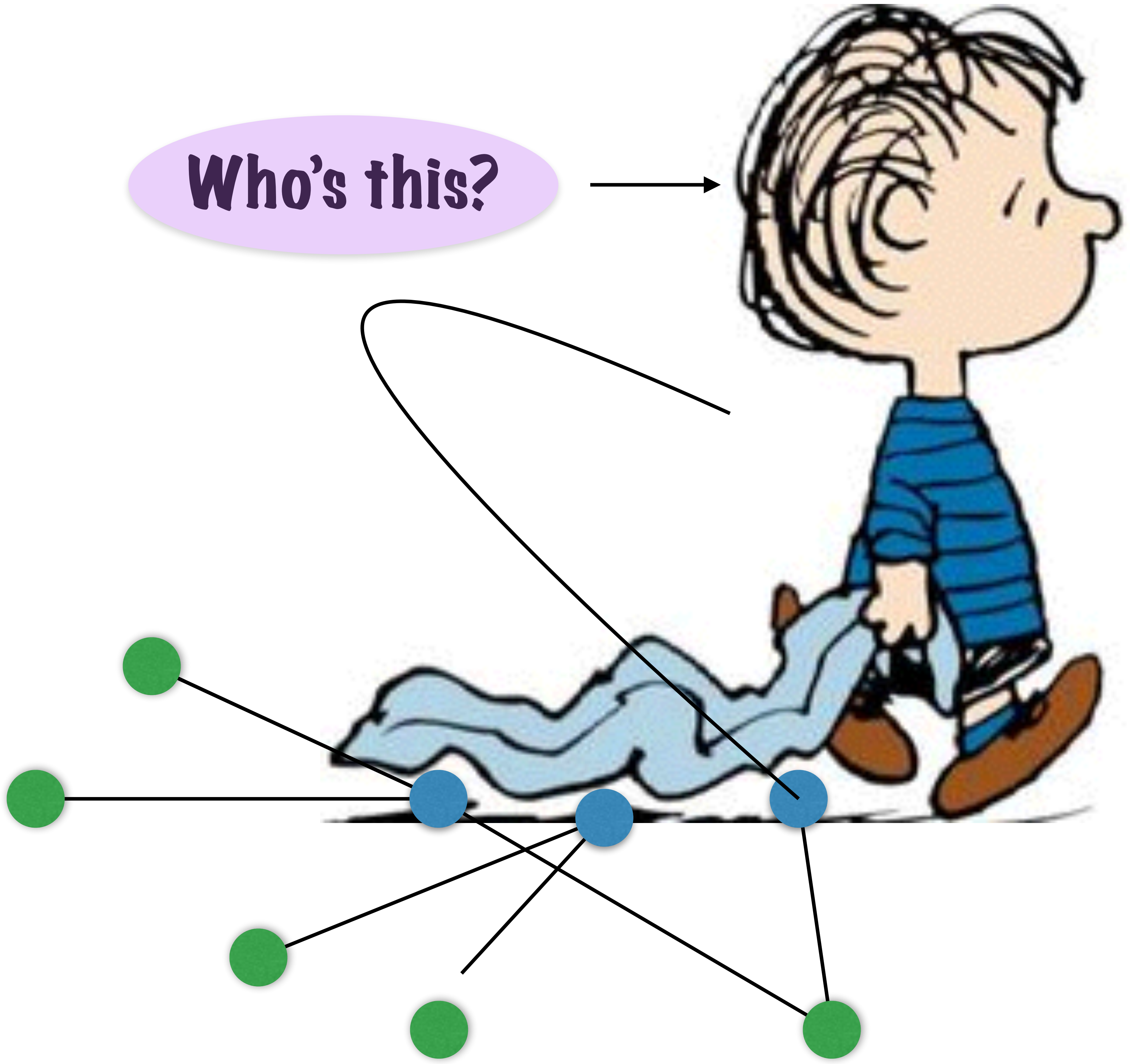
Approximation algorithms, vertex cover, and linear programming

Who's this?



$$\begin{aligned}
&\min c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
&\text{ such that} \\
&\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\
\cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\
\forall i: \quad 0 \leq x_i \leq 1 & \\
\forall i: \quad x_i \ \underline{\text{real number}} &
\end{cases}
\end{aligned}$$

What's that?

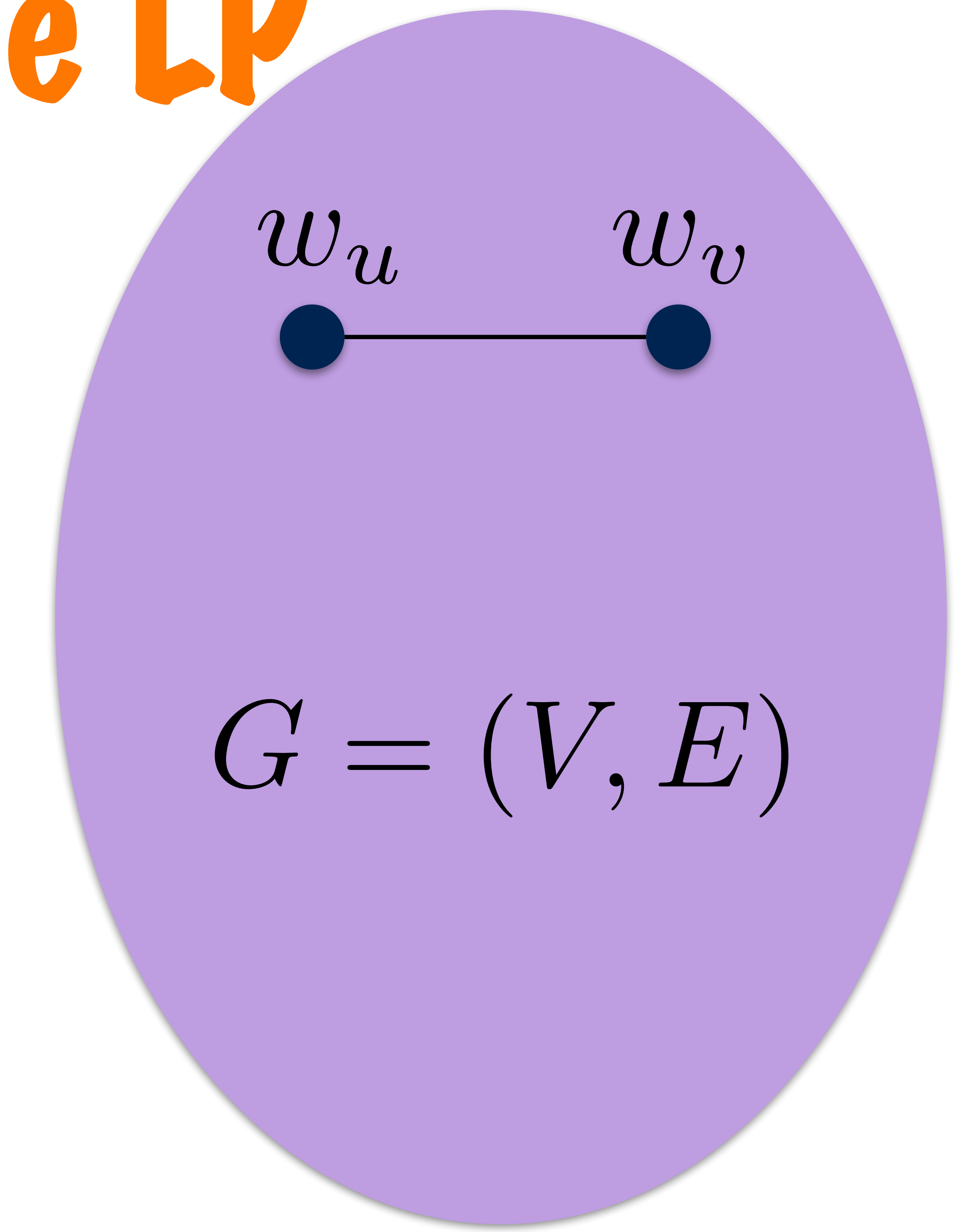
Property of the LP

Constraints:

$$\forall u \in V : 0 \leq x_u \leq 1$$

$$\forall \{u, v\} \in E : x_u + x_v \geq 1$$

Objective: $\min \sum_u w_u x_u$

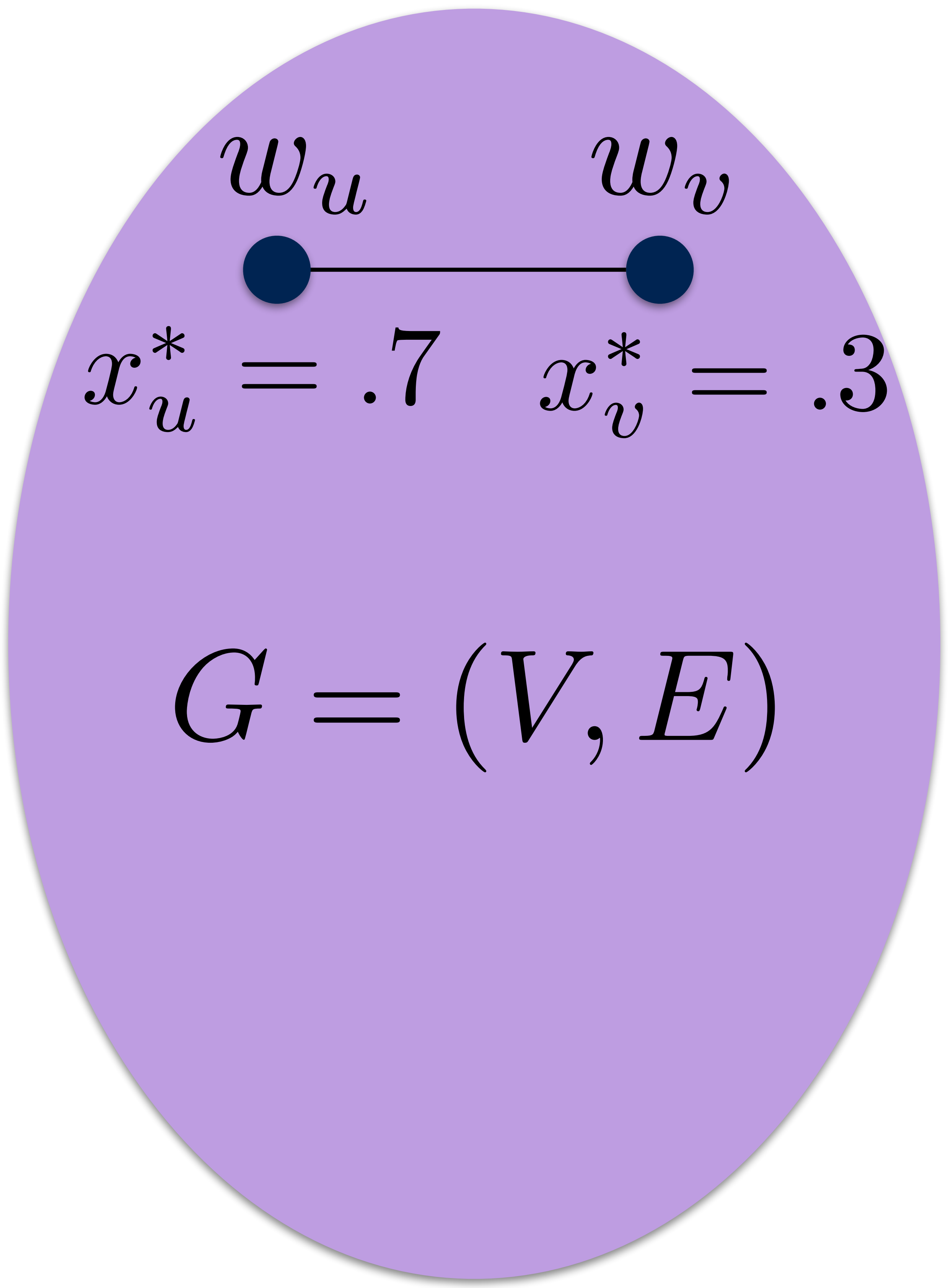


Theorem:

there exists an optimal solution s.t. every coordinate is in {0, .5, 1} and there is a polynomial-time algorithm to construct it

1. Solve the LP

$$\implies (x_u^*)_{u \in V} \text{ such that}$$
$$\forall u \in V : 0 \leq x_u^* \leq 1$$
$$\forall \{u, v\} \in E : x_u^* + x_v^* \geq 1$$
$$\sum_u w_u x_u^* \text{ minimum}$$



2. Freeze all variables with value in {0, .5, 1}

3. While some variables are not frozen

$$L = \{u : .5 < x_u^* < 1\}$$

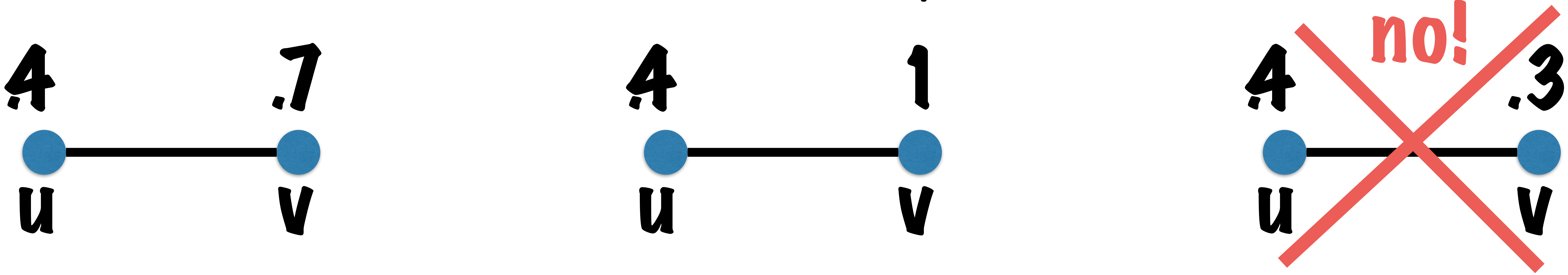
$$S = \{u : 0 < x_u^* < .5\}$$

Observe:

if u is in S and uv is an edge

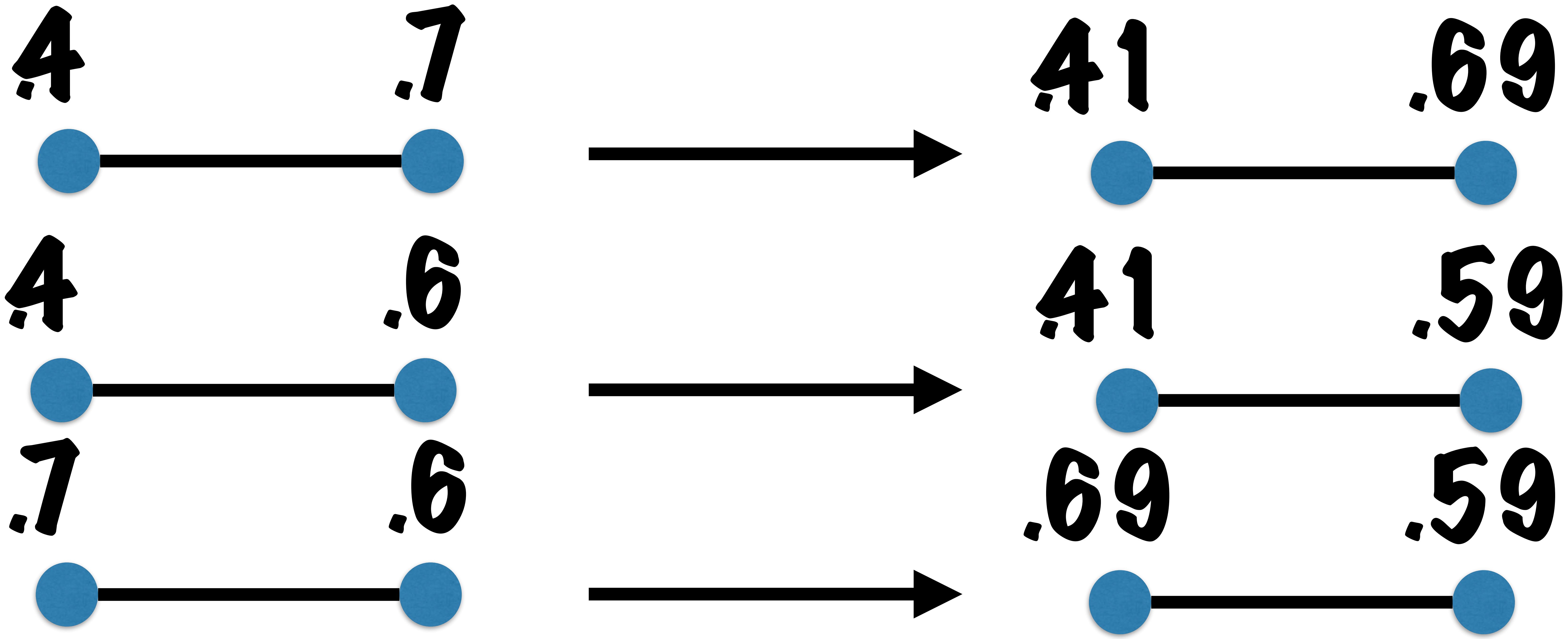
then

v is in L or $x_v^* = 1$



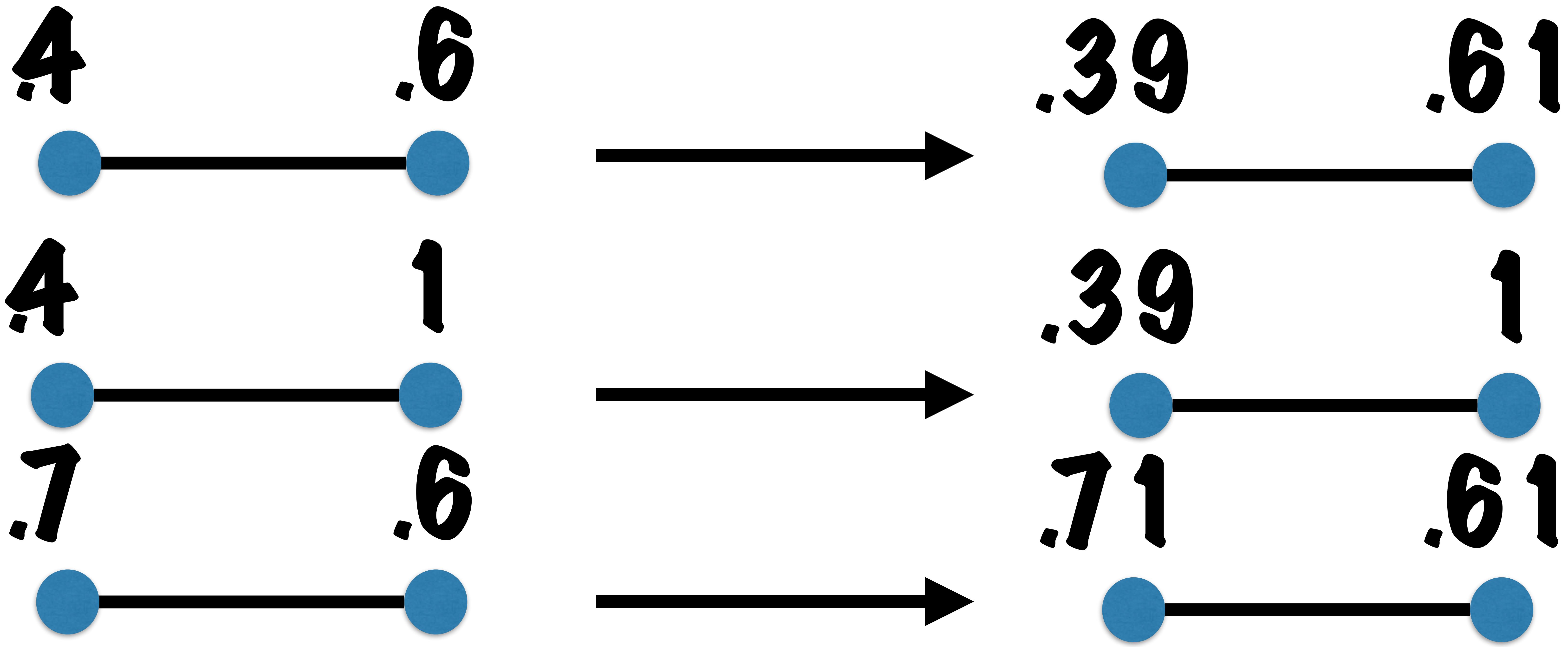
$$\mathbf{y_u} = \begin{cases} x_u^* & \text{if } u \text{ frozen} \\ x_u^* + \epsilon & \text{if } u \in S \\ x_u^* - \epsilon & \text{if } u \in L \end{cases}$$

Observe: for ϵ small, it is still feasible.



$$\mathbf{z_u} = \begin{cases} x_u^* & \text{if } u \text{ frozen} \\ x_u^* - \epsilon & \text{if } u \in S \\ x_u^* + \epsilon & \text{if } u \in L \end{cases}$$

Observe: for ϵ small, it is still feasible.



$$\mathbf{y_u} = \begin{cases} x_u^* & \textbf{if } u \textbf{ frozen} \\ x_u^* + \epsilon & \textbf{if } u \in S \\ x_u^* - \epsilon & \textbf{if } u \in L \end{cases} \qquad \mathbf{z_u} = \begin{cases} x_u^* & \textbf{if } u \textbf{ frozen} \\ x_u^* - \epsilon & \textbf{if } u \in S \\ x_u^* + \epsilon & \textbf{if } u \in L \end{cases}$$

Since y feasible and x* optimal: $\sum \mathbf{w_u y_u} \geq \sum \mathbf{w_u x_u^*}$

Since z feasible and x* optimal: $\sum \mathbf{w_u z_u} \geq \sum \mathbf{w_u x_u^*}$

But observe: $(\sum \mathbf{w_u y_u} + \sum \mathbf{w_u z_u})/2 = \sum \mathbf{w_u x_u^*}$

So: $\sum_\mathbf{u} \mathbf{w_u y_u} = \sum_\mathbf{u} \mathbf{w_u z_u} = \sum_\mathbf{u} \mathbf{x_u^*}$

y and z are also optimal solutions

increase ϵ until something happens:

$$\mathbf{y_u} = \begin{cases} x_u^* & \textbf{if } u \textbf{ frozen} \\ x_u^* + \epsilon & \textbf{if } u \in S \\ x_u^* - \epsilon & \textbf{if } u \in L \end{cases} \qquad \mathbf{z_u} = \begin{cases} x_u^* & \textbf{if } u \textbf{ frozen} \\ x_u^* - \epsilon & \textbf{if } u \in S \\ x_u^* + \epsilon & \textbf{if } u \in L \end{cases}$$

reaches .5 reaches .5 reaches 0 reaches 1

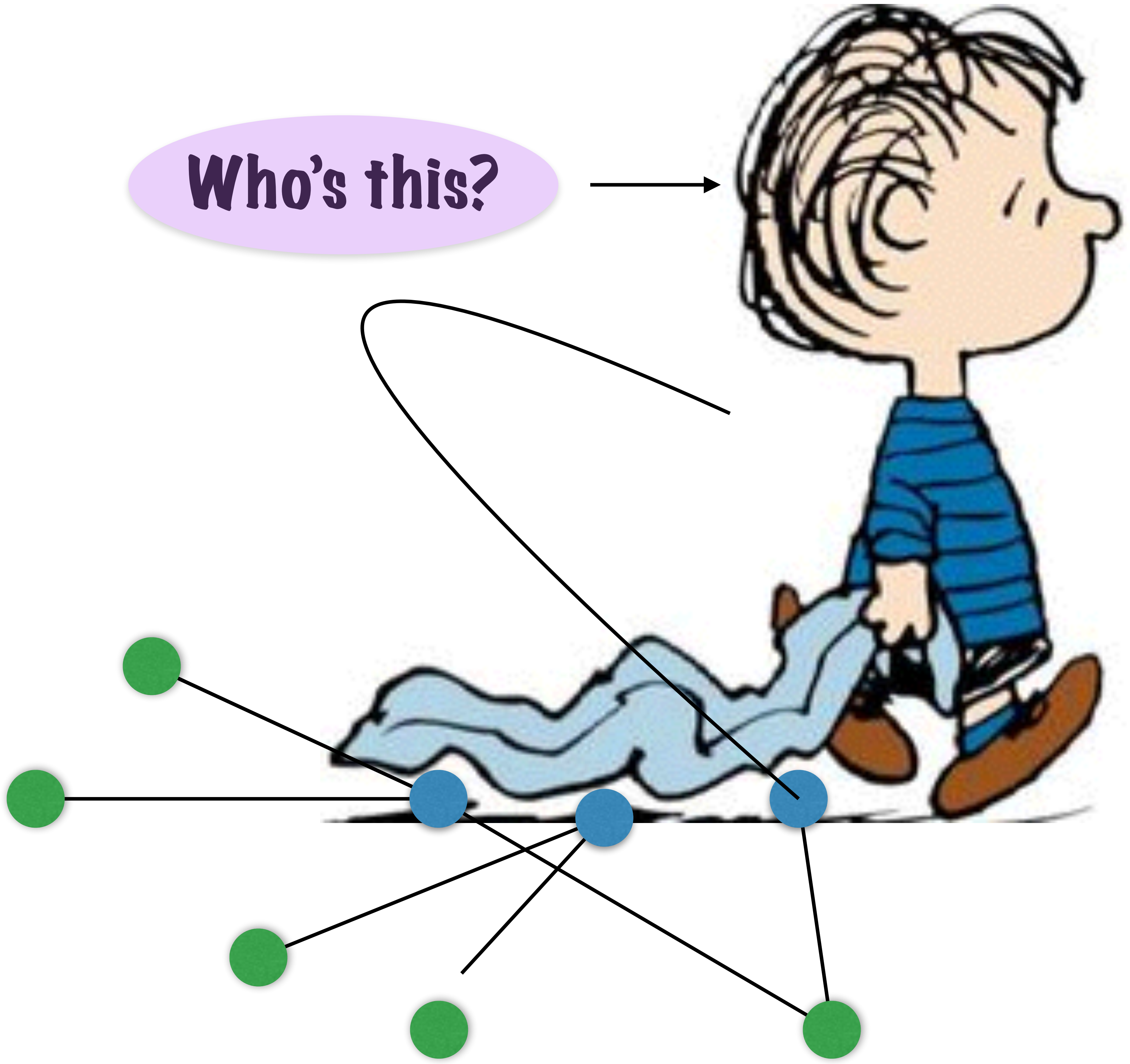
Freeze the variable that reached 0, .5, or 1

$$\mathbf{x}^* \leftarrow \mathbf{y} \text{ or } \mathbf{z}$$

Repeat...

QED

Approximation algorithms, vertex cover, and linear programming



$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

such that

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

What's that?