Welcome to Adaptive Array Signal Processing (5SSC0)

by Ruud van Sloun (lecturer), Iris Huijben (instructor), Julian Merkofer (instructor), and Vincent van de Schaft (instructor).

Motivation AASP example: Skype

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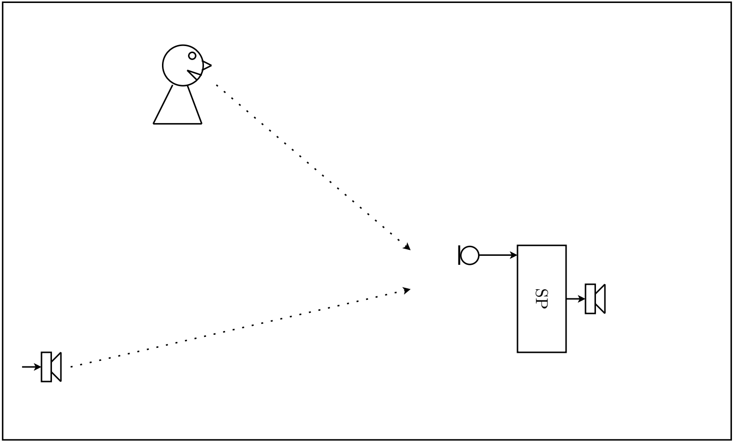

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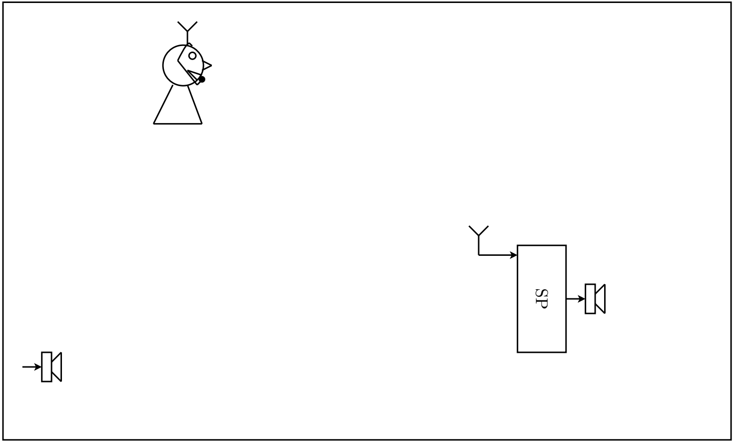

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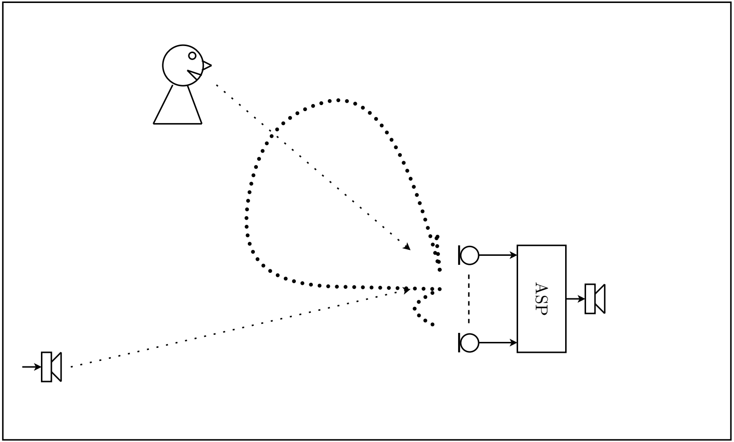

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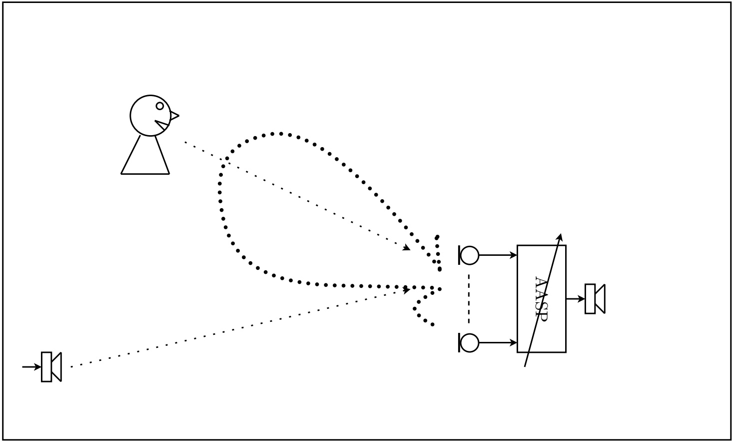

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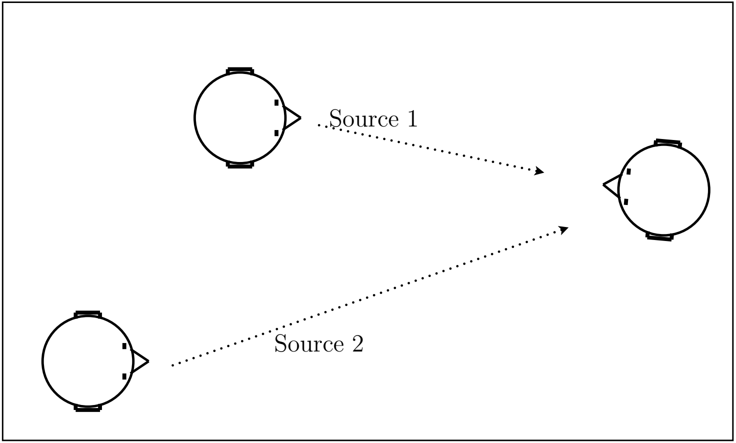

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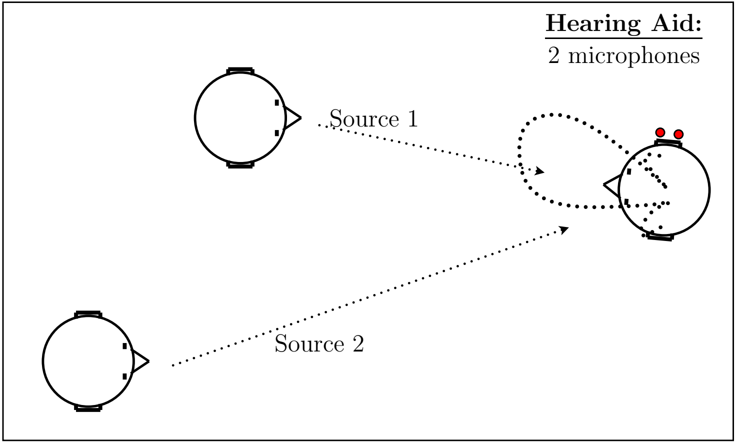

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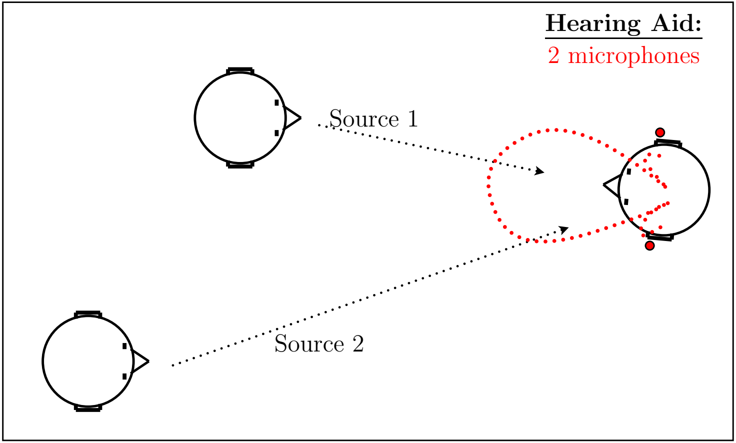

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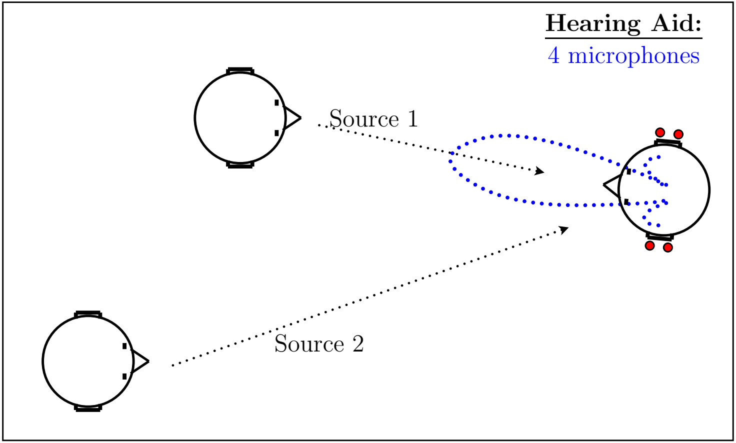

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Register for a group in Canvas

Deadlines and credits

Code	Deadline / Date	Credits
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1C	March 20, 09:00	10
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Oral	tbd (April 10-21)	50
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Adaptive Signal Processing

(Part IA)

Content part I

Focus on single channel adaptive algorithms using FIR structures

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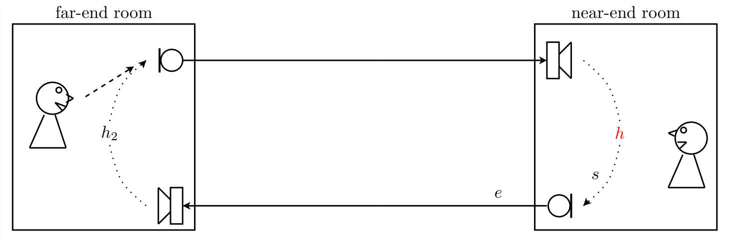
- Applications
- Minimum Mean Squared Error (MMSE)
- Constrained MMSE
- Least Squares (LS)
- Steepest Descent Algorithm (SGD)
- LMS variants: (Complex) (N)LMS, Constrained LMS
- Newton algorithm
- Recursive Least Squares (RLS)
- Frequency Domain Adaptive Filter (FDAF)
- Summary

Applications: Acoustic Echo Cancelation



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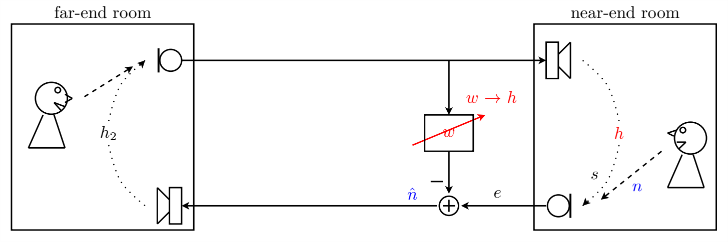

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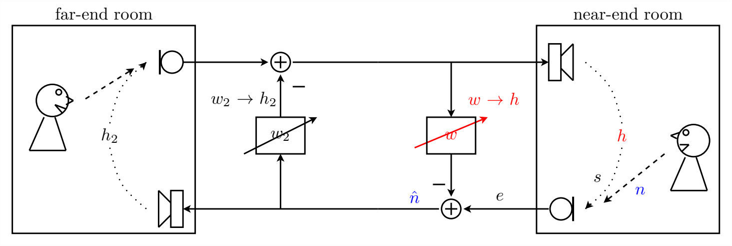

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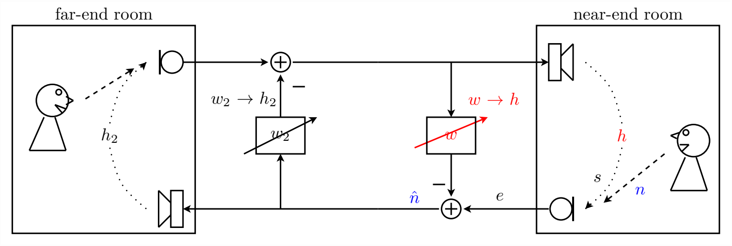
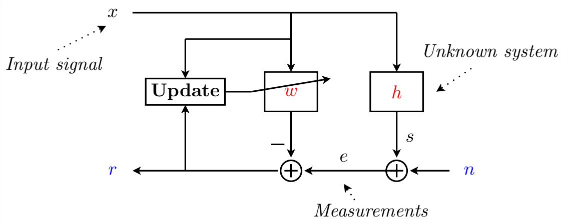
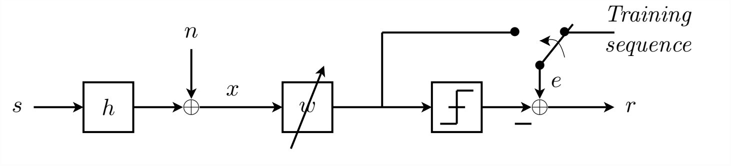




System Identification: $w \rightarrow h \Rightarrow r \rightarrow n$

Applications: Equalization



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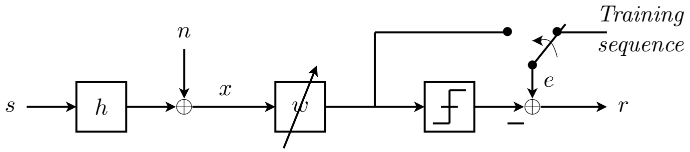
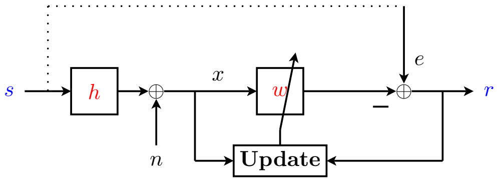
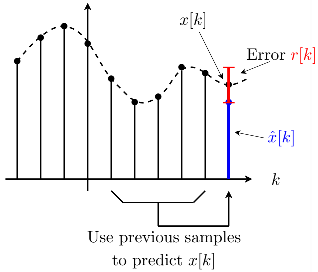




Signal correction/ Inverse modelling: $w \to h^{-1} \Rightarrow r \to s$

Applications: Signal prediction



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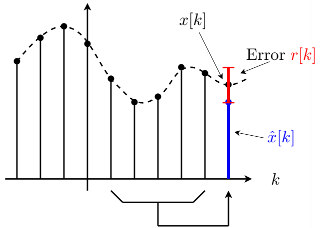
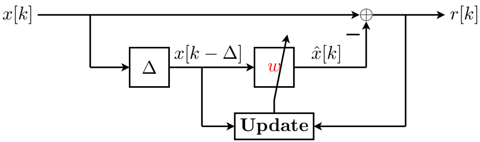
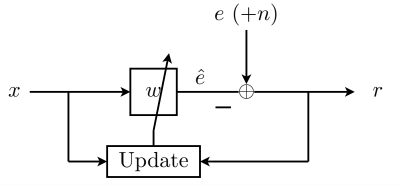




Signal prediction: Predict $x[k]$ from $x[k-\Delta]$

General Adaptive Filter model



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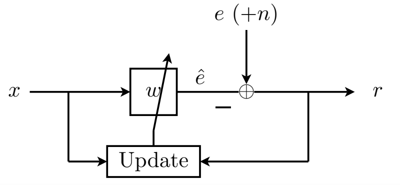

Notes:

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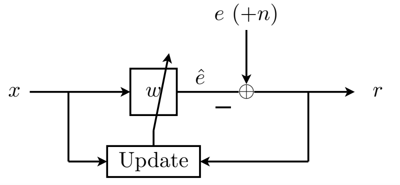

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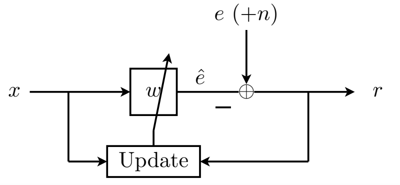

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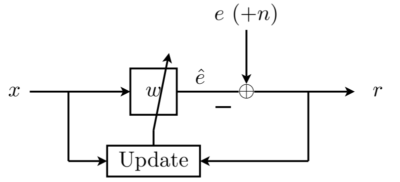

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- Pragmatic choices:
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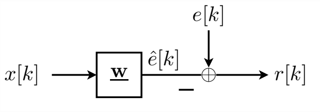
  - Filter w: FIR
- Calculation of weight of filter w:
  - Use quadratic cost function: $J = f(r^2)$
  - First fixed weights (MMSE, LS), then adaptive

Fixed weights: MMSE

General Minimum Mean Squared Error (MMSE) model:



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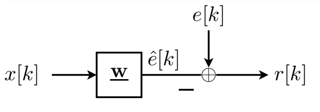

Goal:

Given N samples $\underline{x}[k] = (x[k], x[k-1], \cdots, x[k-N+1])^t$ calculate coefficients fixed filter $\underline{w} = (w_0, w_1, \cdots, w_{N-1})^t$ such that Mean Squared Error (MSE)
$J = E\left\{r^2[k]\right\} = E\{(e[k] - \hat{e}[k])^2\}$ is minimized.

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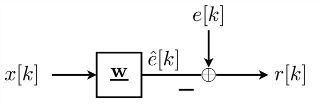

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MMSE Optimization problem:

$$\boxed{\begin{array}{c}\text{Given FIR samples } x[k-i] \text{ for } i = 0, 1, \cdots N-1 \\ \underline{w}_o = \arg\min_{\underline{w}} \left(E\left\{r^2[k]\right\}\right)\end{array}}$$

Fixed weights: MMSE

$$
\begin{aligned}
J &= E\{(e[k] - \underline{w}^t \cdot \underline{x}[k]) \cdot (e[k] - \underline{x}^t[k] \cdot \underline{w})\} \\
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with cross correlation $\rho_{ex}[\tau] = E\{e[k]x[k-\tau]\}$:

$$
\underline{r}_{ex} = E\{e[k]\underline{x}[k]\} = (\rho_{ex}[0], \rho_{ex}[1], \cdots, \rho_{ex}[N-1])^t
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and autocorrelation: $\rho_x[\tau] = E\{x[k]x[k-\tau]\} = rho_x[-\tau]$

$$
\mathrm{R}_x = E\{\underline{x}[k]\underline{x}^t[k]\} = \begin{pmatrix}
\rho_x[0] & \rho_x[1] & \cdots & \rho_x[N-1] \\
\rho_x[1] & \rho_x[0] & \cdots & \rho_x[N-2] \\
\vdots & \vdots & \vdots & \vdots \\
\rho_x[N-1] & \rho_x[N-2] & \cdots & \rho_x[0]
\end{pmatrix}
$$

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General expression: $\boxed{J = J_{min} + (\underline{w} - \underline{w}_o)^t \cdot \mathrm{R}_x \cdot (\underline{w} - \underline{w}_o)}$

Fixed weights: MMSE

$$J = E\{e^2[k]\} - \underline{w}^t \underline{r}_{ex} - \underline{r}_{ex}^t \underline{w} + \underline{w}^t \mathrm{R}_x \underline{w}$$

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Fixed weights: MMSE

$$J = E\{e^2[k]\} - \underline{w}^t \underline{r}_{ex} - \underline{r}_{ex}^t \underline{w} + \underline{w}^t R_x \underline{w}$$

$$\Rightarrow \text{ Optimum: } \underline{\nabla} = \frac{dJ}{d\underline{w}} = -2(\underline{r}_{ex} - R_x \underline{w}) = \underline{0}$$

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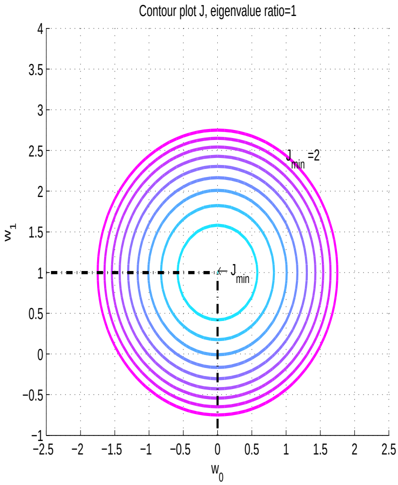
From general expression $\Rightarrow$ J quadratic in $\underline{w}$ thus $\underline{w}_o$ really minimum

Fixed weights: MMSE

Contour plots $J = J_{min} + (\underline{w} - \underline{w}_o)^t \cdot R_x \cdot (\underline{w} - \underline{w}_o)$



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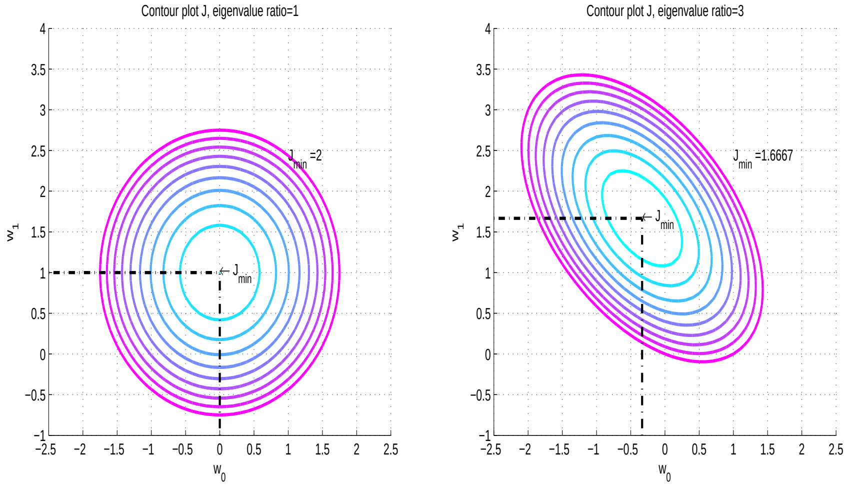

Eigenvalues: see Appendix

Two MMSE variants

Two MMSE variants

- **Complex MMSE:**

Setup with complex signals and weights

Similar result as before:

$$\boxed{\underline{\mathrm{w}}_o = \mathrm{R}_x^{-1} \cdot \underline{\mathrm{r}}_{e^* x}}$$

with $\underline{\mathrm{r}}_{e^* x} = E\{e^*[k]\underline{\mathrm{x}}[k]\}$ and $\mathrm{R}_x = E\{\underline{\mathrm{x}}[k] \cdot \underline{\mathrm{x}}^h[k]\}$ (h=hermetian)

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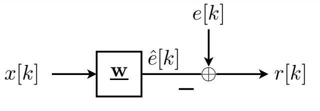
- **Constrained MMSE:**

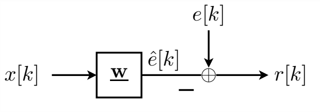
Setup with set of constraints on weights

Constrained MMSE



Goal: Minimize $E\{r^2[k]\}$ subject to M constraints

Constrained MMSE



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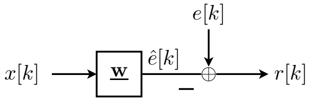
Example: $$\sum_{i=0}^{N-1} w_i = 1$$

Constrained MMSE



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$$\sum_{i=0}^{N-1} w_i = 1 \quad \Leftrightarrow \quad (1, 1, \cdots, 1)^t \cdot \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{pmatrix} = 1$$

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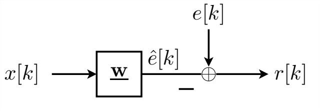

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$$\begin{aligned} \underline{c}_1^t \cdot \underline{w} &= f_1 \\ \underline{c}_2^t \cdot \underline{w} &= f_2 \\ &\vdots \\ \underline{c}_M^t \cdot \underline{w} &= f_M \end{aligned}$$

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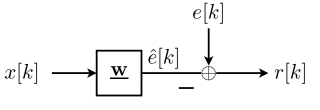

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Constrained MMSE

Some notes on solving $\mathsf{C}^t \cdot \underline{\mathsf{f}} = \underline{\mathsf{w}}$

Constrained MMSE

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$$\begin{aligned} M \times 1 \text{ constraint vector} &\quad : \quad \underline{\mathsf{f}} = (f_1, f_2, \cdots, f_M)^t \\ N \times M \text{ constraint matrix} &\quad : \quad \mathsf{C} = (\underline{\mathsf{c}}_1, \underline{\mathsf{c}}_2, \cdots, \underline{\mathsf{c}}_M)^t \end{aligned}$$

Note: M independent constraints $\Rightarrow$ C full rank

Constrained MMSE

Some notes on solving $\mathsf{C}^t \cdot \underline{\mathsf{f}} = \underline{\mathsf{w}}$

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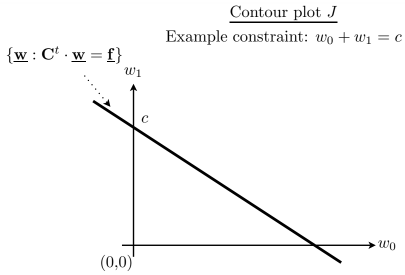

Constrained MMSE

Solutions of $C^t \cdot \underline{w} = \underline{f}$

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  $\Rightarrow$ *No degrees of freedom left for MMSE*

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  Appendix: Generalized inverse $(\mathsf{C}^t)^{\dagger} = \mathsf{C} \cdot (\mathsf{C}^t \cdot \mathsf{C})^{-1}$

  $\Rightarrow$ *$N - M$ degrees of freedom left over for MMSE*

Constrained MMSE

Solutions of $\mathrm{C}^t \cdot \underline{\mathrm{w}} = \underline{\mathrm{f}}$

- Case $N = M$:

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  $\Rightarrow$ *$N - M$ degrees of freedom left over for MMSE*

- Case $N < M$:

  $\Rightarrow$ *Conflicting solutions*

  $\Rightarrow$ Choose e.g. minimum norm solution

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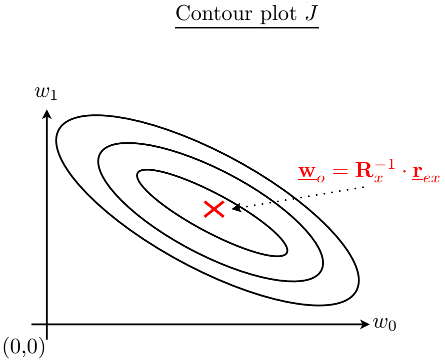

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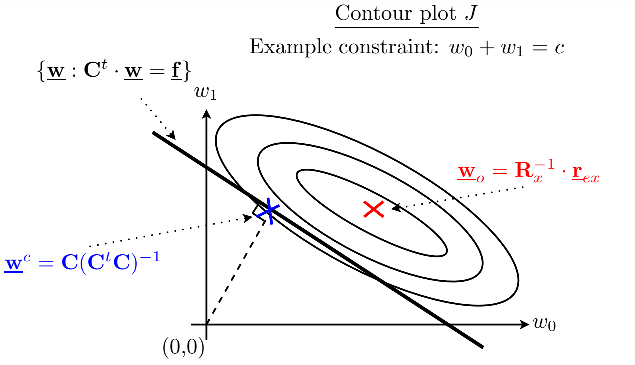

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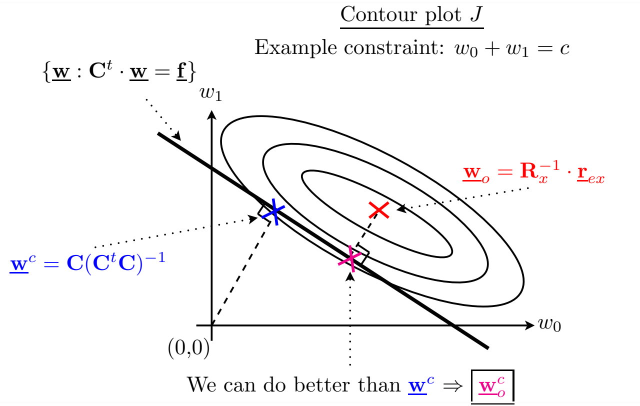

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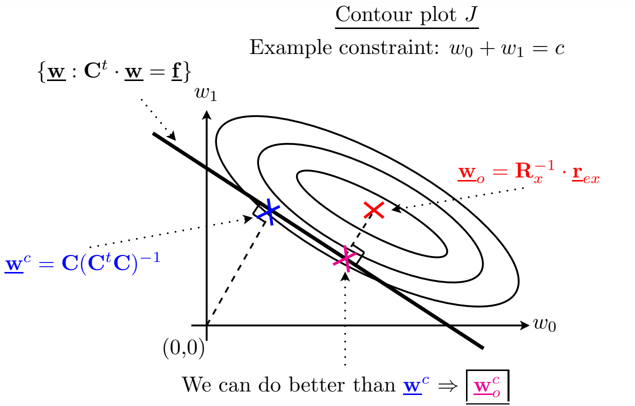

Use $N - M$ degrees of freedom to improve result: $\underline{w}^c \Rightarrow \underline{w}_o^c$

Constrained MMSE

Use Lagrange multipliers

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Performance index:

$$\begin{aligned} J^c &= E\{r^2\} + \underline{\lambda}^t(\mathrm{C}^t\underline{\mathrm{w}} - \underline{\mathrm{f}}) \\ &= E\{e^2\} - \underline{\mathrm{w}}^t\underline{\mathrm{r}}_{ex} - \underline{\mathrm{r}}_{ex}^t\underline{\mathrm{w}} + \underline{\mathrm{w}}^t\mathrm{R}_x\underline{\mathrm{w}} + \underline{\lambda}^t(\mathrm{C}^t\underline{\mathrm{w}} - \underline{\mathrm{f}}) \end{aligned}$$

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$$\frac{\mathrm{d}J^c}{\mathrm{d}\underline{\mathrm{w}}} = \underline{0} \;\Rightarrow\; \underline{\mathrm{w}}_o^c = \mathrm{R}_x^{-1}\underline{\mathrm{r}}_{ex} - \frac{1}{2}\mathrm{R}_x^{-1}\mathrm{C}\underline{\lambda}$$

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Furthermore in optimum: $\mathrm{C}^t\underline{\mathrm{w}}_o^c = \underline{\mathrm{f}}$

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Combine last two equations:

$$\Rightarrow \quad \underline{\lambda} = 2(\mathrm{C}^t\mathrm{R}_x^{-1}\mathrm{C})^{-1}(\mathrm{C}^t\mathrm{R}_x^{-1}\underline{\mathrm{r}}_{ex} - \underline{\mathrm{f}})$$

Constrained MMSE

$$\Rightarrow \quad \boxed{\underline{w}_o^c = \underline{w}_o + R_x^{-1}C(C^tR_x^{-1}C)^{-1}(\underline{f} - C^t\underline{w}_o)}$$

with

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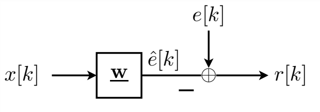
Check: $$C^t\underline{w}_o^c = C^t\underline{w}_o + (C^tR_x^{-1}C)(C^tR_x^{-1}C)^{-1}(\underline{f} - C^t\underline{w}_o) = \underline{f}$$

Least Squares (LS)



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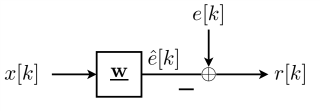

Different quadratic cost functions:

- Mean Square Error (MSE):

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$\Rightarrow$ Minimum MSE (MMSE) = Wiener

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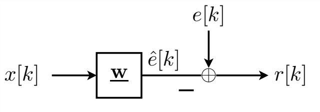

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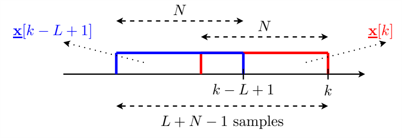
- **Least Square (LS):** If statistical information is not available

$\Rightarrow$

Use criterion based on data (thus without $E\{\cdot\}$)

LS

Collect L (≥ 1) data vectors $\underline{x}[k-i]$ (each of length N)



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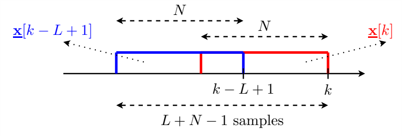

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LS

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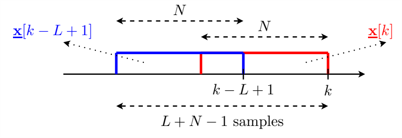

<u>Available data (for $i = 0, 1, \cdots, L-1$):</u>

- *Input signal samples/ vectors* $\underline{x}[k-i]$

$$\underline{x}^t[k-i] = (x[k-i], x[k-i-1], \cdots, x[k-i-N+1])^t$$

- *Reference signal samples:* $e[k-i]$
- *Residual signal samples:* $r[k-i] = e[k-i] - \underline{x}^t[k-i] \cdot \underline{w}$

LS

Notation:

$$X[k] = \begin{pmatrix} \underline{x}^t[k] \\ \underline{x}^t[k-1] \\ \vdots \\ \underline{x}^t[k-L+1] \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{pmatrix}$$

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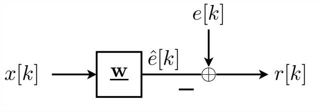
Simplified notation (skip time indices):

$$\underline{r} = \underline{e} - X \cdot \underline{w}$$

LS



LS problem formulation:

$$\boxed{\underline{w}_{ls,o} = \arg\min_{\underline{w}} |\underline{e} - X \cdot \underline{w}|^2}$$

LS

$$J_{ls} = \sum_{i=0}^{L-1} r^2[k-i] = \underline{r}^t \cdot \underline{r} = (\underline{e}^t - \underline{w}^t X^t) \cdot (\underline{e} - X\underline{w})$$

LS

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\end{aligned}
$$

Minimum by setting gradient equal to zero:

$$
\frac{\mathrm{d}J_{ls}}{\mathrm{d}\underline{w}} = \underline{\nabla}_{ls} = -2(X^t \underline{e} - X^t X \cdot \underline{w}) = \underline{0}
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With $\overline{\mathrm{R}} = \mathrm{X}^t \mathrm{X}$ and $\underline{\bar{\mathrm{r}}} = \mathrm{X}^t \underline{\mathrm{e}}$

LS

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\begin{aligned}
J_{ls} &= \sum_{i=0}^{L-1} r^2[k-i] = \underline{r}^t \cdot \underline{r} = (\underline{e}^t - \underline{w}^t X^t) \cdot (\underline{e} - X\underline{w}) \\
&= \underline{e}^t \underline{e} + \underline{w}^t X^t X \underline{w} - \underline{w}^t X^t \underline{e} - \underline{e}^t X \underline{w}
\end{aligned}
$$

Minimum by setting gradient equal to zero:

$$
\frac{\mathrm{d} J_{ls}}{\mathrm{d}\underline{w}} = \underline{\nabla}_{ls} = -2(X^t \underline{e} - X^t X \cdot \underline{w}) = \underline{0}
$$

With $\overline{R} = X^t X$ and $\underline{\overline{r}} = X^t \underline{e}$

$\Rightarrow$ **Normal Equations** $\boxed{\overline{R}_x \cdot \underline{w} = \underline{\overline{r}}_{ex}}$

$\Rightarrow$ **Wiener filter** $\boxed{\underline{w}_{ls,o} = \overline{R}_x^{-1} \cdot \underline{\overline{r}}_{ex}}$

LS: Correspondence with MMSE

Use time-averaging (ergodicity):

$$\hat{\mathrm{R}}_x = \frac{1}{L}\sum_{i=0}^{L-1} \underline{\mathrm{x}}[k-i]\cdot\underline{\mathrm{x}}^t[k-i] = \frac{1}{L}\mathrm{X}^t\cdot\mathrm{X} = \frac{1}{L}\overline{\mathrm{R}}_x$$

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with $\hat{\mathrm{R}}_x$ estimate of R_x and $\hat{\underline{\mathrm{r}}}_{ex}$ estimate of $\underline{\mathrm{r}}_{ex}$

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Finally note that for ergodic processes:

$$\lim_{L\to\infty}\frac{1}{L}\overline{\mathrm{R}}_x = \mathrm{R}_x \;;\; \lim_{L\to\infty}\frac{1}{L}\overline{\underline{\mathrm{r}}}_{ex} = \underline{\mathrm{r}}_{ex} \;;\; \lim_{L\to\infty}\underline{\mathrm{w}}_{ls} = \underline{\mathrm{w}}_{mmse}$$

Steepest gradient descent (SGD)

Problem: Optimal Wiener involves R_x^{-1}

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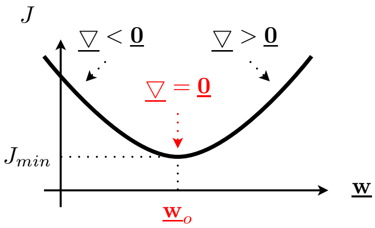
Problem: Optimal Wiener involves R_x^{-1}

To avoid this inversion, estimate optimum *iteratively*

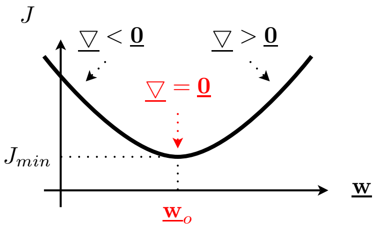
Goal: Decrease J each new iteration

SGD



J
$\underline{\nabla} < \underline{\mathbf{0}}$
$\underline{\nabla} > \underline{\mathbf{0}}$
$\underline{\nabla} = \underline{\mathbf{0}}$
J_{min}
$\underline{\mathbf{w}}$
$\underline{\mathbf{w}}_o$

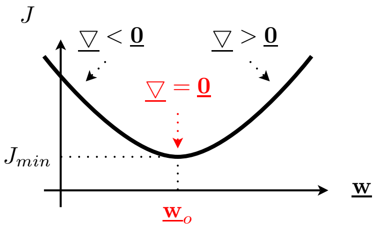
SGD



SGD principle: Update in negative gradient direction

$\Leftrightarrow \underline{w} \doteq \underline{w} - \alpha \underline{\nabla}$ with adaptation constant $\alpha \geq 0$

SGD



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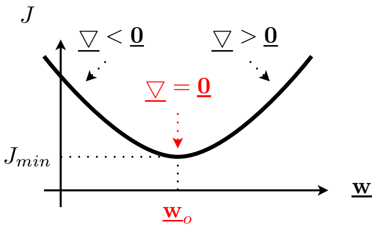

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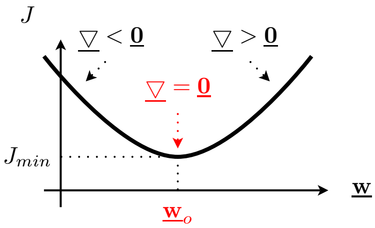

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Notes: 1) No matrix inversion needed! 2) Usually $\underline{w}[0] = \underline{0}$

SGD

SGD converges to Wiener solution:

$$\boxed{\lim_{k\to\infty} \underline{\mathrm{w}}[k] \simeq \mathrm{R}_x^{-1} \cdot \underline{\mathrm{r}}_{ex}}$$

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For exact proof we need **stability analysis**

Stability SGD

Define difference weight vector: $\underline{\mathrm{d}}[k] = \underline{\mathrm{w}}[k] - \underline{\mathrm{w}}_o$

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\end{aligned}
$$

Recursion:

$$
\underline{d}[k] = (I - 2\alpha R_x) \cdot \underline{d}[k-1] = \cdots = (I - 2\alpha R_x)^k \cdot \underline{d}[0]
$$

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Note:

When stable $\Rightarrow \underline{d}[\infty] = \underline{0} \Rightarrow \underline{w}[\infty] \simeq$ Wiener

Stability SGD

<u>How</u> do weights converge:

Stability SGD

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Use eigenvalue decomposition (see Appendix):

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$$
\begin{aligned}
\Rightarrow \quad (I - 2\alpha R_x)^k &= (QQ^h - 2\alpha Q\Lambda Q^h)^k \\
&= Q(I - 2\alpha\Lambda)^k Q^h
\end{aligned}
$$

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$$\Rightarrow \quad (\mathsf{I} - 2\alpha\mathsf{R}_x)^k = (\mathsf{Q}\mathsf{Q}^h - 2\alpha\mathsf{Q}\Lambda\mathsf{Q}^h)^k \\ = \mathsf{Q}(\mathsf{I} - 2\alpha\Lambda)^k\mathsf{Q}^h$$

Change of variables: $\underline{\mathsf{D}}[k] = \mathsf{Q}^h \cdot \underline{\mathsf{d}}[k]$

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Both matrices $\mathbf{I}$ and Λ diagonal $\Rightarrow$ Stable iff:

$$|1 - 2\alpha\lambda_i| < 1 \quad \Leftrightarrow \quad 0 < \alpha < \frac{1}{\lambda_i} \;\text{ for } i = 0, 1, \cdots, N-1$$

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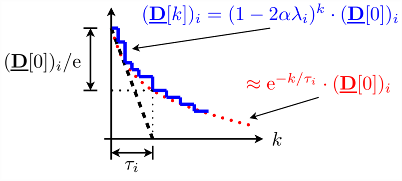
Convergence SGD



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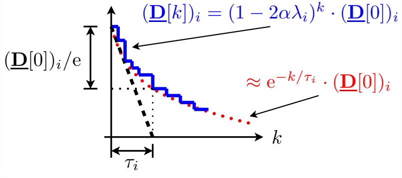

$$\mathrm{e}^{-k/\tau_i} \cdot (\underline{\mathrm{D}}[0])_i \approx (1 - 2\alpha\lambda_i)^k \cdot (\underline{\mathrm{D}}[0])_i \quad \Rightarrow$$

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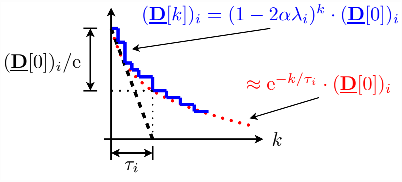

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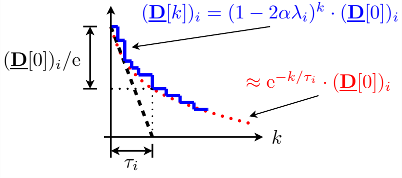
Convergence SGD



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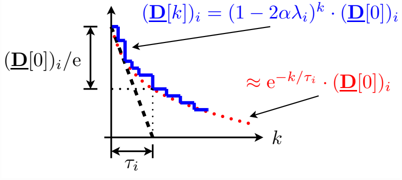

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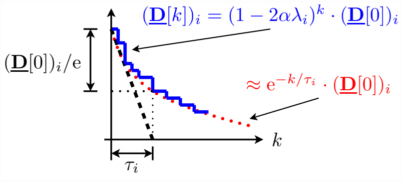
Overall time constant depends on eigenvalue spread $\Gamma_x = \lambda_{max}/\lambda_{min}$. Thus, the larger Γ_x the longer it takes for adaptation.

Convergence SGD



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Q: What happens for white noise?

Convergence SGD

Example with $\Gamma_x = \lambda_{max}/\lambda_{min} = 3$

Convergence SGD

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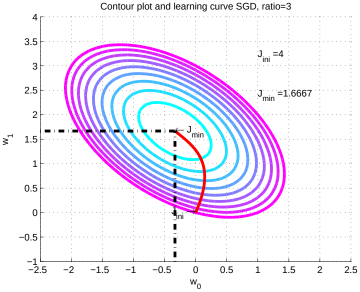
Learning curve in contour plot J



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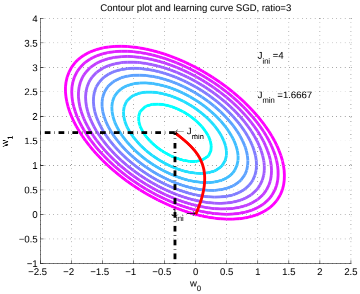
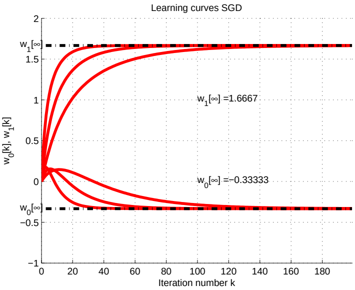


Learning curves for different α



Least Mean Square (LMS)

Motivation: SGD not practical. Gradient assumes **known** R_x and $\underline{r}_{ex}$

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LMS principle: Use instantaneous estimate of gradient:

$$\begin{aligned}\underline{\hat{\bigtriangledown}}[k] &= -2\left(e[k]\underline{\mathrm{x}}[k] - \underline{\mathrm{x}}[k]\underline{\mathrm{x}}^t[k]\underline{\mathrm{w}}[k]\right) \\ &= -2\underline{\mathrm{x}}[k]\left(e[k] - \underline{\mathrm{x}}^t[k]\underline{\mathrm{w}}[k]\right) = -2\underline{\mathrm{x}}[k]r[k]\end{aligned}$$

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With $\underline{w} \doteq \underline{w} - \underline{\hat{\nabla}} \Rightarrow$ LMS algorithm (Widrow, 1975):

$$\boxed{\begin{array}{lcl} k = 0 & : & \underline{w}[0] = \underline{0} \quad \text{(usually)} \\ k > 0 & : & \hat{e}[k] = \underline{w}^t[k] \cdot \underline{x}[k] \\ & & r[k] = e[k] - \hat{e}[k] \\ & & \underline{w}[k+1] = \underline{w}[k] + 2\alpha\underline{x}[k]r[k] \end{array}}$$

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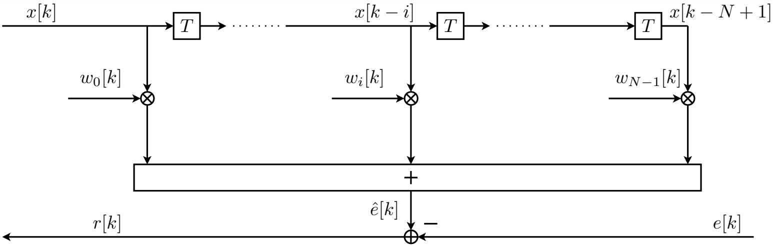
Note: $\underline{w}^t[k] \cdot \underline{x}[k]$ is "convolution" and $\underline{x}[k]r[k]$ "correlation"

LMS

Realization scheme LMS algorithm:



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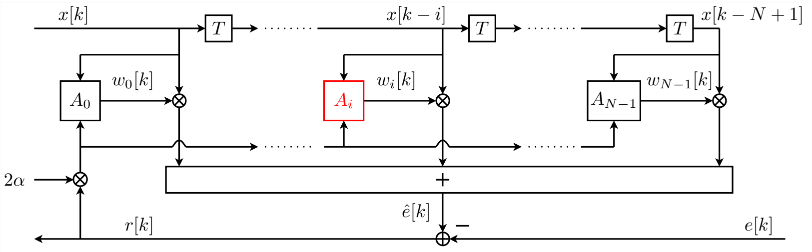

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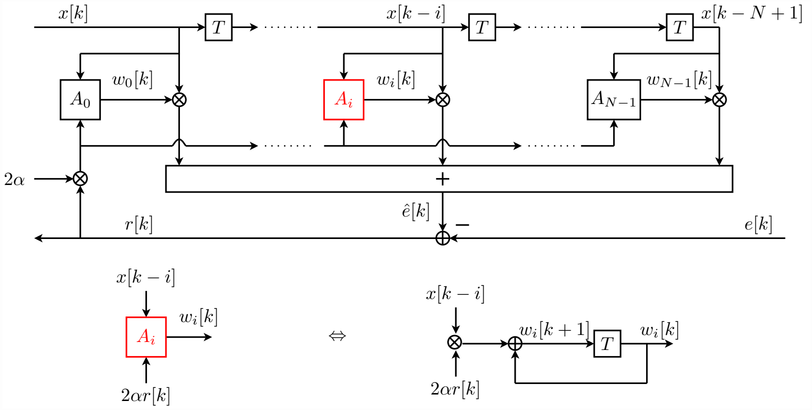

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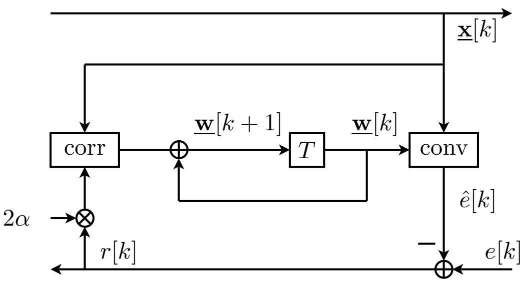
Simplified realization scheme LMS algorithm:



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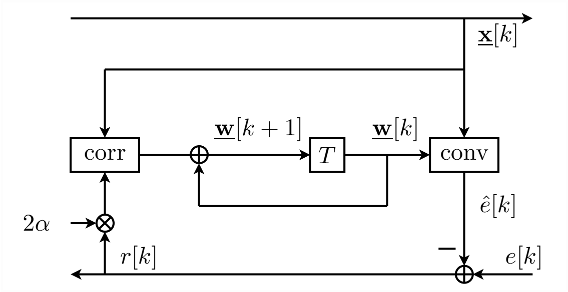

Notes:

- Simple, robust algorithm, complexity $O(2N)$
- LMS tries to "decorrelate" signals x and r
- In contrast to SGD: Weights fluctuate around optimal values

Two LMS variants

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- **NLMS:** LMS with normalization by $\sigma_x^2 = E\{x^2[k]\}$:

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- $\hat{\sigma}_x^2[k] = \beta\hat{\sigma}_x^2[k-1] + (1-\beta)\frac{\underline{x}^t[k]\underline{x}[k]}{N}$ with $0 < \beta < 1$
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- **Complex LMS:** LMS for complex signals and weights:

$$\boxed{\underline{w}[k+1] = \underline{w}[k] + 2\alpha\underline{x}[k]r^*[k]}$$

Newton

Convergence gradient based algorithms relies on coloration input:

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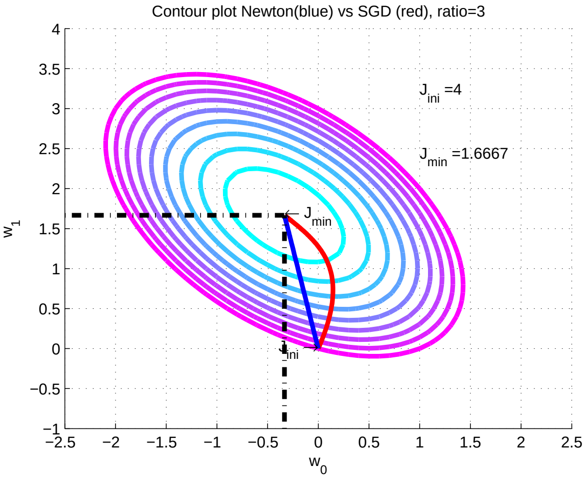
- Newton $\equiv$ SGD with white noise input!

Newton

Learning curves in contour plot: Newton vs. SGD



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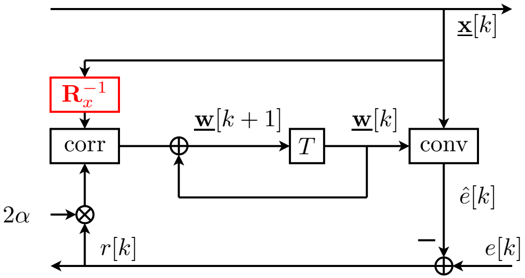

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$\Rightarrow$ Complexity Newton algorithm huge

$\Rightarrow$ Need for efficient solution with estimate of R_x

$\Rightarrow$ Different algorithms, e.g. RLS, FDAF, etc.

Recursive Least Squares (RLS)

For data block length L fixed, Least Squares problem becomes:

$$\min_{\underline{\mathrm{w}}[k]} \left|\underline{\mathrm{e}}[k] - \mathrm{X}[k] \cdot \underline{\mathrm{w}}[k]\right|^2 \quad \Rightarrow \quad \underline{\mathrm{w}}_{LS}[k] = \left(\mathrm{X}^t[k]\mathrm{X}[k]\right)^{-1} \left(\mathrm{X}^t[k]\underline{\mathrm{e}}[k]\right)$$

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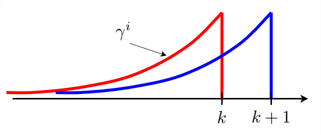

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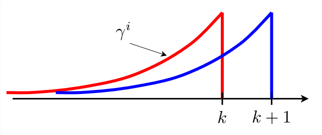

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- Decorrelation takes place in algorithm

Frequency Domain Adaptive Filter (FDAF)

FDAF: Alternative for LMS/Newton and RLS

Frequency Domain Adaptive Filter (FDAF)

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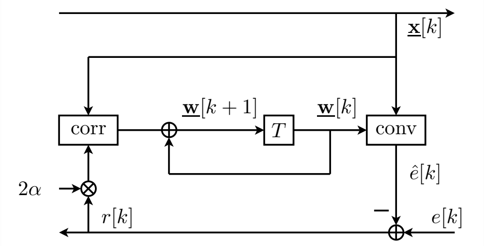
First step of derivation: Translate LMS to frequency domain



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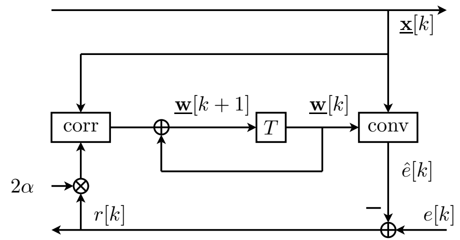

LMS weight update:

$$\underline{w}[k+1] = \underline{w}[k] + 2\alpha\underline{x}[k]r[k]$$

Filter output:

$$\hat{e}[k] = \underline{x}^t[k] \cdot \underline{w}[k]$$

FDAF

Transform vectors to frequency domain:

FDAF

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$$\begin{aligned} \mathsf{F} \cdot \underline{\mathsf{x}}[k] = \underline{\mathsf{X}}[k] &= (X_0[k], X_1[k], \cdots, X_{N-1}[k])^t \\ \mathsf{F}^{-1} \cdot \underline{\mathsf{w}}[k] = \underline{\mathsf{W}}[k] &= (W_0[k], W_1[k], \cdots, W_{N-1}[k])^t \end{aligned}$$

with DFT matrix F, which has properties: $\mathsf{F} = \mathsf{F}^t$ and $\mathsf{F}^{-1} = \frac{1}{N}\mathsf{F}^*$

FDAF

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\end{aligned}
$$

with DFT matrix F, which has properties: $\mathsf{F} = \mathsf{F}^t$ and $\mathsf{F}^{-1} = \frac{1}{N}\mathsf{F}^*$

Apply filter operation in frequency domain:

$$
\hat{e}[k] = \sum_{i=0}^{N-1} x[k-i] \cdot w_i[k] = \underline{x}^t \cdot \underline{w}[k]
$$

FDAF

Transform vectors to frequency domain:

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Apply filter operation in frequency domain:

$$\begin{aligned} \hat{e}[k] &= \sum_{i=0}^{N-1} x[k-i] \cdot w_i[k] = \underline{x}^t \cdot \underline{w}[k] \\ &= \underline{x}^t[k](\mathsf{F} \cdot \mathsf{F}^{-1})\underline{w}[k] = (\mathsf{F}\underline{x}[k])^t \cdot \left(\mathsf{F}^{-1}\underline{w}[k]\right) \end{aligned}$$

FDAF

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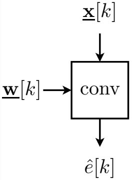
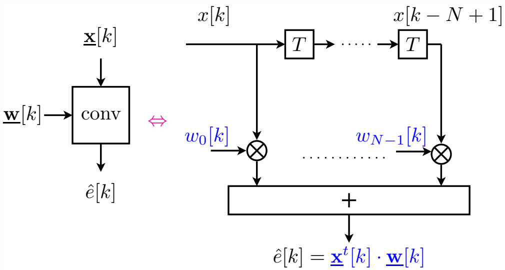
$$\begin{aligned} \hat{e}[k] &= \sum_{i=0}^{N-1} x[k-i] \cdot w_i[k] = \underline{x}^t \cdot \underline{w}[k] \\ &= \underline{x}^t[k](\mathsf{F} \cdot \mathsf{F}^{-1})\underline{w}[k] = (\mathsf{F}\underline{x}[k])^t \cdot \left(\mathsf{F}^{-1}\underline{w}[k]\right) \\ &= \underline{X}^t[k] \cdot \underline{W}[k] = \sum_{l=0}^{N-1} X_l[k] W_l[k] \end{aligned}$$

FDAF

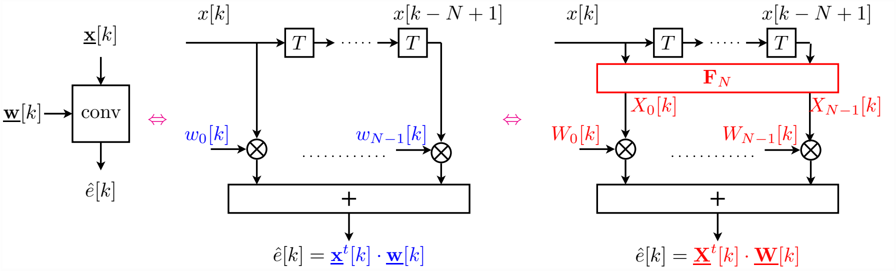


FDAF



$\underline{\mathbf{x}}[k]$
$\underline{\mathbf{w}}[k]$
conv
$\hat{e}[k]$
$\Leftrightarrow$
$x[k]$
T
T
$x[k-N+1]$
$w_0[k]$
$w_{N-1}[k]$
$+$
$\hat{e}[k] = \underline{\mathbf{x}}^t[k] \cdot \underline{\mathbf{w}}[k]$

FDAF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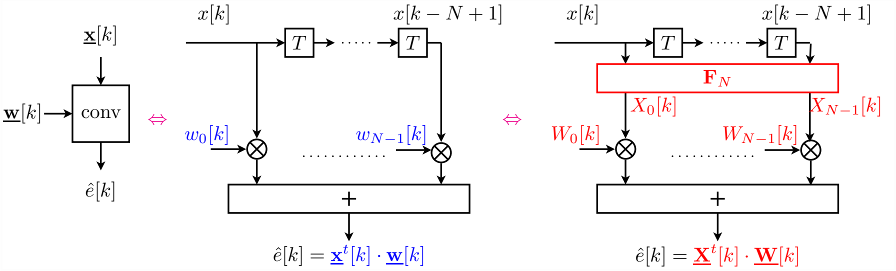
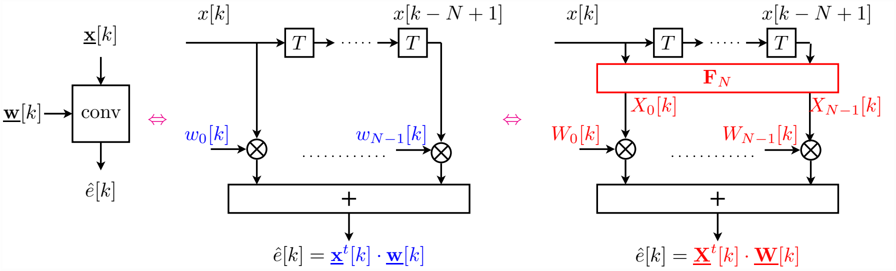


Notes:

- Inverse DFT in definition of weights $W_l[k]$

FDAF



Notes:

- Inverse DFT in definition of weights $W_l[k]$
- Use DFT symmetry to reduce complexity

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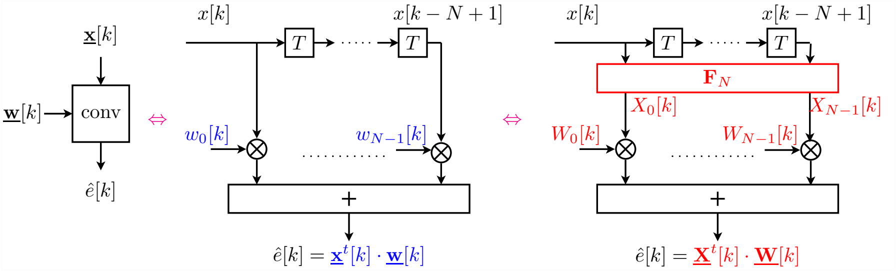

Notes:

- Inverse DFT in definition of weights $W_l[k]$
- Use DFT symmetry to reduce complexity
- For large N: Frequency bins "uncorrelated"

FDAF

Transform LMS to frequency domain (multiply update algorithm by F^{-1}*)*

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$$\mathsf{F}^{-1} \cdot \underline{w}[k+1] \quad = \quad \mathsf{F}^{-1} \cdot \underline{w}[k] + 2\alpha \mathsf{F}^{-1} \cdot \underline{x}[k] r[k]$$

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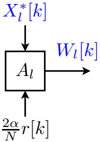

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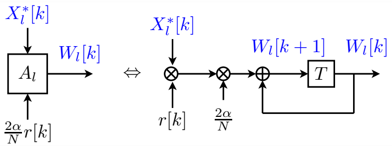

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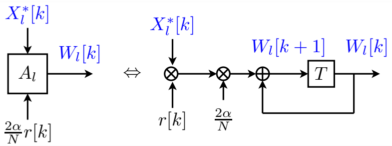

Improve convergence by Power normalization:

$$
P_l = \frac{1}{N} E\{|X_l[k]|^2\}
$$

FDAF

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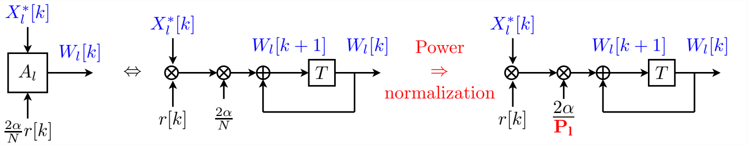

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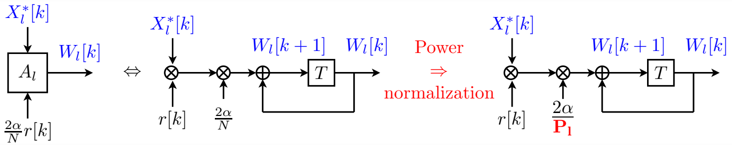

Improve convergence by Power normalization:

$$P_l = \frac{1}{N} E\{|X_l[k]|^2\} \quad \text{e.g.:} \quad \hat{P}_l[k+1] = \beta \hat{P}_l[k] + (1-\beta) \frac{|X_l[k]|^2}{N}$$

FDAF

FDAF algorithm: $\underline{\mathrm{W}}[k+1] = \underline{\mathrm{W}}[k] + 2\alpha \mathrm{P}^{-1} \underline{\mathrm{X}}^{*}[k] r[k]$

with $\mathrm{P} = \mathrm{diag}\{\underline{\mathrm{P}}\}$ and $(\underline{\mathrm{P}})_l = P_l = \frac{1}{N} E\{|X_l[k]|^2\}$

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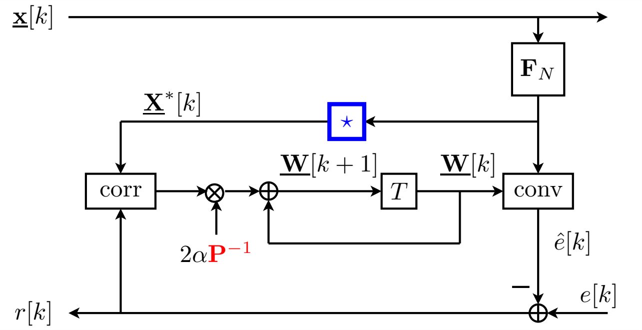

FDAF

Average behavior FDAF:

With $\underline{\mathrm{D}} = \mathrm{F}^{-1}\underline{\mathrm{d}} = \mathrm{F}^{-1}(\underline{\mathrm{w}} - \underline{\mathrm{w}}_o) = \underline{\mathrm{W}} - \underline{\mathrm{W}}_o$ FDAF becomes:

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Different bins 'uncorrelated' $\Rightarrow \frac{E\{\underline{\mathrm{X}}^*[k]\underline{\mathrm{X}}^t[k]\}}{N} \approx \mathrm{P}$

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$$\Rightarrow E\{\underline{\mathrm{D}}[k+1]\} \approx (1-2\alpha)\cdot E\{\underline{\mathrm{D}}[k]\} \Rightarrow \lim_{k\to\infty} E\{\underline{\mathrm{D}}[k]\} = \underline{0}$$

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FDAF converges to Wiener solution: $\lim_{k\to\infty} E\{\underline{\mathrm{W}}[k]\} = \underline{\mathrm{W}}_o = \mathrm{F}^{-1}\underline{\mathrm{w}}_o$

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Notes:

- DFT (FFT) is fixed transform: Easy but not exact $\left(\frac{E\{\underline{\mathrm{X}}^*[k]\underline{\mathrm{X}}^t[k]\}}{N} \approx \mathrm{P}\right)$

FDAF

Average behavior FDAF:

With $\underline{\mathrm{D}} = \mathrm{F}^{-1}\underline{\mathrm{d}} = \mathrm{F}^{-1}(\underline{\mathrm{w}} - \underline{\mathrm{w}}_o) = \underline{\mathrm{W}} - \underline{\mathrm{W}}_o$ FDAF becomes:

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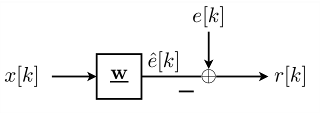
FDAF converges to Wiener solution: $\lim\limits_{k\to\infty} E\{\underline{\mathrm{W}}[k]\} = \underline{\mathrm{W}}_o = \mathrm{F}^{-1}\underline{\mathrm{w}}_o$

Notes:

- DFT (FFT) is fixed transform: Easy but not exact $\left(\frac{E\{\underline{\mathrm{X}}^*[k]\underline{\mathrm{X}}^t[k]\}}{N} \approx \mathrm{P}\right)$
- FDAF equivalent to NLMS with white noise input

Summary



	MMSE	LS
Auto correlation	$\mathbf{R}_x = E\{\underline{x}[k] \cdot \underline{x}^t[k]\}$	$\bar{\mathbf{R}}_x = \mathbf{X}^t \cdot \mathbf{X}$
Cross correlation	$\underline{r}_{ex} = E\{e[k] \cdot \underline{x}[k]\}$	$\underline{\bar{r}}_{ex} = \mathbf{X}^t \cdot \underline{e}$
Error J	$E\{r^2[k]\}$	$\sum_{i=0}^{L-1} r^2[k-i]$
Criterion	$\min_{\underline{w}}\{E\{r^2[k]\}\}$	$\min_{\underline{w}} \lvert\underline{e} - \mathbf{X} \cdot \underline{w}\rvert^2$
Opt. solution $\underline{w}_o$	$\mathbf{R}_x^{-1} \cdot \underline{r}_{ex}$	$\bar{\mathbf{R}}_x^{-1} \cdot \underline{\bar{r}}_{ex}$
Min. error J_{min}	$E\{e^2\} - \underline{r}_{ex}^t \mathbf{R}_x^{-1} \underline{r}_{ex}$	$\underline{e}^t \underline{e} - \underline{\bar{r}}_{ex}^t \bar{\mathbf{R}}_x^{-1} \underline{\bar{r}}_{ex}$

Summary

Set of constraints: $\mathrm{C}^t \cdot \underline{\mathrm{w}} = \underline{\mathrm{f}}$

Solution for $N \geq M$: $\underline{\mathrm{w}}^c = \mathrm{C}(\mathrm{C}^t\mathrm{C})^{-1}\underline{\mathrm{f}}$

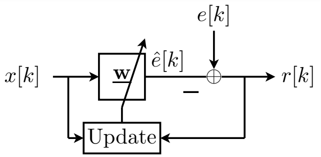
Solution for $N > M$ with MMSE:

$$\underline{\mathrm{w}}_o^c = \underline{\mathrm{w}}_o + \mathrm{R}_x^{-1}\mathrm{C}(\mathrm{C}^t\mathrm{R}_x^{-1}\mathrm{C})^{-1}(\underline{\mathrm{f}} - \mathrm{C}^t\underline{\mathrm{w}}_o)$$

Similar result:

$$\underline{\mathrm{w}}_o^c = \mathrm{R}_x^{-1}\mathrm{C}(\mathrm{C}^t\mathrm{R}_x^{-1}\mathrm{C})^{-1}\underline{\mathrm{f}}$$

Summary



Simple adaptive algorithms (no decorrelation):

$$\begin{aligned} \text{SGD} &: \quad \underline{w}[k+1] = \underline{w}[k] + 2\alpha(\underline{r}_{ex} - R_x\underline{w}[k]) \\ \text{(complex)(N)LMS} &: \quad \underline{w}[k+1] = \underline{w}[k] + \frac{2\alpha}{\hat{\sigma}_x^2}\underline{x}[k]r^*[k] \end{aligned}$$

Constrained LMS: $C^t \cdot \underline{w} = \underline{f}$

$$\underline{w}[k+1] = \tilde{P} \cdot \{\underline{w}[k] + 2\alpha\underline{x}[k]r[k]\} + C\left(C^t \cdot C\right)^{-1}\underline{f}$$

with

$$\tilde{P} = I - C\left(C^t \cdot C\right)^{-1} C^t \text{ and } \underline{w}[0] = C\left(C^t \cdot C\right)^{-1}\underline{f}$$

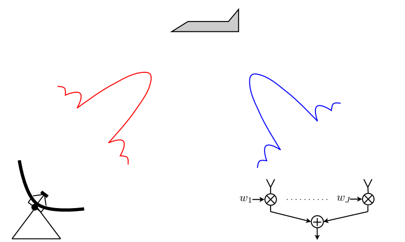
Summary

<u>*Algorithms with improved convergence:*</u>

$$
\begin{aligned}
\text{LMS/Newton} &: \quad \underline{\mathrm{w}}[k+1] = \underline{\mathrm{w}}[k] + 2\alpha \mathrm{R}_x^{-1}\underline{\mathrm{x}}[k]r[k] \\
\text{Newton} &: \quad \underline{\mathrm{w}}[k+1] = \underline{\mathrm{w}}[k] + 2\alpha \mathrm{R}_x^{-1} \cdot (\underline{\mathrm{r}}_{ex} - \mathrm{R}_x\underline{\mathrm{w}}[k]) \\
\text{RLS} &: \quad \underline{\mathrm{g}}[k+1] = \frac{\overline{\mathrm{R}}_x^{-1}[k]\underline{\mathrm{x}}[k+1]}{\gamma^2 + \underline{\mathrm{x}}^t[k+1]\overline{\mathrm{R}}_x^{-1}[k]\underline{\mathrm{x}}[k+1]} \\
&\quad\ \overline{\mathrm{R}}_x^{-1}[k+1] = \gamma^{-2}\left(\overline{\mathrm{R}}_x^{-1}[k] - \underline{\mathrm{g}}[k+1] \cdot \underline{\mathrm{x}}^t[k+1]\overline{\mathrm{R}}_x^{-1}[k]\right) \\
&\quad\ \overline{\underline{\mathrm{r}}}_{ex}[k+1] = \gamma^2\overline{\underline{\mathrm{r}}}_{ex}[k] + \underline{\mathrm{x}}^t[k+1] \cdot e[k+1] \\
&\quad\ \underline{\mathrm{w}}[k+1] = \overline{\mathrm{R}}_x^{-1}[k+1] \cdot \underline{\mathrm{r}}_{ex}[k+1]
\end{aligned}
$$

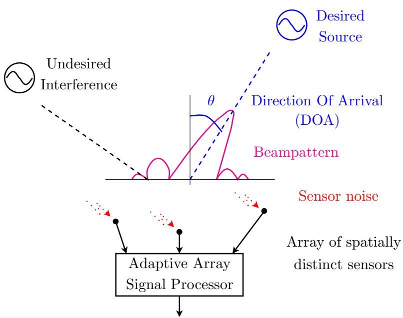
Array Signal Processing (Part IB)

Introduction



Parabolic dish antenna
(*contiuous aperture*)

Sensor array antenna
(*discrete spatial aperture*)



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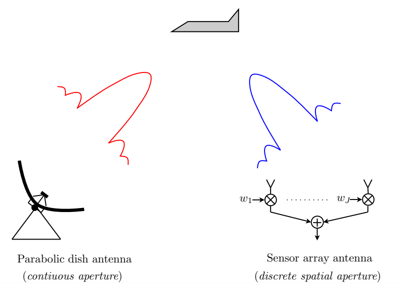

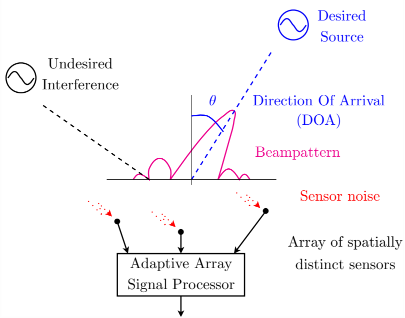

Beamforming: Spatio temporal filtering to either direct of block the radiation or reception of signals in specified directions

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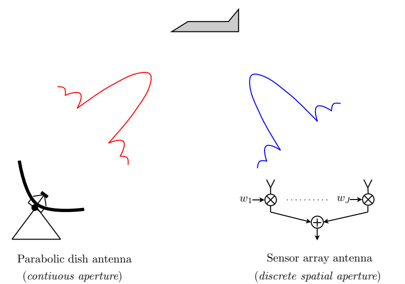

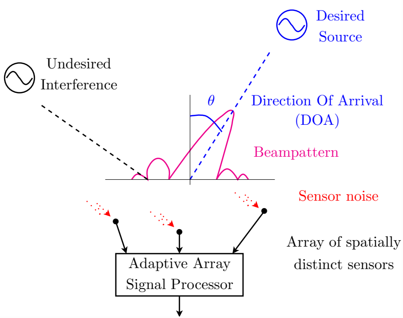

Beamforming: Spatio temporal filtering to either direct of block the radiation or reception of signals in specified directions

Result Beamforming = **Spatial filtering**: Separate signals with possible overlapping frequencies but from different directions

Different scenario's (content first part):

- Bandwidth source
- Array geometry
- Far field vs. near field
- Direction Of Arrival (DOA)
- Discrete-time signal representation
- Array signal model
- ASP unit
- Spatial/ temporal filtering
- Broadband signals

Scenario: Bandwidth source

Analytical representation: $s(t) = A(t)e^{j(\omega_o t + \phi(t))}$

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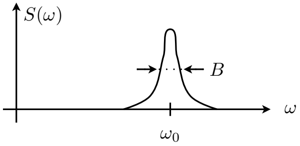
Narrowband: $A(t)$ and $\phi(t)$ vary slower than $e^{j\omega_0 t}$



Narrowband: $|\tau| \ll 1/B$

$A(t-\tau) \approx A(t) = 1$ (usually) $\Rightarrow$

$\phi(t-\tau) \approx \phi(t) = 0$ (usually)

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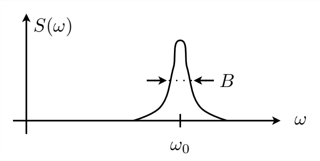

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$\Rightarrow s(t-\tau) = A(t-\tau)e^{j\phi(t-\tau)}e^{j\omega_0(t-\tau)} \approx e^{-j\omega_o\tau} \cdot s(t)$

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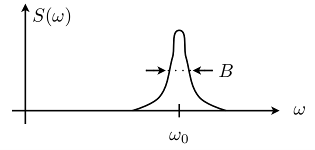

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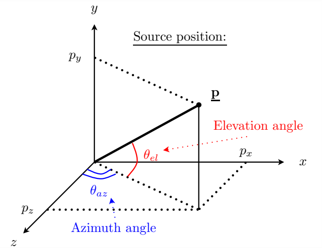
Thus for narrowband: Time delay $\equiv$ phase shift

In this course mainly narrowband

Scenario: Array geometry



$$
\begin{aligned}
\underline{p} &= (p_x, p_y, p_z)^t \\
p_x &= ||\underline{p}|| \sin(\theta_{az}) \cos(\theta_{el}) \\
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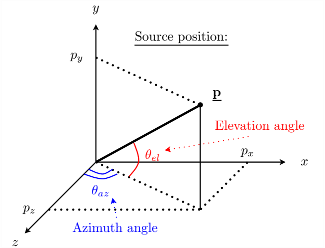

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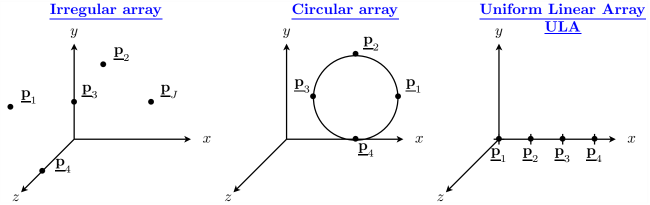
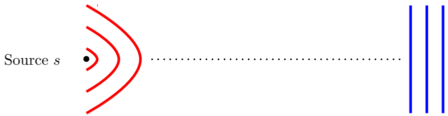
Array can be uniform, nonuniform, linear, circular, $\cdots$



In this course mainly: **ULA**

Scenario: Far field vs. near field

Array aperture: Volume (1D length) that collects incoming signal



Near field

- Distance source - array $\ll$ Array aperture
- Sherical wavefront

Far field

- Distance source - array $\gg$ Array aperture
- Plane wavefront

Scenario: Far field vs. near field

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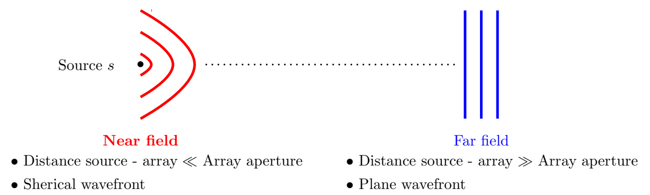

Near field: Propagation for single frequency source

$$s(t, \underline{p}) = \frac{A}{||\underline{p}||^2} e^{j\omega\left(t - \frac{||\underline{p}||}{c}\right)} \text{ with } \omega = 2\pi f \text{ and } f = \frac{c}{\lambda}$$

$\lambda =$ wavelength, $c =$ speed in medium (≈ 334 [m/sec] for sound in air)

$\Rightarrow$ Amplitude decays proportional to distance from source

Scenario: Far field vs. near field

In this course mainly far field:

$$s(t, \underline{p}_i) = A e^{\mathrm{j}\omega(t-\tau_i)} = A e^{\mathrm{j}\omega(t-\frac{\underline{v}^t \cdot \underline{p}_i}{c})} = A e^{\mathrm{j}(\omega t - \underline{k}^t \cdot \underline{p}_i)}$$

with direction vector $\underline{v}$, wave number vector $\underline{k} = \frac{\omega}{c} \cdot \underline{v}$

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- ✓ $s(t, \underline{p}_i)$ describes propagation as function of both time and space
- ✓ Information is preserved while propagating

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$\Rightarrow$ Reconstruction band limited signal over all space and time by either:

- Temporally sampling at given location in space
- Spatially sampling at given instant of time
- **Combination**

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⇒ Reconstruction band limited signal over all space and time by either:

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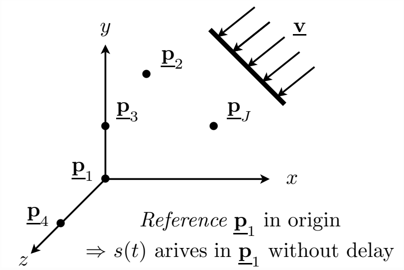
Spatially sampling:

$\boxed{\text{Basis for all aperture and sensor array processing techniques}}$

Scenario: Far field vs. near field



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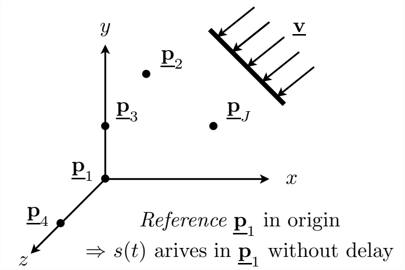

At position $\underline{p}_i$: $s(t - \tau_i) = s(t)\mathrm{e}^{-\mathrm{j}\omega\tau_i}$

with delay $\tau_i = \frac{\underline{v}^t \cdot \underline{p}_i}{c}$ and $\underline{v}$ is direction vector

$\omega = 2\pi f$, $f = \frac{c}{\lambda}$, $\lambda =$ wavelength, $c =$ speed in medium

Scenario: Direction Of Arrival (DOA)

Location is 3D quantity. In practice often 2D

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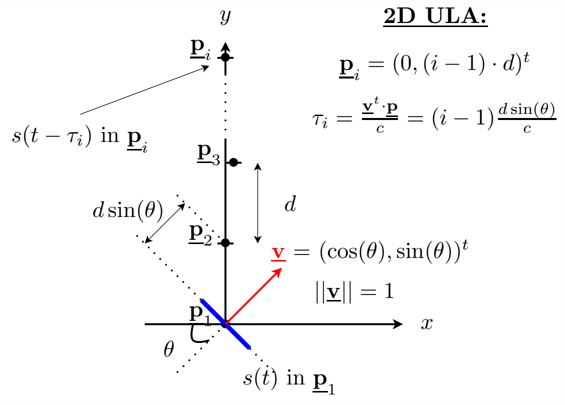
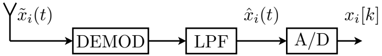
Example: Narrow band, far field 2D DOA for ULA:



Scenario: Discrete-time signal representation



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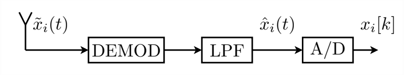

- ✓ Analog sensor signal at sensor i: $\tilde{x}_i(t)$
- ✓ Ideal demodulation and LPF results in baseband signal: $\hat{x}_i(t)$
- ✓ After A/D (complex valued) discrete-time signal: $x_i[k]$
- ✓ Analog signal at $\underline{\mathrm{p}}_i$ for narrow band, far field case:

$$\hat{x}_i(t) = s(t - \tau_i) = s(t)\mathrm{e}^{-\mathrm{j}\omega\tau_i} \quad \text{with} \quad \tau_i = \frac{\underline{\mathrm{v}}^t \cdot \underline{\mathrm{p}}_i}{c}$$

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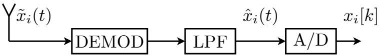

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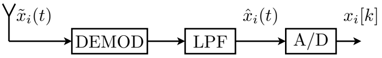
- ✓ Discrete-time signal at $\underline{p}_i$ for ULA-case:

$$s[k]\mathrm{e}^{-\mathrm{j}\omega\tau_i} = s[k]\cdot\mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d\sin(\theta)}{\lambda}} = s[k]\cdot a_i(\theta) \text{ with } a_i(\theta) = \mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d\sin(\theta}{\lambda}}$$

Scenario: Discrete-time signal representation



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$$s[k]e^{-j\omega\tau_i} = s[k]\cdot e^{-j2\pi(i-1)\frac{d\sin(\theta)}{\lambda}} = s[k]\cdot a_i(\theta) \text{ with } a_i(\theta) = e^{-j2\pi(i-1)\frac{d\sin(\theta}{\lambda}}$$

Note: In fact $a_i(\theta)$ also depends on ω

Scenario: Array signal model

$$\begin{aligned}
\text{Array sensor vector} \quad &: \quad \underline{\mathrm{x}}[k] = (x_1[k], x_2[k], \cdots x_J[k])^t \\
\text{Noise vector} \quad &: \quad \underline{\mathrm{n}}[k] = (n_1[k], n_2[k], \cdots n_J[k])^t \\
\text{Steering vector} \quad &: \quad \underline{\mathrm{a}}[k] = (a_1(\theta), a_2(\theta), \cdots a_J(\theta))^t \\
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Case: Noise observation, P sources, J sensors

Scenario: Array signal model

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Case: Noise observation, P sources, J sensors

$$x_i[k] = \sum_{p=1}^{P} a_i(\theta_p) s_p[k] + n_i[k]$$

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Case: Noise observation, P sources, J sensors

$$x_i[k] = \sum_{p=1}^{P} a_i(\theta_p) s_p[k] + n_i[k] \quad \Leftrightarrow \quad \boxed{\underline{\mathrm{x}}[k] = \mathrm{A} \cdot \underline{\mathrm{s}}[k] + \underline{\mathrm{n}}[k]}$$

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$$\begin{aligned}
J \times P \text{ steering matrix A} &= (\underline{a}(\theta_1), \underline{a}(\theta_2), \cdots, \underline{a}(\theta_P)) \\
P \times 1 \text{ signal vector } \underline{s}[k] &= (s_1[k], s_2[k], \cdots, s_P[k])^t
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Case: Noise observation, P sources, J sensors

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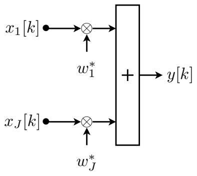
Covariance structure: $\boxed{\mathrm{R}_x = E\{\underline{x} \cdot \underline{x}^h\} = \mathrm{A}\mathrm{R}_s\mathrm{A}^h + \mathrm{R}_n}$

with $\mathrm{R}_s = E\{\underline{s} \cdot \underline{s}^h\}$ and $\mathrm{R}_n = E\{\underline{n} \cdot \underline{n}^h\} = \sigma_n^2 \mathbf{I}$

Scenario: ASP unit

Case: Single complex weight for each sensor



$$y[k] = \sum_{i=1}^{J} w_i^* x_i[k] = \underline{w}^h \cdot \underline{x}[k]$$

$$\underline{x}[k] = (x_1[k], \cdots, x_J[k])t$$

$$\underline{w} = (w_1, \cdots, w_J)^t$$

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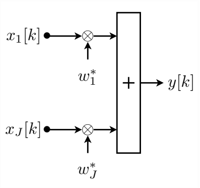

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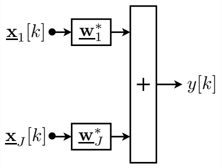
Case: FIR filter for each sensor



$\underline{\mathbf{w}}_i$: FIR filter with N weights

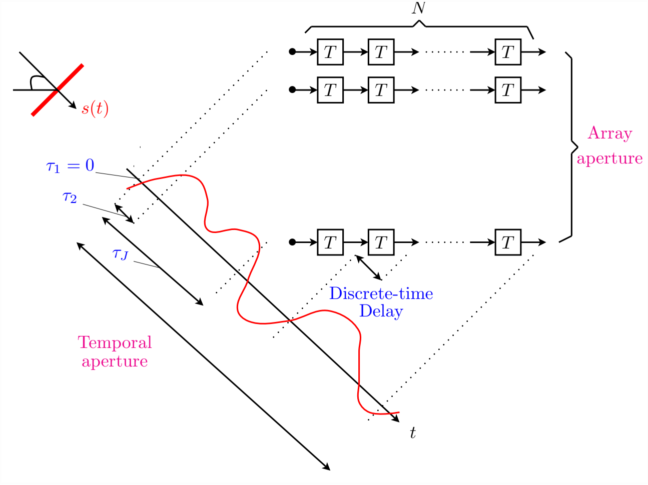
$$y[k] = \sum_{i=1}^{J} \underline{w}_i^h \cdot \underline{x}_i[k] = \underline{w}^h \cdot \underline{x}[k]$$

$$\underline{x}[k] = (\underline{x}_1[k], \cdots, \underline{x}_J[k])t$$

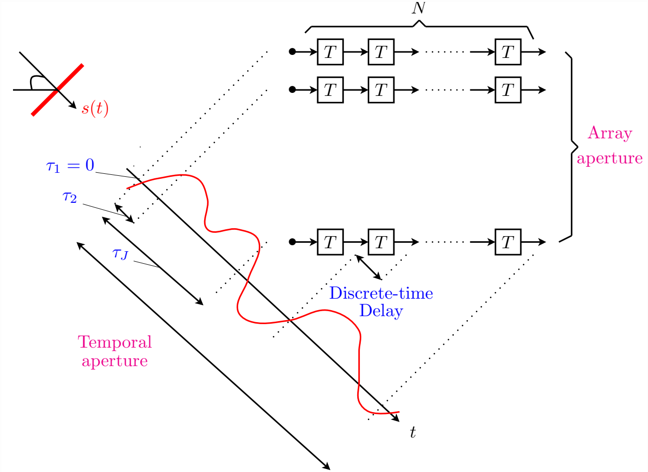
$$\underline{w} = (\underline{w}_1, \cdots, \underline{w}_J)^t$$

$$\underline{w}_i = (w_{i,1}, \cdots, w_{i,N})^t$$

Scenario: Spatial/ temporal filtering

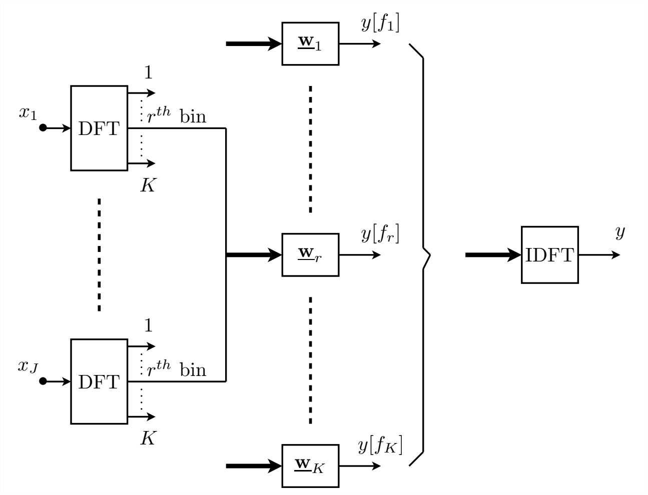


Scenario: Spatial/ temporal filtering



Note: FIR filtering effect both temporal and spatial response

Scenario: Broadband signals



ULA Beampattern

Main properties

ULA Beampattern

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- Single source $s(t) = e^{j\omega t}$
- Frequency relations: $\omega = 2\pi f = 2\pi \frac{c}{\lambda}$, with wavenumber λ and speed of propagation c (≈ 343 [m/sec])
- Direction Of Arrival (DOA): θ
- Far field, thus plane wavefront
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- ULA with distance d [m] between sensors
- J omnidirectional sensors $\Rightarrow$ array aperture $= J \cdot d$ [m]
- Model:$\underline{x}[k] = \underline{a}(\theta) \cdot s[k]$ (No noise, no interference)
- ASP unit: Single complex weight fw_i or each sensor

$$\Rightarrow \quad y[k] = \sum_{i=1}^{J} w_i^* x_i[k] = \underline{w}^h \cdot \underline{x}[k] = \underline{w}^h \cdot \underline{a}(\theta) \cdot s[k]$$

with $(\underline{a}(\theta))_i = \mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d\sin(\theta)}{\lambda}}$

ULA Beampattern

Array response : $r(\theta) = \underline{w}^h \cdot \underline{a}(\theta)$

Other names: Angular response or directivity pattern

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Notes:

- Array response: Response to unit-amplitude plane wave front from direction θ
- Non ideal sensor characteristics can be incorporated
- Weights effect both temporal and spatial response
- Vector space interpretation: Angle between $\underline{w}$ and $\underline{a}$
- To evaluate beampattern: Choose all weight equal to one, thus $\underline{w} = (1, 1, \cdots, 1)^t$

ULA Beampattern: The equation

$$B(\theta) = \frac{1}{J^2}\left|\underline{1}^t \cdot \underline{\mathrm{a}}(\theta)\right|^2$$

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\begin{aligned}
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\end{aligned}
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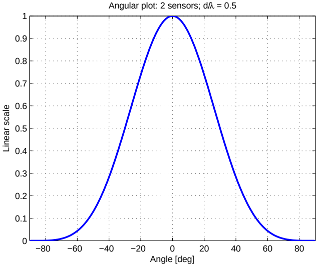
Main parameters:

- DOA θ
- Ratio $\frac{d}{\lambda}$ (everything scales with wavelength)
- Number of sensors J
- Element spacing d
- Array aperture $L = J \cdot d$

ULA Beampattern: Example $J = 2, \frac{d}{\lambda} = \frac{1}{2}$

Linear plot:



Polar plot:

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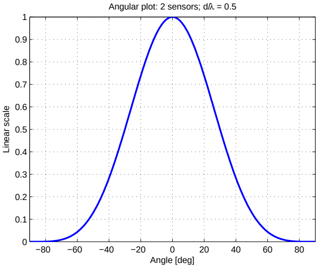

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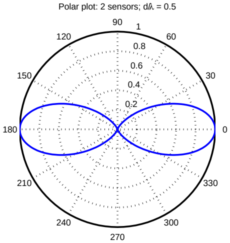

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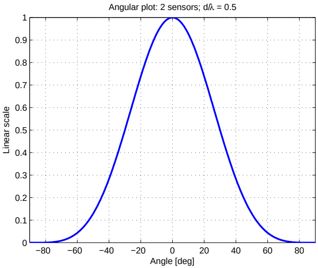

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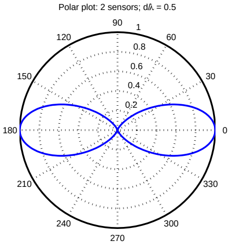
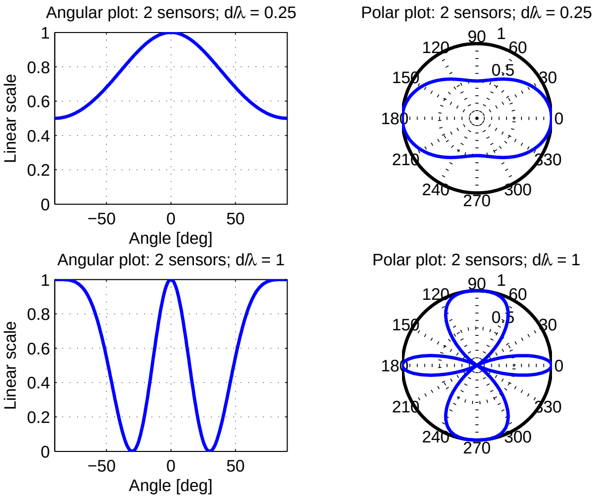


Some preliminary conclusions:

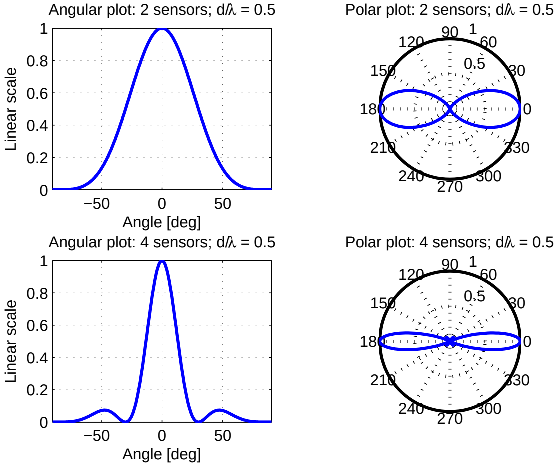
- Ambiguity between 'front' (line of sight) and 'back': $B(\theta) = B(\pi - \theta)$
- Zeroes if numerator of $B(\theta) = 0 \Rightarrow \theta = \arcsin(i \cdot \frac{1}{J} \cdot \frac{\lambda}{d})$
- For $\frac{d}{\lambda} = \frac{1}{2} \Rightarrow$ Zeroes at $\theta = \pm\frac{\pi}{2}$ and mainlobe (3dB) beamwidth: 60^o

ULA Beampattern: $J = 2$, variable $\frac{d}{\lambda}$



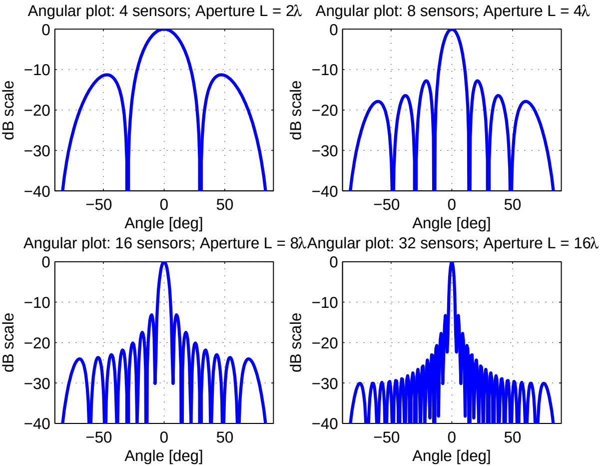
ULA Beampattern: Variable # sensors J

$$L = J \cdot d \text{ with } J \uparrow \text{ and fixed } \frac{d}{\lambda} = \frac{1}{2}$$



ULA Beampattern: dB scale, variable L, $\frac{d}{\lambda} = \frac{1}{2}$

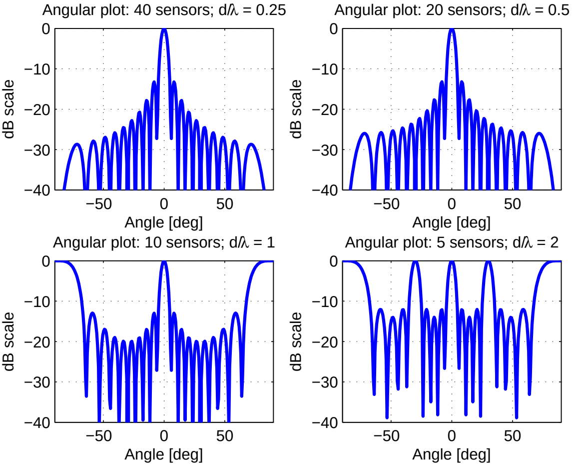
Aperture size $L = J \cdot d$



Note for assignment: Usually maximum at 0 [dB]

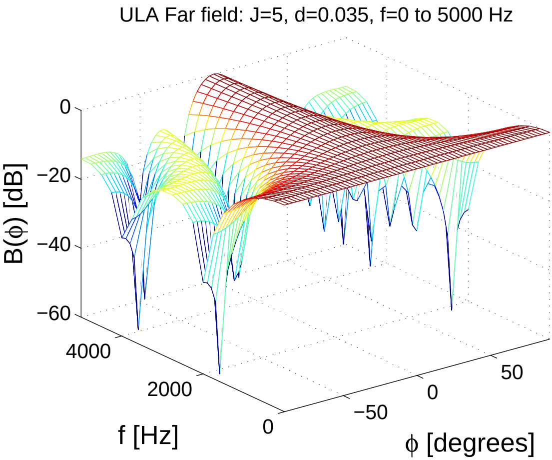
ULA Beampattern: dB scale, variable d, fixed L

Fixed $L = J \cdot d = 10\lambda$



ULA Beampattern: Frequency dependency

Example: $$d = \frac{\lambda_{min}}{2} = \frac{c}{2 \cdot f_{max}} \approx 3.5[\mathrm{cm}]$$



ULA Beampattern: DFT view

With $u = \frac{d\sin(\theta)}{\lambda} \Rightarrow$ ULA beampattern becomes:

$$B(u) = \frac{1}{J^2}|\underline{\mathrm{w}}^h \cdot \underline{\mathrm{a}}(u)|^2 = \frac{1}{J^2}|\sum_{p=0}^{J-1} w_p^* \mathrm{e}^{-\mathrm{j}2\pi p u}|^2$$

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Note: Since unambiguous angles $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, to avoid aliasing

$$-\frac{1}{2} \leq u \leq \frac{1}{2} \quad \Leftrightarrow \quad d \leq \frac{\lambda}{2}$$

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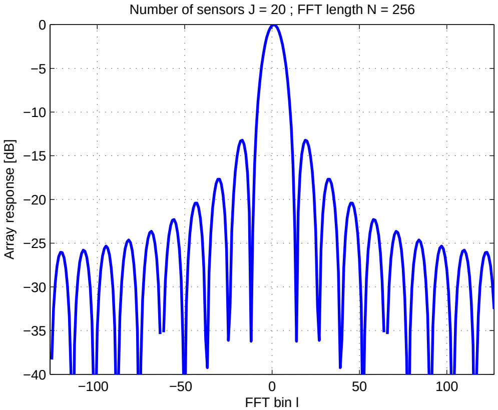
Zero padded DFT: With $N \geq J$

$$B_l = \frac{1}{J^2}|\sum_{p=0}^{J-1} w_p^* e^{-j\frac{2\pi}{N}pl}|^2$$

with $l = N \cdot u = N \cdot \frac{d\sin(\theta)}{\lambda}$

ULA Beampattern: DFT view



Compute corresponding angle via: $\theta = \arcsin\left(\frac{l}{N} \cdot \frac{\lambda}{d}\right)$

Data dependent beamforming

Data dependent beamforming

Conventional approaches:

- Beam steering
- Tapering
- Null steering
- Array response design

Beam steering

Purpose:

Compensate propagation path length differences of direct path from source to each sensor resulting is properly aligned direct path signals at the output

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Other name: **Delay and Sum Beamformer (DSB)**

Beam steering

Desired source signal $s(t)$, of single frequency $f_d = \frac{c}{\lambda}$, at DOA θ_0:

$$\Rightarrow x_i[k] = s[k] \cdot a_i(\theta_0) = s[k] \cdot \mathrm{e}^{-\mathrm{j}2\pi f_d \tau_i}$$

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ULA delay at sensor i:

$\tau_i = (i-1) \cdot \frac{d \sin(\theta_0)}{c} \Rightarrow$

$x_i[k] =$
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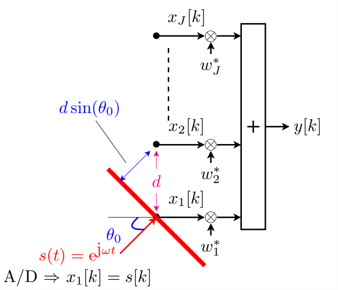

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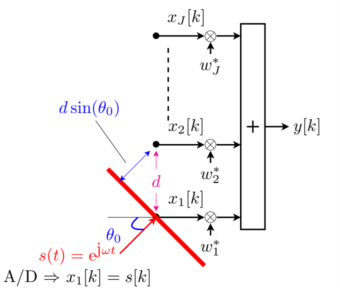

In order to properly align desired source (DOA $= \theta_0$) at output:

$$\boxed{w_i^* = e^{+j2\pi(i-1)\frac{d \sin(\theta_0)}{\lambda}}} \quad \Rightarrow \quad y[k] = J \cdot s[k]$$

Beam steering

Resulting beampattern shifted/ rotated over θ_0:

$$B(\theta) = \frac{1}{J^2} \left| \underline{\mathrm{w}}^h \cdot \underline{\mathrm{a}}(\theta) \right|^2 = \frac{1}{J^2} \left| \sum_{i=1}^{J} \mathrm{e}^{-\mathrm{j}2\pi f_d (i-1) \frac{d}{c} (\sin(\theta) - \sin(\theta_0))} \right|^2$$

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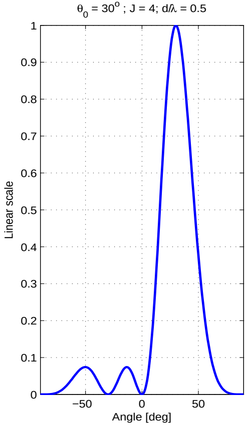
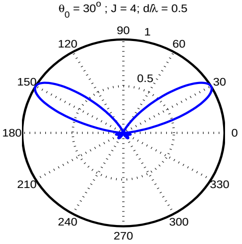
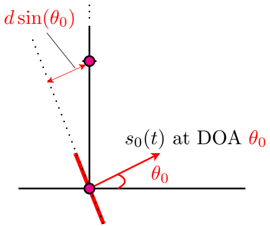




Beam steering

Electronic vs mechanical beamsteering

Electronic beamsteering



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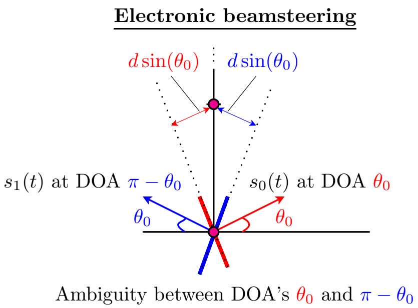

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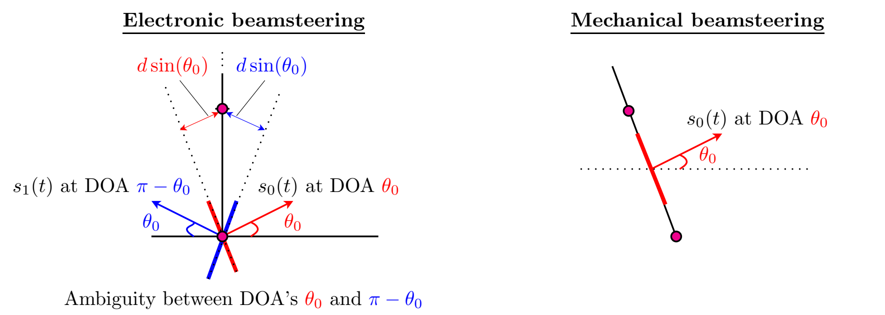

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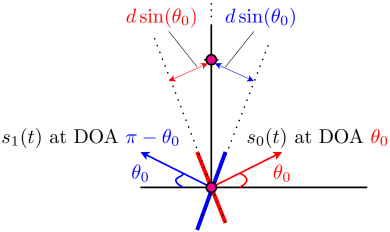
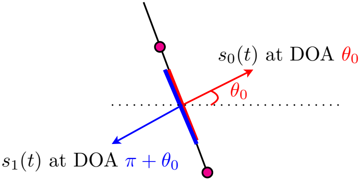


Ambiguity between DOA's θ_0 and $\pi - \theta_0$

Mechanical beamsteering



Ambiguity between DOA's θ_0 and $\pi + \theta_0$

Beam steering: matched filter

Another view to beamsteering for spatially white noise

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Another view to beamsteering for spatially white noise

$$\begin{aligned} y[k] &= \underline{w}^h \cdot \underline{x}[k] = \underline{w}^h \cdot (\underline{a}(\theta)s[k] + \underline{n}[k]) \\ &= \underline{w}^h \cdot \underline{a}(\theta)s[k] + \underline{w}^h \cdot \underline{n}[k] = y_s[k] + y_n[k] \end{aligned}$$

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Define in- and output SNR as: $\mathrm{SNR}_{in} = \frac{\sigma_s^2}{\sigma_n^2} \qquad \mathrm{SNR}_o = \frac{P_s}{P_n}$

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Thus SNR_o maximized by choosing $\underline{w} = \beta \cdot \underline{a}(\theta)$, with β some constant

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Conclusion:

Spatial filter that maximizes DOA $\equiv$ **Matched filter** ($=$ max SNR_o)

Beam steering

Notes on beamsteering (delay and sum beamforming (DSB):

- Source location (or DOA) required
- Position sensors must be known
- DSB aligns only direct path
- Difference between electronic vs mechanical beamsteering
- In sense of max SNR_o, spatial matched filter optimum for spatially white noise

Beam steering: Discrete-time domain

Steering delay $\tau = \frac{d\sin(\theta)}{c}$; Sample rate $f_s = \frac{1}{T_s} \Rightarrow$

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Possible steering to: $\theta_s = \arcsin\left(\frac{c \cdot \alpha \cdot T_s}{d}\right)$ with $|\alpha| = 0, 1, \cdots$

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$d = \frac{\lambda}{2}$, $\lambda = \frac{c}{f_0}$ and $f_s = 2\gamma f_0 \Rightarrow \theta_s = \arcsin\left(\frac{\alpha}{\gamma}\right) \Leftrightarrow |\alpha| \leq \gamma$

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If more directions needed: Use interpolation and/or fractional delays

Beam steering: Tapering

Goal:

Control shape of response i.e. to form beam. Thus window (weighted) sensor signals to compromise between resolution (main lobe width) and leakage (side lobe level) $\Rightarrow \underline{\mathrm{w}}_t = \underline{\mathrm{t}} \odot \underline{\mathrm{w}}$ *with* $\underline{\mathrm{t}}$ *taper window and* $\odot$ *element by element multiplication*

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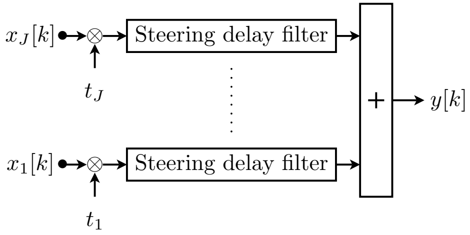

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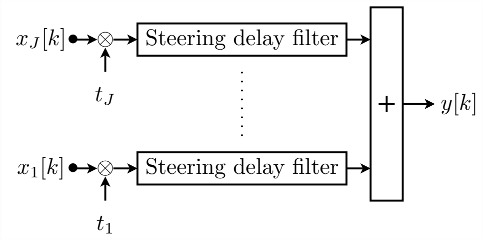

Notes:

- Taper weights used to shape beampattern
- Filters approximate delays (linear phase over frequency band of interest)

Null-steering

Goal: *Calculate J weights to meet M constraints, e.g. to amplify the desired, and to attenuate the undesired sources*

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Case: $M < J$ (*Less constraints then weights*)

$$\boxed{\underline{w} = (A^h)^\dagger \cdot \underline{r}_d = A \cdot \left(A^h \cdot A\right)^{-1} \cdot \underline{r}_d}$$

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Example: *Null signal at 90° with 2 sensor ULA at distance half wavelength. Thus:* $J = 2$, $M = 1$, $\theta_u = \pi/2$, $r(\theta_u) = 0$, $d/\lambda = 1/2$

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Note: For this solution we obtain a "line through origin"

$$A^h \cdot \underline{w} = \underline{r}_d \quad \Leftrightarrow \quad (1, -1) \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \quad \Rightarrow \quad w_1 = w_2$$

Since $J = 2$ and $M = 1$: One degree of freedom left!
This can be used e.g. to overcome the solution $w_1 = w_2 = 0$

Null-steering

Example: *Two (complex) plane waves. One desired at 0º, other undesired at 30º, ULA with 3 sensors at distance half wavelength* $\Rightarrow J = 3$, $M = 2, \theta_d = 0$, $\theta_u = \pi/6$, $r(\theta_d) = 1$, $r(\theta_u) = 0$, $d/\lambda = 1/2$

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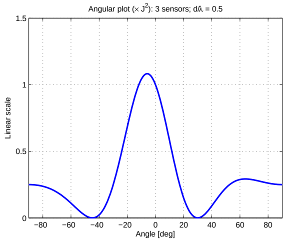

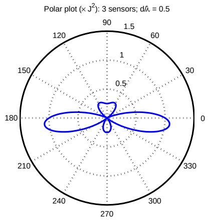

Null-steering: Conclusions

Null steering used to cancel plane waves arriving from **known** *directions*

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DOA of desired/undesired signals, position of sensors

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Null steering used to cancel plane waves arriving from **known** *directions*

Required knowledge:
DOA of desired/undesired signals, position of sensors

Performance:
SNR **not** maximized, but nulls can be put in DOA's of interferences

Properties:

- Result not robust to frequency jammer
- J weights can set maximum J predefined conditions
- Needs much a priori information
- $M < J$: Add extra constraints (e.g. minimize output power)
- Use FIR filters for broadband

Array response design: $M > J$

Procedure: $\quad \underline{w} = \arg\min_{\underline{w}}\{E\}$

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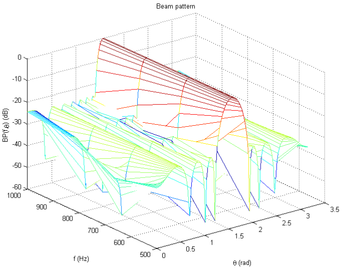
Example: $B(\theta) = 1$ *at* $\theta = \pi/2$ *and* < -25 *[dB] outside this area*



Beamforming: overview

Data independent (conventional approach): **(Part IB)**

- **Beamsteering (DSB, phased array)**
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Data dependent (statistical optimum): (Part IC)

- **Minimum Sidelobe Canceller**
- **Wiener**
- **Maximum SNR**
- **Linear Constraint Minimum Variance**
- **Generalized Sidelobe Canceller**

Summary part II

Far field, narrowband source, direction vector $\underline{v}$, at position $\underline{p}_i$:

$s[k]e^{-j\omega\tau_i}$ with $\tau_i = \frac{\underline{v}^t \cdot \underline{p}_i}{c}$, $\omega = 2\pi f$, $f = \frac{c}{\lambda}$

For ULA-case at sensor i: $s[k] \cdot a_i(\theta)$ with $a_i(\theta) = e^{-j2\pi(i-1)\frac{d\sin(\theta)}{\lambda}}$

Array response/ Angular response/ Directivity pattern:

$r(\theta) = \underline{w}^h \cdot \underline{a}(\theta)$

Beampattern: $B(\theta) = \frac{1}{J^2}|r(\theta)|^2$

For ULA with inter element distance d and $\underline{w} = \underline{1}$:

$$B(\theta) = \frac{1}{J^2}|\underline{1}^t \cdot \underline{a}(\theta)|^2 = \frac{1}{J^2}\left|\frac{1 - e^{-jJ2\pi\frac{d}{\lambda}\sin(\theta)}}{1 - e^{-j2\pi\frac{d}{\lambda}\sin(\theta)}}\right|^2$$

Summary part II

Beamsteering (shifted/ rotated over θ_0):

$$B(\theta) = \frac{1}{J^2}\left|\underline{\mathrm{w}}^h \cdot \underline{\mathrm{a}}(\theta)\right|^2 = \frac{1}{J^2}\left|\sum_{i=1}^{J} \mathrm{e}^{-\mathrm{j}2\pi f_d (i-1)\frac{d}{c}(\sin(\theta)-\sin(\theta_0))}\right|^2$$

Constrained beamforming:
With $J \times 1$ weight vector $\underline{\mathrm{w}} = (w_1, \cdots, w_J)^t$ set up M constraints:

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DOA + Optimum and Adaptive Array Signal Processing

(Part IC)

DOA estimation

Goal:

Estimate Direction Of Arrival (DOA) of sources (and interferences) from noisy observations, in order to locate and/or track these sources

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Main techniques based on:

1. Maximizing power of steered beamformer
2. High-resolution spectral estimation concepts
3. Employing time-difference of arrival (*not in this course*)

Conventional DOA

1. Maximizing power of steered beamformer:

E.g. noisy observation, one source, J sensors:

$$\underline{x}[k] = \underline{a}(\theta_p) \cdot s[k] + \underline{n}[k] \quad \Rightarrow \quad R_x = \sigma_s^2 \underline{a}(\theta_p)\underline{a}^h(\theta_p) + \sigma_n^2 I$$

$$y[k] = \underline{w}^h \cdot \underline{x}[k] \quad \Rightarrow \quad P_y = E\{|y[k]|^2\} = \underline{w}^h \cdot R_x \cdot \underline{w}$$

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Note: In practice estimate $\hat{R}_x = \dfrac{1}{T}\displaystyle\sum_{k=1}^{T} \underline{x}[k] \cdot \underline{x}^h[k]$

Conventional DOA

Example: ULA

$d/\lambda = 0.5$; $P = 4$ sources (at -20,0,40 and 70 degrees); $J = 8$ sensors

High-resolution DOA

High-resolution DOA technique based on signal subspace (see Appendix) method:

Spectral MUSIC = MUltiple Signal Classification

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Covariance structure:

$$\mathrm{R}_x = E\{\underline{\mathrm{x}} \cdot \underline{\mathrm{x}}^h\} = \mathrm{A}\mathrm{R}_s\mathrm{A} + \mathrm{R}_n$$

$$\text{with} \quad \mathrm{R}_s = E\{\underline{\mathrm{s}} \cdot \underline{\mathrm{s}}^h\} = \mathrm{diag}\{\sigma_{s_1}^2, \cdots, \sigma_{s_P}^2\} \quad \text{and} \quad \mathrm{R}_n = \sigma_n^2 \mathrm{I}$$

High-resolution DOA

Recall from appendix:

$$R_x = U_x \Lambda_x U_x^h = U_s \Lambda_{s,n} U_s^h + U_n \Lambda_n U_x^n$$
$$U_x = (\underline{u}_1, \cdots, \underline{u}_J) \; ; \; \Lambda_x = \mathrm{diag}\{\lambda_{x_1}, \cdots, \lambda_{x_J}\}$$

High-resolution DOA

Recall from appendix:

$$\mathrm{R}_x = \mathrm{U}_x \Lambda_x \mathrm{U}_x^h = \mathrm{U}_s \Lambda_{s,n} \mathrm{U}_s^h + \mathrm{U}_n \Lambda_n \mathrm{U}_x^n$$

$$\mathrm{U}_x = (\underline{\mathrm{u}}_1, \cdots, \underline{\mathrm{u}}_J) \; ; \; \Lambda_x = \mathrm{diag}\{\lambda_{x_1}, \cdots, \lambda_{x_J}\}$$

<u>Signal subspace:</u>

$$\mathrm{U}_s = (\underline{\mathrm{u}}_1, \cdots, \underline{\mathrm{u}}_P) \; ; \; \Lambda_{s,n} = \mathrm{diag}\{\lambda_{s_1} + \sigma_n^2, \cdots, \lambda_{s_P} + \sigma_n^2\}$$

High-resolution DOA

Recall from appendix:

$$\mathrm{R}_x = \mathrm{U}_x \Lambda_x \mathrm{U}_x^h = \mathrm{U}_s \Lambda_{s,n} \mathrm{U}_s^h + \mathrm{U}_n \Lambda_n \mathrm{U}_x^n$$
$$\mathrm{U}_x = (\underline{\mathrm{u}}_1, \cdots, \underline{\mathrm{u}}_J) \ ; \ \Lambda_x = \mathrm{diag}\{\lambda_{x_1}, \cdots, \lambda_{x_J}\}$$

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Noise subspace:

$$\mathrm{U}_n = (\underline{\mathrm{u}}_{P+1}, \cdots, \underline{\mathrm{u}}_J) \ ; \ \Lambda_n = \mathrm{diag}\{\sigma_n^2, \cdots, \sigma_n^2\}$$

High-resolution DOA

Recall from appendix:

$$\mathrm{R}_x = \mathrm{U}_x \Lambda_x \mathrm{U}_x^h = \mathrm{U}_s \Lambda_{s,n} \mathrm{U}_s^h + \mathrm{U}_n \Lambda_n \mathrm{U}_x^n$$
$$\mathrm{U}_x = (\underline{\mathrm{u}}_1, \cdots, \underline{\mathrm{u}}_J) \; ; \; \Lambda_x = \mathrm{diag}\{\lambda_{x_1}, \cdots, \lambda_{x_J}\}$$

Signal subspace:

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Noise subspace:

$$\mathrm{U}_n = (\underline{\mathrm{u}}_{P+1}, \cdots, \underline{\mathrm{u}}_J) \; ; \; \Lambda_n = \mathrm{diag}\{\sigma_n^2, \cdots, \sigma_n^2\}$$

Properties:

$$\mathrm{U}_s \perp \mathrm{U}_n \quad \Leftrightarrow \quad \mathrm{U}_s^h \cdot \mathrm{U}_n = 0 \quad \Leftrightarrow \quad \mathrm{U}_n^h \cdot \mathrm{U}_s = 0$$

High-resolution DOA

How obtain DOA's from this?

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\begin{aligned}
\mathsf{R}_x &= \mathsf{U}_s \Lambda_{s,n} \mathsf{U}_s^h + \mathsf{U}_n \Lambda_n \mathsf{U}_n^h \\
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High-resolution DOA

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$$\begin{aligned} R_x &= U_s \Lambda_{s,n} U_s^h + U_n \Lambda_n U_n^h \quad \Rightarrow \quad R_x U_n = \sigma_n^2 U_n \\ R_x &= A R_s A^h + \sigma_n^2 I \quad \Rightarrow \quad R_x U_n = A R_s A^h U_n + \sigma_n^2 U_n \end{aligned}$$

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\end{aligned}
$$

$\Rightarrow \mathrm{A} \mathrm{R}_s \mathrm{A}^h \mathrm{U}_n = 0.$

High-resolution DOA

How obtain DOA's from this?

$$\begin{aligned} \mathrm{R}_x &= \mathrm{U}_s \Lambda_{s,n} \mathrm{U}_s^h + \mathrm{U}_n \Lambda_n \mathrm{U}_n^h \quad \Rightarrow \quad \mathrm{R}_x \mathrm{U}_n = \sigma_n^2 \mathrm{U}_n \\ \mathrm{R}_x &= \mathrm{A} \mathrm{R}_s \mathrm{A}^h + \sigma_n^2 \mathrm{I} \quad \Rightarrow \quad \mathrm{R}_x \mathrm{U}_n = \mathrm{A} \mathrm{R}_s \mathrm{A}^h \mathrm{U}_n + \sigma_n^2 \mathrm{U}_n \end{aligned}$$

$\Rightarrow \mathrm{A}\mathrm{R}_s\mathrm{A}^h\mathrm{U}_n = 0.$ Together with $\mathrm{A}\mathrm{R}_s$ full rank

$$\Rightarrow \quad \mathrm{A}^h \mathrm{U}_n = 0 \quad \Leftrightarrow \quad \mathrm{U}_n^h \mathrm{A} = 0$$

High-resolution DOA

How obtain DOA's from this?

$$\begin{aligned} R_x &= U_s \Lambda_{s,n} U_s^h + U_n \Lambda_n U_n^h \quad \Rightarrow \quad R_x U_n = \sigma_n^2 U_n \\ R_x &= A R_s A^h + \sigma_n^2 I \quad \Rightarrow \quad R_x U_n = A R_s A^h U_n + \sigma_n^2 U_n \end{aligned}$$

$\Rightarrow A R_s A^h U_n = 0.$ Together with $A R_s$ full rank

$$\Rightarrow \quad A^h U_n = 0 \quad \Leftrightarrow \quad U_n^h A = 0$$

Result: Obtain desired DOA's by solving θ from:

$$\boxed{\begin{array}{lll} \underline{u}_i^h \cdot \underline{a}(\theta_p) = 0 & \text{for} & \underline{u}_i \in U_n = \{u_{P+1}, \cdots, \underline{u}_J\} \\ & & \underline{a}(\theta_p) \in A = \{\underline{a}(\theta_1), \cdots, \underline{a}(\theta_P)\} \end{array}}$$

High-resolution DOA

Use this result in "Spectral MUSIC" cost function:

$$C_{SM}(\theta) \quad = \quad \sum_{\underline{u}_i \in U_n} \left| \underline{u}_i^h \underline{a}(\theta) \right|^2 = \underline{a}^h(\theta) \left(\sum_{\underline{u}_i \in U_n} \underline{u}_i \underline{u}_i^h \right) \underline{a}(\theta)$$

High-resolution DOA

Use this result in "Spectral MUSIC" cost function:

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&= \underline{a}^h(\theta) \left(U_n U_n^h \right) \underline{a}(\theta) = \underline{a}^h(\theta) \left(P_n \right) \underline{a}(\theta)
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Use this result in "Spectral MUSIC" cost function:

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with projection matrix $P_n = U_n U_n^h$

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with projection matrix $P_n = U_n U_n^h$

$P_n \underline{a}(\theta)$ is projection of $\underline{a}(\theta)$ on noise subspace U_n.

High-resolution DOA

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&= \underline{a}^h(\theta) \left(U_n U_n^h \right) \underline{a}(\theta) = \underline{a}^h(\theta) \left(P_n \right) \underline{a}(\theta)
\end{aligned}
$$

with projection matrix $P_n = U_n U_n^h$

$P_n \underline{a}(\theta)$ is projection of $\underline{a}(\theta)$ on noise subspace U_n.

Conclusion: C_{SM} is innerproduct of $\underline{a}(\theta)$ and projection of $\underline{a}(\theta)$ on U_n

$$\boxed{C_{SM} = 0 \ \text{ only true for DOA's } \theta_p \text{ with } \ p = 1, \cdots, P}$$

High-resolution DOA

Define pseudo-spectrum:

$$P_{SM}(\theta) = \frac{||\underline{\mathrm{a}}(\theta)||^2}{C_{SM}} = \frac{||\underline{\mathrm{a}}(\theta)||^2}{\underline{\mathrm{a}}^h(\theta)\mathrm{U}_n\mathrm{U}_n^h\underline{\mathrm{a}}(\theta)} = \frac{J}{\underline{\mathrm{a}}^h(\theta)\mathrm{P}_n\underline{\mathrm{a}}(\theta)}$$

High-resolution DOA

Define pseudo-spectrum:

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Notes:

- Minimizing $C_{SM}(\theta) \quad \Leftrightarrow \quad$ Maximizing $P_{SM}(\theta)$
- Sharp peaks (high resolution) in vicinity of source DOA's θ_p, $p = 1, \cdots, P$
- In practice: $\hat{\mathrm{R}}_x \Rightarrow \hat{\mathrm{U}}_s$ not completely orthogonal to $\hat{\mathrm{U}}_n$
- P_{SM} averages $J - P$ pseudo spectra of individual noise vectors
- 'Pseudo' in name since no info about real power

High-resolution DOA

Spectral-MUSIC algorithm

High-resolution DOA

Spectral-MUSIC algorithm

1. Compute/ estimate R_x

High-resolution DOA

Spectral-MUSIC algorithm

1. Compute/ estimate R_x
2. Compute EVD of R_x and split signal- noise subspace:

$$R_x = U_x \Lambda_x U_x^h = U_s \Lambda_{s,n} U_s^h + U_n \Lambda_n U_n^h$$

High-resolution DOA

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3. Compute projection matrix: $\mathsf{P}_n = \mathsf{U}_n \mathsf{U}_n^h$

High-resolution DOA

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4. Evaluate pseudo spectrum:

$$P_{SM}(\theta) = \frac{J}{\underline{\mathsf{a}}^h(\theta) \mathsf{P}_n \underline{\mathsf{a}}(\theta)}$$

High-resolution DOA

Spectral-MUSIC algorithm

1. Compute/ estimate R_x
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4. Evaluate pseudo spectrum:

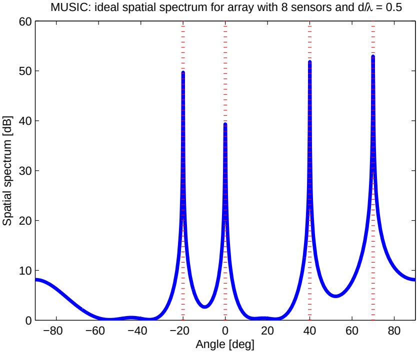
$$P_{SM}(\theta) = \frac{J}{\underline{\mathrm{a}}^h(\theta) \mathrm{P}_n \underline{\mathrm{a}}(\theta)}$$

5. Source DOA's θ_p for $p = 1, \cdots, P$:
   Locate P sharpest peaks in $P_{SM}(\theta)$

High-resolution DOA

Example: ULA

$d/\lambda = 0.5$; $P = 4$ sources (at -20,0,40 and 70 degrees); $J = 8$ sensors



Note: This example exploits $J - P = 4$ *noise sources*

Minimum Mean Squared Error (Wiener)

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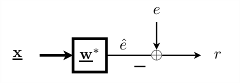
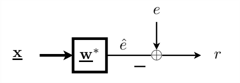


$$
\begin{aligned}
\underline{w}_{mse} &= arg\min_{\underline{w}}\{J\} \\
J &= E\{|r|^2\}
\end{aligned}
$$

Minimum Mean Squared Error (Wiener)



$$\underline{w}_{mse} = \arg\min_{\underline{w}}\{J\}$$
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$$\Rightarrow \quad \underline{w}_{mse} = R_x^{-1}\underline{r}_{e^*x} \quad \text{and} \quad J_{min} = E\{|e|^2\} - \underline{r}_{e^*x}^h R_x^{-1}\underline{r}_{e^*x}$$

$$\text{with} \quad R_x = E\{\underline{x}\cdot\underline{x}^h\} \quad \underline{r}_{e^*x} = E\{\underline{x}\cdot e^*\}$$

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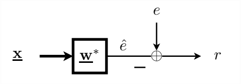

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Necessary knowledge: R_x and $\underline{\mathrm{r}}_{e^*x}$ (both from measurements)

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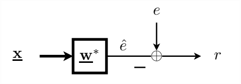

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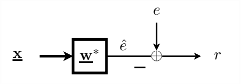

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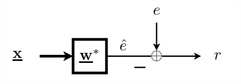

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$$\Rightarrow \quad \underline{w}_{mse} = \beta\cdot\underline{a} \;\; \text{and} \;\; J_{min} = \beta\cdot\sigma_n^2 \;\; \text{with} \;\; \beta = \frac{(\sigma_s^2/\sigma_n^2)}{1 + J\cdot(\sigma_s^2/\sigma_n^2)}$$

Minimum Mean Squared Error (Wiener)

Thus for ULA, one source, narrowband, farfield:

- $\underline{w}_{mse} = \beta \cdot \underline{a}$, which is equivalent to matched filter result, which maximizes SNR
- $J_{min} \approx \frac{1}{J} \cdot \sigma_n^2 \Rightarrow$ SNR impoved approx. by factor J

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Conclusion MMSE

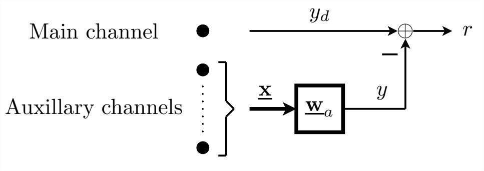
\+ Simple

\+ Direction of desired signal may be unknown

\- Must generate reference signal

Multiple Sidelobe Canceller (Applebaum)

Use auxillary channel to cancel interference in main channel



$$\underline{w}_{opt} = arg \min_{\underline{w}_a} \{ |y_d - \underline{w}_a^h \cdot \underline{x}|^2 \}$$

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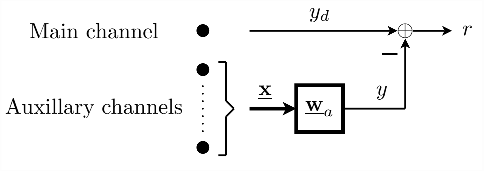

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Main assumption MSC: Interference assumed to be present in both main and auxillary channels. Desired signal strongly present in main channel, but **below noise level** in auxillary channels

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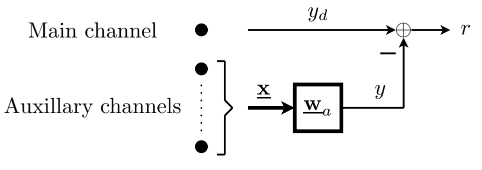

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$$\Rightarrow \quad \boxed{\underline{w}_{opt} = R_x^{-1} \cdot \underline{r}_{xy_d^*}} \quad \text{and} \quad \boxed{P_{out} = \sigma_{y_d}^2 - \underline{r}_{xy_d}^h R_x^{-1} \underline{r}_{xy_d}}$$

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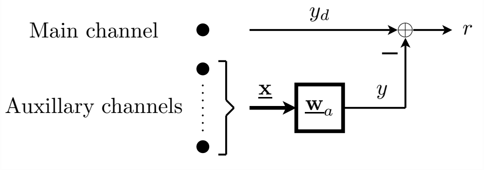

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Conclusion MSC: Simple, but requires desired signal below noise level in auxillay channels, otherwise ...

Linear Constrained Minimum Variance (Frost)

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- **MSC:** Signal cancelling
- **Max SINR:** Needs signal and noise covariance estimate
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Design philosophy LCMV:

Design weight vector by minimizing average output power subject to M constraints that filter response remains constant at some specific frequencies of interest

Linear Constrained Minimum Variance

Average output power:

$$P_y = E\{|y|^2\} = \underline{w}^h \cdot E\{\underline{x}\underline{x}^h\} \cdot \underline{w} = \underline{w}^h \cdot \mathrm{R}_x \cdot \underline{w}$$

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Solution similar to results in part I:

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Minimum Variance Distortionless Response

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Conclusions MVDR/ LCMV:

+ Flexible and general constraints possible
- Computation constraint weight vector not trivial
- As signal extractor: sensitive to errors in DOA
- Problems with correlated signals

Generalized Sidelobe Canceller (Griffith-Jim)

GSC: Alternative formulation of LCMV, illustrates relationship between MSC and LCMV. Mechanism to change constrained minimization problem in unconstrained one

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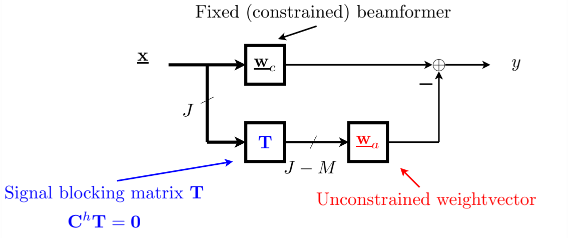
$$\frac{d}{d\underline{w}_a} = \underline{0} \;\Rightarrow\; -2T^h R \underline{w}_c + 2T^h R T \underline{w}_a = \underline{0} \;\Rightarrow$$

$$\boxed{\underline{w}_a = \left(T^h R T\right)^{-1} T^h R \underline{w}_c}$$

Generalized Sidelobe Canceller



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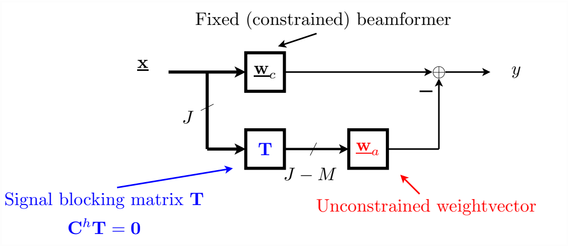

Note: Blocking matrix can be constructed by any orthogonalization procedure (e.g. Gramm- Schmidt or QR- decomposition)

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Beampattern of this fixed filter part:

$$|r(\omega, \theta)| = |\underline{\mathrm{w}}_c^h \cdot \underline{\mathrm{a}}(\omega, \theta)| = \left|\frac{1}{J}\sum_{i=1}^{J} \mathrm{e}^{\mathrm{j}(i-1)\frac{d}{c}(\omega_0 \sin(\theta_0) - \omega \sin(\theta))}\right|$$

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For $\omega = \omega_0 = 2\pi f_0 = 2\pi \frac{c}{\lambda_0}$ beampattern becomes:

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Furthermore construction of blocking matrix:

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(= amplitude response of m^{th} column blocking matrix)

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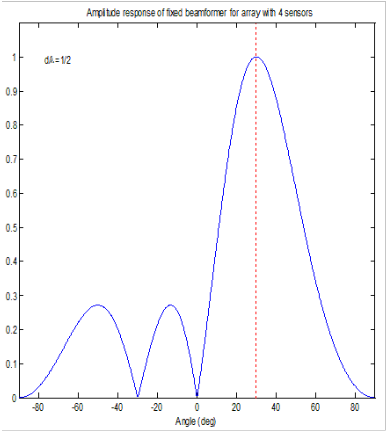
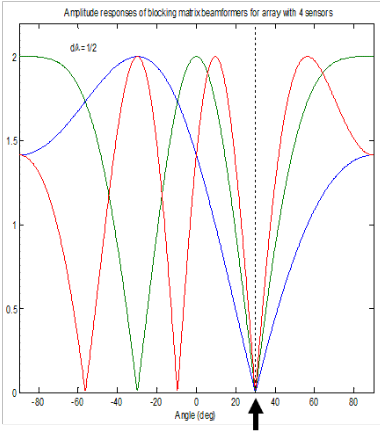
$|r(\theta)|$



$\theta_{desired}$

$|r_m(\theta)|$



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Adaptive Array Signal Processing

Why adaptive?

In most previous results knowledge of SOS needed.
These statistics are usually unknown and/or time-varying
With ergodic assumption $\rightarrow$ SOS can be estimated from available data samples $\rightarrow$ adaptive solutions!

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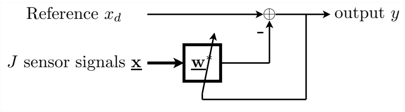

Wiener: $\underline{w}_{opt} = arg\min_{\underline{w}} \left\{ E\{|y|^2\} \right\}$

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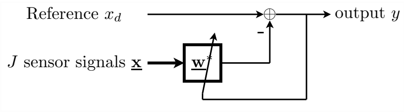

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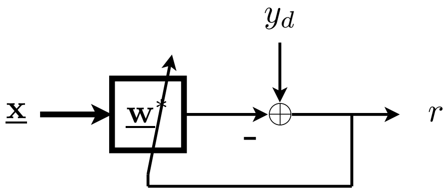
Final value : $\lim_{k \to \infty} E\{\underline{w}[k]\} = \underline{w}_{opt}$

Adaptive MMSE (Wiener)



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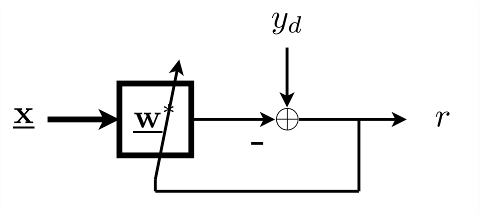

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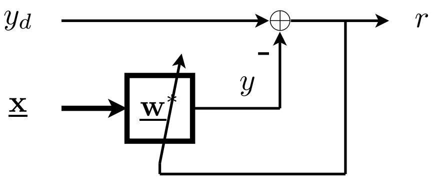
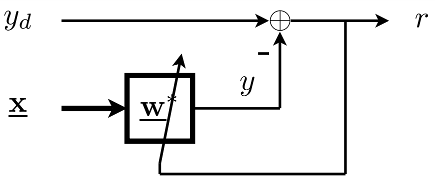


Adaptive MSC



$$\textbf{Optimal Wiener} \quad : \quad \underline{w}_{opt} = arg \min_{\underline{w}} \left\{ E\{|r|^2\} \right\} = R_x^{-1} \cdot \underline{r}_{xy_d^*}$$

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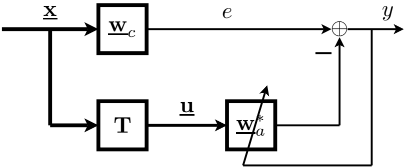

$$
\begin{aligned}
\underline{w}_c &= C \cdot \left(C^h C\right)^{-1} \underline{r}_d \\
C^h T &= 0 \\
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\end{aligned}
$$

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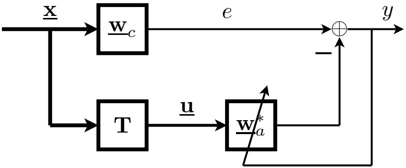

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Appendix AASP

Content appendix

- Eigenvalue problem
- Generalized inverse
- Projection matrix
- Matrix inversion lemma
- Signal subspace techniques

Eigenvalue problem

Procedure: With eigenvalues λ_i and eigenvectors $\underline{q}_i$:

$$\mathsf{R} \cdot \underline{q}_i = \lambda_i \cdot \underline{q}_i \quad \Rightarrow \quad (\mathsf{R} - \lambda_i \mathsf{I}) \cdot \underline{q}_i = \underline{0} \;\text{ for } i = 0, 1, \cdots, N-1$$

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With $Q = (\underline{q}_0, \cdots, \underline{q}_{N-1})$ and $\Lambda = diag\{\lambda_0, \cdots, \lambda_{N-1}\}$

$$R \cdot Q = Q \cdot \Lambda \qquad \Rightarrow \qquad R = Q \Lambda Q^{-1}$$

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Main result:

Diagonalization: $\boxed{\mathrm{Q}^h \mathrm{R} \mathrm{Q} = \Lambda \quad \Leftrightarrow \quad \mathrm{R} = \mathrm{Q} \Lambda \mathrm{Q}^h}$

Eigenvalue problem

Example MA(1):

$x[k] = i[k] + ai[k-1]$ with $E\{i[k]\} = 0$ and $E\{i^2[k]\} = \sigma_i^2 \Rightarrow$

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Notes:

- Vector $\underline{q}_0$ orthogonal to $\underline{q}_1$ since $\underline{q}_0^t \cdot \underline{q}_1 = 0$
- For white noise ($a = 0$): $\Lambda = \mathrm{I}$
- For MA(1) with $N > 2$: R is tri-diagonal

Eigenvalue problem

Example: Eigenvalues and psd for white noise, MA(1) and AR(1)

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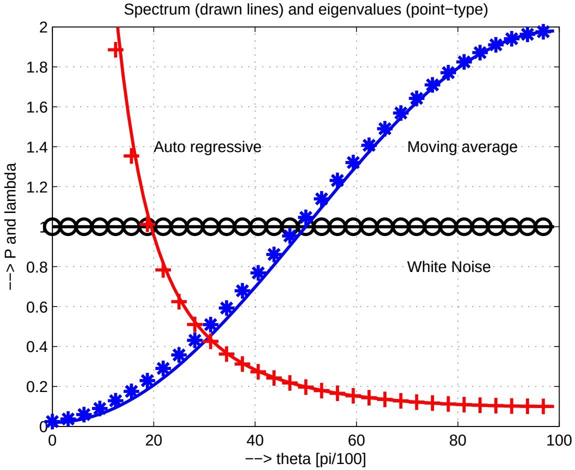

Eigenvalue problem

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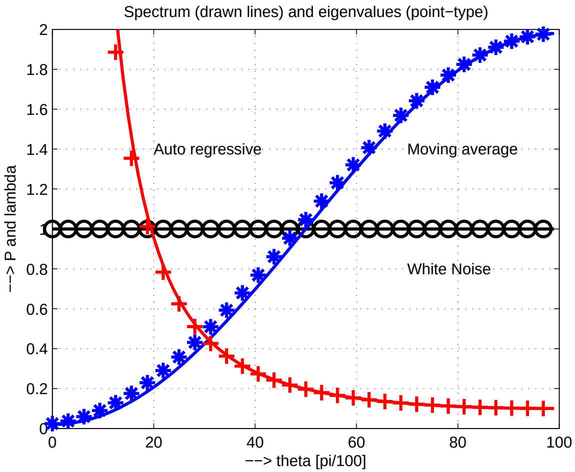

Generalized inverse

Goal:

For known $M \times N$ matrix A *and $M \times 1$ vector* $\underline{\mathrm{b}}$, *solve linear set of M equations and N unknowns:*

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$$\begin{aligned} \mathsf{A}^{\dagger} &= \mathsf{A}^{h} \left(\mathsf{A}\mathsf{A}^{h}\right)^{-1} && \text{for} \quad M < N \\ \mathsf{A}^{\dagger} &= \mathsf{A}^{-1} && \text{for} \quad M = N \\ \mathsf{A}^{\dagger} &= \left(\mathsf{A}^{h}\mathsf{A}\right)^{-1} \mathsf{A}^{h} && \text{for} \quad M > N \end{aligned}$$

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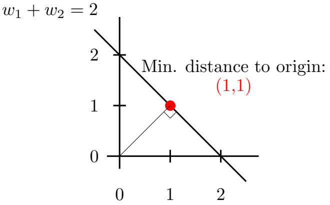

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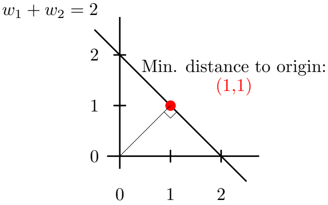

Conclusion:

† results in solution with smallest Euclidian norm ("minimum distance to the origin (0,0)")

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Case $M = N$: $\underline{w} = A^{-1} \cdot \underline{b}$ (A must be invertable)

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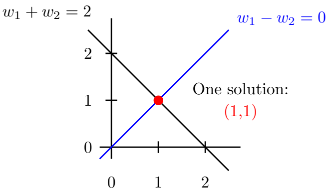

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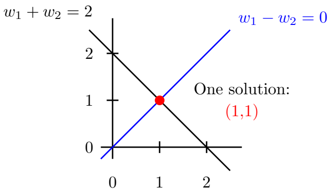

Conclusion: † *results in unique solution*

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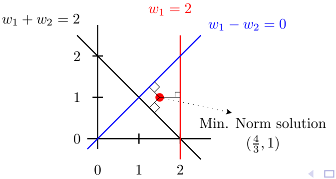

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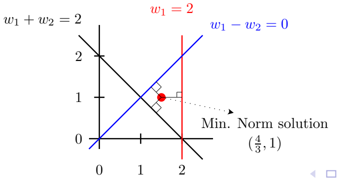

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General: $N \times M$ matrix V, with linearly independent columns

Projection $(N \times 1)$ $\underline{\mathrm{b}}$ onto M-dim subspace spanned by columns V:

$$\hat{\underline{\mathrm{b}}} = \mathrm{P}_V \cdot \underline{\mathrm{b}} \quad \text{and} \quad \underline{\mathrm{b}}^{\perp} = (\mathbf{I} - \mathrm{P}_V) \cdot \underline{\mathrm{b}}$$

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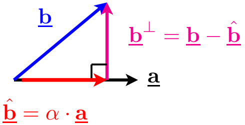
Simple example:



$$\underline{\hat{\mathrm{b}}}^h \cdot \underline{\mathrm{b}}^\perp = 0 \;\Rightarrow\; \alpha = (\underline{\mathrm{a}}^h\underline{\mathrm{a}})^{-1}\underline{\mathrm{a}}^h \cdot \underline{\mathrm{b}}$$

$$\Rightarrow\; \underline{\hat{\mathrm{b}}} = \mathrm{P}_a \cdot \underline{\mathrm{b}} \quad \text{and} \quad \underline{\mathrm{b}}^\perp = (\mathrm{I} - \mathrm{P}_a) \cdot \underline{\mathrm{b}}$$

with $\mathrm{P}_a = \underline{\mathrm{a}}(\underline{\mathrm{a}}^h\underline{\mathrm{a}})^{-1}\underline{\mathrm{a}}^h$

Matrix inversion lemma

Matrix dimensions: A: $N \times N$; B: $N \times M$; C: $M \times M$; D: $M \times N$

$$\boxed{(\mathrm{A} + \mathrm{BCD})^{-1} = \mathrm{A}^{-1} - \mathrm{A}^{-1}\mathrm{B}\left(\mathrm{DA}^{-1}\mathrm{B} + \mathrm{C}^{-1}\right)^{-1}\mathrm{DA}^{-1}}$$

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<u>Special case:</u> (RLS-like)

$\mathrm{B} = \underline{x}$: $N \times 1$; $\mathrm{C} = 1$: 1×1 and $\mathrm{D} = \underline{x}^h$

$$\left(\mathrm{A} + \underline{x}\underline{x}^h\right)^{-1} = \mathrm{A}^{-1} - \frac{\mathrm{A}^{-1}\underline{x}\underline{x}^h\mathrm{A}^{-1}}{1 + \underline{x}^h\mathrm{A}^{-1}\underline{x}}$$

Signal subspace techniques

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Covariance structure:

$$\mathrm{R}_x = E\{\underline{x} \cdot \underline{x}^h\} = \mathrm{A}\mathrm{R}_s\mathrm{A}^h + \mathrm{R}_n$$

with $\mathrm{R}_s = E\{\underline{s}\underline{s}^h\} = \mathrm{diag}\{\sigma_{s_1}^2, \cdots, \sigma_{s_P}^2\}$; $\mathrm{R}_n = \sigma_n^2 \mathrm{I}$

and $J \times P$ steering matrix $\mathrm{A} = (\underline{a}(\theta_1), \cdots, \underline{a}(\theta_P))$

Signal subspace techniques

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$$\Rightarrow \quad \left(\text{AR}_s\text{A}^h\right) \cdot \text{U}_s = \text{U}_s \cdot \Lambda_s$$

with $\text{U}_s = (\underline{\text{u}}_1, \cdots, \underline{\text{u}}_P)$; $\Lambda_s = \text{diag}\{\lambda_{s_1}, \cdots, \lambda_{s_P}\}$

Signal subspace techniques

Eigenvalue decomposition input signal (use $\underline{u}_i$ for $i = 1, \cdots, J$):

$$R_x \cdot \underline{u}_i = \left(AR_sA^h\right) \cdot \underline{u}_i + \sigma_n^2 I \cdot \underline{u}_i = (\lambda_{s_i} + \sigma_n^2) \cdot \underline{u}_i$$

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Thus eigenvalues can be divided into two groups:

$$\lambda_{x_i} = \begin{cases} \lambda_{s_i} + \sigma_n^2 & \text{for } i = 1, \cdots, P \\ \sigma_n^2 & \text{for } i = P+1, \cdots, J \end{cases}$$

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With $U_x = (\underline{u}_1, \cdots, \underline{u}_J)$ and $\Lambda_x = \mathrm{diag}\{\lambda_{x_1}, \cdots, \lambda_{x_J}\}$ we can write:

$$\begin{aligned} R_x &= U_x \Lambda_x U_x^h = \sum_{i=1}^{J} \lambda_{x_i} \underline{u}_i \underline{u}_i^h = \sum_{i=1}^{P} (\lambda_{s_i} + \sigma_n^2) \underline{u}_i \underline{u}_i^h + \sum_{i=P+1}^{J} \sigma_n^2 \underline{u}_i \underline{u}_i^h \\ &= U_s \Lambda_{s,n} U_s^h + U_n \Lambda_n U_n^h \end{aligned}$$

Signal subspace techniques

$$\textbf{Signal subspace} \quad : \quad \mathrm{U}_s = (\underline{\mathrm{u}}_1, \cdots, \underline{\mathrm{u}}_P)$$

$$\Lambda_{s,n} = \mathrm{diag}\{\lambda_{s_1} + \sigma_n^2, \cdots, \lambda_{s_P} + \sigma_n^2\}$$

$$\textbf{Noise subspace} \quad : \quad \mathrm{U}_n = (\underline{\mathrm{u}}_{P+1}, \cdots, \underline{\mathrm{u}}_J)$$

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Signal subspace techniques

$$\begin{aligned} \textbf{Signal subspace} \quad &: \quad \mathrm{U}_s = (\underline{\mathrm{u}}_1, \cdots, \underline{\mathrm{u}}_P) \\ &\quad\ \ \Lambda_{s,n} = \mathrm{diag}\{\lambda_{s_1} + \sigma_n^2, \cdots, \lambda_{s_P} + \sigma_n^2\} \\ \textbf{Noise subspace} \quad &: \quad \mathrm{U}_n = (\underline{\mathrm{u}}_{P+1}, \cdots, \underline{\mathrm{u}}_J) \\ &\quad\ \ \Lambda_n = \mathrm{diag}\{\sigma_n^2, \cdots, \sigma_n^2\} \end{aligned}$$

$$\begin{aligned} \text{Since} \qquad & \mathrm{U}_x = (\underline{\mathrm{u}}_1, \cdots, \underline{\mathrm{u}}_J) = (\mathrm{U}_s, \mathrm{U}_n) \\ \text{and} \qquad & \text{all eigenvectors orthogonal} \quad \underline{\mathrm{u}}_i \perp \underline{\mathrm{u}}_j \end{aligned}$$

$$\Rightarrow \quad \boxed{\mathrm{U}_s \perp \mathrm{U}_n \quad \Leftrightarrow \quad \mathrm{U}_s^h \cdot \mathrm{U}_n = 0 \Leftrightarrow \quad \mathrm{U}_n^h \cdot \mathrm{U}_s = 0}$$

Signal subspace techniques

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Conclusion:

Any vector from signal subspace is orthogonal to noise subspace

End

Appendix