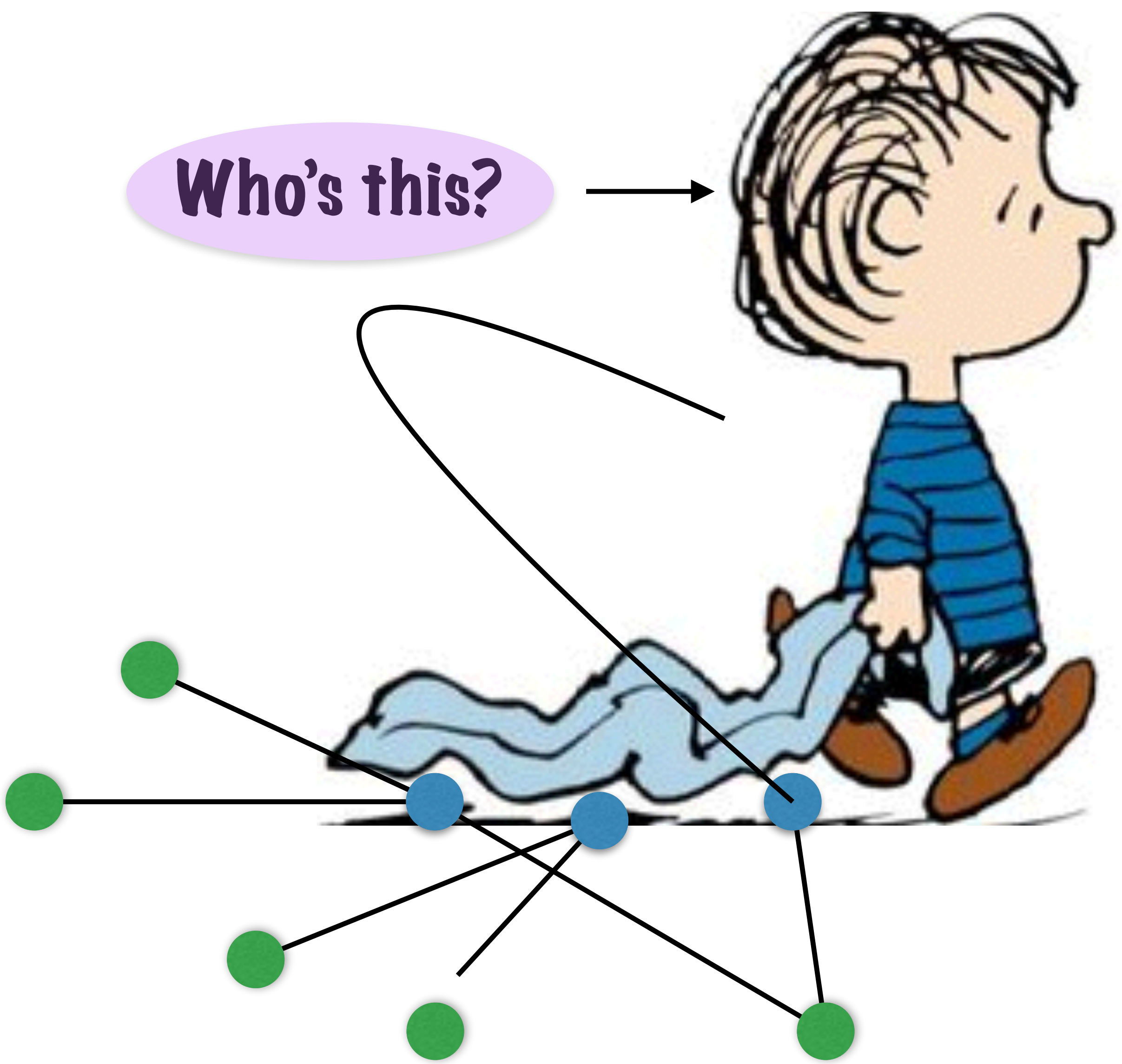
Approximation algorithms, vertex cover, and linear programming

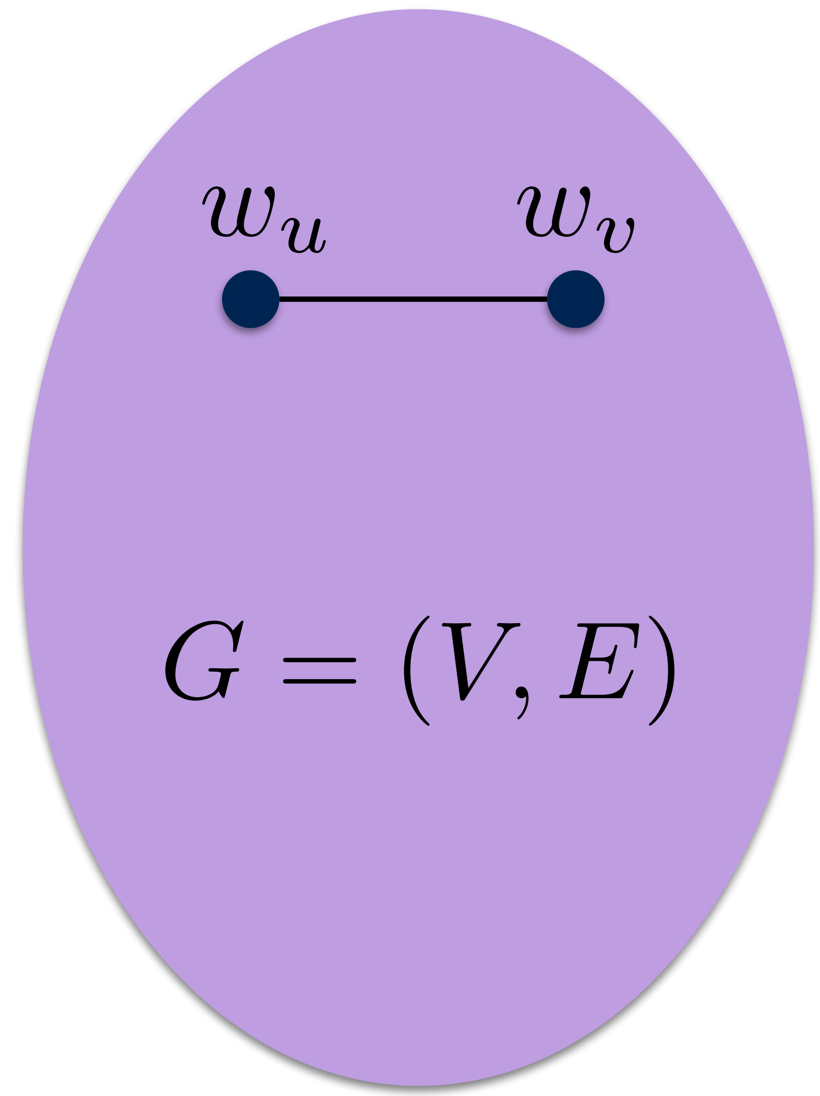
Who's this?



$$
\begin{aligned}
&\min c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
&\text{such that} \\
&\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\
\cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\
\forall i : \quad 0 \leq x_i \leq 1 & \\
\forall i : \quad x_i \ \underline{\text{real number}} &
\end{cases}
\end{aligned}
$$

What's that?

Using the LP (1/3)



Constraints:

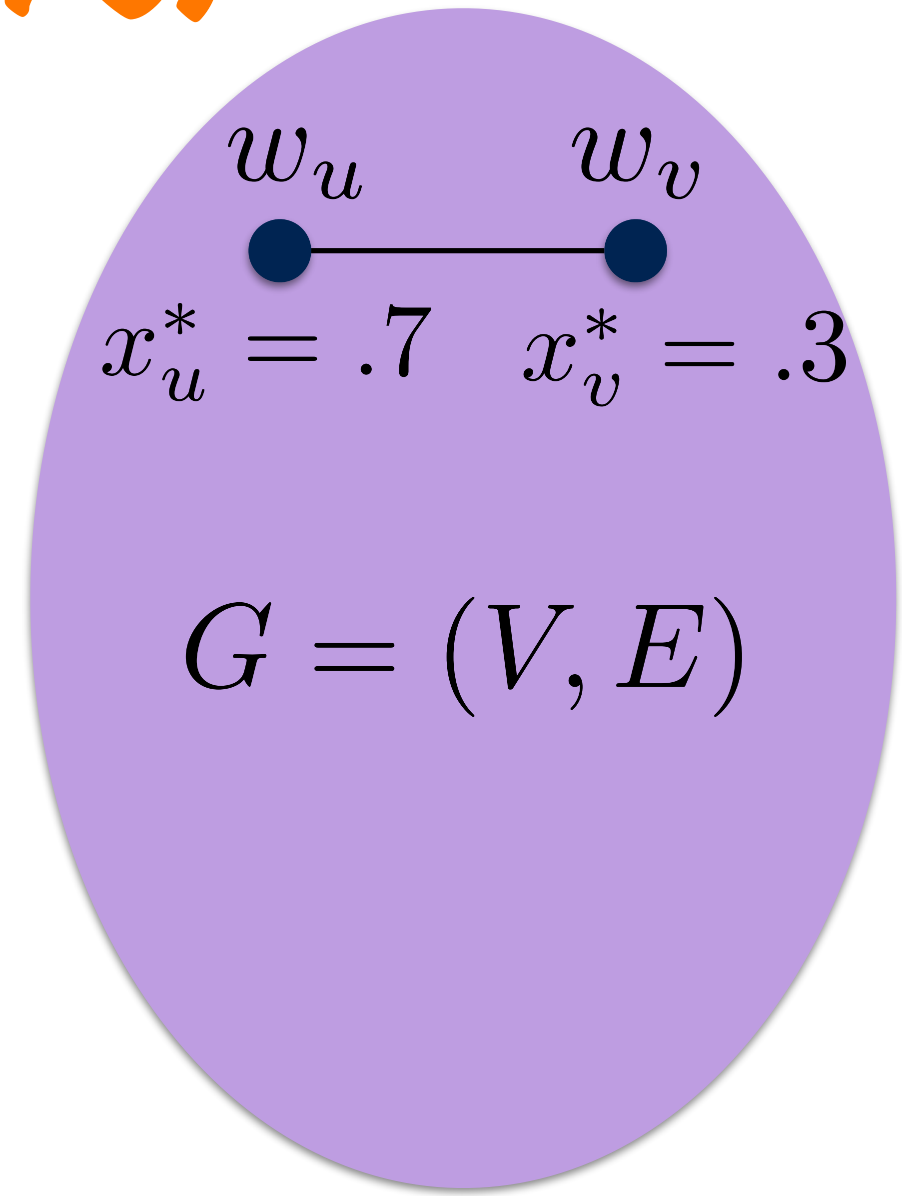
$$\forall u \in V : 0 \leq x_u \leq 1$$

$$\forall \{u, v\} \in E : x_u + x_v \geq 1$$

Objective: $\min \sum_u w_u x_u$

Using the LP (2/3)

1. Solving the LP



$\implies (x_u^*)_{u \in V}$ such that

$\forall u \in V : 0 \leq x_u^* \leq 1$

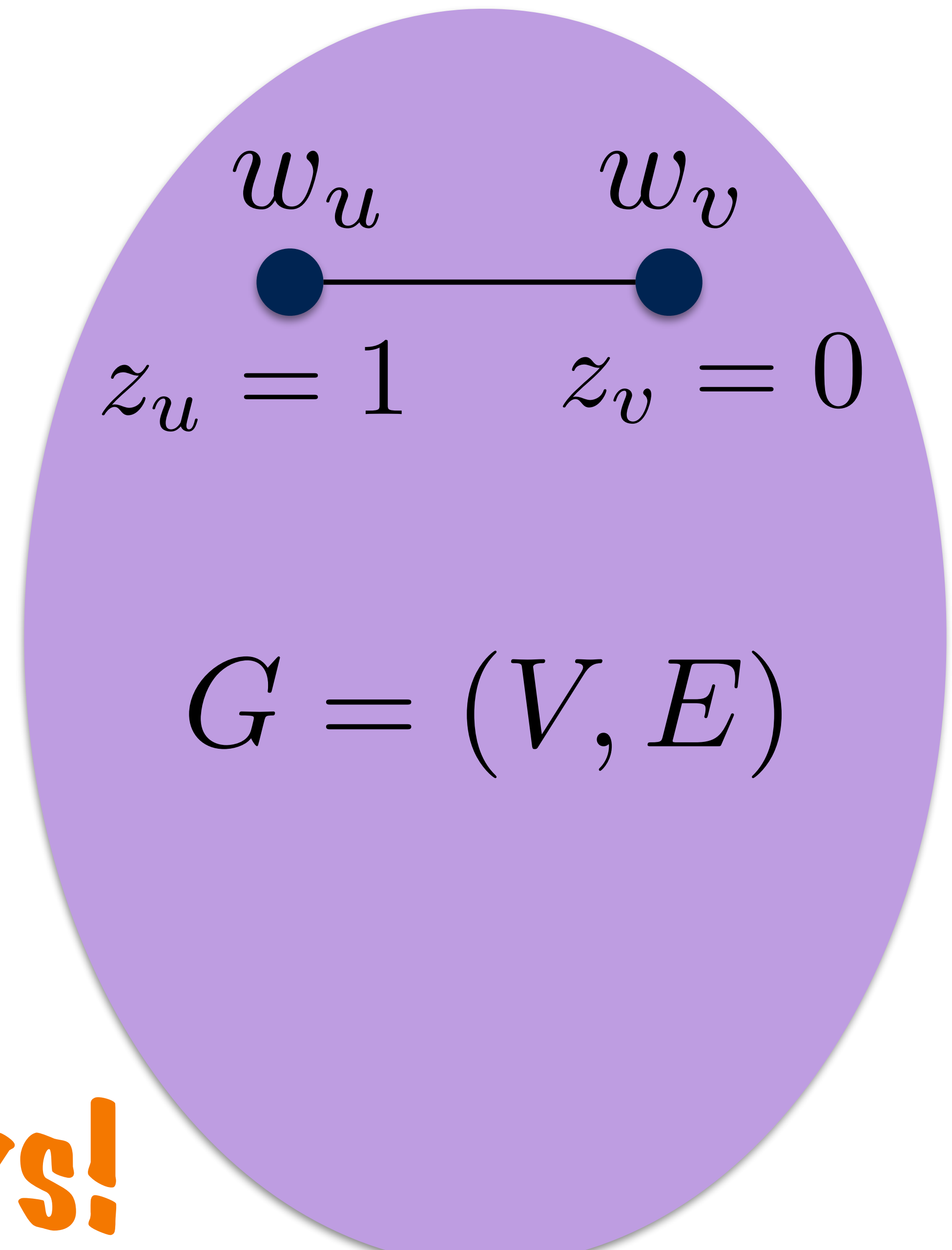
$\forall \{u, v\} \in E : x_u^* + x_v^* \geq 1$

$\sum_u w_u x_u^*$ minimum

Using the LP (3/3)
2. Rounding the LP solution

$$\implies (z_u)_{u \in V} \text{ defined by}$$

$$z_u = \begin{cases} 1 & \text{if } x_u^* \geq .5 \\ 0 & \text{otherwise} \end{cases}$$



We are back to integers!

Runtime

1. Solve the LP

$(x_u^*)_{u \in V}$ such that

$\forall u \in V : 0 \leq x_u^* \leq 1$

$\forall \{u, v\} \in E : x_u^* + x_v^* \geq 1$

$\sum_u w_u x_u^*$ minimum

Polynomial time

2. Round the LP solution

$\implies (z_u)_{u \in V}$ defined by

Linear time

$$z_u = \begin{cases} 1 & \text{if } x_u^* \geq .5 \\ 0 & \text{otherwise} \end{cases}$$

3. Output

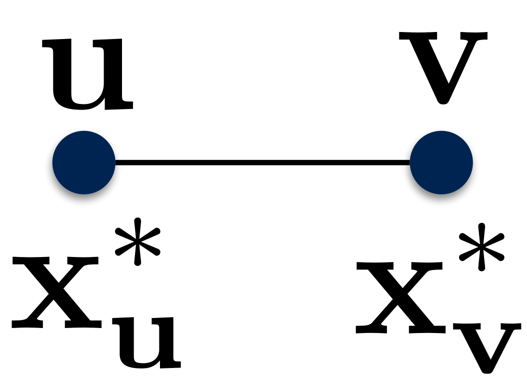
$\{\mathbf{u} \in \mathbf{V} \text{ such that } \mathbf{z_u} = \mathbf{1}\}$

Correctness

... is it a vertex cover?

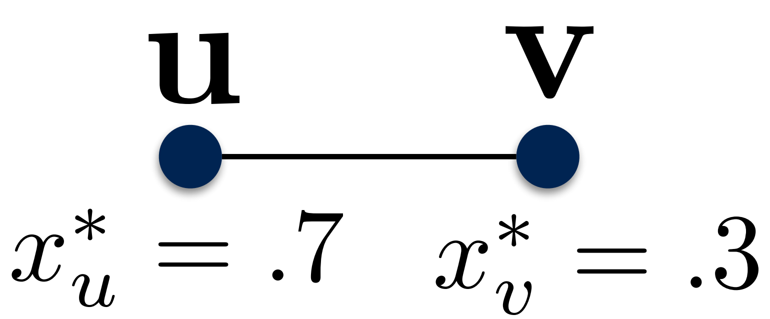
Does output cover all edges?

$$\{\mathbf{u}, \mathbf{v}\} : \mathbf{x}^*_{\mathbf{u}} + \mathbf{x}^*_{\mathbf{v}} \geq 1$$



Does output cover all edges?

$$\mathbf{x_u^* + x_v^* \geq 1}$$



$$\mathbf{x_u^* \geq .5} \text{ or } \mathbf{x_v^* \geq .5}$$

Does output cover all edges?

$$\implies (z_u)_{u \in V} \text{ defined by}$$

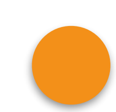
$x_u^* = .7 \quad x_v^* = .3$

$$z_u = \begin{cases} 1 & \text{if } x_u^* \geq .5 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{x_u^* \geq .5 \text{ or } x_v^* \geq .5}$$

$\mathbf{z_u = 1} \quad \mathbf{z_v = 0}$

u is in output



u

$$\mathbf{z_u = 1 \text{ or } z_v = 1}$$

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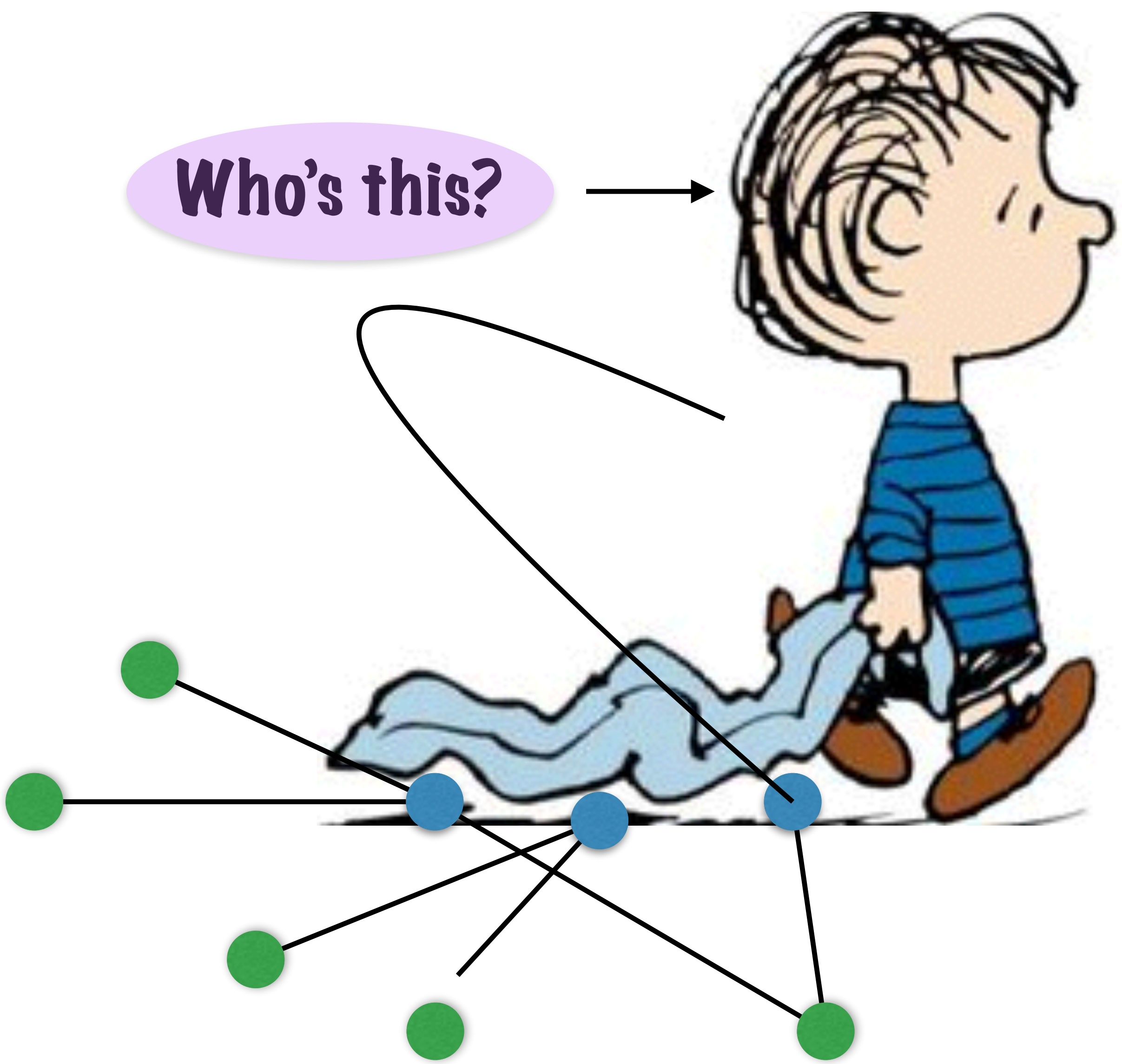

$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
such that
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq b_m \\ \forall i: \quad 0 \leq x_i \leq 1 & \\ \forall i: \quad x_i \ \underline{\text{real number}} & \end{cases}$$

What's that?