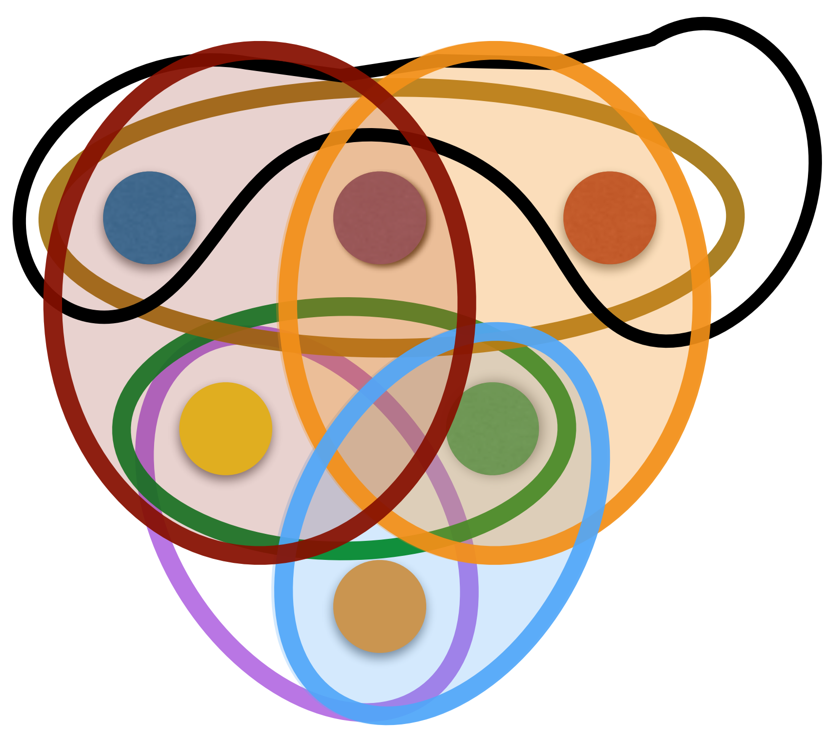
Set cover, linear programming and randomized rounding



Is the output a cover?

Maybe, maybe not

Number of elements covered:

$$\sum_{\mathbf{e}} 1(\mathrm{e\ covered})$$

On average:

$$\sum_{\mathbf{e}} \Pr[\mathrm{e\ covered}]$$

Consider an element e.
With what probability
is it covered by output?

$$\Pr[\textbf{e covered}] = \Pr[\textbf{there exists } S \textbf{ in output: } \mathbf{e} \in \mathbf{S}]$$

$$\Pr[\textbf{there exists } S \textbf{ in output: } \mathbf{e} \in \mathbf{S}] = \mathbf{1} - \Pr[\textbf{for all } S \textbf{ containing } e\textbf{: } \mathbf{S} \textbf{ not in output}]$$

Independence

If independence:

$$\Pr[\mathbf{A\ and\ B}] = \Pr[\mathbf{A}] \times \Pr[\mathbf{B}]$$

$$\Pr[\textbf{for all } S \textbf{ containing } e\textbf{:}\ \textbf{S not in output}] = \prod_{\mathbf{S:e\in S}} \Pr[\textbf{S not in output}]$$

$$\Pr[\mathbf{S\ not\ in\ output}] = \mathbf{1 - x_S}$$

Together

$$\Pr[\mathbf{e\ covered}] = \mathbf{1} - \prod_{\mathbf{S:e\in S}} (\mathbf{1} - \mathbf{x_S})$$

Algebra

$$\mathbf{X} = \mathbf{e}^{\ln \mathbf{X}}$$

$$\ln(\mathbf{XY}) = \ln \mathbf{X} + \ln \mathbf{Y}$$

$$\prod_{\mathbf{S}:\mathbf{e}\in\mathbf{S}} (\mathbf{1} - \mathbf{x}_{\mathbf{S}}) = \mathbf{e}^{\sum_{\mathbf{S}:\mathbf{e}\in\mathbf{S}} \ln(\mathbf{1}-\mathbf{x}_{\mathbf{S}})}$$

Algebra

$$\ln(\mathbf{1} - \mathbf{X}) \leq -\mathbf{X}$$

$$\mathbf{e}^{\sum_{\mathbf{S}:\mathbf{e}\in\mathbf{S}} \ln(\mathbf{1}-\mathbf{x}_\mathbf{S})} \leq \mathbf{e}^{-\sum_{\mathbf{S}:\mathbf{e}\in\mathbf{S}} \mathbf{x}_\mathbf{S}}$$

Use LP constraint

$$\sum_{S:e\in S} x_S \geq 1$$

$$e^{-\sum_{S:e\in S} x_S} \leq e^{-1}$$

Combining

$$\Pr[\text{e covered}] \geq 1 - 1/e$$

Average number of elements covered:

$$\#(\text{elements})(1 - 1/e)$$

$$1 - 1/e = 0.63\ldots$$

Recap

Randomized rounding gives
collection of sets
with average cost
at most OPT
and covering on average
63% of the elements.

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