A 3/2-approximation for 4-colourable Graphs. In this exercise, we propose to derive a 3/2-approximation algorithm for a more restricted class of graphs. We recall that a graph is said to be *4-colourable* if given a set of 4 colours, it is possible to assign a colour of the set to each vertex in such a way that for each edge (u, v) of the graph, u and v receive different colours. A 4-colouring of G is an assignment of colours to the vertices of G such that for each edge (u, v) of the graph, u and v receive different colours. Consider the Linear Program for Vertex Cover that was described during the lectures and assume that we obtained a solution for this program such that the value of each variable is either 0, 1/2 or 1. Namely, we have an **optimal** fractional assignment of the variables X such that for each $x_v \in X$, $x_v \in \{0, 1/2, 1\}$. We denote by $\mathrm{val}(X)$ the objective value of the assignment X for the LP. [1]
We consider a 4-colouring C of the vertices of G. Let $V^{1/2} = \{v \mid x_v = 1/2\}$, i.e: the set of vertices v such that $x_v = 1/2$ and $V^1 = \{v \mid x_v = 1\}$, i.e the set of vertices v such that $x_v = 1$. Similarly, $V^0 = \{v \mid x_v = 0\}$. Moreover let $V_0^{1/2}, V_1^{1/2}, V_2^{1/2}, V_3^{1/2}$ the set of vertices of V that have colour 0, 1, 2, and 3 respectively in C, i.e: $V_0^{1/2} = \{v \mid x_v = 1/2, C(v) = 0\}$. Finally, we assume that $|V_0^{1/2}| \leq |V_1^{1/2}| \leq |V_2^{1/2}| \leq |V_3^{1/2}|$.

A Rounding Procedure. We propose to define a rounding procedure for this assignment. We build a solution S. For each variable $x_v = 1$, v is added to the solution S. $x_v = 1/2$ and such that $C(v) \neq 3$, v is added to S. Otherwise, v is discarded.

Approximation Ratio.
Question 1. Give a relation between the value $\mathrm{val}(X)$ the cardinality of the sets $V^{1/2}$ and V^1.
Question 2. Give a tight lower bound on the cardinality of $V_3^{1/2}$ based on the cardinality of $V^{1/2}$.
Question 3. Deduce from Question 2 an upper bound on the cardinality of $|V_0^{1/2}| + |V_1^{1/2}| + |V_2^{1/2}|$ based on the cardinality of $V^{1/2}$.
Question 4. Combine Questions 1 and 3 to give an upper bound on the number of vertices in S based on $\mathrm{val}(X)$.
Question 5. Combine Question 4 and the property of $\mathrm{val}(X)$ to conclude on the approximation ratio of the rounding procedure. Recall that $\mathrm{val}(X)$ is the value of the optimal fractional solution to the LP.

Correctness We now show that S is a correct vertex cover. Namely, we want to show that for each edge (u, v), u or v (or both of them) are in S. We will proceed by contradiction and assume that neither u or v are in S.
Question 6. Suppose that $u \in V^0$, to which set does v belong? Recall that X is a solution to the LP.
Question 7. Deduce from the previous question to which set do u and v be-

[1] such a solution is an extreme point solution of the LP and can be found in polynomial time.

long.
Question 8. If u and v belong to $V^{1/2}$, to which subset of V do they belong if they do not belong to S?
Question 9. Recall that C is a 4-colouring. Explain the contradiction.
Question 10. Give an example of a well-known class of graph that is 4-colourable.