Statistical signal processing (5CTA0)

Exercise Bundle

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Part 1: Random variables and random signals

1.1 *Probability and random variables*

Exercise 1.1: Ricardo's offers customers two kinds of pizza crust, Roman (R) and Neapolitan (N). All pizza's have cheese but not all pizza's have tomato sauce. Roman pizza's can have tomato sauce or they can be white (W); Neapolitan pizza's always have tomato sauce. It is possible to order a Roman pizza with mushrooms (M) added. A neapolitan pizza can contain mushrooms or onions (O) or both, in addition to the tomato sauce and cheese. Draw a Venn diagram that shows the relationship among the ingredients N, M, O, T, and W in the menu of Ricardo's pizzeria.

Exercise 1.2: A company has a model of telephone usage. It classifies all calls as either long (l), if they last more than three minutes, or brief (b). It also observes whether calls carry voice (v), data (d), or fax (f). This model implies an experiment in which the procedure is to monitor a call and the observation consists of the type of call, v, d, or f , and the length, l or b. The sample space has six outcomes S = $\{lv, bv, ld, bd, lf, bf\}$. In this problem, each call is classifed in two ways: by length and by type. Using L for the event that a call is long and B for the event that a call is brief, L, B is an event space. Similarly, the voice (V), data (D) and fax (F) classification is an event space V, D, F. The sample space can be represented by a table in which the rows and columns are labeled by events and the intersection of each row and column event contains a single outcome. The corresponding table entry is the probability of that outcome. Given the sample space representade by the table below,

	V	D	F
L	0.3	0.12	0.15
B	0.2	0.08	0.15

Find the probability of a long call Pr[L].

Exercise 1.3: Suppose a cellular telephone is equally likely to make zero handoffs (H_0), one handoff (H_1), or more than one handoff (H_2). Also, a caller is either on foot (F) with probability 5/12 or in a vehicle (V).

(a) Given the preceding information, find three ways to fill in the following probability table:

	H_0	H_1	H_2
F			
V			

(b) Suppose we also learn that 1/4 of all callers are on foot making calls with no handoffs and that 1/6 of all callers are vehicle users making calls with a single handoff. Given these additional facts, find all possible ways to fill in the table of probabilities.

Exercise 1.4: Is it possible for A and B to be independent events yet satisfy $A = B$?

Exercise 1.5: A company has three machines B_1, B_2, and B_3 for making 1 kΩ resistors. It has been observed that 80% of resistors produced by B_1 are within 50 Ω of the nominal value. Machine B_2 produces 90% of resistors within 50 Ω of the nominal value. The percentage for machine B_3 is 60%. Each hour, machine B_1 produces 3000 resistors, B_2 produces 4000 resistors, and B_3 produces 3000 resistors. All of the resistors are mixed together at random in one bin and packed for shipment. What is the probability that the company ships a resistor that is within 50 Ω of the nominal value?

Exercise 1.6: In Exercise **1.5** about a shipment of resistors from the factory, we learned that:

- The probability that a resistor is from machine B3 is Pr[B3] = 0.3.
- The probability that a resistor is acceptable, i.e., within 50Ω of the nominal value,is Pr[A] = 0.78.
- Given that a resistor is from machine B3, the conditional probability that it is acceptable is Pr[$A|B3$] = 0.6.

What is the probability that an acceptable resistor comes from machine B3?

Exercise 1.7: Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time. What is $\Pr[-|H]$, the conditional probability that a person tests negative given that the person does have the HIV virus? What is $\Pr[H|+]$, the conditional probability that a randomly chosen person has the HIV virus, given that the person tests positive?

Exercise 1.8: Let X have the binomial PMF

$$p_X(x) = \binom{4}{x}(1/2)^4. \tag{1}$$

(a) Find the standard deviation of X.

(b) What is $\Pr[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X]$, the probability that X is within one standard deviation of the expected value?

Exercise 1.9: This problem outlines the steps needed to show that the Gaussian PDF integrates to unity. For a Gaussian(μ,σ) random variable W, we will show that

$$I = \int_{-\infty}^{\infty} f_W(w)dw = 1. \tag{2}$$

(a) Use the substitution $x = (w - \mu)/\sigma$ to show that

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx. \tag{3}$$

(b) Show that

$$I^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dxdy. \tag{4}$$

(c) Change to polar coordinates to show that $I^2 = 1$.

Exercise 1.10: X and Y are identically distributed random variables with $\mathrm{E}[X] = \mathrm{E}[Y] = 0$ and covariance $\mathrm{Cov}[X, Y] = 3$ and correlation coefficient $\rho_{X,Y} = 1/2$. For nonzero constants a and b, $U = aX$ and $V = bY$.

(a) Find $\mathrm{Cov}[U, V]$.

(b) Find the correlation coefficient $\rho_{U,V}$.

(c) Let $W = U + V$. For what values of a and b are X and W uncorrelated?

Exercise 1.11: Random variables X and Y have joint PDF

$$p_{X,Y}(x, y) = \begin{cases} 5x^2/2 & -1 \leq x \leq 1; 0 \leq y \leq x^2, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Answer the following questions.

(a) What are $\mathrm{E}[X]$ and $\mathrm{Var}[X]$?

(b) What are $\mathrm{E}[Y]$ and $\mathrm{Var}[Y]$?

(c) What is $\mathrm{Cov}[X, Y]$?

(d) What is $\mathrm{E}[X + Y]$?

(e) What is $\mathrm{Var}[X + Y]$?

Exercise 1.12: In a weekly lottery, each \$1 ticket solds adds 50 cents to the jackpot that starts at \$1 million before any tickets are sold. The jackpot is announced each morning to encourage people to play. On the morning of the ith day before the drawing, the current value of the jackpot J_i is announced.

On that day, the number of tickets sold, N_i, is a Poisson random variable with expected value, J_i. Thus, six days before the drawing, the morning jackpot starts at \$1 million and N_6 tickets are sold that day. On the day of the drawing, the announced jackpot is J_0 dollars and N_0 tickets are sold before the evening drawing. What is the expected value of J, the value of the jackpot the instant before the drawing?

Hint: Use conditional expectations.

Exercise 1.13: An n-dimensional Gaussian vector $\mathbf{W}$ has a block diagonal covariance matrix

$$\mathbf{C_W} = \begin{bmatrix} \mathbf{C_X} & \mathbf{0} \\ \mathbf{0} & \mathbf{C_Y} \end{bmatrix}, \tag{6}$$

where $\mathbf{C_X}$ is $m \times m$, $\mathbf{C_Y}$ is $(n-m) \times (n-m)$. Show that $\mathbf{W}$ can be written in terms of component vectors $\mathbf{X}$ and $\mathbf{Y}$ in the form

$$\mathbf{W} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}, \tag{7}$$

such that $\mathbf{X}$ and $\mathbf{Y}$ are independent Gaussian random vectors.

1.2 *Stochastic processes and random signals*

Exercise 1.14: Let $g(x)$ be a deterministic function. If $X(t)$ is a stationary random process, is $Y(t) = g(X(t))$ a stationary process?

Exercise 1.15: $X(t)$ and $Y(t)$ are independent wide sense stationary processes with expected values μ_X and μ_Y and auto-correlation functions $R_x(\tau)$ and $R_y(\tau)$, respectively. Let $W(t) = X(t)Y(t)$.

(a) Find $E[W(t)]$ and $R_W(t,\tau)$ and show that $W(t)$ is wide sense stationary.

(b) Are $W(t)$ and $X(t)$ jointly wide sense stationary?

Exercise 1.16: A stationary Gaussian process $X(t)$ is observed at times t_1 and t_2 to form the random vector $\boldsymbol{X} = \begin{bmatrix} X(t_1) & X(t_2) \end{bmatrix}'$ with expected value $E[\boldsymbol{X}] = \mathbf{0}$ and covariance matrix $\boldsymbol{C}_X = \begin{bmatrix} \sigma_1^2 & 1 \\ 1 & \sigma_2^2 \end{bmatrix}$. What is the range of valid values (if any) of σ_1^2 and σ_2^2?

Exercise 1.17: Given a Gaussian process $X(t)$, identify which of the following, if any, are Gaussian proccesses.

(a) $2X(t)$,

(b) $X(t/2)$,

(c) $X(t)/2$,

(d) $X(t) - X(t-1)$,

(e) $X(2t)$.

Exercise 1.18: Let $w[n]$ be a zero-mean, uncorrelated Gaussian random sequence with variance $\sigma^2[n] = 1$.

(a) Characterize the random sequence $w[n]$.

(b) Define $x[n] = w[n] + w[n-1], -\infty < n < \infty$. Determine the mean and autocorrelation of $x[n]$. Also characterize $x[n]$.

Exercise 1.19: Given the following autocorrelation sequences. Find the PSD and verify that this function is real and nonnegative

(a) $\rho[\tau] = 3 - |\tau|$ for $|\tau| \leq 3$ and $\rho[\tau] = 0$ elsewhere.

(b) $\rho[\tau] = 2(-0.6)^{|\tau}| + \delta[\tau]$.

Exercise 1.20: The random process $x[k]$ is generated by filtering innovation $i[k]$ (white noise sequence with zero mean and unit variance) with filter $H(z) = (1 - \frac{1}{3}z^{-1})/(1 - \frac{1}{2}z^{-1})$.

(a) Calculate autocorrelation function $\rho[\tau]$ of process $x[k]$.

(b) Calculate psd of $x[k]$ in two different ways.

Exercise 1.21: Given the following autocorrelation sequences. Find the PSD and verify that this function is real and nonnegative

(a) $\rho[\tau] = 3 - |\tau|$ for $|\tau| \leq 3$ and $\rho[\tau] = 0$ elsewhere.

(b) $\rho[\tau] = 2(-0.6)^{|\tau}| + \delta[\tau]$.

1.3 *Rational signal models*

Exercise 1.22: Calculate the autocorrelation function of the random process $x[k]$ that is generated by filtering innovation $i[k]$ (white noise sequence with zero mean and variance $\sigma_i^2 = \frac{1}{4}$) with the (all-pass) filter $H(z) = (1 - 2z^{-1})/(1 - \frac{1}{2}z^{-1})$.

Exercise 1.23: Process $x[k]$ is generated by the following difference equation: $x[k] = i[k] + 0.1i[k-1] - 0.2i[k-2]$, where $i[k]$ is stationary white noise (zero mean, unit variance).

(a) Is the process $x[k]$ stationary? Why?

(b) Is the model (filter) minimum-phase? Why?

(c) Determine the autocorrelation of the process $x[k]$.

Exercise 1.24: Apply spectral factorization to the following psd:

$$P(e^{j\theta}) = \frac{5 - 4\cos(\theta)}{10 - 6\cos(\theta)} \tag{8}$$

Give an expression for the innovation filter $L(z)$, the variance σ_i^2 of the innovation signal, the compression gain G and give a realization scheme.

Exercise 1.25: Given the autocorrelation function $\rho[2\tau] = (\frac{1}{3})^{|\tau|}$ and $\rho[2\tau + 1] = 0$. Give an expression for the innovation filter $L(z)$, the variance σ_i^2 of the innovation signal and draw a realizationscheme.

Exercise 1.26: The random process $x[k]$ is generated by filtering the white noise sequence $n[k]$ (zero mean and variance σ_n^2) with filter $H(z)$. This filter $H(z)$ has one (real) pole in $z = b = 0.9$ and two (complex conjugated) zeros at $z = a \pm ja$, with $a = 0.8$. Design a stable whitening filter $\Gamma(z)$ with input $x[k]$ and output innovation $i[k]$ in such a way that $\sigma_i^2 = \sigma_n^2$ *Note:* You may neglect the fact that $H(z)$ is non causal.

Exercise 1.27: Given the innovation filter $L(z) = (1 - az^{-2})/(1 - bz^{-1})$ with input the innovation $i[k]$ (white noise with zero mean and variance σ_i^2).

(a) Calculate the autocorrelation for $a = \frac{1}{4}, b = -\frac{1}{2}$ and $\sigma_i^2 = 2$

(b) Given the autocorrelation function $\rho[\tau] = (\frac{1}{3})^{|\tau|}$. Calculate the parameters a, b and σ_i^2.

Exercise 1.28: Consider a random signal $x[n] = s[n] + v[n]$, where $v[n] \sim WGN(0, 1)$ and $s[n]$ is the AR(1) process $s[n] = 0.9s(n-1) + w[n]$, where $w[n] \sim$ WGN(0, 0.64). The signals $s[n]$ and $v[n]$ are uncorrelated. Determine and plot the autocorrelation $r_s[l]$ and the PSD $P_s(e^{j\omega})$ of $s[n]$.

Part 2: Estimation theory

2.1 *Cramer-Rao Lower Bound*

Exercise 2.1: For a 2×2 Fisher information matrix

$$\boldsymbol{I}(\boldsymbol{\theta}) = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{9}$$

which is positive definite:

(a) Show that

$$[\boldsymbol{I}^{-1}(\boldsymbol{\theta})]_{11} = \frac{c}{ac-b^2} \geq \frac{1}{a} = \frac{1}{[\boldsymbol{I}(\boldsymbol{\theta})]_{11}} \tag{10}$$

(b) What does this say about estimating a parameter when a second parameter is either known or unknown ?

(c) When does equality hold and why ?

Hint: If A is a 2x2 positive definite matrix, then the the diagonal entries A_{ii} are real and non-negative and det(A)>0.

2.2 *Maximum likelihood estimator*

Exercise 2.2: If we observe N IID samples from a Bernoulli experiment (coin toss) with the probabilities

$$\begin{aligned} Pr\{x[n]=1\} &= p \\ Pr\{x[n]=0\} &= 1-p \end{aligned} \tag{11}$$

find the MLE of p

2.3 *MVUE for linear models*

Exercise 2.3: For N IID observations from a $\mathcal{U}[0,\theta]$[1] PDF find the MLE of θ.

2.4 *Least-square estimation*

Exercise 2.4: Consider the signal $x[n] = A + w[n]$, with $w[n]$ zero-mean uncorrelated noise with variance $\sigma^2[n]$. Using weights $1/\sigma^2[n]$ find the weighted LSE for A, and calculate its mean and variance.

2.5 *Bayesian estimation*

Exercise 2.5: The data $x[n]$ for $n = 0, 1, \ldots, N-1$ are observed, each sample having the conditional PDF

$$p(x[n]|\theta) = \begin{cases} \exp\Big(-\big(x[n]-\theta\big)\Big) & x[n] \geq \theta \\ 0 & x[n] < \theta, \end{cases} \tag{12}$$

and conditioned on θ the observations are independent. The prior PDF is

$$p(\theta) = \begin{cases} \exp(-\theta) & \theta \geq 0 \\ 0 & \theta < 0, \end{cases} \tag{13}$$

Find the MMSE estimator of θ.
Note: Two variables x and y are conditionally independent given z, when $p(x,y|z) = p(x|z)p(y|z)$.

[1] $\mathcal{U}[x,y]$ is a uniform distribution in the range (x,y)

Exercise 2.6: A quality assurance inspector has the job of monitoring the resistance values of manufactured resistors. He does so by choosing a resistor from a batch and measuring its resistance with an ohmmeter. He knows that the ohmmeter is of poor quality and imparts an error to the measurement which he models as a $\mathcal{N}(0,1)$ random variable. Hence, he takes N independent measurements. Also, he knows that the resistors should be 100 ohms. Due to manufacturing tolerances, however, they generally are in error by ϵ, where $\epsilon \sim \mathcal{N}(0, 0.011)$.

(a) What is the likelihood distribution $p(\boldsymbol{x}|R)$ in this case ?

Given the fact that the posterior is equal to

$$p(R|\boldsymbol{x}) = \frac{1}{\sqrt{2\pi\sigma^2_{R|x}}} \exp\left[-\frac{1}{2\sigma^2_{R|x}}(R - \mu_{R|x})^2\right]. \tag{14}$$

With

$$\sigma^2_{R|\boldsymbol{x}} = \frac{1}{\frac{N}{\sigma^2_w} + \frac{1}{\sigma^2_R}} \qquad \mu_{R|\boldsymbol{x}} = \left(\frac{N}{\sigma^2_w}\overline{\boldsymbol{x}} + \frac{\mu_R}{\sigma^2_R}\right)\sigma^2_{R|\boldsymbol{x}} \tag{15}$$

Answer the following questions:

(b) If the inspector chooses a resistor, what would be the MMSE estimator for estimating the resistance $\hat{R}$, based on N measurements $\boldsymbol{x} = (x[1], x[2], \ldots, x[N])$ of the ohmmeter ?

(c) How many ohmmeter measurements N are necessary to ensure that a MMSE estimator of the resistance R yields the correct resistance to 0.1 ohms *on the average* or as he continues to choose resistors throughout the day?

(d) How many measurements would he need if he did not have any prior knowledge about the manufacturing tolerances?

Exercise 2.7: In fitting a line through experimental data we assume the model

$$x[n] = A + Bn + w[n] \qquad -M \leq n \leq M \tag{16}$$

where $w[n]$ is WGN with variance σ^2. If we have some prior knowledge of the slope B and intercept A such as

$$\begin{bmatrix} A \\ B \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \begin{bmatrix} \sigma^2_A & 0 \\ 0 & \sigma^2_B \end{bmatrix}\right) \tag{17}$$

find the MMSE estimator of A and B as well as the minimum Bayesian MSE. Assume that A,B are independent of $w[n]$. Which parameter will benefit most from the prior knowledge?

Exercise 2.8: X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x) & 0 \leq x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

(a) What is $f_{X|Y}(x \mid y)$?

(b) What is $\hat{x}_{MMSE}(y)$, the minimum mean square error estimate of X given $Y = y$?

(c) What is $f_{Y|X}(y \mid x)$?

(d) What is $\hat{x}_{MMSE}(x)$, the minimum mean square error estimate of Y given $X = x$?

Part 3: Spectral Estimation

3.1 *Introduction to spectral estimation*

Exercise 3.1: Given $x[n] = \frac{1}{2} \cdot \sin(2\pi f_1 \cdot n) + \sin(2\pi f_2 \cdot n)$ with relative frequency $f_i = f_i'/f_s$ and $f_1' = 11\ Hz$, $f_2' = 17\ Hz$ and sample frequency $f_s = 50\ Hz$. The used data window has length $N = 16$. Furthermore we eventually apply zero padding with M zeroes. De DFT used has length L, with $L = N + M$. We want to evaluate if this length L DFT is able to discriminate between the spectral peaks of the two sinus waves.

(a) First we do not apply zero padding, thus $M = 0$ and $L = N = 16$. Reason out for which indices k of the DFT frequency bins you expect the peaks of the two sinus waves. Can you discriminate between these different peaks?

(b) Now we apply zero padding with $M = 112$, thus the DFT length is $L = M + N = 112 + 16 = 128$. Do the same as in a). Explain why the discrimination between the two spectral peaks has not been increased by the zero padding? How can you increase the spectral discrimination?

3.2 *Non-parametric spectral estimation*

Exercise 3.2: Given the biased estimate $\hat{\rho}[\tau]$ for the autocorrelation:

$$\hat{\rho}[\tau] = \begin{cases} \frac{1}{N} \sum_{k=\tau}^{N-1} x[k]x[k-\tau] & 0 \leq \tau \leq N-1 \\ \hat{\rho} & -(N-1) \leq \tau \leq 0 \\ 0 & \text{elsewhere} \end{cases} \tag{19}$$

Show that the Periodogram $\hat{P}(e^{j\theta})$ can be expressed as the FTD of $\hat{\rho}[\tau]$.

3.3 *Parametric spectral estimation*

Exercise 3.3: From a random process $x[k]$ we know the autocorrelation function $\rho[\tau] = 2^{-|\tau|} + \delta[\tau]$.

(a) Calculate the psd $P(e^{j\theta})$ of this process.

(b) Assume the autocorrelation can be estimated perfectly and the correlogram spectral estimate is formed by using the first three lags:

$$\hat{\rho}[\tau] = \begin{cases} \rho[\tau] & \text{for} \quad \tau = 0, \pm 1, \pm 2 \\ 0 & \text{elsewhere} \end{cases} \tag{20}$$

Give an expression for this correlogram spectral estimate $\hat{P}_{cor}(e^{j\theta})$.

(c) Calculate a one-step FIR predictor with one coefficient for this process.

(d) Use the result of c) to model this process with an AR(1) model and give the spectral estimate $\hat{P}_{ar1}(e^{j\theta})$.

(e) Compare in one (hand made) plot $P(e^{j\theta})$ with $\hat{P}_{cor}(e^{j\theta})$ and $\hat{P}_{ar1}(e^{j\theta})$.

Part 4: Detection theory

4.1 *Hypothesis testing*

Exercise 4.1: For the DC level in WGN detection problem assume that we wish to have $P_{FA} = 10^{-4}$ and $P_D = 0.99$. If the SNR is $10 \log_{10} \left(A^2/\sigma^2 \right) = -30dB$, determine the necessary number of samples N.

Hint: To calculate the inverse of the Gaussian CDF $Q^{-1}(x)$ for any x you can use the Table 1, or, in MATLAB, the function *norminv(x)*

x	$Q^{-1}(x)$	x	$Q^{-1}(x)$	x	$Q^{-1}(x)$	x	$Q^{-1}(x)$	x	$Q^{-1}(x)$
0.001	-3.090	0.201	-0.838	0.401	-0.251	0.601	0.256	0.801	0.845
0.021	-2.034	0.221	-0.769	0.421	-0.199	0.621	0.308	0.821	0.919
0.041	-1.739	0.241	-0.703	0.441	-0.148	0.641	0.361	0.841	0.999
0.061	-1.546	0.261	-0.640	0.461	-0.098	0.661	0.415	0.861	1.085
0.081	-1.398	0.281	-0.580	0.481	-0.048	0.681	0.470	0.881	1.180
0.101	-1.276	0.301	-0.522	0.501	0.003	0.701	0.527	0.901	1.287
0.121	-1.170	0.321	-0.465	0.521	0.053	0.721	0.586	0.921	1.412
0.141	-1.076	0.341	-0.410	0.541	0.103	0.741	0.646	0.941	1.563
0.161	-0.990	0.361	-0.356	0.561	0.154	0.761	0.710	0.961	1.762
0.181	-0.912	0.381	-0.303	0.581	0.204	0.781	0.776	0.981	2.075

Table 1: Inverse Gaussian CDF

Exercise 4.2: Design a *perfect detector* for the problem

$$\mathcal{H}_0 : x[0] \sim \mathcal{U}[-c, c] \tag{21}$$
$$\mathcal{H}_1 : x[0] \sim \mathcal{U}[1-c, 1+c], \tag{22}$$

with $c > 0$, by choosing an appropriate value for c. Where $\mathcal{U}[a, b]$ denotes a uniform PDF on the interval $[a, b]$, and a perfect detector has $P_{FA} = 0$ and $P_D = 1$.