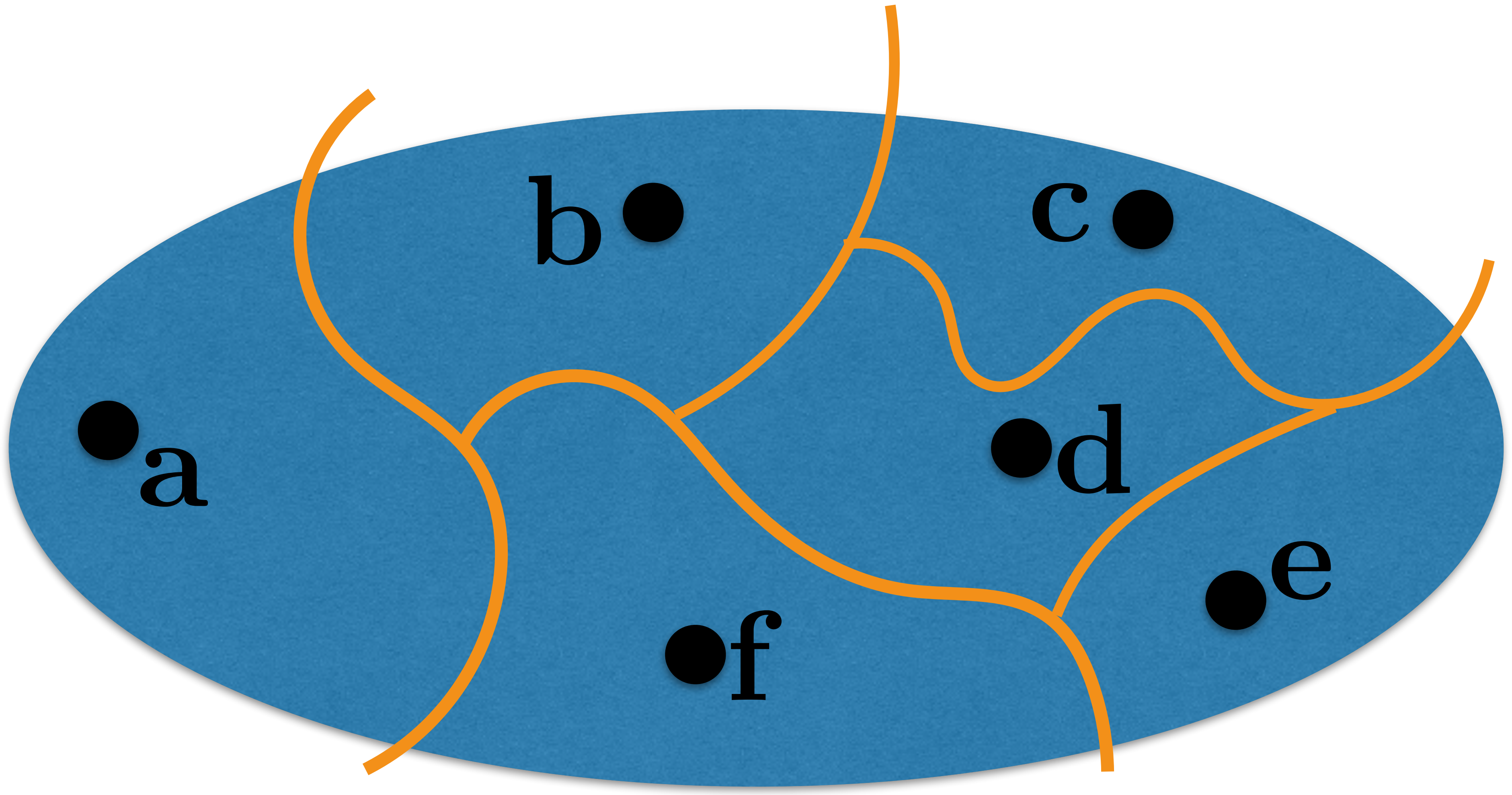
Multiway cut, linear programming and randomized rounding



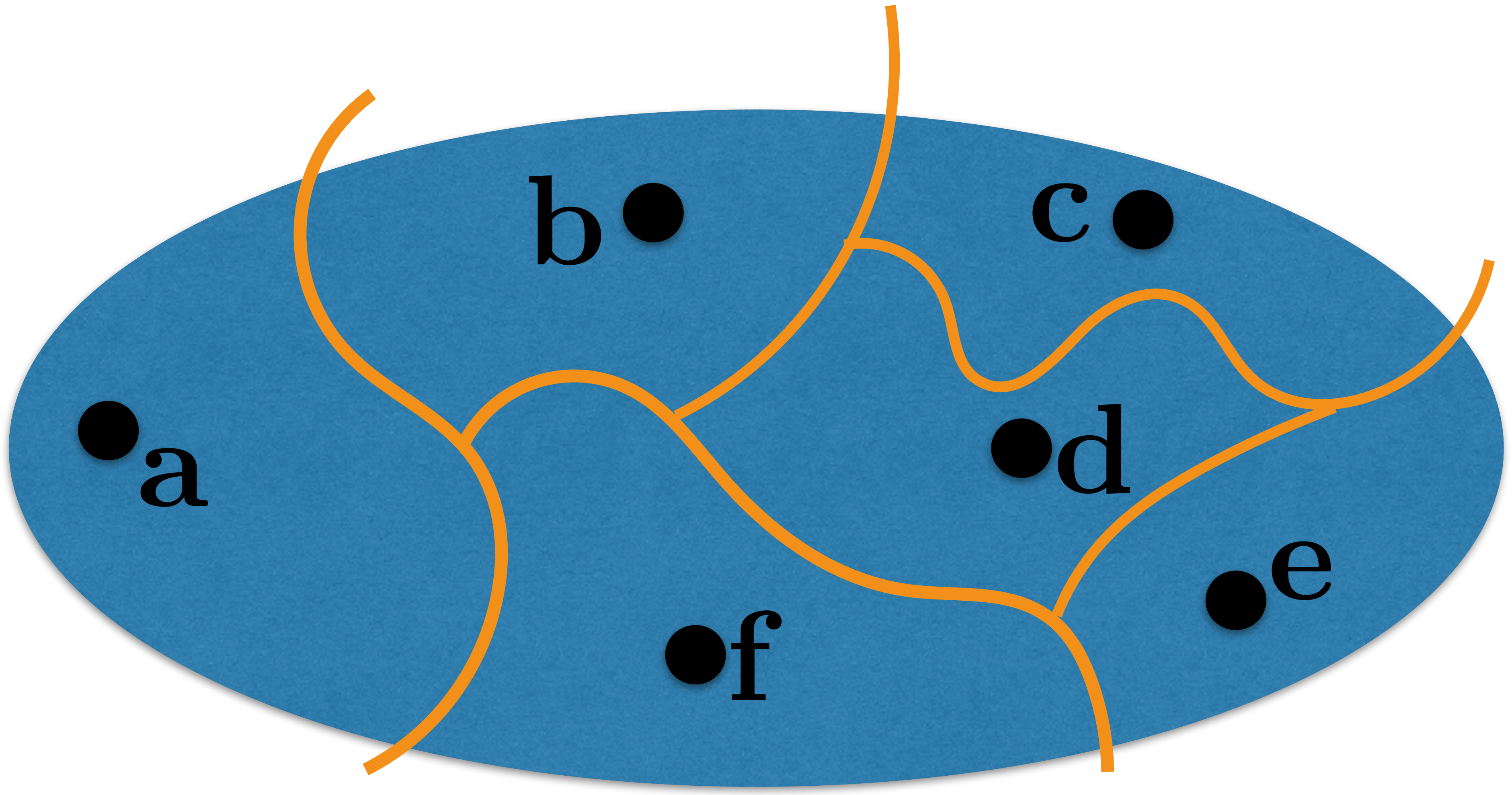
Problem

Let k be a fixed integer.

Given: graph G with edge weights,
k vertices called "terminals"

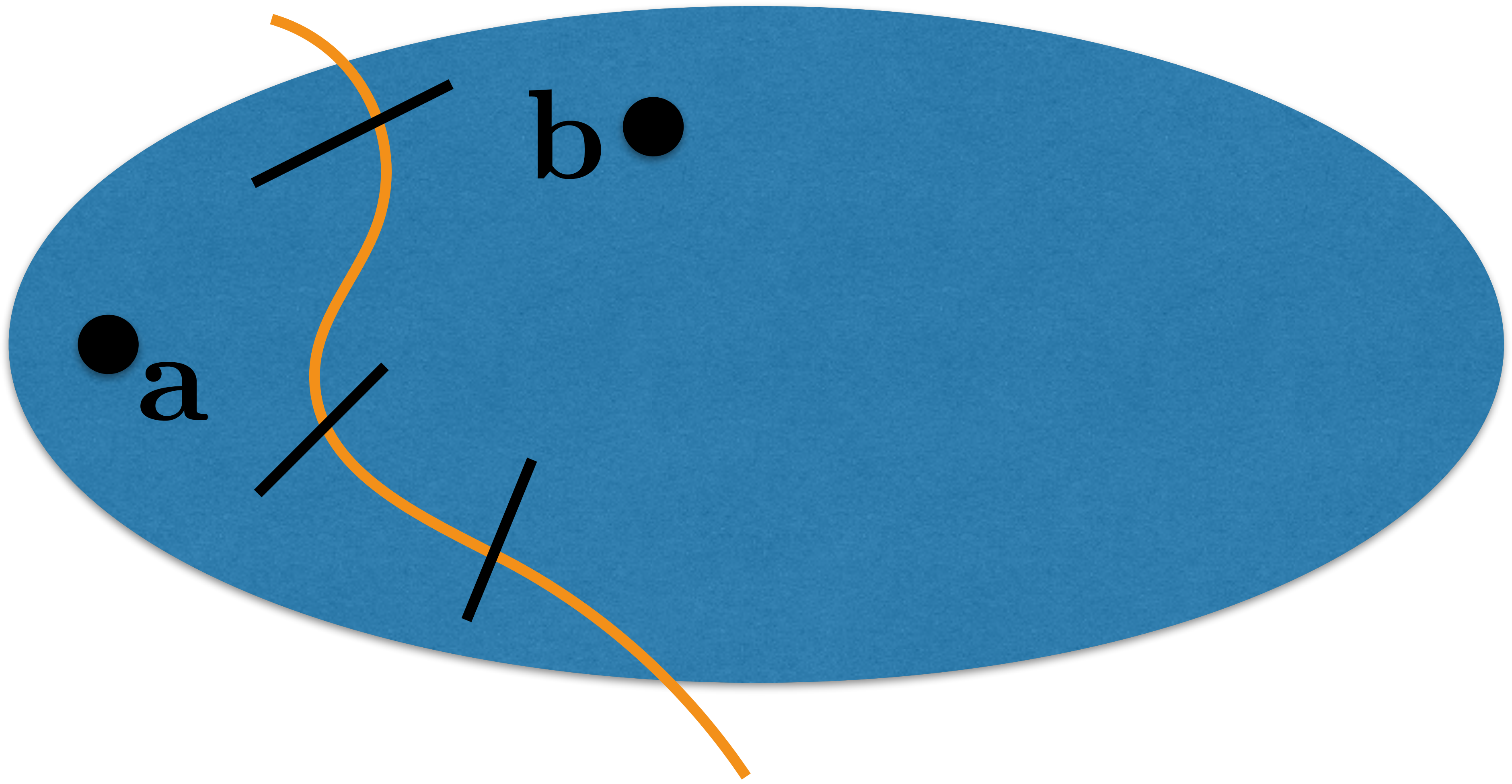
Find: subset F of edges such that Removing F disconnects the terminals from one another

Goal: minimize weight of F



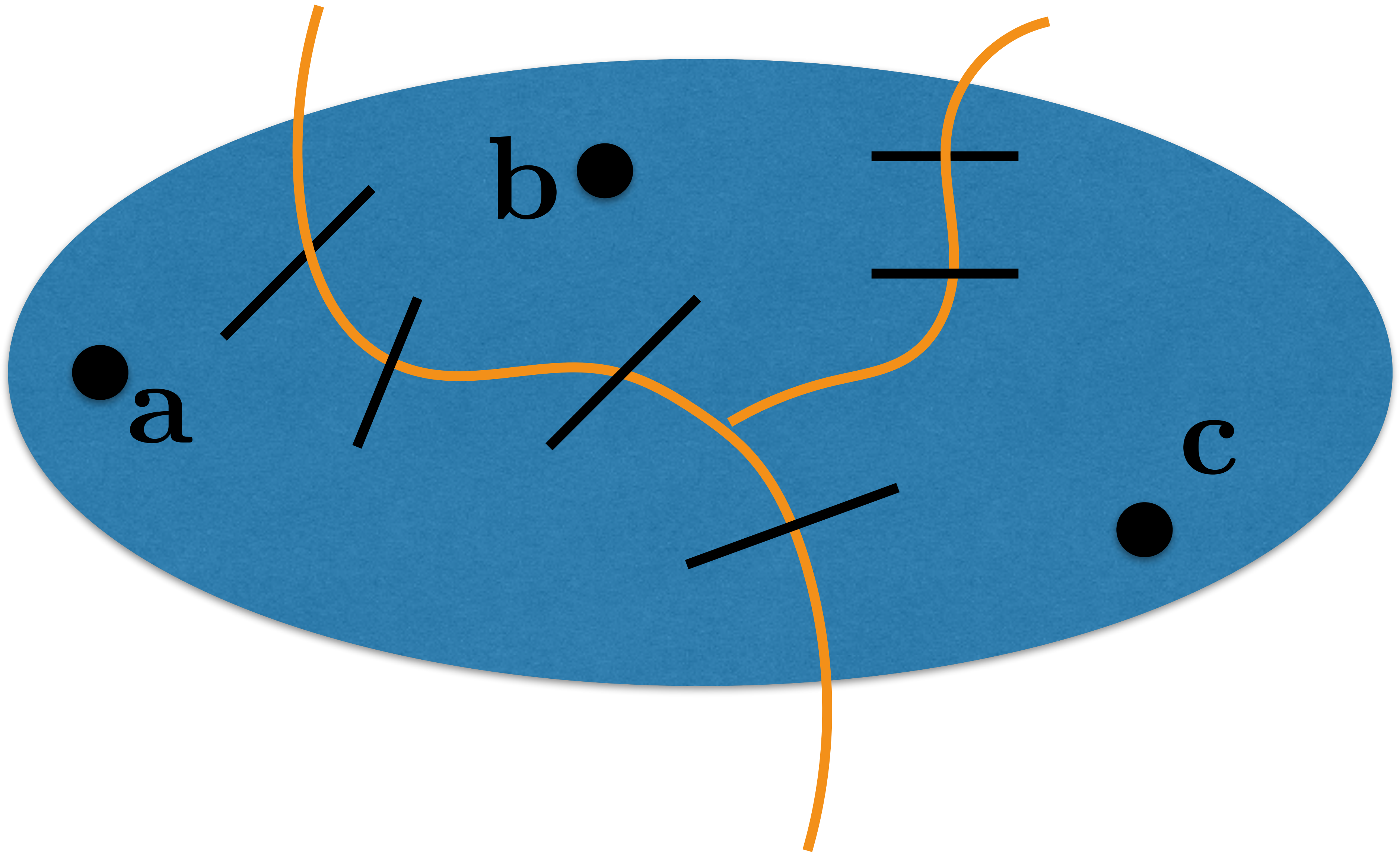
k=2

min cut in P



k=3

NP-hard



Simple algorithm for k=3

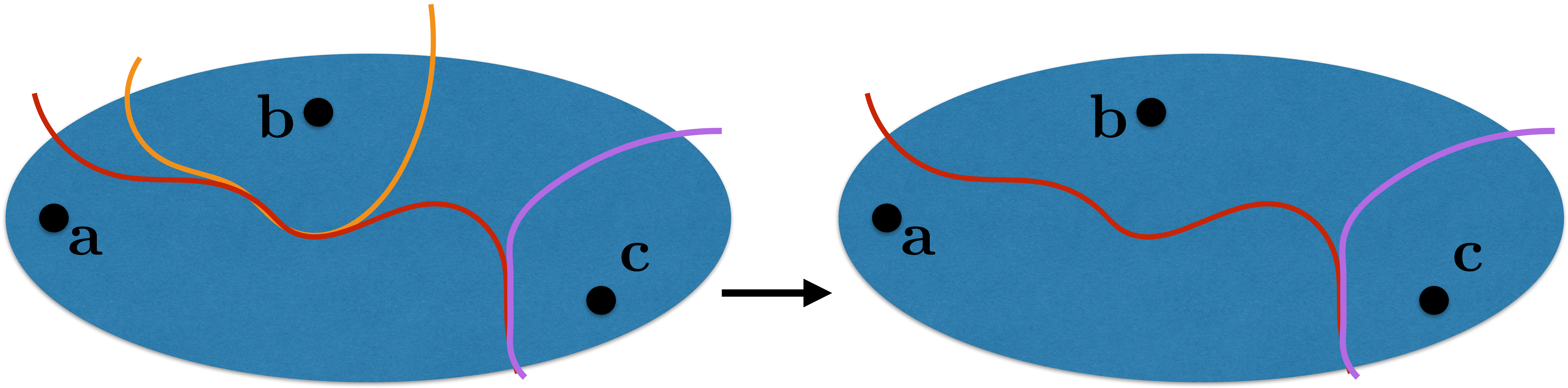
Terminals: a,b,c

Output the two smallest of

{Mincut(a,{b,c}),

Mincut(b,{c,a}),

Mincut(c,{a,b})}



Analysis

- It takes polynomial time…
- It's a correct multiway cut:
a,b,c are separated from one another
- But how good is it?

Cost of output

Output is at most

(2/3) (Mincut(a,bc)+Mincut(b,ac)+Mincut(c,ab))

- OPT is at least Mincut(a,bc)
- OPT is at least Mincut(b,ac)
- OPT is at least Mincut(c,ab)

So OPT is at least

(Mincut(a,bc)+Mincut(b,ac)+Mincut(c,ab))/3

Alg is a 2 approximation

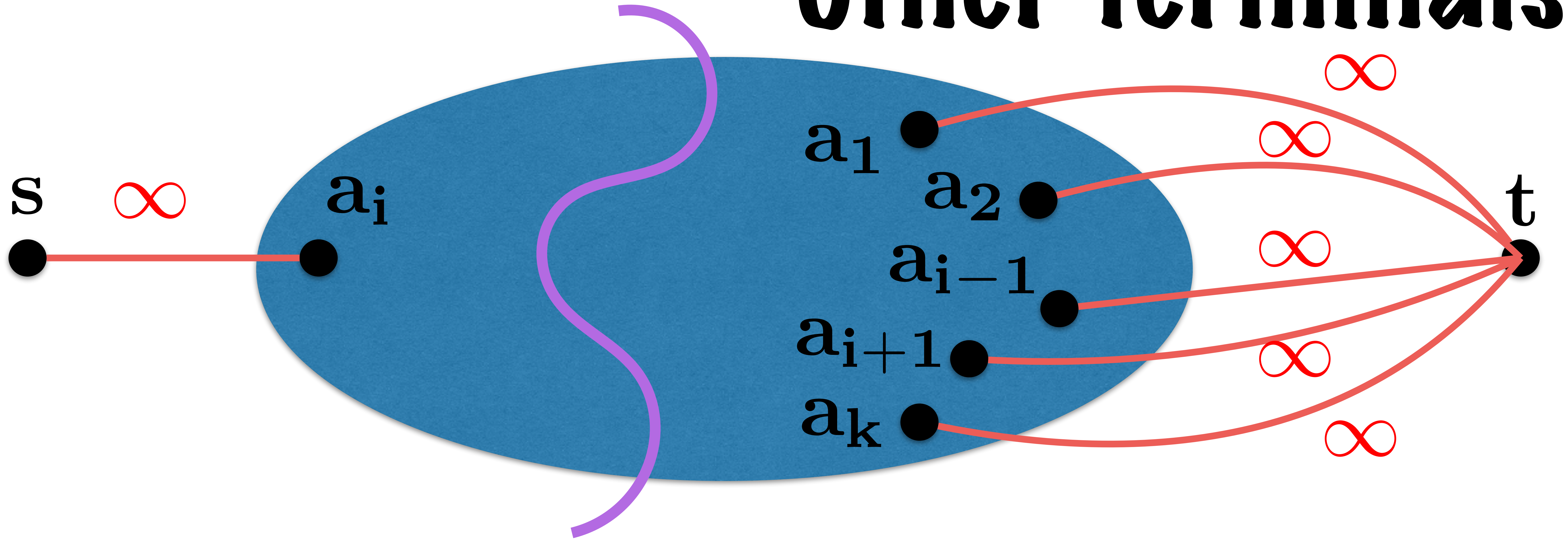
Extension: algorithm for k

Terminals: a1,a2,...,ak

Output the k-1 smallest of

{Mincut(ai,{a1,...,ai-1,ai+1,... ak}): i=1,2,...,k}

To compute the min cut separation of ai from the other terminals:



Mincut(s,t)

Analysis of output

The k-1 smallest cuts cost less than a random choice of k-1, so

$$\mathrm{Cost(Output)} \leq \frac{k-1}{k} \sum_i \mathrm{Mincut}(a_i, \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k\})$$

Analysis of OPT

Let F(i)={edges of OPT that separate ai from some aj}

Consider e in OPT

e separates some ai from some aj

e belongs to F(i) and to F(j)

so

$$\sum_i \mathrm{Cost}(F_i) = 2\mathrm{OPT}$$

F(i)={edges of OPT that separate ai from some aj}

F(i) separates ai from all other terminals so

$$\mathrm{Cost}(\mathrm{F_i}) \geq \mathrm{Mincut}(\mathrm{a_i}, \{\mathrm{a_1}, \ldots, \mathrm{a_{i-1}}, \mathrm{a_{i+1}}, \ldots, \mathrm{a_k}\})$$

Together

$$
\begin{aligned}
&\mathbf{Cost(Output)} \\
&\leq \tfrac{\mathbf{k-1}}{\mathbf{k}} \textstyle\sum_{\mathbf{i}} \mathbf{Cost(F_i)} \\
&\leq \mathbf{2(1-\tfrac{1}{k})OPT}
\end{aligned}
$$

k=2: Optimal
k=3: it's a 4/3 approximation

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