SLT-4

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4.1

$s(\theta) = H\theta$

$\theta = \begin{bmatrix} A \\ B \end{bmatrix}$, $s_n = A + B(-1)^n$

$n = 0, 1, 2, \dots, N-1$

$H = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & -1 \end{bmatrix}$ this is wrong

$S(\theta) = H\theta$

$S(\theta) = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{N-1} \end{bmatrix}$

$J(\theta) = \sum_{n=0}^{N-1} e^2(\theta) \rightarrow J(\theta) = \sum_{n=0}^{N-1} (x_n - s_n(\theta))^2 \quad (1)$

$\hat{\theta}_{LS} = \underset{\theta}{\arg\min}\, J(\theta) \quad (2)$

$J(\theta) = \sum_{n=0}^{N-1} (x_n - H\theta)^T . (x_n - H\theta)$

$J(\theta) = \sum_{n=0}^{N-1} (x_n^T - H^T\theta^T) . (x_n - H\theta)$

$J(\theta) = \sum_{n=0}^{N-1} x_n^T x_n - x_n^T . H\theta - H^T\theta^T x_n - H^T\theta^T . H\theta$

H is symmetric

$J(\theta) = \sum_{n=0}^{N-1} x_n^T . x_n - x_n^T . H\theta - H^T\theta^T x_n - \theta^T . H^T . H . \theta$

$\frac{\partial}{\partial \theta} J(\theta) = -x_n^T . H - H^T x_n - 2H^T H\theta \overset{!}{=} 0$

$-2x_n^T H - 2\hat{\theta} H^T H \overset{!}{=} 0$

$\hat{\theta}_{LS} = (H^T H)^{-1} H^T . X$

$\hat{\theta}_{LS} = \left(\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & 1 & -1 & \dots & -1 \end{bmatrix}}_{2\times N} . \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & -1 \end{bmatrix}}_{N\times 2} \right)^{-1} . \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & 1 & -1 & \dots & -1 \end{bmatrix}}_{2\times N} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}}_{N\times 1}$

$= \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \left(\begin{matrix} x_1 \\ \sum_{n=0}^{N-1} x_n (-1)^n \end{matrix} \right)$

$\hat{\theta}_{LS} = \begin{bmatrix} \sum_{n=0}^{N-1} x_n (-1)^n \\ x_1 - \sum_{n=0}^{N-1} x_n (-1)^n \end{bmatrix} \rightarrow \begin{matrix} \hat{A} = \sum_{n=0}^{N-1} x_n (-1)^n \\ \hat{B} = x_1 - \sum_{n=0}^{N-1} x_n (-1)^n \end{matrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$ → the observation matrix

$H^T H = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$

$= I . N$

$(H^T H)^{-1} = \frac{1}{N} I$

$\hat{\theta}_{LS} = \frac{1}{N} H^T . X$

$\hat{\theta}_{LS} = \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix} = \begin{bmatrix} \frac{1}{N} \sum_{n=0}^{N-1} x_n \\ \frac{1}{N} \sum (-1)^n x_n \end{bmatrix}$

4.2

a) $p(\theta|x) = \begin{cases} \exp(-(\theta - x)), & \theta \geq x \\ 0, & \theta < x \end{cases}$

$p(\theta|x) = \frac{p(x|\theta) p(\theta)}{p(x)}$

$\hat{\theta}_{MMSE} = \int \theta\, p(\theta|x)\, d\theta$,

$\hat{\theta}_{MMSE} = \int \theta . e^{(-\theta - x))} . d\theta$

$= -(\theta + 1) . e^{x-\theta}$

$= e^x . \left(-(\theta+1) e^{-\theta}\right) \Big|_{-\infty}^{+\infty}$

$\hat{\theta}_{MMSE} = (x+1)$

b) $\hat{\theta}_{MAP} = \underset{\theta}{\arg\max}\, p(\theta|x)$

$\ln(p(\theta|x)) = -(\theta - x)$

$\frac{\partial \ln(p(\theta|x))}{\partial \theta} = -1$

$\theta_{MAP} = x$

4.3

$Y = X + N$

$p(x|y) = \frac{p(y|x) p(x)}{p(y)}$

$p(x) = \frac{1}{\sqrt{2\pi \sigma_x^2}} \exp\left(-\frac{1}{2\sigma_x^2} (x[n] - \mu_x)^2\right)$

$p(y|x) = \frac{1}{(2\pi \sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma} (y[n] - x[n])^2\right)$

$\frac{p(y|x) . p(x)}{p(y)} = \frac{p(y|x) . p(x)}{\int p(y|x) . p(x) . dx}$

$P_r[X = 1|Y] = \frac{P_r[Y|X=1] P_r[X=1]}{P_r[Y]}$

$P_r[Y|X=1] = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y-1)^2}{2\sigma^2}\right)$

$P[X] = \begin{cases} \frac{1}{2} & X = 1 \\ \frac{1}{2} & X = -1 \end{cases}$

$P[Y] = \sum P_r[Y|X] P[X]$

$= P_r[Y|X=1] P_r[X=1] + P_r[Y|X=-1] P(X=-1)$

$P_r[X=1|Y] = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} . \exp\left(-\frac{(y-1)}{2\sigma^2}\right) \frac{1}{2}}{\frac{1}{2} . \frac{1}{\sqrt{2\pi\sigma^2}} \left(e^{\frac{-(y-1)^2}{2\sigma^2}} + e^{\frac{-(y+1)^2}{2\sigma^2}} \right)}$

$P_r[X=-1|Y] = \frac{e^{-\frac{(y+1)^2}{2\sigma^2}}}{\left(e^{-\frac{(y-1)^2}{2\sigma^2}} + e^{-\frac{(y+1)^2}{2\sigma^2}} \right)}$

b) $\hat{X}_{MMSE} = \int X P_r[X = x|Y]\, dx$

$= \sum x P(X = x|Y)$

$= \frac{\exp\left(-\frac{(y-1)^2}{2\sigma}\right) - \exp\left(-\frac{(y+1)^2}{2\sigma^2}\right)}{\exp\left(\frac{-(y-1)^2}{2\sigma}\right) + \exp\left(-\frac{(y+1)^2}{2\sigma}\right)}$

$(y-1)^2 = y^2 - 2y + 1$

$e^{-\frac{(y-1)^2}{2\sigma^2}} - e^{-\frac{(y+1)^2}{2\sigma^2}} = e^{-\frac{(y^2+1)}{2\sigma^2}} e^{\frac{(2y)}{2\sigma^2}} - e^{-\frac{(y^2+1)}{2\sigma^2}} . e^{-\frac{(2y)}{2\sigma^2}}$

$+ \quad$ for denominator $\quad +$

$= \frac{\exp\left(\frac{y}{\sigma^2}\right) - \exp\left(-\frac{y}{\sigma^2}\right)}{\exp\left(\frac{y}{\sigma^2}\right) + \exp\left(-\frac{y}{\sigma^2}\right)}$

$= \tanh\left(\frac{y}{\sigma^2}\right)$

c) $\hat{X}_{MAP} = ?$

$X_{MAP} = \underset{x \in \{-1, 1\}}{\arg\max}\, p(x|y)$

ratio $\frac{P[X = 1|Y = y]}{P[X = -1|Y = y]} = \frac{\exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)}{\exp\left(-\frac{(y+1)^2}{2\sigma^2}\right)}$

$= \exp\left(\frac{(y+1)^2 - (y-1)^2}{2\sigma^2}\right)$

$= \exp\left(\frac{4y}{2\sigma^2}\right)$

$\hat{X}_{MAP} = \begin{cases} 1 & y > y_{th} \\ -1 & y < y_{th} \end{cases}$ with $y_{th} = 0$