Solutions of Problem set 3
Optimal control and reinforcement learning 4SC000, TU/e

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Linear quadratic regulation

Problem 1.1 We can apply HJB with

$$A = -a = -1, B = 1, Q = 1, R = \gamma = 0.1, S = 0, Q_T = 0 \tag{1}$$

We first find the solution $P(t)$ from the nonlinear differential equation

$$P(T) = Q_T = 0 \tag{2}$$
$$\dot{P}(t) = -(P(t)A + A^\top P(t) - (P(t)B + S)R^{-1}(B^\top P(t) + S^\top) + Q) = 2P(t) + 10P(t)^2 - 1 \tag{3}$$

and then compute the policy

$$u(t) = K(t)x(t) \tag{4}$$
$$K(t) = -R^{-1}(B^\top P(t) + S^\top) = -10P(t) \tag{5}$$

To solve the nonlinear differential equation we note that

$$\dot{P}(t) = (2P(t) + 10P(t)^2 - 1)$$

is equivalent to

$$\frac{1}{(2P(t) + 10P(t)^2 - 1)}\dot{P}(t) = 1$$

which in turn is equivalent to

$$\frac{d}{dt}g(P(t)) = 1 \tag{6}$$

where $g(p)$ is a primitive of $1/(2p + 10p - 1)$, which using Matlab (int.m) is given by

$$g(p) = -\frac{\text{atanh}\frac{(10p+1)}{\sqrt{(11)}}}{\sqrt{11}}$$

where atanh is the inverse of the function $\tanh(x) = \frac{e^h - e^{-h}}{e^h + e^{-h}}$.

Integrating on both sides of (6) we obtain

$$\frac{\text{atanh}\frac{(10P(t)+1)}{\sqrt{(11)}}}{\sqrt{11}} = a - t \tag{7}$$

where a is a constant. This constant is obtain from the boundary condition $P(t) = 0$ when $t = 10$. Thus

$$a = 10 + \frac{\text{atanh}\frac{(1)}{\sqrt{(11)}}}{\sqrt{11}}.$$

From (7) we obtain

$$p(t) = \frac{\sqrt{11}\tanh(\sqrt{11}(a-t)) - 1}{10}$$

(ii) The infinite horizon solution is given by

$$0 = -(PA + A^\top P - (PB+S)R^{-1}(B^\top P + S^\top) + Q) = 2P + 10P^2 - 1 \tag{8}$$

$$P = \frac{-2 \pm \sqrt{(2^2 - 4 * 10 * -1)}}{2 * 10} = \frac{-2 \pm \sqrt{44}}{20} \tag{9}$$

The solution should be positive, thus $P = \frac{-2+\sqrt{44}}{20}$ and $K = \frac{2-\sqrt{44}}{2}$

Problem 1.2 Standard linear quadratic regulation problem with $A = a = -\frac{3}{2}$, $A = b = 3$, $Q = q = 4$, $R = r = 9$. The optimal policy is

$$u(t) = k(t)x(t)$$

where

$$k(t) = -\frac{b}{r}p(t) = -\frac{1}{3}p(t)$$

and $p(t)$ satisfies the continuous-time Riccati equation

$$\dot{p}(t) = -(2ap(t) + q - p(t)^2\frac{b^2}{r}) = p(t)^2 - 3p(t) - 4 = (p(t) - 1)(p(t) + 4)$$

with terminal constraint $p(1) = 0$ (since the terminal cost is zero). Using the suggestion

$$p(t) = \frac{4ce^{5t} + 1}{1 - ce^{5t}}$$

and imposing $p(1) = 0$ we obtain $4ce^5 + 1 = 0 \to c = -\frac{1}{4}e^{-5}$ from which we conclude

$$u(t) = -\frac{1}{3}(\frac{-e^{5(t-1)} + 1}{1 + \frac{1}{4}e^{5(t-1)}})x(t) \quad t \in [0, 1)$$

Problem 1.3 Not provided.

Problem 1.4 (i) Standard LQR problem with $A = 1$, $B = 1$, $Q = 1$, $R = 1$

$$u(t) = Kx(t)$$

where

$$K = -R^{-1}B^\intercal P = -P$$

where P satisfies the algebraic Riccati equation

$$0 = A^\intercal P + PA + Q - PBR^{-1}B^\intercal P$$

which in this case is equivalent to

$$0 = 2P + 1 - P^2.$$

The roots are $P = 1 + \sqrt{2}$ and $P = 1 - \sqrt{2}$, thus we take the positive one $P = 1 + \sqrt{2}$.

The cost is then

$$px_0^2 = (1 + \sqrt{2})x_0^2$$

. (ii) Euler method

$$x_{k+1} = x_k + \tau(x_k + u_k) = (1 + \tau)x_k + \tau u_k)$$

and the zero order approximation is

$$\sum_{k=0}^{\infty} \tau x_k^2 + \tau u_k^2.$$

Obtaining the optimal control policy is a standard problem solved with the algebraic Riccati equation (discrete-time)

$$0 = A^\intercal PA - P + Q - (A^\intercal PB)(B^\intercal PB + R)^{-1}(B^\intercal PA)$$

In this case we have $A = (1 + \tau)$, $B = \tau$, Q $= \tau$, $R = \tau$. Thus we need to solve

$$0 = (1 + \tau)^2 P - P + \tau - \frac{((1 + \tau)P\tau)^2}{P\tau^2 + \tau}$$

This is equivalent to

$$\begin{aligned}
0 &= (\tau - P)(P\tau^2 + \tau) + \tau(1 + \tau)^2 P \\
0 &= \tau - (1 + \tau^2)P - P^2\tau + (1 + \tau)^2 P \\
0 &= 1 + 2P - P^2
\end{aligned}$$

The roots are $P = 1 + \sqrt{2}$ and $P = 1 - \sqrt{2}$, thus we take the positive one $P = 1 + \sqrt{2}$. The optimal policy is

$$u_k = Kx_k$$

where

$$K = -\frac{(1 + \tau)P\tau}{P\tau^2 + \tau} = -\frac{(1 + \tau)P}{P\tau + 1}$$

The cost is then the same as in the continuous-time case.

$$x_0^2 P = x_0^2(1 + \sqrt{2}).$$

(iii) The cost is the same and taking the limit as $\tau \to 0$ we obtain

$$K = -\frac{(1 + \tau)P}{P\tau + 1} \to -P = -(1 + \sqrt{2})$$

which is the same as in the continuous-time case.

Problem 1.5 Not provided.

Linear systems with terminal state constraints

Problem 2.1

i.

Define $X = (x, v)$ then the state space model will be

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{10}$$

Cost parameters are

$$Q = 0, \quad S = 0, \quad R = 1 \tag{11}$$

The Hamiltonian

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\alpha & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \alpha \end{bmatrix} \tag{12}$$

for $\alpha = 0$,

$$e^{Ht} = \begin{bmatrix} 1 & t & \frac{t^3}{6} & -\frac{t^2}{2} \\ 0 & 1 & \frac{t^2}{2} & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{bmatrix} \rightarrow e^{H} = \begin{bmatrix} 1 & 1 & \frac{1}{6} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{13}$$

Then defining $\Lambda = (\lambda_1, \lambda_2)$

$$\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = e^{Ht} \begin{bmatrix} X(0) \\ \Lambda(0) \end{bmatrix} \rightarrow \begin{bmatrix} X(1) \\ \Lambda(1) \end{bmatrix} = e^{H} \begin{bmatrix} X(0) \\ \Lambda(0) \end{bmatrix}$$

For $v(1) = 0$

$$\begin{aligned} \frac{1}{6}\lambda_1(0) - \frac{1}{2}\lambda_2(0) &= 1 \\ \frac{1}{2}\lambda_1(0) - \lambda_2(0) &= 0 \end{aligned} \rightarrow \lambda_1(0) = -12,\ \lambda_2(0) = -6$$

Hence,

$$\begin{aligned} x(t) &= 3t^2 - 2t^3 \\ v(t) &= 6t - 6t^2 \\ u(t) &= 6 - 12t \end{aligned}$$

For $v(1) = 1$,

$$\begin{aligned} \frac{1}{6}\lambda_1(0) - \frac{1}{2}\lambda_2(0) &= 1 \\ \frac{1}{2}\lambda_1(0) - \lambda_2(0) &= 1 \end{aligned} \rightarrow \lambda_1(0) = -6,\ \lambda_2(0) = -4$$

Hence,

$$\begin{aligned} x(t) &= 2t^2 - t^3 \\ v(t) &= 4t - 3t^2 \\ u(t) &= 4 - 6t \end{aligned}$$

ii.

For $\alpha = 0.5$

$$e^{Ht} = \begin{bmatrix} 1 & 2-2e^{-t/2} & 4e^{t/2}-4e^{-t/2}-4t & 4-2e^{t/2}-2e^{-t/2} \\ 0 & e^{-t/2} & 2e^{-t/2}+2e^{t/2}-4 & e^{-t/2}-e^{t/2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2-2e^{t/2} & e^{t/2} \end{bmatrix} \tag{14}$$

$$e^{H} = \begin{bmatrix} 1 & 2-2e^{-1/2} & 4e^{1/2}-4e^{-1/2}-4 & 4-2e^{1/2}-2e^{-1/2} \\ 0 & e^{-1/2} & 2e^{-1/2}+2e^{1/2}-4 & e^{-1/2}-e^{1/2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2-2e^{1/2} & e^{1/2} \end{bmatrix}$$

For $v(1) = 0$

$$(4e^{1/2}-4e^{-1/2}-4)\lambda_1(0) + (4-2e^{1/2}-2e^{-1/2})\lambda_2(0) = 1$$
$$(2e^{-1/2}+2e^{1/2}-4)\lambda_1(0) + (e^{-1/2}-e^{1/2})\lambda_2(0) = 0$$

$$\lambda_1(0) = \frac{e^{1/2}+1}{12e^{1/2}-20}$$
$$\lambda_2(0) = \frac{e^{1/2}-1}{6e^{1/2}-10}$$

Hence,

$$x(t) = \frac{e^{1/2}+1}{12e^{1/2}-20}(4e^{t/2}-4e^{-t/2}-4t) + \frac{e^{1/2}-1}{6e^{1/2}-10}(4-2e^{t/2}-2e^{-t/2})$$
$$v(t) = \frac{e^{1/2}+1}{12e^{1/2}-20}(2e^{-t/2}+2e^{t/2}-4) + \frac{e^{1/2}-1}{6e^{1/2}-10}(e^{-t/2}-e^{t/2})$$
$$u(t) = -\frac{e^{1/2}-2e^{t/2}+1}{6e^{1/2}-10}$$

iii.

(a) Find $\Lambda(0)$ in terms of $X(0)$

$$\begin{array}{r} \frac{1}{6}\lambda_1(0) - \frac{1}{2}\lambda_2(0) + x(0) + v(0) = 1 \\ \frac{1}{2}\lambda_1(0) - \lambda_2(0) + v(0) = 0 \end{array} \rightarrow \begin{array}{l} \lambda_1(0) = -12 + 12x(0) + 6v(0) \\ \lambda_2(0) = -6 + 6x(0) + 4v(0) \end{array}$$

since $S = 0$,

$$u(t) = -R^{-1}B^T\Lambda(t) = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \Lambda(0)$$
$$= 6 - 12t + 6(2t-1)x(0) + (6t-4)v(0) \tag{15}$$

(b)

$$\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = e^{H(t-s)} \begin{bmatrix} X(s) \\ \Lambda(s) \end{bmatrix} \rightarrow \begin{bmatrix} X(1) \\ \Lambda(1) \end{bmatrix} = e^{H(1-s)} \begin{bmatrix} X(s) \\ \Lambda(s) \end{bmatrix}$$

and

$$e^{H(1-s)} = \begin{bmatrix} 1 & (1-s) & \frac{(1-s)^3}{6} & -\frac{(1-s)^2}{2} \\ 0 & 1 & \frac{(1-s)^2}{2} & -(1-s) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -(1-s) & 1 \end{bmatrix}$$

therefore

$$\frac{(1-s)^3}{6}\lambda_1(s) - \frac{(1-s)^2}{2}\lambda_2(s) + x(s) + (1-s)v(s) = 1$$
$$\frac{(1-s)^2}{2}\lambda_1(s) - (1-s)\lambda_2(s) + v(s) = 0$$

which results in

$$\lambda_1(s) = -\frac{12}{(1-s)^3} + \frac{12}{(1-s)^3}x(s) + \frac{6}{(1-s)^2}v(s)$$
$$\lambda_2(s) = -\frac{6}{(1-s)^2} + \frac{6}{(1-s)^2}x(s) + \frac{4}{1-s}v(s)$$

Therefore

$$\begin{aligned} u(t) = -R^{-1}B^T\Lambda(t) &= -\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -(t-s) & 1\end{bmatrix}\Lambda(s) \\ &= \frac{6}{(1-s)^2}(\frac{-2(t-s)}{1-s}+1) + \frac{6}{(1-s)^2}(\frac{2(t-s)}{1-s}-1)x(s) \\ &\qquad + \frac{2}{1-s}(\frac{3(t-s)}{1-s}-2)v(s) \end{aligned} \tag{16}$$

(c) A policy is a function which maps states to the first action of the optimal path which leads to the last stage. Thus the optimal policy is obtained by making $t = s$, i.e,. $u(t) = \mu(t, x(t), v(t)$ where

$$\mu(t,x,v) = \frac{6}{(1-t)^2} - \frac{6}{(1-t)^2}x - \frac{4}{1-t}v$$

Problem 2.2 We have a continuous-time optimal control problem. In particular we have a linear system and quadratic cost without a final constraint. We aim to find the optimal input $u(t)$ for $t \in [0,T]$ for a finite horizon cost for $T = 5$.

$$A = \begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}, B = \begin{bmatrix}0\\ 1\end{bmatrix}, Q = diag([1,2]), R = 10, S = 0, Q_T = \begin{bmatrix}10 & 0\\ 0 & 0\end{bmatrix} \tag{17}$$

We use PMP to compute the solution. Minimizing

$$\int_0^5 x_1(t)^2 + 2x_2(t)^2 + 10u(t)^2 + 10x_1(5)^2$$

is equivalent to minimizing

$$\frac{1}{2}\int_0^5 x_1(t)^2 + 2x_2(t)^2 + 10u(t)^2 + 10x_1(5)^2.$$

From the derivations of Lecture 6 we can conclude that the optimal input and the associated state and co-state are given by

$$\dot{\lambda}(t) = -A^\top\lambda(t) - Qx(t) \tag{18}$$
$$\lambda(T) = Q_T x(T) \tag{19}$$
$$u(t) = -R^{-1}B^\top\lambda(t) \tag{20}$$
$$\dot{x}(t) = Ax(t) + Bu(t) \tag{21}$$

The boundary conditions are

$$x(0) = \begin{bmatrix}1 & 2\end{bmatrix}^\intercal \quad \lambda(T) = \frac{\partial}{\partial x}1/2x^\intercal Q_T x_{|x=x(T)} = \begin{bmatrix}10x_1(T)\\ 0\end{bmatrix}$$

```
clear all
Ac = [2 1;0 1];
Bc = [0;1];
Qc  = [1 0; 0 2];
Rc  = 10;
QT  = [10 0;0 0];
T   = 5;
x0  = [1 2]';
n   = size(Ac,2);
H = expm( [Ac  -Bc*inv(Rc)*Bc';-Qc -Ac']*T);
H11 = H(1:n,1:n); H12 = H(1:n,n+1:2*n);
H21 = H(n+1:2*n,1:n); H22 = H(n+1:2*n,n+1:2*n);
% 2 obtain lambda0
% [I;QT]xT = [H11;H21][x0] + [H12;H22]lambda0
xTlambda0 = [eye(size(Ac)) -H12;QT -H22]\([H11;H21]*x0);
```

Figure 1: Matlab code to answer Problem 2.2

Then, we have

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix}}_{H} \begin{bmatrix} x \\ \lambda \end{bmatrix}. \tag{22}$$

$$\begin{bmatrix} x_1(T) \\ x_2(T) \\ 10x_1(T) \\ 0 \end{bmatrix} = e^{HT} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \lambda_1(0) \\ \lambda_2(0) \end{bmatrix} \rightarrow \begin{bmatrix} x_1(T) \\ x_2(T) \\ \lambda_1(0) \\ \lambda_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -h13 & -h14 \\ 0 & 1 & -h23 & -h24 \\ 10 & 0 & -h33 & -h34 \\ 0 & 0 & -h43 & -h44 \end{bmatrix}^{-1} \begin{bmatrix} h11 & h12 \\ h21 & h22 \\ h31 & h32 \\ h41 & h42 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{23}$$

We can compute by filling in $T = 5$, $x_1(0) = 1$, $x_2(0) = 2$, and using Matlab (see Figure 1 we get

$$\begin{bmatrix} x_1(T) \\ x_2(T) \\ \lambda_1(0) \\ \lambda_2(0) \end{bmatrix} = \begin{bmatrix} -0.0122 \\ -1.7237 \\ 631.7717 \\ 245.7611 \end{bmatrix} \tag{24}$$

The optimal input is given by

$$u(t) = -R^{-1}B^\intercal \lambda(t) = -\frac{1}{10}\begin{bmatrix} 0 & 1 \end{bmatrix} \lambda(t)$$

where

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = e^{Ht} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \lambda_1(0) \\ \lambda_2(0) \end{bmatrix} \tag{25}$$

Problem 2.3 [see also written solutions to Sample Exam 1 Y14-15 Problem 6]

$$\min \int_0^2 u(t)^2 dt + 10x_1(2)$$

is equivalent to

$$\min \frac{1}{2}\left(\int_0^2 u(t)^2 dt + 10x_1(2)^2\right)$$

which can be written in the form

$$\min \frac{1}{2}\left(\int_0^2 x(t)^\top Q x(t) + u(t)^\top R u(t) dt + x(2)^\top Q_T x(2)\right)$$

with $Q = 0$, $R = 1$, $Q_T = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$ for

$$\dot{x} = Ax + Bu, \ A = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From PMP, we know that

$$u(t) = -R^{-1}B^\top \lambda(t)$$

where the co-state satisfies

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix}}_{H} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

The boundary conditions are $x_2(2) = 0, x(0) = [1, 1]^\top$, $\lambda_1(2) = \frac{d}{dx_1} \frac{1}{2} 10x_1^2 \Big|_{x_1 = x_1(2)} = 10x_1(2)$.

So $\lambda(0), \lambda_2(2), x_1(2)$ are free/unknown. To obtain $x(t)$ and $\lambda(t)$, we impose the boundary conditions in

$$\begin{bmatrix} x(2) \\ \lambda(2) \end{bmatrix} = e^{H2} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}.$$

Finding $\lambda(0)$ is sufficient to obtain

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Ht} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}.$$

We can use the suggestion to compute e^{Ht} since H has the upper triangular structure for $Q = 0$. Furthermore, e^{At} again has the upper triangular structure and we can use the suggestion again.

$$e^{At} = \begin{bmatrix} e^{-3t} & \frac{1}{2}(e^{-t} - e^{-3t}) \\ 0 & e^{-t} \end{bmatrix}$$

$$e^{-A^\top t} = \begin{bmatrix} e^{3t} & 0 \\ \frac{1}{2}(e^t - e^{3t}) & e^t \end{bmatrix}$$

$$\int_0^t e^{-As} BR^{-1}B^\top e^{A^\top s} ds = \int_0^t \begin{bmatrix} \frac{1}{2}(e^s - e^{3s}) \\ e^{-s} \end{bmatrix} \begin{bmatrix} \frac{1}{2}(e^s - e^{3s}) & e^s \end{bmatrix} = \begin{bmatrix} \frac{1}{8}e^{2t} - \frac{1}{8}e^{4t} + \frac{1}{24}e^{6t} - \frac{1}{24} & \frac{1}{4}e^{2t} - \frac{1}{8}e^{4t} - \frac{1}{8} \\ \frac{1}{4}e^{2t} - \frac{1}{8}e^{4t} - \frac{1}{8} & \frac{1}{2}e^{2t} - \frac{1}{2} \end{bmatrix}$$

In numerical values (also by numerical integration) we find

$$e^{A2} = \begin{bmatrix} 0.002478 & 0.06643 \\ 0 & 0.1353 \end{bmatrix}$$

$$\int_0^2 e^{-As} BR^{-1}B^\top e^{A^\top s} ds = \int_0^t \begin{bmatrix} \frac{1}{2}(e^s - e^{3s}) \\ e^{-s} \end{bmatrix} \begin{bmatrix} \frac{1}{2}(e^s - e^{3s}) & e^s \end{bmatrix} \begin{bmatrix} -3208 & 179.5 \\ 179.5 & -13.400 \end{bmatrix}$$

We multiply the integral by e^{A2} and we get

$$e^{H2} = \begin{bmatrix} 0.002478 & 0.06643 & 3.9757 & 0.445 \\ 0 & 0.1353 & 24.299 & -1.8134 \\ 0 & 0 & 403.4 & 0 \\ 0 & 0 & 198.02 & 7.39 \end{bmatrix}$$

Now first solve

$$\begin{bmatrix} x_1(2) \\ 0 \\ 10x_1(2) \\ \lambda_2(2) \end{bmatrix} = e^{H2} \begin{bmatrix} 1 \\ 1 \\ \lambda_1(0) \\ \lambda_2(0) \end{bmatrix}.$$

to find $x_1(2) = \frac{1}{10}403.4\lambda_1(0)$. Thus, we need to solve

$$\begin{bmatrix} \frac{1}{10}403.4\lambda_1(0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0.002478 & 0.06643 & 3.9757 & 0.445 \\ 0 & 0.1353 & 24.299 & -1.8134 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \lambda_1(0) \\ \lambda_2(0) \end{bmatrix}.$$

and solve the set of equations (bring all terms with λ to left side and the rest to the right and multiply the matrix inverse of the left and side) to find

$$\begin{bmatrix} \lambda_1(0) \\ \lambda_2(0) \end{bmatrix} = \begin{bmatrix} 0.0008434 \\ 0.8593 \end{bmatrix}.$$

Now from

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Ht} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}.$$

we have that $\lambda(t) = e^{-A^\top t}\lambda(0) = \begin{bmatrix} e^{3t} & 0 \\ \frac{1}{2}(e^t - e^{3t}) & e^t \end{bmatrix} \lambda(0)$

Since

$$u(t) = -R^{-1}B^\top \lambda(t) = -\lambda_2(t) = -\frac{1}{2}(e^t - e^{3t})0.0008434 - e^t 0.8593$$

is the optimal control input.

Problem 2.4 [Problem 2.5, see also written solutions to Sample Exam 2 Y14-15 Problem 5]

Standard problem in scalar form $a = 3, b = 1, q = 7, r = 1$, without terminal cost.

Optimal control is given by $u(t) = -r^{-1}b^\top \lambda(t)$. For

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a & -b^2r^{-1} \\ -q & -a \end{bmatrix}}_{H} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

We find the Hamiltonian and its eigendecomposition

$$H = \begin{bmatrix} 3 & -1 \\ -7 & -3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

with eigenvalues $\lambda_{1,2} = \frac{3-3\pm\sqrt{(3-3)^2-4((-3)3-7)}}{2} = \pm\frac{\sqrt{(64)}}{2} = \pm 4$. $\lambda_1 = 4, \lambda_2 = -4$ and eigenvectors $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} \frac{1}{7} \\ 1 \end{bmatrix}$.

Then

$$e^{Ht} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

where

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} = \frac{1}{-1-\frac{1}{7}} \begin{bmatrix} 1 & -\frac{1}{7} \\ -1 & -1 \end{bmatrix}$$

With boundary conditions $x(0) = 1, x(1) = 0$ we can solve the top equation of

$$\begin{bmatrix} x(1) \\ \lambda(1) \end{bmatrix} = e^{H1} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}.$$

where $x(1)$ and $x(0)$ are known, to find

$$\lambda(0) = 7.0027.$$

The bottom equation of

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Ht} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}.$$

provides then the exact solution to $\lambda(t)$ which is use in the optimal input

$$u(t) = -r^{-1}b^\top \lambda(t) = -\lambda(t) = -7e^{-4*s}/8 + 7e^{4*s}/8 - (7e^{-4*s}/8 + e^{4*s}/8)7.0027$$

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Pontryagin's maximum principle, minimum time optimal control problems

Problem 3.1 [see also the slides 29-31 of `III_2`]

$$f(t, x(t), u(t)) = A(t)x(t) + B(t)u(t)$$
$$g(t, x(t), u(t)) = x(t)^\top Q(t)x(t) + u(t)^\top Ru(t)$$
$$g_T(x(T)) = x(t)^\top Q_T x(T)$$

The Hamiltonian is given by

$$H(t, x(t), u(t), \lambda(t)) = \lambda(t)^\top f(\cdot) + g(\cdot)$$

Since Q_T is diagonal, the boundary condition for the final costates are given by

$$\lambda_j(T) = \frac{\partial(\frac{1}{2}q_j x_j^2)}{\partial x_j} |_{x=x(T)} = q_j x_j(T).$$

Since all time-varying parameters are dependent on t but not on u, x, λ, all derivatives of the Hamiltonian w.r.t. u, x, λ are the same for the time-varying case as for the time-invariant case in the slides (apart from the notation with '(t)').

Problem 3.2 Solution not provided.

Problem 3.3

(i) (a) For

$$\xi = \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix}, \ u = \begin{bmatrix} a_x \\ a_y \end{bmatrix},$$

The dynamics and cost function are given by

$$f(\xi, u) = \begin{bmatrix} v_x \\ v_y \\ a_x \\ a_y \end{bmatrix}, \ g(\xi, u) = \gamma(\begin{bmatrix} a_x \\ a_y \end{bmatrix}^\top \begin{bmatrix} a_x \\ a_y \end{bmatrix} - 1), \ g_T(\xi) = -x(T).$$

The Hamiltonian is given by

$$\lambda = \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_{v_x} \\ \lambda_{v_y} \end{bmatrix}, \ H(\xi, u, \lambda) = \lambda^\top f(\xi, u) + g(\xi, u).$$

We are given that

$$\xi(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \xi(T) = \begin{bmatrix} free \\ h = 1 \\ free \\ 0 \text{ (path parallel to x-axis)} \end{bmatrix}.$$

For the free variables $x(T), v_x(T)$, we use that

$$\lambda_x(T) = [\frac{\partial g_T(\xi)}{\partial x}]^\top_{\xi=\xi(T)} = -1.$$

$$\lambda_{v_x}(T) = [\frac{\partial g_T(\xi)}{\partial v_x}]^\top_{\xi=\xi(T)} = 0.$$

Also, we have that the optimal input $\begin{bmatrix} a_x \\ a_y \end{bmatrix}$ satisfies

$$0 = [\frac{\partial}{\partial u} H]^\top = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^\top \lambda + 2\gamma \begin{bmatrix} a_x \\ a_y \end{bmatrix} \implies u = \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \frac{-1}{2\gamma} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^\top \lambda = \frac{-1}{2\gamma} \begin{bmatrix} \lambda_{v_x} \\ \lambda_{v_y} \end{bmatrix}.$$

Now we have all constraints.

Now we consider the dynamics of the co-state

$$\dot{\lambda}(t) = -[\frac{\partial}{\partial \xi} H]^\top = \begin{bmatrix} 0 \\ 0 \\ -\lambda_x \\ -\lambda_y \end{bmatrix}$$

which can be integrated to find

$$\begin{bmatrix} \lambda_x(t) \\ \lambda_y(t) \\ \lambda_{v_x}(t) \\ \lambda_{v_y}(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ -c_1 t + c_3 \\ -c_2 t + c_4 \end{bmatrix}$$

where c_1, c_2, c_3, c_4 are to be determined.

From the terminal constraints, we directly determine that

$$c_1 = \lambda_x(T) = -1 \text{ and } 0 = \lambda_{v_x}(T) = -c_1 T + c_3 = 3 + c_3 \implies c_3 = -3.$$

From the system dynamics with optimal input, we find that

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} v_y \\ \frac{-1}{2\gamma}\lambda_{v_y} \end{bmatrix} = \begin{bmatrix} v_y \\ \frac{c_2 t - c_4}{2\gamma} \end{bmatrix}.$$

Integrating this gives

$$v_y(t) = \frac{1}{2\gamma}(\frac{c_2}{2}t^2 - c_4 t) + \kappa_1 = \frac{1}{2\gamma}(\frac{c_2}{2}t^2 - c_4 t),$$

where $\kappa_1 = 0$ by $v_y(0) = 0$. We integrate again to find

$$y(t) = \frac{1}{2\gamma}(\frac{c_2}{6}t^3 - \frac{c_4}{2}t^2) + \kappa_2 = \frac{1}{2\gamma}(\frac{c_2}{6}t^3 - \frac{c_4}{2}t^2),$$

where $\kappa_2 = 0$ by $y(0) = 0$.

By $y(T) = 1, v_y(T) = 0$ and $T = 3$, we have that

$$0 = c_2\frac{9}{2} - c_4 3,\ 1 = \frac{1}{2\gamma}(c_2\frac{9}{2} - c_4\frac{9}{2}) \implies c_4 = c_2\frac{3}{2},\ 2\gamma = c_2(\frac{9}{2} - \frac{27}{4}) = -c_2(\frac{9}{4}) \implies c_2 = -\frac{8\gamma}{9},\ c_4 = -\frac{4\gamma}{3}.$$

We find that

$$\begin{bmatrix} \lambda_x(0) \\ \lambda_y(0) \\ \lambda_{v_x}(0) \\ \lambda_{v_y}(0) \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{8\gamma}{9} \\ -3 \\ -\frac{4\gamma}{3} \end{bmatrix}.$$

Now, we have all dynamics.

Since the system is linear, we can compute the Hamiltonian matrix and e^{HT} to find $x(T)$. Alternatively, one can use ode-solvers.

By numerical or symbolic integration, we can compute $\int_0^T ||u(t)||^2 dt$ where

$$||u(t)||^2 = \begin{bmatrix} a_x \\ a_y \end{bmatrix}^\top \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \frac{1}{4\gamma^2}((-c_1 t + c_3)^2 + (-c_2 t + c_4)^2).$$

For $\gamma = a$: $x(T) \approx 9$, $\quad \int_0^T ||u(t)||^2 dt = 85/9 = 9.44 > T$,

For $\gamma = b$: $x(T) = 9/2$, $\quad \int_0^T ||u(t)||^2 dt = 97/36 = 2.69 < T$

(b) Use e.g. bisection to search between a and b: $\gamma = \frac{9}{2\sqrt{23}} = 0.9381$, $x(T) = \sqrt{23} = 4.79$ Alternatively, one can compute the primitive.

ii (a) Case C is a hard constraint and I is a soft constraint. The set of solutions to I contains the solution of C. Hence, the optimal solution to I must be at least as good as the solution to C.

(b)[4 points] We solve the same problem as before but now with $u(t) = \beta(t)$ as input variable and nonlinear dynamics. The dynamics and cost function are given by

$$f(\xi, u) = \begin{bmatrix} v_x \\ v_y \\ cos(\beta) \\ sin(\beta) \end{bmatrix}, \; g(\xi, u) = 0, \; g_T(\xi) = x(T).$$

Also, we have that the optimal input satisfies

$$0 = [\frac{\partial}{\partial u} H]^\top \implies \beta(t) = \tan^{-1}(\frac{\lambda_{v_y}}{\lambda_{v_x}}).$$

Where the co-state has the same dynamics as before but now with different c_1, c_2, c_3, c_4.

Again from the terminal constraints, we directly determine that

$$c_1 = \lambda_x(T) = -1 \;\text{ and } 0 = \lambda_{v_x}(T) = -c_1 T + c_3 = 3 + c_3 \implies c_3 = -3.$$

From the system dynamics with optimal input, we find that

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} v_y \\ \sin(\tan^{-1}(\frac{\lambda_{v_y}}{\lambda_{v_x}})) \end{bmatrix} = \begin{bmatrix} v_y \\ \sin(\tan^{-1}(\frac{-c_2 t + c_4}{t-3})) \end{bmatrix}.$$

Integrating this analytically is much harder. One can use the shooting method. Possibly also use that $\sin(\tan^{-1}(x)) = \frac{x}{\sqrt{1+x^2}}$.

The values of the constants are found to be $c_2 = 0.61$ and $c_4 = 1.22$.

One can then solve forward in time or integrate twice over $a_x(t)$ to find $x(T) = 4.25$.

(c) We can directly see the connection to (b) where this problem has been already solved:

$$a_y = \frac{1}{2}\frac{(\frac{-c_2t+c_4}{t-3})}{\frac{1}{2}\sqrt{1+(\frac{-c_2t+c_4}{t-3})^2}} = \frac{1}{2}\frac{-c_2t+c_4}{\frac{1}{2}\sqrt{(t-3)^2+(-c_2t+c_4)^2}}$$

γ is the normalization factor for the input at (i) resulting from $\sin(\tan^{-1}(x))$.

$$\gamma(t) = \frac{1}{2}\sqrt{(3-t)^2+(c_2t-c_4)^2}$$

Problem 3.4 Solution not provided.

Problem 3.5 Let us start by considering a change of variables for the control input

$$\bar{u}(t) = \frac{1}{2}u(t).$$

Then we have $\bar{u}(t) \in [-1, 1]$ and

$$\dot{x} = Ax + B\bar{u}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Then, this is a standard minimal time optimal control problem. We start by writing the Hamiltonian

$$H = \lambda^\intercal(Ax + Bu) + 1$$

and writing the conditions for optimality we obtain

$$\dot{\lambda}(t) = -[\frac{\partial H}{\partial x}]^\intercal \to \dot{\lambda}(t) = -A^\intercal\lambda(t)$$
$$u(t) = \text{argmin}_{u\in[-1,1]}H \to u(t) = -\text{sign}(\lambda(t)^\intercal B) = -\text{sign}(\lambda_1(t))$$

where $\text{sign}(a) = -1$ if $a < 0$ and $\text{sign}(a) = 1$ is $a > 0$. Since

$$\lambda(t) = \exp^{-A^\intercal t}\lambda(0) \to e^2t\begin{bmatrix} 1 & -t \\ 1 & 1 \end{bmatrix}\lambda(0)$$

we conclude that for any initial condition $\lambda(0)$, $\lambda_1(t)$ changes sign at most once (it may not change sign) and therefore the same holds for the control input. Then it suffices to

(i) Find the set of initial conditions for which the origin $x(T) = 0$ is achieved after a time T for the control input $u(t) = 1$, $t \in [0, T]$ and for the control input $u(t) = -1$, $t \in [0, T]$. For these initial conditions we know that this is the optimal control input that reaches the origin in minimal time.

(ii) Given an initial condition different from the one compute in (i) we know there to reach the origin we will need one switch: we have to compute the initial value of $u(t)$ for that initial condition (either 1 or -1).

To pursue (i) for $u(t) = 1$ we have to find initial conditions x_0 such that

$$0 = e^{AT}x_0 + \int_0^T e^{A(T-s)}Bds$$

which is equivalent to finding x_0 such that

$$x_0 = \int_T^0 e^{-As} B ds$$

Solving for the given A and B we get

$$x_0 = \begin{bmatrix} 1/2 - e^{2T}/2 \\ (e^{2T}(2^T - 1))/4 + 1/4 \end{bmatrix}$$

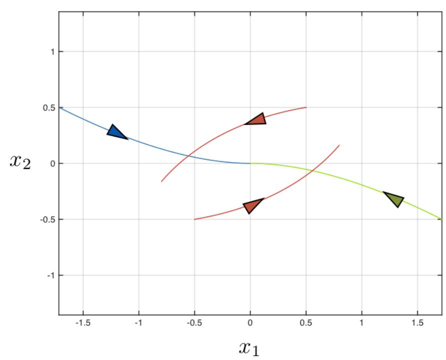
If we had considered $u(t) = -1$ we would get

$$x_0 = -\begin{bmatrix} 31/2 - e^{2T}/2 \\ (e^{2T}(2^T - 1))/4 + 1/4 \end{bmatrix}$$

These are the switching curves which are depicted in the figure in blue (for $u(t) = 1$) and in green (for $u(t) = -1$). For a given initial condition not in the switching curve the response when $u(t) = 1$ is

$$x(t) = e^{At}x_0 + \int_0^T e^{A(T-s)} B ds = e^{-2}t \begin{bmatrix} 1 & 0 \\ t & 0 \end{bmatrix} (x_0 + \begin{bmatrix} e^{2t} - 1 \\ -(e^{2t}(2t-1))/2 - 1/2 \end{bmatrix})$$

22 By inspection we conclude that for an initial condition below the switching curves we should use $u(t) = 1$ (and then switch to $u(t) = -1$ when the trajectories reach the green switching curve) for an initial condition above the switching we should use $u(t) = -1$ (and then switch to $u(t) = 1$ when the trajectories reach the blue switching curve). This is illustrated in red for the initial conditions $(0.5, 0.5)$ and $-(0.5, 0.5)$



Problem 3.6 Solution not provided.

Linear quadratic control, separation principle

Problem 4.1 (a) For a linear system in the canonical form the equations of the Kalman filter are

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(t)(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = \mathbb{E}[x(0)],$$

$$L(t) = \Phi(t)C^{\intercal}V^{-1},$$

where

$$\dot{\Phi}(t) = A\Phi(t) + \Phi(t)A^{\intercal} + W - \Phi(t)C^{\intercal}V^{-1}C\Phi(t), \quad \Phi(0) = \mathbb{E}[(x(0) - \hat{x}(0))(x(0) - \hat{x}(0))^{\intercal}]$$

where V and W are zero-mean Gaussian white noise processes with $\mathbb{E}[v(t)v(t+\tau)^{\intercal}] = V\delta(\tau)$, $\mathbb{E}[w(t)w(t+\tau)^{\intercal}] = W\delta(\tau)$. For the given system we have

$$A = -3/2 \quad B = 3 \quad C = 1 \quad W = 4 \quad V = 3 \quad T = 1 \quad \Phi(0) = 1$$

The fact that $W = 4$ follows from the fact that the given system can be written as

$$\dot{x}(t) = -\frac{3}{2}x(t) + 3u(t) + \bar{w}(t), \quad x(0) = x_0, \quad t \in \mathbb{R}_{\geq 0}.$$

where

$$\bar{w}(t) = 2w(t)$$

and $\mathbb{E}[\bar{w}(t)\bar{w}(t+\tau)] = 4\mathbb{E}[\bar{w}(t)\bar{w}(t+\tau)] = 4\delta(\tau)$. This leads to

$$\dot{\hat{x}}(t) = -\frac{3}{2}\hat{x}(t) + 3u(t) + L(t)(y(t) - \hat{x}(t)) \quad \hat{x}(0) = \bar{x}_0, \quad t \in [0,1].$$

and

$$L(t) = \frac{\Phi(t)}{3}$$

and

$$\dot{\Phi}(t) = -3\Phi(t) + 4 - \frac{1}{3}\Phi(t)^2, \quad \Phi(0) = 1/2$$

or equivalently

$$\dot{\Phi}(t) = -\frac{1}{3}(\Phi(t)^2 + 9\Phi(t) - 12) = -\frac{1}{3}(\Phi(t) + 10.1789)(\Phi(t) - 1.1789)$$

The hint suggests how to obtain the solution to this equation: make $k = -\frac{1}{3}$, $\alpha_1 = 10.1789$, $\alpha_2 = -1.1789$ in the formula $\Phi(t) = \frac{\alpha_2 c e^{k(\alpha_2 - \alpha_1)t} - \alpha_1}{1 - ce^{k(\alpha_2 - \alpha_1)t}}$. To obtain c simply note that for $t = 0$, $\Phi(0) = 1/2$ and therefore

$$1/2 = \frac{\alpha_2 c - \alpha_1}{1 - c}$$

from which

$$c = \frac{1/2 + \alpha_1}{\alpha_2 + 1/2} = -15.7295$$

This leads to $\Phi(t) = \frac{18.5437e^{3.7859t} - 10.1789}{1 + 15.7295e^{3.7859t}}$ for $t \in [0,1]$.

(b) the equations of the Stationary Kalman filter

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = \mathbb{E}[x(0)],$$

where

$$L = \Phi C^\intercal V^{-1},$$
$$0 = A\Phi + \Phi A^\intercal + W - \Phi C^\intercal V^{-1} C\Phi.$$

For the parameters of the given system we get

$$\dot{\hat{x}}(t) = -\frac{3}{2}\hat{x}(t) + 3u(t) + L(y(t) - \hat{x}(t)) \quad \hat{x}(0) = \bar{x}_0, \quad t \in \mathbb{R}_{\geq 0}$$

where

$$\Phi = 1.1789$$
$$L = \frac{1.1789}{3}$$

Problem 4.2 The Kalman filter is given by

$$\dot{\hat{x}}(t) = \begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix} \hat{x}(t) + L(y(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(t)), \quad \hat{x}(0) = \mathbb{E}[x(0)],$$

The state covariance matrix is $W = B_w I B_w^\intercal = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ and the noise covariance matrix is $V = 3$. Using the kalman.m instruction we get

$$L = \begin{bmatrix} 6.6178 \\ -0.0444 \end{bmatrix}$$

Problem 4.3 This is a standard LQG problem. The optimal policy is given by

$$u(t) = K(t)\hat{x}(t)$$

where $\hat{x}(t)$ is the Kalman filter estimate, which was already computed in 2.1. The gains are given by $K(t) = -R^{-1}B^\intercal P(t)$ and $P(t)$ is given by the continuous-time Riccati equations

$$\begin{aligned} \dot{P}(t) &= -(A^\intercal P(t) + P(t)A + Q - P(t)BR^{-1}B^\intercal P(t)), \quad t \in [0, T] \\ P(T) &= Q_T \end{aligned}$$

for the parameters given ($P(1) = 0$ since the terminal cost is zero), we have

$$\dot{P}(t) = -(3^\intercal P(t) + 4 - P(t)^2) = -(P(t) - 1)(P(t) + 4) \quad P(1) = 0,$$

whose solution is $P(t) = \frac{-ce^{-5t} - 4}{1 - ce^{-5t}}$ for a given constant c such that $P(1) = 0$. This leads to $c = -4e^5$ and

$$K(t) = -\frac{1}{3}\frac{4e^{-5(t-1)} - 4}{1 + 4e^{-5(t-1)}}$$

Problem 4.4 The optimal policy is

$$u(t) = K\hat{x}(t)$$

where

$$\dot{\hat{x}}(t) = A\hat{x}(t) + BK\hat{x}(t) + L(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = \mathbb{E}[x(0)],$$

and L and K can be obtained with the matlab instructions lqr.m and kalman.m using the parameters

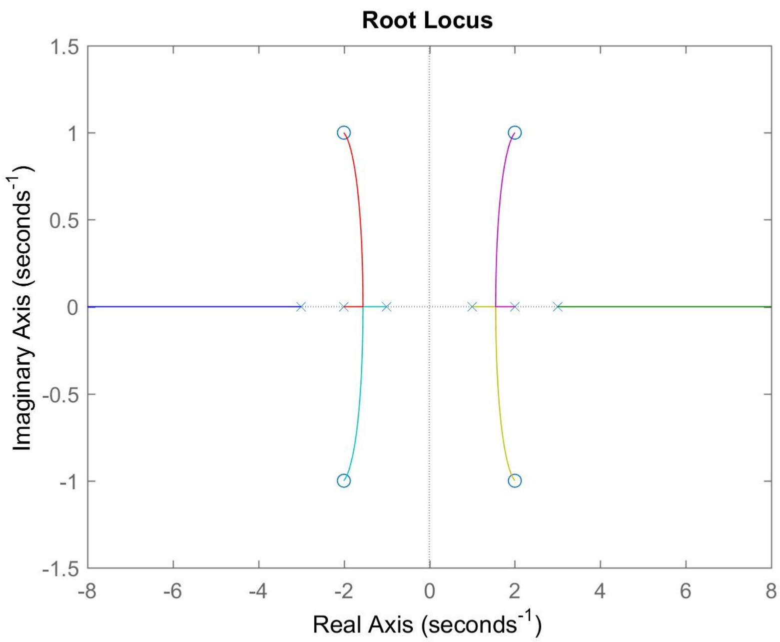
$$
\begin{aligned}
A &= \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} \\
B &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
C &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\
Q &= I \\
R &= 0.1 \\
W &= I \\
V &= 0.1
\end{aligned}
$$

This leads to

$$
\begin{aligned}
K &= \begin{bmatrix} 1.7495 & 6.9495 \end{bmatrix} \\
L &= \begin{bmatrix} 7.1767 \\ 9.3097 \end{bmatrix}
\end{aligned}
$$

Root square locus and loop transfer recovery

Problem 5.1



1. This is a standard root square locus problem. Due to the frequency domain equality the following holds

$$\underbrace{(1-K(sI-A)^{-1}B)(1-K(-sI-A)^{-1}B)}_{\text{eigenvalues of } A+BK \text{ and their symmetric w.r.t. im. axis}} = 1 + \underbrace{\frac{1}{\rho}}_{\text{"}gain\text{"}\,\bar{k}\, for\, r.s.locus} \underbrace{C(sI-A)^{-1}BC(-sI-A)^{-1}B}_{:= \frac{n(s)}{d(s)} \text{"open-loop t.f." for root square locus}} \tag{26}$$

We have

$$C(sI-A)^{-1}B = \frac{s+\frac{1}{2}}{s^3+4s^2+s-6} = \frac{s+\frac{1}{2}}{(s+3)(s+2)(s-1)}.$$

and therefore

$$\frac{n(s)}{d(s)} = \frac{s+\frac{1}{2}}{s^3+4s^2+s-6}\frac{-s+\frac{1}{2}}{-s^3+4s^2-s-6} = \frac{-s^2+\frac{1}{4}}{(-s^2+9)(-s^2+4)(-s^2+1)}.$$

Using the matlab function rlocus.m with $\frac{n(s)}{d(s)}$ results in the plot in the figure (the closed-loop eigenvalues of the lqr design are the stable ones in the plot). Checking the eigenvalues in the plot corresponding to the gains $1/\rho$ for $\rho \in \{1, 10, 100\}$ we obtain

	$\rho = 1$	$\rho = 10$	$\rho = 100$
closed-loop eigenvalues	-0.9851	-0.8908	-0.6526
	-2.0639	$-2.6047 + 0.4257i$	$-3.0622 + 1.6094i$
	-2.9614	$-2.6047 - 0.4257i$	$-3.0622 + 1.6094i$

2. From the FDE (see equation (26)) we see that by picking C and ρ one can select the closed-loop eigenvalues. When $\rho \to 0$, the gain of the root square locus converges to infinity $\frac{1}{\rho} \to \infty$. One of the rules of the root locus is that a subset of the closed loop poles/eigenvalues, i.e., the roots of

$$1 + \bar{k}\frac{n(s)}{d(s)} = 0$$

converge to the open-loop zeros (roots of $n(s) = 0$) when the gain $\bar{k}$ approaches infinity and the remaining closed loop poles/eigenvalues converge to infinity. The zeros of the open-loop t.f., i.e., the number of zeros of

$$\underbrace{C(sI - A)^{-1}BC(-sI - A)^{-1}B = 0}$$

are at most 4, since the number of zeros of $C(sI - A)^{-1}B$ are at most 2 since (A, B, C) is a third order system. Moreover, the zeros corresponding to $C(-sI - A)^{-1}B$ are symmetric of the zeros of $C(sI - A)^{-1}B$ w.r.t the imaginary axis. Therefore, if we pick the zeros of $C(sI - A)^{-1}B$ (or of $C(-sI - A)^{-1}B$) to correspond to $-2 \pm i$ we know that two stable eigenvalues will eventually converge to these values. Since there are at most 3 stable closed-loop eigenvalues and the third one must converge to infinity, it must converge to minus infinity along the real axis. Letting $C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$ we get $C(sI - A)^{-1}B = \frac{c_1 + c_2 s + c_3 s^2}{s^3 + 4s^2 + s - 6}$ Then, we need

$$c_1 + c_2 s + c_3 s^2 = (s + 2 + i)(s + 2 - i) = s^2 + 4s + 5$$

which can be obtained by making $c_3 = 1$, $c_2 = 4$, $c_1 = 5$. If we had picked the zeros of $C(sI - A)^{-1}B$ we would get $C(-sI - A)^{-1}B = \frac{c_1 - c_2 s + c_3 s^2}{s^3 - 4s^2 + s - 6}$ So picking either

$$C = \begin{bmatrix} 5 & -4 & 1 \end{bmatrix}$$

or

$$C = \begin{bmatrix} 5 & 4 & 1 \end{bmatrix}$$

leads to the desired closed-loop poles/eigenvalues.

Problem 5.2 Solution not provided.