

# Statistical signal processing 5CTA0

Estimation theory - linear signal models

# Linear signal model

- Many signals can be modeled by linear signal models
- For this class of signal models, efficient estimators exist
- Signal model:

$$\mathbf{s}(\boldsymbol{\theta}) = \mathbf{H}\boldsymbol{\theta}$$

 $\mathbf{H}$  is the so-called observation matrix

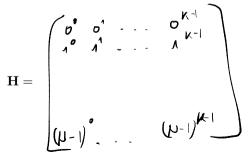


#### Example I

■ Polynomial of degree K-1,

$$s_n(\theta) = \theta_0 n^0 + \theta_1 n^1 + \dots + \theta_{K-1} n^{K-1}$$
 for  $0 \le n \le N-1$ 

Observation matrix:





# Efficient estimator for linear models and additive Gaussian noise

Observation model:

 $\mathbf{x} = \mathbf{G}(\mathbf{0}) / \mathbf{x} = \mathbf{H}\mathbf{\theta} + \mathbf{w}$ 

Joint PDF:

$$p(\mathbf{x}, \boldsymbol{\theta}) = \frac{1}{|2\pi \mathbf{C}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^T \mathbf{C}^{-1}(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})\right)$$

Linear signal models

Efficient estimator:

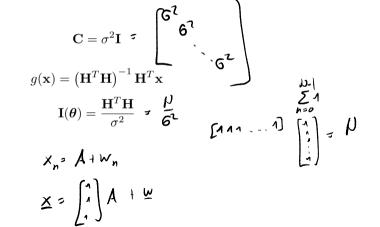
$$g(\mathbf{x}) = (\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{C}^{-1} \mathbf{x} = \mathbf{G}^T (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \mathbf{x}$$

$$\mathbf{G}^T \mathbf{G}^T \mathbf{$$



# Special case: white Gaussian noise

- Covariance matrix:
- Efficient estimator



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- Signal composed of P sines and cosines with known frequency but unknown amplitudes
- Signal is embedded in white Gaussian noise
- Observation model

$$x_{n} = \sum_{p=0}^{P-1} a_{p} \cos \left(2\pi \frac{k_{p}}{N}n\right) + \sum_{p=0}^{P-1} b_{p} \sin \left(2\pi \frac{k_{p}}{N}n\right) + w_{n}$$

Linear signal models



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$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$$

Observation matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{c}_{k_0} & \dots & \mathbf{c}_{k_{(P-1)}} & \mathbf{s}_{k_0} & \dots & \mathbf{s}_{k_{(P-1)}} \end{bmatrix}$$

$$\mathbf{c}_{k_p} = \begin{bmatrix} \cos\left(2\pi\frac{k_p}{N}0\right) & \cos\left(2\pi\frac{k_p}{N}1\right) & \dots & \cos\left(2\pi\frac{k_p}{N}(N-1)\right) \end{bmatrix}^T$$

$$\mathbf{s}_{k_p} = \begin{bmatrix} \sin\left(2\pi\frac{k_p}{N}0\right) & \sin\left(2\pi\frac{k_p}{N}1\right) & \dots & \sin\left(2\pi\frac{k_p}{N}(N-1)\right) \end{bmatrix}^T$$

■ Parameter vector

$$\theta = \begin{bmatrix} a_0 & \dots & a_{(P-1)} & b_0 \dots b_{(P-1)} \end{bmatrix}^{\mathsf{T}}$$

$$0 \le k_2 \le \mathcal{V}^{-1}$$

$$\mathcal{V}^{\mathsf{T}} \mathcal{V} = \frac{\mathcal{V}}{2} \mathcal{I}$$

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Efficient estimator

$$\hat{\boldsymbol{\theta}} = \left( \underbrace{\mathbf{H}^T \mathbf{H}}_{\mathbf{Z}} \right)^{-1} \mathbf{H}^T \mathbf{x} = \frac{2}{N} \begin{bmatrix} \mathbf{c}_{k_0}^T \\ \vdots \\ \mathbf{c}_{k_{(P-1)}}^T \\ \mathbf{s}_{k_0}^T \\ \vdots \\ \mathbf{s}_{k_{(P-1)}}^T . \end{bmatrix} \mathbf{x}$$

$$\hat{a}_p = \frac{2}{N} \sum_{n=0}^{N-1} x_n \cos\left(2\pi \frac{k_p}{N}n\right) \qquad \hat{b}_p = \frac{2}{N} \sum_{n=0}^{N-1} x_n \sin\left(2\pi \frac{k_p}{N}n\right)$$

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- Signal composed of P cosines with known frequency but unknown amplitude and phase
- Observation model

$$x_n = \sum_{p=0}^{P-1} c_p \cos\left(2\pi \frac{k_p}{N}n - \varphi_p\right) + w_n$$

■ Model is nonlinear in the unknown phase



■ Trigonometric identities

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha - \beta) + \cos(\alpha + \beta)\right)$$

$$\pm \sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta)\right)$$

$$\cos(\alpha)\cos(\beta) + \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

Transformed model

$$x_n = \sum_{p=0}^{r-1} \underbrace{c_p \cos(\varphi_p)}_{a_p} \cos\left(2\pi \frac{k_p}{N}n\right) + \underbrace{c_p \sin(\varphi_p)}_{b_p} \sin\left(2\pi \frac{k_p}{N}n\right) + w_n$$
$$c_p = \sqrt{a_p^2 + b_p^2} \qquad \varphi_p = \arctan(b_p, a_p)$$