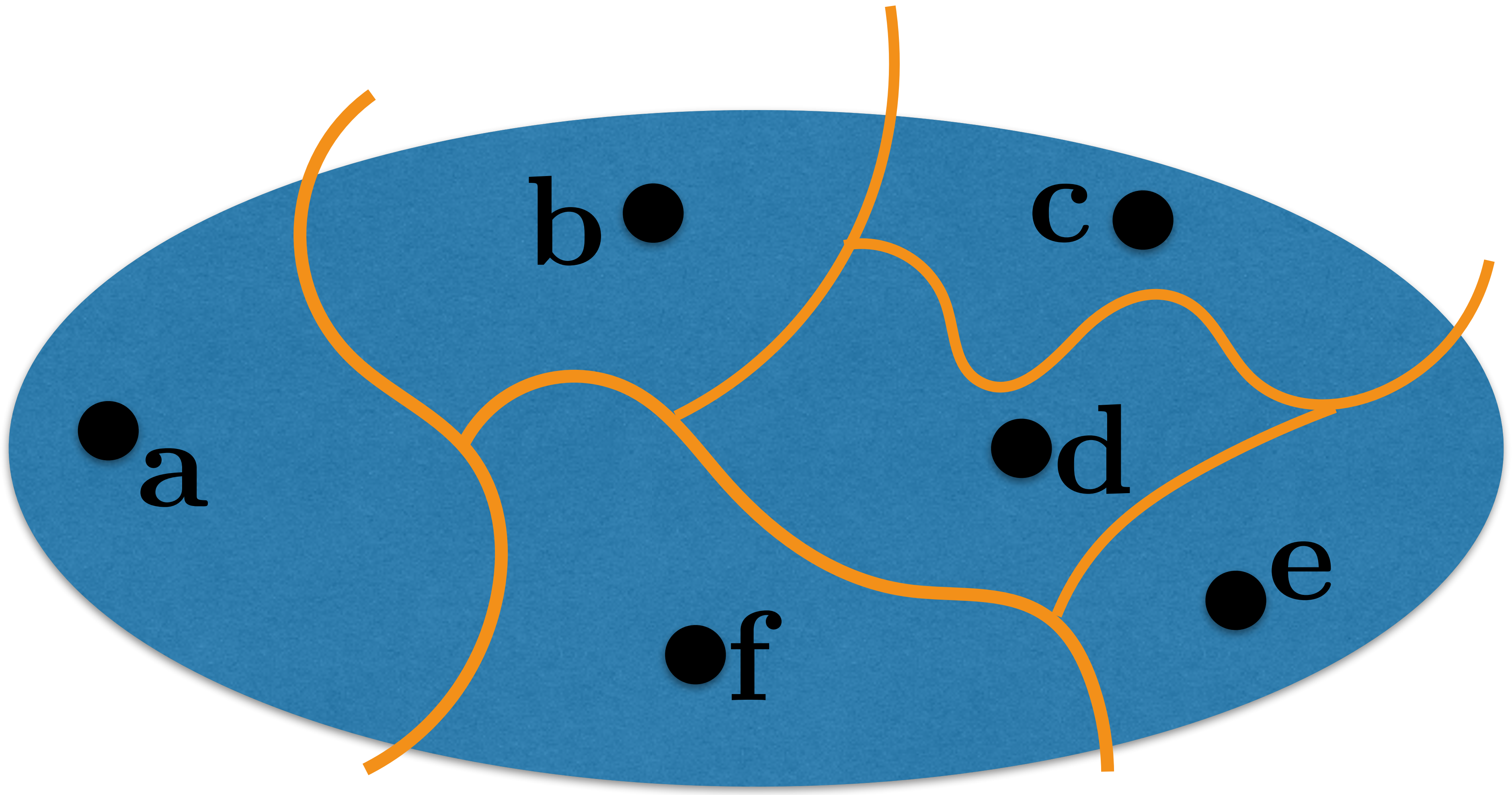
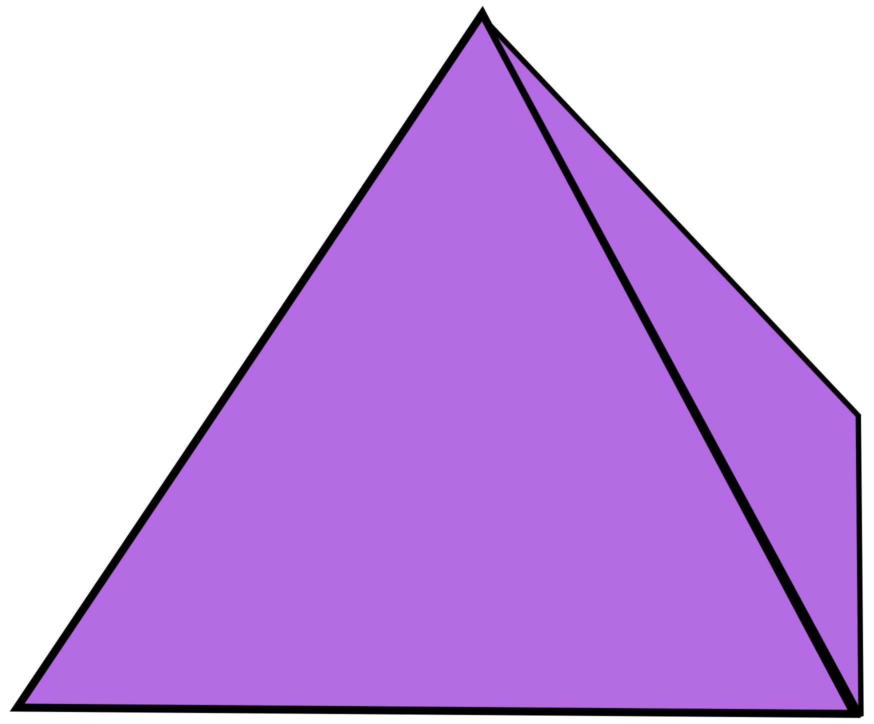
Multiway cut, linear programming and randomized rounding



Algorithm

$$\mathbf{d(u,v) = \frac{1}{2}|x_u - x_v|_1}$$

LP relaxation: embed vertices in k-simplex with terminals at corners



$$\min \sum_{\mathbf{uv}\in \mathbf{E}} \mathbf{c_{uv} d(u,v)} :$$

$$\mathbf{u \in \text{k-simplex} \quad \forall u}$$

$$\mathbf{x_i \geq 0, \quad \sum_i x_i = 1}$$

random ordering $\mathbf{a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(k)}}$ $\quad \mathbf{r \in [0,1]}$

$\mathbf{for\ i = 1, \ldots, k-1:}$

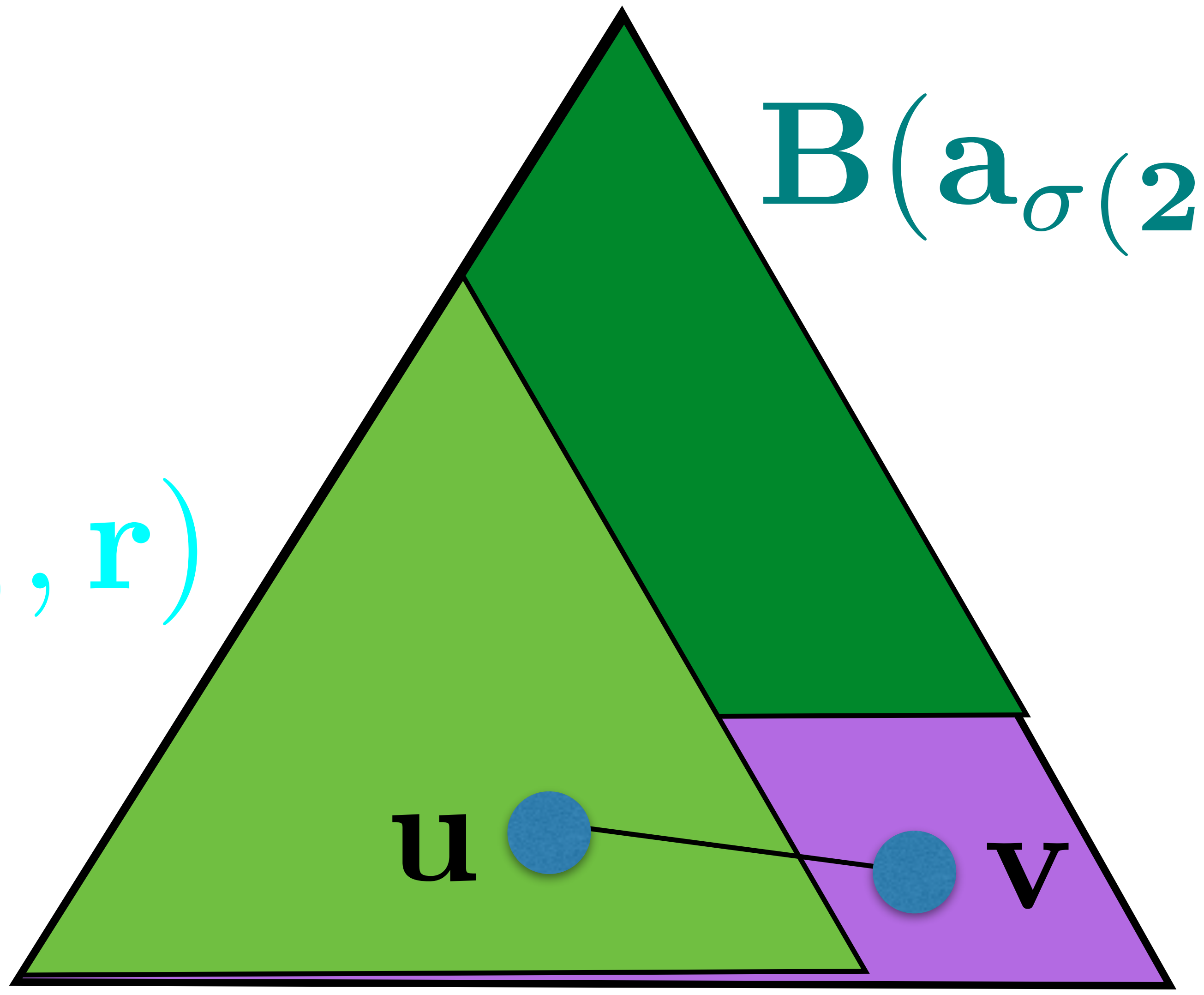
$\quad$ assign to $\mathbf{a_{\sigma(i)}}$ unassigned vertices of $\mathbf{B(a_{\sigma(i)}, r)}$

assign rest to $\mathbf{a_{\sigma(k)}}$

Analysis

$$\mathbf{E}[\mathbf{Output}] = \sum_{\mathbf{uv} \in \mathbf{E}} \mathbf{c_{uv}} \Pr(\mathbf{uv} \in \ \mathbf{Output})$$

$$\mathbf{a} = \mathbf{a}_\sigma(\mathbf{1}), \mathbf{b} = \mathbf{a}_{\sigma(\mathbf{2})}, \mathbf{c} = \mathbf{a}_{\sigma(\mathbf{3})}$$

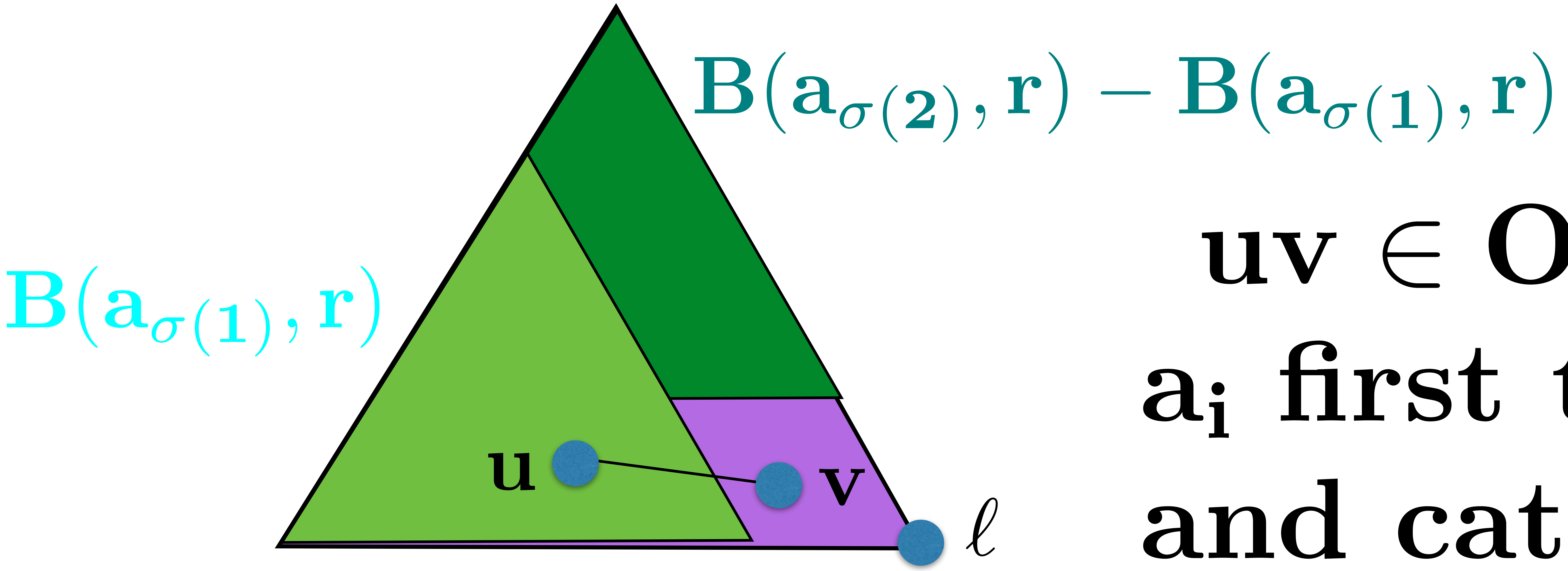


$\mathbf{uv} \in \ \mathbf{Output\ \ iff:}$

$$\min(\mathbf{d}(\mathbf{a}, \mathbf{u}), \mathbf{d}(\mathbf{a}, \mathbf{v})) < \mathbf{r} < \max(\mathbf{d}(\mathbf{a}, \mathbf{u}), \mathbf{d}(\mathbf{a}, \mathbf{v})) \ \ \mathbf{or}$$

$$[\mathbf{r} < \min(\mathbf{d}(\mathbf{a}, \mathbf{u}), \mathbf{d}(\mathbf{a}, \mathbf{v})) \ \mathbf{and}$$

$$\min(\mathbf{d}(\mathbf{b}, \mathbf{u}), \mathbf{d}(\mathbf{b}, \mathbf{v})) < \mathbf{r} < \max(\mathbf{d}(\mathbf{b}, \mathbf{u}), \mathbf{d}(\mathbf{b}, \mathbf{v}))]$$



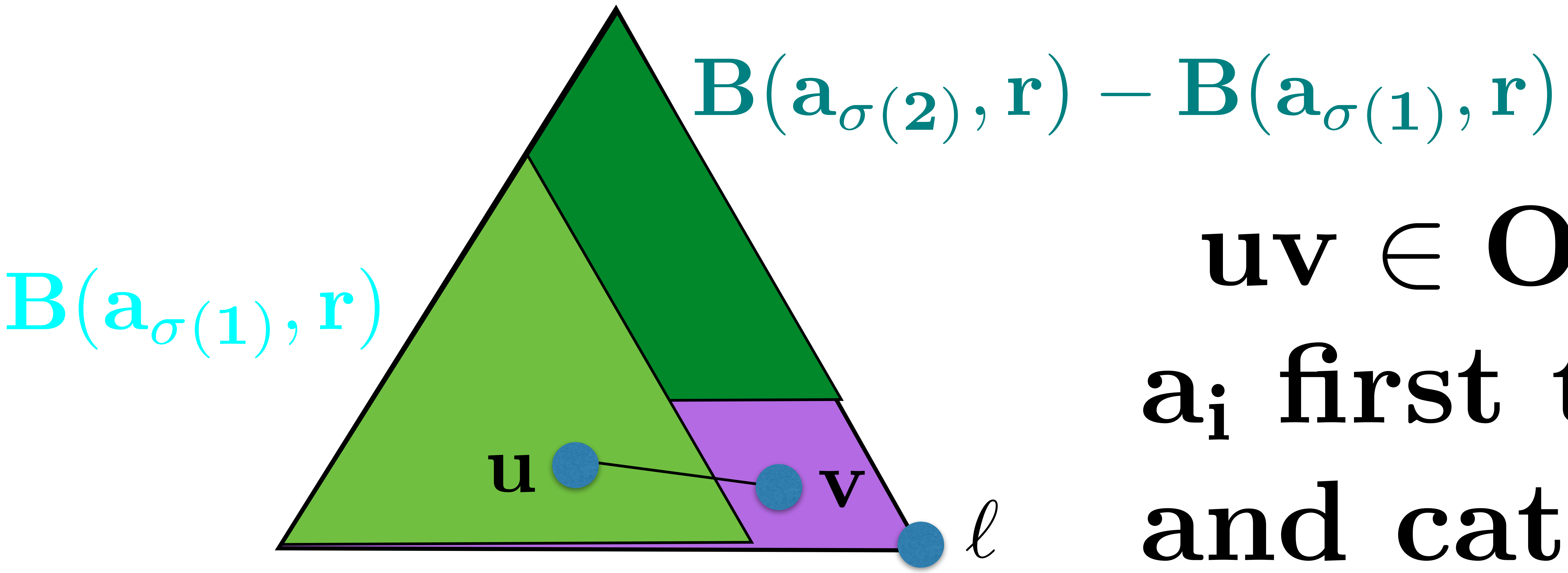
$\mathbf{uv \in Output:}$

$\mathbf{a_i}$ **first to catch u or v and catches exactly one**

$$\ell = \mathbf{arg}\min\{\min(\mathbf{d(a_i, u), d(a_i, v)})\}$$

$\mathbf{i} \neq \ell :$ **i precedes** ℓ **and** $\mathbf{B(a_i, r)}$ **separates them**

$$\mathbf{\frac{1}{2}|d(a_i, u) - d(a_i, v)|}$$



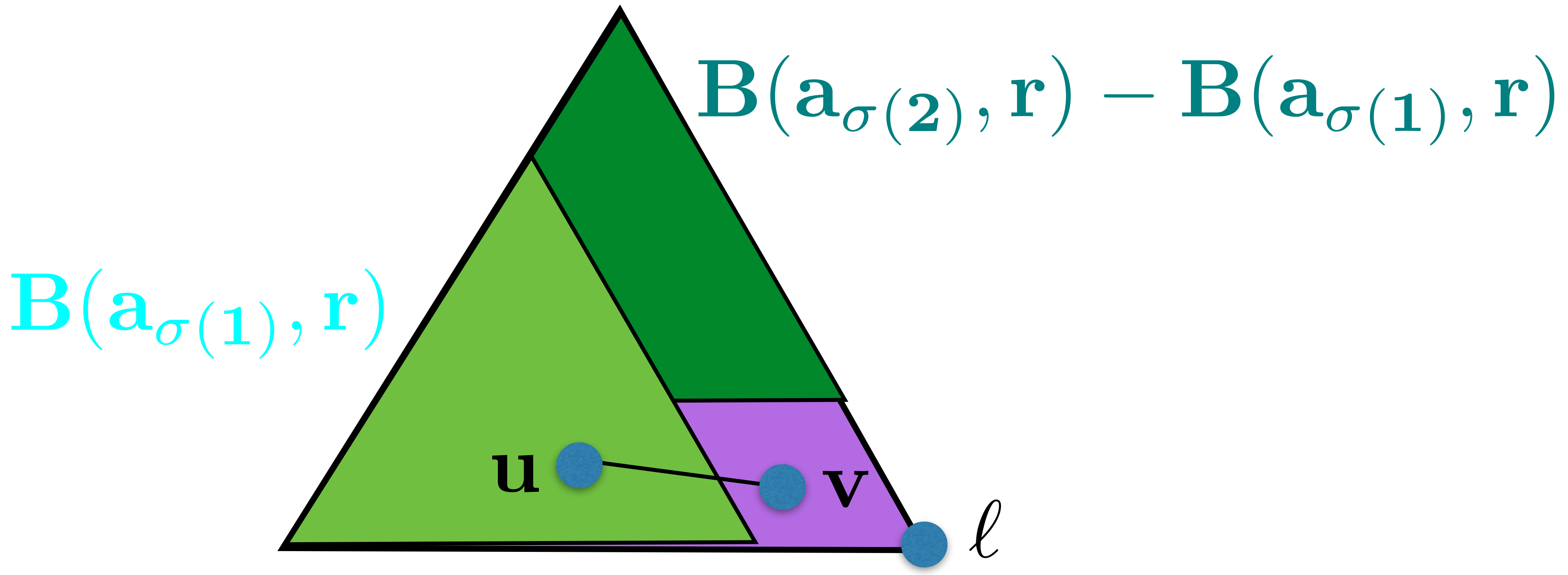
$\mathbf{uv} \in$ Output:

$\mathbf{a_i}$ first to catch u or v and catches exactly one

$$\ell = \mathbf{arg}\min\{\min(\mathbf{d}(\mathbf{a_i}, \mathbf{u}), \mathbf{d}(\mathbf{a_i}, \mathbf{v}))\}$$

$$\mathbf{i} = \ell : \ \ell \text{ \textbf{not last and} } \mathbf{B}(\mathbf{a}_\ell, \mathbf{r}) \text{ \textbf{separates them}}$$

$$(\mathbf{1} - \tfrac{\mathbf{1}}{\mathbf{k}})|\mathbf{d}(\mathbf{a}_\ell, \mathbf{u}) - \mathbf{d}(\mathbf{a}_\ell, \mathbf{v})|$$

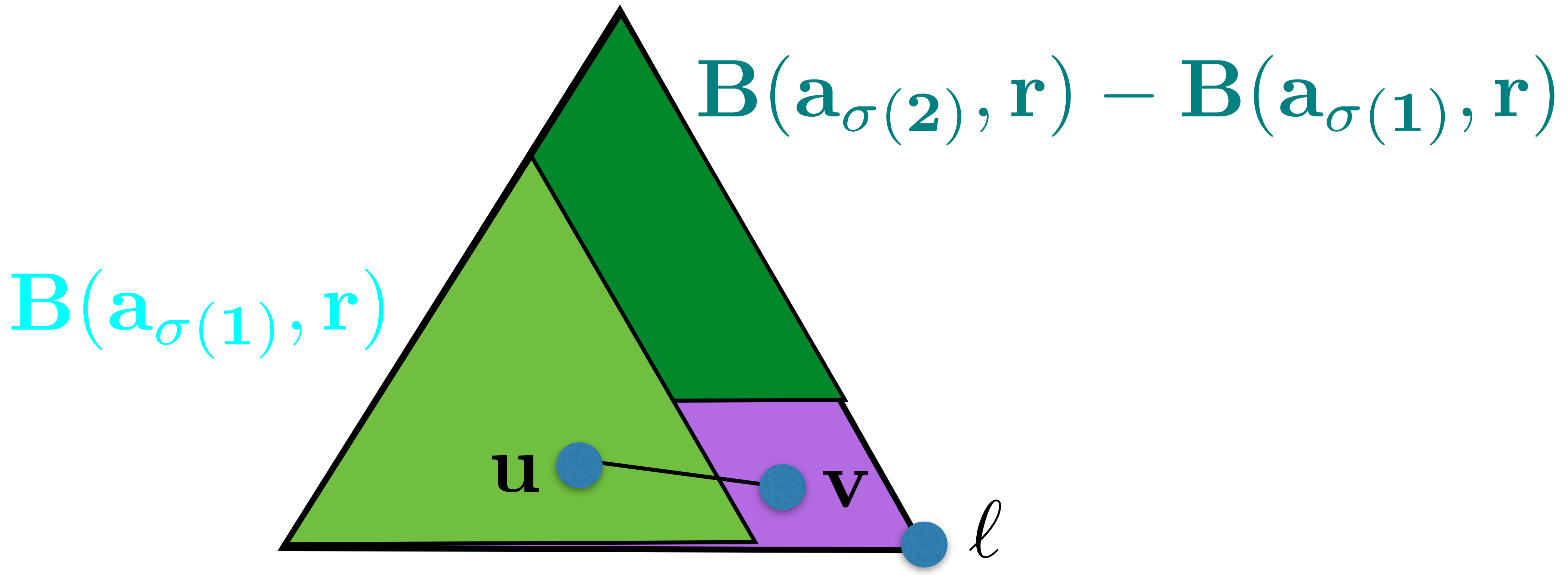


Together

$$\mathbf{(1 - \tfrac{1}{k})|d(a_\ell, u) - d(a_\ell, v)| + \textstyle\sum_{i \neq \ell} \tfrac{1}{2}|d(a_i, u) - d(a_i, v)|}$$

$$\mathbf{= \tfrac{1}{2}(1 - \tfrac{2}{k})|d(a_\ell, u) - d(a_\ell, v)| + \textstyle\sum_i \tfrac{1}{2}|d(a_i, u) - d(a_i, v)|}$$

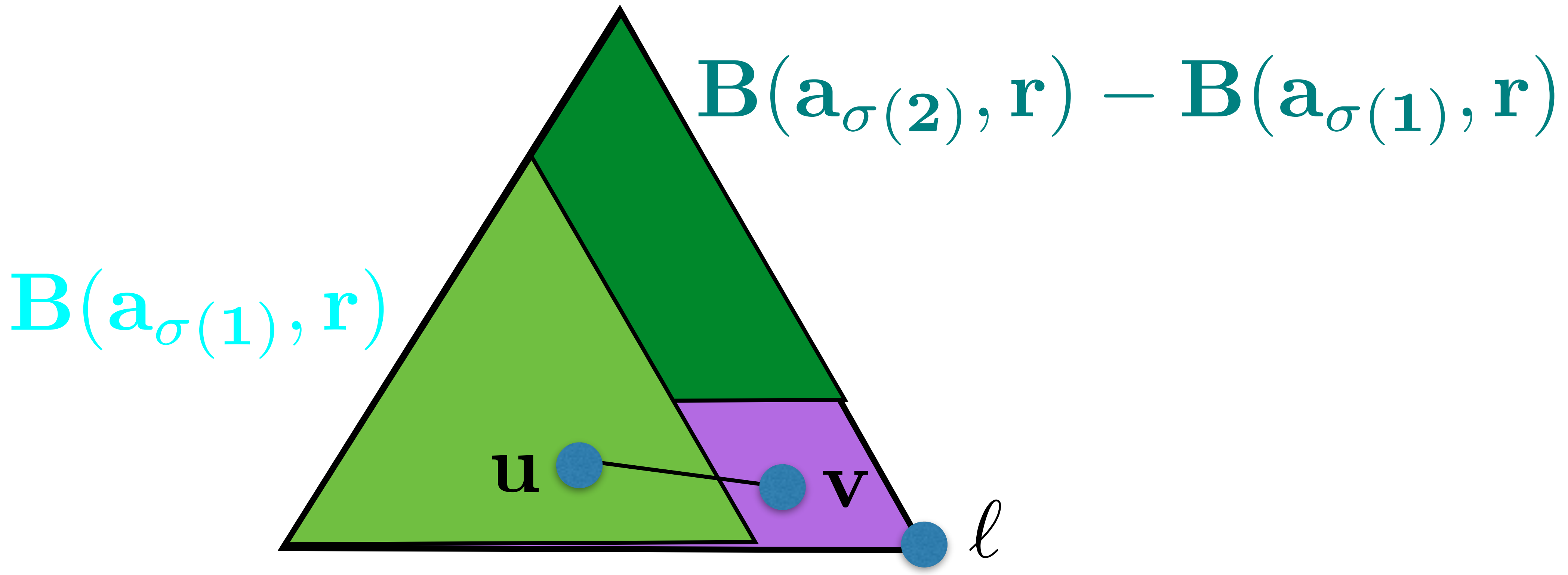
$$\mathbf{= \tfrac{1}{2}(1 - \tfrac{2}{k})|u_\ell - v_\ell| + \textstyle\sum_i \tfrac{1}{2}|u_i - v_i|}$$



$$\sum_{\mathbf{u}} \mathbf{u_i} = \sum_{\mathbf{i}} \mathbf{v_i} = \mathbf{1}$$

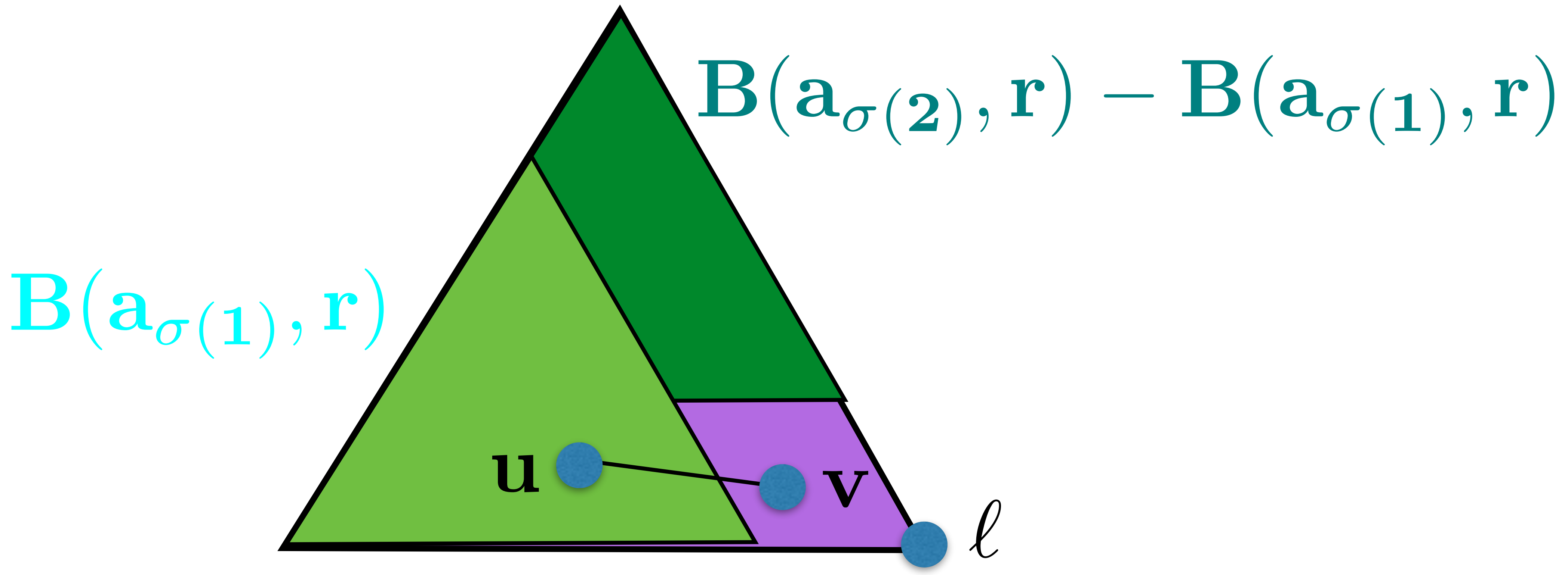
so

$$|\mathbf{u}_\ell - \mathbf{v}_\ell| \leq \frac{\mathbf{1}}{\mathbf{2}} \sum_{\mathbf{i}} |\mathbf{u_i} - \mathbf{v_i}|$$



Together

$$\left(\frac{1}{2} - \frac{1}{k}\right)|u_\ell - v_\ell| + \sum_i \frac{1}{2}|u_i - v_i|$$
$$\leq \sum_i \left(\frac{3}{4} - \frac{1}{2k}\right)|u_i - v_i|$$
$$= \left(\frac{3}{2} - \frac{1}{k}\right) \cdot \frac{1}{2}|u - v|_1$$

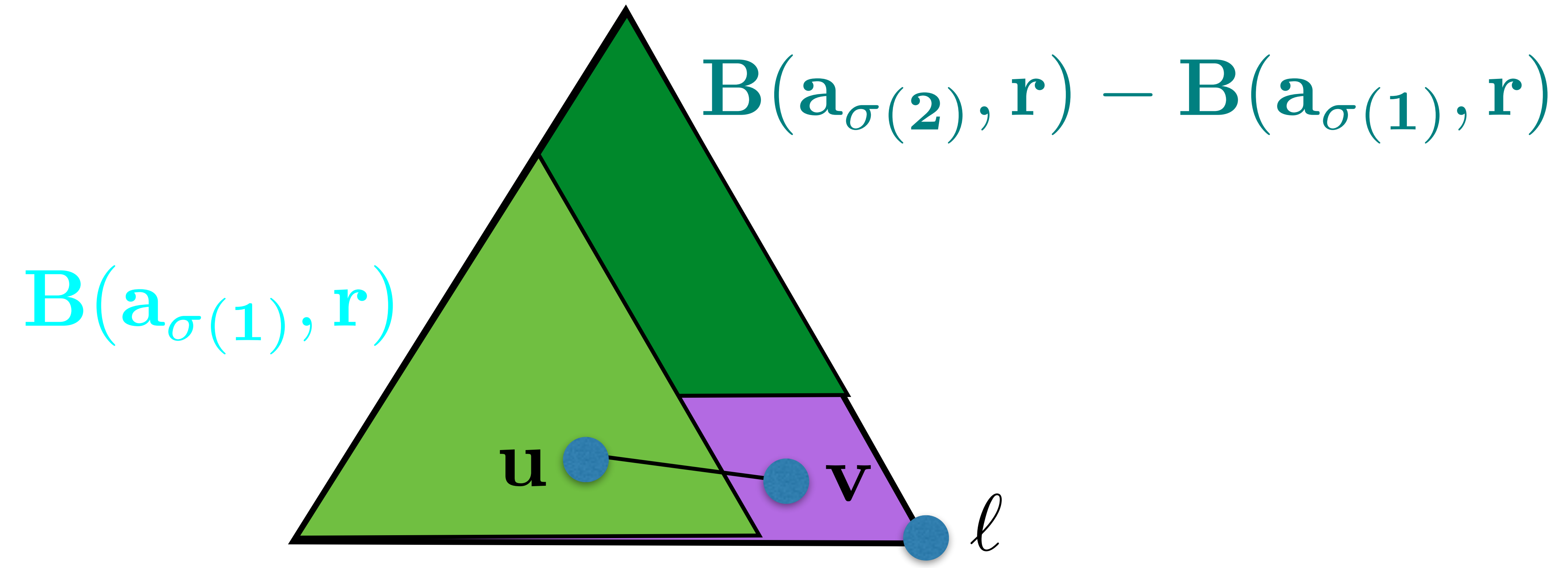


Sum

$$\mathbf{E[Output]} = \sum_{\mathbf{uv \in E}} \mathbf{c_{uv}} \Pr(\mathbf{uv} \in \mathbf{Output})$$

$$\leq \sum_{\mathbf{uv \in E}} \mathbf{c_{uv}} \left(\frac{\mathbf{3}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{k}}\right) \cdot \frac{\mathbf{1}}{\mathbf{2}} |\mathbf{u} - \mathbf{v}|_{\mathbf{1}}$$

$$\leq \left(\frac{\mathbf{3}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{k}}\right) \cdot \mathbf{OPT}$$



Linear programming and randomized rounding give a 3/2 - 1/k approximation for multicut

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