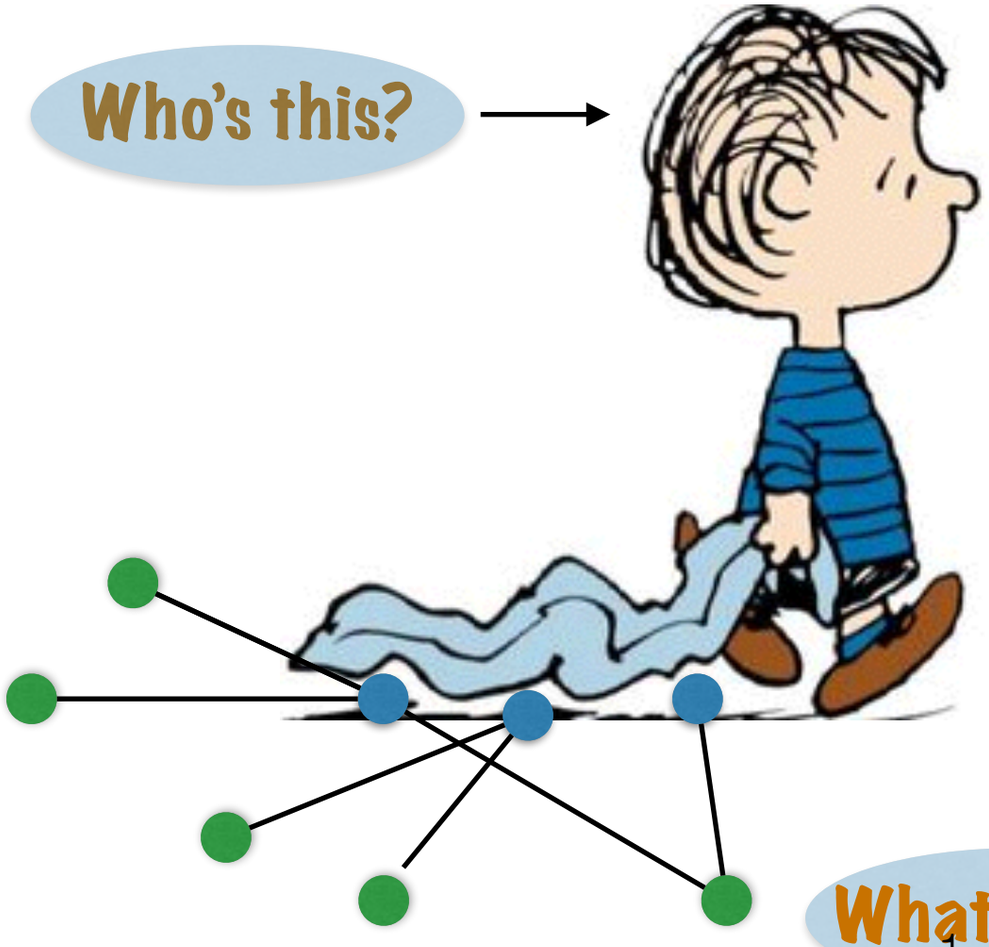
Approximation algorithms, vertex cover, and linear programming



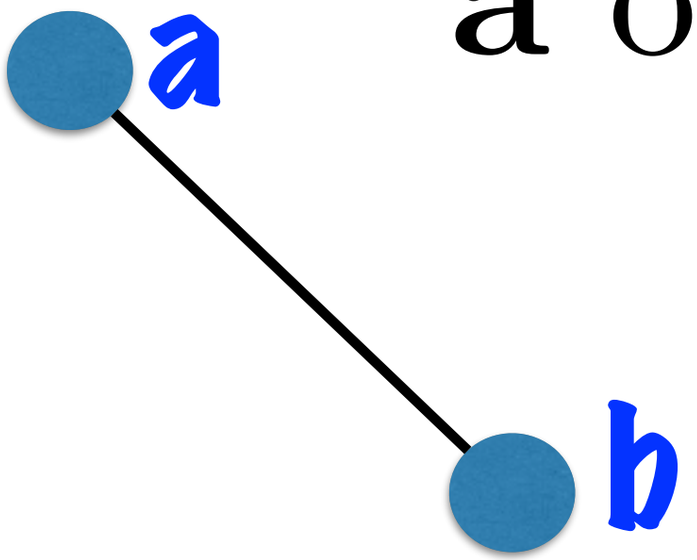
$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
such that
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 \\ \forall i: \quad x_i \text{ real number} \end{cases}$$

What's that?

Variables

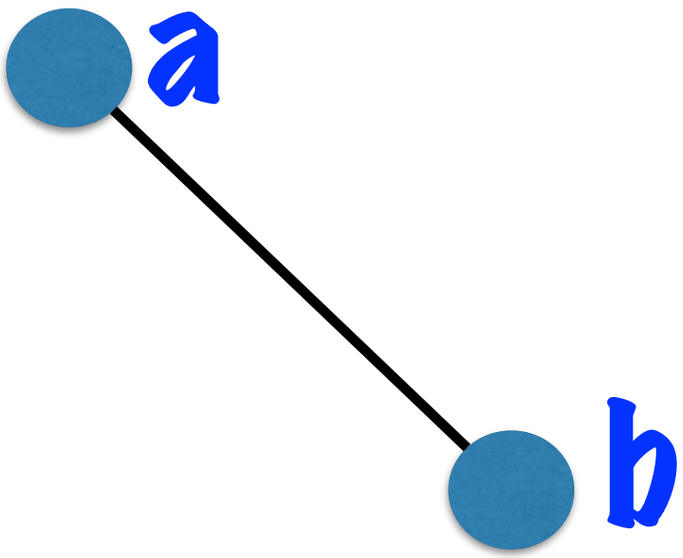
$\{\mathbf{a}, \mathbf{b}\} \in \mathbf{E}$:

$\mathbf{a}$ or $\mathbf{b}$ must be in cover



$$x_a = \begin{cases} 1 & \text{if } a \text{ in cover} \\ 0 & \text{otherwise} \end{cases}$$

Constraints



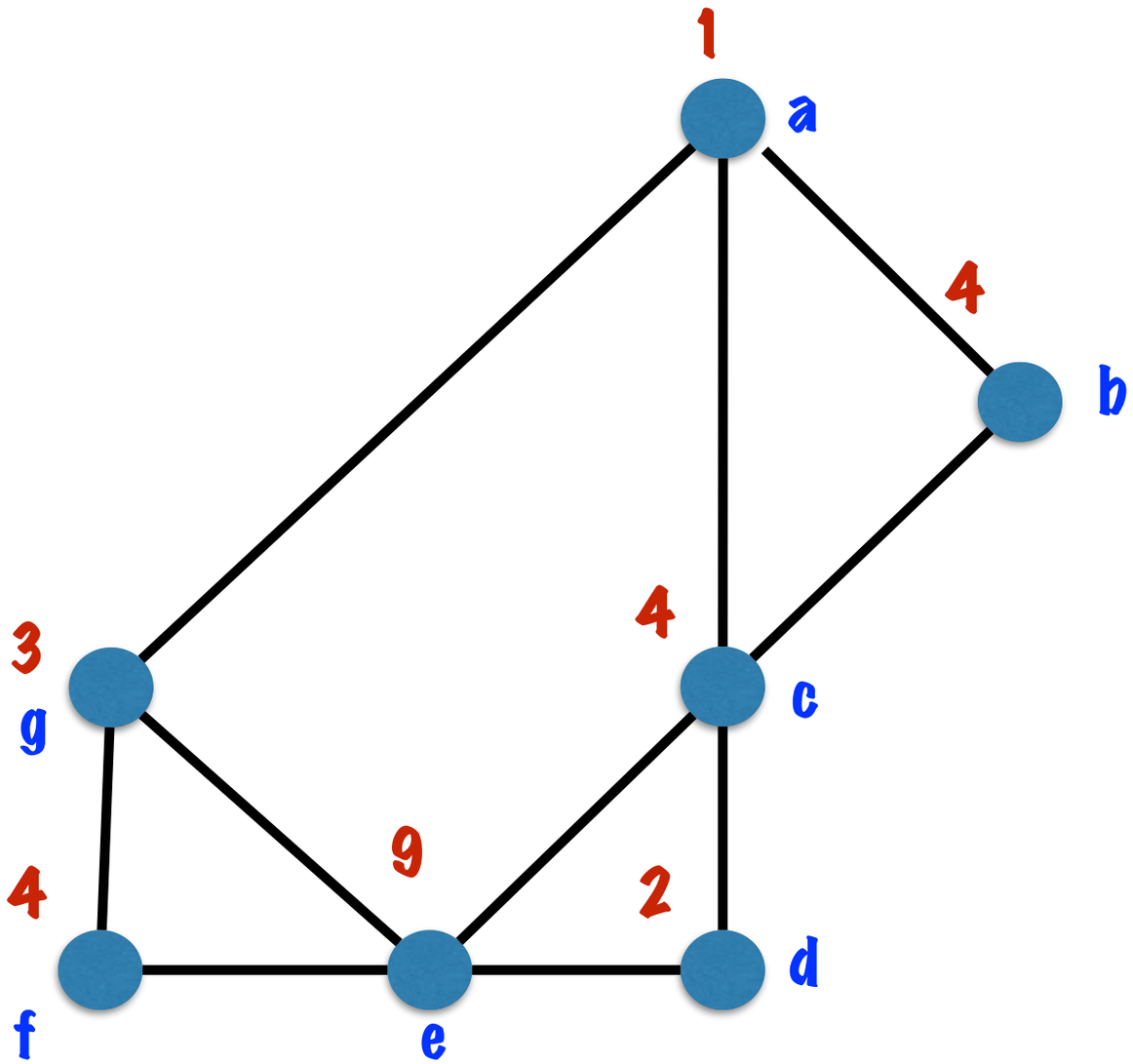
x_a	x_b	edge covered?	$x_a + x_b$
0	0	no	0
1	0	yes	1
0	1	yes	1
1	1	yes	2

$$\{a, b\} \text{ covered}$$

$$\Leftrightarrow$$

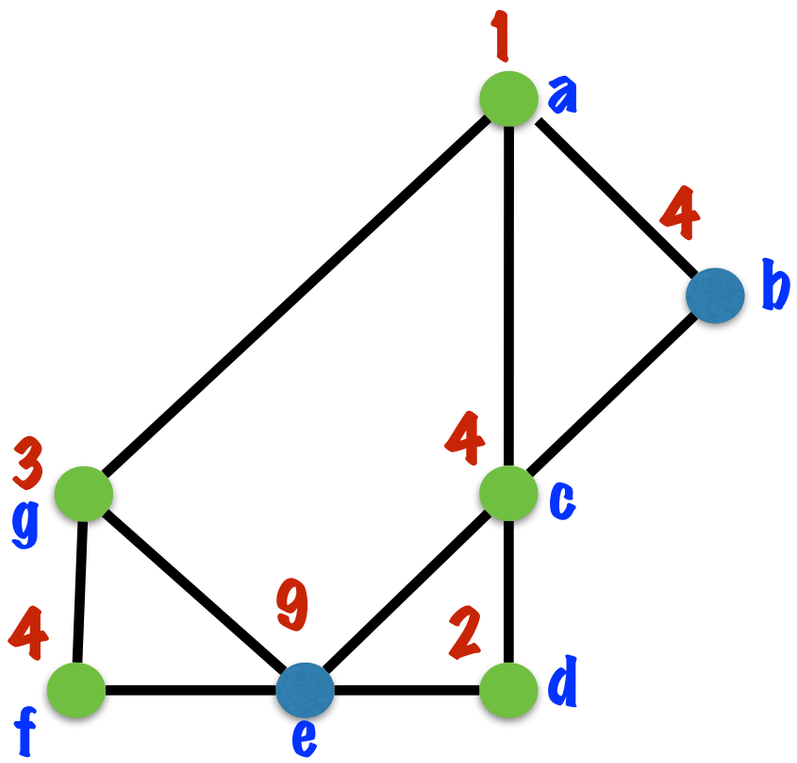
$$x_a + x_b \geq 1$$

Objective



$$\begin{cases} x_a + x_b & \geq 1 \\ x_a + x_c & \geq 1 \\ x_a + x_g & \geq 1 \\ x_b + x_c & \geq 1 \\ x_c + x_d & \geq 1 \\ x_c + x_e & \geq 1 \\ x_d + x_e & \geq 1 \\ x_e + x_f & \geq 1 \\ x_e + x_g & \geq 1 \\ x_f + x_g & \geq 1 \end{cases}$$

$$\min x_a + 4x_b + 4x_c + 2x_d + 9x_e + 4x_f + 3x_g$$



$$\begin{cases} x_a + x_b \geq 1 \\ x_a + x_c \geq 1 \\ x_a + x_g \geq 1 \\ x_b + x_c \geq 1 \\ x_c + x_d \geq 1 \\ x_c + x_e \geq 1 \\ x_d + x_e \geq 1 \\ x_e + x_f \geq 1 \\ x_e + x_g \geq 1 \\ x_f + x_g \geq 1 \end{cases}$$

$\{a, c, d, f, g\}$ vertex cover

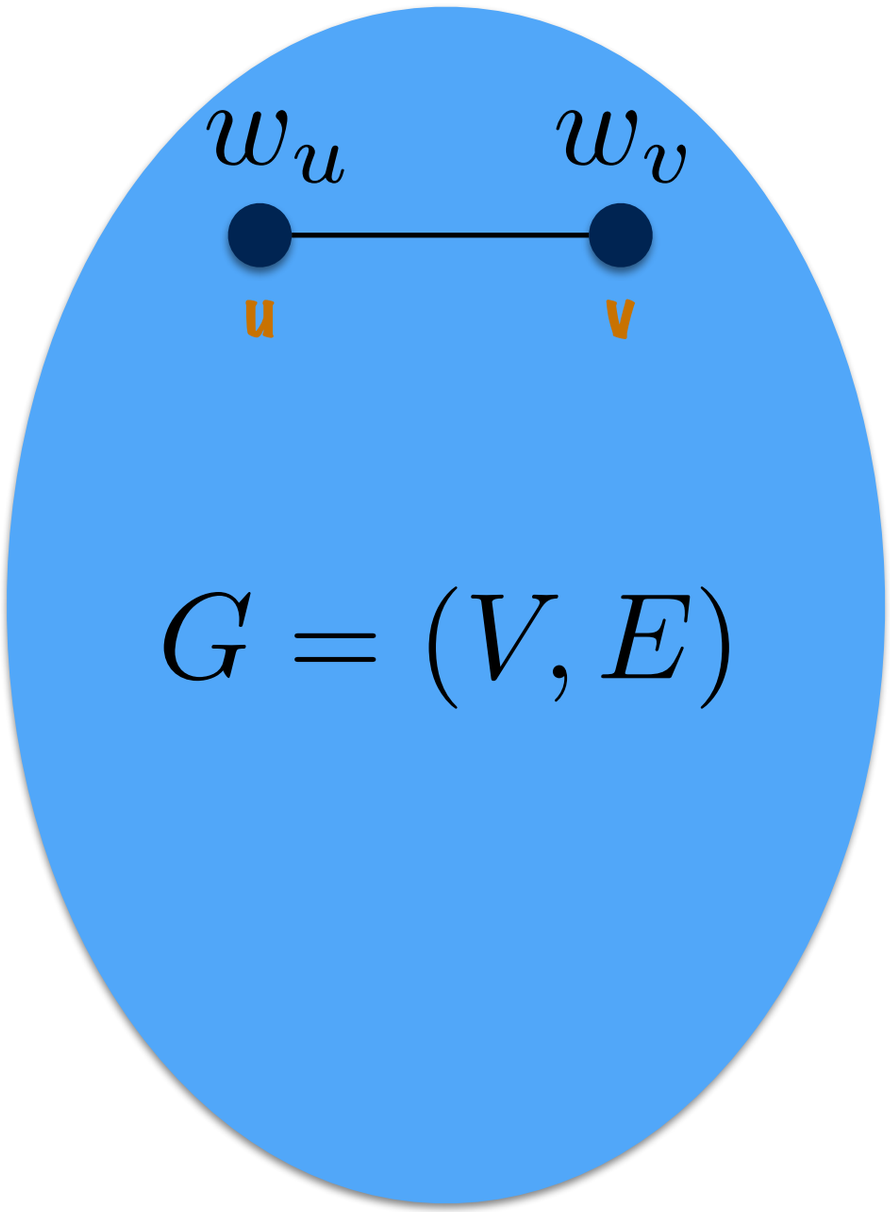
$$\min x_a + 4x_b + 4x_c + 2x_d + 9x_e + 4x_f + 3x_g$$

$x_a = x_c = x_d = x_f = x_g = 1$

$x_b = x_e = 0$

satisfies all constraints value $= 14$

in general



Constraints:

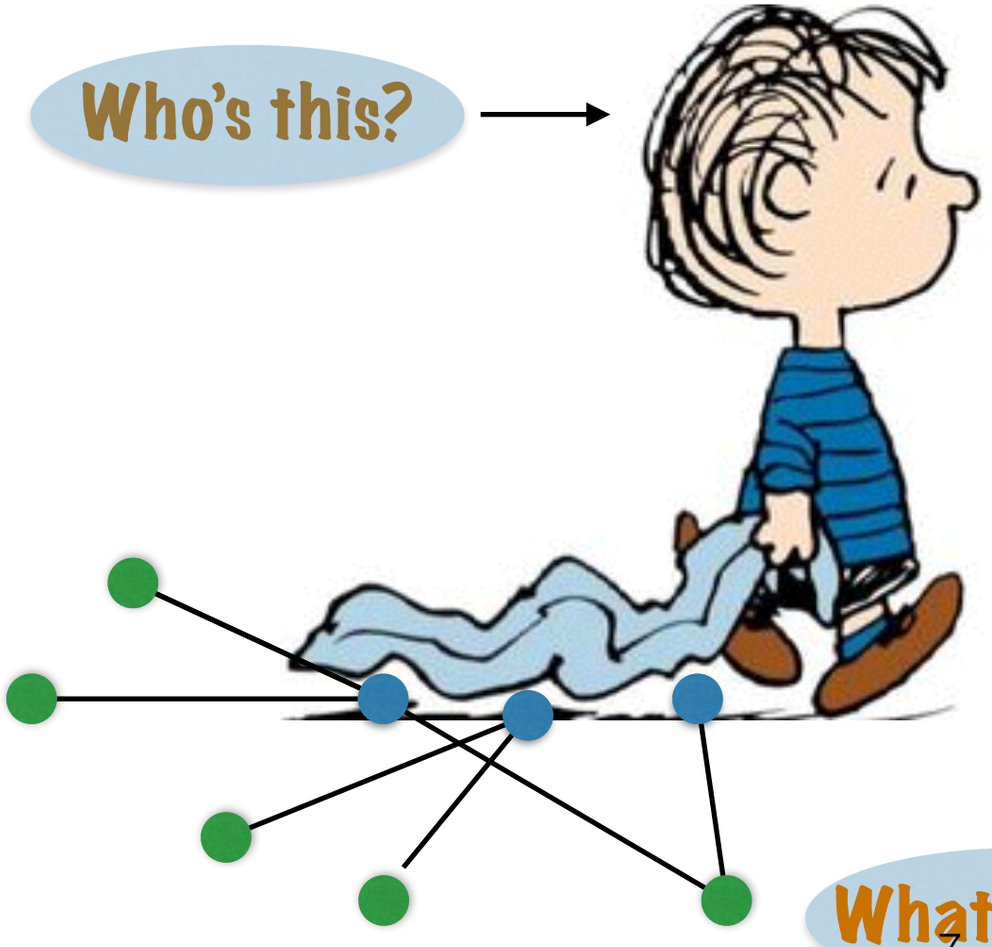
$\forall u \in V : x_u = 0 \text{ or } 1$

$\forall \{u, v\} \in E : x_u + x_v \geq 1$

Objective: $\min \sum_u w_u x_u$

integer program

Approximation algorithms, vertex cover, and linear programming



$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
such that
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \text{ real number} & \end{cases}$$

What's that?