SLT-2

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Exercise 2.4

Given random variables X and $Y = AX + b$

$$A = \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$$

$$\mu_X = \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix} \quad R_{XX} = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \quad C_X = \begin{bmatrix} 1/12 & 1/24 \\ 1/24 & 1/12 \end{bmatrix}$$

Compute μ_Y, R_Y, C_Y, R_{XY}, C_{XY}, $\rho_{X,Y}$, $\rho_{Y,X}$.

$$\mu_Y = E[Y] = E[AX + b] = A\,E[X] + b = A\mu_X + b$$

$$\mu_Y = \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$$

$$\mu_Y = \begin{bmatrix} 1/2 \\ 3 \\ 3 \end{bmatrix}$$

$$R_Y = E[YY^T] = E[(AX+b)(AX+b)^T] = E[(AX+b)(X^TA^T + b^T)]$$

$$R_Y = E[AXX^TA^T + AXb^T + bX^TA^T + bb^T]$$

$$= AE[XX^T]A^T + AE[X]b^T + bE[X^T]A^T + bb^T$$

$$\rightarrow A R_X A^T + A\mu_X b^T + b\mu_X^T A^T + bb^T$$

$$= \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \end{bmatrix}$$

$$R_Y = \begin{bmatrix} 1/4 & 11/4 & 11/2 \\ 11/4 & 5/2 & 19/6 \\ 11/2 & 19/6 & 17/6 \end{bmatrix}$$

$$C_Y = E[(Y-\mu_Y)(Y-\mu_Y)^T]$$

$$= E[(AX + b - A\mu_X - b)(AX + b - A\mu_X - b)^T]$$

$$= E[A(X - \mu_X)(A(X - \mu_X))^T]$$

$$= E[A(X-\mu_X)(X^T - \mu_X^T)A^T]$$

$$= A\,E[(X-\mu_X)(X^T-\mu_X^T)]\,A^T$$

$$A\,C_X\,A^T$$

$$= \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1/12 & 1/24 \\ 1/24 & 1/12 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$C_Y = \begin{bmatrix} 1/12 & 5/12 & 11/6 \\ 5/12 & 5/2 & 19/6 \\ 11/6 & 19/6 & 3/2 \end{bmatrix}$$

$$R_{XY} = E[XY^T] = E[X(AX+b)^T] = E[X(X^TA^T) + Xb^T]$$

$$R_{XY} = E[XX^T]A^T + E[X]b^T$$

$$R_{XY} = R_X A^T + \mu_X b^T$$

$$R_{YX} = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \end{bmatrix}$$

$$R_{YX} = \begin{bmatrix} 1/4 & 11/6 & 11/3 \\ 1/2 & 5/3 & 10/3 \end{bmatrix}$$

$$C_{XY} = E[(X-\mu_X)(Y-\mu_Y)^T]$$

$$= E[(X-\mu_X)(AX+b-A\mu_X-b)^T]$$

$$= E[(X-\mu_X)(A(X-\mu_X))^T]$$

$$= E[(X-\mu_X)(X-\mu_X)^TA^T]$$

$$= E[(X-\mu_X)(X-\mu_X)^T]A^T$$

$$C_{XY} = C_X A^T$$

$$C_{XY} = \begin{bmatrix} 1/12 & 1/24 \\ 1/24 & 1/12 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$C_{XY} = \begin{bmatrix} 1/12 & 5/12 & 1/3 \\ 1/24 & 1/2 & 5/12 \end{bmatrix}$$

$$S_{X,Y,S} = \frac{Cov[X,Y]}{\sqrt{Var[X]\cdot Var[Y]}} = \frac{E[(Y-\mu_Y)(Y_1-\mu_Y)]}{\sqrt{1/6 \cdot 3/2}} = \frac{1/6}{\sqrt{\frac{3}{12}}}$$

$$S_{X,Y,S} = 2/3$$

$$S_{X,Y,1} = \frac{Cov[X,Y_1]}{\sqrt{Var[X]\cdot Var[Y]}} = \frac{1/6b}{\sqrt{1/6 \cdot 3/18}}$$

$$= \frac{4b\,1/6}{\sqrt{b}} = 1/2$$

$$S_{X,Y_1} = 1/2$$

Exercise 2.3:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad C_Y = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \qquad C_X = QC_YQ^T$$

$$E[X] = \mu_X = 0 \qquad \forall\, N(0, \sigma_2)$$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad Y = QX \qquad \sigma_1^2 = \sigma_2^2$$

a)

$$C_Y = QC_XQ^T$$

$$C_Y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$C_Y = \begin{bmatrix} \cos^2\theta\cdot\sigma_1^2 + \sin^2\theta\cdot\sigma_2^2 & \sin\theta\cdot\cos\theta\cdot\sigma_1^2 - \sin\theta\cdot\cos\theta\cdot\sigma_2^2 \\ \sin\theta\cdot\cos\theta\cdot\sigma_1^2 - \sin\theta\cdot\cos\theta\cdot\sigma_2^2 & \sin^2\theta\cdot\sigma_1^2 + \cos^2\theta\cdot\sigma_2^2 \end{bmatrix}$$

we know that $\sigma_1^2 = \sigma_2^2$

$$\Rightarrow \boxed{C_Y = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}} \rightarrow C_{Y_1Y_2} = 0$$

$$\rightarrow \rho_{Y_1Y_2} = 0$$

b)

$$C_Y = \begin{bmatrix} \cos^2\theta\cdot\sigma_1^2 + \sin^2\theta\cdot\sigma_2^2 & \sin\theta\cdot\cos\theta\cdot\sigma_1^2 - \sin\theta\cdot\cos\theta\cdot\sigma_2^2 \\ \sin\theta\cdot\cos\theta\cdot\sigma_1^2 - \sin\theta\cdot\cos\theta\cdot\sigma_2^2 & \sin^2\theta\cdot\sigma_1^2 + \cos^2\theta\cdot\sigma_2^2 \end{bmatrix}$$

Therefore

$$\sin\theta\cdot\cos\theta\cdot\sigma_1^2 - \sin\theta\cdot\cos\theta\cdot\sigma_2^2 = 0$$

$$\sin\theta\cdot\cos\theta\,(\sigma_1^2 - \sigma_2^2) = 0$$

$$\boxed{\text{we know } \sigma_1 \geq \sigma_2}$$

$$\theta = 0,\ \pi/2,\ \pi,\ 3\pi/2$$

$$\theta = 2n\pi \text{ or } \frac{k\pi}{2}$$

and $n = 0, 1, 2, \ldots$

$k = 0, 1, 2, \ldots$

2.2

a) $X(t) = A$, $A \sim U[0,1]$

$$E[X(t)] = \frac{(a+b)}{2} = 1/2 \qquad \rightarrow \sqrt{\frac{1}{12}(1-0)^2} = \frac{1}{\sqrt{12}}$$

$$Var[X(t)] = \frac{(b-a+1)^2 - 1}{12} = \frac{2^2-1}{12} = \frac{3}{12} = 1/4$$

$t = n_1 - n_2 \rightarrow t = 0$ (homogeneity)

$$r_X[0] = \sigma_X^2 + |\mu_X|^2$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$r_X[0] = 1/2$$

(same as covariance)

first and second order statistics are constant

$X(t) \rightarrow$ WSS

if they independent constant mean/variance hold based non autocorrelation

$\rightarrow$ strict stationary

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} r(u)\,du = \mu_X = E[X(t)]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\,dt = \lim_{T\to\infty} \frac{1}{2T} At \Big|_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{1}{2T}(AT + AT)$$

$$= \lim_{T\to\infty} \frac{AT}{T} = \lim_{T\to\infty} A = \underline{A} = E[X(t)]$$

for ergodic only look one process

$$A = 1/2$$

$$X(t) = A, \quad A \sim U[0,1] \text{ is ergodic}$$

2.3 b)

$$Y_n = A\cos(\omega_0 n), \quad A \sim N(0,1), \quad A \text{ and } \cos(\omega_0 n) \text{ are independent} \Rightarrow \text{WSS}$$

$$E[Y_n] = E[A\cos(\omega_0 n)]$$

$$E[Y_n] = E[A]E[\cos(\omega_0 n)]$$

$$E[Y_n] = 0$$

$$Var[Y_n] = Var[A\cos(\omega_0 n)]$$

$$Var[Y_n] = Var[A]\,Var[\cos(\omega_0 n)]$$

$$\text{var} \qquad \text{var} \qquad -1 \leq \cos(\omega_0 n) \leq 1$$

$$Var[Y_n] = \underline{1} \quad \square$$

$t = n_1 - n_2 \quad t = 0 \quad n_1 = n_2$

$$r_X[0] = \cancel{1^2 + 0} = \underline{1}$$

first and second order statistics are constant $\rightarrow$ WSS

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\cos(\omega_0 n)\,dn = 0$$

$$\lim_{T\to\infty} \frac{1}{2T} \frac{A}{\omega_0} \left(\sin(\omega_0 n)\right)_{-T}^{T} = E[Y_n]$$

$$\lim_{T\to\infty} \frac{1}{2T} \frac{A}{\omega_0} \left(\sin(\omega T) - \sin(-\omega T)\right)$$

$$\lim_{T\to\infty} \frac{1}{2T} \frac{A}{\omega_0} \cdot 2\sin(\omega_0 T)$$

$$\lim_{T\to\infty} \frac{A}{\omega_0} \frac{\sin(\omega_0 T)}{T} = 0 = E[Y_n]$$

X_n is ergodic $\square$

n_1 and n_2 $\xrightarrow{\text{lag}}$ collinearity correlation

its ergodic process

Exercise 2.4

Given random WSS process

If it is a LTI system, it should be both homogeneous and additive.

$$y[n] = \frac{1}{2}y[n-1] + x[n] + \frac{1}{3}x[n-1]$$

$$\text{output} \quad \text{output} \quad \text{input} \quad \text{input}$$

$$r_x = 0.5^{|n|} \qquad P_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x[k]e^{-j\omega k}$$

$$r_y[k] = r_x[k] * r_h[k] \qquad \text{is WSS and WSS}$$

$$P_x(e^{j\omega}) = \frac{1 - 0.5^2}{1 + 0.5^2 - 2.0.5\cos\omega}$$

$n = 0$

$$y[0] = \frac{1}{2}y[-1] + x[0] + \frac{1}{3}x[-1]$$

$$y[0] = x[0] \qquad \frac{1}{3}x[-1]$$

I didn't have time

I couldn't make it...

$n = 1$

$$y[1] = \frac{1}{2}y[0] + x[1] + \frac{1}{3}x[0]$$

$$y[1] = \frac{5}{6}x[0] + x[1]$$

$n = 2$

$$y[2] = \frac{1}{2}y[1] + x[2] + \frac{1}{3}x[1]$$

$$y[2] = \frac{5}{12}x[0] + \frac{5}{6}x[1] + x[2] + \frac{1}{3}x[1]$$

$$y[2] = \frac{5}{12}x[0] + \frac{5}{6}x[1] + x[2] \qquad x[1] + \frac{1}{3}x[1] = \frac{4}{3}x[1] + \frac{1}{2}x[2]$$

$n = 3$

$$y[3] = \frac{1}{2}y[2] + x[3] + \frac{1}{3}x[2]$$

$$= \frac{5}{24}x[0] + \frac{5}{12}x[1] + \frac{5}{6}x[2] + x[3] + \frac{1}{3}x[2]$$

$$= \frac{5}{24}x[0] + \frac{5}{12}x[1] + \frac{5}{6}x[2] + x[3]$$

$$y[n] = \begin{cases} & n \geq 0 \\ 0 & , n < 0 \end{cases}$$

$$P_y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_y[k]e^{-j\omega k} \qquad \left|H(e^{j\omega})\right|^2 =$$

z transform

$$Y(z)\left(1 - \frac{1}{2}z^{-1}\right) = X(z)\left(1 + \frac{1}{3}z^{-1}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$P_y = |H(z)|^2$$

$$P_y(z) = \frac{\left|1 + \frac{1}{3}z^{-1}\right|^2}{\left|1 - \frac{1}{2}z^{-1}\right|^2}$$

$$P_y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_y[k]e^{-j\omega k} \qquad \text{ex 2.3}$$

$$a = 0.5 \qquad \frac{3/4}{5/4 - \cos\theta}$$

$$\left|H(e^{j\omega})\right|^2 = \left|\frac{1 + \frac{1}{3}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}\right|^2$$

$$= \frac{1 + 1/3\cos\theta}{1 + 1/4 - \cos\theta}$$

$$P_y(e^{j\omega}) = P_y(e^{j\omega}) = \left|H(e^{j\omega})\right|^2$$

$$= \frac{3/4 + 1/3\cos\theta}{5/4 - 1/4\cos\theta}$$

long

$$\left|H(z)\right|^2 =$$