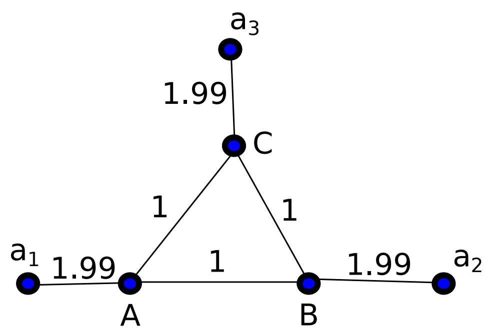
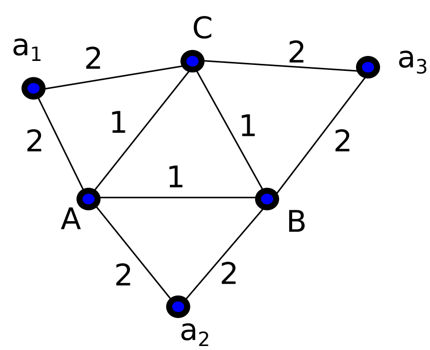
1. Consider the simple algorithm described at the first lecture of session 5. Consider the following graph, what is the cut output by the algorithm? What is the optimal cut? Conclude.



2. Consider the linear program relaxation described during the lectures. Consider the following graph, what is the value of the fractional optimal solution? What is the optimal cut? Conclude.



3. Consider the linear program for multiway cut described during the lectures and the distances defined by a fractional optimal solution. We describe a new rounding procedure:

   (a) Output $\leftarrow \emptyset$.

   (b) Pick at random $r \in [0, 1/2]$.

   (c) For each edge (u, v), add (u, v) to Output if there exists a terminal a_i such that $d(u, a_i) \leq r \leq d(v, a_i)$.

   Show that the cut that is output by the algorithm is of cost at most 2OPT.

4. We present a different relaxation for the multiway cut problem. Consider an undirected graph $G = (V, E)$ and a set of terminal $\{a_1, \ldots, a_k\}$. Compute the following directed graph $H = (V, E')$ where $E' = \{< u, v >| (u, v) \in E\}$. Namely for each edge $(u, v) \in E$, there are two arcs $< u, v >$ and $< v, u >$ in E'. Additionally, each arc $< u, v >$ has the same weight

$w_{(u,v)}$ as the edge (u, v). Let $\mathcal{P}$ be the collection of all the directed paths starting at a terminal a_i and ending at a terminal a_j with $i < j$.

We now define the linear program. Each variable d_e correspond to an arc of E'.

$$\min \sum_{e \in E'} d_e w_e$$

subject to:

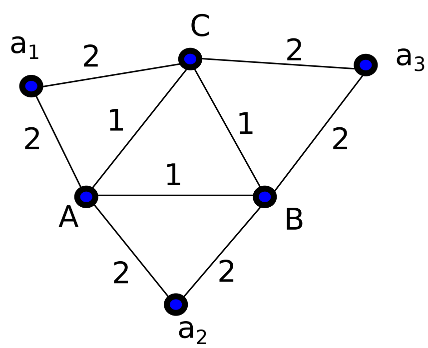
$$\sum_{e \in P} d_e \geq 1, \forall P \in \mathcal{P}$$

$$d_e \in \{0, 1\}$$

(a) Argue that any (integral) solution to this linear program can be converted into a solution to the multiway cut problem of same cost.

(b) Argue that any multiway cut of G can be converted into a solution to this linear program of same cost.

(c) Consider the following graph. What is the value of the optimal fractional solution on this graph? What is the value of the optimal multiway cut? Conclude.



5. We define a slightly different way of rounding the fractional optimal solution of the LP described during the lecture.

   (a) Pick at random $r \in (0, 1)$.

   (b) Pick at random σ in the two following permutations: $(a_1, a_2, \ldots, a_{k-1}, a_k)$ and $(a_{k-1}, a_{k-2}, \ldots, a_1, a_k)$.

   (c) For each i in $1, \ldots, k-1$:

      i. Add each vertex v such that $x_{v,\sigma(i)} \geq r$ to the cluster of $a_{\sigma(i)}$.

   (d) Add the remaining vertices to the cluster of $a_{\sigma(k)}$.

   (e) Output $\leftarrow$ the edges between the vertices of different clusters.

   We aim at showing that this rounding procedure provides the same approximation guarantee than the one described during the lectures.