MAS Ü11

1) $(\Omega, \mathcal{S}, P)$ ... Wahrscheinlichkeitsraum, $\mathcal{A} \subseteq \mathcal{S}$ ... σ-Algebra, $1 \leq p$, $q := \frac{p}{p-1}$

$X \in \mathcal{L}_p(\Omega, \mathcal{S}, P)$, $Y \in \mathcal{L}_p(\Omega, \mathcal{A}, P)$

zz: $Y = \mathbb{E}(X \mid \mathcal{A}) \Leftrightarrow \forall Z \in \mathcal{L}_q(\Omega, \mathcal{A}, P): \int Y Z \, dP = \int X Z \, dP$

$\Leftarrow$ zz: Y ist $\mathcal{A}$ messbar und $\forall A \in \mathcal{A}: \mathbb{E}[X \mathbb{1}_A] = \int_A X \, dP = \int_A Y \, dP = \mathbb{E}[Y \mathbb{1}_A]$

Da $Y \in \mathcal{L}_p(\Omega, \mathcal{A}, P)$ ist Y $\mathcal{A}$-messbar.

Sei $A \in \mathcal{A}$ bel. $\Rightarrow \mathbb{1}_A \in \mathcal{L}_q(\Omega, \mathcal{A}, P)$, da $\int_\Omega \mathbb{1}_A^q \, dP = \int_A dP = P(A) \leq 1 < \infty$ $\quad Z = \mathbb{1}_A$

$\Rightarrow \int Y \mathbb{1}_A \, dP = \int X \mathbb{1}_A \, dP \quad \Rightarrow \mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[Y \mathbb{1}_A] \quad \Rightarrow Y = \mathbb{E}(X \mid \mathcal{A})$

$\Rightarrow$

$$g: \mathcal{L}_q(\Omega, \mathcal{A}, P) \to \mathbb{R}, \quad Z \mapsto \int Z \cdot X \, dP \qquad \text{ist linear und stetig}$$

$\Rightarrow \exists! \, h \in \mathcal{L}_q(\Omega, \mathcal{A}, P) : \forall Z \in \mathcal{L}_q(\Omega, \mathcal{A}, P): g(Z) = \langle h, Z \rangle$ nach Riesz

also $\forall Z \in \mathcal{L}_q(\Omega, \mathcal{A}, P): \int Z \cdot X \, dP = \int Z \cdot h \, dP$

$\Rightarrow \forall A \in \mathcal{A}: \int \mathbb{1}_A \cdot X \, dP = \int \mathbb{1}_A \cdot h \, dP \quad \Rightarrow h = \mathbb{E}(X \mid \mathcal{A}) = Y$

$\Rightarrow \forall Z \in \mathcal{L}_q(\Omega, \mathcal{A}, P): \int Z \cdot X \, dP = \int Z \cdot Y \, dP$

$\square$

MAS Ü11

3) $(\Omega, \mathcal{S})$ ... Messraum, $\mathcal{F}$ ... Familie von Wahrscheinlichkeitsmaßen

$\mathcal{A} \subseteq \mathcal{S}$ .. Sub-σ-Algebra heißt suffizient für $\mathcal{F} :\Leftrightarrow \forall A \in \mathcal{S}\ \exists h_A$ ... $\mathcal{A}$-messbare Fkt.

mit $h_A = \mathbb{E}_P[1\!\!1_A \mid \mathcal{A}]\ \forall P \in \mathcal{F}$

Sei $\Omega = \{1,2,3,4\}$, $\mathcal{S} = 2^\Omega$, $\mathcal{F} = \{P_1, P_2\}$ wobei $P_1(A) = 1\!\!1_A(1)$, $P_2(A) = 1\!\!1_A(4)$

$\mathcal{A}_1 = \{\emptyset, \Omega, \{1,3\}, \{2,4\}\}$, $\mathcal{A}_2 = \{\emptyset, \Omega, \{1,2\}, \{3,4\}\}$

zz: $\mathcal{A}_1, \mathcal{A}_2$ sind suffizient für $\mathcal{F}$

$\mathcal{A}_1$ Sei $h_A := P_1(\{1,3\} \cap A) 1\!\!1_{\{1,3\}} + P_2(\{2,4\} \cap A) 1\!\!1_{\{2,4\}}$ ist $\mathcal{A}_1$-messbar

$= 1\!\!1_{\{1,3\} \cap A}(1) 1\!\!1_{\{1,3\}} + 1\!\!1_{\{2,4\} \cap A}(4) 1\!\!1_{\{2,4\}} = 1\!\!1_A(1) 1\!\!1_{\{1,3\}} + 1\!\!1_A(4) 1\!\!1_{\{2,4\}}$

$\Rightarrow h_A(t) = \begin{cases} 1, & (1 \in A \wedge t \in \{1,3\}) \vee (4 \in A \wedge t \in \{2,4\}) \\ 0, & \text{sonst} \end{cases}$

zz: $\forall A \in \mathcal{P}(\Omega)$: $h_A = \mathbb{E}_{P_1}[1\!\!1_A \mid \mathcal{A}_1] \wedge h_A = \mathbb{E}_{P_2}[1\!\!1_A \mid \mathcal{A}_1]$

Sei $B \in \mathcal{A}_1$ bel.

$$\mathbb{E}_{P_1}[1\!\!1_A 1\!\!1_B] = \int_{A \cap B} dP_1 = P_1(A \cap B) = 1\!\!1_{A \cap B}(1)$$

$$\mathbb{E}_{P_1}[h_A 1\!\!1_B] = \int_B h_A \, dP_1 = 1\!\!1_A(1) \int_B 1\!\!1_{\{1,3\}}(t) \, dP_1(t) + 1\!\!1_A(4) \underbrace{\int_B 1\!\!1_{\{2,4\}}(t) \, dP_1(t)}_{=0}$$

$$= 1\!\!1_A(1) P_1(B) + 0 = 1\!\!1_A(1) 1\!\!1_B(1) = 1\!\!1_{A \cap B}(1)$$

$$\mathbb{E}_{P_2}[1\!\!1_A 1\!\!1_B] = \int_{A \cap B} dP_2 = 1\!\!1_{A \cap B}(4)$$

$$\mathbb{E}_{P_2}[h_A 1\!\!1_B] = 1\!\!1_A(1) \underbrace{\int_B 1\!\!1_{\{1,3\}}(t) \, dP_2(t)}_{=0} + 1\!\!1_A(4) \int_B 1\!\!1_{\{2,4\}}(t) \, dP_2(t) = 1\!\!1_A(4) P_2(B) = 1\!\!1_{A \cap B}(4)$$

$\mathcal{A}_2$ $h_A = P_1(\{1,2\} \cap A) 1\!\!1_{\{1,2\}} + P_2(\{3,4\} \cap A) 1\!\!1_{\{3,4\}} = 1\!\!1_A(1) 1\!\!1_{\{1,2\}} + 1\!\!1_A(4) 1\!\!1_{\{3,4\}}$ ist $\mathcal{A}_2$-messbar

$$\mathbb{E}_{P_1}[1\!\!1_A 1\!\!1_B] = \int_{A \cap B} dP_1 = 1\!\!1_{A \cap B}(1)$$

$$\mathbb{E}_{P_1}[h_A 1\!\!1_B] = 1\!\!1_A(1) \int_B 1\!\!1_{\{1,2\}}(t) \, dP_1(t) + 1\!\!1_A(4) \underbrace{\int_B 1\!\!1_{\{3,4\}}(t) \, dP_1(t)}_{=0} = 1\!\!1_A(1) P_1(B) = 1\!\!1_{A \cap B}(1)$$

$$\mathbb{E}_{P_2}[1\!\!1_A 1\!\!1_B] = 1\!\!1_{A \cap B}(4)$$

$$\mathbb{E}_{P_2}[h_A 1\!\!1_B] = 1\!\!1_A(1) \underbrace{\int_B 1\!\!1_{\{1,2\}}(t) \, dP_2(t)}_{=0} + 1\!\!1_A(4) \int_B 1\!\!1_{\{3,4\}}(t) \, dP_2(t) = 1\!\!1_A(4) P_2(B) = 1\!\!1_{A \cap B}(4)$$

$\mathcal{A}_1 \cap \mathcal{A}_2 = \{\emptyset, \Omega\}$ suffizient für $\mathcal{F}$? h_A ... $\mathcal{A}_1 \cap \mathcal{A}_2$-messbar $\Rightarrow h_A = c$ ... konstant

Für $B = \Omega$ gilt: $\int_\Omega 1\!\!1_{\{1\}} dP_1 = \int_\Omega c \, dP_1 \wedge \int_\Omega 1\!\!1_{\{1\}} dP_2 = \int_\Omega c \, dP_2 \Rightarrow 1 = P_1(\{1\}) = c = P_2(\{1\}) = 0$ ↯

MAS Ü11

4) N … SG auf $\mathbb{N}$ verteilt mit $P[N=k] = p_k$, $(X_i)_{i\in\mathbb{N}}$ … Folge, u.a. von N, mit VFen F_i

$\mathbb{E}(X_i) = \mu_i$, $\mathbb{V}(X_i) = \sigma_i^2$; $Z = X_N$ … Auswahlgröße, zufällig $X_0, X_1, X_2, \dots$ an der N-ten Stelle wird beobachtet.

zz: VF von Z $F_Z(x) = \sum_{k\geq 0} p_k F_k(x)$ $\wedge$ falls $\exists$ M_{X_k} auf $(-\delta,\delta)\ \forall k = 0,1,2,\dots \Rightarrow M_Z(t) = \sum_{k\geq 0} p_k M_{X_k}(t)$

existiert, falls Summe endlich

$$F_Z(x) = P(Z\leq x) = P(X_N \leq x) = \sum_{k=0}^{\infty} P(X_N \leq x \mid N=k)\cdot P(N=k)$$
$$= \sum_{k\geq 0} P(X_k \leq x)\cdot p_k = \sum_{k\geq 0} p_k \cdot F_k(x)$$

$$M_Z(t) = \mathbb{E}(e^{tZ}) = \mathbb{E}(e^{tX_N}) = \sum_{k\geq 0} \mathbb{E}(e^{tX_k} \mid N=k)\, P(N=k)$$
$$= \sum_{k\geq 0} p_k \cdot \mathbb{E}(e^{tX_k}) = \sum_{k\geq 0} p_k \cdot M_{X_k}(t) \qquad \text{falls endliche Summe} \Rightarrow M_Z(t) < \infty$$

$N \sim Pois(\theta)$ mit Rate $\theta > 0$, $X_i \sim N(0,i)$, $X_0 = 0$

ges: $M_Z(t)$

$$p_k = P(N=k) = \frac{\theta^k}{k!} e^{-\theta}$$
$$M_{X_0}(t) = \mathbb{E}(e^{tX_0}) = \mathbb{E}(e^0) = \mathbb{E}(1) = 1 \left(= e^{\frac{1}{2}\cdot 0\cdot t^2}\right)$$
$$M_{X_k}(t) = e^{\frac{1}{2}kt^2}$$
$$\Rightarrow M_Z(t) = \sum_{k\geq 0} p_k M_{X_k}(t) = \sum_{k\geq 0} \frac{\theta^k}{k!} e^{-\theta}\cdot e^{\frac{1}{2}\cdot k\cdot t^2} = e^{-\theta} \sum_{k\geq 0} \frac{1}{k!}\left(\theta\cdot e^{\frac{1}{2}t^2}\right)^k$$
$$= e^{-\theta} \exp\left(\theta\cdot e^{\frac{1}{2}t^2}\right) = \exp\left(\theta\left(e^{\frac{1}{2}t^2}-1\right)\right)$$

$\square$

(5) $$\frac{d}{dt} M_{X_k}(t) = e^{\frac{1}{2}kt^2}\cdot \frac{1}{2}k\cdot 2t = k\cdot t\cdot e^{\frac{1}{2}kt^2}$$
$$\frac{d}{dt} M_Z(t) = \exp\left(\theta\left(e^{\frac{1}{2}t^2}-1\right)\right)\cdot \theta\cdot e^{\frac{1}{2}t^2-1}\cdot \frac{1}{2}2t = \theta\cdot t\cdot e^{\frac{1}{2}t^2-1}\cdot \exp\left(\theta\cdot e^{\frac{1}{2}t^2-1}\right) = \theta\cdot t\cdot \exp\left(\frac{1}{2}t^2-1+\theta e^{\frac{1}{2}t^2-1}\right)$$

$$\sum_{k=0}^{\infty} p_k \frac{d}{dt} M_{X_k}(t) = \sum_{k=1}^{\infty} \frac{\theta^k}{k!} e^{-\theta}\cdot k\cdot t\cdot e^{\frac{1}{2}kt^2} = e^{-\theta}\cdot t\cdot \sum_{k=1}^{\infty} \frac{\theta^k}{(k-1)!} e^{\frac{1}{2}kt^2} =$$
$$= \theta\cdot e^{\frac{1}{2}t^2}\cdot e^{-\theta}\cdot t\cdot \sum_{k=1}^{\infty} \frac{1}{(k-1)!}\left(\theta\cdot e^{\frac{1}{2}t^2}\right)^{k-1} = \theta\cdot t\cdot e^{\frac{1}{2}t^2-\theta}\cdot \sum_{k=0}^{\infty} \frac{1}{k!}\left(\theta\cdot e^{\frac{1}{2}t^2}\right)^k$$
$$= \theta\cdot t\cdot e^{\frac{1}{2}t^2-\theta}\cdot \exp\left(\theta\cdot e^{\frac{1}{2}t^2}\right) = \theta\cdot t\cdot \exp\left(\frac{1}{2}t^2-\theta+\theta e^{\frac{1}{2}t^2}\right) \neq \frac{d}{dt} M_Z(t)$$

MASÜM

5) N … SG auf $\mathbb{N}$, $P[N=k]=p_k$ $\quad X_i$ … Folge, u.a. von N $\quad \mathbb{E}X_i=\mu_i$; $\mathbb{V}X_i=\sigma_i^2$

X_i hat VF F_i, $\quad Z=X_N$ … Auswahlgröße

beschränkter Fall: $\exists n\, \forall k>n: p_k=0$

zz: $\mathbb{E}(Z)=\sum_{k\geq 0} p_k\mu_k \wedge \mathbb{V}(Z)=\mathbb{E}(\sigma_N^2)+\mathbb{V}(\mu_N)$

$$\mathbb{E}(Z)=\frac{d}{dt}M_Z(t)\Big|_{t=0}=\frac{d}{dt}\sum_{k=0}^{n}p_k M_{X_k}(t)\Big|_{t=0}\overset{\Delta}{=}\sum_{k=0}^{n}p_k\frac{d}{dt}M_{X_k}(t)\Big|_{t=0}$$

$$=\sum_{k=0}^{n}p_k\,\mathbb{E}(X_k)=\sum_{k=0}^{n}p_k\mu_k$$

$$\mathbb{E}(Z^2)=\frac{d^2}{dt^2}M_Z(t)\Big|_{t=0}=\frac{d^2}{dt^2}\sum_{k=0}^{n}p_k M_{X_k}(t)\Big|_{t=0}\overset{\Delta}{=}\sum_{k=0}^{n}p_k\frac{d^2}{dt^2}M_{X_k}(t)\Big|_{t=0}=\sum_{k=0}^{n}p_k\,\mathbb{E}(X_k^2)$$

$$\mathbb{V}(Z)=\mathbb{E}(Z^2)-\mathbb{E}(Z)^2=\sum_{k=0}^{n}p_k\,\mathbb{E}(X_k^2)-\Big(\sum_{k=0}^{n}p_k\mu_k\Big)^2$$

$* \ \sigma_i^2=\mathbb{V}X_i=\mathbb{E}(X_i^2)-\mathbb{E}(X_i)^2=\mathbb{E}(X_i^2)-\mu_i^2$
$\Rightarrow \mathbb{E}(X_i^2)=\sigma_i^2+\mu_i^2$

$$\overset{*}{=}\sum_{k=0}^{n}p_k\sigma_i^2+\sum_{k=0}^{n}p_k\mu_i^2-\Big(\sum_{k=0}^{n}p_k\mu_k\Big)^2$$

$$=\mathbb{E}(\sigma_N^2)+\mathbb{E}(\mu_N^2)-(\mathbb{E}(\mu_N))^2=\mathbb{E}(\sigma_N^2)+\mathbb{V}(\mu_N)$$

Δ Im unbeschränkten Fall möglicherweise problematisch.

$N\sim \mathrm{Pois}(\theta)$, $X_i\sim N(0,i)$, $X_0\sim 0$

ges: $\mathbb{V}(Z)$

Mit obiger Formel (eigentlich nicht beschränkt): $\mathbb{V}(Z)=\mathbb{E}(\sigma_N^2)+\mathbb{V}(\mu_N)$

$$\mathbb{E}(\sigma_N^2)=\sum_{k=0}p_k\sigma_i^2=\sum_{k=1}^{\infty}\frac{\theta^k}{k!}e^{-\theta}\cdot k=\theta e^{-\theta}\sum_{k=1}^{\infty}\frac{\theta^{k-1}}{(k-1)!}=\theta\cdot e^{-\theta}\sum_{k=0}^{\infty}\frac{\theta^k}{k!}=\theta e^{-\theta}e^{\theta}=\theta$$

$\mathbb{V}(\mu_N)=0$, da $\forall k\in\mathbb{N}: \mu_k=0$ also konstant

$\Rightarrow \mathbb{V}(Z)=\theta$

Mit moment erzeugender Funktion: $\mathbb{V}(Z)=\mathbb{E}(Z^2)-\mathbb{E}(Z)^2=\frac{d^2}{dt^2}M_Z(t)\big|_{t=0}-\big(\frac{d}{dt}M_Z(t)\big|_{t=0}\big)^2$

$$=\frac{d^2}{dt^2}\exp(\theta\cdot e^{\frac{1}{2}t^2-1})\Big|_{t=0}-\Big(\frac{d}{dt}\exp(\theta\cdot e^{\frac{1}{2}t^2-1})\Big|_{t=0}\Big)^2$$

$$=\theta\exp(\theta\cdot e^{-1}-2)(e)-0=\theta e^{\frac{\theta}{e}-2+1}=\theta e^{\frac{\theta}{e}-1}\neq\theta$$, da siehe unten (4)

$$\frac{d}{dt}\exp(\theta\cdot e^{\frac{1}{2}t^2-1})=\exp(\theta\cdot e^{\frac{1}{2}t^2-1})\cdot\theta\cdot\exp(\tfrac{1}{2}t^2-1)\cdot\tfrac{1}{2}\cdot 2\cdot t=\theta t\exp(\theta\cdot e^{\frac{1}{2}t^2-1})\cdot e^{\frac{1}{2}t^2-1}$$

$$\frac{d^2}{dt^2}\exp(\theta\cdot e^{\frac{1}{2}t^2-1})=\theta\exp(\theta\cdot e^{\frac{t^2}{2}-1}+\frac{t^2}{2}-2)(\theta t^2e^{\frac{t^2}{2}}+et^2+e)$$

MAS ÜM

6) X_i... Folge identisch verteilter SG , $X_i \sim X$, $\mathbb{E}X < \infty$

X_i paarweise unkorreliert zz: $p-\lim_{n\to\infty} \overline{X}_n = p-\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \mathbb{E}X$

[Tschebyscheff-Ungleichung $P(|X-\mu_x| \geq k) \leq \frac{\sigma_x^2}{k^2}$ falls $\mu_x = \mathbb{E}(X)$, $\sigma_x^2 = \mathbb{V}(X) < \infty$, $k>0$

$\mathbb{E}(\overline{X}_n) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}X_i = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X) = \mathbb{E}(X) = \mu_x < \infty$

$\sigma_{\overline{X}_n}^2 = \mathbb{V}(\overline{X}_n) = \frac{1}{n^2}\mathbb{V}(\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}(X_i) = \frac{1}{n^2}(n\cdot\mathbb{V}(X)) = \frac{\mathbb{V}(X)}{n} = \frac{\sigma_x^2}{n}$

unkorreliert $\Rightarrow \mathbb{V}(X+Y) = \mathbb{V}(X)+\mathbb{V}(Y)+2\underbrace{\mathrm{Cov}(X,Y)}_{=0} = \mathbb{V}(X)+\mathbb{V}(Y)$

$P(|\overline{X}_n - \mu_x| \geq k) \leq \frac{\sigma_x^2}{n}\frac{1}{k^2} \xrightarrow{n\to\infty} 0 \quad \forall k>0$

$\Rightarrow p-\lim_{n\to\infty} \overline{X}_n = \mu_x = \mathbb{E}X$

$\square$

[Def $p-\lim_{n\to\infty} X_n = X \iff \forall \varepsilon > 0: \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$

X_i, X_j... unkorreliert $\Rightarrow 0 = \mathbb{C}(X_i, X_j) = \mathbb{E}(X_i \cdot X_j) - \mathbb{E}(X_i)\cdot\mathbb{E}(X_j) = \mathbb{E}(X^2) - \mu_x^2$

$\Rightarrow \mathbb{E}(X^2) = \mu_x^2 < \infty$

$\Rightarrow \mathbb{V}(X) < \infty$ (eigentlich sogar $\mathbb{V}(X) = 0$)

bei $\mathbb{V}(X) = 0 \Rightarrow \mathbb{V}(\overline{X}_n) = 0 \Rightarrow \overline{X}_n \equiv \frac{1}{n}\sum_{i=1}^{n} X_i \equiv \mathbb{E}X$ also auch p-lim