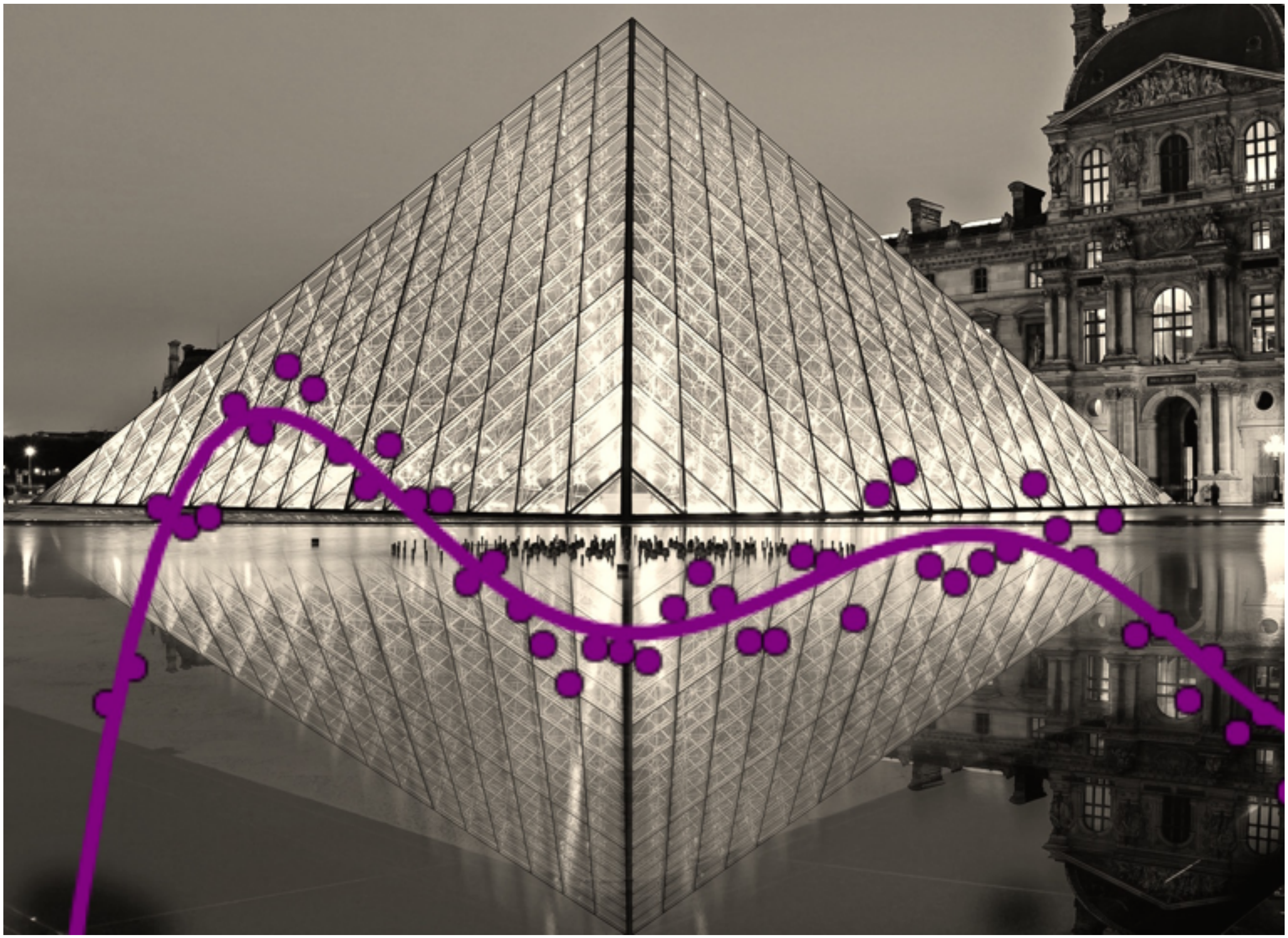
Approximation Algorithms

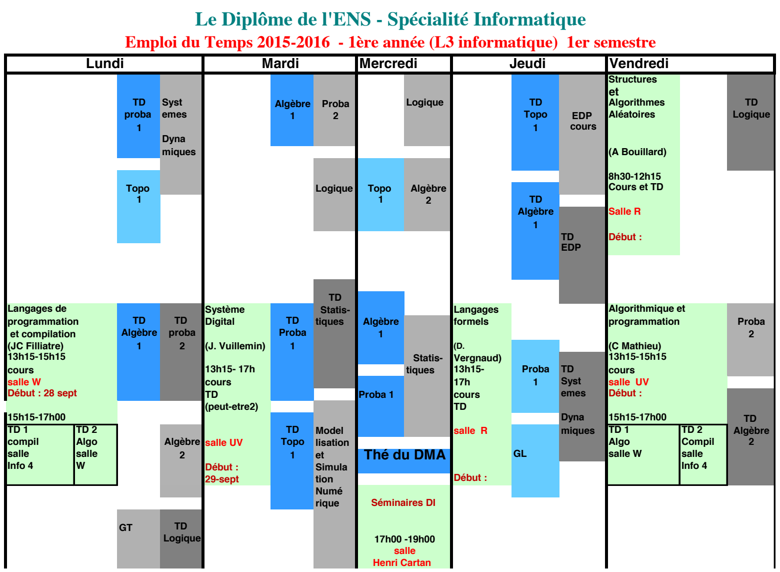


Combinatorial optimization:

- Scheduling classes
- Planning delivery routes for trucks
- ...

Most are NP-hard

What's that?



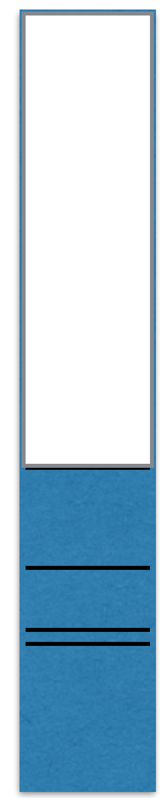
Dealing with NP-hard problems

- Give up?
- Roll up our sleeves?
- Try something, hoping for luck?
- Do a rough but good enough job

In polynomial time, we find a solution whose value is provably within a "small" factor of the optimal solution

Designing approximation algorithms

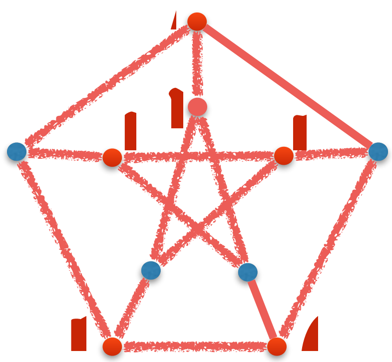
- Real-life problems are too complex
- Study idealized problems
- Theorems: new algorithmic and structural insights



The interface with real life

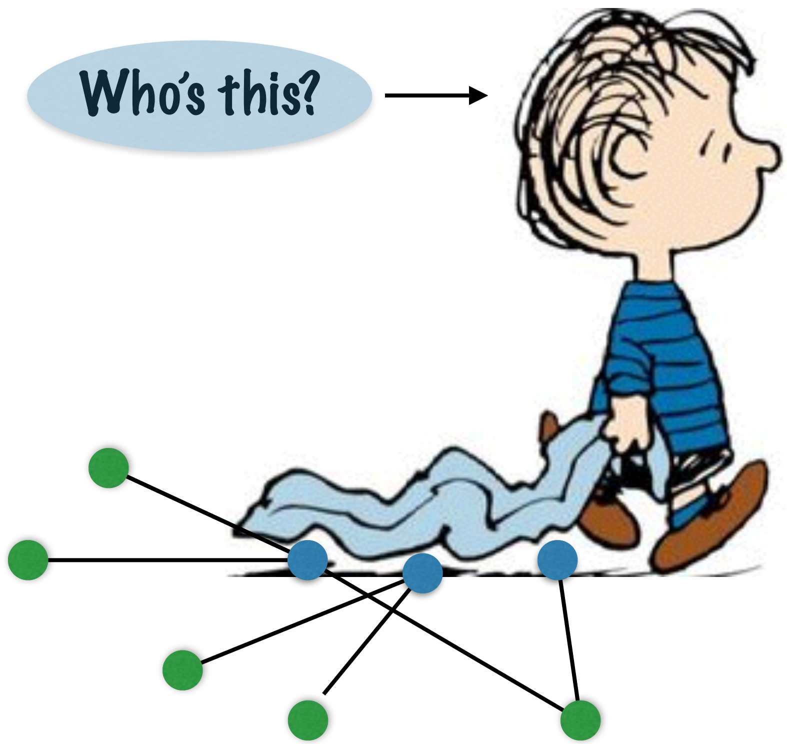
A course with two parts
Approximation algorithms, Part I

- Vertex cover
- Knapsack
- Bin packing
- Set cover
- Multiway cut



Approximation algorithms, vertex cover, and linear programming

Who's this?

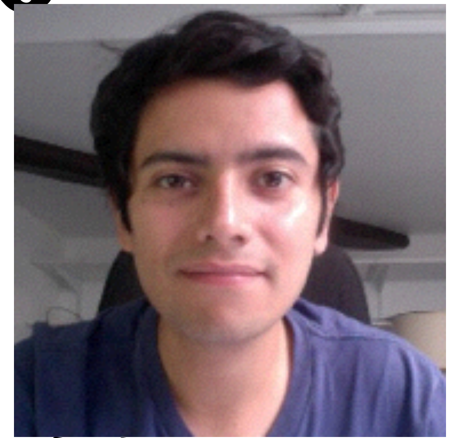


What's that?

$$
\begin{aligned}
&\min c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
&\text{such that} \\
&\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\
\cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\
\forall i: \quad 0 \leq x_i \leq 1 & \\
\forall i: \quad x_i \ \underline{\text{real number}} &
\end{cases}
\end{aligned}
$$

Teaching staff behind the scenes

- Vincent Cohen-Addad
- Frederik Mallmann-Trenn
- Victor Verdugo

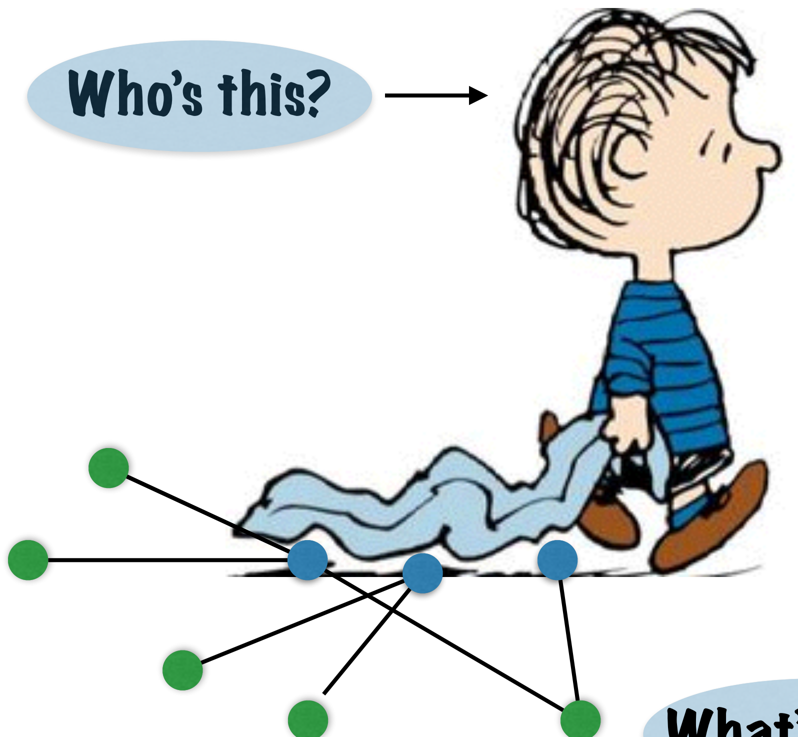


Film director: Nordine Méziane

Cameraman: Jovanny Parvedy

Approximation algorithms, vertex cover, and linear programming

Who's this?



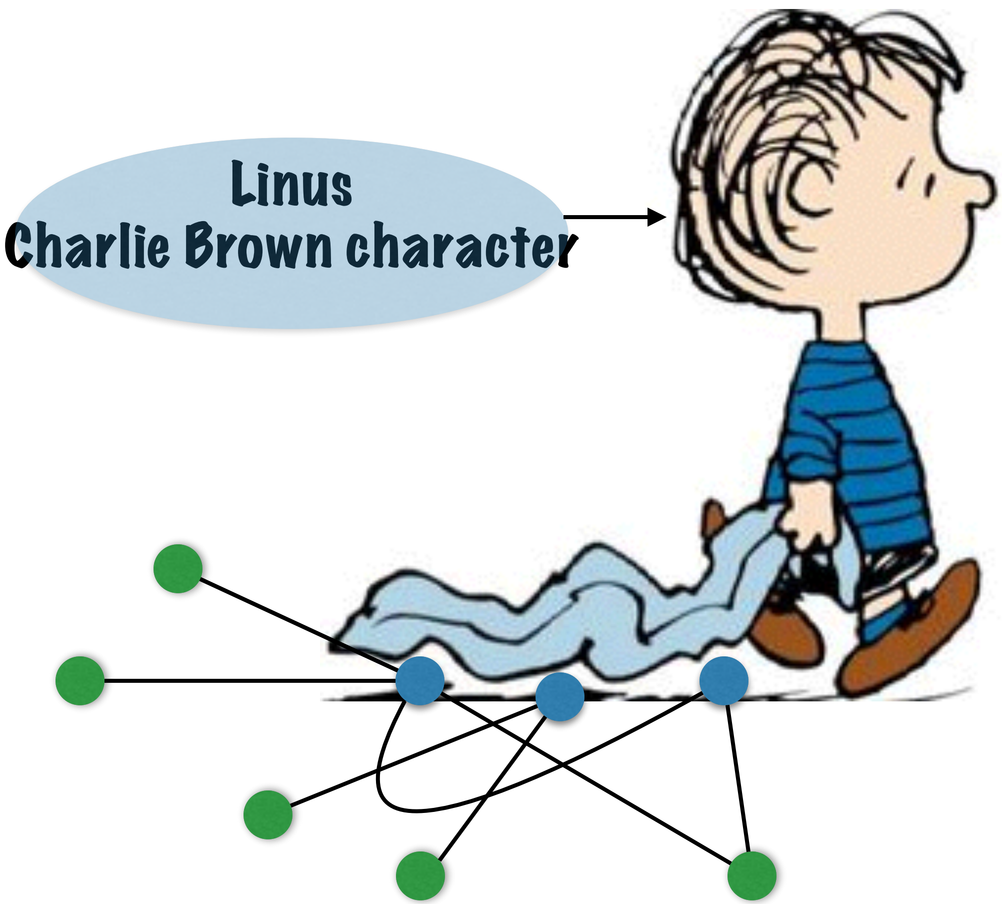
$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

such that

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

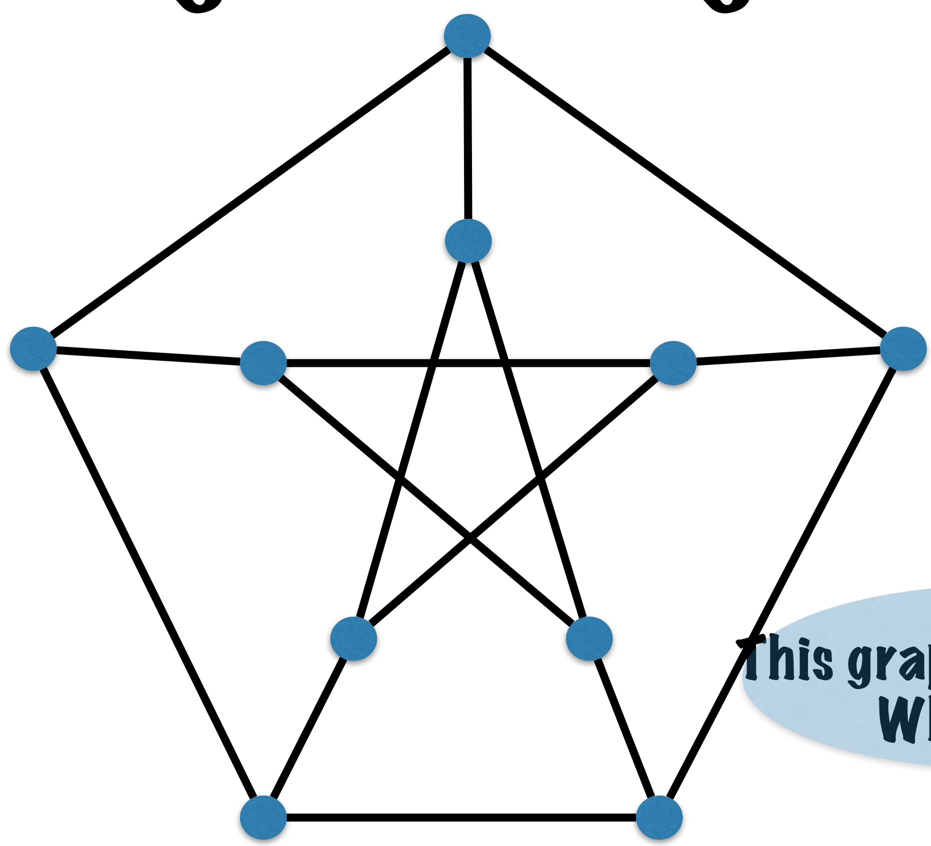
What's that?

Vertex Cover

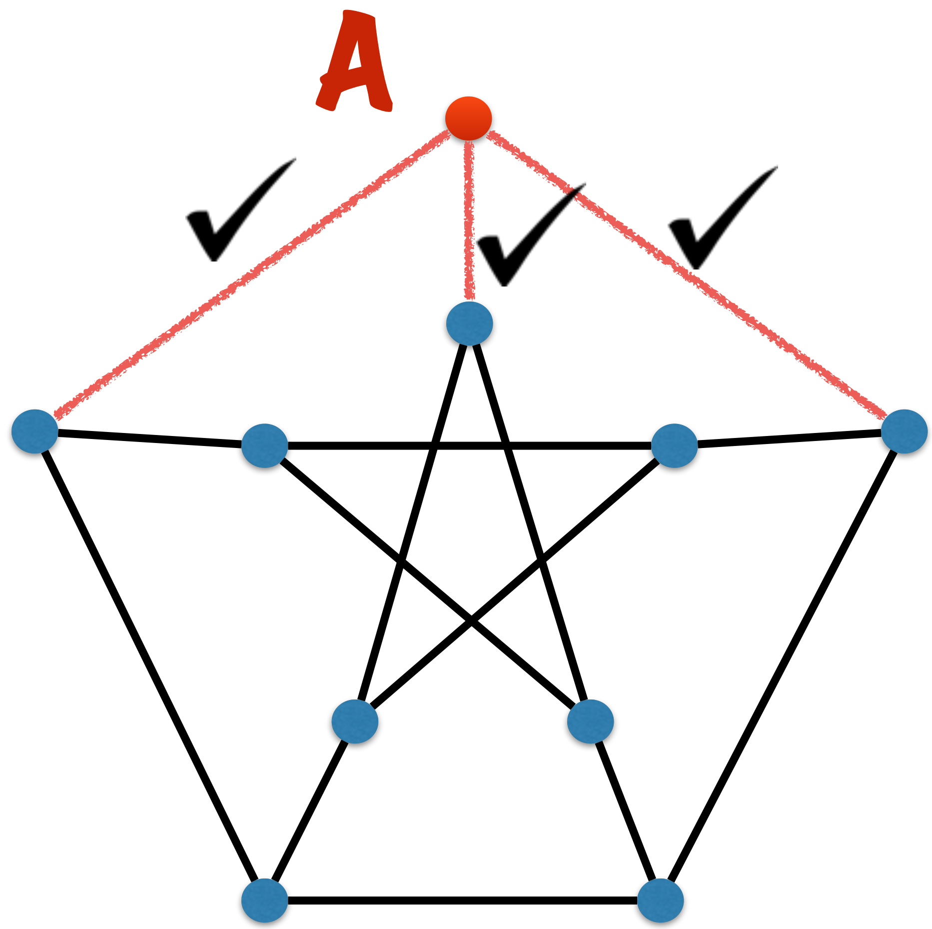


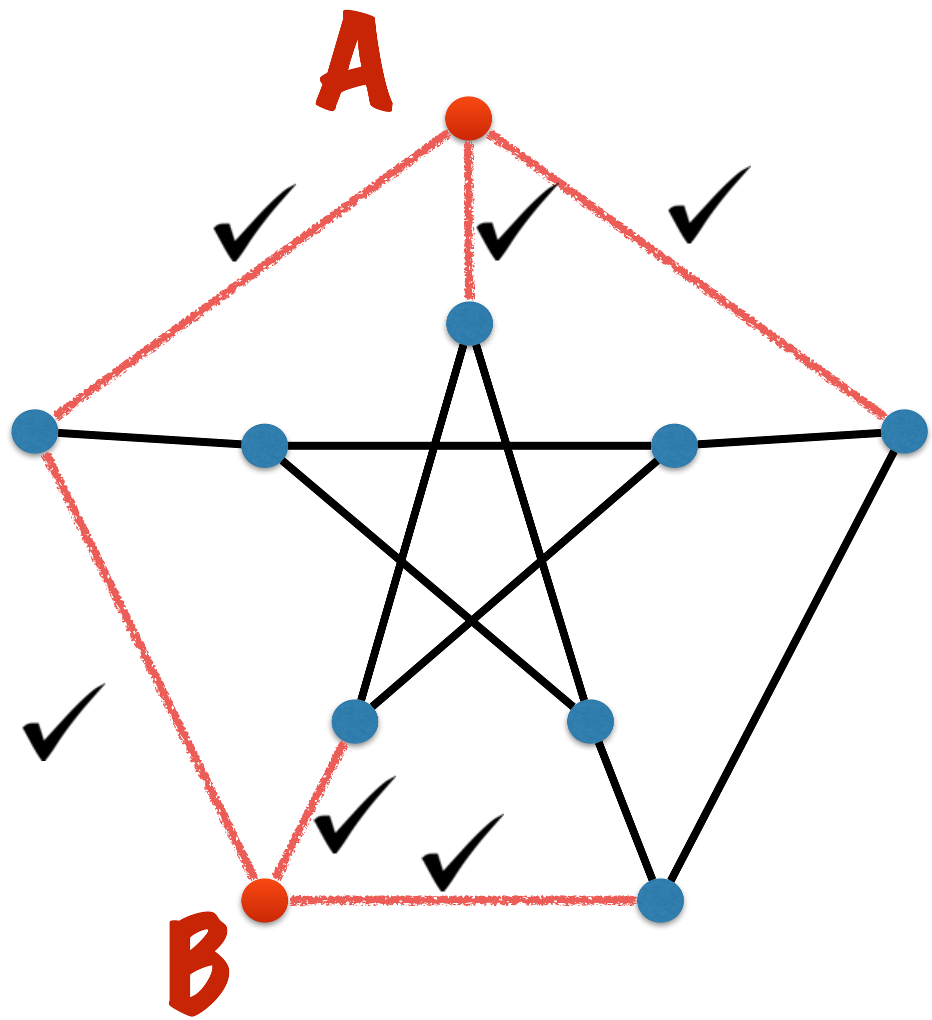
Vertex cover

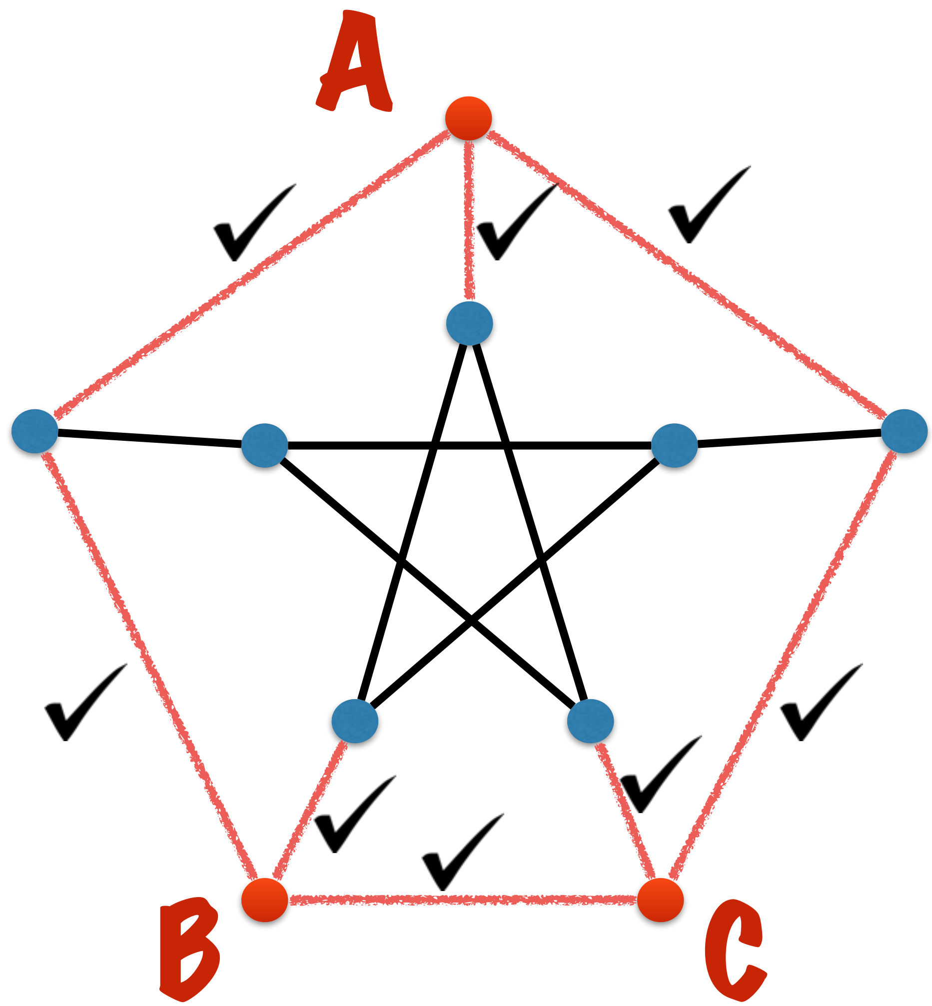
Given graph with vertex weights, cover edges with lightest vertices

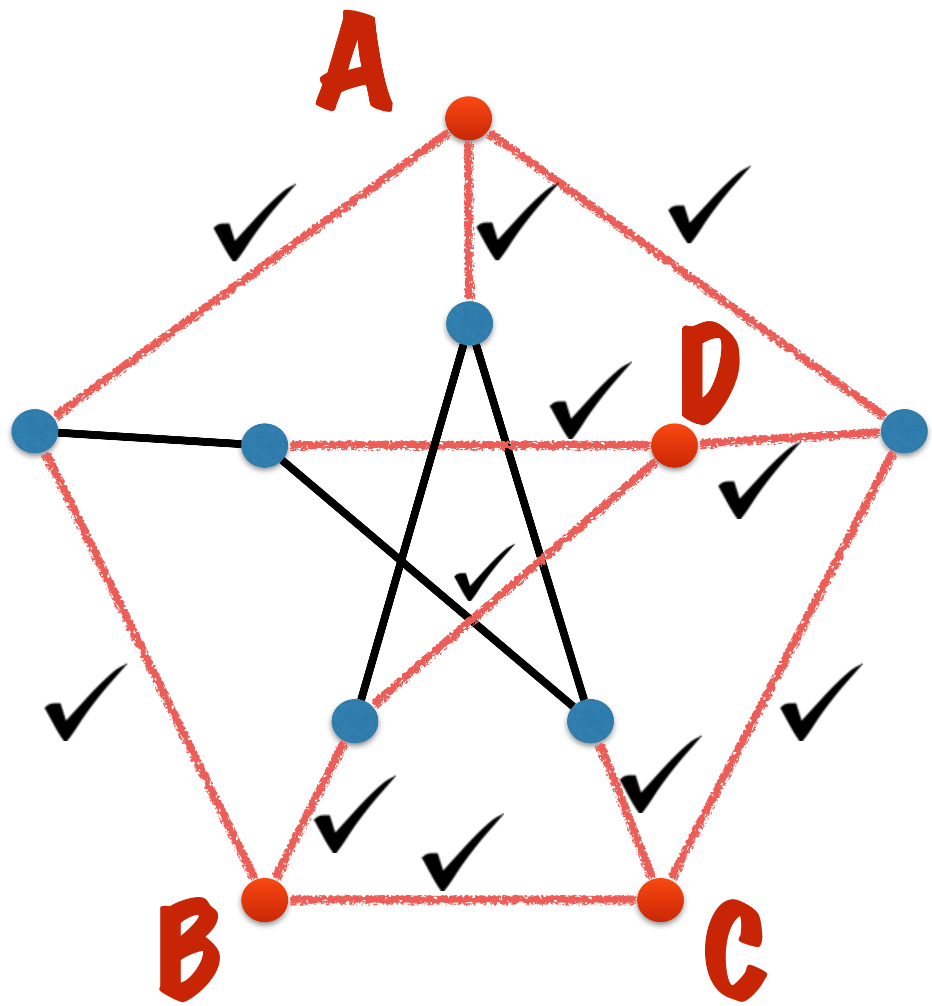


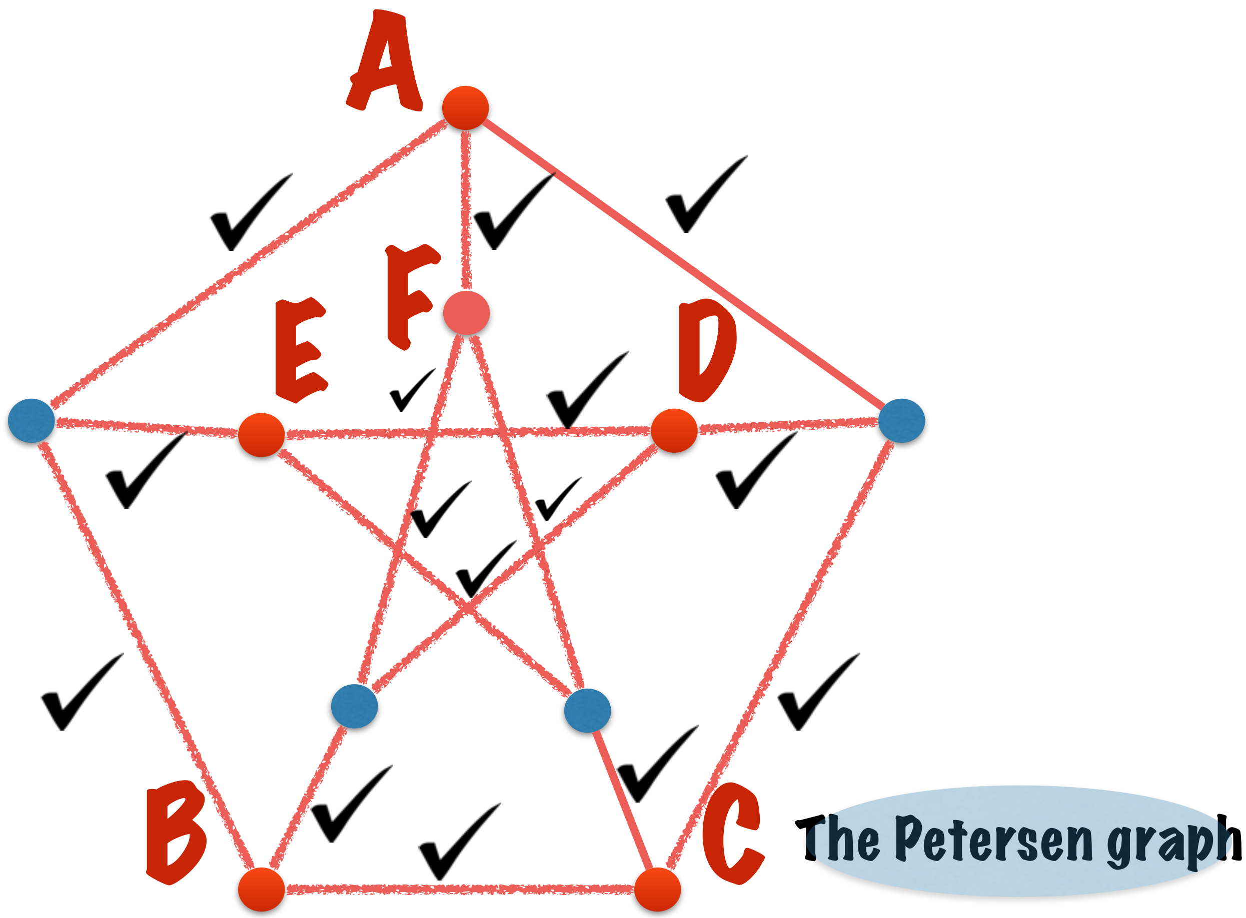
This graph has a name.
What is it?


A

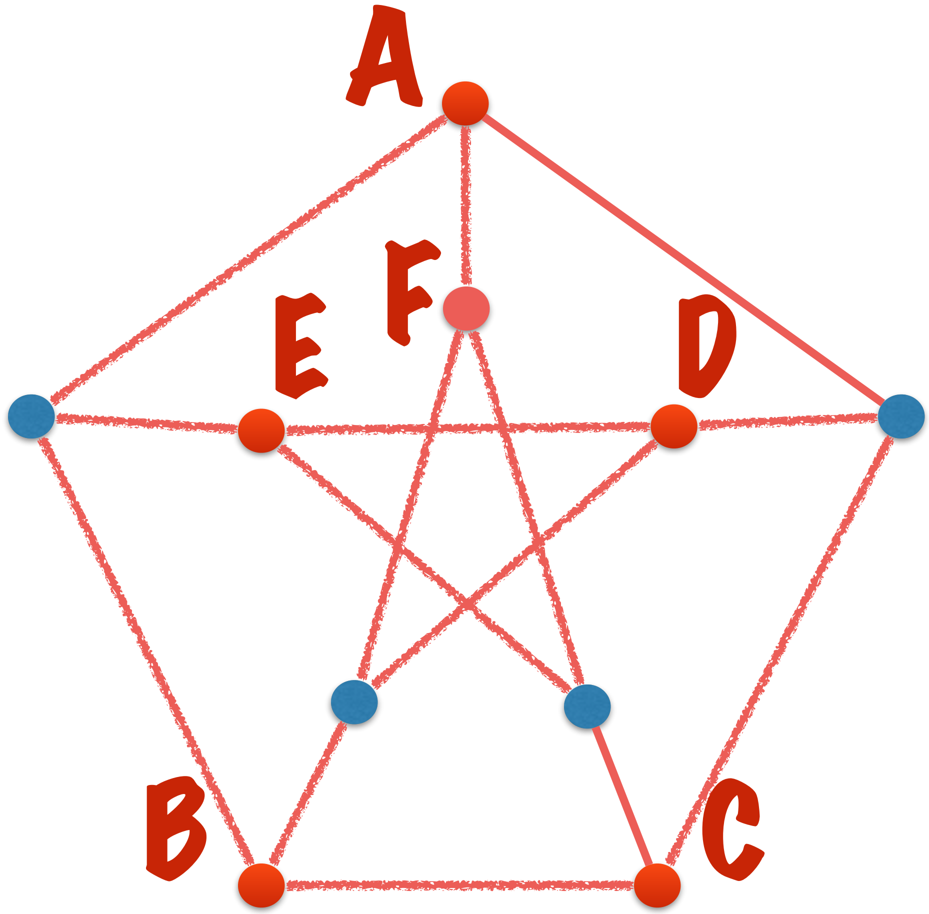

A
B


A
B
C


A
D
B
C



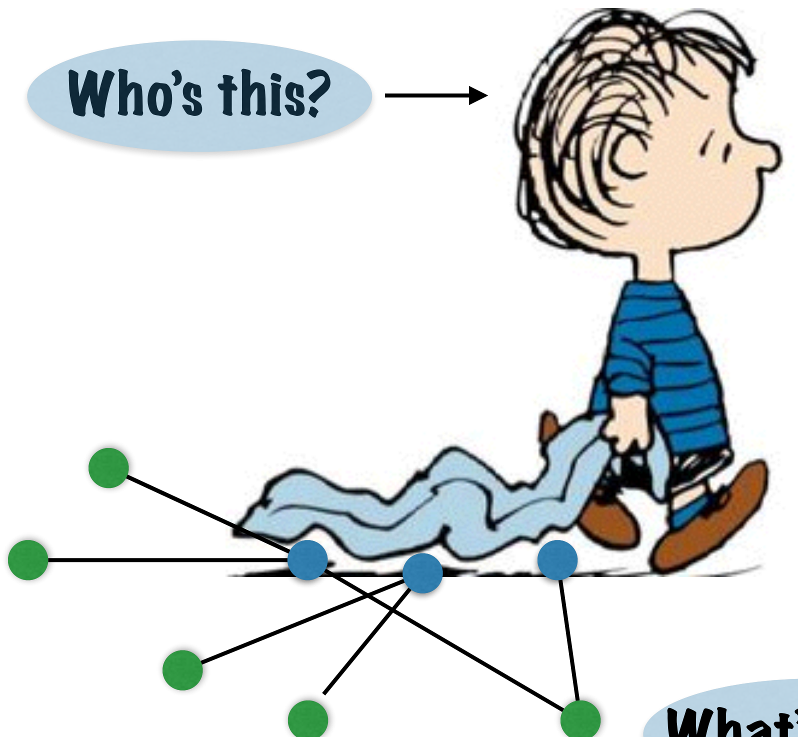
Vertex weights: 1.
Cover edges with 6 vertices.
Optimal : cannot cover with 5



Given graph with vertex weights, cover edges with lightest vertices

Approximation algorithms, vertex cover, and linear programming

Who's this?



$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

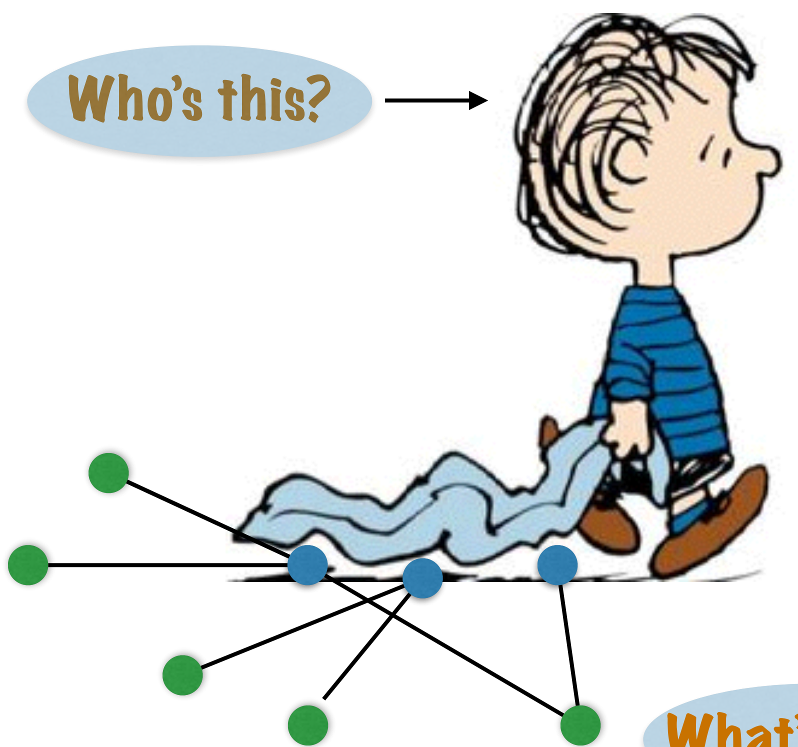
such that

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

What's that?

Approximation algorithms, vertex cover, and linear programming

Who's this?



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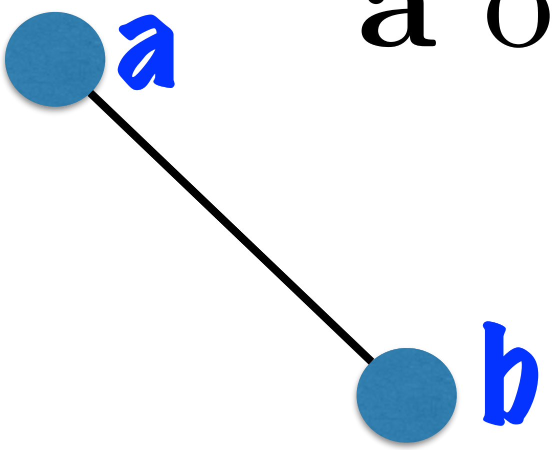
\forall i : \quad 0 \leq x_i \leq 1 & \\
\forall i : \quad x_i \ \underline{\text{real number}} &
\end{cases}
\end{aligned}
$$

What's that?

Variables

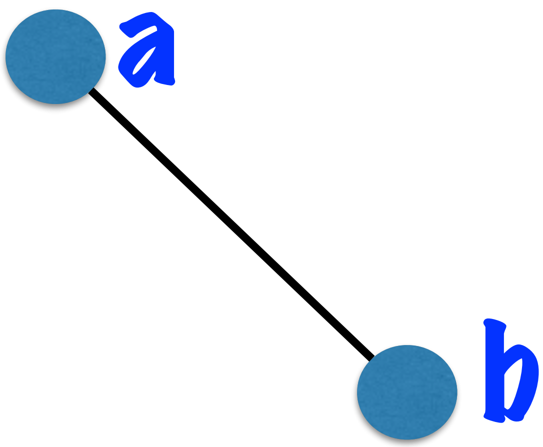
$$\{\mathbf{a},\mathbf{b}\} \in \mathbf{E}:$$

$$\mathbf{a} \text{ or } \mathbf{b} \text{ must be in cover}$$



$$x_a = \begin{cases} 1 & \text{if } a \text{ in cover} \\ 0 & \text{otherwise} \end{cases}$$

Constraints



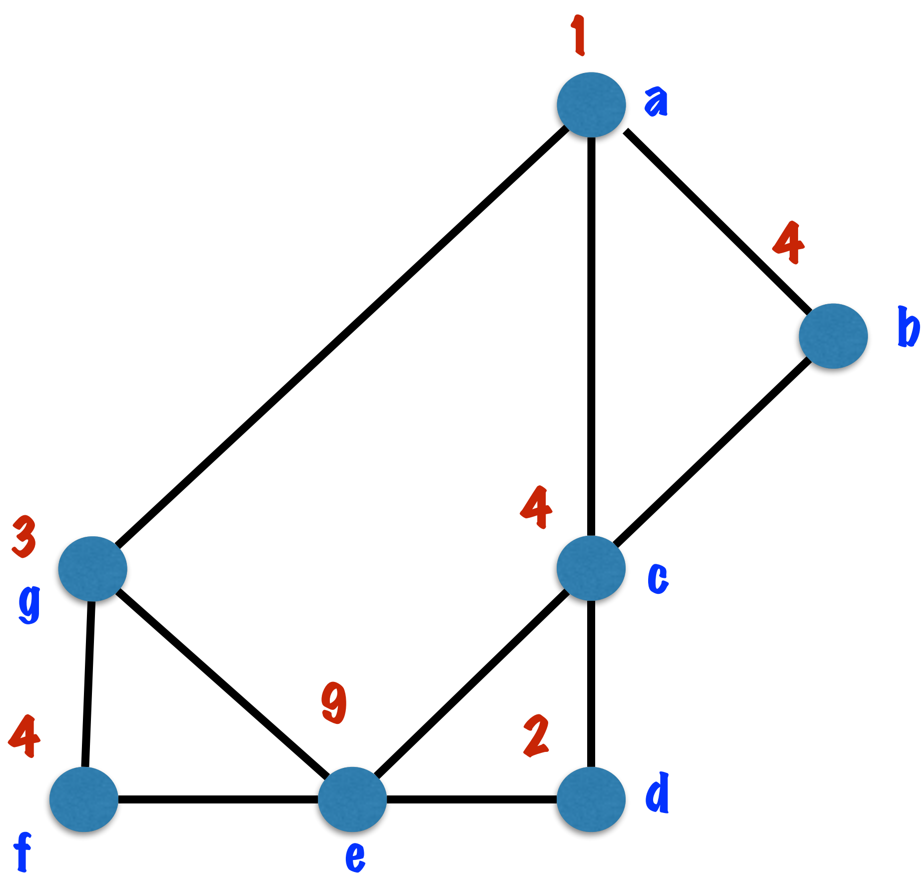
x_a	x_b	edge covered?	$x_a + x_b$
0	0	no	0
1	0	yes	1
0	1	yes	1
1	1	yes	2

$$\{a, b\} \text{ covered}$$

$$\Leftrightarrow$$

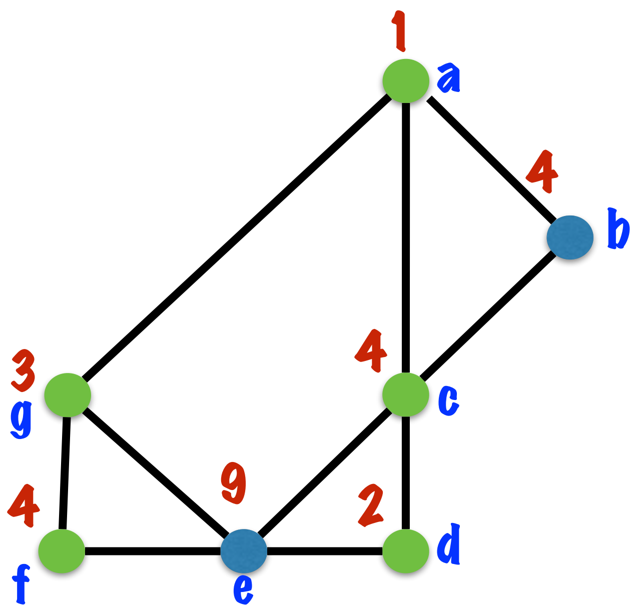
$$x_a + x_b \geq 1$$

Objective



$$\begin{cases} x_a + x_b & \geq 1 \\ x_a + x_c & \geq 1 \\ x_a + x_g & \geq 1 \\ x_b + x_c & \geq 1 \\ x_c + x_d & \geq 1 \\ x_c + x_e & \geq 1 \\ x_d + x_e & \geq 1 \\ x_e + x_f & \geq 1 \\ x_e + x_g & \geq 1 \\ x_f + x_g & \geq 1 \end{cases}$$

$$\min x_a + 4x_b + 4x_c + 2x_d + 9x_e + 4x_f + 3x_g$$



$$\begin{cases} x_a + x_b & \geq 1 \\ x_a + x_c & \geq 1 \\ x_a + x_g & \geq 1 \\ x_b + x_c & \geq 1 \\ x_c + x_d & \geq 1 \\ x_c + x_e & \geq 1 \\ x_d + x_e & \geq 1 \\ x_e + x_f & \geq 1 \\ x_e + x_g & \geq 1 \\ x_f + x_g & \geq 1 \end{cases}$$

$\{a, c, d, f, g\}$ vertex cover

$$\min x_a + 4x_b + 4x_c + 2x_d + 9x_e + 4x_f + 3x_g$$

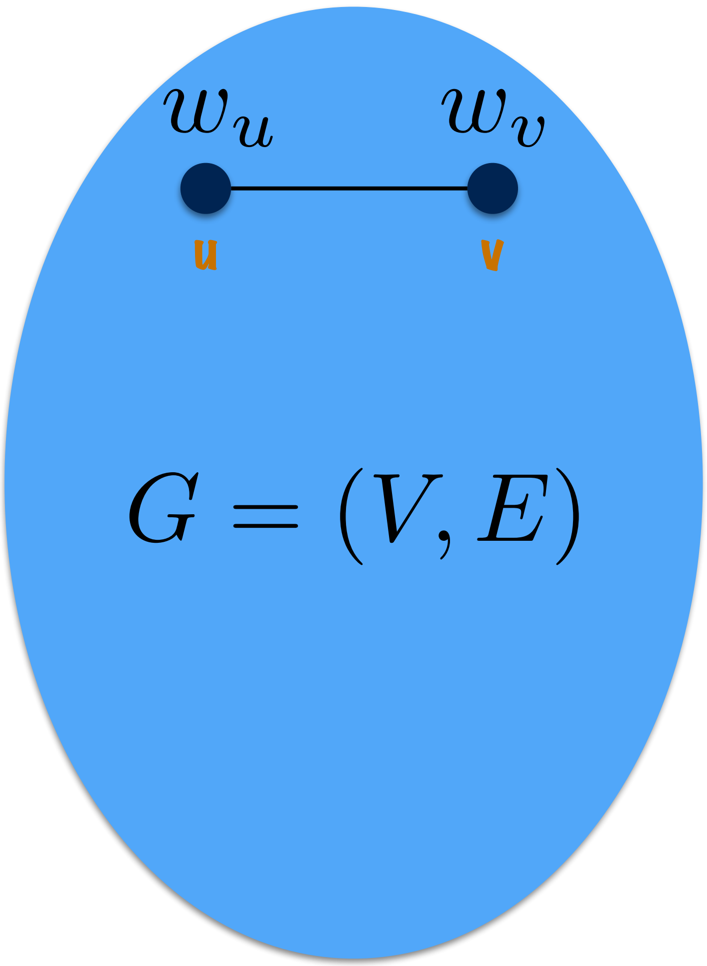
$x_a = x_c = x_d = x_f = x_g = 1$

$x_b = x_e = 0$

satisfies all constraints

value $= 14$

in general



Constraints:

$\forall u \in V : x_u = 0 \text{ or } 1$

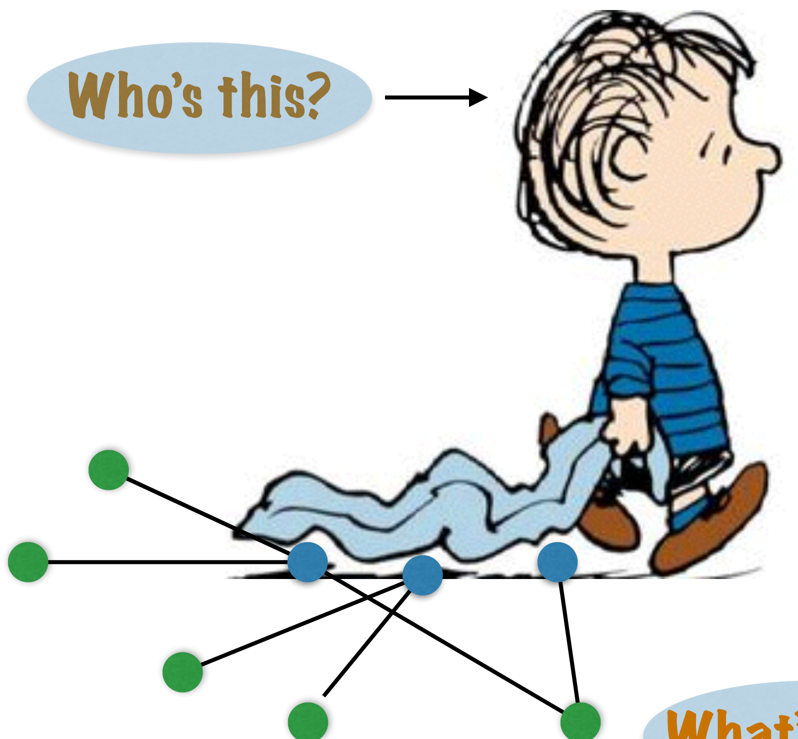
$\forall \{u, v\} \in E : x_u + x_v \geq 1$

Objective: $\min \sum_u w_u x_u$

integer program

Approximation algorithms, vertex cover, and linear programming

Who's this?



$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

such that

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

What's that?

Approximation algorithms, vertex cover, and linear programming

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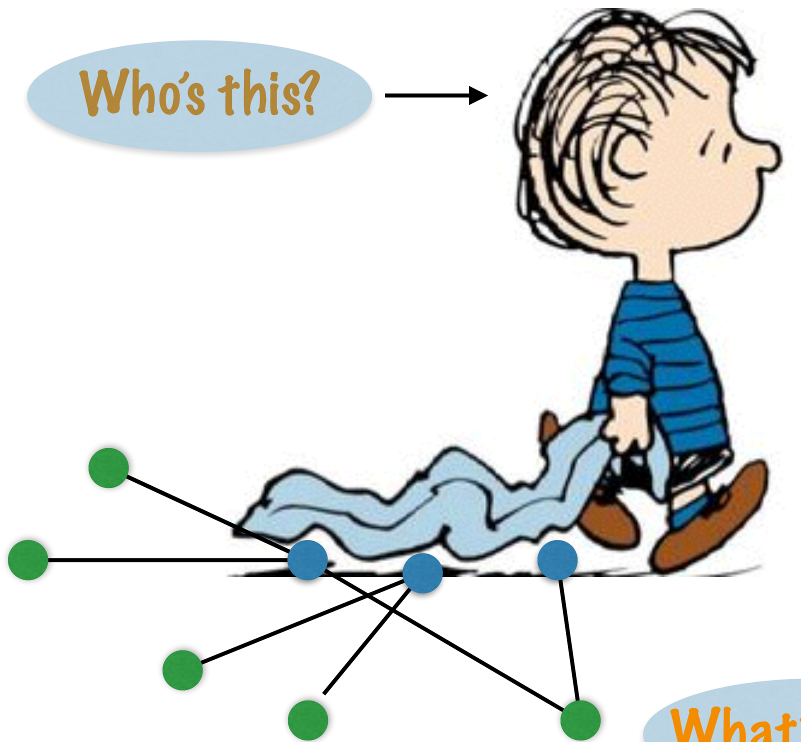

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\forall i: \quad 0 \leq x_i \leq 1 & \\
\forall i: \quad x_i \ \underline{\text{real number}} &
\end{cases}
\end{aligned}
$$

What's that?

Integer program

$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

such that

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \text{ integer} & \end{cases}$$

NP-hard

Linear program

$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

such that

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

polynomial time

Two ways to present

$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

such that

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

Linear program

$$\min c \cdot x \text{ such that } \begin{cases} Ax \geq b \\ x \in [0,1]^n \\ x \text{ vector of } \mathbb{R}^n \end{cases}$$

Same linear program

Integer vs. linear programs

IP

$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
such that
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \text{ integer} & \end{cases}$$

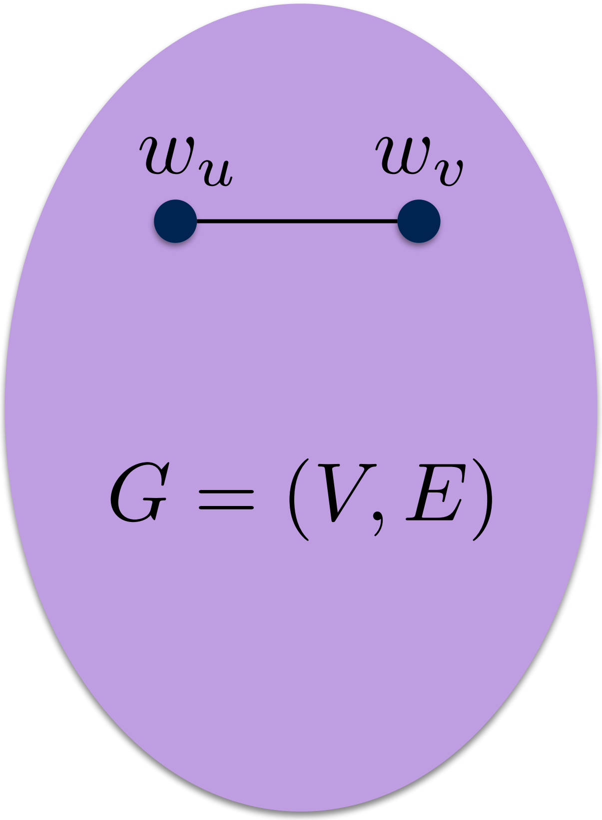
NP-hard

LP

$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
such that
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

polynomial time

Vertex cover linear program



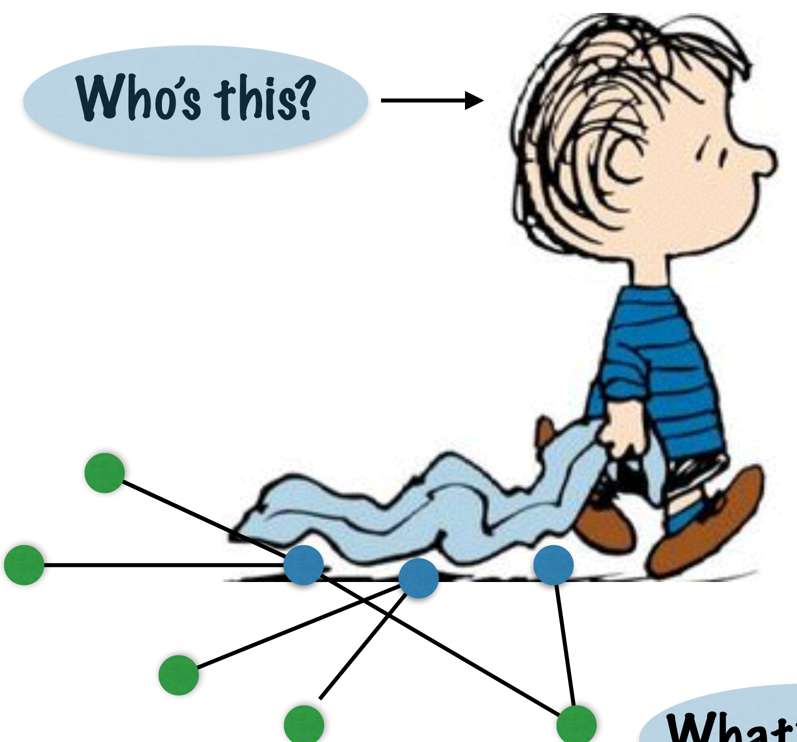
Constraints:

$$\forall u \in V : 0 \leq x_u \leq 1$$

$$\forall \{u, v\} \in E : x_u + x_v \geq 1$$

Objective: $\min \sum_u w_u x_u$

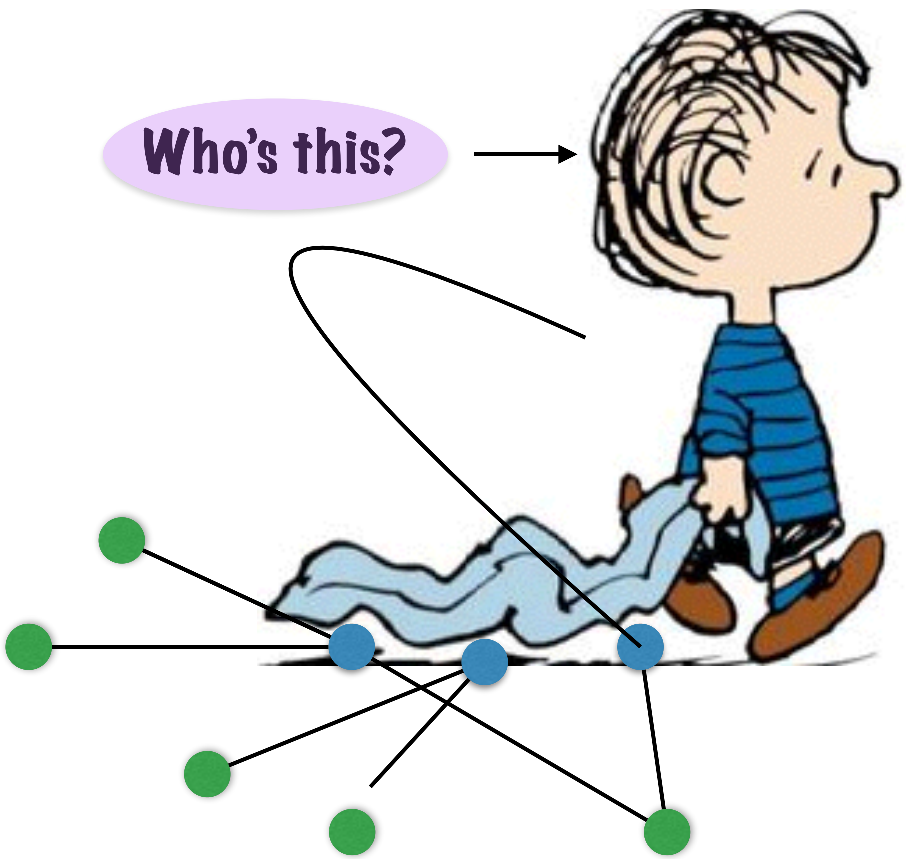
Approximation algorithms, vertex cover, and linear programming



$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$\text{such that}$$
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

What's that?

Approximation algorithms, vertex cover, and linear programming



$$
\min c_1x_1 + c_2x_2 + \cdots + c_nx_n
$$
$$
\text{such that}
$$
$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\
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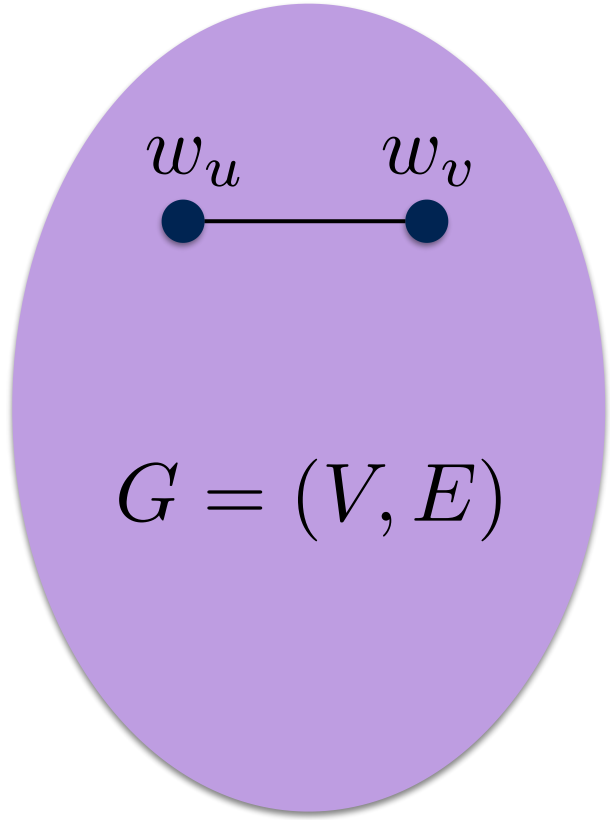
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\
\forall i: \quad 0 \leq x_i \leq 1 & \\
\forall i: \quad x_i \ \underline{\text{real number}} &
\end{cases}
$$

What's that?

Using the LP (1/3)



Constraints:

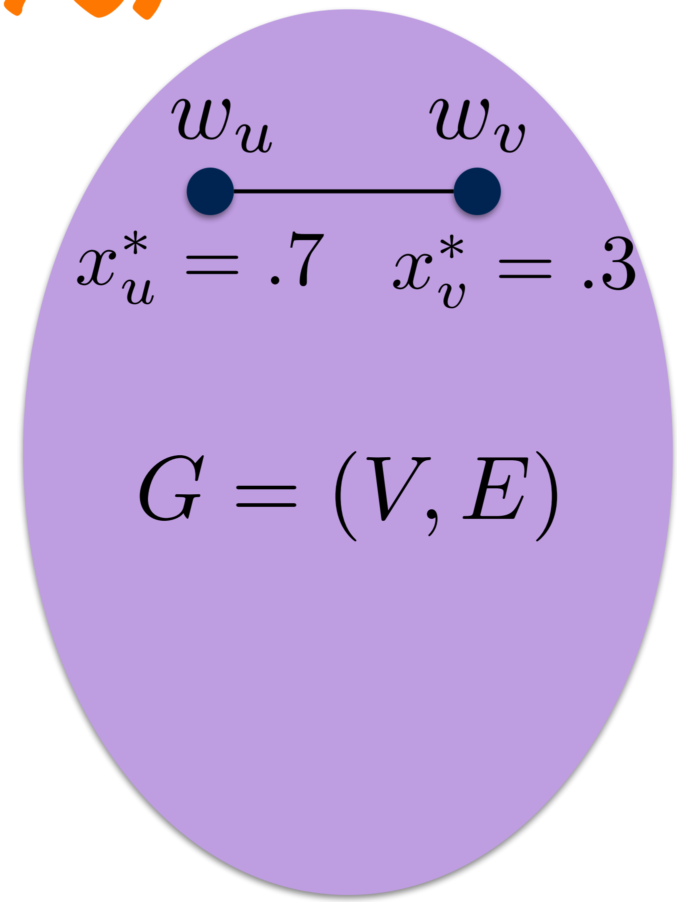
$$\forall u \in V : 0 \leq x_u \leq 1$$

$$\forall \{u, v\} \in E : x_u + x_v \geq 1$$

Objective: $\min \sum_u w_u x_u$

Using the LP (2/3)

1. Solving the LP



$\implies (x_u^*)_{u \in V}$ such that

$\forall u \in V : 0 \leq x_u^* \leq 1$

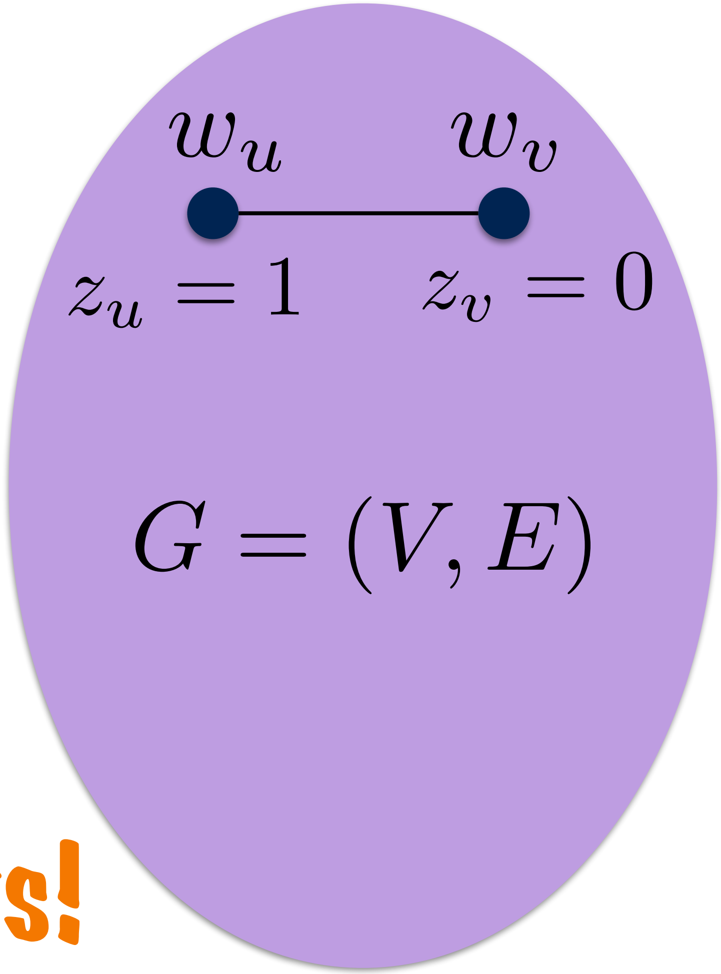
$\forall \{u, v\} \in E : x_u^* + x_v^* \geq 1$

$\sum_u w_u x_u^*$ minimum

Using the LP (3/3)
2. Rounding the LP solution

$$\Longrightarrow \quad (z_u)_{u \in V} \text{ defined by}$$

$$z_u = \begin{cases} 1 & \text{if } x_u^* \geq .5 \\ 0 & \text{otherwise} \end{cases}$$



We are back to integers!

Runtime

1. Solve the LP

$(x_u^*)_{u \in V}$ such that

$\forall u \in V : 0 \leq x_u^* \leq 1$

$\forall \{u, v\} \in E : x_u^* + x_v^* \geq 1$

$\sum_u w_u x_u^*$ minimum

Polynomial time

2. Round the LP solution

$\implies (z_u)_{u \in V}$ defined by

Linear time

$$z_u = \begin{cases} 1 & \text{if } x_u^* \geq .5 \\ 0 & \text{otherwise} \end{cases}$$

3. Output

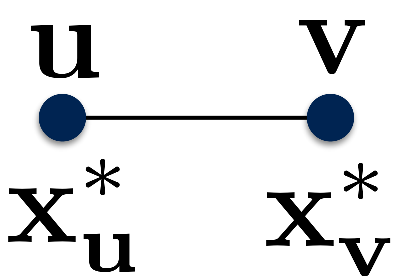
$\{\mathbf{u} \in \mathbf{V} \text{ such that } \mathbf{z_u} = \mathbf{1}\}$

Correctness

… is it a vertex cover?

Does output cover all edges?

$$\{\mathbf{u}, \mathbf{v}\} : \mathbf{x}_{\mathbf{u}}^* + \mathbf{x}_{\mathbf{v}}^* \geq 1$$



Does output cover all edges?

$$\mathbf{x_u^* + x_v^* \geq 1}$$

$x_u^* = .7 \quad x_v^* = .3$

$$\mathbf{x_u^* \geq .5} \text{ or } \mathbf{x_v^* \geq .5}$$

Does output cover all edges?

$$\implies (z_u)_{u \in V} \text{ defined by}$$

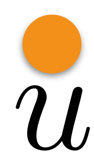
$$z_u = \begin{cases} 1 & \text{if } x_u^* \geq .5 \\ 0 & \text{otherwise} \end{cases}$$

$x_u^* = .7 \quad x_v^* = .3$

$$\mathbf{x_u^* \geq .5 \text{ or } x_v^* \geq .5}$$

$\mathbf{z_u = 1 \quad z_v = 0}$

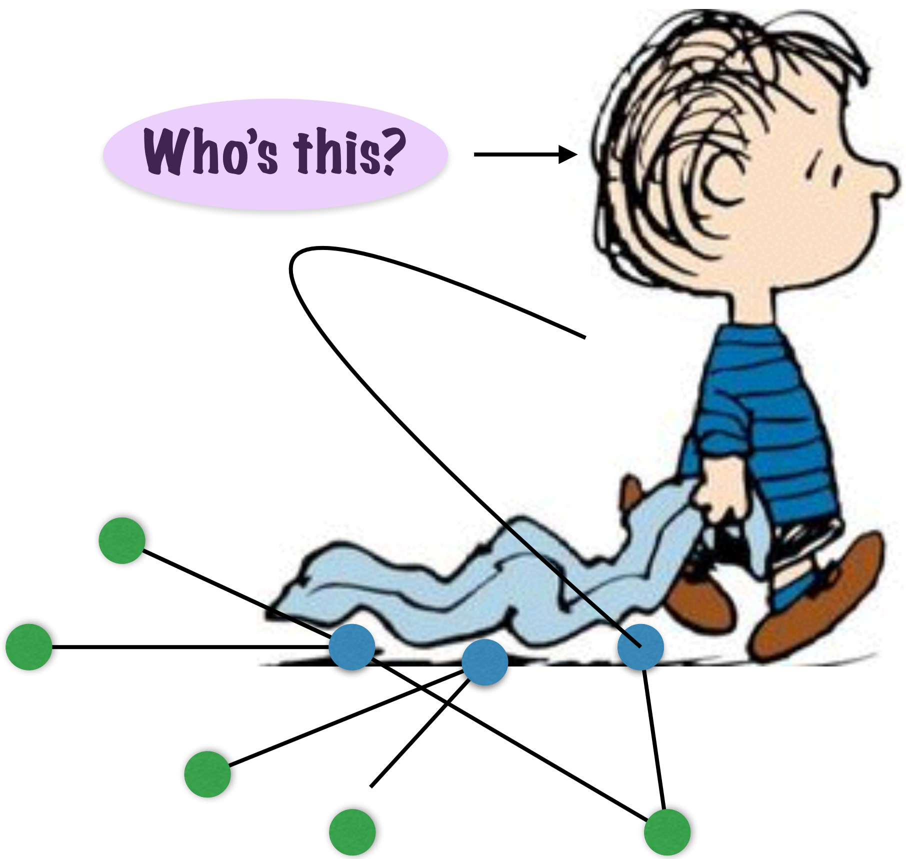
u is in output



$$\mathbf{z_u = 1 \text{ or } z_v = 1}$$

Approximation algorithms, vertex cover, and linear programming

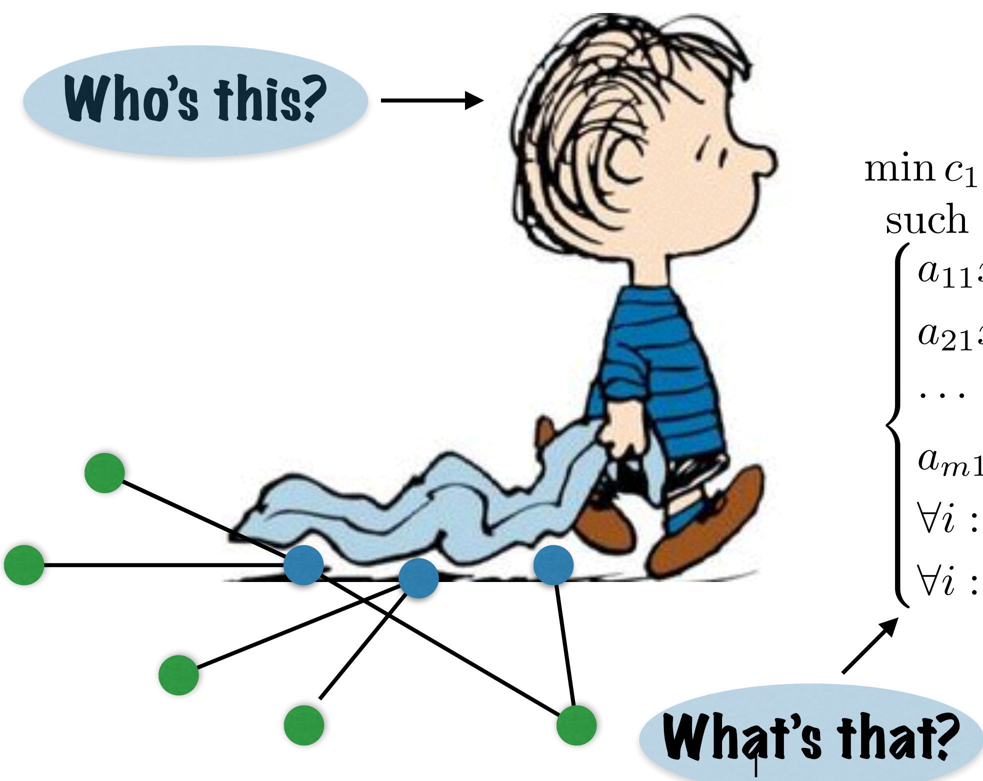
Who's this?



$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
such that
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

What's that?

Approximation algorithms, vertex cover, and linear programming



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$$

Quality of output?

2. Round the LP solution

$$\mathbf{z_u} \in \{0, 1\}$$

3. Output $\{\mathbf{u} : \mathbf{z_u} = 1\}$

Output cost $= \sum_{\mathbf{u}} \mathbf{w_u z_u}$

1. Solve the LP

$$(\mathbf{x_u^*})$$

2. Round the LP solution

$$\mathbf{z_u} = \begin{cases} 1 & \text{if } \mathbf{x_u^*} \geq \mathbf{1/2} \\ 0 & \text{otherwise} \end{cases}$$

Observe: $\mathbf{z_u} \leq \mathbf{2x_u^*}$

1. Solve the LP

IP min: x(u)=1 iff u in optimum vertex cover

$$\min \sum_{\mathbf{u}} \mathbf{w_u x_u}$$

$$\mathbf{x_u} + \mathbf{x_v} \geq 1$$

$$\mathbf{x_u} \in \{0, 1\}$$

LP min: x*(u)

$$\min \sum_{\mathbf{u}} \mathbf{w_u x^*_u}$$

$$\mathbf{x^*_u} + \mathbf{x^*_v} \geq 1$$

$$\mathbf{0} \leq \mathbf{x^*_u} \leq 1$$

The LP is a relaxation of the IP

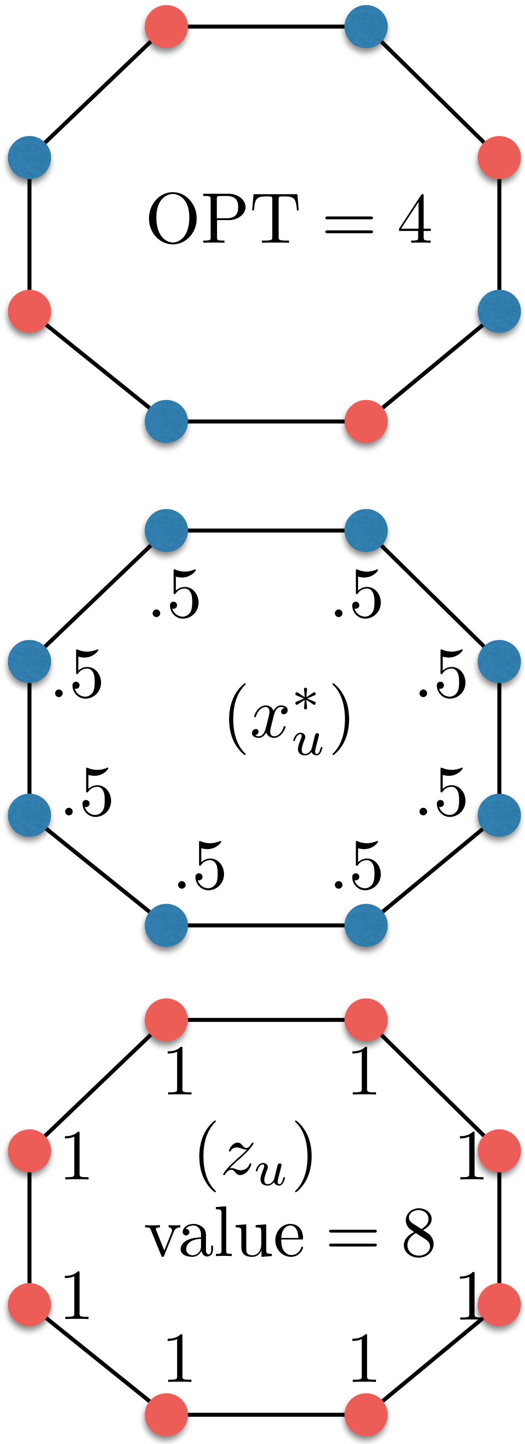
$$\sum_{\mathbf{u}} \mathbf{w_u x^*_u} \leq \mathrm{OPT}$$

Combine

$$\text{Output cost } = \sum_{\mathbf{u}} \mathbf{w_u z_u}$$
$$\leq \mathbf{2} \sum_{\mathbf{u}} \mathbf{w_u x_u^*}$$
$$\leq \mathbf{2}\mathrm{OPT}$$

Thm: output is a vertex cover of value at most 2 OPT

Is the analysis tight?



Yes

How good is that?

Typical performance (hearsay):
within 10% of optimum

How do we know?

Can compare output value to

$$\sum_{\mathbf{u}} \mathbf{w}_{\mathbf{u}} \mathbf{x}^*_{\mathbf{u}}$$

Approximation algorithms, vertex cover, and linear programming

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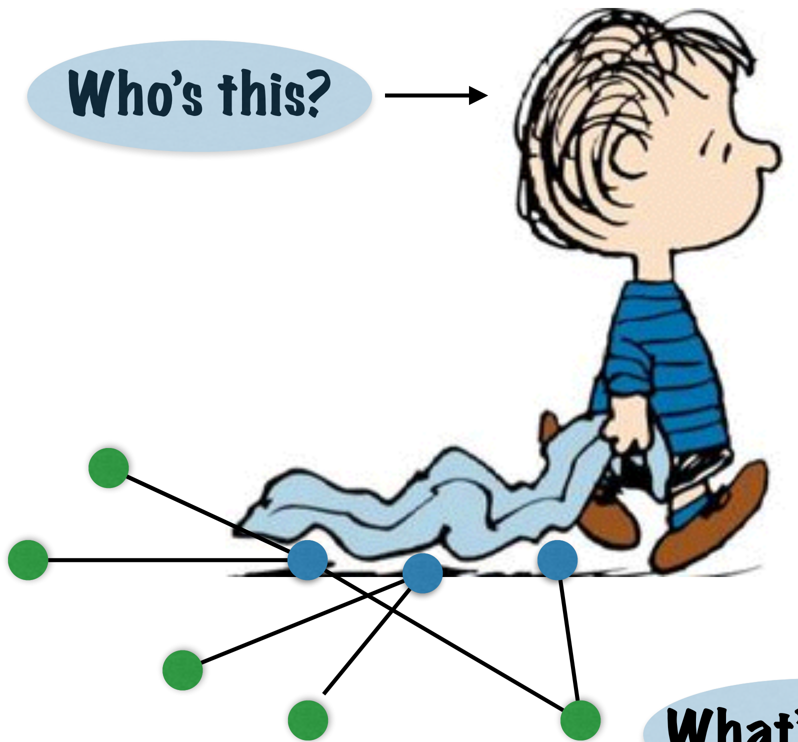

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What's that?

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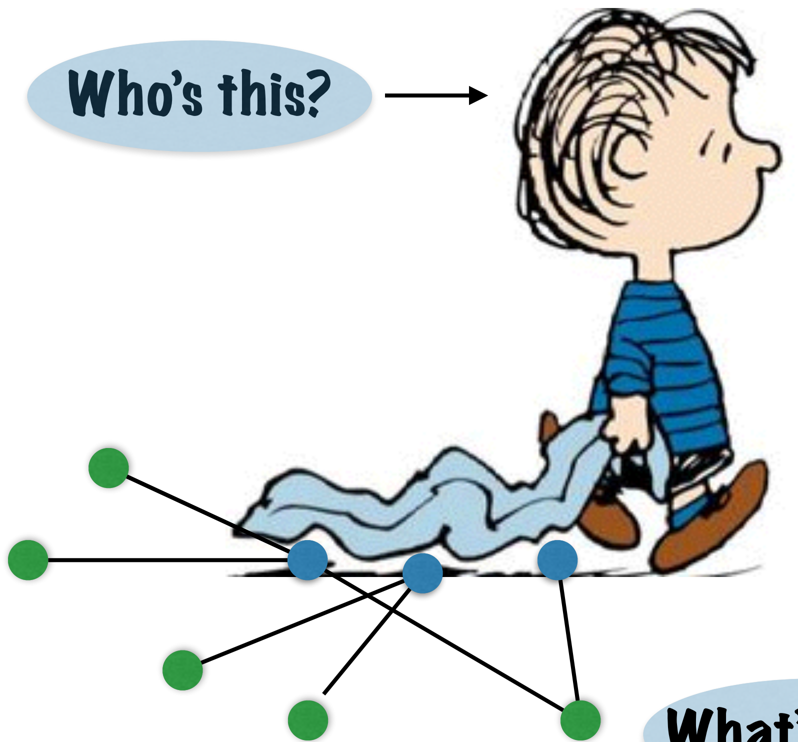

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What's that?

The method

Meta approximation algorithm

- Find IP
- Solve LP relaxation
- Round solution to integers

Analysis

- correct: does it satisfy conditions?
- efficient: polynomial runtime?
- good: value of output solution within factor of optimal value?

How good is it?

Method

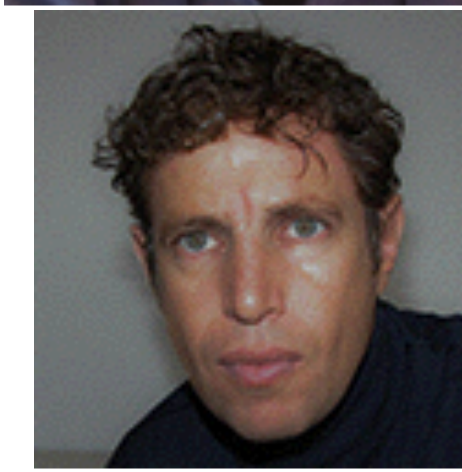
- Output can be related to LP value
- OPT can be bounded by LP value
- Combine

Message: for analysis, focus on LP value.

Karp (1972)
NP-complete

Khot
Regev
(2003)
Conditional <2 hard

Papadimitriou
Yannakakis:
1.0001 hard (1991)



Dinur
Safra:
1.36 hard (2002)

Approximation algorithms, vertex cover, and linear programming

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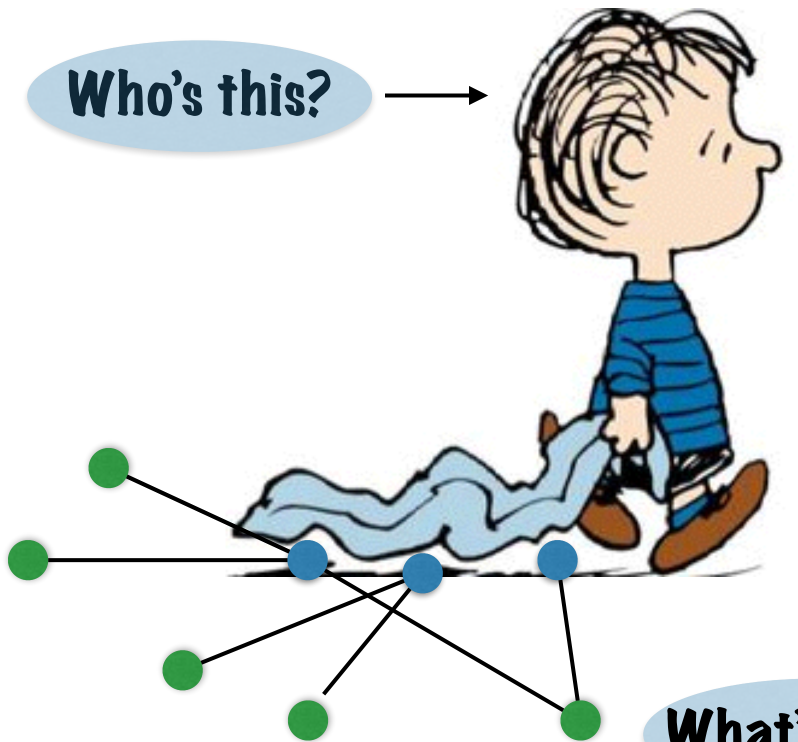

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What's that?

Approximation algorithms, vertex cover, and linear programming

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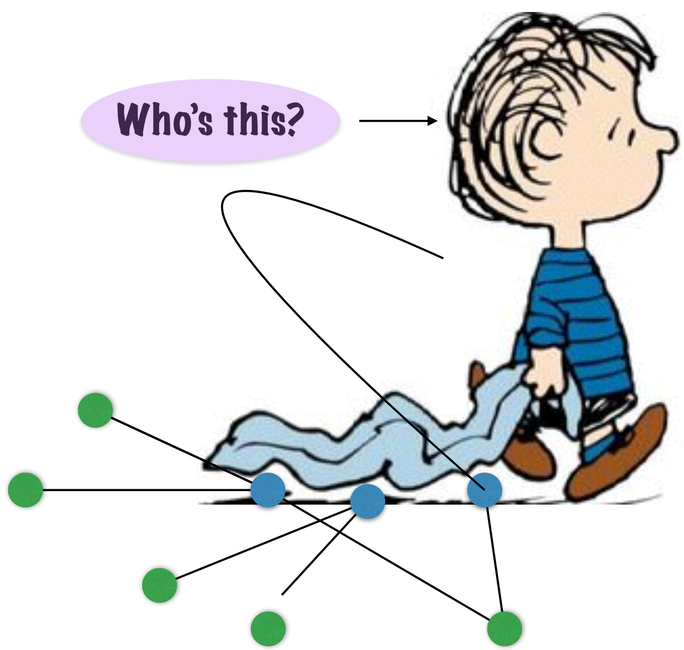

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What's that?

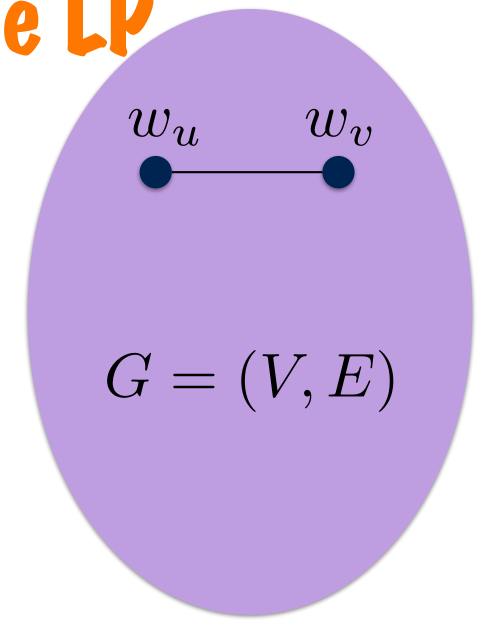
Property of the LP

Constraints:

$$\forall u \in V : 0 \leq x_u \leq 1$$

$$\forall \{u, v\} \in E : x_u + x_v \geq 1$$

Objective: $\min \sum_u w_u x_u$

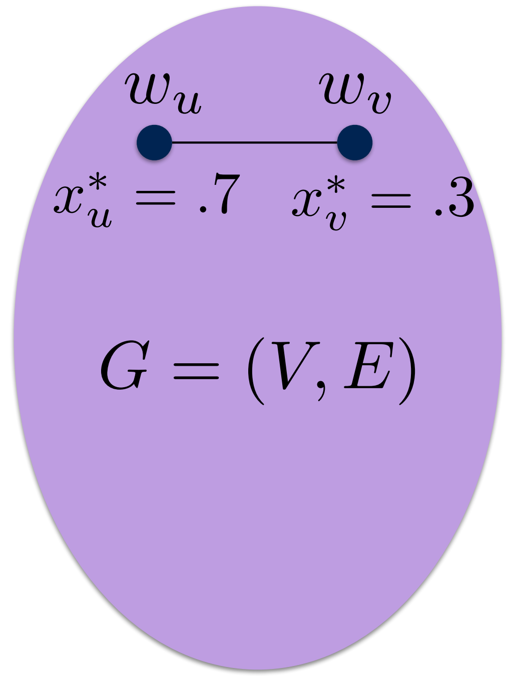


Theorem:

there exists an optimal solution
s.t. every coordinate is in {0, .5, 1}
and there is a polynomial-time algorithm to construct it

1. Solve the LP

$$\Longrightarrow \ (x_u^*)_{u \in V} \text{ such that}$$
$$\forall u \in V : 0 \le x_u^* \le 1$$
$$\forall \{u, v\} \in E : x_u^* + x_v^* \ge 1$$
$$\sum_u w_u x_u^* \text{ minimum}$$



2. Freeze all variables with value in {0, .5, 1}

3. While some variables are not frozen

$$\mathbf{L} = \{\mathbf{u} : .5 < \mathbf{x}^*_\mathbf{u} < 1\}$$

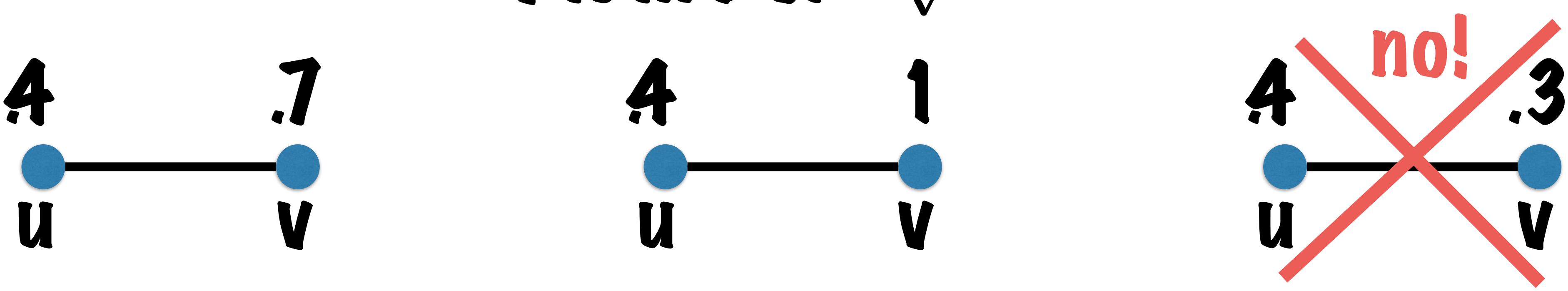
$$\mathbf{S} = \{\mathbf{u} : 0 < \mathbf{x}^*_\mathbf{u} < .5\}$$

Observe:

if u is in S and uv is an edge

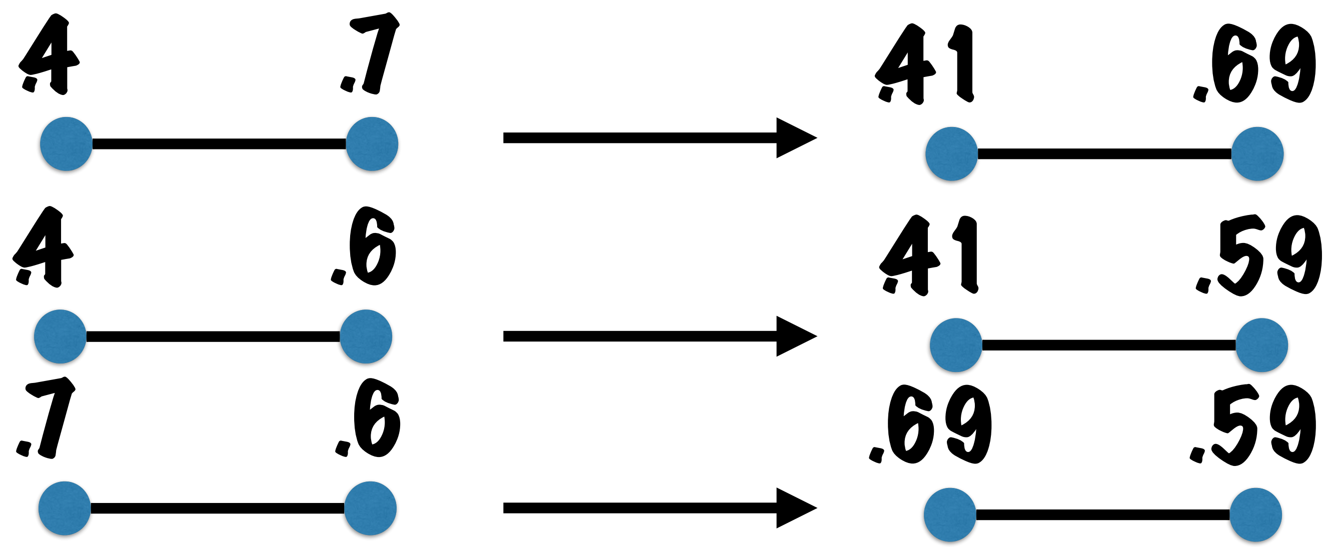
then

v is in L or $\mathbf{x}^*_\mathbf{v} = 1$



$$\mathbf{y_u} = \begin{cases} x_u^* & \textbf{if } u \textbf{ frozen} \\ x_u^* + \epsilon & \textbf{if } u \in S \\ x_u^* - \epsilon & \textbf{if } u \in L \end{cases}$$

Observe: for ϵ small, it is still feasible.



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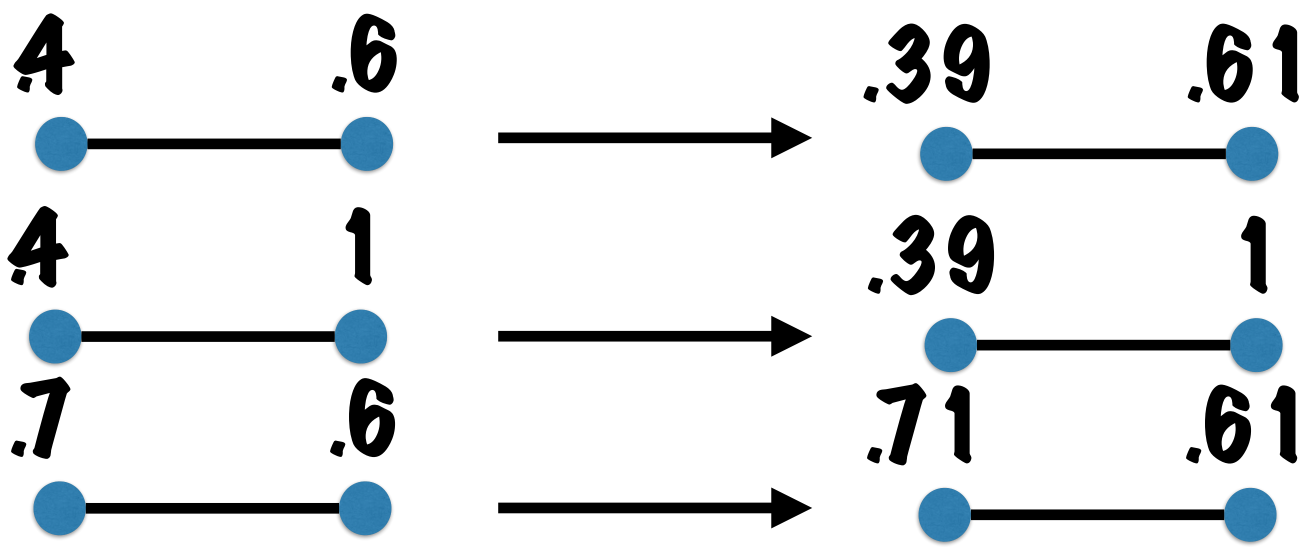

$$\mathbf{y_u} = \begin{cases} x_u^* & \textbf{if } u \textbf{ frozen} \\ x_u^* + \epsilon & \textbf{if } u \in S \\ x_u^* - \epsilon & \textbf{if } u \in L \end{cases} \qquad \mathbf{z_u} = \begin{cases} x_u^* & \textbf{if } u \textbf{ frozen} \\ x_u^* - \epsilon & \textbf{if } u \in S \\ x_u^* + \epsilon & \textbf{if } u \in L \end{cases}$$

Since y feasible and x* optimal: $\sum \mathbf{w_u y_u} \geq \sum \mathbf{w_u x_u^*}$

Since z feasible and x* optimal: $\sum \mathbf{w_u z_u} \geq \sum \mathbf{w_u x_u^*}$

But observe: $(\sum \mathbf{w_u y_u} + \sum \mathbf{w_u z_u})/\mathbf{2} = \sum \mathbf{w_u x_u^*}$

So: $\sum_\mathbf{u} \mathbf{w_u y_u} = \sum_\mathbf{u} \mathbf{w_u z_u} = \sum_\mathbf{u} \mathbf{x_u^*}$

y and z are also optimal solutions

increase ϵ until something happens:

$$\mathbf{y_u} = \begin{cases} x_u^* & \text{if } u \text{ frozen} \\ x_u^* + \epsilon & \text{if } u \in S \\ x_u^* - \epsilon & \text{if } u \in L \end{cases} \qquad \mathbf{z_u} = \begin{cases} x_u^* & \text{if } u \text{ frozen} \\ x_u^* - \epsilon & \text{if } u \in S \\ x_u^* + \epsilon & \text{if } u \in L \end{cases}$$

reaches .5 reaches .5 reaches 0 reaches 1

Freeze the variable that reached 0, .5, or 1

$$\mathbf{x}^* \leftarrow \mathbf{y} \text{ or } \mathbf{z}$$

Repeat...

QED

Approximation algorithms, vertex cover, and linear programming

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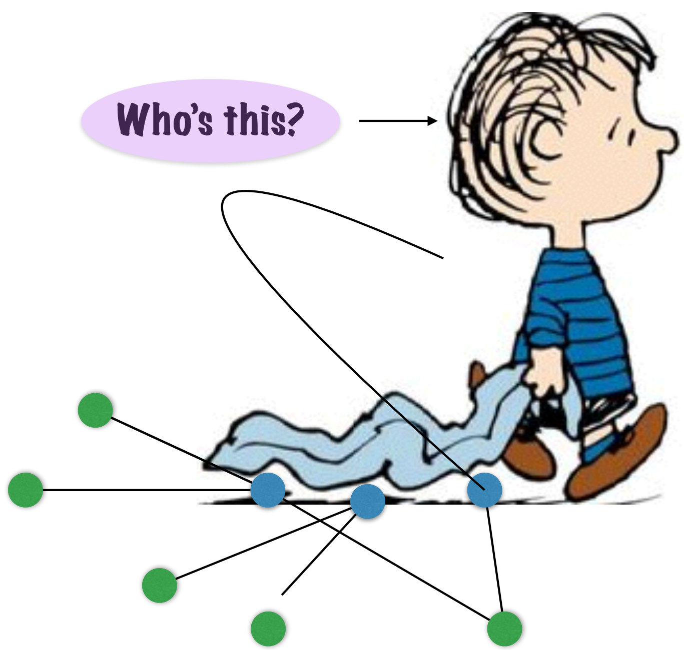

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What's that?