ -e^{-t} + \frac{t}{2} e^{-2} \end{pmatrix} + e^{-t}\begin{pmatrix} 2t \\ \frac{t+1}{t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + e^{-2}\left(\frac{t^2}{2} - 2 \right) \cancel{- 2t e^{-t}} + t^2 e^{-2} \cancel{+ 2t e^{-t}} \\ -\frac{1}{t} e^{-t} + \frac{1}{2} e^{-2} + \frac{t+1}{t} e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + e^{-2}\frac{t^2}{2} - 2e^{-2} + t^2 e^{-2} \\ e^{-t}\left(\frac{\cancel{-1} + t \cancel{+1}}{t} \right) + \frac{1}{2} e^{-2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + e^{-2}\frac{t^2}{2} - 2e^{-2} + t^2 e^{-2} \\ e^{-t} + \frac{1}{2} e^{-2} \end{pmatrix}$$

---

c') $x_p(t) = Y(t) \int_{t_0}^{t} Y^{-1}(s) b(s) \, ds$

$$= \begin{pmatrix} e^{-t_0}\left(\cancel{t^3 t_0} + \frac{t^3}{t_0} - \cancel{t^3 t_0} - t t_0 \right) \\ e^{-t}(\cancel{t^2} - \cancel{t^2} - 1) + e^{-t_0}\left(\cancel{t t_0} + \frac{t}{t_0} - \cancel{t t_0} \right) \end{pmatrix} = \begin{pmatrix} e^{-t_0}\left(\frac{t^3}{t_0} - t t_0 \right) \\ -e^{-t} + e^{-t_0}\frac{t}{t_0} \end{pmatrix}$$