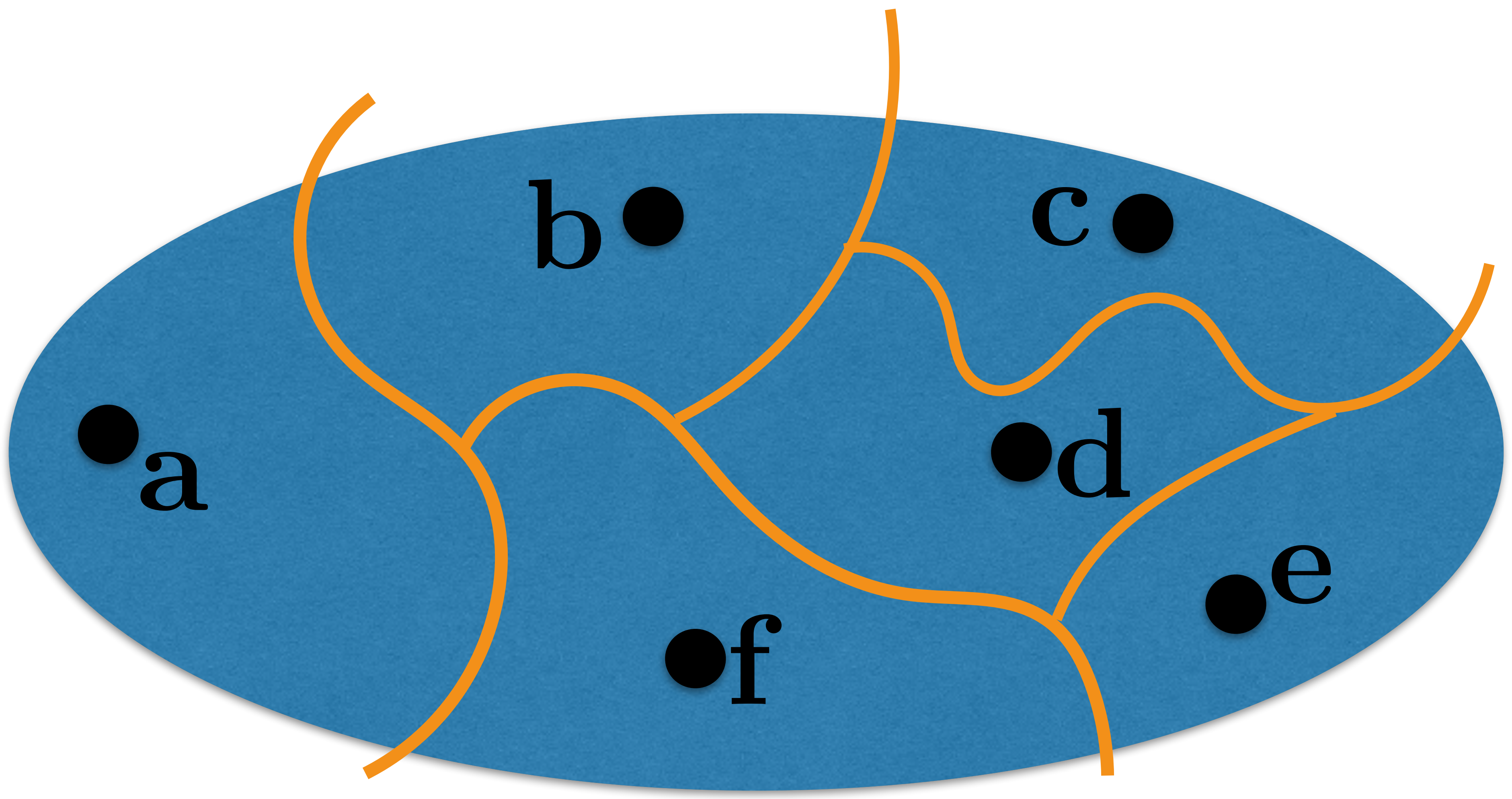
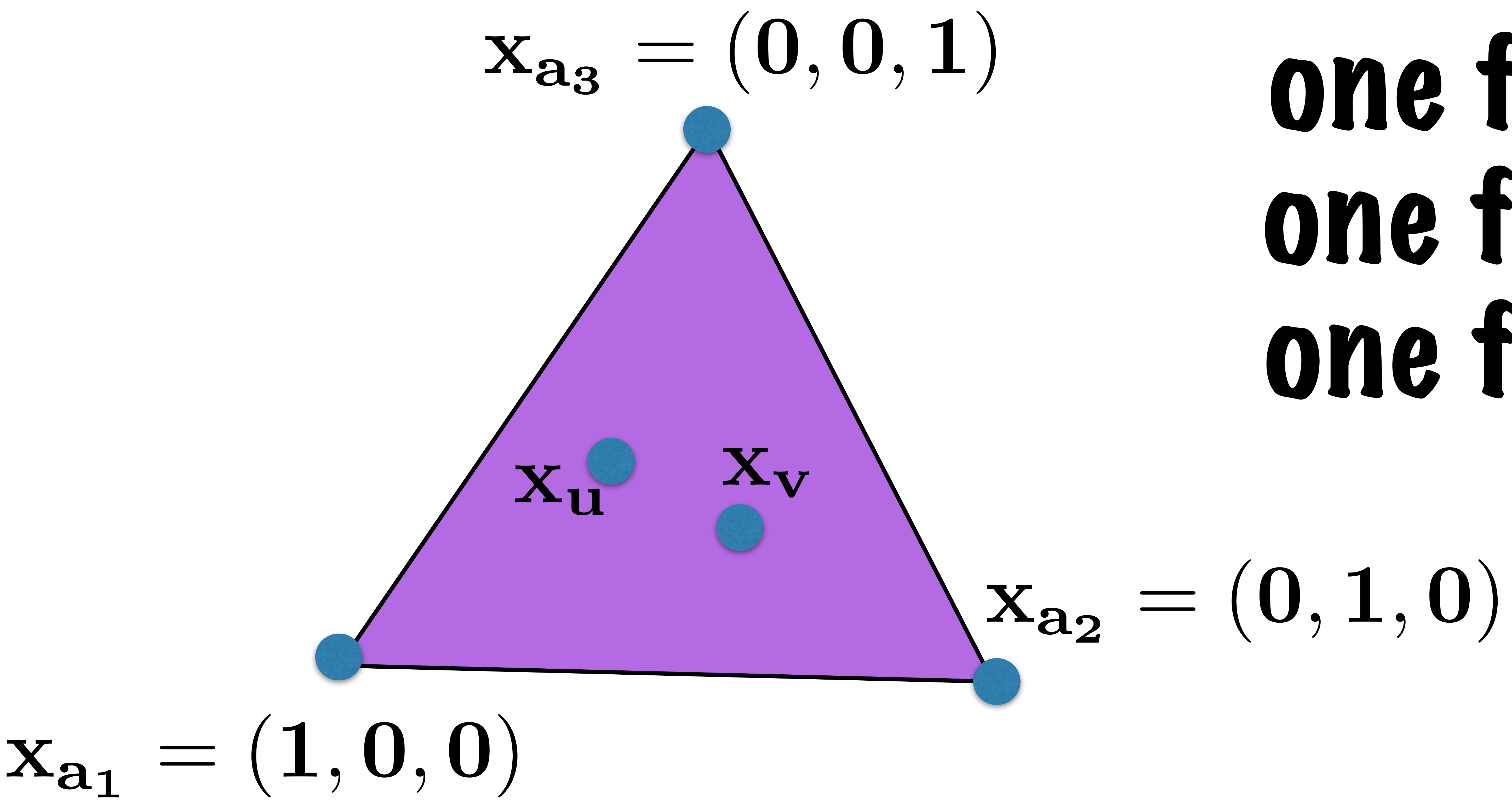
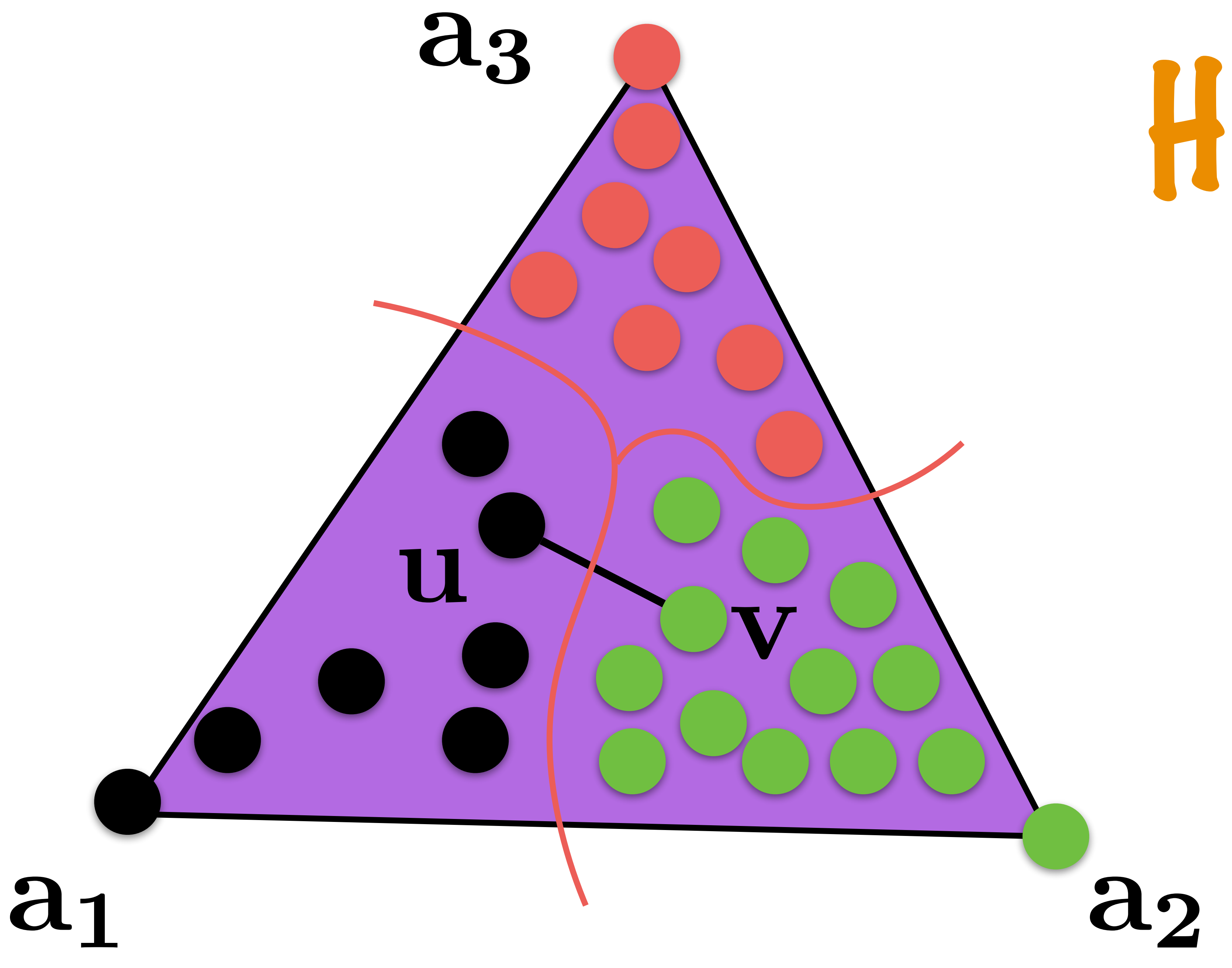
Multiway cut, linear programming and randomized rounding



How do we round?



Partition triangle into three areas:
one for (0,0,1)
one for (1,0,0)
one for (0,1,0)



How do we round?

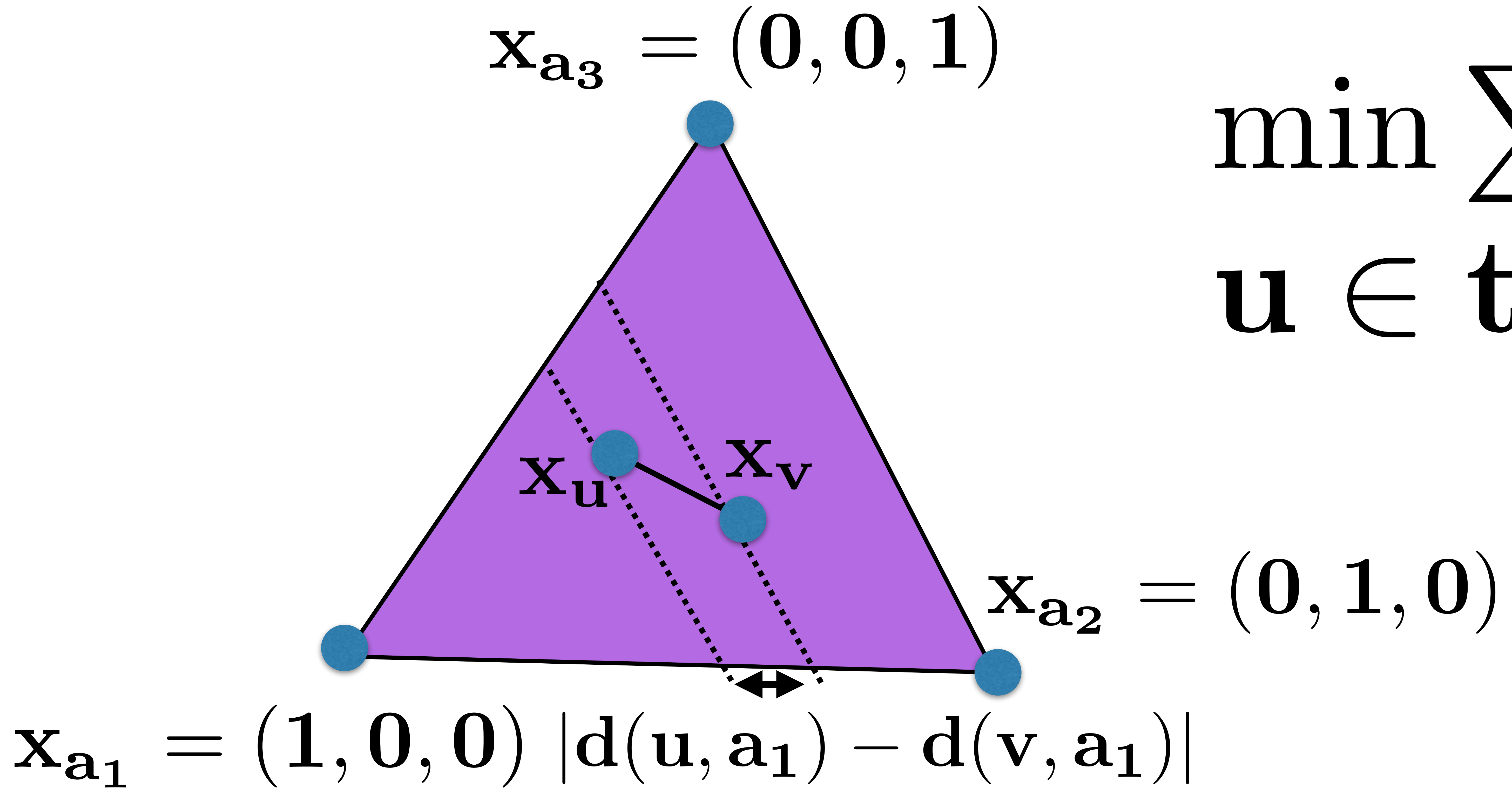
Vertices ● go with a1
Vertices ● go with a2
Vertices ● go with a3
Pay cost of edges across

Good rounding = small cost choice of partition of triangle into three areas

LP relaxation

$$\mathbf{d(u,v)} = \tfrac{1}{2}|\mathbf{x_u} - \mathbf{x_v}|_1$$

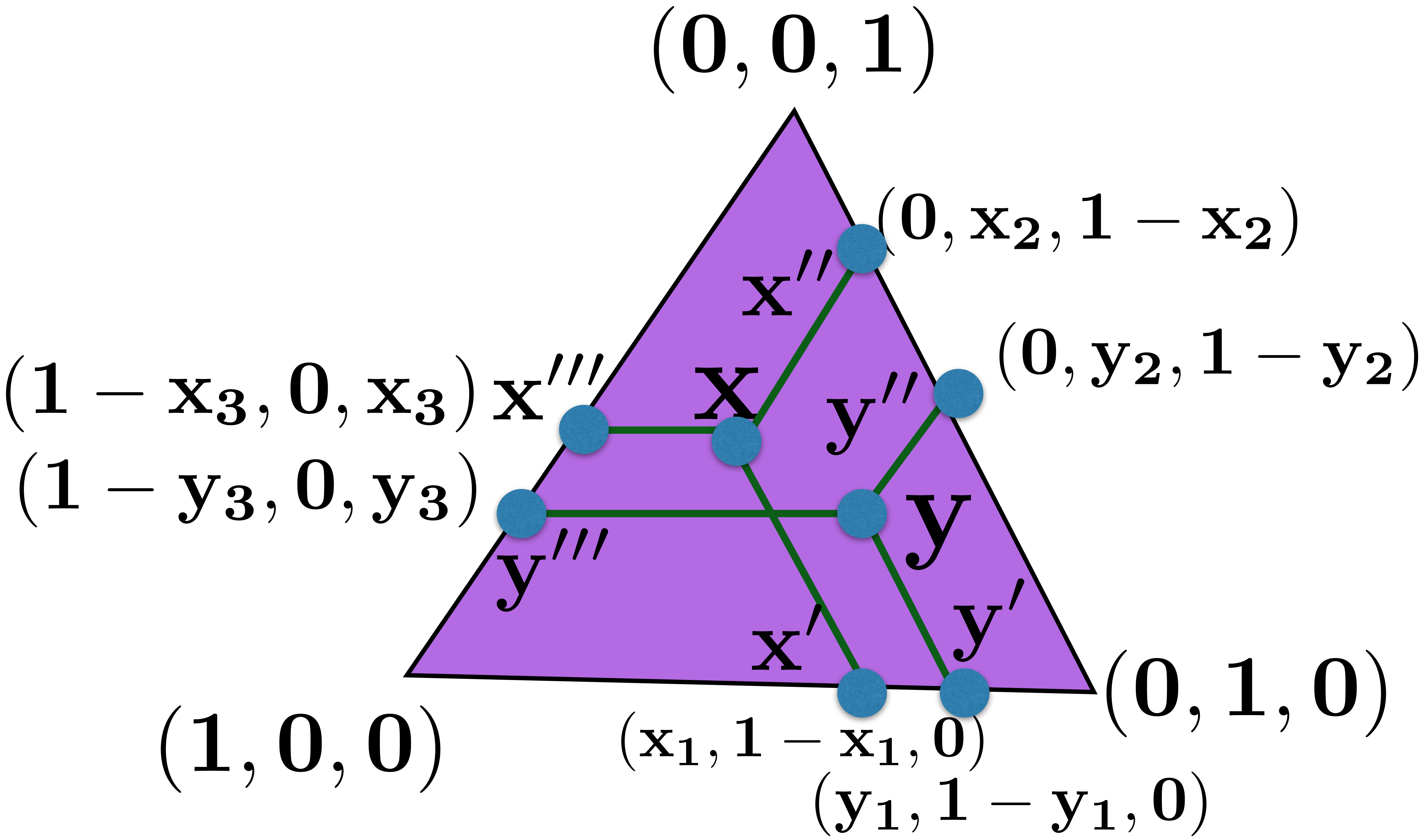
Place vertices in triangle, min lengths of edge projections on sides



$$\min \sum_{\mathbf{uv}\in\mathbf{E}} \mathbf{c_{uv}}\mathbf{d(u,v)} :$$

$$\mathbf{u} \in \mathbf{triangle} \quad \forall \mathbf{u}$$

How do we round?



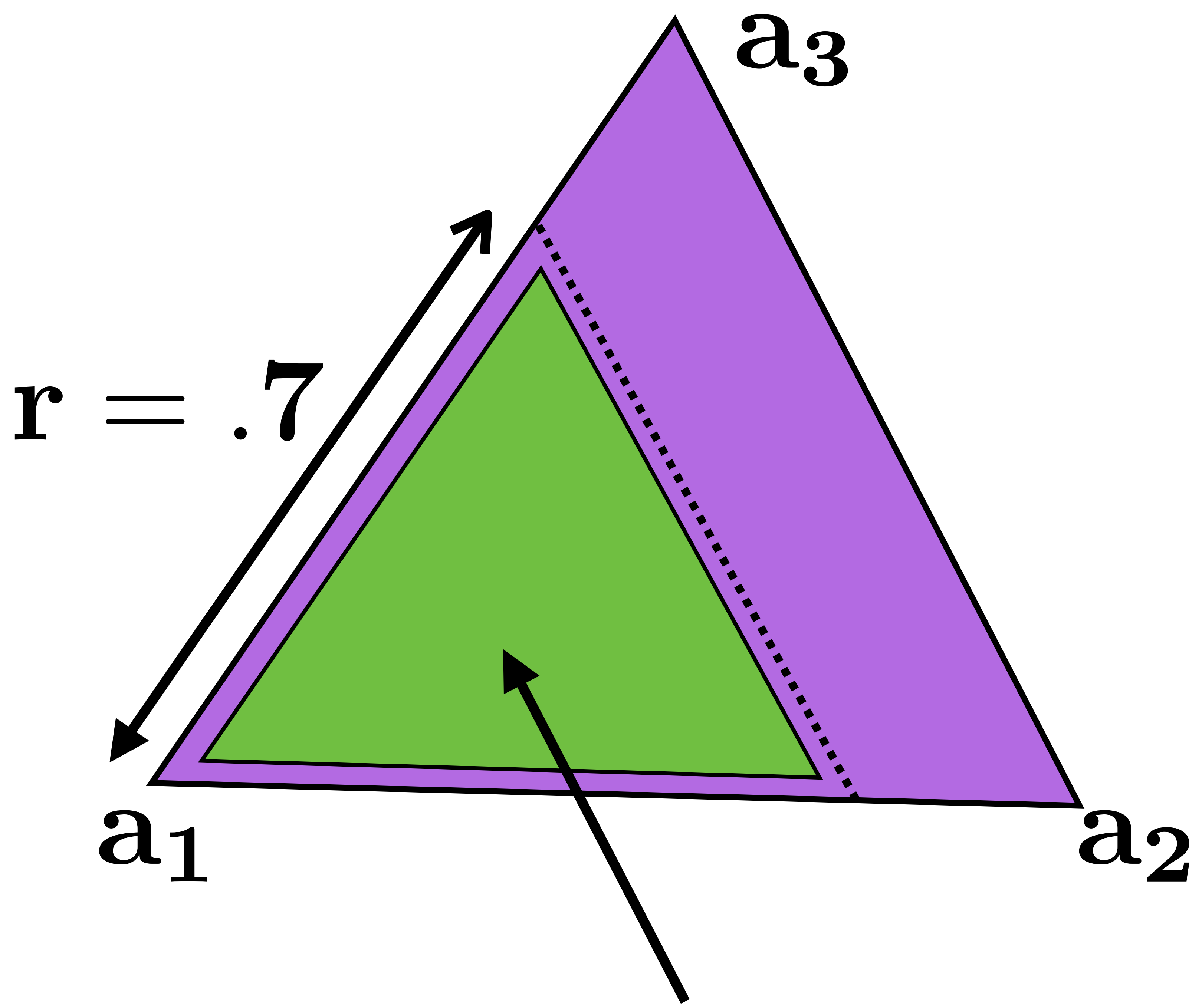
$$d(x', y') = |x_1 - y_1|$$

$$\begin{aligned} d(u,v) &= \tfrac{1}{2}(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|) \\ &= \tfrac{1}{2}(d(x',y') + d(x'',y'') + d(x''',y''')) \end{aligned}$$

Using balls in l1 metric

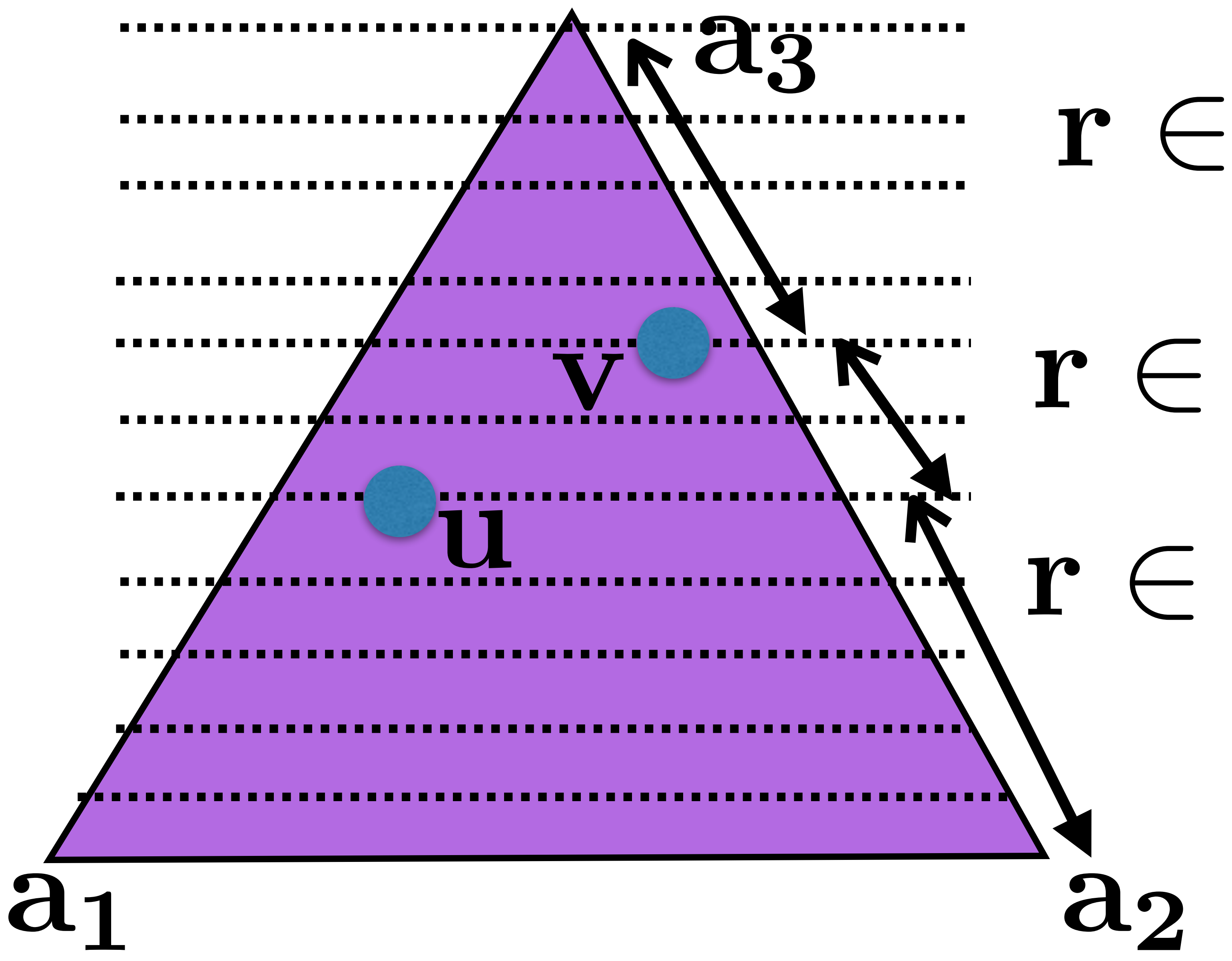
$$\mathbf{d(u, v) = \tfrac{1}{2}|u - v|_1}$$

$$\mathbf{d(a_i, a_j) = 1}$$



$$\mathbf{B(a_1, r) = \{u : d(a_1, u) \leq r\}}$$

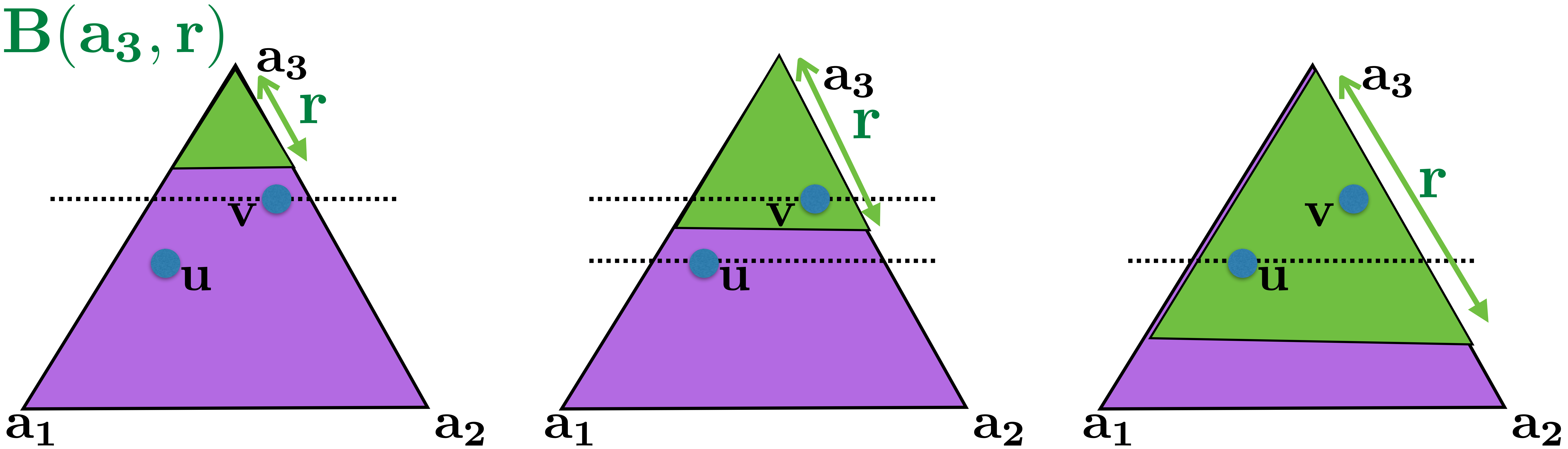
Pick random r, assign B(a3,r) to a3



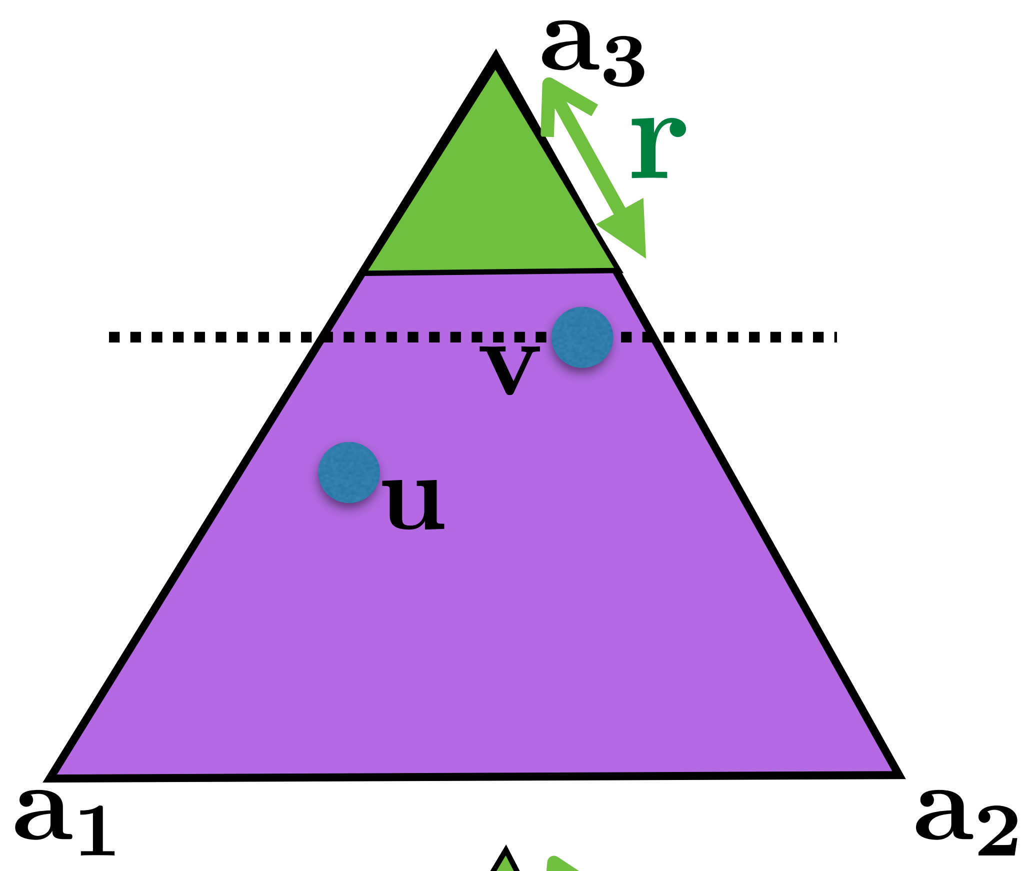
$$\mathbf{r} \in [\mathbf{0}, \mathbf{d}(\mathbf{v}, \mathbf{a_3})]$$

$$\mathbf{r} \in [\mathbf{d}(\mathbf{v}, \mathbf{a_3}), \mathbf{d}(\mathbf{u}, \mathbf{a_3})]$$

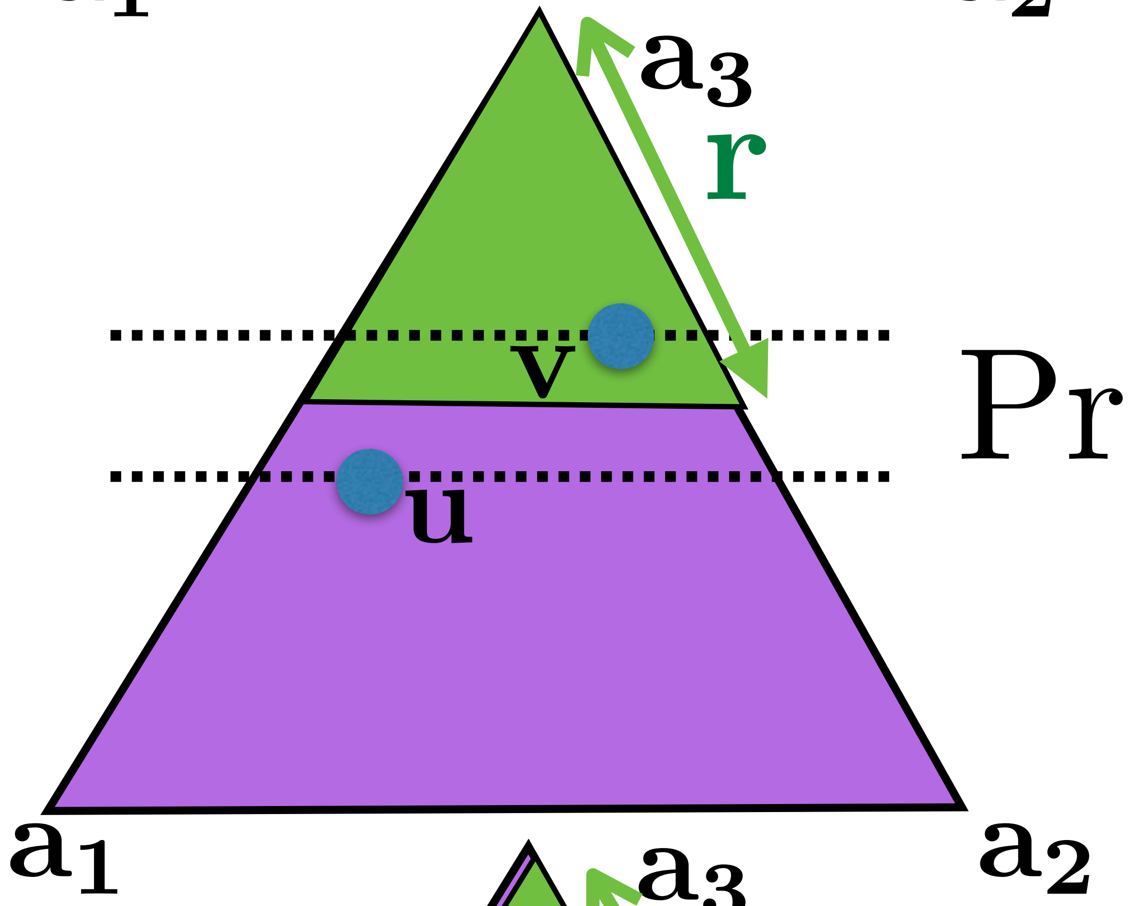
$$\mathbf{r} \in [\mathbf{d}(\mathbf{u}, \mathbf{a_3}), \mathbf{1}]$$



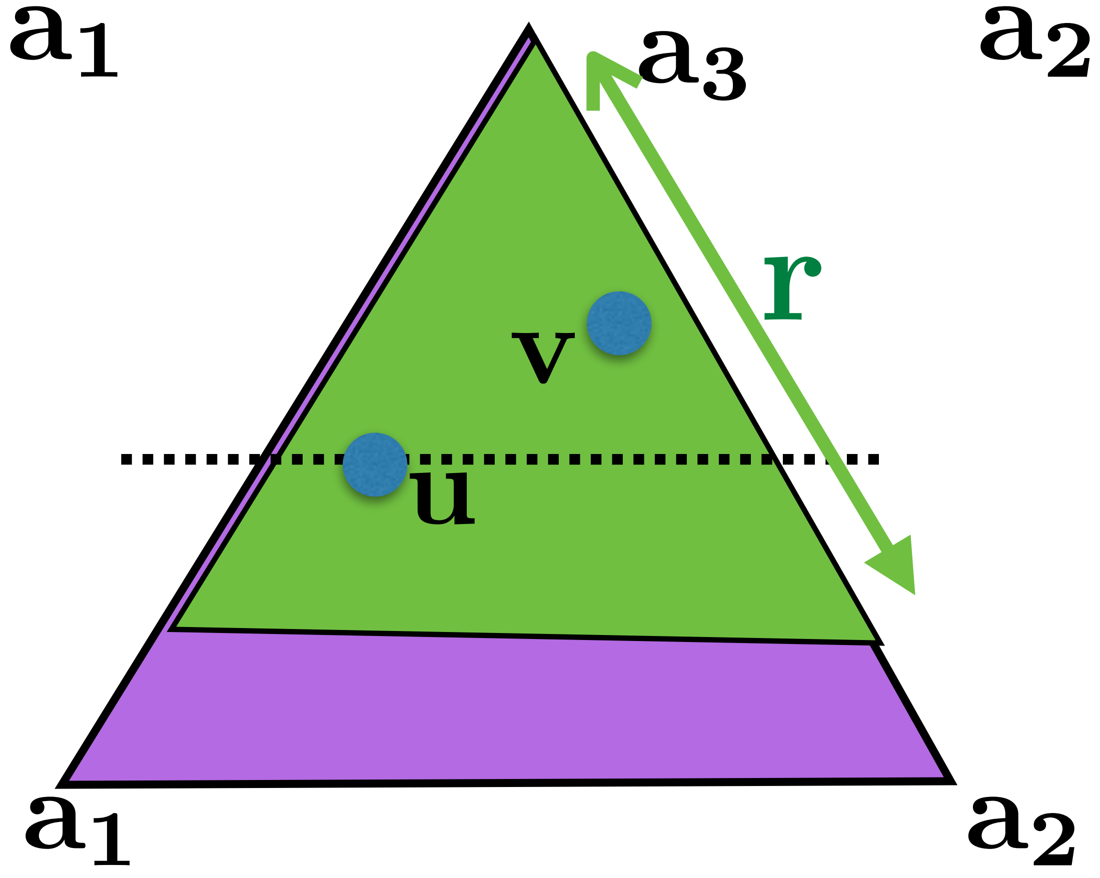
Pick random r, assign B(a3,r) to a3



$$\Pr[\mathbf{u}, \mathbf{v} \notin \mathbf{B}(\mathbf{a_3}, \mathbf{r})] = \Pr[\mathbf{r} < \mathbf{d}(\mathbf{a_3}, \mathbf{v})] = \mathbf{d}(\mathbf{a_3}, \mathbf{v})$$



$$\begin{aligned}\Pr[\mathbf{u}, \mathbf{v} \text{ separated by } \mathbf{B}(\mathbf{a_3}, \mathbf{r})] &= \mathbf{d}(\mathbf{a_3}, \mathbf{u}) - \mathbf{d}(\mathbf{a_3}, \mathbf{v}) \\ &= |\mathbf{u_3} - \mathbf{v_3}|\end{aligned}$$



$$\Pr[\mathbf{u}, \mathbf{v} \in \mathbf{B}(\mathbf{a_3}, \mathbf{r})] = \mathbf{1} - \mathbf{d}(\mathbf{a_3}, \mathbf{u})$$

Consider terminals in random order

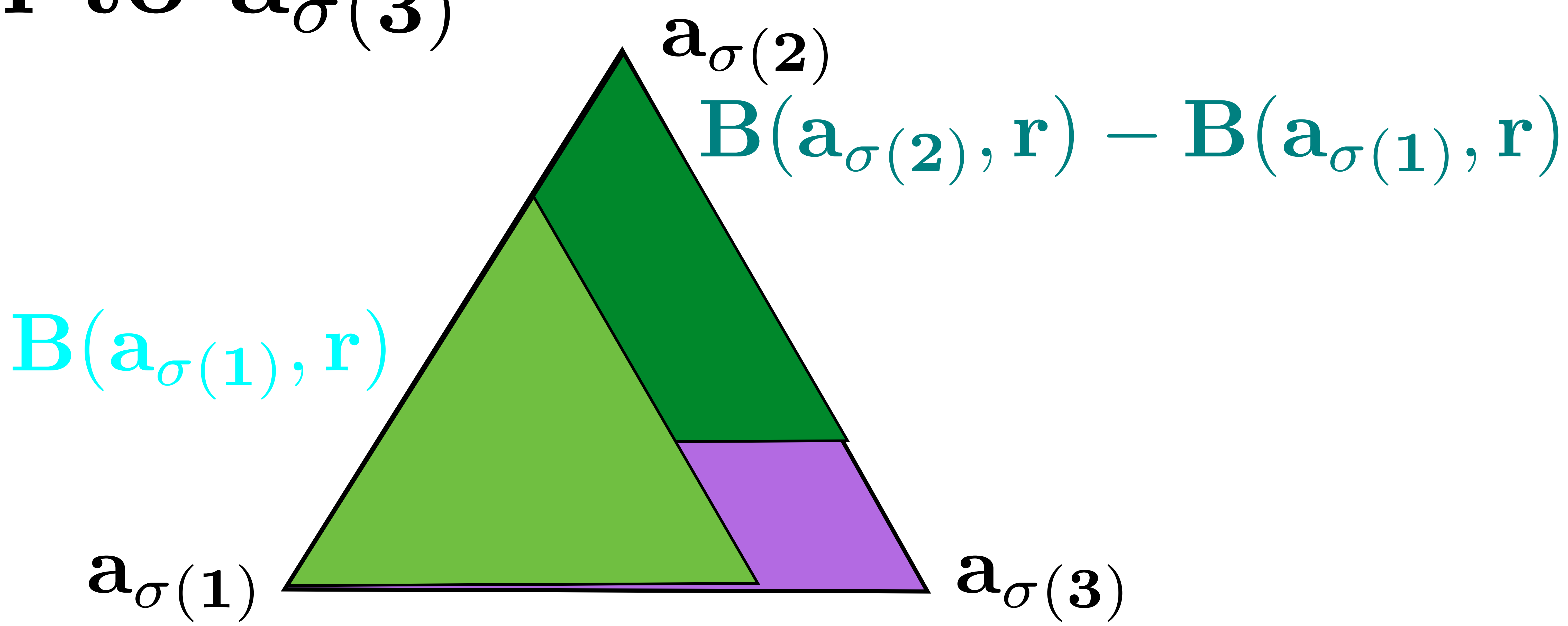
$$\mathbf{a}_{\sigma(1)}, \mathbf{a}_{\sigma(2)}, \mathbf{a}_{\sigma(3)}$$

Assigning u

$\mathbf{if\ d(a_{\sigma(1)}, u) < r\ then\ assign\ to\ a_{\sigma(1)}}$

$\mathbf{else\ if\ d(a_{\sigma(2)}, u) < r\ then\ assign\ to\ a_{\sigma(2)}}$

$\mathbf{else\ assign\ to\ a_{\sigma(3)}}$



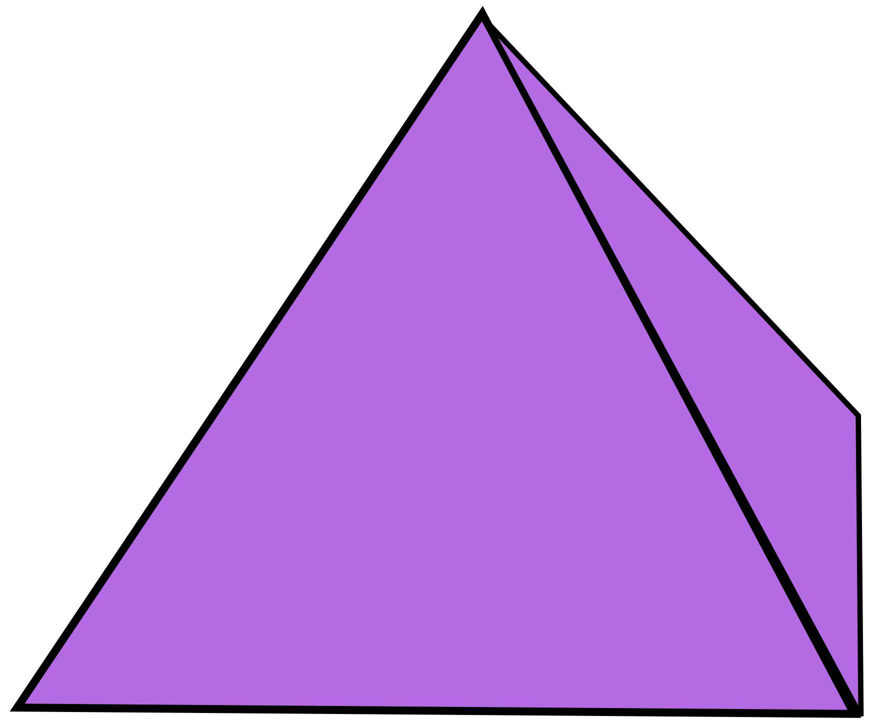
Full Algorithm for 3-way cut

Solve relaxation: embed vertices in triangle
Pick random permutation of terminals
-assign to first terminal a all vertices in B(a,r) where r is random uniform in [0,1]
-assign to second terminal b all unassigned vertices in B(b,r)
-assign to third terminal remaining vertices

Algorithm

$\mathbf{d(u,v) = \frac{1}{2}|x_u - x_v|_1}$

LP relaxation: embed vertices in k-simplex with terminals at corners



$$\min \sum_{uv\in E} c_{uv} d(u,v) :$$

$$u \in \text{k-simplex} \quad \forall u$$

$x_i \geq 0, \ \sum_i x_i = 1$

random ordering $a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(k)}$ $\qquad r \in [0,1]$

for $i = 1, \ldots, k-1$:

  assign to $a_{\sigma(i)}$ unassigned vertices of $B(a_{\sigma(i)}, r)$

assign rest to $a_{\sigma(k)}$

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