Frequency domain design with LQG

Assignment 3, 4SC000, TU/e, 2022-2023

The Linear Quadratic Gaussian design framework provides an output feedback stabilizing controller for stabilizable and detectable plants under mild conditions. While it is essentially a time-domain framework, it has a frequency domain interpretation through the Parseval's theorem. Through this interpretation we can design a controller to shape frequency domain functions of interest. This is achieved by augmenting the state of the system with auxiliary states corresponding to filters whose purpose is clear in the frequency domain and penalizing the states and inputs appropriately in the cost function.

Let us start by recalling the time-domain LQG framework using a slightly different notation from the one used in the lectures and considering single-input single-output systems. The LQG framework considers a linear system

$$\begin{aligned} x_{k+1} &= Ax_k + B_2 u_k + B_1 w_k \\ y_k &= C_2 x_k + D_{21} w_k \end{aligned}$$

driven by zero-mean independent and identically distributed Gaussian disturbances w_k with unitary covariance $\mathbb{E}[w_k w_k^\intercal] = I$, where we now suppose that $y_k \in \mathbb{R}$ and $u_k \in \mathbb{R}$. We pick matrices $B_1 = \begin{bmatrix} W^{\frac{1}{2}} & 0 \end{bmatrix}$, $D_{21} = \begin{bmatrix} 0 & V^{\frac{1}{2}} \end{bmatrix}$, leading to $\mathbb{E}[(B_1 w_k)(B_1 w_k)^\intercal] = W$, $\mathbb{E}[(D_{21} w_k)(D_{21} w_k)^\intercal] = V$ and thus recovering the usual framework. Consider a performance input

$$z_k = C_1 x_k + D_{12} u_k$$

and the cost

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}[z_k^\intercal z_k]$$

We pick $C_1 = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \end{bmatrix}^\intercal$ and $D_{12} = \begin{bmatrix} 0 & R^{\frac{1}{2}} \end{bmatrix}^\intercal$, and therefore the running cost is as usual $z_k^\intercal z_k = x_k^\intercal Q x_k + u_k^\intercal R u_k$. The optimal output feedback controller is the LQG controller, consisting of a state estimator

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} + Bu_k \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + L(y_k - C\hat{x}_{k|k-1}) \end{aligned}$$

with a linear feedback law, either

$$u_k = K\hat{x}_{k|k},$$

if non-stricly proper controllers are allowed or

$$u_k = K\hat{x}_{k|k-1},$$

if only strictly proper controllers are allowed. Let us assume the latter. To simplify notation, let us set $\bar{x}_k := \hat{x}_{k|k-1}$. This leads to the simple Luenberger controller structure

$$\begin{aligned} \bar{x}_{k+1} &= A_c \bar{x}_k + B_c y_k \\ u_k &= C_c \bar{x}_k \end{aligned}$$

with

$$A_c = (A + B_2K - \tilde{L}C_2), \quad B_c = \tilde{L}, \quad C_c = K. \tag{1}$$

The expressions for the gains are

$$K = -(R + B^\intercal PB)^{-1}B^\intercal PA$$

with P the unique positive definite solution to the Riccati equation

$$P = A^\intercal PA + Q - A^\intercal PB(R + B^\intercal PB)^{-1}B^\intercal PA$$

and $\tilde{L} = AL$ with

$$L = \Theta C^\intercal(C\Theta C^\intercal + V)^{-1}$$

and with Θ the unique positive semi-definite solution to the Riccati equation

$$\Theta = A\Theta A^\intercal + W - A\Theta C^\intercal(V + C\Theta C^\intercal)^{-1}C\Theta A.$$

The cost is given by

$$\text{trace}(PW) + \text{trace}(X\Theta) \tag{2}$$

with

$$X = A^\intercal PB(R + B^\intercal PB)^{-1}B^\intercal PA.$$

The closed-loop system is

$$\begin{aligned} \xi_{k+1} &= A_{\text{cl}}\xi_k + B_{\text{cl}}w_k \\ z_k &= C_{\text{cl}}\xi_k \end{aligned} \tag{3}$$

with

$$A_{cl} = \begin{bmatrix} A & B_2C_c \\ B_cC_2 & A_c \end{bmatrix} \quad B_{cl} = \begin{bmatrix} B_1 \\ B_cD_{21} \end{bmatrix} \quad C_{cl} = \begin{bmatrix} C_1 & D_{12}C_c \end{bmatrix}.$$

The crucial link between time-domain and frequency domain analysis is made by noticing that the following three costs are identical, for a general controller (A_c, B_c, C_c) that yields the closed loop system stable (A_{cl} is Schur):

(i) $\lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} \mathbb{E}[z_k^\intercal z_k] = \lim_{T\to\infty} \mathbb{E}[z_T^\intercal z_T]$

(ii) $\sum_{i=1}^{n_w}\sum_{k=0}^{\infty}(z_k^i)^\intercal z_k^i$ where z_k^i is the output response of (3) when the input is $w_k^i = \mathsf{e}_i\delta_k$ with e_i a vector of zeros except at position i where it equals 1.

(iii) $\|H\|_{h_2}^2 = \frac{1}{2\pi}\int_0^{2\pi} \text{trace}(H(e^{j\omega})^* H(e^{j\omega}))d\omega$ where $H(z) = C_{cl}(zI - A_{cl})^{-1}B_{cl}$.

While (i) and (ii) are two time-domain costs, (iii) is a frequency domain cost. The connection between (ii) and (iii) is obtained through Parseval's theorem. Thus attempting to find a controller that minimizes (i) or (ii) is equivalent to attempting to find a controller that minimizes the frequency domain cost (iii). These statements are established in Proposition 1 in the appendix.

In order to obtain a connection with known frequency domain functions such as the sensitivity, complementary sensitivity and control sensitivity, we set

$$C_1 = \begin{bmatrix} C_2 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_2 & 0 \end{bmatrix}, \quad \sigma = V^{\frac{1}{2}}, \quad \rho = R^{\frac{1}{2}}$$

which leads to

$$D_{21} = \begin{bmatrix} 0 & \sigma \end{bmatrix}, W = B_2B_2^\top, Q = C_2^\top C_2, D_{12} = \begin{bmatrix} 0 \\ \rho \end{bmatrix}.$$

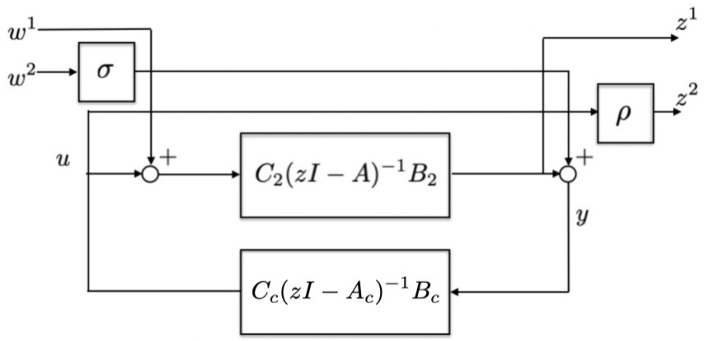

Figure 1: Standard Two-port Frequency-domain Framework

This choice corresponds to the diagram in Figure 1. We define the loop transfer function to be

$$L(z) = P(z)K(z)$$

with $P(z) := C_2(zI - A)^{-1}B_2$ the process transfer function and $K(z) := C_c(zI - A_c)^{-1}B_c$ the controller transfer function. Moreover, we define the sensitivity, complementary sensitivity, the control sensitivity, and the plant sensitivity as $S(z) = \frac{1}{1-L(z)}$, $C(z) = \frac{L(z)}{1-L(z)}$, $N(z) = \frac{K(z)}{1-L(z)}$, $M(z) = \frac{P(z)}{1-L(z)}$ respectively. These are known as the 'gang of four'. Then

$$H(z) = \begin{bmatrix} M(z) & \sigma C(z) \\ \rho C(z) & \rho\sigma N(z) \end{bmatrix}$$

and

$$\|H\|_{h_2}^2 = \|M(z)\|_{h_2}^2 + V\|C(z)\|_{h_2}^2 + R\|C(z)\|_{h_2}^2 + RV\|N(z)\|_{h_2}^2$$

corresponding to the weighted sum of the areas below three graphs, with $\omega \in [0, 2\pi)$

$$\left(\omega, \left|M\left(e^{j\omega}\right)\right|^2\right), \left(\omega, \left|C\left(e^{j\omega}\right)\right|^2\right), \left(\omega, \left|N\left(e^{j\omega}\right)\right|^2\right)$$

which in turn correspond to the squared norm of the complementary sensitivity, control sensitivity and process sensitivity.

Assignment 3.1 Program a matlab function

```
[h2,h2M,h2C,h2N] = freqdecomposeLQGcost(A,B2,C2,rho,sigma);
```

which provides `h2`$= \|H\|_{h_2}^2$, `h2M`$= \|M(z)\|_{h_2}^2$, `h2C`$= \|C(z)\|_{h_2}^2$, `h2N`$=\|N(z)\|_{h_2}^2$ given A, B_2, C_2, ρ and σ.

The framework is insufficient to shape the functions of interest. First it does not consider the complementary sensitivity; second it penalizes equality the frequencies of the functions of interest. In order to shape the loop transfer functions of interest such as $C(z)$, $N(z)$ and $M(z)$ one can add frequency shaping filters as depicted in Figure 2. In Figure 2 an extra output $z3 = y$ is added to take sensitivity $S(z)$ into account. By penalizing now $\lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} \mathbb{E}[z_k^\intercal z_k] = \lim_{T\to\infty} \mathbb{E}[z_T^\intercal z_T]$, with $z_k = [z_{1,k}\ z_{2,k}\ z_{3,k}]^\intercal$ and including the shaping filter states in the information available for feedback a new LQG problem is formulated leading to a new controller (A_C, B_C, C_c), in general different from the one considered previously. The transfer function from

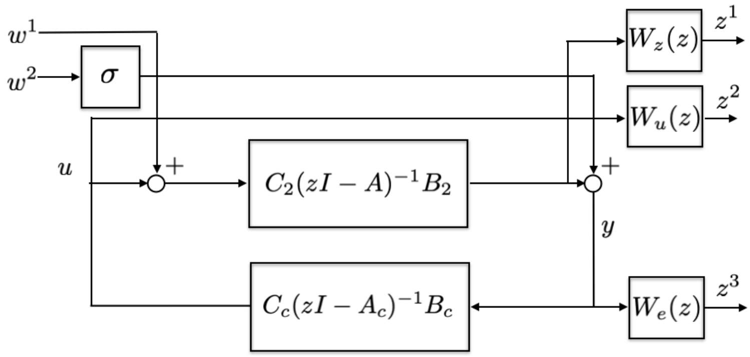

Figure 2: Standard Two-port Frequency-domain Framework

$w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^\intercal$ to $z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^\intercal$ is

$$H(z) = \begin{bmatrix} W_z(z)M(z) & \sigma W_z(z)C(z) \\ W_u(z)C(z) & \sigma W_u(z)N(z) \\ W_e(z)M(z) & \sigma W_e(z)S(z) \end{bmatrix}$$

and the following decomposition holds

$$\begin{aligned} \|H\|_{h_2}^2 &= \|W_z(z)M(z)\|_{h_2}^2 + V\,\|W_z(z)C(z)\|_{h_2}^2 + \|W_u(z)C(z)\|_{h_2}^2 \\ &\quad + V\,\|W_u(z)N(z)\|_{h_2}^2 + \|W_e(z)M(z)\|_{h_2}^2 + V\,\|W_e(z)S(z)\|_{h_2}^2 \end{aligned}$$

By properly choosing the frequency shaping filters $W_z(z)$, $W_u(z)$ and $W_e(z)$ one can obtain the desired shapes for the transfer functions of interest $(C(z), S(z), ...)$. For simplicity, assume that $W_u(z) = \rho$, $W_z(z) = 1$, and $W_e(z) = C_e(zI - A_e)^{-1}B_e$ for given matrices A_e, B_e, C_e. By choosing $W_e(z)$ to be a low pass filter with cut off frequency ω_e we can shape the sensitivity function so that is small for frequencies below ω_e.

Assignment 3.2 Program a matlab function

```
[h2,h2M,h2C,h2N,h2S,h2WM,h2WS] =
freqdecomposeshapedLQGcost(A,B2,C2,rho,sigma,Ae,Be,Ce);
```

which provides `h2`= $\|H\|_{h_2}^2$, `h2M`= $\|M(z)\|_{h_2}^2$, `h2C`= $\|C(z)\|_{h_2}^2$, `h2N`=$\|N(z)\|_{h_2}^2$, `h2S`=$\|S(z)\|_{h_2}^2$, `h2WS`=$\|W_e(z)M(z)\|_{h_2}^2$, `h2WS`=$\|W_e(z)S(z)\|_{h_2}^2$ given A, B_2, C_2, ρ, σ, A_e, B_e, C_e, and D_e.

Proposition 1 *The three costs (i), (ii), and (iii) are identical for any stabilizing controller (A_c, B_c, C_c).*

Proof.

We start by showing how to compute cost (i). Let $\Psi_k := \mathbb{E}[\xi_k \xi_k^\intercal]$. Then

$$\Psi_{k+1} = \mathbb{E}[(A_{\mathrm{cl}}\xi_k + B_{\mathrm{cl}}w_k)(A_{\mathrm{cl}}\xi_k + B_{\mathrm{cl}}w_k)^\intercal] = A_{\mathrm{cl}}\Psi_k A_{\mathrm{cl}}^\intercal + B_{\mathrm{cl}}B_{\mathrm{cl}}^\intercal$$

Since A_{cl} is Schur stable this iteration will converge to the solution of the following Lyapunov equation

$$\bar{\Psi} = A_{\mathrm{cl}}\bar{\Psi}A_{\mathrm{cl}}^\intercal + B_{\mathrm{cl}}B_{\mathrm{cl}}^\intercal$$

which is given by

$$\bar{\Psi} = \sum_{k=0}^{\infty} A_{\mathrm{cl}}B_{\mathrm{cl}}B_{\mathrm{cl}}^\intercal (A_{\mathrm{cl}}^\intercal)^k.$$

Thus,

$$\lim_{T\to\infty} \mathbb{E}[z_T^\intercal z_T] = \mathrm{trace}(C_{\mathrm{cl}}\Psi C_{\mathrm{cl}}^\intercal). \tag{4}$$

Let us now see that an identical expression is obtained for cost (ii). We have that

$$z_k^i = \begin{cases} C_{\mathrm{cl}}A_{\mathrm{cl}}^{k-1}B_{\mathrm{cl}}\mathsf{e}_\mathsf{i} \text{ if } k \geq 1 \\ 0 \text{ otherwise }, \end{cases}$$

so that

$$\sum_{i=1}^{n_w}\sum_{k=0}^{\infty}(z_k^i)^\intercal z_k^i = \mathrm{trace}(C_{\mathrm{cl}}\Psi C_{\mathrm{cl}}^\intercal).$$

The equivalence between cost (ii) and cost (iii) results from the Parseval's relation. For each scalar impulsive input $w_k^i = \mathsf{e}_i\delta_k$ we can define the scalar output z_k^{ij} and parseval's relation ensures that

$$\sum_{k=0}^{\infty}(z_k^{ij})^2 = \frac{1}{2\pi}\int_0^{2\pi} H_{ji}(e^{j\omega})^* H_{ji}(e^{j\omega}))d\omega$$

where $H_{ji}(z) = \mathsf{e}_j^\intercal H(z)\mathsf{e}_i$ is the transfer function from input i to output j. Thus,

$$\begin{aligned}\sum_{i=1}^{n_w}\sum_{k=0}^{\infty}(z_k^i)^\intercal z_k^i &= \sum_{j=1}^{n_z}\sum_{i=1}^{n_w}\sum_{k=0}^{\infty}(z_k^{ij})^2 = \sum_{j=1}^{n_z}\sum_{i=1}^{n_w}\frac{1}{2\pi}\int_0^{2\pi}(H_{ji}(e^{j\omega})^* H_{ji}(e^{j\omega}))d\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{trace}(H(e^{j\omega})^* H(e^{j\omega}))d\omega\end{aligned}$$

concluding the proof. ■