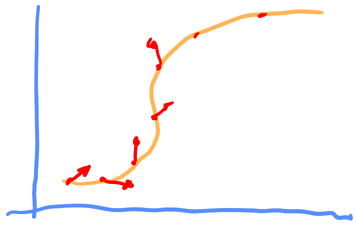
Time-optimal trajectories for robot arms

Assignment 1, 4SC000, TU/e, 2022-2023

Consider a 2 dimensional path described by a continuous and differential function of its arclength resulting from the concatenation of straight lines and circles with different radii. One way to define this path is through the solution of the differential equation

$$\frac{d}{ds}x(s) = \cos(\theta(s)), \quad \frac{d}{ds}y(s) = \sin(\theta(s)), \quad \frac{d}{ds}\theta(s) = \omega(s),$$



with initial condition $\begin{bmatrix} x(0) & y(0) & \theta(0) \end{bmatrix}^\intercal = \begin{bmatrix} x_0 & y_0 & \theta_0 \end{bmatrix}^\intercal$ with

$$\omega(s) = \omega_i, \quad s \in [s_i, s_{i+1}),$$

and $i \in \{0, 1, \ldots, L-1\}$. Then, when $\omega_i \neq 0$

$$\boxed{\begin{aligned} x(s) &= x(s_i) + \frac{1}{\omega_i}\big(\sin(\theta(s_i) + \omega_i(s - s_i)) - \sin(\theta(s_i))\big), \\ y(s) &= y(s_i) - \frac{1}{\omega_i}\big(\cos(\theta(s_i) + \omega_i(s - s_i)) - \cos(\theta(s_i))\big), \\ \theta(s) &= \theta(s_i) + \omega_i(s - s_i), \qquad s \in [s_i, s_{i+1}) \end{aligned}}$$

$\dot{x} = Ax + Bu + v$

Cost function: $\min \int_0 x^\intercal Q x + u^\intercal R u$

where ω_i is the curvature of the circle (reciprocal of the radius) and when $\omega_i = 0$,

$$\begin{aligned} x(s) &= x(s_i) + \cos(\theta(s_i))(s - s_i) \\ y(s) &= y(s_i) + \sin(\theta(s_i))(s - s_i), \qquad s \in [s_i, s_{i+1}). \end{aligned}$$

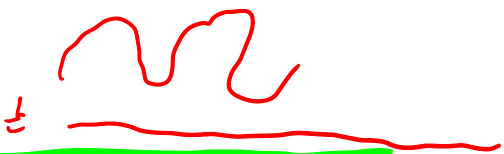
Note that s is the arclength of the path and that s_L is the total path length.

The problem considered here is to assign a differentiable function $s = \beta(t), t \in [0, T]$ from a given class $\beta \in \mathcal{S}$ such $\beta(T) = s_L$ and T in minimal. [1] This is equivalent to choosing an optimal-time trajectory for a unicycle model. Three important constraints for β are that the linear and centrifugal accelerations must be bounded as well as the velocity

$$|\ddot{s}| \leq L_1, \quad \omega_i \dot{s}^2 \leq L_2, \quad \dot{s} \leq L_3. \qquad \text{S .. Kurvenlänge} \qquad (1)$$

A function $s(t)$ belongs to $\mathcal{S}$ if it is the solution of

$$\begin{aligned} \dot{s}(t) &= v(t) \\ \dot{v}(t) &= a(t) \end{aligned}$$



with $\begin{bmatrix} s(0) & v(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $\begin{bmatrix} s(T) & v(T) \end{bmatrix} = \begin{bmatrix} s_L & 0 \end{bmatrix}$ when $a(t)$ is a piecewise affine function taking the form

Cost function

$$a(t) = \sum_{k=0}^{h-1} p_k(t) 1_{[k\tau,(k+1)\tau)}(t), \quad p_k(t) = c_{1,k} + c_{2,k}(t - k\tau)$$

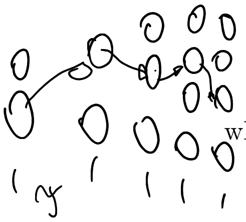
[1] That is for any other map $s = \tilde{\beta}(t), t \in [0, \tilde{T}]$, from the same class $\tilde{\beta} \in \mathcal{S}$ and $\beta(\tilde{T}) = s_L$, then $\tilde{T} \geq T$ must hold.

$p_k(t) = c_{1,k} + c_{2,k}(t - k\tau)$

$1_{[k\tau,(k+1)\tau)}(t) = \begin{cases} 1 & t \in A \\ 0 & t \notin A \end{cases}$

searched optimal

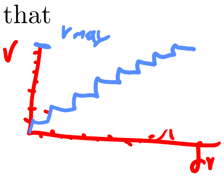
$a_k = [(1,k + c_{2,k}(t-k\tau)); \ldots]$ $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}^{[k\tau,(k+1)\tau)}$



where $1_A(t) = 1$ if $t \in A$ and $1_A(t) = 0$ if $t \notin A$. The coefficients $c_{1,k}$, $c_{2,k}$, must be such that

$$\boxed{v(k\tau) \in \mathcal{V}, \quad \mathcal{V} := \{0, \delta_v, \ldots, \delta_v(M_v - 1), \delta_v M_v\}}$$
$$s(k\tau) \in \mathcal{S}, \quad \mathcal{S} := \{0, \delta_s, \ldots, \delta_s(M_s - 1), \delta_s M_s\}$$

Speedsteps v



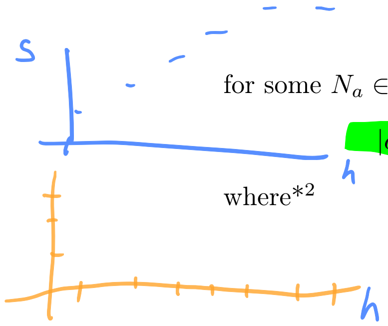
Limitations

and such that (1) is met. Assume, for simplicity, that

!!

$$\boxed{\delta_s = \delta_v \tau, \quad \delta_v M_v = L_3, \quad \delta_s M_s = s_L, \quad L_1 \tau = N_a \delta_v}$$

constraint



for some $N_a \in \mathbb{N}$. The following constraints, for $k \in \{0, 1, \ldots, h-1\}$,

$$|c_{1,k}| \leq L_1, |c_{1,k} + c_{2,k}\tau| \leq L_1, \quad \omega_i v_{\text{intmax},k}^2 \leq L_2, \quad v_{\text{intmax},k} \leq L_3, \tag{2}$$

where*[2]

either solve or
or s_{k+1} & v_{k+1}

$$\boxed{v_{\text{intmax},k}} = \begin{cases} v(k\tau) - \dfrac{c_{1,k}^2}{2c_{2,k}} & \text{if } c_{2,k} < 0 \,\&\, c_{1,k} > 0 \\ v(k\tau) \text{ otherwise} \end{cases}$$

ensure that $|\ddot{s}| \leq L_1$, $\omega_i \dot{s}^2 \leq L_2$, $\dot{s} \leq L_3$ are met for every $t \in [0, T]$ (assuming $v(T) = 0$). Let $v_k = \frac{v(k\tau)}{\delta_v} \in \{0, 1, \ldots, M_v\}$, $s_k = \frac{s(k\tau)}{\delta_s} \in \{0, 1, \ldots, M_s\}$. Then

1 $(v_{k+1} - v_k) = \frac{1}{\delta_v}(c_{1,k}\tau + c_{2,k}\frac{\tau^2}{2})$

$s_{k+1} - v_k - s_k = \frac{1}{\delta_s}(c_{1,k}\frac{\tau^2}{2} + c_{2,k}\frac{\tau^3}{6})$

$$v_{k+1} = v_k + \frac{1}{\delta_v}(c_{1,k}\tau + c_{2,k}\frac{\tau^2}{2})$$
$$s_{k+1} = s_k + v_k + \frac{1}{\delta_s}(c_{1,k}\frac{\tau^2}{2} + c_{2,k}\frac{\tau^3}{6}).$$

(c)

ges c

Since there is a one-to-to map between the $c_{1,k}$, $c_{2,k}$ and the v_{k+1}, s_{k+1} we can pick rather the latter. However, the v_{k+1}, s_{k+1} cannot be chosen arbitrarily, they must be such that (2) is met.

Assignment 1.1 Program a matlab function

step (eng)

$L_1 = \frac{N_a \delta_v}{\tau}$

```
[c] = timeoptimalpathspeed(omegai,si,tau,deltav,L2,Na,Mv);
```

slope speed

which provides the parameters of the optimal acceleration input

constraint

$L_3 = M_v \delta_v$

: set up cost fn
: cp end pole

) solve v_k, s_k
) map v_{k+1}, s_{k+1} to $c_{1,k}$, $c_{2,k}$

$$c = \begin{bmatrix} c_{1,0} & c_{1,1} & \ldots & c_{1,h-1} & c_{1,h-1} \\ c_{2,0} & c_{2,1} & \ldots & c_{2,h-1} & c_{2,h-1} \end{bmatrix},$$

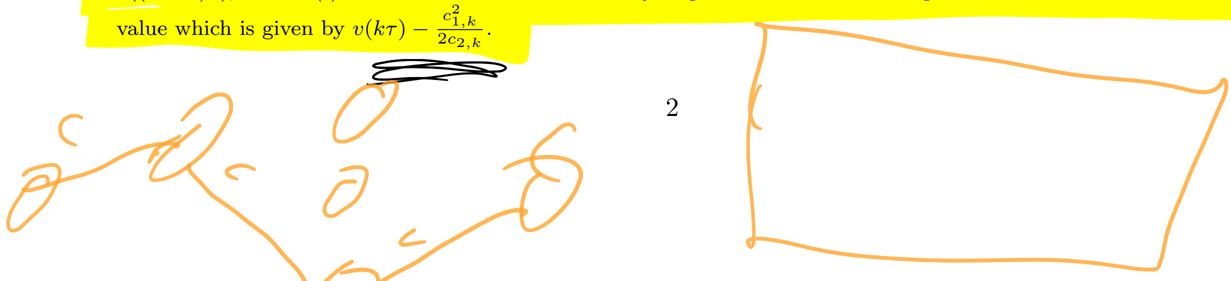
with h the optimal time $(T = \tau h)$, given omegai $= [\omega_0 \;\; \omega_1 \;\; \ldots \;\; \omega_{L-1}]$, si $= [s_0 \;\; s_1 \;\; \ldots \;\; s_L]$ characterizing the curve, and τ, δ_v, L_2, N_a, M_v.

Suppose now that we wish to move the end effector of a planar (no gravity forces) robot-arm with two links along a pre-defined two-dimensional path in minimum time. The robot arm is depicted in Figure 1, where the two dimensional position (x, y) and the joint angles θ_1 and θ_2 are defined, as well as the arm lengths L_1 and L_2. For convenience, assume that $L_1 \geq L_2$.

The robot-arm is assumed to be initially at rest and also should reach the end of the path with zero velocity. The direct kinematics of the robot is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

[2]when $c_{2,k} < 0$, $c_{1,k} > 0$, $v(t)$ in the interval $(k\tau, (k+1)\tau)$, might be larger than the values $v(k\tau)$ and $v((k+1)\tau)$; since $v(t)$ in this interval is described by a quadratic function it is possible to take the maximum value which is given by $v(k\tau) - \frac{c_{1,k}^2}{2c_{2,k}}$.



$[M_v, M_s, h]$

$$v_{k+1} = v_k + \frac{1}{\delta_v}\left(c_{1,k}\, J + c_{2,k} \cdot \frac{J^2}{2}\right)$$

$$s_{k+1} = s_k + v_k + \frac{1}{\delta_s}\left(c_{1,k} \cdot \frac{J^2}{2} + c_{2,k} \cdot \frac{J^3}{6}\right)$$

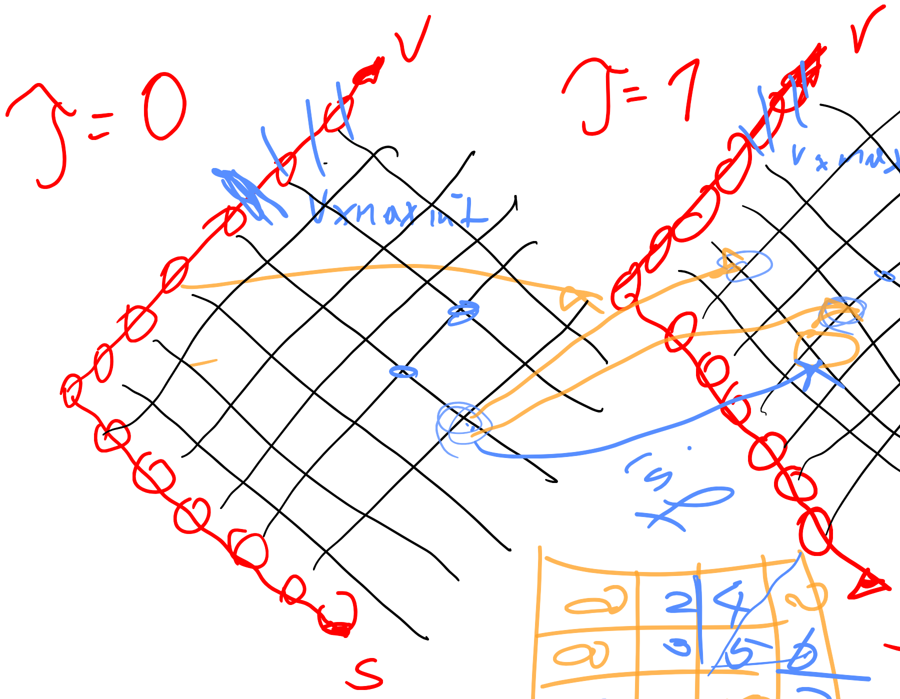
$$c_{1,k}\, J + c_{2,k} \cdot \frac{J^2}{2} = (v_{k+1} - v_k) \cdot \delta_v$$

$$c_{1,k} \frac{J^2}{2} + c_{2,k} \frac{J^3}{6} = (s_{k+1} - s_k - v_k) \cdot \delta_s$$

$$\Rightarrow c_{1,k} = \frac{(v_{k+1} - v_k) \cdot \delta_v}{J} - c_{2,k} \frac{J}{2}$$

$$\frac{(v_{k+1} - v_k) \cdot \delta_v}{\cancel{J}} \cdot \frac{J^{\cancel{2}}}{2} - c_{2,k} \cdot \frac{J^3}{4} + c_{2,k} \cdot \frac{J^3}{6} = (s_{k+1} - s_k - v_k)\delta_s$$

$$c_{2,k} = \frac{(s_{k+1} - s_k - v_k)\,\delta_s - (v_{k+1} - v_k) \cdot \delta_v \cdot \frac{J}{2}}{\left(\frac{J^3}{6} - \frac{J^3}{4}\right)}$$



$V_k \geq 0$

$V_k = \frac{v(k,J)}{J}$

$v_k = \{ set(\delta v \cdot M_v) \}$

$max(\langle v_k \rangle) = M_v$

$s_k = \frac{s(k,J)}{\delta s}$

$\boxed{s_k \geq 0}$

∞	2	4	∞
∞	0	5	6
1	∞	∞	2

$V_{int\,max} \leq L_3 = M_v \cdot \delta v$

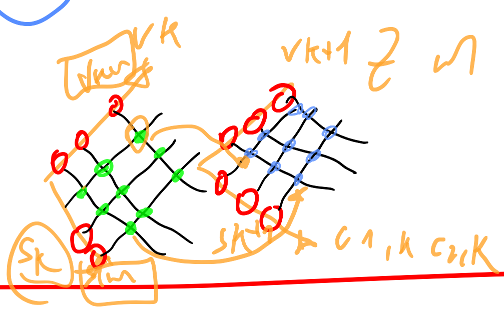
$w_i \cdot v_{int\,max}^2 \leq L_2$ if $w_i > 0$

$v_{int\,max} \leq \sqrt{\frac{L_2}{w_i}}$

$C_{1,K} \leq \frac{N_a dv}{J} = L_1$

① möglich Grid aufstellen (mit Limits)

② cost z möglichen Transitions zuweisen



$s_{k+1} = 10$, $v_k = 13$

$s_k = 9$, $v_k = 9$ $\Rightarrow C_{1,K}$ $C_{2,K}$

③ DP anwenden um optimum zu finden.

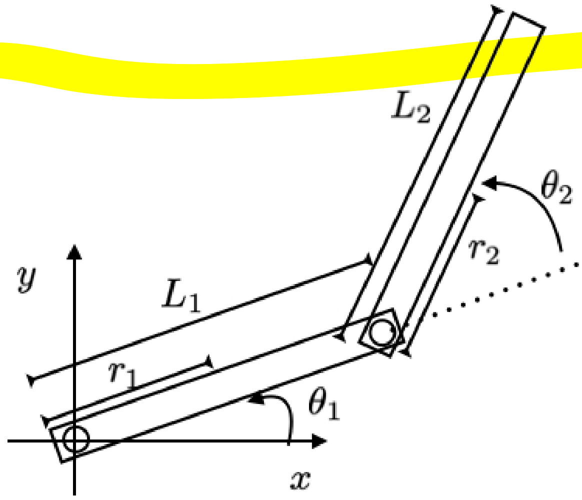


Figure 1: Diagram of the simple robot arm

Given x and y the corresponding angles θ_1 and θ_2 are not unique. Thus we assume that $\theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and that (x, y) are such that $\theta_2 \in (0, \pi)$. Moreover, we assume that $x > 0$. The inverse kinematics of the robot are (see Appendix 1 for the derivation):

$$\theta_2 = \alpha_2(x, y), \quad \alpha_2(x, y) = \arccos(\frac{x^2 + y^2 - (L_1^2 + L_2^2)}{2L_1L_2}),$$
$$\theta_1 = \alpha_1(x, y), \quad \alpha_1(x, y) = \arctan(\frac{y}{x}) - \arcsin(\frac{L_2}{\sqrt{x^2 + y^2}} \sin(\theta_2))$$

The dynamics model is well-known [3]

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c_{121}\dot{\theta}_2 & c_{211}\dot{\theta}_1 + c_{221}\dot{\theta}_2 \\ c_{112}\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

where

$$m_{11} = m_1 r_1^2 + I_1 + m_2(L_1^2 + r_2^2 + 2L_1 r_2 \cos(\theta_2)) + I_2$$
$$m_{12} = m_{21} = m_2(L_1 r_2 \cos(\theta_2) + r_2^2) + I_1$$
$$m_{22} = m_2 r_2^2 + I_2$$
$$c_{121} = c_{211} = c_{221} = -m_2 L_1 r_2 \sin(\theta_2), c_{112} = -c_{121}$$

where τ_1 and τ_2 are the torques applied at joints 1 and 2, I_1, I_2 are the moments of inertia of joints 1 and 2, m_1 and m_2 are the masses of of joints 1 and 2, and r_1, r_2 are the distances to the centers of mass of joints 1 and 2, as depicted in Figure 1. The numerical values for matrices M and N are obtained from the numerical values for the parameters of the links such as masses, lengths and moments of inertia and are given in Appendix 2. A crucial constraint that must be met is that

$$|\tau_1| \leq \bar{\tau}, \quad |\tau_2| \leq \bar{\tau}$$

[3] see, e.g., `https://www.youtube.com/watch?v=zRdL_v7JcHc`

for some given bound $\bar{\tau}$. This imposes a constraint on the accelerations $\ddot{\theta}_1$, $\ddot{\theta}_2$ that depends of the values of θ_2, $\dot{\theta}_1$, $\dot{\theta}_2$. In turn this imposes a constraint on $\ddot{s}$ which depends on $\dot{s}$, s. Rather than testing this condition for every time t for the same of simplicity it will only be tested at times $t = k\tau$, i.e.,

$$|\tau_1(k\tau)| \leq \bar{\tau}(k\tau), \quad |\tau_2(k\tau)| \leq \bar{\tau} \tag{3}$$

Assignment 1.2 Program a matlab function

```
[c] = timeoptimalarmspeed(omegai,si,tau,deltav,taubar,Na,Mv,xi0);
```

with the same inputs, except for taubar instead of L2 and xi0$=\begin{bmatrix} x(0) & y(0) & \cos(\theta(0)) \end{bmatrix}^\intercal$, and outputs as in Assignment 1.1 but that considers the new constraints of the robot arm (3) rather than (2).

Appendix 1: Inverse kinematics

Let

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(\frac{y}{x}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

be the angular coordinates of x and y. Note that if we draw an extension of the first arm (with length L_1) with length $L_2\cos(\theta_2)$ we get that

$$r^2 = (L_1 + L_2\cos(\theta_2))^2 + (L_2\sin(\theta_2))^2$$

from which

$$r^2 = L_1^2 + L_2^2 + 2L_1L_2\cos(\theta_2)$$

and

$$\cos(\theta_2) = \frac{r^2 - (L_1^2 + L_2^2)}{2L_1L_2}.$$

Thus,

$$\theta_2 = \arccos(\frac{r^2 - (L_1^2 + L_2^2)}{2L_1L_2}) = \arccos(\frac{x^2 + y^2 - (L_1^2 + L_2^2)}{2L_1L_2})$$

Then we get that $\theta = \theta_1 + \bar{\theta}$ where $\bar{\theta}$ due to $L_1 \geq L_2$ and where

$$R\sin(\bar{\theta}) = L_2\sin(\theta_2).$$

Thus,

$$\bar{\theta} = \arcsin(\frac{L_2}{R}\sin(\theta_2))$$

and

$$\theta_1 = \arctan(\frac{y}{x}) - \arcsin(\frac{L_2}{\sqrt{x^2 + y^2}}\sin(\theta_2)).$$