SLT 7

Tuesday, 18 October 2022 12:46

7.1

$$p(x|H_0) = \begin{cases} \lambda_0 e^{-\lambda_0 x} & \text{for } x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

$$p(x|H_1) = \begin{cases} \lambda_1 e^{-\lambda_1 x} & \text{for } x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

$\lambda_0 > \lambda_1 > 0$ and that N observations where each observations is IID.

$$\prod^{N-1} p(x_n, H)$$

$$= \lambda^N \cdot e^{-\sum \lambda_1 x} = \lambda^N \cdot e^{-\lambda_0 N}$$

$$L(x) = \frac{\lambda_1^N \cdot e^{-\lambda_1 N}}{\lambda_0 e^{-\lambda N}}$$

a)

$$L(x) = \frac{p(x|H_1)}{p(x|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

b)

for $x \geq 0$

$$L(x) = \frac{\lambda_1 e^{-\lambda_1 \cdot x}}{\lambda_0 e^{-\lambda_0 \cdot x}} = \lambda$$

$$\frac{\lambda_1}{\lambda_0} \exp(-\lambda_1 x + \lambda_0 x) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

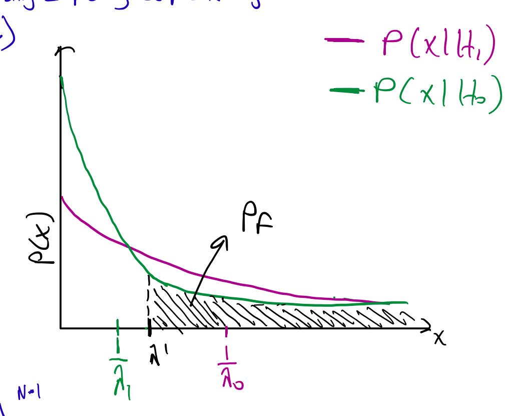
$$\exp(x \cdot (\lambda_0 - \lambda_1)) \quad \frac{\lambda \cdot \lambda_0}{\lambda_1}$$

$$x \cdot (\lambda_0 - \lambda_1) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{\lambda \cdot \lambda_0}{\lambda_1}\right)$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln \lambda + \ln \lambda_0 - \ln \lambda_1}{(\lambda_0 - \lambda_1)} = \lambda'$$

We can see that we can make a decision only comparing sample x against the threshold λ'.

c)



d)

$N=1$

$$P_F = \int_{\lambda'}^{\infty} \lambda_0 \cdot e^{-\lambda_0 \cdot x} \cdot dx$$

$$0.1 = \int_{\lambda'}^{\infty} 0.5\, e^{-0.5 \cdot x} \cdot dx \rightarrow \underline{\lambda' = 4.60}$$

e)

$$P_D = \int_{\lambda'}^{\infty} \lambda_1 e^{-\lambda_1 \cdot x} \cdot dx$$

$$\int_{4.60}^{\infty} 0.2 \cdot e^{-0.2x} \cdot dx = 0.398$$

$$\underline{P_D = 0.398}$$

7.2

$$p(x|H_0) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

$$p(x|H_1) = \begin{cases} \lambda_1 e^{-\lambda_1 x} & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

a) If $x < 0$

$$L(x) = \frac{p(x|H_1)}{p(x|H_0)} = \frac{0}{\frac{1}{2\pi} \cdot e^{-\frac{x^2}{2}}} \geq \lambda$$

$$\underline{L(x) = 0 \geq \lambda}$$

we cannot compare the sample x against λ'

If $x \geq 0$

$$L(x) = \frac{\lambda_1 e^{-\lambda_1 \cdot x}}{\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}} \geq \lambda$$

$$\sqrt{2\pi} \cdot \lambda_1 \exp\left(-\lambda_1 x - \frac{x^2}{2}\right) \geq \lambda$$

$$\ln \sqrt{2\pi} \cdot \lambda_1 + \frac{\lambda_1^2}{2} - \frac{(x+\lambda_1)^2}{2} \geq \ln \lambda$$

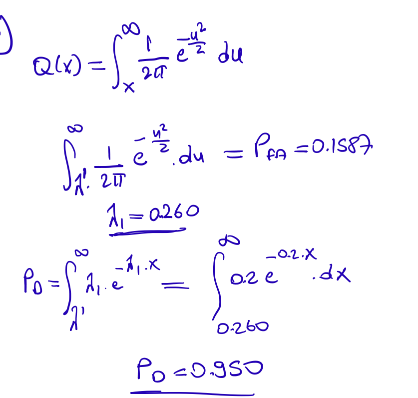
$$(x+\lambda_1)^2 = 2\left(\ln \lambda - \ln \sqrt{2\pi}\, \lambda_1 - \frac{\lambda_1^2}{2}\right)$$

$$(x+\lambda_1)^2 = 2 \cdot \ln \lambda - \ln 2\pi - 2 \cdot \ln \lambda_1 - \lambda_1^2$$

$$x + \lambda_1 = (2 \cdot \ln \lambda - \ln 2\pi - 2 \cdot \ln \lambda_1 - \lambda_1^2)^{1/2}$$

$$\underline{x \geq (2 \cdot \ln \lambda - \ln 2\pi - 2 \cdot \ln \lambda_1 - \lambda_1^2)^{1/2} - \lambda_1 = \lambda'}$$

b)



c)

$$Q(x) = \int_x^{\infty} \frac{1}{2\pi} e^{-\frac{u^2}{2}} du$$

$$\int_{\lambda'}^{\infty} \frac{1}{2\pi} e^{-\frac{u^2}{2}} \cdot du = P_{FA} = 0.1587$$

$$\underline{\lambda_1 = 0.260}$$

$$P_D = \int_{\lambda'}^{\infty} \lambda_1 \cdot e^{-\lambda_1 \cdot x} = \int_{0.260}^{\infty} 0.2\, e^{-0.2 \cdot x} \cdot dx$$

$$\underline{P_D = 0.950}$$

Exercise 7.3

$$\frac{p(x|H_1)}{p(x|H_0)} \geq \frac{(C_{1,0} - C_{0,0}) P_0}{(C_{0,1} - C_{1,1}) P_1} = \frac{P_0}{P_1} = L(x)$$

$$R = P_0 \int_{X_1} p(x|H_0) dx + P_1 \int_{X_0} p(x|H_1) dx$$

$$L(x) = \frac{\cancel{\frac{1}{(2\pi\sigma^2)^{N/2}}} \cdot \exp\left(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x_n^2\right)}{\cancel{\frac{1}{(2\pi\sigma^2)^{N/2}}} \cdot \exp\left(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x_n - A)^2\right)}$$

$$L(x) = \exp\left(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x_n^2 + \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x_n - A)^2\right) \cdot$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1} x_n^2 - \left(\sum_{n=0}^{N-1} x_n^2 - 2x_n A + A^2\right)\right)\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1} 2x_n A - A^2\right)\right)$$

$$\bar{x} = \frac{1}{N} \cdot \sum_{n=0}^{N-1} x_n$$

$$N \cdot \bar{x} = \sum_{n=0}^{N-1} x_n$$

$$\exp\left(-\frac{1}{2\sigma^2}\left(NA^2 - 2NA\bar{x}\right)\right) \geq 2$$

$$-\frac{1}{2\sigma^2}\left(N \cdot A^2 - 2NA \cdot \bar{x}\right) \geq \ln(2)$$

b)

$$N \cdot A^2 - 2 \cdot N \cdot A \cdot \bar{x} \geq -2\sigma^2 \ln \lambda$$

$$\bar{x} \geq \frac{2\sigma^2 \ln 2 + N \cdot A^2}{2 \cdot N} = \lambda'$$

c)

$A = 2$, $\sigma^2 = 1$, $N = 1$

$$\frac{2 \cdot 1 \cdot \ln 2 + 1 \cdot 4}{2} = \lambda'$$

$$\lambda' = 2.693$$

$$Q(x) = 2 \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

$$\rightarrow P_{FA} = 0.003539$$

d)

$$P_D = \int_{2.69}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-2)^2}{2}} \cdot dx$$

$$P_D = 0.244108$$

$$P_M = 1 - 0.244108$$

$$\underline{P_M = 0.756}$$

$$y = \frac{x - \mu}{\sigma} = \frac{x-2}{1} \Rightarrow x = y + 2$$

$$P_D = P_r[x > \lambda' | H_1] = P_r[y + 2 > \lambda' | H_1]$$

$$P_D = P_r[y > \lambda' - 2 | H_1] = \int_{\lambda-2}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} dy = Q(\lambda' - 2)$$

$$P_M = (1 - Q(\lambda' - 2)) \frac{1}{3} \rightarrow \text{bayes}$$

$$P_M = \{1 - [1 - Q(2 - \lambda')]\} \frac{1}{3}$$

$$Q = (0{,}7) \overset{\lambda-2}{\approx} 0{,}2420 \frac{1}{3}$$

e)

$$R = P_0 \underbrace{\int_{X_1} p(x|H_0) dx}_{P_{FA}} + P_1 \cdot \underbrace{\int_{X_0} P(x|H_1) dx}_{P_M}$$

$$R = 0.760$$

Quiz 7

H_0: a call is a voice call $\rightarrow H_0$ if $T \leq t_0$

H_1: a call is data call

$E[V] = 3$

$E[D] = \mu_D > 3$

$V \sim \exp(1/3)$

$D \sim \exp(\mu_D)$

$H_0 : x \sim p(x; H_0)$

$H_1 : x \sim p(x; H_1)$

$$P_{FA} = \int_{X_1} p(x; H_0) dx, \quad P_D = \int_{X_1} p(x; H_1) dx$$

$P_{TN} = 1 - P_{FA}$

$P_M = 1 - P_D$

$$\int_{t_0}^{\infty} \frac{1}{3} \cdot e^{-\frac{1}{3}x} dx$$

Known signal noise

$H_0 : x_n = w_n$

$H_1 : x_n = s_n + w_n$

$$p(x; H_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x_n^2\right)$$

$$p(x; H_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x_n - s_n)^2\right)$$

Likelihood ratio

$$L(x) = \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1} (x_n - s_n)^2 - \sum_{n=0}^{N-1} x_n^2\right)\right) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

Log-likelihood ratio test:

$$\ln L(x) = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} x_n \cdot s_n - \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} s_n^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln \lambda$$

$$\underbrace{\sum_{n=0}^{N-1} x_n \cdot s_n}_{\text{this is a correlation}} \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \cdot \ln \lambda + \frac{1}{2} \sum_{n=0}^{N-1} s_n^2 = \lambda'$$

Correlation can be expressed as a convolution

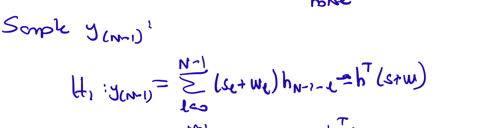
$$y_n = \sum_{l=0}^{N-1} x_l h_{n-l} \rightarrow \text{filter response}$$

· **Filter**: $h_n = s_{N-1-n}$ ← flipped version of known signal

Sample at $N-1$

$$y_{N-1} = \sum_{l=0}^{N-1} x_l \cdot s_{(N-1+l-N+1)}$$

$$= \sum_{l=0}^{N-1} x_l \cdot s_l$$



Matched filter SNR

$$SNR = \frac{E^2[y(N-1); H_1]}{Var[y(N-1); H_0]}$$

→ energy of the signal

→ energy of the noise

• Sample $y_{(N-1)}$:

$$H_1 : y_{(N-1)} = \sum_{l=0}^{N-1} (s_l + w_l) h_{N-1-l} = h^T (s + w)$$

$$H_0 : y_{(N-1)} = \sum_{l=0}^{N-1} w_l \cdot h_{N-1-l} = h^T w$$

signal ← $s = [s_0, s_1, \ldots, s_{(N-1)}]^T$

noise ← $w = [w_0, w_1, \ldots, w_{N-1}]^T$

filter ← $h = [h_{(N-1)}, h_{(N-2)}, \ldots, h_0]^T$

$$SNR = \frac{E^2[y(N-1); H_1]}{Var[y(N-1); H_0]}$$

→ energy of the signal

→ energy of the noise

$$SNR = \frac{(h^T s)^2}{\sigma^2 h^T \cdot h}$$

+ Cauchy-Schwarz inequality

$$(h^T \cdot s)^2 \leq (h^T \cdot h)(s^T \cdot s)$$

Equality: $h = c \cdot s$

$$\boxed{\text{Matched filter is just flipped version of the signal}}$$