Problem set 3

Continuous-time optimal control problems

Optimal control and reinforcement learning, TU/e, 2022-2023

Outline

Linear quadratic regulation

Linear systems with terminal state constraints

Pontryagin's maximum principle, minimum time optimal control problems

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Linear quadratic regulation

Problem 1.1 Consider a first order linear differential equation

$$\dot{x}(t) = -\alpha x(t) + u(t), \quad x(0) = x_0, \quad t \in \mathbb{R}_{\geq 0},$$

where $x(t)$ denotes the state and $u(t)$ denotes the control input. Performance is measured by the following cost

$$\int_0^T x(t)^2 + \gamma u(t)^2 dt$$

to be minimized. Suppose that $\alpha = 1$ and $\gamma = 0.1$.

(i) Provide the optimal policy for the control input $u(t)$, $t \in [0, T]$, when $T = 10$.

(ii) Provide the optimal policy for the control input $u(t)$, $t \in \mathbb{R}_{\geq 0}$, when $T = \infty$.

Problem 1.2 Consider the following system

$$\dot{x}(t) = -\frac{3}{2}x(t) + 3u(t), \quad t \in \mathbb{R}_{\geq 0}.$$

Provide the optimal policy for $u(t)$ in the interval $[0, 1]$ that minimizes the cost

$$\int_0^1 4x(t)^2 + 9u(t)^2 dt$$

for every initial condition $x(0)$ by solving the continuous-time Riccati equations.

[Hint: Use the fact that the solution to the differential equation $\dot{p} = k(p + \alpha_1)(p + \alpha_2)$ for real constants k, α_1, α_2 takes the general form $p(t) = \frac{\alpha_2 c e^{k(\alpha_2 - \alpha_1)t} - \alpha_1}{1 - c e^{k(\alpha_2 - \alpha_1)t}}$ where c is a constant.]

Problem 1.3 Consider the following system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in \mathbb{R}_{\geq 0}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\intercal$, $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^\intercal$ and

$$A = \begin{bmatrix} \frac{5}{2} & \alpha \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Performance is measured by the following cost

$$\int_0^\infty 12x_1(t)^2 + x_2(t)^2 + \frac{1}{2}u_1(t)^2 + 3u_2(t)^2 dt$$

to be minimized.

(i) Provide the optimal policy for the control input $u(t)$, $t \in \mathbb{R}_{\geq 0}$, when $\alpha = 0$.

(ii) Provide the optimal policy for the control input $u(t)$, $t \in \mathbb{R}_{\geq 0}$, when $\alpha = 2$, using Matlab. [Hint: use lqr.m or care.m]

Problem 1.4 Consider the following problem

$$\min_u \int_0^\infty x(t)^2 + u(t)^2 dt$$

where

$$\dot{x}(t) = x(t) + u(t), \quad t \in \mathbb{R}_{\geq 0}.$$

(i) Provide the optimal policy and the optimal cost (as a function of x_0).

(ii) Suppose that

$$u(t) = u_k, \quad t \in [k\tau, (k+1)\tau).$$

Approximate the dynamic model by the Euler's method, and the cost using a zero-order approximation. Provide the optimal policy for $\{u_k\}_{k\in\mathbb{N}_0}$ for the resulting state decision problem as a function of τ.

(iii) Let $\tau \to 0$ and show that the optimal solution obtained in (i) is recovered.

Problem 1.5 Consider an optimal control problem with cost function (to be minimized)

$$\int_0^T x(t)^\intercal Q x(t) + u(t)^\intercal R u(t) dt + x(T)^\intercal Q(T) x(T) \tag{1}$$

for $Q > 0$, $R > 0$, and dynamic model

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \in [0, T],$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$. The optimal policy for this problem is given by

$$u(t) = K(t)x(t), \quad t \in [0, T]. \tag{2}$$

where

$$K(t) = -R^{-1}B^\intercal P(t) \tag{3}$$

and $P(t)$ is a symmetric matrix for every $t \in [0, T]$ such that

$$\begin{aligned} -\dot{P}(t) &= A^\intercal P(t) + P(t)A + Q - P(t)BR^{-1}B^\intercal P(t) \\ P(T) &= Q_T \end{aligned} \tag{4}$$

(i) Suppose that $n = 2$ and consider the following notation

$$P(t) = \begin{bmatrix} p_1(t) & p_2(t) \\ p_2(t) & p_3(t) \end{bmatrix}, \quad Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix}, \quad Q_T = \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad R = r_1.$$

Show that for $t \in [0, T]$

$$\begin{aligned} \begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \\ \dot{p}_3(t) \end{bmatrix} &= \begin{bmatrix} -\left(2a_1p_1(t) + 2a_3p_2(t) + q_1 - \frac{1}{r_1}(p_1(t)b_1 + p_2(t)b_2)^2\right) \\ -\left((a_1 + a_4)p_2(t) + a_3p_3(t) + p_1(t)a_2 + q_2 - \frac{1}{r_1}(p_1(t)b_1 + p_2(t)b_2)(p_2(t)b_1 + p_3(t)b_2)\right) \\ -\left(2a_2p_2(t) + 2a_4p_3(t) + q_3 - \frac{1}{r_1}(p_2(t)b_1 + p_3(t)b_2)^2\right) \end{bmatrix} \\ \begin{bmatrix} p_1(T) \\ p_2(T) \\ p_3(T) \end{bmatrix} &= \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \end{aligned}$$

(ii) Consider the kronecker product between two matrices

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{bmatrix}$$

and let ν denote the operator that transforms a matrix into a column vector $\nu(A) = \nu([a_1 \, \dots \, a_n]) = [a_1^\intercal \, \dots \, a_n^\intercal]^\intercal$. Show that for $t \in [0, T]$

$$\begin{aligned} \nu(\dot{P}(t)) &= -(I \otimes A^\intercal + A^\intercal \otimes I)\nu(P(t)) - \nu(Q) + \frac{1}{r_1}(P(t)B) \otimes (P(t)B) \\ \nu(P(T)) &= \nu(Q_T) \end{aligned}$$

[Hint: Use the fact that $\nu(ABC) = (C^\intercal \otimes A)\nu(B)$ for matrices wih compatible dimension.]

(iii) Confirm (i) specializing the conclusions obtained in (ii).

Linear systems with terminal constraints

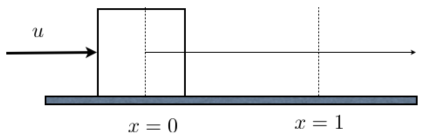
Problem 2.1 Suppose that we wish to move a mass from position $x = 0$ to position $x = 1$ in 1 seconds while minimizing the control effort measured by

$$\int_0^1 u(t)^2 dt,$$

where $u(t)$ denotes the force applied to the mass. The position of the mass evolves according to

$$\ddot{x}(t) = -\alpha\dot{x}(t) + u(t)$$

where $-\alpha\dot{x}(t)$, $\alpha > 0$, models friction. The initial velocity $v(t) = \dot{x}(t)$ at time zero is assumed to be $v(0) = 0$.



(i) Suppose that $\alpha = 0$. Compute the optimal control input $u(t)$ and the corresponding position $x(t)$ and velocity $v(t)$ in the interval $[0, 1]$ when the desired terminal velocity equals zero ($v(1) = 0$) and when it equals one ($v(1) = 1$).

(ii) Consider now that $\alpha = \frac{1}{2}$. Compute the optimal control input and the corresponding position and velocity in the interval $[0, 1]$ when the desired terminal velocity equals zero.

(iii) Consider again that $\alpha = 0$ and that the desired terminal velocity is $v(1) = 0$.

(a) Compute the optimal control input in the interval $[0, 1]$ as a function of $(x(0), v(0)$.

(b) Compute the optimal control input in the interval $[s, 1]$ as a function of $(x(s), v(s))$.

(c) Based on your answer to (iii).(b) provide the optimal control policy.

[Note: You can use the Matlab functions *inv.m*, *expm.m* and to compute inverses and matrix exponentials. In particular these functions can be applied to obtain symbolic expressions (e.g. *syms t; inv(A*t)*; computes the inverse of a matrix as a function of t)]

Problem 2.2 Provide the optimal control input for the following problem

$$\min_u \int_0^5 x_1(t)^2 + 2x_2(t)^2 + 10u(t)^2 dt + 10x_1(5)^2$$

where

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad t \in [0, T]$$

and $x(0) = [1 \ \ 2]^\intercal$ (you can use Matlab to compute matrix exponentials and inverses).

Problem 2.3 Consider the following dynamic model

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad t \in \mathbb{R}_{\geq 0},$$

where $x(0) = [1 \;\; 1]^\intercal$. Find the control input $u(t)$, $t \in [0, 2]$ which minimizes

$$\int_0^2 u(t)^2 dt + 10x_1(2)^2$$

and achieves $x_2(2) = 0$.

[Hint: if you decide to follow the approach which requires computing the exponential of matrices, the following facts may be convenient:

$$\begin{aligned} e^{\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} t} &= \begin{bmatrix} e^{Xt} & e^{Xt} \int_0^t e^{-Xs} Y e^{Zs} ds \\ 0 & e^{Zt} \end{bmatrix} \\ e^{X^\intercal t} &= (e^{Xt})^\intercal \\ e^{-Xt} &= e^{Xs}{}_{|s=-t} \end{aligned}$$

for matrices X, Y, Z with compatible dimension.]

Problem 2.4 Consider the following optimal control problem

$$\min \frac{1}{2} (\int_0^T 7x(t)^2 + u(t)^2 dt)$$

where

$$\dot{x}(t) = 3x(t) + u(t), \quad x(0) = 1, \quad t \in \mathbb{R}_{\geq 0}.$$

(i) Suppose that $T = 1$. Provide the optimal control input $u(t)$, $t \in [0, 1]$, which minimizes the cost and achieves $x(1) = 0$.

(ii) Suppose that $T = \infty$. Provide the optimal control policy.

Pontryagin's maximum principle, minimum time optimal control problems

Problem 3.1 The Pontryagin's maximum principle extends to time-varying systems

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \in \mathbb{R}_{\geq 0}, \quad x(0) = x_0$$

with cost

$$\int_0^T g(t, x(t), u(t))dt + g_T(x(T))$$

to be minimized. The necessary conditions for $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ to be an optimal solution for the problem are

$$\begin{aligned} \dot{\lambda}(t) &= -[\frac{\partial}{\partial x} H(t, x(t), u(t), \lambda(t))]^\intercal \\ \dot{x}(t) &= [\frac{\partial}{\partial \lambda} H(t, x(t), u(t), \lambda(t))]^\intercal \\ \frac{\partial}{\partial u} H(t, &x(t), u(t), \lambda(t)) = 0 \end{aligned}$$

with boundary condition

$$\lambda(T) = [\frac{\partial g_T(x)}{\partial x}]^\intercal_{x=x(T)} \tag{5}$$

where $H(t, x(t), u(t), \lambda(t)) = \lambda(t)^\intercal f(t, x(t), u(t)) + g(t, x(t), u(t))$ and $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))$, $t \in [0, T]$ is the co-state. Moreover, if some (or all) of the components of the state are constrained at the terminal time $x_i(T) = c_i$, for $i \in \mathcal{C} \subseteq \{1, 2, \ldots, n\}$, then the variables $\lambda_i(T)$ are free whereas the constraint (5) still holds for the remaining variables $\lambda_j(T)$, for $j \in \{1, 2, \ldots, n\} \setminus \mathcal{C}$, i.e.,

$$\lambda_j(T) = [\frac{\partial g_T(x)}{\partial x_j}]^\intercal_{x=x(T)}, \quad j \in \{1, 2, \ldots, n\} \setminus \mathcal{C}.$$

Apply the Pontryagin's maximum principle to establish that the optimality conditions for the problem

$$\min \frac{1}{2}\Big(\int_0^T x(t)^\intercal Q(t) x(t) + u(t)^\intercal R(t) u(t) dt + x(T)^\intercal Q_T x(T)\Big)$$

for positive definite $R(t)$ and $Q(t)$ and diagonal $Q_T = \mathrm{diag}(q_1, q_2, \ldots, q_n)$, subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in \mathbb{R}_{\geq 0}, \quad x(0) = x_0$$

with inital constraint $x(0) = x_0$ and terminal constraints $x_i(T) = c_i$, for $i \in \mathcal{C} \subseteq \{1, 2, \ldots, n\}$ are

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R(t)^{-1}B(t)^\intercal \\ -Q(t) & -A(t)^\intercal \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \tag{6}$$

and

$$u(t) = -R(t)^{-1}B(t)^\intercal \lambda(t)$$

with terminal constraints

$$x_i(T) = c_i, \text{ for } i \in \mathcal{C}$$

and

$$\lambda_j(T) = q_j x_j(T), \text{ for } j \in \{1, \ldots, n\} \setminus \mathcal{C}.$$

Problem 3.2 [1] Consider a particle of mass m, acted upon by a thrust force of magnitude ma. We assume planar motion and use an inertial coordinate system x, y to locate the particle; the velocity components

[1] adapted from Applied Optimal Control, Bryson, Ho, 1975, Sec 2.4

of the particle are u, v. The thrust-direction angle $\beta(t)$ is the control variable for the system. The equations of motion are

$$\begin{aligned}\dot{u} &= a\cos(\beta)\\ \dot{v} &= a\sin(\beta)\\ \dot{x} &= u\\ \dot{y} &= v\end{aligned}$$

where the thrust acceleration a is assumed to be a known function of time.

(i) Using the Pontryagin's maximum principle, show that if we wish to optimize a function that depends only on the state at the terminal time T, then the optimal control input $\beta(t)$ takes the form

$$\beta(t) = \mathrm{atan}\left(\frac{-c_2 t + c_4}{-c_1 t + c_3}\right)$$

for some constants c_i, $i \in \{1, 2, 3, 4\}$.

(ii) Suppose that the initial position of the particle at time $t = 0$ is $(y(0), x(0)) = (0, 0)$ and the initial velocity is zero. We wish to transfer the particle to a path parallel to the x-axis, a distance h away, in a given time T, arriving with the maximum value of $u(T)$. We do not care what the final x coordinate is. Compute the optimal control input and corresponding state.

Problem 3.3 [2] Consider a particle of mass m, acted upon by a thrust force $u = (u_x, u_y) = m(a_x, a_y)$, where $m = 1$ is the mass, and described by the equations of motion

$$\begin{aligned}\dot{x} &= v_x\\ \dot{y} &= v_y\\ \dot{v}_x &= a_x\\ \dot{v}_y &= a_y\end{aligned}$$

where v_x, v_y are the velocity components of the particle and x, y are the coordinates of the particle's position in an inertial coordinate system. The initial position of the particle at time $t = 0$ is $(x(0), y(0)) = (0, 0)$ and the initial velocity is zero. We wish to transfer the particle to a path parallel to the x-axis, a distance $h = 1$ away, in a given time T, arriving with the maximum value of $x(T)$, i.e., we are interested in the problem

$$\min -x(T). \tag{7}$$

The final velocity along x, $v_x(T)$, is not specified (it is free). Figure 1 illustrates the problem setting. Consider $T = 3$.

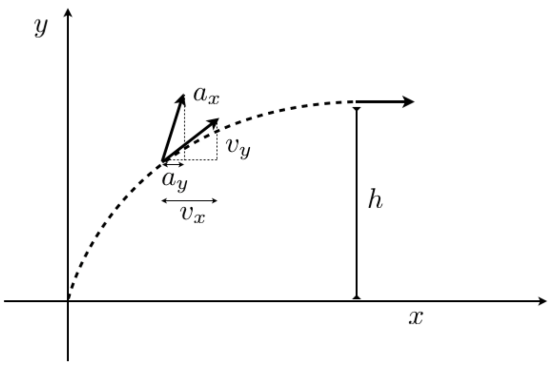


Figure 1: Particle acted by a thrust force

[2] Suggestion: Solve Problems 3.1 and 3.2 before solving Problem 3.3.

(i) Suppose that the energy of the control input should satisfy, at least approximately, the constraint

$$\int_0^T \|u(t)\|^2 dt = T. \tag{8}$$

(a) We start by penalizing violations of the constraint in the cost function, i.e., considering the problem

$$\min \gamma(\int_0^T \|u(t)\|^2 dt - T) - x(T). \tag{9}$$

Solve this problem for $\gamma = a$ and $\gamma = b$, $a = \frac{1}{2}$, $b = 1$ and conclude that for $\gamma = a$, $\int_0^T \|u(t)\|^2 dt > T$ and for $\gamma = b$, $\int_0^T \|u(t)\|^2 dt < T$.

(b) Find γ such that the solution to the problem (9) satisfies $\int_0^T \|u(t)\|^2 dt = T$ and provide the value of $x(T)$ corresponding to the optimal solution. [Note: Both an analytical exact solution and a numerical approximate solution will be considered correct.] [3]

(ii) Suppose now that instead of the constraint (8), the magnitude of the control input must satisfy the constraint

$$\|u(t)\|^2 = 1. \tag{10}$$

(a) Let $-x_C^*(T)$ be the optimal value achieved for the problem of minimizing (7) subject to (10) and let $-x_I^*(T)$ be the optimal value achieved for the problem of minimizing (7) subject to (8), which was obtained in (i).(b). Argue that

$$x_C^*(T) \leq x_I^*(T).$$

(b) Consider the following parameterization of the control input in terms of the angle $\beta(t) \in [-\pi, \pi)$,

$$u(t) = (\cos(\beta(t)), \sin(\beta(t))).$$

Find the optimal function $\beta(t)$ in the interval $t \in [0, T]$ for the problem (7) using the Pontryagin's maximum principle as in Problem 3.2. Provide the value of $x(T)$ corresponding to the optimal solution.

(c) Find a time-varying penalty $\gamma(t)$ such that the solution to the problem

$$\min \int_0^T \gamma(t)(\|u(t)\|^2 - 1)dt - x(T).$$

meets the constraint $\|u(t)\|^2 = 1$, in which case $x(T)$ must coincide with the optimal solution obtained in (ii).(b) (and therefore this is an alternative method to obtain the optimal solution). [Hint: Write the optimality conditions as in Problem 3.1 and find $\gamma(t)$ such that $\|u(t)\|^2 = 1$ and such that all the constraints in the problem are met].

Problem 3.4 We wish to find the curve $y(x)$ of length $L = \frac{5}{2}$ with fixed end points $y(0) = y(2) = 0$ with maximal area

$$\int_0^2 y(x)dx.$$

The length of the curve is given by $\int_0^2 \sqrt{1 + (y'(x))^2}dx$ and if $y(x)$ is an optimal curve for this problem it is also an optimal curve for the following problem

$$\int_0^2 y(x)dx + \gamma(\int_0^2 \sqrt{1 + (y'(x))^2}dx - L),$$

[3] For a numerical approach you can consider the function $\phi : [a, b] \to \mathbb{R}$ mapping γ into $\int_0^T u(t)^2 dt - T$ where $u(t)$ is the solution to the problem (9) considered in (i).(a). From (i).(a) we have $\phi(a) > 0$, $\phi(b) < 0$. Assume that ϕ is continuous and monotone. Apply three iterations of the bisection method (https://en.wikipedia.org/wiki/Bisection_method) to find approximations $\gamma_1 = \frac{a+b}{2}$, γ_2, γ_3 to the solution $\bar{\gamma}$ such that $\phi(\bar{\gamma}) = 0$. Alternatively you can solve the problem with Matlab for a dense grid of values $\gamma_k \in [a, b]$, plot the function ϕ and obtain an accurate approximation $\bar{\gamma}$ such that $\phi(\bar{\gamma}) = 0$.

for some constant γ. Find the optimal curve and the corresponding γ.

Problem 3.5 Consider the following minimal time problem

$$\min T$$

for the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in \mathbb{R}_{\geq 0}, x(0) = x_0$$

with $u(t) \in [-2, 2]$ and terminal constraint

$$x(T) = 0,$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Explain how to compute the optimal control input $u(t)$ given an initial condition x_0.

Problem 3.6 Suppose that we wish to drive to rest $(p(0), \dot{p}(0)) = (0, 0))$ a mass-spring system

$$\ddot{p}(t) = -p(t) + \frac{1}{2}u(t), \quad t \in \mathbb{R}_{\geq 0}$$

in minimal time, where the control input is constrained to the set $u(t) \in [-2, 2]$. Explain how to compute the optimal control input $u(t)$ given an initial position $p(0)$ and an initial velocity $\dot{p}(0)$ in the set $\{(p(0), \dot{p}(0)) \in \mathbb{R}^2 : \|(p(0), \dot{p}(0))\| \leq 2\}$.

Linear quadratic control, separation principle

Problem 4.1 Consider the following system

$$\dot{x}(t) = -\frac{3}{2}x(t) + 3u(t) + 2w(t), \quad x(0) = x_0, \quad t \in \mathbb{R}_{\geq 0}.$$

with output

$$y(t) = x(t) + v(t)$$

where v and w are zero-mean Gaussian white noise processes with $\mathbb{E}[v(t)v(t+\tau)] = 3\delta(\tau)$, $\mathbb{E}[w(t)w(t+\tau)] = 1\delta(\tau)$. Assume that the initial state is unknown and follows a Gaussian distribution with mean $\bar{x}_0 = -1$, and variance $\mathbb{E}[(x_0 - \bar{x}_0)^2] = \frac{1}{2}$.

i) Provide the Kalman filter to estimate the state in the interval $t \in [0, 1]$. [Hint: use the hint given for Problem 1.2]

ii) Provide the stationary Kalman filter to estimate the state.

Problem 4.2 Provide the stationary Kalman filter to estimate the state of

$$\dot{x}(t) = Ax(t) + B_w w(t), \quad t \in \mathbb{R}_{\geq 0},$$

where

$$A = \begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix}, \quad B_w = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

given the measurements

$$y(t) = Cx(t) + v(t)$$

where $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ v and w are zero-mean Gaussian white noise processes with $\mathbb{E}[v(t)v(t+\tau)] = 3\delta(\tau)$, $\mathbb{E}[w(t)w(t+\tau)] = I\delta(\tau)$ [Hint: use Kalman.n or care.m]

Problem 4.3 Consider the following system

$$\dot{x}(t) = -\frac{3}{2}x(t) + 3u(t) + 2w(t), \quad t \in \mathbb{R}_{\geq 0}.$$

with output

$$y(t) = x(t) + v(t)$$

where v and w are zero-mean Gaussian white noise processes with $\mathbb{E}[v(t)v(t+\tau)] = 3\delta(\tau)$, $\mathbb{E}[w(t)w(t+\tau)] = 1\delta(\tau)$. Assume that the initial state is unknown and follows a Gaussian distribution with mean $\bar{x}_0 = -1$, and variance $\mathbb{E}[(x_0 - \bar{x}_0)^2] = \frac{1}{2}$. Provide the optimal policy for $u(t)$ in the interval $[0, 1]$ that minimizes

$$\mathbb{E}[\int_0^1 4x(t)^2 + 9u(t)^2 dt].$$

[Hint: Use the fact that the solution to the differential equation $\dot{p} = k(p + \alpha_1)(p + \alpha_2)$ for real constants k, α_1, α_2 takes the general form $p(t) = \frac{\alpha_2 c e^{k(\alpha_2 - \alpha_1)t} - \alpha_1}{1 - c e^{k(\alpha_2 - \alpha_1)t}}$ where c is a constant.]

Problem 4.4 Provide the optimal policy for the following problem

$$\min_u \lim_{T \to \infty} \frac{1}{T}\mathbb{E}[\int_0^T x(t)^\intercal x(t) + 0.1u(t)^2]$$

$$\dot{x}(t) = Ax(t) + Bu(u) + w(t), \quad t \in \mathbb{R}_{\geq 0},$$

where

$$A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

given the measurements

$$y(t) = Cx(t) + v(t)$$

where $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, v and w are zero-mean Gaussian white noise processes with $\mathbb{E}[v(t)v(t+\tau)] = 0.1\delta(\tau)$, $\mathbb{E}[w(t)w(t+\tau)] = I\delta(\tau)$ [Hint: Use Matlab function kalman.m and lqr.m]

Root square locus and loop transfer recovery

Problem 5.1 Consider the following problem

$$\min \int_0^\infty y(t)^2 + \rho u(t)^2 dt$$

for

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and C and ρ can be seen as tunning knobs to shape the eigenvalues of the closed-loop system $\dot{x}(t) = (A + BK)x(t)$ where K are the optimal gains of the optimal policy $u = Kx$ for the problem.

1. Suppose that $C = \begin{bmatrix} \frac{1}{2} & 1 & 0 \end{bmatrix}$. Plot the closed-loop eigenvalues in the complex plane as a function of $\rho \in (0, \infty)$ (root locus) and indicate the values of the closed-loop eigenvalues for $\rho \in \{1, 1/10, 1/100\}$.
2. Pick C such that two closed-loop eigenvalues converge to $-2 \pm i$ when $\rho \to 0$ and the third closed-loop eigenvalue approaches minus infinity along the real axis. Are the values C and $-C$ the only values that meet these specifications? Why?

Problem 5.2 Consider a process described by the transfer function

$$t(s) = \frac{s^2 + 3s + 2}{s^3 + 7s^2 - 48s - 180}$$

which can be written in the standard state-space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + v(t) \end{aligned}$$

where A, B, C, are such that $t(s) = C(sI - A)^{-1}B$ and $w(t)$, $v(t)$ are Gaussian white noise model process disturbances and output noise. Design an LQG controller, by picking the matrices Q, R of the cost

$$\lim_{T \to \infty} \int_0^\infty \frac{1}{T} \mathbb{E}[x(t)^\intercal Q x(t) + u(t)^\intercal R u(t)] dt$$

and the matrices $W = \mathbb{E}[w(t)w(t)^\intercal]$, $V = \mathbb{E}[n(t)n(t)^\intercal]$, to guarantee the downward and upward gain margins $\text{GM}^- = 3/5$, $\text{GM}^+ = 2$, respectively and the negative and positive phase margins $\text{PM}^- = -40$, $\text{PM}^+ = 40$, respectively.