Statistical signal processing (5CTA0)

Exercise Bundle

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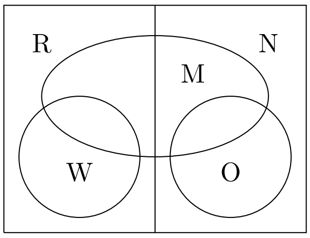
Part 1: Random variables and random signals

1.1 *Probability and random variables*

Solution 1.1: At Ricardo's, the pizza crust is either Roman (R) or Neapolitan (N). To draw the Venn diagram as shown below, we make the following observations:

- The set R, N is a partition so we can draw the Venn diagram with this partition.
- Only Roman Pizza's can be white. Hence $W \subset R$.
- Only a Neapolitan pizza can have onions. Hence $O \subset N$.
- Both Neapolitan and Roman pizza's can have mushrooms so that event M straddles the R, N partition.
- The Neapolitan pizza can have both mushrooms and onions so $M \cap O$ cannot be empty.
- The problem statement does not preclude putting mushrooms on a white Roman pizza. Hence the intersection $W \cap M$ should not be empty.



Solution 1.2: As a consequence of probability axioms

$$\Pr[A] = \sum_{i=1}^{m} \Pr[A \cap B_i]. \tag{1}$$

Thus we can apply the Theorem in previous equation to find the probability of a long call as

$$\Pr[L] = \Pr[\mathrm{LV}] + \Pr[\mathrm{LD}] + \Pr[\mathrm{LF}] = 0.57. \tag{2}$$

Solution 1.3:

(a) For convenience, let $p_i = \Pr[FH_i]$ and $q_i = \Pr[VH_i]$. Using this shorthand, the siz unknowns $p_0, p_1, p_2, q_0, q_1, q_2$ fill the table as

	H_0	H_1	H_2
F	p_0	p_1	p_2
V	q_0	q_1	q_2

However, we are given a number of facts:

$$\begin{array}{ll} p_0 + q_0 = 1/3 & p_1 + q_1 = 1/3 \\ p_2 + q_2 = 1/3 & p_0 + p_1 + p_2 = 5/12 \end{array} \tag{3}$$

Other facts, such as $q_0 + q_1 + q_2 = 7/12$, can be derived from these facts. Thus, we have four equations and six unknowns, choosing p_0 and p_1 will specify the other unknowns. Unfortunately, arbitrary choices for either p_0 or p_1 will lead to negative values for the other probabilities. In terms of p_0 and p_1, the other unknowns are

$$\begin{array}{ll} q_0 = 1/3 - p_0 & p_2 = 5/12 - (p_0 + p_1), \\ q_1 = 1/3 - p_1 & q_2 = p_0 + p_1 - 1/12. \end{array} \tag{4}$$

Because the probabilities must be nonnegative, we see that

$$0 \le p_0 \le 1/3, \tag{5}$$

$$0 \le p_1 \le 1/3, \tag{6}$$

$$1/12 \le p_0 + p_1 \le 5/12. \tag{7}$$

Although there are an infinite number of solutions, three possible solutions are:

$$\begin{array}{lll} p_0 = 1/3, & p_1 = 1/12, & p_2 = 0, \\ q_0 = 0, & q_1 = 1/4, & q_2 = 1/3 \end{array} \tag{8}$$

and

$$\begin{array}{lll} p_0 = 1/4, & p_1 = 1/12, & p_2 = 1/12, \\ q_0 = 1/12, & q_1 = 3/12, & q_2 = 3/12 \end{array} \tag{9}$$

and

$$\begin{array}{lll} p_0 = 0, & p_1 = 1/12, & p_2 = 1/3, \\ q_0 = 1/3, & q_1 = 3/12, & q_2 = 0 \end{array} \tag{10}$$

(b) In terms of the p_i,q_i notation, the new facts are $p_0 = 1/4$ and $q_1 = 1/6$. These extra facts uniquely specify the probabilities. In this case,

$$\begin{array}{lll} p_0 = 1/4, & p_1 = 1/6, & p_2 = 0, \\ q_0 = 1/12, & q_1 = 1/6, & q_2 = 1/3 \end{array} \tag{11}$$

Solution 1.4: This problem asks whether A and B can be independent events, yet satisfy $A = B$? By definition, events A and B are independent if and only if $\Pr[AB] = \Pr[A]\Pr[B]$. We can see that if $A = B$, that is they are the same set, then

$$\Pr[AB] = \Pr[AA] = \Pr[A] = \Pr[B]. \tag{12}$$

Thus, for A and B to be the same set and also independent,

$$\Pr[A] = \Pr[AB] = \Pr[A]\Pr[B] = (\Pr[A])^2. \tag{13}$$

There are two ways that this requirement can be satisfied:

- $\Pr[A] = 1$, implying $A = B = S$.
- $\Pr[A] = 0$, implying $A = B = \phi$.

Solution 1.5: Let A = resistor is within 50Ω of the nominal value. Using the resistor accuracy information to formulate a probability model, we write

$$\Pr[A|B1] = 0.8, \Pr[A|B2] = 0.9, \Pr[A|B3] = 0.6 \tag{14}$$

The production figures state that 3000 + 4000 + 3000 = 10,000 resistors per hour are produced. The fraction from machine B1 is $\Pr[B1] = 3000/10{,}000 = 0.3$. Similarly, $\Pr[B2] = 0.4$ and $\Pr[B1] = 0.3$. Now it is a simple matter to apply the law of total probability to find the accuracy probability for all resistors shipped by the company:

$$\begin{aligned} \Pr[A] = \Pr[A|B1]\Pr[B1] + \Pr[A|B2]\Pr[B2] + \Pr[A|B3]\Pr[B3] = \\ = (0.8)(0.3) + (0.9)(0.4) + (0.6)(0.3) = 0.78. \end{aligned} \tag{15}$$

For the whole factory, 78% of resistors are within 50Ω of the nominal value.

Solution 1.6: Now we are given the event A that a resistor is within 50Ω of the nominal value, and we need to find $\mathrm{Pr}P[B_3|A]$. Using Bayes' theorem, we have

$$Pr[B_3|A] = \frac{Pr[A|B_3]Pr[B_3]}{Pr[A]}. \tag{16}$$

Since all of the quantities we need are given in the problem description, our answer is

$$Pr[B_3|A] = (0.6)(0.3)/(0.78) = 0.23. \tag{17}$$

Similarly we obtain $P[B_1|A] = 0.31$ and $P[B_2|A] = 0.46$. Of all resistors within 50Ω of the nominal value, only 23% come from machine B_3 (even though this machine produces 30% of all resistors). Machine B_1 produces 31% of the resistors that meet the 50Ω criterion and machine B_2 produces 46% of them.

Solution 1.7: The $\mathrm{Pr}[-|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV, but tests positive for the disease, is called a false-positive result and has probability $\mathrm{Pr}[+|H^c]$. Since the test is correct 99% of the time,

$$\mathrm{Pr}[-|H] = \mathrm{Pr}[+|H^c] = 0.01. \tag{18}$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$\mathrm{Pr}[H|+] = \frac{\mathrm{Pr}[+, H]}{\mathrm{Pr}[+]} = \frac{\mathrm{Pr}[+, H]}{\mathrm{Pr}[+, H] + \mathrm{Pr}[+, H^c]}. \tag{19}$$

We can use Bayes' formula to evaluate these joint probabilities.

$$\begin{aligned} \mathrm{Pr}[H|+] &= \frac{\mathrm{Pr}[+|H]\mathrm{Pr}[H]}{\mathrm{Pr}[+|H]\mathrm{Pr}[H] + \mathrm{Pr}[+|H^c]\mathrm{Pr}[H^c]} \\ &= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} \\ &= 0.0194. \end{aligned} \tag{20}$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Solution 1.8:

(a) The expected value of X is

$$\begin{aligned} \mathrm{E}[X] &= \sum_{x=0}^{4} x P_X(x) \\ &= 0\binom{4}{0}\frac{1}{2^4} + 1\binom{4}{1}\frac{1}{2^4} + 2\binom{4}{2}\frac{1}{2^4} + 3\binom{4}{3}\frac{1}{2^4} + 4\binom{4}{4}\frac{1}{2^4} \\ &= [4 + 12 + 12 + 4]/2^4 = 2. \end{aligned} \tag{21}$$

The expected value of X^2 is

$$\begin{aligned} \mathrm{E}[X^2] &= \sum_{x=0}^{4} x^2 P_X(x) \\ &= 0^2\binom{4}{0}\frac{1}{2^4} + 1^2\binom{4}{1}\frac{1}{2^4} + 2^2\binom{4}{2}\frac{1}{2^4} + 3^2\binom{4}{3}\frac{1}{2^4} + 4^2\binom{4}{4}\frac{1}{2^4} \\ &= [4 + 24 + 36 + 16]/2^4 = 5. \end{aligned} \tag{22}$$

The variance of X is

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = 5 - 2^2 = 1. \tag{23}$$

Thus, X has standard deviation $\sigma_X = \sqrt{\mathrm{Var}[X]} = 1$

(b) The probability that X is within one standard deviation of its expected value is

$$\begin{aligned} \mathrm{P}[\mu_X - \sigma_X \le X \le \mu_X + \sigma_X] &= \mathrm{P}[2-1 \le X \le 2+1] \\ &= \mathrm{P}[1 \le X \le 3]. \end{aligned} \tag{24}$$

This calculation is easy using the PMF of X:

$$\mathrm{P}[1 \le X \le 3] = P_X(1) + P_X(2) + P_X(3) = 7/8. \tag{25}$$

Solution 1.9: First we note that since W has an $N[\mu, \sigma^2]$ distribution, the integral we wish to evaluate is

$$I = \int_{-\infty}^{\infty} f_W(w)dw = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(w-\mu)^2/2\sigma^2} dw. \tag{26}$$

(a) Using the substitution $x = (w-\mu)/\sigma$, we have $dx = dw/\sigma$ and

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx. \tag{27}$$

(b) When we write I^2 as the product of integrals, we use y to denote the other variable of integration so that

$$\begin{aligned} I^2 &= (\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx)(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dy) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dxdy. \end{aligned} \tag{28}$$

(c) By changing the polar coordinates, $x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$ so that

$$\begin{aligned} I^2 &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{r^2/2} rdrd\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} -e^{r^2} 2 \Big|_0^{\infty} d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1. \end{aligned} \tag{29}$$

Solution 1.10:

(a) Since X and Y have zero expected value, $\mathrm{Cov}[X,Y] = \mathrm{E}[XY] = 3$, $\mathrm{E}[U] = a\mathrm{E}[X] = 0$ and $\mathrm{E}[V] = b\mathrm{E}[Y] = 0$. It follows that

$$\begin{aligned} \mathrm{Cov}[U,V] &= \mathrm{E}[UV] \\ &= \mathrm{E}[abXY] \\ &= ab\mathrm{E}[XY] = ab\mathrm{Cov}[X,Y] = 3ab. \end{aligned} \tag{1}$$

(b) We start by observing that $\mathrm{Var}[U] = a^2\mathrm{Var}[X]$ and $\mathrm{Var}[V] = b^2\mathrm{Var}[Y]$. It follows that

$$\begin{aligned} \rho_{U,V} &= \frac{\mathrm{Cov}[U,V]}{\sqrt{\mathrm{Var}[U]\mathrm{Var}[V]}} \\ &= \frac{ab\mathrm{Cov}[X,Y]}{\sqrt{a^2\mathrm{Var}[X]b^2\mathrm{Var}[Y]}} = \frac{ab}{\sqrt{a^2b^2}}\rho_{X,Y} = \frac{1}{2}\frac{ab}{|ab|}. \end{aligned} \tag{2}$$

Not that $ab/|ab|$ is 1 if a and b have the same sign or is -1 if they have opposite signs.

(c) Since $\mathrm{E}[X] = 0$,

$$\begin{aligned}
\mathrm{Cov}[X, W] &= \mathrm{E}[XW] - \mathrm{E}[X]\mathrm{E}[U] \\
&= \mathrm{E}[XW] \\
&= \mathrm{E}[X(aX + bY)] \\
&= a\mathrm{E}[X^2] + b\mathrm{E}[XY] \\
&= a\mathrm{Var}[X] + b\mathrm{Cov}[X, Y].
\end{aligned} \tag{3}$$

Since X and Y are identically distributed, $\mathrm{Var}[X] = \mathrm{Var}[Y]$ and

$$\frac{1}{2} = \rho_{X,Y} = \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} = \frac{\mathrm{Cov}[X, Y]}{\mathrm{Var}[X]} = \frac{3}{\mathrm{Var}[X]}. \tag{4}$$

This implies $\mathrm{Var}[X] = 6$. From (3), $\mathrm{Cov}[X, W] = 6a + 3b = 0$, or $b = -2a$.

Solution 1.11: The joint PDF of X and Y and the region of nonzero probability are

$$f_{X,Y}(x, y) = \begin{matrix} 5x^2/2 & -1 \le x \le 1, 0 \le y \le x^2, \\ 0 & \text{otherwise} \end{matrix}. \tag{5}$$

(a) The first moment of X is

$$\mathrm{E}[X] = \int_{-1}^{1} \int_{0}^{x^2} x \frac{5x^2}{2} dy dx = \int_{-1}^{1} \frac{5x^5}{2} dx = \left. \frac{5x^6}{12} \right|_{-1}^{1} = 0. \tag{6}$$

Since $\mathrm{E}[X] = 0$, the variance of X and the second moment are both

$$\mathrm{Var}[X] = \mathrm{E}[X^2] = \int_{-1}^{1} \int_{0}^{x^2} x^2 \frac{5x^2}{2} dy dx = \left. \frac{5x^7}{14} \right|_{-1}^{1} = \frac{10}{14}. \tag{7}$$

(b) The first and second moments of Y are

$$\mathrm{E}[Y] = \int_{-1}^{1} \int_{0}^{x^2} y \frac{5x^2}{2} dy dx = \frac{5}{14}, \tag{8}$$

$$\mathrm{E}[Y^2] = \int_{-1}^{1} \int_{0}^{x^2} y^2 \frac{5x^2}{2} dy dx = \frac{5}{26}. \tag{9}$$

Therefore, $\mathrm{Var}[Y] = 5/26 - (5/14)^2 = .0576$.

(c) Since $\mathrm{E}[X] = 0, \mathrm{Cov}[X, Y] = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y] = \mathrm{E}[XY]$. Thus,

$$\mathrm{Cov}[X, Y] = \mathrm{E}[XY] = \int_{-1}^{1} \int_{0}^{x^2} xy \frac{5x^2}{2} dy dx = \int_{-1}^{1} \frac{5x^2}{4} dx = 0. \tag{10}$$

(d) The expected value of the sum $X + Y$ is

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y] = \frac{5}{14}. \tag{11}$$

(e) By Theorem 5.12, the variance of $X + Y$ is

$$\begin{aligned}
\mathrm{Var}[X + Y] &= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}[X, Y] \\
&= 5/7 + 0.0576 = 0.7719
\end{aligned} \tag{12}$$

Solution 1.12: Since 50 cents of each dollar ticket is added to the jackpot,

$$J_{i-1} = J_i + \frac{N_i}{2}. \tag{13}$$

Given $J_i = j, N_i$ has a Poisson distribution with mean j. It follows that $\mathrm{E}[N_i|J_i = j] = j$ and that $\mathrm{Var}[N_i|J_i = j] = j$. This implies

$$\mathrm{E}[N_i^2|J_i = j] = \mathrm{Var}[N_i|J_i = j] + (\mathrm{E}[N_i|J_i = j])^2 = j + j^2. \tag{14}$$

In terms of the conditional expectations given J_i, these facts can be written as

$$\mathrm{E}[N_i|J_i] = J_i, \quad \mathrm{E}[N_i^2|J_i] = J_i + J_i^2. \tag{15}$$

This permits us to evaluate the moments of J_{i-1} in terms of the moments of J_i. Specifically,

$$\mathrm{E}[J_{i-1}|J_i] = \mathrm{E}[J_i|J_i] + \frac{1}{2}\mathrm{E}[N_i|J_i] = J_i + \frac{J_i}{2} = \frac{3J_i}{2} \tag{16}$$

Using the iterated expectation, this implies

$$\mathrm{E}[J_{u-1}] = \mathrm{E}[\mathrm{E}[J_{i-1}|J_i]] = \frac{3}{2}\mathrm{E}[J_i] \tag{17}$$

We can use this to calculate $\mathrm{E}[J_i]$ for all i. Since the jackpot starts at 1 million dollars, $J_6 = 10^6$. This implies

$$\mathrm{E}[J_i] = (3/2)^{6-i}10^6. \tag{18}$$

Solution 1.13: As given in the problem statement, we define the m-dimensional vector $\mathbf{X}$, the n-dimensional vector $\mathbf{Y}$ and $\mathbf{W} = \begin{bmatrix}\mathbf{X}' \\ \mathbf{Y}'\end{bmatrix}'$. Note that $\mathbf{W}$ has expected value

$$\boldsymbol{\mu}_{\mathbf{W}} = \mathrm{E}[\mathbf{W}] = \mathrm{E}\begin{bmatrix}\mathbf{X} \\ \mathbf{Y}\end{bmatrix} = \begin{bmatrix}\mathrm{E}[\mathbf{X}] \\ \mathrm{E}[\mathbf{Y}]\end{bmatrix} = \begin{bmatrix}\boldsymbol{\mu}_{\boldsymbol{X}} \\ \boldsymbol{\mu}_{\boldsymbol{Y}}\end{bmatrix}. \tag{19}$$

The covariance matrix of $\mathbf{W}$ is

$$\begin{aligned}
\mathbf{C}_{\mathbf{W}} =& \mathrm{E}[(\mathbf{W} - \boldsymbol{\mu}_{\boldsymbol{W}})(\mathbf{W} - \boldsymbol{\mu}_{\boldsymbol{W}})'] = \\
&= \mathrm{E}\left[\begin{bmatrix}\mathbf{X} - \boldsymbol{\mu}_{\boldsymbol{X}} \\ \mathbf{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}}\end{bmatrix}[(\mathbf{X} - \boldsymbol{\mu}_{\boldsymbol{X}})' \quad (\mathbf{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}})']\right] \\
&= \begin{bmatrix}\mathrm{E}[(\mathbf{X} - \boldsymbol{\mu}_{\boldsymbol{X}})(\mathbf{X} - \boldsymbol{\mu}_{\boldsymbol{X}})'] & \mathrm{E}[(\mathbf{X} - \boldsymbol{\mu}_{\boldsymbol{X}})(\mathbf{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}})'] \\ \mathrm{E}[(\mathbf{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}})(\mathbf{X} - \boldsymbol{\mu}_{\boldsymbol{X}})'] & \mathrm{E}[(\mathbf{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}})(\mathbf{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}})']\end{bmatrix} \\
&= \begin{bmatrix}\mathbf{C}_{\mathbf{X}} & \mathbf{C}_{\mathbf{XY}} \\ \mathbf{C}_{\mathbf{YX}} & \mathbf{C}_{\mathbf{Y}}\end{bmatrix}.
\end{aligned} \tag{20}$$

The assumption that $\mathbf{X}$ and $\mathbf{Y}$ are independent implies that

$$\mathbf{C}_{\mathbf{XY}} = \mathrm{E}[(\mathbf{X} - \boldsymbol{\mu}_{\boldsymbol{X}})(\mathbf{Y}' - \boldsymbol{\mu}_{\boldsymbol{Y}}')] = \mathrm{E}[(\mathbf{X} - \boldsymbol{\mu}_{\boldsymbol{X}})]\mathrm{E}[(\mathbf{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}})'] = \mathbf{0}. \tag{21}$$

This also implies that $\mathbf{C}_{\mathbf{YX}} = \mathbf{C}_{\mathbf{XY}}' = \mathbf{0}$. Thus

$$\mathbf{C}_{\mathbf{W}} = \begin{bmatrix}\mathbf{C}_{\mathbf{X}} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\mathbf{Y}}\end{bmatrix}, \tag{22}$$

1.2 *Stochastic processes and random signals*

Solution 1.14: Since $X(t)$ is stationary, we have $\Pr(X(t+\tau)=x)=\Pr(X(t)=x)$. Then we have

$$\begin{aligned}\Pr(Y(t+\tau)=y) &= \Pr(g(X(t+\tau))=y) && (23)\\ &= \sum_{x \text{ s.t. } g(x)=y} \Pr(X(t+\tau)=x) = \sum_{x \text{ s.t. } g(x)=y} \Pr(X(t)=x) && (24)\\ &= \Pr(g(X(t))=y) = \Pr(Y(t))=y) && (25)\end{aligned}$$

Solution 1.15:

(a) Since $X(t)$ and $Y(t)$ are independent processes,

$$E[W(t)] = E[X(t)Y(t)] = E[X(t)]E[Y(t)] = \mu_X \mu_Y. \quad (1)$$

In addition,

$$R_W(t,\tau) = R_X(\tau)R_Y(\tau). \quad (2)$$

We can conclude that $W(t)$ is wide sense stationary.

(b) To examine whether $W(t)$ and $X(t)$ are jointly wide sense stationary, we calculate

$$R_{WX}(t,\tau) = E[W(t)X(t+\tau)] = E[X(t)Y(t)X(t+\tau)]. \quad (3)$$

By independence of $X(t)$ and $Y(t)$,

$$R_{WX}(t,\tau) = E[X(t)X(t+\tau)]E[Y(t)] = \mu_Y R_X(\tau). \quad (4)$$

Since $W(t)$ and $X(t)$ are both wide sense stationary and since $R_{WX}(t,\tau)$ depends only on the time difference τ, we can conclude from Defininition 13.18 that $W(t)$ and $X(t)$ are jointly wide sense stationary.

Solution 1.16: For the $X(t)$ process to be stationary, we must have $f_{X(t_1)}(x) = f_{X(t_2)}(x)$. Since $X(t_1)$ and $X(t_2)$ are both Gaussian and zero mean, this requires that

$$\sigma_1^2 = \mathrm{Var}[X(t_1)] = \mathrm{Var}[X(t_2)] = \sigma_2^2. \quad (1)$$

In addition the correlation coefficient of $X(t_1)$ and $X(t_2)$ must satisfy

$$|\rho_{X(t_1),X(t_2)}| \le 1. \quad (2)$$

This implies

$$\rho_{X(t_1),X(t_2)} = \frac{\mathrm{Cov}|X(t_1),X(t_2)|}{\sigma_1\sigma_2} = \frac{1}{\sigma_2^2} \le 1. \quad (3)$$

Thus $\sigma_1^2 = \sigma_2^2 \ge 1$.

Solution 1.17: Not available.

Solution 1.18: Note that the variance of $w[n]$ is a constant.

(a) Since uncorrelatedness implies independence for Gaussian random variables, $w[n]$ is an independent random sequence. Since its mean and variance are constants, it is at least stationary in the first order. Furthermore, we have

$$r_w[n_1, n_2] = \sigma^2 \delta[n_1 - n_2] = \delta[n_1 - n_2]. \tag{4}$$

Hence $w[n]$ is also a WSS random process.

(b) The mean of $x[n]$ is zero for all n since $w[n]$ is a zero-mean process. Consider

$$\begin{aligned} r_w[n_1, n_2] &= Ex[n_1]x[n_2] = E(w[n_1] + w[n_1 - 1])(w[n_2] + w[n_2 - 1]) = \\ &= r_w[n_1, n_2] + r_w[n_1, n_2 - 1] + r_w[n_1 - 1, n_2] + r_w[n_1 - 1, n_2 - 1] \\ &= \sigma^2\delta[n_1 - n_2] + \sigma^2\delta[n_1 - n_2 + 1] + \sigma^2\delta[n_1 - n_2 + 1] + \sigma^2\delta[n_1 - 1 - n_2 + 1] \\ &= 2\delta[n_1 - n_2] + \delta[n_1 - n_2 + 1] + \delta[n_1 - n_2 - 1]. \end{aligned} \tag{5}$$

Clearly, $r_w[n_1, n_2]$ is a function of $n1 - n2$. Hence

$$r_w[l] = 2\delta[l] + \delta[l + 1] + \delta[l - 1]. \tag{6}$$

Therefore, $x[n]$ is a WSS sequence. However, it is not an independent random sequence since both $x[n]$ and $x[n + 1]$ depend on $w[n]$.

Solution 1.19:

(a)

$$\begin{aligned} P(\mathrm{e}^{\mathrm{j}\theta}) &= \sum_{\tau=-\infty}^{\infty} \rho[\tau]\mathrm{e}^{-\mathrm{j}\tau\theta} \\ &= \sum_{\tau=-3}^{3} (3 - |\tau|)\mathrm{e}^{-\mathrm{j}\tau\theta} \\ &= \sum_{\tau=-3}^{-1} (3 + \tau)\mathrm{e}^{-\mathrm{j}\tau\theta} + \sum_{\tau=1}^{3}(3 - \tau)\mathrm{e}^{-\mathrm{j}\tau\theta} + (3 - 0)\mathrm{e}^{-\mathrm{j}0\theta} \\ &= \mathrm{e}^{\mathrm{j}2\theta} + 2\mathrm{e}^{\mathrm{j}\theta} + 2\mathrm{e}^{-\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}2\theta} + 3 \\ &= 3 + 4\cos(\theta) + 2\cos(2\theta) \end{aligned} \tag{1}$$

(b)

$$\begin{aligned} P(\mathrm{e}^{\mathrm{j}\theta}) &= \sum_{\tau=-\infty}^{\infty} \rho[\tau]\mathrm{e}^{-\mathrm{j}\tau\theta} \\ &= \sum_{\tau=-\infty}^{\infty} \delta[\tau]\mathrm{e}^{-\mathrm{j}\tau\theta} + 2\sum_{\tau=-\infty}^{\infty} (-0.6)^{|\tau|}\mathrm{e}^{-\mathrm{j}\tau\theta} \\ &= 1 + 2\left(\sum_{\tau=-\infty}^{0} (-0.6)^{-\tau}\mathrm{e}^{-\mathrm{j}\tau\theta} + \sum_{\tau=0}^{\infty}(-0.6)^{\tau}\mathrm{e}^{-\mathrm{j}\tau\theta} - 1\right) \\ &= 1 + 2\left(\sum_{\tau=0}^{\infty}(-0.6)^{\tau}\mathrm{e}^{\mathrm{j}\tau\theta} + \sum_{\tau=0}^{\infty}(-0.6)^{\tau}\mathrm{e}^{-\mathrm{j}\tau\theta} - 1\right) \\ &= 1 + 2\left(\frac{1}{1 + 0.6\mathrm{e}^{\mathrm{j}\theta}} + \frac{1}{1 + 0.6\mathrm{e}^{-\mathrm{j}\theta}} - 1\right) \\ &= 1 + 2\left(\frac{0.64}{1.36 - 1.2\cos(\theta)}\right) = 1 + \frac{1.28}{1.36 - 1.2\cos(\theta)} \end{aligned} \tag{2}$$

Both functions of (a) and (b) are indeed real and ≥ 0.

Solution 1.20:

(a) Use $X(Z) = H(Z)I(Z)$, so $x[k] - 1/2x[k-1] = i[k] - 1/3i[k-1]$. The difference equation of this signal is given by $x[k] = i[k] - \frac{1}{3}i[k-1] + \frac{1}{2}x[k-1]$. From this the following autocorrelation can be calculated:

$$\rho[0] = \frac{28}{27}; \qquad \rho[1] = \rho[-1] = \frac{5}{27}; \qquad \rho[\tau] = \left(\frac{1}{2}\right)^{|\tau|-1} \rho[1] \qquad \text{for } |\tau| \geq 2$$

$$\begin{aligned}
E[x[k]x[k]] &= E[x[k]i[k]] - 1/3E[x[k]i[k-1]] + 1/2E[x[k]x[k-1]] && (3)\\
&= E[(i[k] - 1/3i[k-1] + 1/2x[k-1])i[k]] && (4)\\
&\quad - 1/3E[(i[k] - 1/3i[k-1] + 1/2x[k-1])i[k-1]] + 1/2E[x[k]x[k-1]] && (5)\\
&= E[i[k]i[k]] - 1/3E[i[k-1]i[k]] + 1/2E[x[k-1]i[k]] && (6)\\
&\quad - 1/3E[i[k]i[k-1]] + 1/9E[i[k-1]i[k-1]] - 1/6E[x[k-1]i[k-1]] && (7)\\
&\quad + 1/2E[x[k]x[k-1]] && (8)\\
&= 1 - 0 + 0 - 0 + 1/9 - 1/6 + 1/2E[x[k]x[k-1]] \quad \text{fill in from next} && (9)\\
&= 17/18 + 1/2(-1/3 + 1/2E[x[k]x[k]]) && (10)\\
3/4E[x[k]x[k]] &= 14/18 && (11)\\
E[x[k]x[k]] &= 28/27 && (12)
\end{aligned}$$

$$\begin{aligned}
E[x[k]x[k+1]] &= E[x[k]i[k+1]] - 1/3E[x[k]i[k]] + 1/2E[x[k]x[k]] && (13)\\
&= E[(i[k] - 1/3i[k-1] + 1/2x[k-1])i[k+1]] && (14)\\
&\quad - 1/3E[(i[k] - 1/3i[k-1] + 1/2x[k-1])i[k]] + 1/2E[x[k]x[k]] && (15)\\
&= E[i[k]i[k+1]] - 1/3E[i[k-1]i[k+1]] + 1/2E[x[k-1]i[k+1]] && (16)\\
&\quad - 1/3E[i[k]i[k]] + 1/9E[i[k-1]i[k]] - 1/6E[x[k-1]i[k]] + 1/2E[x[k]x[k]] && (17)\\
&= 0 - 0 + 0 - 1/3 + 0 - 0 + 1/2E[x[k]x[k]] \quad \text{fill in from previous} && (18)\\
&= -1/3 + 14/27 = 5/27 = E[x[k]x[k-1]] && (19)
\end{aligned}$$

assume $\tau > 1$

$$\begin{aligned}
E[x[k]x[k+\tau]] &= E[x[k]i[k+\tau]] - 1/3E[x[k]i[k+\tau-1]] + 1/2E[x[k]x[k+\tau-1]] && (20)\\
&= 0 - 0 + 1/2E[x[k]x[k+\tau-1]] && (21)\\
&= 5/27 \cdot (1/2)^{\tau-1} && (22)
\end{aligned}$$

Since the autocorrelation function is symmetric, we have $E[x[k]x[k+\tau]] = 5/27 \cdot (1/2)^{|\tau|-1}$ for $|\tau| \geq 2$.

(b) First approach is via Wiener-Khintchine:

$$
\begin{aligned}
P_x(\mathrm{e}^{\mathrm{j}\theta}) &= \sum_{\tau=-\infty}^{\infty} \rho[\tau]\mathrm{e}^{-\mathrm{j}\tau\theta} \\
&= \rho[0] + \sum_{\tau=-\infty}^{-1} \left(\frac{1}{2}\right)^{|\tau|-1} \rho[1]\mathrm{e}^{-\mathrm{j}\tau\theta} + \sum_{\tau=1}^{\infty} \left(\frac{1}{2}\right)^{|\tau|-1} \rho[1]\mathrm{e}^{-\mathrm{j}\tau\theta} \\
&= \rho[0] + \sum_{\tau=-\infty}^{0} \left(\frac{1}{2}\right)^{|\tau|} \rho[1]\mathrm{e}^{-\mathrm{j}(\tau-1)\theta} + \sum_{\tau=0}^{\infty} \left(\frac{1}{2}\right)^{|\tau|} \rho[1]\mathrm{e}^{-\mathrm{j}(\tau+1)\theta} \\
&= \rho[0] + \rho[1]\mathrm{e}^{\mathrm{j}\theta} \sum_{\tau=0}^{\infty} \left(\frac{1}{2}\right)^{\tau} \mathrm{e}^{\mathrm{j}\tau\theta} + \rho[1]\mathrm{e}^{-\mathrm{j}\theta} \sum_{\tau=0}^{\infty} \left(\frac{1}{2}\right)^{\tau} \rho[1]\mathrm{e}^{-\mathrm{j}\tau\theta} \\
&= \rho[0] + \rho[1] \left(\frac{\mathrm{e}^{\mathrm{j}\theta}}{1 - \frac{1}{2}\mathrm{e}^{\mathrm{j}\theta}} + \frac{\mathrm{e}^{-\mathrm{j}\theta}}{1 - \frac{1}{2}\mathrm{e}^{-\mathrm{j}\theta}} \right) \\
&= \rho[0] + \rho[1] \left(\frac{-1 + 2\cos(\theta)}{\frac{5}{4} - \cos(\theta)} \right) \\
&= \frac{(-\rho[1] + \frac{5}{4}\rho[0]) + (2\rho[1] - \rho[0])\cos(\theta)}{\frac{5}{4} - \cos(\theta)} = \frac{\frac{10}{9} - \frac{2}{3}\cos(\theta)}{\frac{5}{4} - \cos(\theta)}
\end{aligned} \tag{23}
$$

Other approach is via input- output relation of spectra:

$$
\begin{aligned}
P_x(\mathrm{e}^{\mathrm{j}\theta}) &= P_i(\mathrm{e}^{\mathrm{j}\theta}) \cdot \left| H(\mathrm{e}^{\mathrm{j}\theta}) \right|^2 = \sigma_i^2 \cdot \left| \frac{1 - \frac{1}{3}\mathrm{e}^{-\mathrm{j}\theta}}{1 - \frac{1}{2}\mathrm{e}^{-\mathrm{j}\theta}} \right|^2 \\
&= \sigma_i^2 \cdot \left(\frac{1 - \frac{1}{3}\mathrm{e}^{-\mathrm{j}\theta}}{1 - \frac{1}{2}\mathrm{e}^{-\mathrm{j}\theta}} \right)^* \cdot \left(\frac{1 - \frac{1}{3}\mathrm{e}^{-\mathrm{j}\theta}}{1 - \frac{1}{2}\mathrm{e}^{-\mathrm{j}\theta}} \right), * \text{ complex conjugate} \\
&= \left(\frac{1 - \frac{1}{3}\mathrm{e}^{\mathrm{j}\theta}}{1 - \frac{1}{2}\mathrm{e}^{\mathrm{j}\theta}} \right) \cdot \left(\frac{1 - \frac{1}{3}\mathrm{e}^{-\mathrm{j}\theta}}{1 - \frac{1}{2}\mathrm{e}^{-\mathrm{j}\theta}} \right) \\
&= \frac{1 + \frac{1}{9} - \frac{1}{3}\left(\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}\right)}{1 + \frac{1}{4} - \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}\right)} \\
&= \frac{\frac{10}{9} - \frac{2}{3}\cos(\theta)}{\frac{5}{4} - \cos(\theta)}
\end{aligned} \tag{24}
$$

Solution 1.21:

(a)

$$
\begin{aligned}
P(\mathrm{e}^{\mathrm{j}\theta}) &= \sum_{\tau=-\infty}^{\infty} \rho[\tau]\mathrm{e}^{-\mathrm{j}\tau\theta} \\
&= \sum_{\tau=-3}^{3} (3 - |\tau|)\mathrm{e}^{-\mathrm{j}\tau\theta} \\
&= \sum_{\tau=-3}^{-1} (3 + \tau)\mathrm{e}^{-\mathrm{j}\tau\theta} + \sum_{\tau=1}^{3} (3 - \tau)\mathrm{e}^{-\mathrm{j}\tau\theta} + (3 - 0)\mathrm{e}^{-\mathrm{j}0\theta} \\
&= \mathrm{e}^{\mathrm{j}2\theta} + 2\mathrm{e}^{\mathrm{j}\theta} + 2\mathrm{e}^{-\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}2\theta} + 3 \\
&= 3 + 4\cos(\theta) + 2\cos(2\theta)
\end{aligned} \tag{1}
$$

(b)

$$
\begin{aligned}
P(\mathrm{e}^{\mathrm{j}\theta}) &= \sum_{\tau=-\infty}^{\infty} \rho[\tau]\mathrm{e}^{-\mathrm{j}\tau\theta} \\
&= \sum_{\tau=-\infty}^{\infty} \delta[\tau]\mathrm{e}^{-\mathrm{j}\tau\theta} + 2\sum_{\tau=-\infty}^{\infty} (-0.6)^{|\tau|}\mathrm{e}^{-\mathrm{j}\tau\theta} \\
&= 1 + 2\left(\sum_{\tau=-\infty}^{0} (-0.6)^{-\tau}\mathrm{e}^{-\mathrm{j}\tau\theta} + \sum_{\tau=0}^{\infty}(-0.6)^{\tau}\mathrm{e}^{-\mathrm{j}\tau\theta} - 1\right) \\
&= 1 + 2\left(\sum_{\tau=0}^{\infty}(-0.6)^{\tau}\mathrm{e}^{\mathrm{j}\tau\theta} + \sum_{\tau=0}^{\infty}(-0.6)^{\tau}\mathrm{e}^{-\mathrm{j}\tau\theta} - 1\right) \\
&= 1 + 2\left(\frac{1}{1+0.6\mathrm{e}^{\mathrm{j}\theta}} + \frac{1}{1+0.6\mathrm{e}^{-\mathrm{j}\theta}} - 1\right) \\
&= 1 + 2\left(\frac{0.64}{1.36 - 1.2\cos(\theta)}\right) = 1 + \frac{1.28}{1.36 - 1.2\cos(\theta)}
\end{aligned} \tag{2}
$$

Both functions of (a) and (b) are indeed real and ≥ 0.

1.3 *Rational signal models*

Solution 1.22: The given filter $H(z)$ is all-pass thus we have:

$$P_x(\mathrm{e}^{\mathrm{j}\theta}) = P_i(\mathrm{e}^{\mathrm{j}\theta}) \cdot \left|H(\mathrm{e}^{\mathrm{j}\theta})\right|^2 = \sigma_i^2 \cdot 4 = 1$$

The spectrum of process $x[k]$ is independent of θ and thus white noise: $\rho_x[0] = 1$ and $\rho_x[\tau] = 0$ for $|\tau| \geq 1$.
<u>Alternative:</u>

$$
\begin{aligned}
x[k] &= i[k] - 2i[k-1] + \frac{1}{2}x[k-1] \;\Rightarrow \\
\rho[0] &= \sigma_i + 4\sigma_i + \frac{1}{4}\rho[0] - 2\sigma_i \;\Rightarrow \rho[0] = 1 \\
\rho[1] &= -2\sigma_i + \frac{1}{2}\rho[0] = 0 \\
\rho[2] &= \dots \\
\dots & \quad \dots
\end{aligned}
$$

Solution 1.23:

(a) Signal is stationary since statistical properties, such as average and autocorrelation, are independent of time.

(b) The system function of the signalmodel can be written as:

$$H(z) = 1 + 0.1z^{-1} - 0.2z^{-2} = (1 + 0.5z^{-1}) \cdot (1 - 0.4z^{-1})$$

This system contains, besides two poles at location 0, two zeros at locations 0.4 and -0.5. Both zeros are within the unit circle, thus the system is minimum-phase.

(c) The autocorrelation function is given by:

$$\rho[\tau] = \begin{cases} 1.05 & \text{for } \tau = 0 \\ 0.08 & \text{for } \tau = \pm 1 \\ -0.2 & \text{for } \tau = \pm 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution 1.24:

$$P(z) = \frac{5 - 2(z^{-1} + z)}{10 - 3(z^{-1} + z)} = \frac{c_n}{c_d} \cdot \frac{1 - az^{-1}}{1 - bz^{-1}} \cdot \frac{1 - az}{1 - bz} = \sigma_i^2 \cdot L(z) \cdot L(z^{-1})$$

with contacts c_n, c_d, a and b, above that $|a| < 1$ and $|b| < 1$. In this way the innovation filter $L(z)$ is always minimum phase and the first term equals one, thus $L(z) = 1 + \cdot z^{-1} + \cdots$. Now we obtain the following set of equation:

$$\begin{aligned} c_n(1 - az^{-1})(1 - az) &= 9 - 2(z^{-1} + z) \\ &\Rightarrow (a = \frac{1}{2} \text{ and } c_n = 4) \text{ or } (a = 2 \text{ and } c_n = 1) \\ c_d(1 - bz^{-1})(1 - bz) &= 10 - 3(z^{-1} + z) \\ &\Rightarrow (b = \frac{1}{3} \text{ and } c_d = 9) \text{ or } (b = 3 \text{ and } c_d = 1) \end{aligned}$$

Since we need to choose that solution which results in a minimum phase innovation filter, this results in:

$$P(z) = \frac{4}{9} \cdot \left(\frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}} \right) \cdot \left(\frac{1 - \frac{1}{2}z}{1 - \frac{1}{3}z} \right) = \sigma_i^2 \cdot L(z) \cdot L(z^{-1})$$

Thus $L(z) = (1 - \frac{1}{2}z^{-1})/(1 - \frac{1}{3}z^{-1})$ and $\sigma_i^2 = 4/9$. A realization scheme of this process is given in Fig.1.

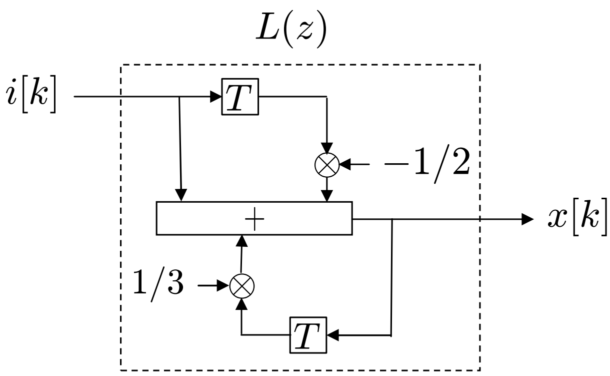


Figure 1: Innovation filter for process $x[k]$ of exercise 3.6

Furthermore the compression gain can be calculated via $G = \sum_{p=0}^{\infty} l_p^2$. In this equation the coefficients l_p are derived from the expression of the innovation filter as an infinite polynomial. In this example this leads to:

$$L(z) = \sum_{p=0}^{\infty} l_p z^{-p} \quad \Rightarrow \quad L(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}} = 1 - (\frac{1}{6}) \sum_{p=1}^{\infty} (\frac{1}{3})^{p-1} z^{-p}$$

thus

$$G = \sum_{p=0}^{\infty} l_p^2 = 1 + (\frac{1}{36}) \sum_{p=1}^{\infty} (\frac{1}{9})^{p-1} = 1 + (\frac{1}{36}) \cdot (\frac{9}{8}) = \frac{33}{32}$$

Alternative:
An alternative expression is $G = \rho[0]/\sigma_i$ with $\rho[0] = \frac{11}{24} \cdot \frac{4}{9}$ and $\sigma_i = 1$.

Solution 1.25:

$$
\begin{aligned}
P(z) &= \sum_{\tau=-\infty}^{\infty} \rho[\tau] z^{-\tau} = \sum_{\tau=-\infty}^{\infty} \rho[2\tau] z^{-2\tau} = \sum_{\tau=-\infty}^{\infty} (\frac{1}{3})^{|\tau|} z^{-2\tau} \\
&= \sum_{\tau=-\infty}^{0} (\frac{1}{3})^{-\tau} z^{-2\tau} + \sum_{\tau=0}^{\infty} (\frac{1}{3})^{\tau} z^{-2\tau} - 1 = \frac{1}{1-\frac{1}{3}z^{2}} + \frac{1}{1-\frac{1}{3}z^{-2}} - 1 \\
&= \frac{8}{9} \cdot \left(\frac{1}{1-\frac{1}{3}z^{-2}}\right) \cdot \left(\frac{1}{1-\frac{1}{3}z^{2}}\right) = \sigma_i^2 \cdot L(z) \cdot L(z^{-1})
\end{aligned}
$$

Thus $\sigma_i^2 = \frac{8}{9}$ and $L(z) = \frac{1}{1-\frac{1}{3}z^{-2}}$. A realization scheme is depicted in Fig. 2.

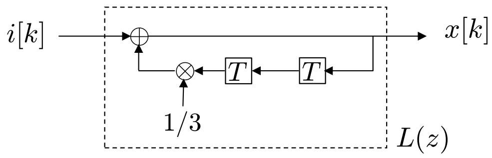


Figure 2: Realization of process $x[k]$ of exercise 3.8

Solution 1.26: Note that the zeros of filter $H(z)$ are outside the unit circle ($r = \sqrt{0.8^2 + 0.8^2} = 1.13$) and thus inverting of this filter would lead to an unstable solution.So first write this filter as the cascade of an all-pass and a minimum phase section as $H(z) = H^a(z) \cdot H^m(z)$ as depicted in Fig. 3.

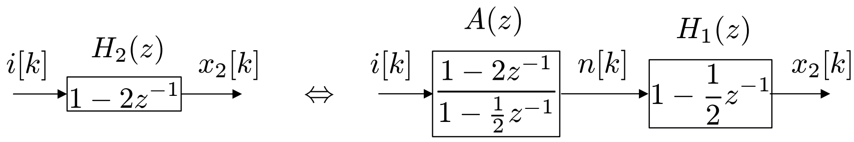


Figure 3: Filter as a cascade of all-pass and minimum phase section

With $\alpha = a + \mathrm{j}a$ ($a = 0.8$), $b = 0.9$ and c some constant we have:

$$
H(z) = H^a(z) \cdot H^m(z) = c \cdot \frac{(1-\alpha z^{-1}) \cdot (1-\alpha^* z^{-1})}{(1-\frac{1}{\alpha^*}z^{-1}) \cdot (1-\frac{1}{\alpha}z^{-1})} \cdot \frac{1}{c} \cdot \frac{(1-\frac{1}{\alpha^*}z^{-1}) \cdot (1-\frac{1}{\alpha}z^{-1})}{1-bz^{-1}}
$$

Since $H^a(z)$ is all-pass signal $i[k]$ is also white noise:

$$
P_i(\mathrm{e}^{\mathrm{j}\theta}) = \left|H^a(\mathrm{e}^{\mathrm{j}\theta})\right|^2 \cdot P_n(\mathrm{e}^{\mathrm{j}\theta}) \qquad \Rightarrow \qquad \sigma_i^2 = c^2 \cdot |\alpha|^4 \cdot \sigma_n^2
$$

Since we need to design the whitening filter such that $\sigma_i^2 = \sigma_n^2$ this results in $c = \pm\frac{1}{|\alpha|^2}$ and the whitening filter:

$$
\Gamma(z) = \frac{1}{H^m(z)} = \frac{1-0.9z^{-1}}{1.28 - 1.2z^{-1} + z^{-2}}
$$

Solution 1.27:

(a)

$$
L(z) = \frac{1-\frac{1}{4}z^{-2}}{1+\frac{1}{2}z^{-1}} = 1 - \frac{1}{2}z^{-1}
$$

Thus the difference equation for the generation of this process equals the following MA(1) description: $x[k] = i[k] - \frac{1}{2}i[k-1]$. From this the autocorrelation results in:

$$
\rho[\tau] = \begin{cases} \frac{5}{2} & \text{for } \tau = 0 \\ -1 & \text{for } \tau = \pm 1 \\ 0 & \text{elsewhere} \end{cases}
$$

(b) From the structure of the autocorrelation function it follows that the process must be AR(1). Thus the difference equation must have the form: $x[k] = i[k] + \alpha \cdot x[k-1]$. Using this description leads to the following autocorrelation function:

$$\rho[\tau] = \frac{\sigma_i^2}{1-\alpha^2} \cdot (\alpha)^{|\tau|} \qquad \Leftrightarrow \qquad (\frac{1}{3})^{|\tau|}$$

From this it follows that $\alpha = \frac{1}{3}$ and $\sigma_i^2 = \frac{8}{9}$ thus the innovation filter must have the following form:

$$L(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} \qquad \Rightarrow \qquad a = 0 \; ; \; b = \frac{1}{3} \text{ and } \sigma_i^2 = \frac{8}{9}$$

Solution 1.28:

(a) Start with

$$w[n] = s[n] * h[n] \tag{3}$$

where,

$$h[n] = (0.9)^n u[n] \tag{4}$$

Hence

$$r_s[l] = h[l] * h[-l] * r_w[l] = 0.64[h[l] * h[-l]] \tag{5}$$

Using

$$[a^l u[n]] * [a^{-l} u[n]] = \frac{a^{|l|}}{1-a^2} \quad |a| < 1 \tag{6}$$

and

$$r_s[l] = \frac{0.64}{0.19}(0.9)^{|l|} = \frac{64}{19}(0.9)^{|l|} \tag{7}$$

The PSD is

$$P_s(e^{j\omega}) = \sum_{-\infty}^{\infty} r_s[l] e^{-j\omega l} = \frac{64}{19}\sum_{0}^{\infty}(0.9)^l e^{-j\omega l} + \sum_{-1}^{-\infty}(0.9)^{-l} e^{-j\omega l} \tag{8}$$

Therefore,

$$P_s(e^{j\omega}) = \frac{0.64}{1.81 - 1.8\cos\omega} \tag{9}$$

The plot is obtained using the Matlab script and is shown in Figure 6.20a.

Part 2: Estimation theory

2.1 *Cramer-Rao Lower Bound*

Solution 2.1:

(a) Positive definite $\rightarrow z^T M z > 0, \quad \forall z \neq 0$ and all eigenvalues are positive.
$\mathbf{I}^{-1}(\theta) = \frac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \rightarrow [\mathbf{I}^{-1}(\theta)]_{11} = \frac{c}{ac-b^2}$
Since $a \geq 0$ and $c \geq 0$ and $b^2 \geq 0$: $\frac{c}{ac-b^2} \geq \frac{c}{ac} = \frac{1}{a}$

(b) When the second parameter is unknown, the variance of the estimation of the first parameter is $\frac{c}{ac-b^2}$. However, when the second parameter is known the variance is $\frac{1}{a}$. This means that if more parameters are unknown, the variance and thus the uncertainty of the estimation of the parameters is bigger.

In other words, the CRLB is almost always increased when we estimate additional parameters.

(c) Equality holds when $b = 0$, which means that the parameters are not correlated.

In other words, the Fisher Information matrix is decoupled, i.e., it is diagonal. In this case, the additional parameter does not affect the CRLB.

2.2 *Maximum likelihood estimator*

Solution 2.2:

$$\begin{aligned}
Pr\{\boldsymbol{x}; p\} &= \prod_{n=0}^{N-1} p^{x[n]}(1-p)^{1-x[n]} \\
&= p^{\sum_{n=0}^{N-1} x[n]}(1-p)^{N-\sum_{n=0}^{N-1} x[n]} \\
\log(Pr\{\boldsymbol{x}; p\}) &= \left(\sum_{n=0}^{N-1} x[n]\right)\log(p) + \left(N - \sum_{n=0}^{N-1} x[n]\right)\log(1-p) \\
\frac{\partial \log(Pr\{\boldsymbol{x}; p\})}{\partial p} &= \frac{\sum_{n=0}^{N-1} x[n]}{p} - \frac{\left(N - \sum_{n=0}^{N-1} x[n]\right)}{1-p} = 0 \\
&= \left(\sum_{n=0}^{N-1} x[n]\right)(1-p) - p\left(N - \sum_{n=0}^{N-1} x[n]\right) = 0 \\
\hat{p}_{MLE} &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]
\end{aligned} \tag{10}$$

Solution 2.2: This is a simple extension of the problem presented in the first example provided in the MLE module. The likelihood function is given as

$$p(\mathbf{x}; \theta) = \begin{cases} \frac{1}{(2\theta)^N}, & \text{if } |x_i| \leq \theta, \quad i = 1, 2, \ldots, N \\ 0, & \text{else.} \end{cases} \tag{11}$$

The likelihood function is zero if at least on observation exceeds θ. However, since we observe the sequence, we know that θ must be larger than or equal to $\max_i |x_i|$. Furthermore, the likelihood function is monotonically decreasing. Thus, the MLE is $\hat{\theta}_{\mathrm{ML}} = \max_i . |x_i|$

Solution 2.2:

(a) The likelihood function is given as

$$p(\mathbf{x},\mathbf{y};\lambda) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2\right) \frac{1}{(2\pi\lambda\sigma^2)^{\frac{N}{2}}} \exp\left(-\frac{1}{2\lambda\sigma^2}\sum_{i=1}^{N}(y_i-\mu)^2\right), \tag{12}$$

and the log-likelihood function is

$$\mathcal{L}(\mathbf{x},\mathbf{y};\lambda) = -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2 - \frac{N}{2}\ln(2\pi\lambda\sigma^2) - \frac{1}{2\lambda\sigma^2}\sum_{i=1}^{N}(y_i-\mu)^2. \tag{13}$$

Equating the derivative of the log-likelihood function with respect to λ with zero yields

$$\frac{N}{2\lambda} = \frac{1}{2\lambda^2\sigma^2}\sum_{i=1}^{N}(y_i-\mu)^2 \tag{14}$$

Solving for λ gives us the MLE estimate, which is

$$\hat{\lambda}_{\text{ML}} = \frac{1}{N\sigma^2}\sum_{i=1}^{N}(y_i-\mu)^2. \tag{15}$$

(It can also be shown that the second derivative of the log-likelihood function is indeed negative.)

(b) The likelihood function for this problem is the equivalent to (12). However, σ^2 is now considered a parameter, and thus, we denote the likelihood function by $p(\mathbf{x},\mathbf{y};\lambda,\sigma^2)$. The derivative with respect to λ equated to zero gives us, as shown in (a),

$$\lambda = \frac{1}{N\sigma^2}\sum_{i=1}^{N}(y_i-\mu)^2. \tag{16}$$

Differentiation of the log-likelihood function with respect to σ^2 yields

$$\begin{aligned}
\frac{\partial\mathcal{L}(\mathbf{x},\mathbf{y};\lambda,\sigma^2)}{\partial\sigma^2} &= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(x_i-\mu)^2 - \frac{N}{2\sigma^2} + \frac{1}{2\lambda\sigma^4}\sum_{i=1}^{N}(y_i-\mu)^2 &(17)\\
&= -\frac{N}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(x_i-\mu)^2 + \frac{1}{2\lambda\sigma^4}\sum_{i=1}^{N}(y_i-\mu)^2 &(18)
\end{aligned}$$

Substituting (16) in (18) and equating the derivative of the log-likelihood function with zero yields

$$\begin{aligned}
-\frac{N}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(x_i-\mu)^2 + \frac{1}{2\lambda\sigma^4}\sum_{i=1}^{N}(y_i-\mu)^2 &= 0 &(19)\\
-\frac{N}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(x_i-\mu)^2 + \frac{N}{2\sigma^2} &= 0 &(20)\\
-\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(x_i-\mu)^2 &= 0 &(21)\\
-\frac{N}{\sigma^2} + \frac{1}{\sigma^4}\sum_{i=1}^{N}(x_i-\mu)^2 &= 0 &(22)
\end{aligned}$$

which after solving for σ^2 yields

$$\hat{\sigma^2}_{\text{ML}} = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2. \tag{23}$$

Substituting (23) in (16) gives us

$$\hat{\lambda}_{\text{ML}} = \frac{\sum_{i=1}^{N}(y_i-\mu)^2}{\sum_{i=1}^{N}(x_i-\mu)^2}. \tag{24}$$

2.3 *MVUE for linear models*

Solution 2.3:

$$p(x) = \begin{cases} \frac{1}{\theta} & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

$$p(\boldsymbol{x};\theta) = \prod_{n=0}^{N-1} p(x[n]) = \begin{cases} \frac{1}{\theta^N} & 0 < x[n] < \theta \quad \forall n = 0,\dots,N-1 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

Clearly, $\max_\theta(p(x;\theta)) = \max_\theta\left(\frac{1}{\theta^N}\right)$, so the likelihood is maximized for minimized θ. However, $\theta > x[n]$ for all $x[n]$. Therefore $\hat{\theta} = \max\big(x[n]\big)$.

Solution 2.3:

(a) The linear model for the problem is given as

$$\mathbf{x} = \mathbf{H}\mathbf{a} + \mathbf{w}, \tag{27}$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ \vdots & \\ -1 & 1 \end{bmatrix} \tag{28}$$

and

$$\mathbf{a} = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} \tag{29}$$

(b) The MVU for linear models in AWGN is

$$\hat{a} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x}. \tag{30}$$

Note that the columns of $\mathbf{H}$ are orthogonal. Thus,

$$\mathbf{H}^T\mathbf{H} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \tag{31}$$

and

$$\left(\mathbf{H}^T\mathbf{H}\right)^{-1} = \begin{bmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{N} \end{bmatrix}. \tag{32}$$

The MVUE is given as

$$\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T = \frac{1}{N}\mathbf{H}^T. \tag{33}$$

2.4 *Least-square estimation*

Solution 2.4: The cost function is given by

$$J(a) = \sum_{n=0}^{N-1} \frac{1}{\sigma^2[n]}(x[n] - A)^2 \tag{34}$$

The minimum of the cost function is obtained by taking the first derivative and setting it to zero as

$$\frac{dJ(a)}{dA} = -2\sum_{n=0}^{N-1}\frac{1}{\sigma^2[n]}(x[n]-A) = 0. \tag{35}$$

Then $\hat{A}$ is given by

$$\hat{A} = \frac{\sum_{n=0}^{N-1}\frac{x[n]}{\sigma^2[n]}}{\sum_{n=0}^{N-1}\frac{1}{\sigma^2[n]}} \tag{36}$$

and the expectation and variance are calculated as

$$\begin{aligned}
E[\hat{A}] =& \frac{\sum_{n=0}^{N-1}\frac{E[x[n]]}{\sigma^2[n]}}{\sum_{n=0}^{N-1}\frac{1}{\sigma^2[n]}} = A \\
\mathrm{var}[\hat{A}] =& \frac{\sum_{n=0}^{N-1}\mathrm{var}\left[\frac{x[n]}{\sigma^2[n]}\right]}{(\sum_{n=0}^{N-1}\frac{1}{\sigma^2[n]})^2} = \frac{\sum_{n=0}^{N-1}\frac{\mathrm{var}[x[n]]}{\sigma^4[n]}}{(\sum_{n=0}^{N-1}\frac{1}{\sigma^2[n]})^2} = \frac{\sum_{n=0}^{N-1}\frac{\sigma^2[n]}{\sigma^4[n]}}{(\sum_{n=0}^{N-1}\frac{1}{\sigma^2[n]})^2} \\
=& \frac{1}{\sum_{n=0}^{N-1}\frac{1}{\sigma^2[n]}}
\end{aligned} \tag{37}$$

2.5 *Bayesian estimation*

Solution 2.5:

$$\begin{aligned}
p(\boldsymbol{x}|\theta) &= \exp\left(-\sum_{n=0}^{N-1}x[n]-\theta\right) \quad \text{if } x[n]\geq\theta \quad \forall n=0,\dots N-1 && (38)\\
&= \exp\big(-N\left(\overline{\boldsymbol{x}}-\theta\right)\big) && (39)\\
p(\theta|\boldsymbol{x}) &= \frac{p(\boldsymbol{x}|\theta)p(\theta)}{\int p(\boldsymbol{x}|\theta)p(\theta)\,\mathrm{d}\theta} && (40)\\
&= \frac{\exp\big(-N\left(\overline{\boldsymbol{x}}-\theta\right)\big)\exp\left(-\theta\right)}{\int_0^{\min(\boldsymbol{x})}\exp\big(-N\left(\overline{\boldsymbol{x}}-\theta\right)\big)\exp\left(-\theta\right)\mathrm{d}\theta} && (41)\\
&= \frac{\exp\big(\theta\left(N-1\right)\big)}{\int_0^{\min(\boldsymbol{x})}\exp\big(\theta\left(N-1\right)\big)\,\mathrm{d}\theta} && (42)\\
&= \frac{\exp\big(\theta\left(N-1\right)\big)}{\left[\frac{\exp\left(\theta(N-1)\right)}{N-1}\right]_0^{\min(\boldsymbol{x})}} && (43)\\
&= \frac{(N-1)\exp\big(\theta\left(N-1\right)\big)}{\exp\big(\min\left(\boldsymbol{x}\right)(N-1)\big)-1} \quad 0\leq\theta\leq\min\left(\boldsymbol{x}\right) && (44)\\
& && (45)
\end{aligned}$$

then,

$$
\begin{aligned}
\hat{\theta} &= E[\theta|\boldsymbol{x}] = \int_0^{\min(\boldsymbol{x})} \theta \frac{(N-1)\exp\big(\theta\,(N-1)\big)}{\exp\big(\min(\boldsymbol{x})\,(N-1)\big)-1}\,\mathrm{d}\theta && (46)\\
&= \frac{(N-1)}{\exp\big(\min(\boldsymbol{x})\,(N-1)\big)-1}\left[\frac{\theta\,(N-1)-1}{(N-1)^2}\exp\big(\theta\,(N-1)\big)\right]_0^{\min(\boldsymbol{x})} && (47)\\
&= \frac{\big(\min(\boldsymbol{x})\,(N-1)-1\big)\exp\big(\min(\boldsymbol{x})\,(N-1)\big)+1}{(N-1)\Big(\exp\big(\min(\boldsymbol{x})\,(N-1)\big)-1\Big)} && (48)\\
&= \frac{\min(\boldsymbol{x})\,(N-1)\exp\big(\min(\boldsymbol{x})\,(N-1)\big)}{(N-1)\Big(\exp\big(\min(\boldsymbol{x})\,(N-1)\big)-1\Big)} - \frac{1}{N-1} && (49)\\
&= \frac{\min(\boldsymbol{x})}{1-\exp\big(-\min(\boldsymbol{x})\,(N-1)\big)} - \frac{1}{N-1} && (50)\\
& && (51)
\end{aligned}
$$

Solution 2.6:

(a) Prior PDF: $\mathcal{N}(100, 0.011)$, with $\mu_R = 100$ and $\sigma_R^2 = 0.011$.
Data model: $x[n] = R + w[n] \quad n = 0, 1, \ldots, N-1$, where $w[n] \sim \mathcal{N}(0,1)$ and iid.

$$
\begin{aligned}
p(\boldsymbol{x}|R) &= \frac{1}{(2\pi\sigma_w^2)^{N/2}}\exp\left(-\frac{1}{2}\sum_{n=0}^{N-1}\left(\frac{x[n]-R}{\sigma_w}\right)^2\right) && (52)\\
&= \frac{1}{(2\pi\sigma_w^2)^{N/2}}\exp\left(-\frac{1}{2\sigma_w^2}\sum_{n=0}^{N-1}x^2[n] - 2RN\overline{\boldsymbol{x}} + NR^2\right) && (53)
\end{aligned}
$$

(b) Then $\hat{R} = E[R|\boldsymbol{x}] = \mu_{R|\boldsymbol{x}}$

(c)

$$
\begin{aligned}
B_{MSE}(\hat{R}) &= E[(R-\hat{R})^2] = \int\int (R-\hat{R})^2 p(\boldsymbol{x},R)\,\mathrm{d}R\,\mathrm{d}\boldsymbol{x} && (54)\\
&= \int\int (R - E[R|\boldsymbol{x}])^2 p(R|\boldsymbol{x})\,\mathrm{d}R \quad p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} && (55)\\
&= \int \sigma_{R|\boldsymbol{x}}^2 p(\boldsymbol{x})\,\mathrm{d}x && (56)\\
&= \frac{1}{\frac{N}{\sigma_w^2}+\frac{1}{\sigma_R^2}} = \frac{1}{\frac{N}{1}+\frac{1}{0.011}} && (57)
\end{aligned}
$$

When $B_{MSE}(\hat{R}) = 0.01$ then $N \to 9.09 = 10$.

(d) Then $\sigma_R^2 \to \infty$ and $B_{MSE}(\hat{R}) \to \frac{\sigma_w^2}{N} = 1/N$ and then $B_{MSE}(\hat{R}) = 0.01$ requires that $N = 100$.

Solution 2.7: This is a Bayesian Linear model of the form $\boldsymbol{x} = \boldsymbol{H}\boldsymbol{\theta} + \boldsymbol{w}$, with $\boldsymbol{\theta} = \begin{bmatrix} A \\ B \end{bmatrix}$. Now we use Theorem 10.3 from the book, with:

$$\mu_\theta = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \tag{58}$$

$$C_\theta = \begin{bmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{bmatrix} \tag{59}$$

$$\boldsymbol{H} = \begin{bmatrix} 1 & -M \\ 1 & -(M-1) \\ \vdots & \vdots \\ 1 & M-1 \\ 1 & M \end{bmatrix} \tag{60}$$

$$C_w = \sigma^2 \boldsymbol{I} \tag{61}$$

We use the alternative form of Thm 10.3:

$$\boldsymbol{C}_\theta H^T = \begin{bmatrix} \sigma_A^2 & \sigma_A^2 & \dots & \sigma_A^2 \\ -M\sigma_B^2 & -(M-1)\sigma_B^2 & \dots & M\sigma_B^2 \end{bmatrix} \tag{62}$$

$$H^T C_w^{-1} H = \frac{1}{\sigma^2} \begin{bmatrix} 2M+1 & 0 \\ 0 & \sum_{n=-M}^{M} n^2 \end{bmatrix} \tag{63}$$

$$\left(C_\theta^{-1} + H^T C_w^{-1} H\right)^{-1} = \begin{bmatrix} \frac{1}{\frac{2M+1}{\sigma^2} + \frac{1}{\sigma_A^2}} & 0 \\ 0 & \frac{1}{\frac{\sum n^2}{\sigma^2} + \frac{1}{\sigma_B^2}} \end{bmatrix} \tag{64}$$

$$H^T C_w^{-1} \left(\boldsymbol{x} - H\mu_\theta\right) = \frac{1}{\sigma^2} \begin{bmatrix} \sum x[n] - (2M+1)\,A_0 \\ \sum nx[n] - \sum n^2 B_0 \end{bmatrix} \tag{65}$$

Therefore,

$$\hat{A} = A_0 + \frac{1}{\sigma^2}\left(\sum x[n] - (2M+1)\,A_0\right) \frac{1}{\frac{2M+1}{\sigma^2} + \frac{1}{\sigma_A^2}} \tag{66}$$

$$\hat{B} = B_0 + \frac{1}{\sigma^2}\left(\sum nx[n] - \sum n^2 B_0\right) \frac{1}{\frac{\sum n^2}{\sigma^2} + \frac{1}{\sigma_B^2}} \tag{67}$$

Also, the minimum Bayesian MSE is

$$B_{MSE}\left(\hat{A}\right) = \left[C_{\theta|\boldsymbol{x}}\right]_{11} = \left(\frac{1}{\sigma_A^2} + \frac{2M+1}{\sigma^2}\right)^{-1} \tag{68}$$

$$B_{MSE}\left(\hat{B}\right) = \left[C_{\theta|\boldsymbol{x}}\right]_{22} = \left(\frac{1}{\sigma_B^2} + \frac{\sum n^2}{\sigma^2}\right)^{-1}. \tag{69}$$

And we can see that $\hat{A}$ has the most benefit from the prior knowledge, since $\hat{B}$ has much stronger profit from the available data.

From Theorem 10.3 (p326) posterior PDF for the Bayesian Linear Model
If $\boldsymbol{x} = \boldsymbol{H\theta} + \boldsymbol{w}$, the the posterior PDF $p(\boldsymbol{\theta}|\boldsymbol{x})$ is Gaussian, with

$$E[\boldsymbol{\theta}|\boldsymbol{x}] = \boldsymbol{\mu}_\theta + \boldsymbol{C}_\theta H^T \left(H\boldsymbol{C}_\theta H^T + \boldsymbol{C}_w\right)^{-1} (\boldsymbol{x} - H\mu_\theta) \tag{70}$$

$$\boldsymbol{C}_{\theta|x} = \boldsymbol{C}_\theta - \boldsymbol{C}_\theta H^T \left(H\boldsymbol{C}_\theta H^T + \boldsymbol{C}_w\right)^{-1} H\boldsymbol{C}_\theta \tag{71}$$

Furthermore on page 328 there is the alternative form of Thm 10.3 , which in this case us much more convenient for the calculations:

$$
\begin{aligned}
E[\boldsymbol{\theta}|\boldsymbol{x}] &= \mu_\theta + \left(C_\theta^{-1} + H^T C_w^{-1} H\right)^{-1} H^T C_w^{-1} (x - H\mu_\theta) && (72)\\
C_{\theta|\boldsymbol{x}} &= \left(C_\theta^{-1} + H^T C_w^{-1} H\right)^{-1} && (73)
\end{aligned}
$$

Alternative Solution 1 (not sure if solvable)
USe the Matrix Inversion Lemma:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1} \tag{74}$$

$$\left(H\boldsymbol{C}_\theta H^T + \boldsymbol{C}_w\right)^{-1} = \boldsymbol{C}_w^{-1} - \boldsymbol{C}_w^{-1} H (H^T \boldsymbol{C}_w^{-1} H + \boldsymbol{C}_\theta^{-1})^{-1} H^T \boldsymbol{C}_w^{-1} \tag{75}$$

Where we calculate

$$
\begin{aligned}
H^T \boldsymbol{C}_w^{-1} H + \boldsymbol{C}_\theta^{-1} &= \begin{bmatrix} 1 & 1 & \dots & 1 \\ -M & -(M-1) & \dots & M \end{bmatrix} \frac{1}{\sigma^2}\boldsymbol{I} \begin{bmatrix} 1 & -M \\ 1 & -(M-1) \\ \vdots & \vdots \\ 1 & M \end{bmatrix} + \frac{1}{\sigma_A^2\sigma_B^2}\boldsymbol{I} && (76)\\
&= \frac{2M+1}{\sigma^2}\boldsymbol{I} + \frac{1}{\sigma_A^2\sigma_B^2}\boldsymbol{I} && (77)\\
&= \frac{(2M+1)(\sigma_A^2\sigma_B^2) + \sigma^2}{\sigma^2\sigma_A^2\sigma_B^2}\boldsymbol{I} && (78)
\end{aligned}
$$

We can then fill this in, and continue to solve

$$
\begin{aligned}
\left(H\boldsymbol{C}_\theta H^T + \boldsymbol{C}_w\right)^{-1} &= \frac{1}{\sigma^2}\boldsymbol{I} - \frac{1}{\sigma^2}\boldsymbol{I}H\frac{\sigma^2\sigma_A^2\sigma_B^2}{(2M+1)(\sigma_A^2\sigma_B^2) + \sigma^2}\boldsymbol{I}H^T\frac{1}{\sigma^2}\boldsymbol{I} && (79)\\
&= \frac{1}{\sigma^2}\boldsymbol{I} - HH^T\frac{\sigma_A^2\sigma_B^2}{\sigma^2((2M+1)(\sigma_A^2\sigma_B^2) + \sigma^2)} && (80)
\end{aligned}
$$

Alternative Solution 2 (not sure if solvable)

$$\left(H\boldsymbol{C}_\theta H^T + \boldsymbol{C}_w\right)^{-1} = \left(\begin{bmatrix} 1 & -M \\ 1 & -(M-1) \\ \vdots & \vdots \\ 1 & M \end{bmatrix} \begin{bmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 \\ -M & -(M-1) & \dots & M \end{bmatrix} + \sigma^2\boldsymbol{I}\right)^{-1} \tag{81}$$

$$= \begin{bmatrix} M^2\sigma_B^2 + \sigma_A^2 + \sigma^2 & M(M-1)\sigma_B^2 + \sigma_A^2 & \dots & -M^2\sigma_B^2 + \sigma_A^2 \\ (M-1)M\sigma_B^2 + \sigma_A^2 & (M-1)^2\sigma_B^2 + \sigma_A^2 + \sigma^2 & \dots & -M(M-1)\sigma_B^2 + \sigma_A^2 \\ (M-2)M\sigma_B^2 + \sigma_A^2 & (M-1)(M-2)\sigma_B^2 + \sigma_A^2 & \dots & -M(M-2)\sigma_B^2 + \sigma_A^2 \\ \vdots & & \dots & \vdots \\ -M^2\sigma_B^2 + \sigma_A^2 & -M(M-1)\sigma_B^2 + \sigma_A^2 & \dots & M^2\sigma_B^2 + \sigma_A^2 + \sigma^2 \end{bmatrix}^{-1} \tag{82}$$

Solution 2.8: The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x) & 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{83}$$

(a) The conditional PDF of X given Y is found by dividing the joint PDF by the marginal with respect to Y. For $y < 0$ or $y > 1$, $f_Y(y) = 0$. For $0 \le y \le 1$,

$$f_Y(y) = \int_0^y 6(y-x)dx = 6xy - 3x^2\big|_0^y = 3y^2 \tag{84}$$

The complete expression for the marginal PDF of Y is

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{85}$$

Thus for $0 < y \le 1$,

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{6(y-x)}{3y^2} & 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{86}$$

(b) The minimum mean square estimator of X given $Y = y$ is

$$\begin{aligned} \hat{x}_{MMSE}(y) = E[X \mid Y = y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)dx \\ &= \int_0^y \frac{6x(y-x)}{3y^2}dx \\ &= \frac{3x^2y - 2x^3}{3y^2}\bigg|_{x=0}^{x=y} \\ &= y/3 \end{aligned} \tag{87}$$

(c) First we must find the marginal PDF for X. For $0 \le x \le 1$,

$$\begin{aligned} f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy &= \int_x^1 6(y-x)dy = 3y^2 - 6xy\big|_{y=x}^{y=1} \\ &= 3 - 6x + 3x^2 \end{aligned} \tag{88}$$

The conditional PDF of Y given X is

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{2(y-x)}{1-2x+x^2} & x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{89}$$

(d) The minimum mean square estimator of Y given X is

$$\begin{aligned} \hat{y}_{MMSE}(x) = E[Y \mid X = x] &= \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)dy \\ &= \int_x^1 \frac{2y(y-x)}{1-2x+x^2}dy \\ &= \frac{(2/3)y^3 - y^2x}{1-2x+x^2}\bigg|_{y=x}^{y=1} \\ &= \frac{2-3x+x^3}{3(1-x)^2} \end{aligned} \tag{90}$$

Perhaps surprisingly, this result can be simplified to

$$\hat{y}_{MMSE}(x) = \frac{x}{3} + \frac{2}{3} \tag{91}$$

Part 3: Spectral Estimation

3.1 *Introduction to spectral estimation*

Solution 3.1:

The signal is given by: $x[n] = \frac{1}{2}\sin(\theta_1) + \sin(\theta_2)$ with $\theta_1 = 0.44\pi$ and $\theta_2 = 0.68\pi$. Since each sine wave has two (complex conjugated) peaks in the spectrum, we may expect two times two peaks in the DFT spectrum. Furthermore the amplitude of the first two peaks should be half the second ones.

(a) With $L = N = 16$ the DFT resolution is $\frac{\pi}{8} = 0.125\pi$. From this we may expect the first peak of each of these two sine waves at the following position:

$$\theta_1 \rightarrow \text{between } k = 3 - 4 \qquad \theta_2 \rightarrow \text{between } k = 5 - 6$$

This is plotted in Fig. 4 (left). From this figure it follows that the spectral peaks of these two sine waves can hardly be separated with this low DFT resolution.

(b) With $L = N = 128$ the DFT resolution is $\frac{\pi}{64} = 0.015625\pi$, which is much better as in case a). Now we can find the first two peaks at the following positions:

$$\theta_1 \rightarrow \text{between } k = 27 - 28 \qquad \theta_2 \rightarrow \text{between } k = 43 - 44$$

The results can be verified in Fig. 4. Note that the zero padding in case b) has indeed increased

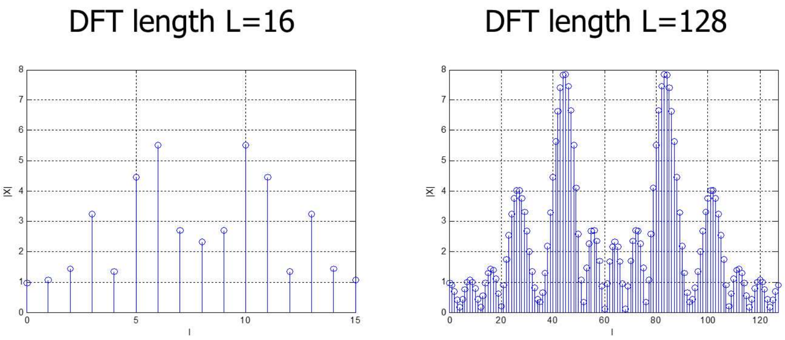


Figure 4: $N = 16$, $f_s = 50Hz$, $f_1 = 11Hz$ and $f_2 = 17Hz$

the DFT resolution but it did not change the spectral shape. The spectral discrimination can only be increased by increasing the length N of the data window.

3.2 *Non-parametric spectral estimation*

Solution 3.2:

$$
\begin{aligned}
\sum_{-\infty}^{\infty} \hat{\rho}[\tau] e^{-j\tau\theta} &= \sum_{\tau=-(N-1)}^{N-1} \left(\frac{1}{N} \sum_{k=|\tau|}^{N-1} x[k]x[k-|\tau|e^{-j\tau\theta} \right) \\
&= \frac{1}{N} \sum_{\tau=-(N-1)}^{N-1} \sum_{k=|\tau|}^{N-1} x[k]x[k-|\tau|e^{-j\tau\theta} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x[k]x[n]e^{-j(k-n)\theta} \\
&= \frac{1}{N} \left(\sum_{k=0}^{N-1} x[k]e^{-jk\theta} \right) \cdot \left(\sum_{n=0}^{N-1} x[k]e^{-jn\theta} \right) \\
&= \frac{1}{N} \left| \sum_{k=0}^{N-1} x[k]e^{-jk\theta} \right|^2 = \hat{P}(e^{j\theta})
\end{aligned}
$$

3.3 *Parametric spectral estimation*

Solution 3.3:

(a) The psd of this process can be calculated as follows:

$$
P(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{\tau=-\infty}^{\infty} \rho[\tau] \mathrm{e}^{-\mathrm{j}\tau\theta} = \frac{2 - \cos(\theta)}{\frac{5}{4} - \cos(\theta)}
$$

This function is plotted in Fig. 5.

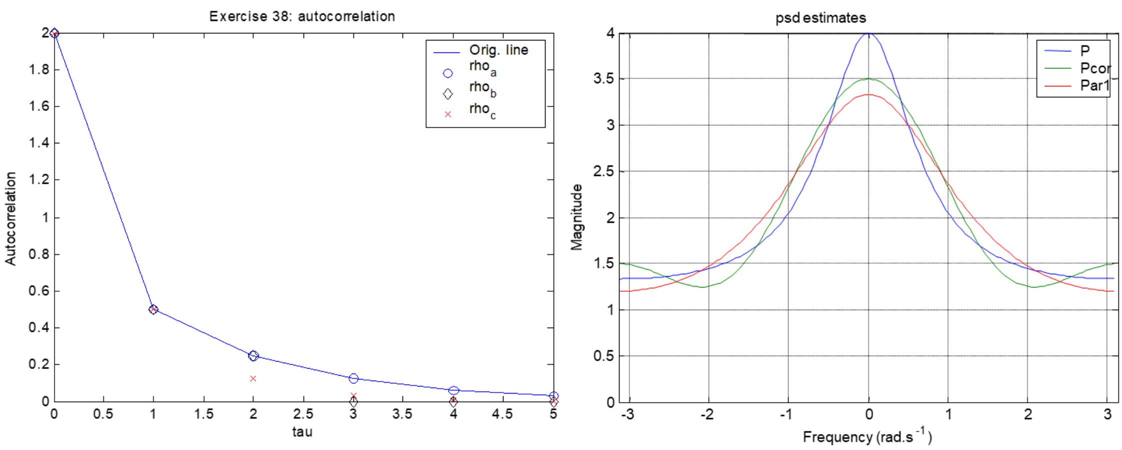


Figure 5: Comparison of different psd estimates

(b) The correlogram uses only five autocorrelation lags:

$$
P_{cor}(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{\tau=-2}^{2} \rho[\tau] \mathrm{e}^{-\mathrm{j}\tau\theta} = 2 + \cos(\theta) + \frac{1}{2}\cos(2\theta)
$$

This function is also plotted in Fig. 5.

(c) The result of FIR prediction with one coefficient are:

$$w_1 = (\rho[0])^{-1} \cdot \rho[1] = \frac{1}{4} \quad \text{and} \quad J_{min} = \rho[0] - w_1 \cdot \rho[1] = \frac{15}{8}$$

(d) Using the results of c) to model the given process as an AR1 process results in the following difference equation:

$$x_{ar1}[k] = i[k] - a_1 \cdot x_{ar1}[k-1] \quad \text{with} \quad a_1 = -w_1 = -\frac{1}{4} \text{ and } E\{i^2[k]\} = \sigma_i^2 = J_{min} = \frac{15}{8}$$

With this the AR1 spectral estimate becomes:

$$P_{ar1}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{\sigma_i^2}{|1 + a_1 \mathrm{e}^{\mathrm{j}\theta}|^2} = \frac{\frac{15}{8}}{\frac{17}{16} - \frac{1}{2}\cos(\theta)}$$

This function is also plotted in Fig. 5.

(e) For a comparison: see Fig. 5.

Part 4: Detection theory

4.1 *Hypothesis testing*

Solution 4.1: See slides:

$$\mathcal{H}_0 : x_n = \quad w_n, \qquad n = 0, 1, \ldots, N-1 \tag{92}$$
$$\mathcal{H}_1 : x_n = \quad A + w_n, \qquad n = 0, 1, \ldots, N-1 \tag{93}$$

Now P_{FA} is $\Pr(\mathcal{H}_1; \mathcal{H}_0)$, that is $\mathcal{H}_1$ is chosen, but $\mathcal{H}_0$ is correct. $P_{FA} = \int_{x:L(x)>\gamma} p(x; \mathcal{H}_0)\, \mathrm{d}x$, with $L(x) = \frac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} > \gamma$. Furthermore $P_D = 1 - P_M = \Pr(\mathcal{H}_1; \mathcal{H}_1)$.
First, we find $L(x)$ and find that $\frac{1}{N}\sum_{n=0}^{N-1} x[n] > \frac{\sigma^2}{NA}\ln\gamma + \frac{A}{2} = \gamma'$.
Now $P_{FA} = \int_{x:\frac{1}{N}\sum_{n=0}^{N-1} x[n]>\gamma'} p(x; \mathcal{H}_0)\, \mathrm{d}x = \mathcal{Q}\left(\frac{\gamma'}{\sqrt{\sigma^2/N}}\right)$
And therefore, $\gamma' = \sqrt{\frac{\sigma^2}{N}}\mathcal{Q}^{-1}\left(P_{FA}\right)$
Furthermore, $P_D = \int_{x:\frac{1}{N}\sum_{n=0}^{N-1} x[n]<\gamma'} p(x; \mathcal{H}_1)\, \mathrm{d}x = \mathcal{Q}\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}}\right) = \mathcal{Q}\left(\mathcal{Q}^{-1}\left(P_{FA}\right) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$
Now we can find that

$$\mathcal{Q}^{-1}\left(P_D\right) = \mathcal{Q}^{-1}\left(P_{FA}\right) - \sqrt{\frac{NA^2}{\sigma^2}} \tag{94}$$
$$\sqrt{\frac{NA^2}{\sigma^2}} = \mathcal{Q}^{-1}\left(P_{FA}\right) - \mathcal{Q}^{-1}\left(P_D\right) \tag{95}$$
$$\frac{NA^2}{\sigma^2} = \left(\mathcal{Q}^{-1}\left(P_{FA}\right) - \mathcal{Q}^{-1}\left(P_D\right)\right)^2 \tag{96}$$
$$N \cdot 10^{-3} = \left(\mathcal{Q}^{-1}\left(P_{FA}\right) - \mathcal{Q}^{-1}\left(P_D\right)\right)^2 \tag{97}$$
$$N = 36546 \tag{98}$$

Solution 4.2: Perfect detector means no overlap between the PDFs. Therefore $c > 1 - c \rightarrow c < \frac{1}{2}$.