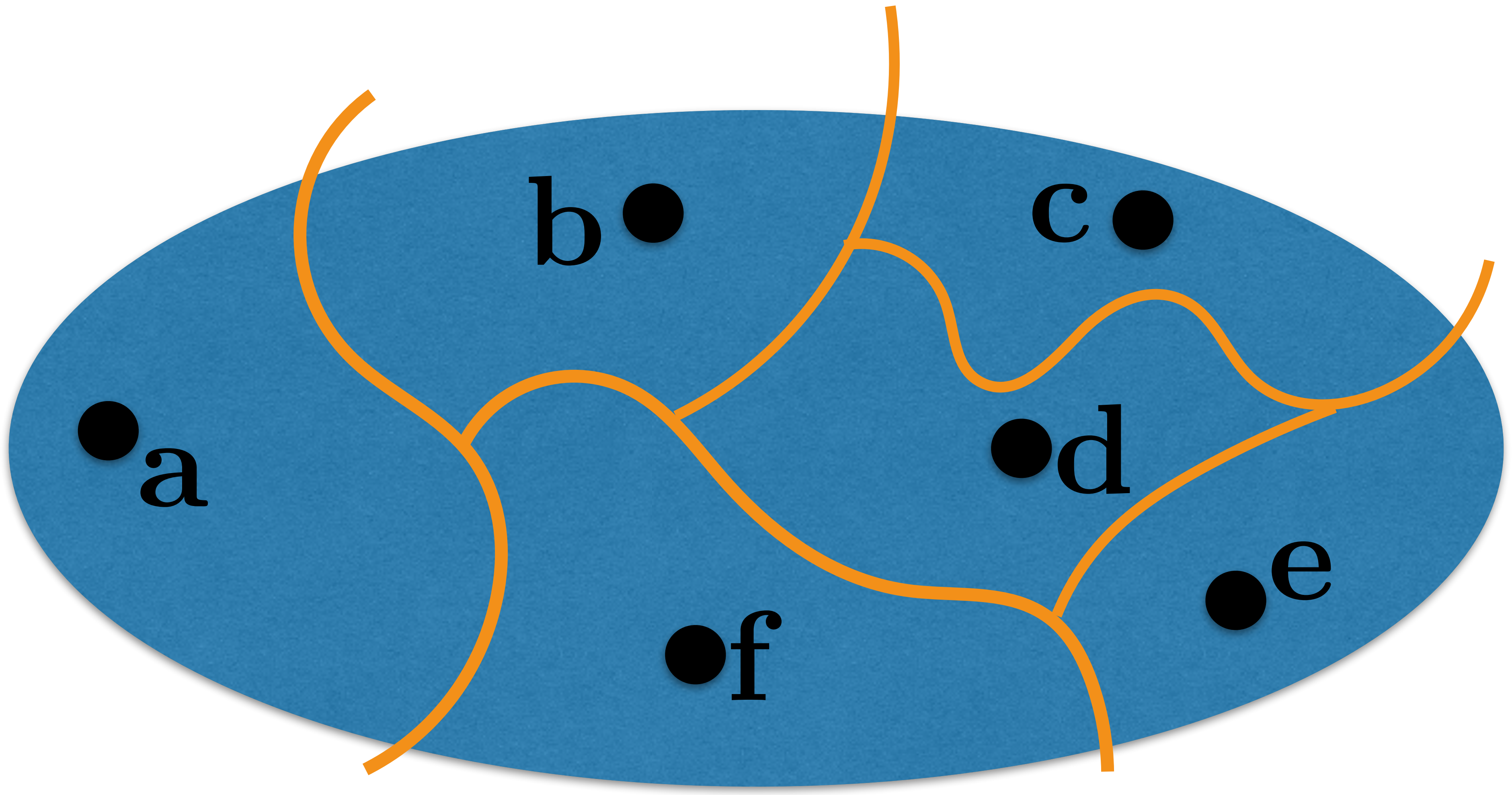
Multiway cut, linear programming and randomized rounding

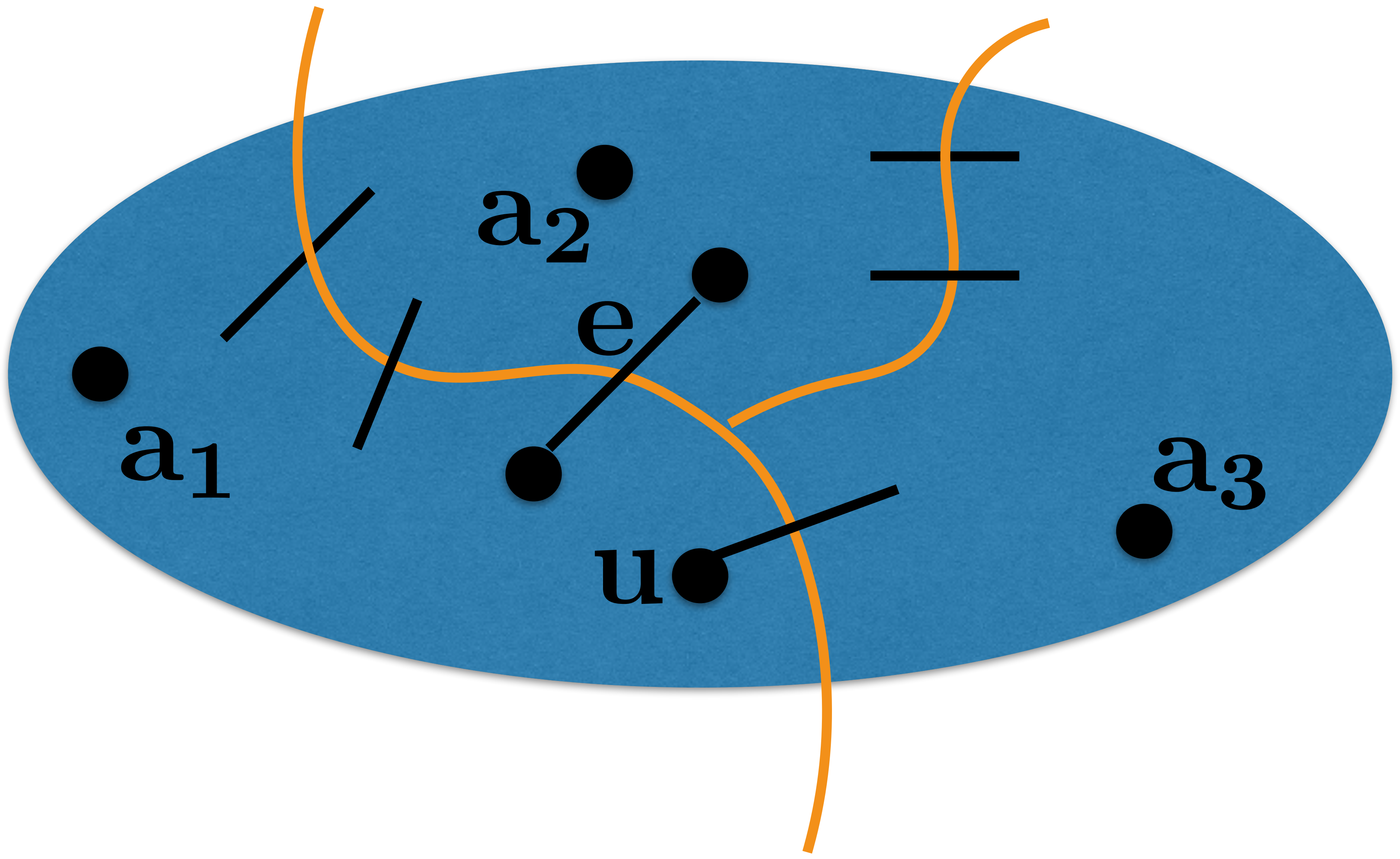


IP model

Variables

$x_{u,i} = 1$ iff vertex u belongs to the cluster of terminal ai.

$z_{e,i} = 1$ iff removal of edge e separates ai from some other terminal



IP model

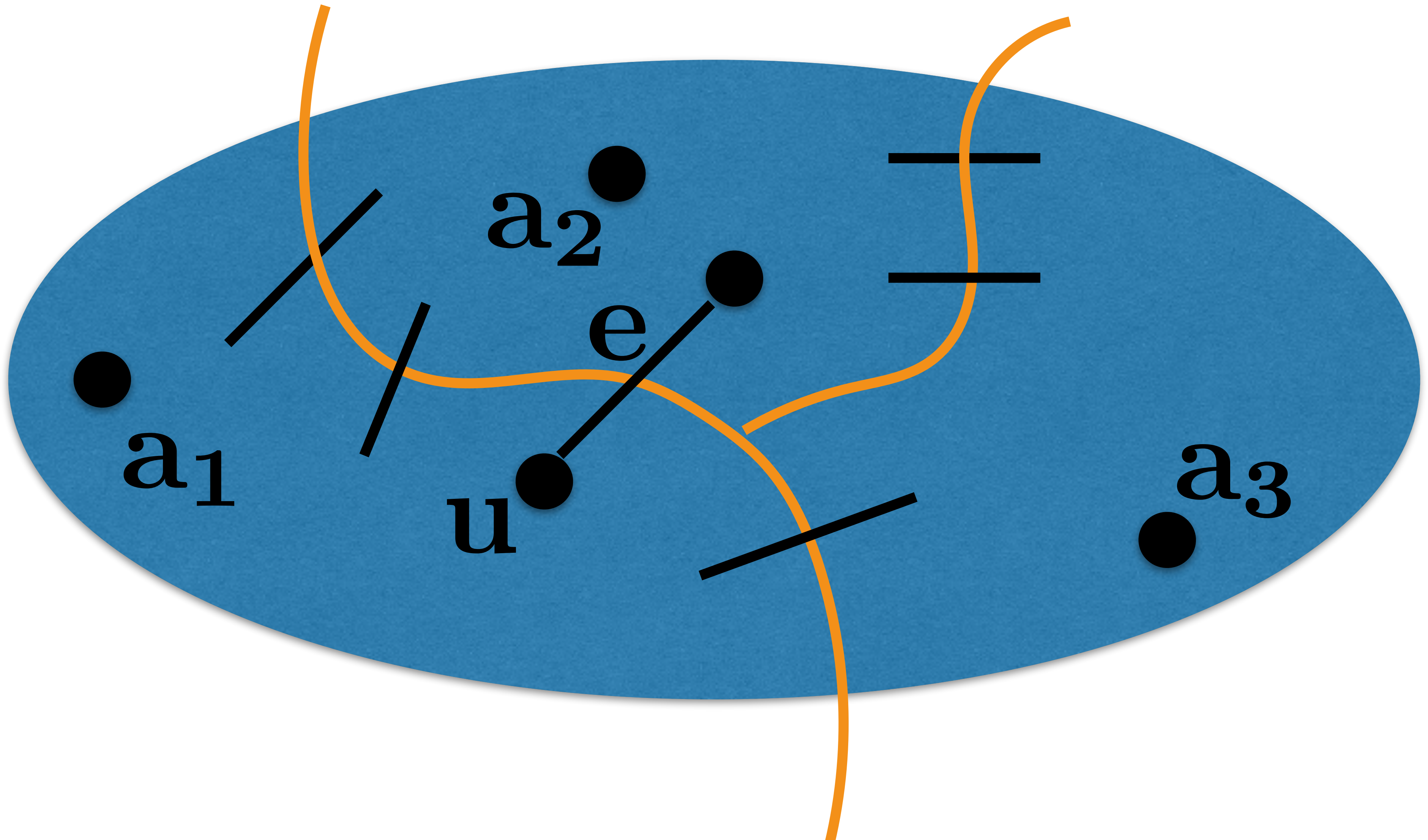
Constraints

$$\mathbf{x_{u,i}, z_{e,i} \in \{0, 1\}}$$

$$\mathbf{x_{a_i,i} = 1 \ (a_i \text{ belongs to its own cluster})}$$

$$\mathbf{\sum_i x_{u,i} = 1 \ (u \text{ belongs to some cluster})}$$

$\mathbf{z_{uv,i} = 1}$ **iff** $\mathbf{x_{u,i} \neq x_{v,i}}$ **???**

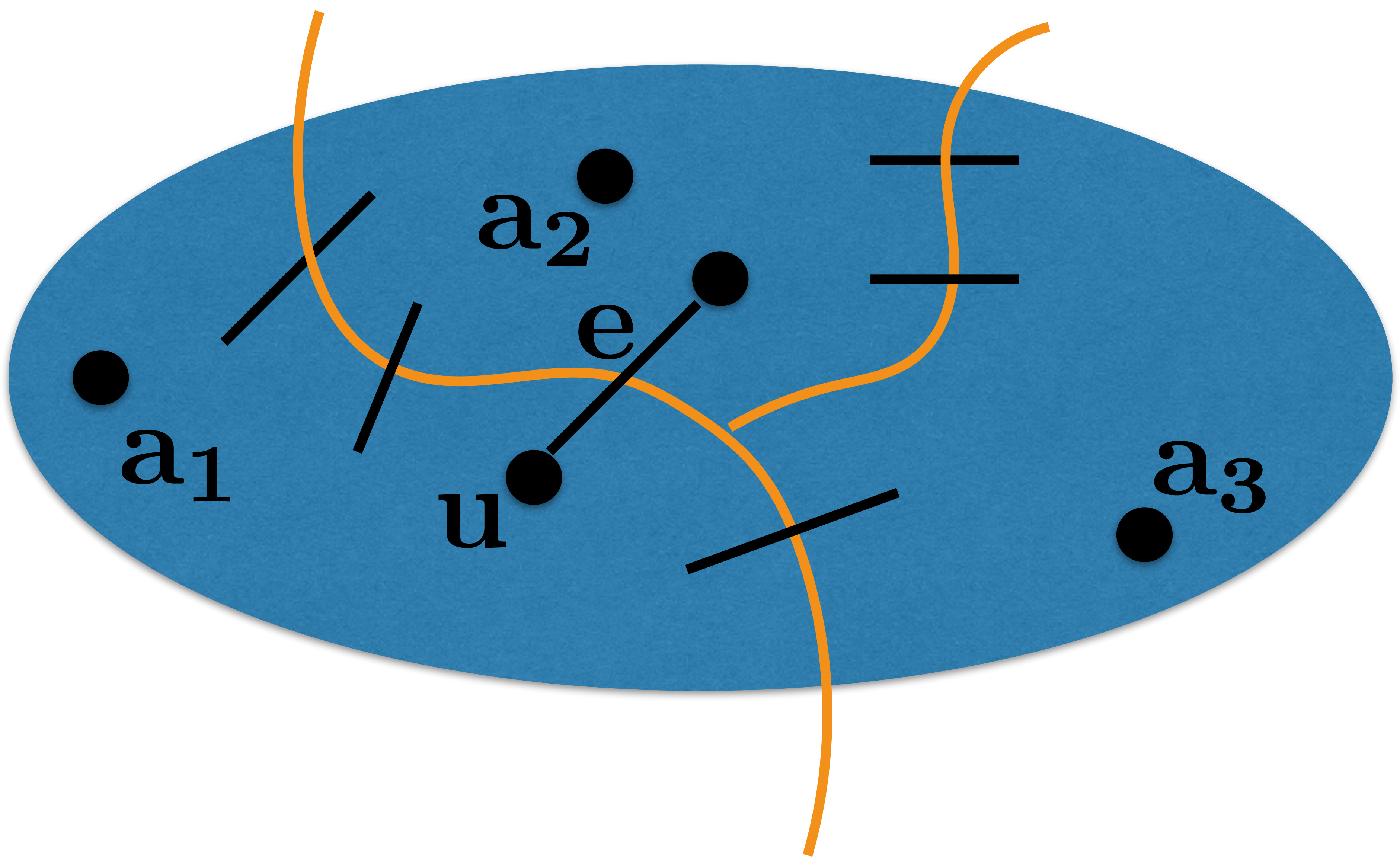


$\mathbf{z_{uv,i} = 1}$ **iff** $\mathbf{x_{u,i} \neq x_{v,i}}$

IP model

Objective

$$\min \quad (\mathbf{1}/\mathbf{2}) \sum_{\mathbf{e,i}} \mathbf{c_e z_{e,i}}$$



$$\mathbf{z_{uv,i} = 1} \quad \textbf{iff} \quad \mathbf{x_{u,i} \neq x_{v,i}}$$

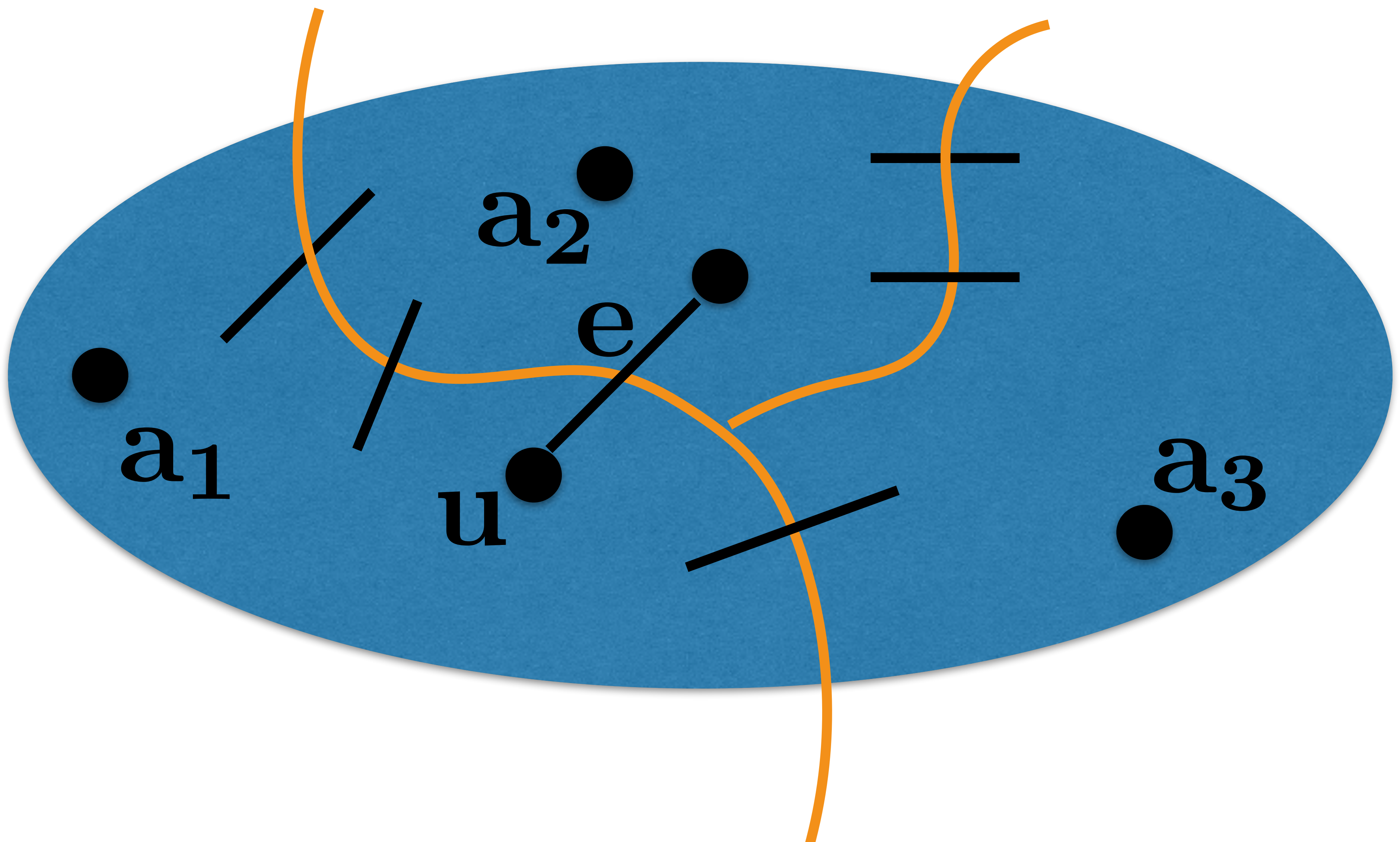
IP model

How do we express that constraint?

$$\mathbf{z_{e,i} \geq |x_{u,i} - x_{v,i}|}$$

$$\mathbf{z_{e,i} \geq x_{u,i} - x_{v,i}}$$

$$\mathbf{z_{e,i} \geq x_{v,i} - x_{u,i}}$$



IP model

$$\mathbf{x_{u,i}}, \mathbf{z_{e,i}} \in \{\mathbf{0}, \mathbf{1}\}$$

$$\mathbf{x_{a_i,i}} = \mathbf{1}$$

$$\sum_{\mathbf{i}} \mathbf{x_{u,i}} = \mathbf{1}$$

$$\mathbf{z_{e,i}} \geq \mathbf{x_{u,i}} - \mathbf{x_{v,i}}$$

$$\mathbf{z_{e,i}} \geq \mathbf{x_{v,i}} - \mathbf{x_{u,i}}$$

$$\min \quad (\mathbf{1}/\mathbf{2}) \sum_{\mathbf{e,i}} \mathbf{c_e z_{e,i}}$$

Linear programming relaxation

$$\mathbf{x_{u,i}, z_{e,i}} \in \{0, 1\} \longrightarrow \mathbf{0 \leq x_{u,i}, z_{e,i} \leq 1}$$

$$\mathbf{x_{a_i,i} = 1}$$

$$\sum_{\mathbf{i}} \mathbf{x_{u,i} = 1}$$

$$\mathbf{z_{e,i} \geq x_{u,i} - x_{v,i}}$$

$$\mathbf{z_{e,i} \geq x_{v,i} - x_{u,i}}$$

$$\min \quad \mathbf{(1/2)} \sum_{\mathbf{e,i}} \mathbf{c_e z_{e,i}}$$

A geometric interpretation

Variables

$\mathbf{x_{u,i}} = 1$ iff vertex u belongs to the cluster of terminal ai.

Vector $\mathbf{x_u} = (\mathbf{x_{u,i}})_\mathbf{i}$

One dimension for each terminal

$$\sum_\mathbf{i} \mathbf{x_{u,i}} = \sum_\mathbf{i} |\mathbf{x_{u,i}}| = |\mathbf{x_u}|_1$$

A geometric interpretation

Vector $\mathbf{x_u} = (\mathbf{x_{u,i}})_\mathbf{i}$

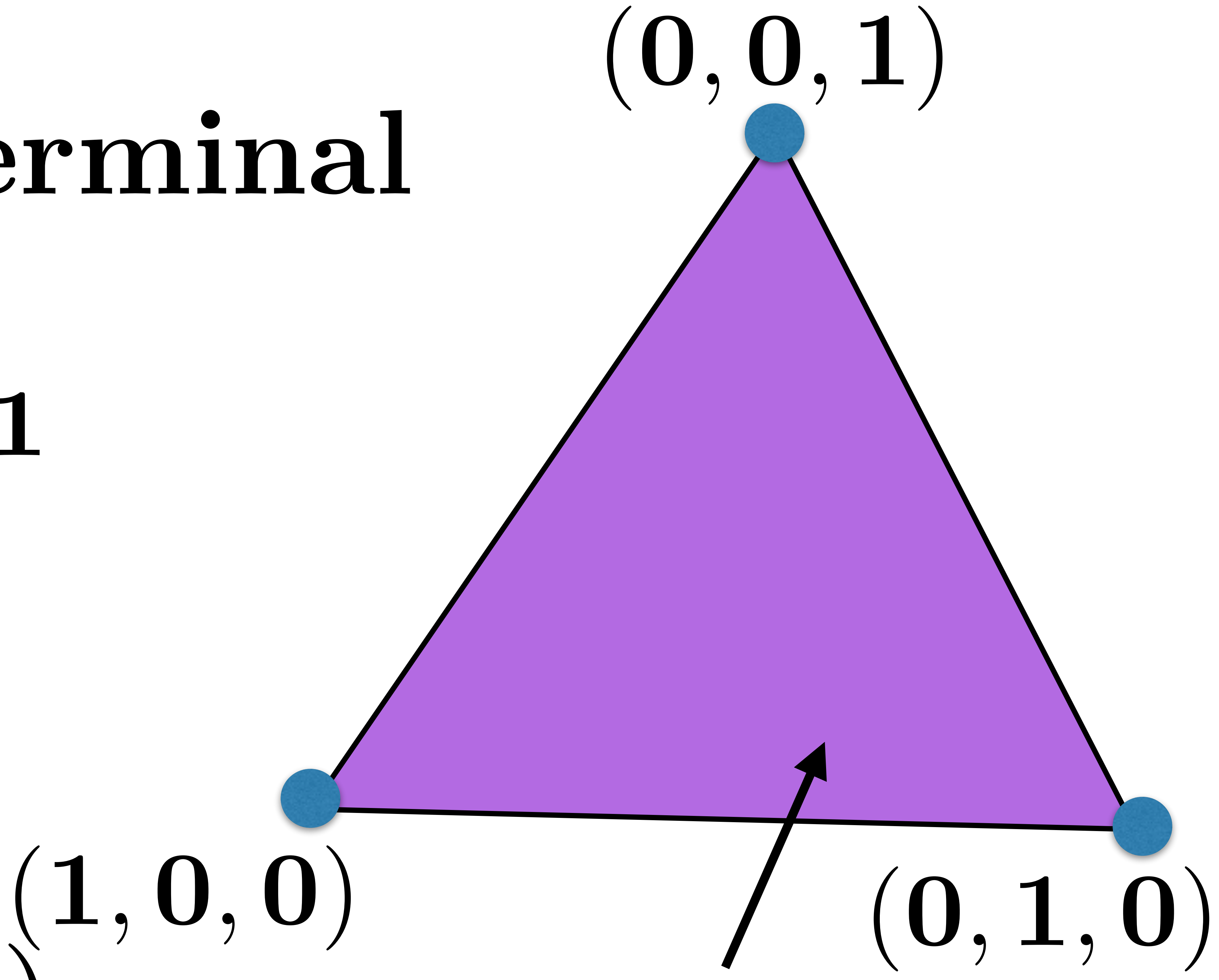
One dimension for each terminal

$\sum_\mathbf{i} \mathbf{x_{u,i}} = \sum_\mathbf{i} |\mathbf{x_{u,i}}| = |\mathbf{x_u}|_\mathbf{1}$

$|\mathbf{x_u}|_\mathbf{1} = \mathbf{1} \quad \forall \mathbf{u}$

$\mathbf{x_u} \in [\mathbf{0}, \mathbf{1}]^\mathbf{k} \forall \mathbf{u}$

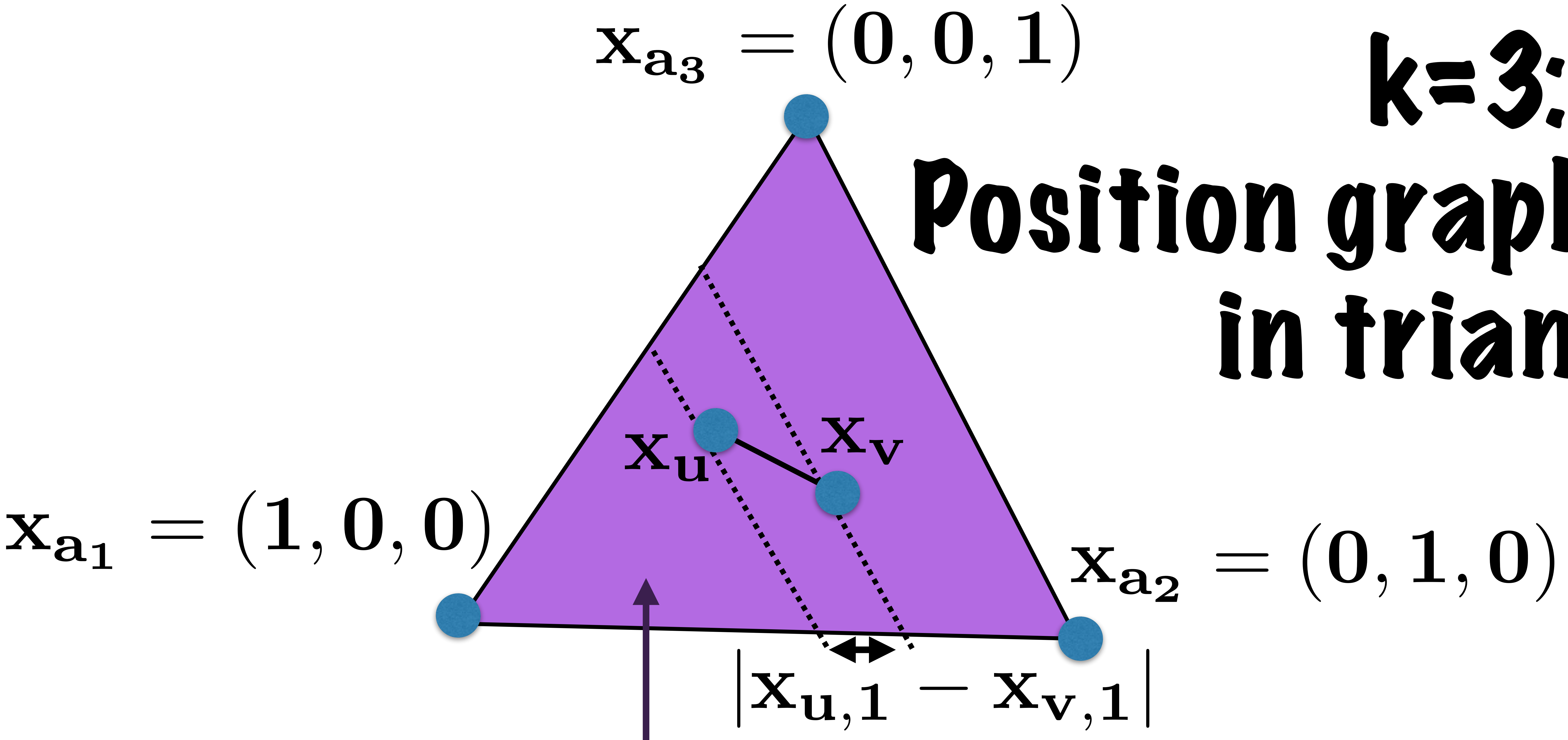
$\mathbf{x_{a_i}} = (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{1}, \mathbf{0}, \ldots \mathbf{0})$



$$\begin{cases} \mathbf{x_1} + \mathbf{x_2} + \mathbf{x_3} = \mathbf{1} \\ \mathbf{x_i} \geq \mathbf{0} \end{cases}$$

$$\min \quad (\mathbf{1}/\mathbf{2}) \sum_\mathbf{uv \in E} \mathbf{c_{uv}} |\mathbf{x_u} - \mathbf{x_v}|_\mathbf{1}$$

$$\min \quad (1/2)\sum_{uv\in E} c_{uv}|x_u - x_v|_1$$



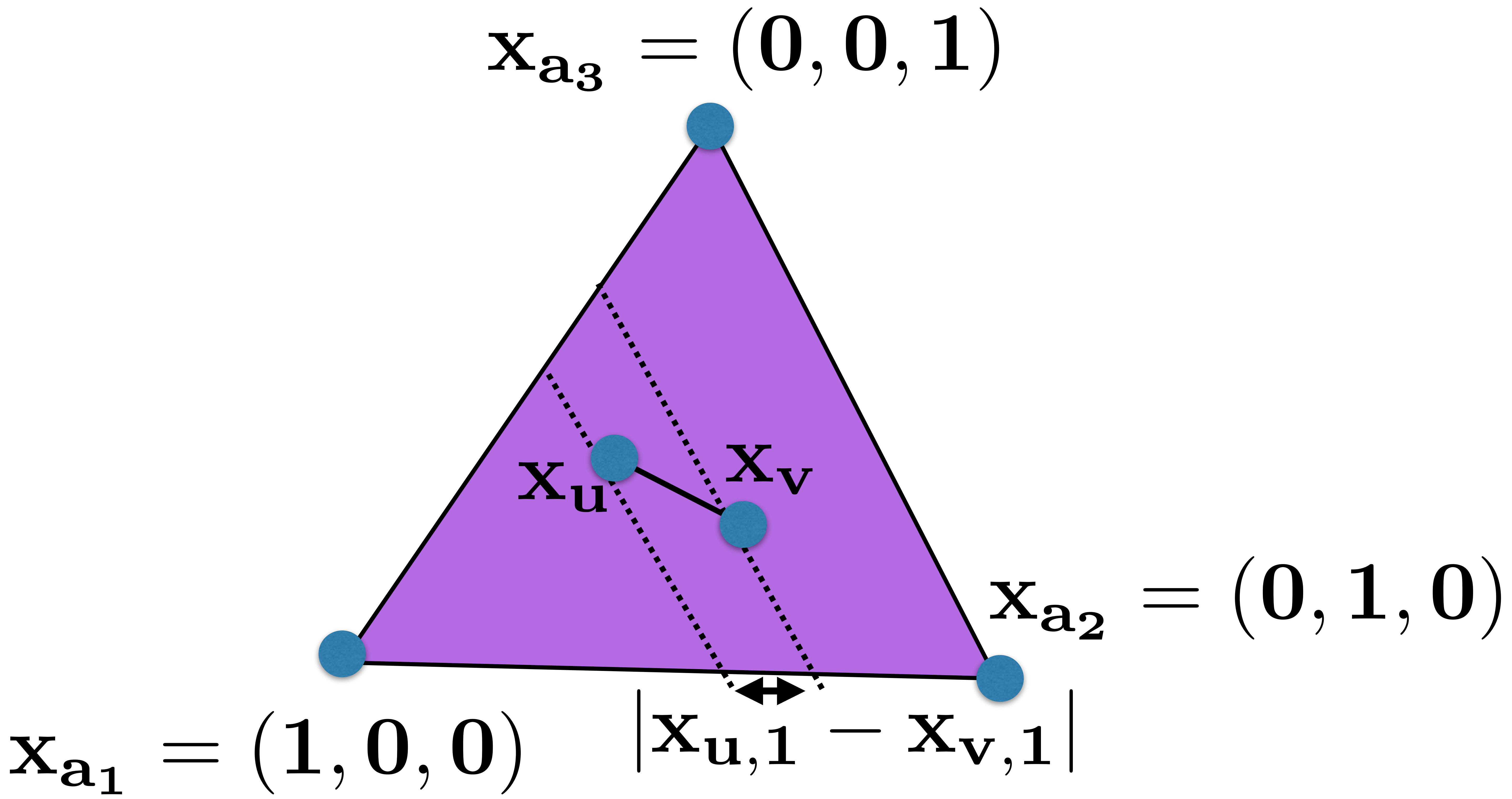
$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_i \geq 0 \end{cases}$$

k=3:
Position graph vertices in triangle

to minimize weighted lengths of projections on sides

A geometric interpretation (k=3)

Position graph vertices in triangle to minimize weighted lengths of projections of graph edges on the sides



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