Statistical signal processing
5CTA0

Estimation theory - linear signal models

Linear signal model

- Many signals can be modeled by linear signal models
- For this class of signal models, efficient estimators exist
- Signal model:

$$\mathbf{s}(\boldsymbol{\theta}) = \mathbf{H}\boldsymbol{\theta}$$

$\mathbf{H}$ is the so-called observation matrix

$$s_n(\theta) = \sum_{k=0}^{K-1} [H]_{n,k} \theta_k$$

Example I

- Polynomial of degree $K-1$,

$$s_n(\boldsymbol{\theta}) = \theta_0 n^0 + \theta_1 n^1 + \cdots + \theta_{K-1} n^{K-1} \quad \text{for } 0 \leq n \leq N-1$$

- Observation matrix:

$$\mathbf{H} = \begin{bmatrix} 0^0 & 0^1 & \cdots & 0^{K-1} \\ 1^0 & 1^1 & \cdots & 1^{K-1} \\ & & & \\ (N-1)^0 & \cdots & & (N-1)^{K-1} \end{bmatrix}$$

Efficient estimator for linear models and additive Gaussian noise

- Observation model:

$$\mathbf{x} = \overset{s(\theta)}{\mathbf{H}\boldsymbol{\theta}} + \overset{\mathcal{N}(\underline{0}, C)}{\mathbf{w}}$$

- Joint PDF:

$$p(\mathbf{x}, \boldsymbol{\theta}) = \frac{1}{|2\pi\mathbf{C}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x} - \overset{s(\theta)}{\mathbf{H}\boldsymbol{\theta}})^T\mathbf{C}^{-1}(\mathbf{x} - \overset{s(\theta)}{\mathbf{H}\boldsymbol{\theta}})\right)$$

- Efficient estimator:

$$g(\mathbf{x}) = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x} = \cancel{\sigma^2}(H^T I H)^{-1} H^T \frac{1}{\cancel{\sigma^2}} x$$

$$= (H^T H)^{-1} H^T x$$

$$\begin{bmatrix} \frac{1}{\sigma^2} & & & \\ & \frac{1}{\sigma^2} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma^2} \end{bmatrix} = \frac{1}{\sigma^2} I$$

Special case: white Gaussian noise

- Covariance matrix:

$$\mathbf{C} = \sigma^2 \mathbf{I} = \begin{bmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \ddots & \\ & & & \sigma^2 \end{bmatrix}$$

- Efficient estimator

$$g(\mathbf{x}) = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x}$$

$$\mathbf{I}(\boldsymbol{\theta}) = \frac{\mathbf{H}^T\mathbf{H}}{\sigma^2} = \frac{N}{\sigma^2}$$

$$[1\ 1\ 1\ \dots\ 1] \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \sum_{n=0}^{N-1} 1 = N$$

$$x_n = A + w_n$$

$$\underline{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} A + \underline{w}$$

Example II - Fourier analysis

- Signal composed of P sines and cosines with known frequency but unknown amplitudes
- Signal is embedded in white Gaussian noise
- Observation model

$$x_n = \sum_{p=0}^{P-1} a_p \cos\left(2\pi \frac{k_p}{N} n\right) + \sum_{p=0}^{P-1} b_p \sin\left(2\pi \frac{k_p}{N} n\right) + w_n$$

Example II - Fourier analysis

- $$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}.$$

- Observation matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{c}_{k_0} & \dots & \mathbf{c}_{k_{(P-1)}} & \mathbf{s}_{k_0} & \dots & \mathbf{s}_{k_{(P-1)}} \end{bmatrix}$$

$$\mathbf{c}_{k_p} = \left[\cos\left(2\pi\frac{k_p}{N}0\right) \quad \cos\left(2\pi\frac{k_p}{N}1\right) \quad \dots \quad \cos\left(2\pi\frac{k_p}{N}(N-1)\right)\right]^T$$

$$\mathbf{s}_{k_p} = \left[\sin\left(2\pi\frac{k_p}{N}0\right) \quad \sin\left(2\pi\frac{k_p}{N}1\right) \quad \dots \quad \sin\left(2\pi\frac{k_p}{N}(N-1)\right)\right]^T$$

- Parameter vector

$$\boldsymbol{\theta} = \begin{bmatrix} a_0 & \dots & a_{(P-1)} & b_0 \dots b_{(P-1)} \end{bmatrix}^T$$

$0 \leq k_p \leq N-1$

$H^T H = \frac{N}{2} I$

Example II - Fourier analysis

- Efficient estimator

$$\hat{\boldsymbol{\theta}} = \underbrace{\left(\mathbf{H}^T\mathbf{H}\right)}_{\frac{N}{2}}^{-1}\mathbf{H}^T\mathbf{x} = \frac{2}{N}\begin{bmatrix} \mathbf{c}_{k_0}^T \\ \vdots \\ \mathbf{c}_{k_{(P-1)}}^T \\ \mathbf{s}_{k_0}^T \\ \vdots \\ \mathbf{s}_{k_{(P-1)}}^T . \end{bmatrix}\mathbf{x}$$

$$\hat{a}_p = \frac{2}{N}\sum_{n=0}^{N-1} x_n \cos\left(2\pi\frac{k_p}{N}n\right) \qquad \hat{b}_p = \frac{2}{N}\sum_{n=0}^{N-1} x_n \sin\left(2\pi\frac{k_p}{N}n\right)$$

Example II - Fourier analysis

- Signal composed of P cosines with known frequency but unknown amplitude and phase
- Observation model

$$x_n = \sum_{p=0}^{P-1} c_p \cos\left(2\pi \frac{k_p}{N} n - \varphi_p\right) + w_n$$

- Model is nonlinear in the unknown phase

Example II - Fourier analysis

- Trigonometric identities

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$$

$$+ \quad \sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha-\beta)$$

- Transformed model

$$x_n = \sum_{p=0}^{P-1} \underbrace{c_p\cos(\varphi_p)}_{a_p}\cos\left(2\pi\frac{k_p}{N}n\right) + \underbrace{c_p\sin(\varphi_p)}_{b_p}\sin\left(2\pi\frac{k_p}{N}n\right) + w_n$$

$$c_p = \sqrt{a_p^2 + b_p^2} \qquad \varphi_p = \arctan(b_p, a_p)$$