Statistical signal processing
5CTA0

Estimation theory - Least squares estimation

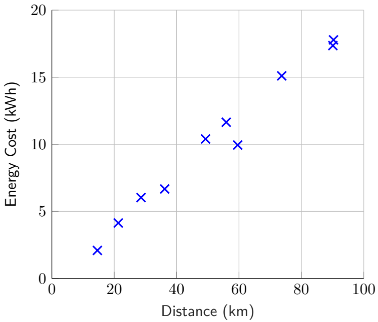
Least squares estimation

- No probabilistic assumption for the observation needed
  - \+ Applicable when statistical characterization of the observations is unknown
  - − No statement about optimality
- Minimize discrepancy between observation and assumed signal model
- Cost function :

$$J(\boldsymbol{\theta}) = \sum_{n=0}^{N-1} (e_n(\boldsymbol{\theta}))^2$$
$$= \sum_{n=0}^{N-1} (x_n - s_n(\boldsymbol{\theta}))^2$$

Example

- Estimation of energy consumption per km of an electrical vehicle

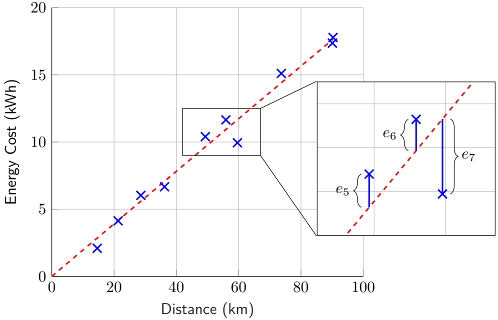


- Signal model:

$$s_n(\theta) = \theta D_n$$

Example

- Estimation of energy consumption per km of an electrical vehicle



- Signal model:

$$s_n(\theta) = \theta D_n$$

Example

$$J(\theta) = \sum_{n=0}^{N-1} e_n^2(\theta)$$

$$\stackrel{!}{=} \sum_{n=0}^{N-1} (x_n - \theta D_n)^2$$

$$\frac{d}{d\theta} J(\theta) = \sum_{n=0}^{N-1} 2(x_n - \theta D_n)(-D_n) \stackrel{!}{=} 0$$

$$\sum_{n=0}^{N-1} x_n D_n = \theta \sum_{n=0}^{N-1} D_n^2$$

$$\hat{\theta}_{LS} = \frac{\sum_{n=0}^{N-1} x_n D_n}{\sum_{n=0}^{N-1} D_n^2}$$

Linear least squares estimator

- Signal model:

$$\mathbf{s}(\boldsymbol{\theta}) = \mathbf{H}\boldsymbol{\theta}$$

- Observation model:

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}.$$

$$J(\theta) = \| x - H\theta \|^2$$

$$= (x - H\theta)^T (x - H\theta)$$

$$= (x^T - \theta^T H^T)(x - H\theta)$$

$$= (x^T x - \theta^T H^T x - x^T H\theta + \theta^T H^T H\theta)$$

$$\frac{\partial}{\partial\theta} J(\theta) = -\cancel{2} H^T x + \cancel{2} H^T H\theta \stackrel{!}{=} 0 \Rightarrow H^T H\theta = H^T x$$

$$\hat{\theta}_{LS} = (H^T H)^{-1} H^T x$$

Geometric interpretation

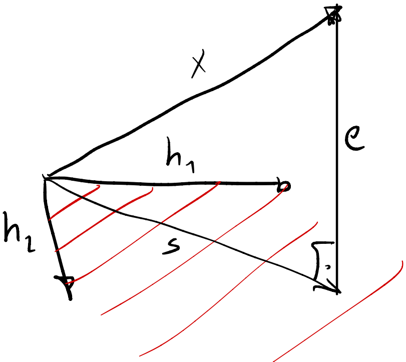
$s(\theta) = H\theta = \sum_{k=1}^{K} \theta_k h_k$

$H \in \mathbb{R}^{N \times K}$

$N > K$

Ex. $N=3, K=2$

$J(\theta) = \|x - H\theta\|^2$



Geometric interpretation

$$s = Px = H(H^T H)^{-1} H^T x \qquad \hat{\theta}_{LS} = (H^T H)^{-1} H^T x$$

$$s = H\theta$$

- Projection matrix:

$$\mathbf{P} = \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T,$$

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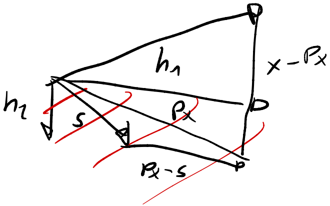
$$\begin{aligned} J(\theta) = \|\mathbf{x} - \mathbf{s}\|^2 &= \|\mathbf{x} - \mathbf{Px} + \mathbf{Px} - \mathbf{s}\|^2 \\ &= \|\mathbf{x} - \mathbf{Px}\|^2 + \|\mathbf{Px} - \mathbf{s}\|^2 - \overbrace{2(\mathbf{x} - \mathbf{Px})^T(\mathbf{Px} - \mathbf{s})}^{0}. \end{aligned}$$



Weighted least squares estimator

- Cost function:

$$J(\boldsymbol{\theta}) = (\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^T \mathbf{W} (\mathbf{x} - \mathbf{H}\boldsymbol{\theta})$$

- Diagonal matrix:

$$J(\boldsymbol{\theta}) = \sum_{n=0}^{N-1} [\mathbf{W}]_{n,n} (x_n - s_n(\boldsymbol{\theta}))^2$$

- Weighted least squares estimator:

$$\hat{\boldsymbol{\theta}}_{\mathsf{WLS}} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{W} \mathbf{x}$$

$$\hat{\theta} = (H^T C_{*}^{-1} H)^{-1} H^T C^{-1} x$$

Best linear unbiased estimator

$\hat{\Theta} = Ax$

- Suppose we have knowledge about the mean $\mathrm{E}[\mathbf{x}] = \mathbf{H}\boldsymbol{\theta}$ and covariance matrix $\mathbf{C}_\mathbf{x}$ of the observation $\mathbf{x}$
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$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{\theta}}_{\mathsf{LS}}] &= \mathrm{E}[(\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{W}\mathbf{x}] \\
&= (\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{W}\,\mathrm{E}[\mathbf{x}] \\
&= (\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{W}\mathbf{H}\boldsymbol{\theta} \\
&= \boldsymbol{\theta}
\end{aligned}
$$

- The weighted least square estimator with $\mathbf{W} = \mathbf{C}_\mathbf{x}^{-1}$ is the best liner unbiased estimator (BLUE)