Statistical signal processing (5CTA0)

Lecture 2, part B

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Electrical Engingeering, Signal Processing Systems group

Part 1: Random variables and Random Signals

Part 1

Random Variables and Random Signals

Lecture 2: Random vectors, Random processes and random signals

Part A: Pairs of random variables
Part B: Random vectors, random processes and random signals

Random vectors, random processes and random signals

Lecture 2, Part B

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Outline

- Random vectors
- Stochastic processes
- Stationarity and ergodicity
- Auto-correlation and power spectral density
- Ideal and approximate signal statistics

Random vectors: introduction

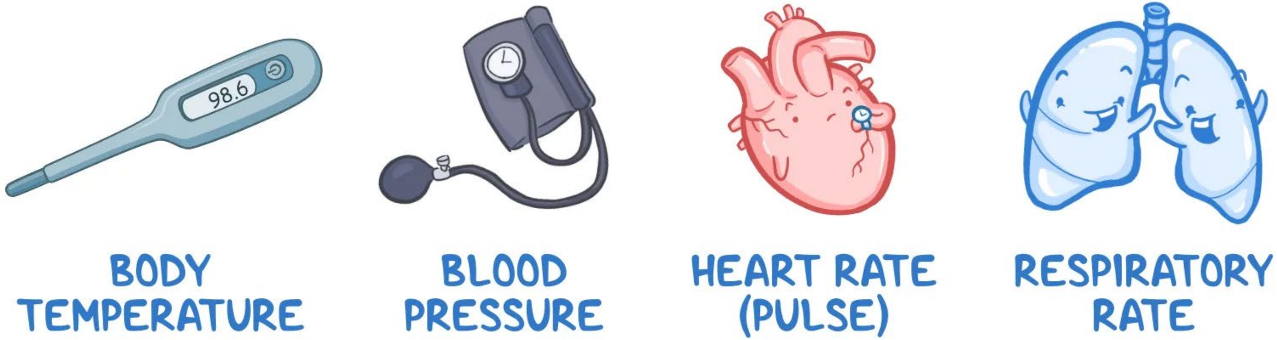
- A stochastic process may involve multiple random variables
  - Multiple random variables associated to the same stochastic process are called **joint** random variables
  - The **probability model** of joint random variables contains properties of the individual random variables and the relationships among them

*Illustrations taken from Osmosis, Elsevier

Multiple random variables

Suppose we have *N* random variables

Example: vital signs measurements



Random vectors

Denoting each random variable X_n, with $n = 1, \ldots, N$, we can group all random variables X_n in a random vector as

$$\mathbf{X} = [X_1, X_2, \ldots, X_N]^\top$$

Random vector

$$\mathbf{x} = [x_1, x_2, \ldots, x_N]^\top$$

Single realization

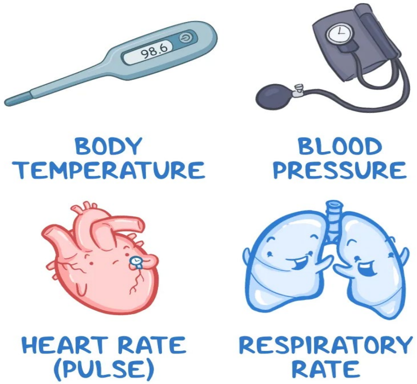
Random vectors

Suppose now you have *2* random vectors containing 2 different types of random variables

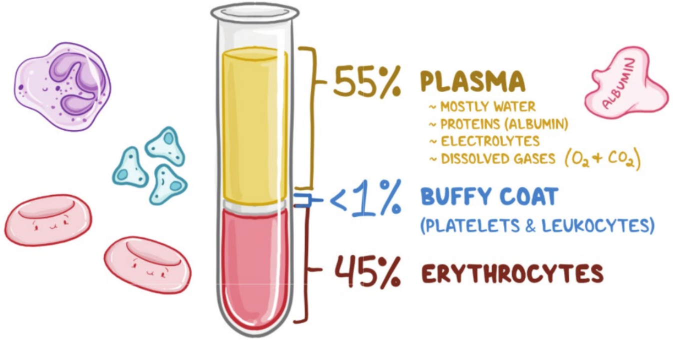
Vital signs

Lab values (blood)

$\mathbf{X} =$



$Y =$



Random vectors

Suppose now you have *2* random vectors containing 2 different types of random variables
Example: vital signs + lab values

$$\mathbf{X} = [X_1, X_2, \ldots, X_N]^\top$$

$$\mathbf{Y} = [Y_1, Y_2, \ldots, Y_M]^\top$$

$$\mathbf{Z} = [\mathbf{X}^\top, \mathbf{Y}^\top]^\top$$

$$\mathbf{Z} = [X_1, X_2, \ldots, X_N, Y_1, Y_2, \ldots, Y_M]^\top$$

Multivariate probability distributions

Denoting each random variable X_n, with $n = 1, \ldots, N$, we can define the **multivariate joint CDF** as

$$P_{\mathbf{X}}(\mathbf{x}) = P_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = \Pr[X_1 \leq x_1,\ldots,X_N \leq x_N].$$

Multiple random variables

Denoting each random variable X_n, with $n = 1, \ldots, N$, we can define the **multivariate joint PMF** and **PDF** as

Joint PMF $$p_{\mathbf{X}}(\mathbf{x}) = p_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = \Pr[X_1 = x_1,\ldots,X_N = x_N]$$

Joint PDF $$p_{\mathbf{X}}(\mathbf{x}) = p_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = \frac{\partial^N P_{X_1,\ldots,X_N}(x_1,\ldots,x_N)}{\partial x_1 \ldots \partial x_N}$$

Marginalization

To obtain the probability function of a random variable of interest, we "marginalize" over all other random variables

Discrete

$$p_{X_2,X_3}(x_2,x_3) = \sum_{x_1 \in S_{X_1}} \sum_{x_4 \in S_{X_4}} \cdots \sum_{x_N \in S_{X_N}} p_{\mathbf{X}}(\mathbf{x})$$

Continuous

$$p_{X_2,X_3}(x_2,x_3) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{X}}(\mathbf{x})\; \mathrm{d}x_1 \mathrm{d}x_4 \cdots \mathrm{d}x_N$$

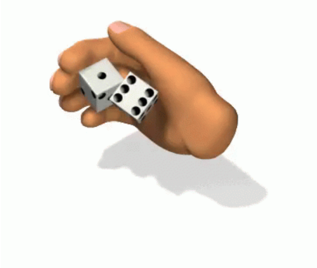
Independence

The random variables $X_1, X_2, \ldots, X_N$ can be regarded as independent if and only if the following factorization holds

$$p_{X_1,X_2,\ldots,X_N}(x_1, x_2, \ldots, x_N) = p_{X_1}(x_1)p_{X_2}(x_2)\ldots p_{X_N}(x_N).$$

Independence: example

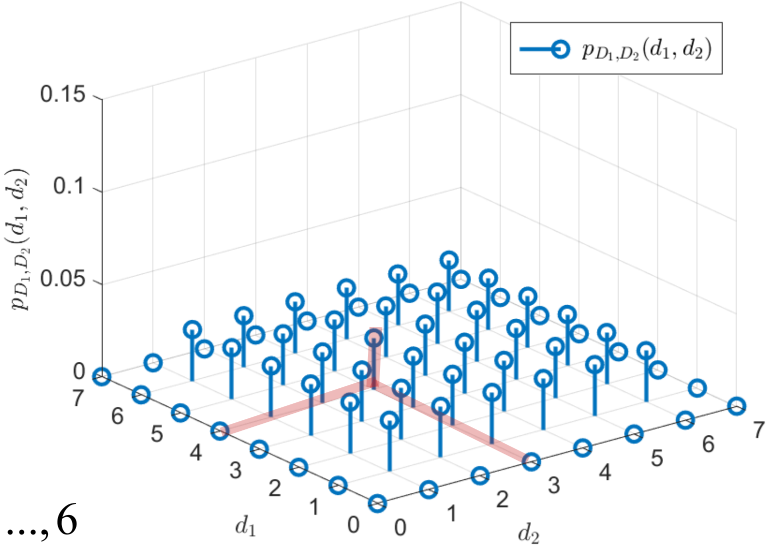
Rolling two dice



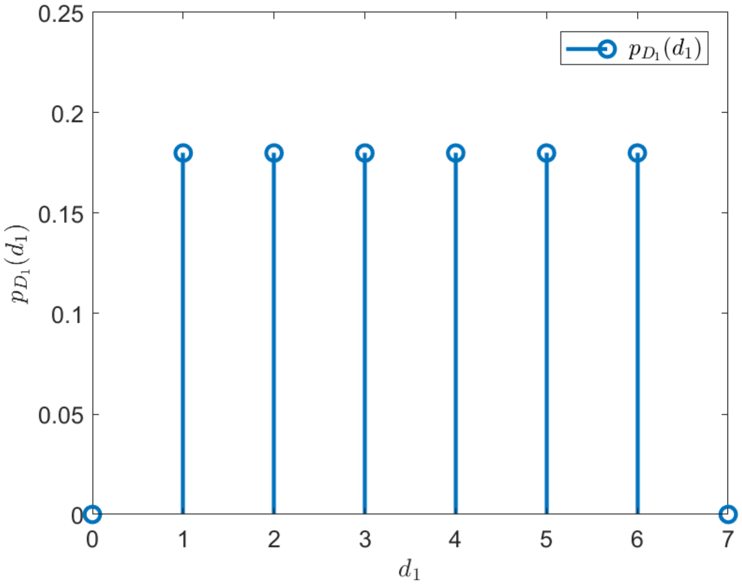
D_1 = number on first die

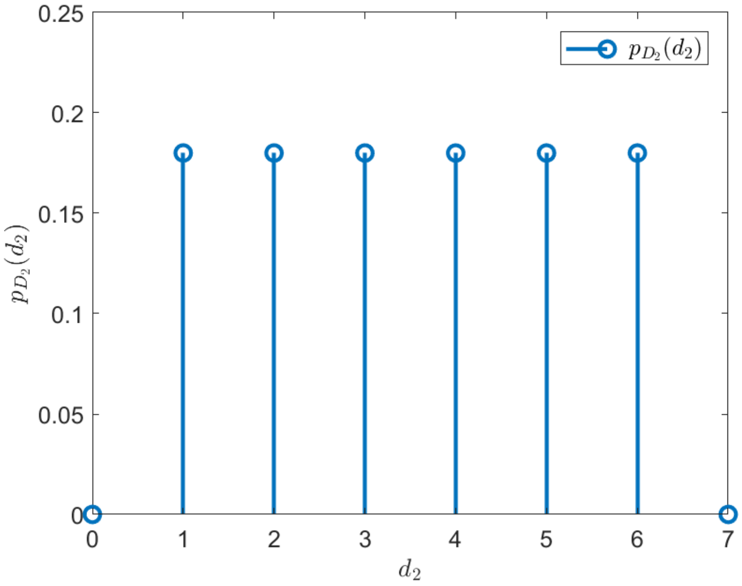
D_2 = number on second die

$$p_{D_1,D_2}(d_1,d_2) = \begin{cases} 1/36 & for \quad d_1 = 1,\ldots,6;\ d_2 = 1,\ldots,6 \\ 0 & elsewhere \end{cases}$$

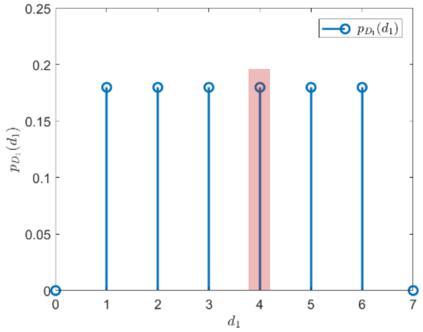


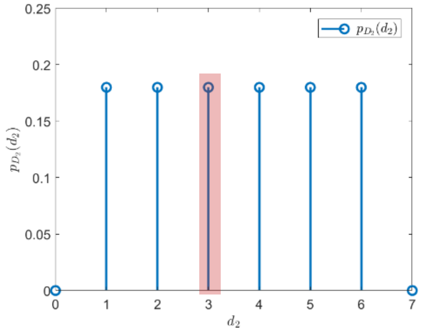
Independence: example

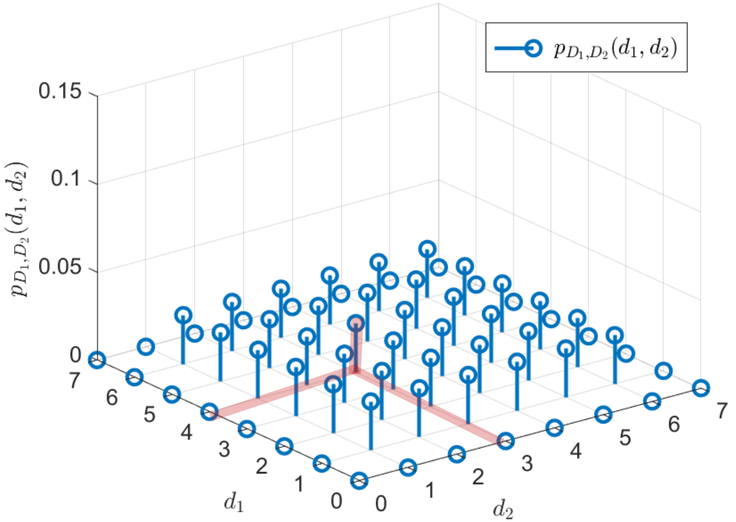




Independence: example





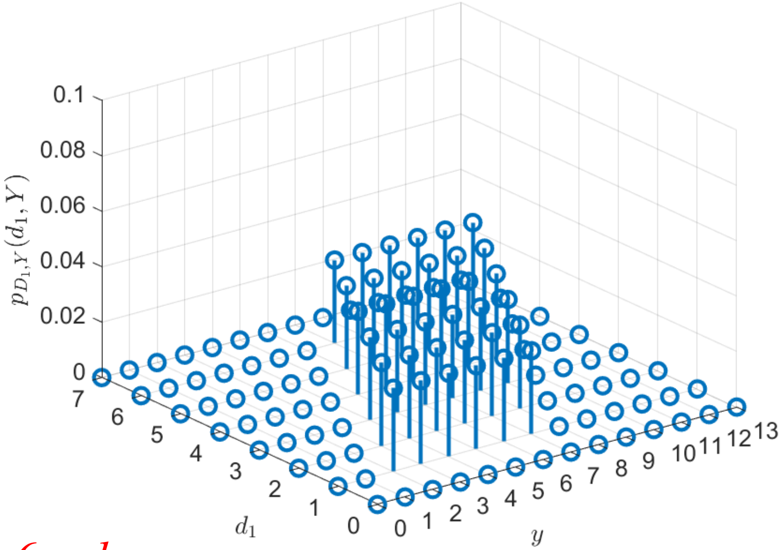


$$p_{D_1,D_2}(d_1=4,d_2=3)=p_{D_1}(d_1=4)\,p_{D_2}(d_2=3)=\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{36}$$

Independence: example

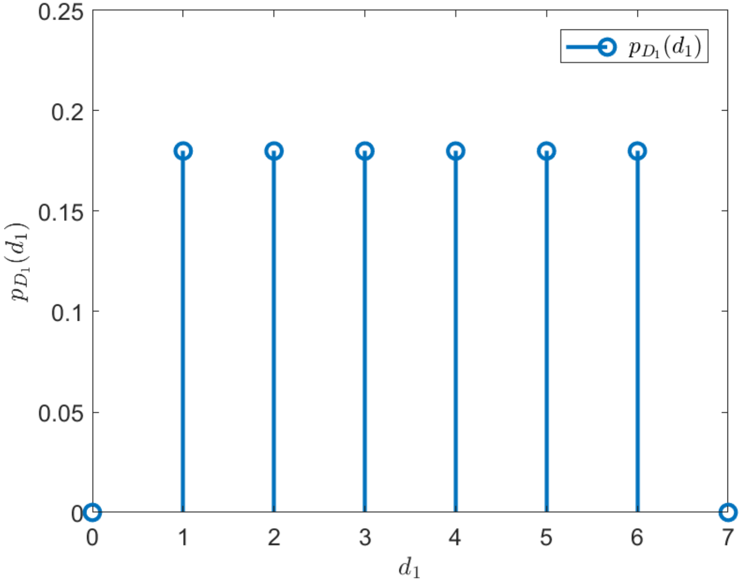
Rolling two dice

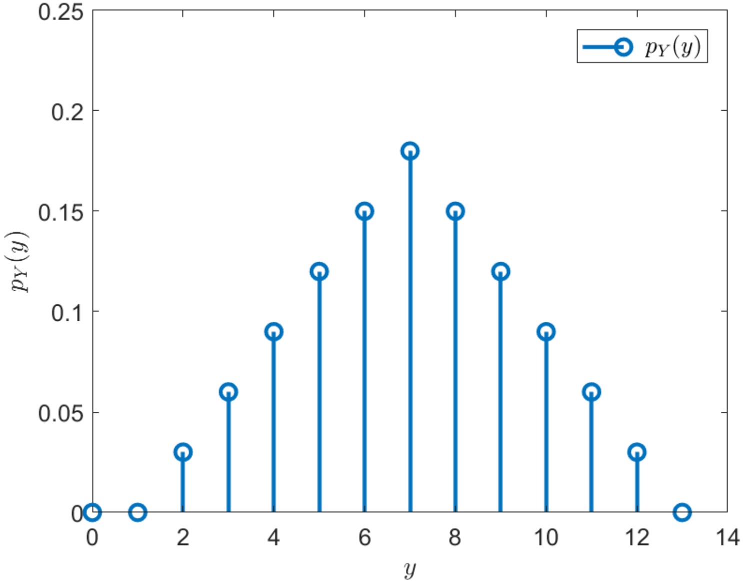
D_1 = number on first die
D_2 = number on second die
$Y = D_1 + D_2$



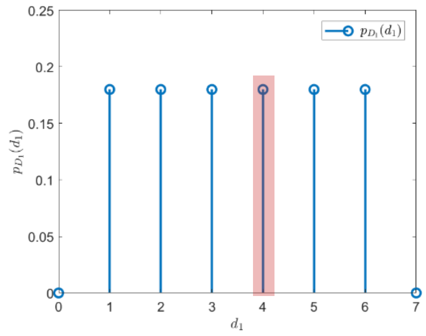
$$p_{D_1,Y}(d_1,y) = 1/36 \quad for \quad d_1 = 1,\ldots,6;\ y = 1+d_1,\ldots,6+d_1$$

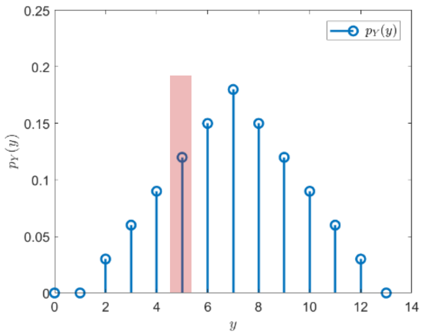
Independence: example

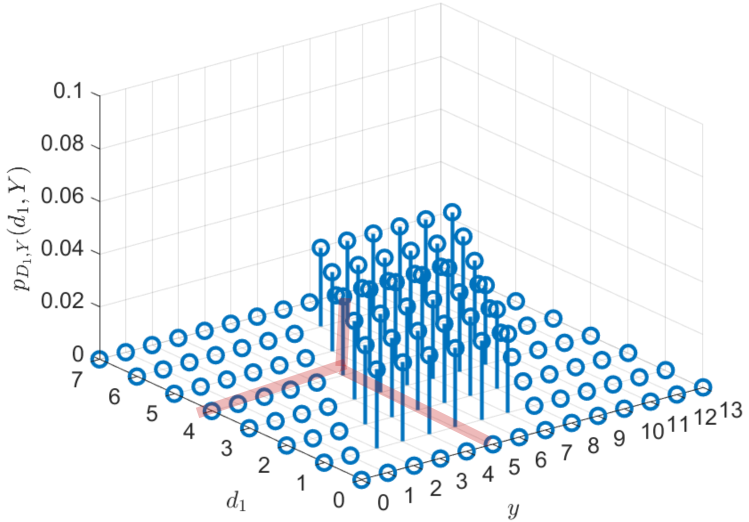




Independence: example







$$p_{D_1,Y}(d_1 = 4, y = 5) = \frac{1}{36}$$

$$p_{D_1,Y}(d_1 = 4, y = 5) \cancel{\neq} p_{D_1}(x_1 = 4)\, p_Y(y = 5) = \frac{1}{6} \cdot \frac{4}{36} \neq \frac{1}{36}$$

D_1 and Y are not independent

Covariance and correlation

The **covariance** is a measure of the correlation between two random variables

$$\mathrm{Cov}[X_1, X_2] = \mathrm{E}[(X_1 - \mu_{X_1})(X_2 - \mu_{X_2})] =$$

$$= \mathrm{E}[X_1 X_2] - \mu_{X_1}\mu_{X_2} = r_{X_1,X_2} - \mu_{X_1}\mu_{X_2}$$

With r_{x_1,x_2} called correlation $\quad r_{X_1,X_2} = \mathrm{E}[X_1 X_2]$

X_1, X_2	Real-valued random variables

Extension to random vector: Cross-covariance matrix

$$\mathbf{C}_{\mathbf{XY}} = \mathrm{E}[(\mathbf{X}-\mu_{\mathbf{X}})(\mathbf{Y}-\mu_{\mathbf{Y}})^\top] = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{bmatrix}$$

$\mathbf{X}, \mathbf{Y}$ uncorrelated if $\mathbf{C}_{\mathbf{XY}} = 0$

$$c_{mn} = \mathrm{E}[(X_m-\mu_n)(Y_n-\mu_n)]$$

Special case $\mathbf{X} = \mathbf{Y}$,
auto-covariance matrix

Extension to random vector: auto-covariance matrix

$$\mathbf{C}_{\mathbf{X}} = \mathrm{E}[(\mathbf{X}-\mu_{\mathbf{X}})(\mathbf{X}-\mu_{\mathbf{X}})^{\top}] = \begin{bmatrix} \sigma_1{}^2 & c_{12} & \cdots & c_{1N} \\ c_{21} & \sigma_2{}^2 & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & \sigma_N{}^2 \end{bmatrix}$$

$$c_{mn} = \mathrm{E}[(X_m - \mu_n)(X_n - \mu_n)] \quad \text{for } m \neq n$$

$$\sigma_m{}^2 = \mathrm{E}[(X_m - \mu_m)^2] \qquad \text{for } m = n$$

Extension to random vector: Cross-correlation matrix

$$\mathbf{R}_{\mathbf{XY}} = \mathrm{E}[\mathbf{X}\mathbf{Y}^\top] = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ r_{21} & r_{22} & \cdots & r_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ r_{N1} & r_{N2} & \cdots & r_{NN} \end{bmatrix}$$

$\mathbf{X}, \mathbf{Y}$ orthogonal if $\mathbf{R}_{\mathbf{XY}} = 0$

$$r_{mn} = \mathrm{E}[X_m Y_n]$$

Special case $\mathbf{X} = \mathbf{Y}$,
auto-correlation matrix

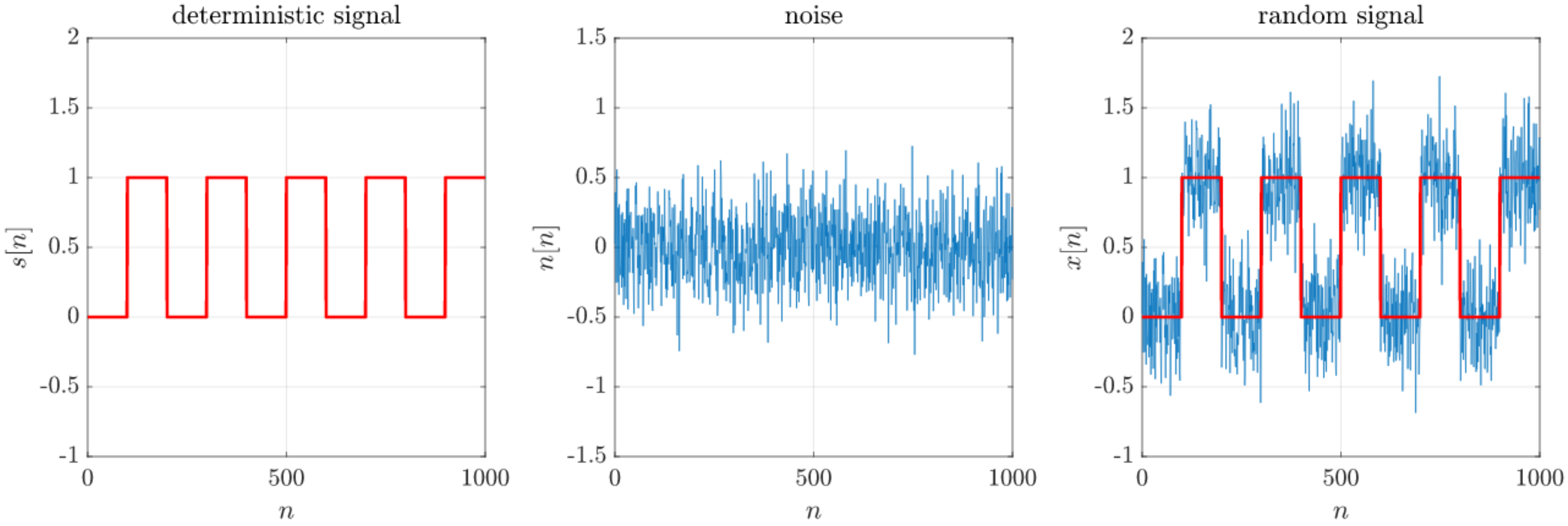
TU/e

Random signals: introduction

Continuous-time $x(t) = s(t) + w(t)$

Discrete-time $x[n] = s[n] + w[n]$

$s(t), s[n]$: deterministic signal
$x(t), x[n]$: random signal
$w(t), w[n]$: noise sequence



Random Processes, random variables, random signals

- Start with an experiment specified by its outcomes ζ forming the sample space $\mathcal{S}$.

- To every outcome ζ_k we assign a time function $x(t, \zeta_k)$, each occurring with a different probability

- The sample space, the probabilities and the time functions constitute together a random process (or random signal)

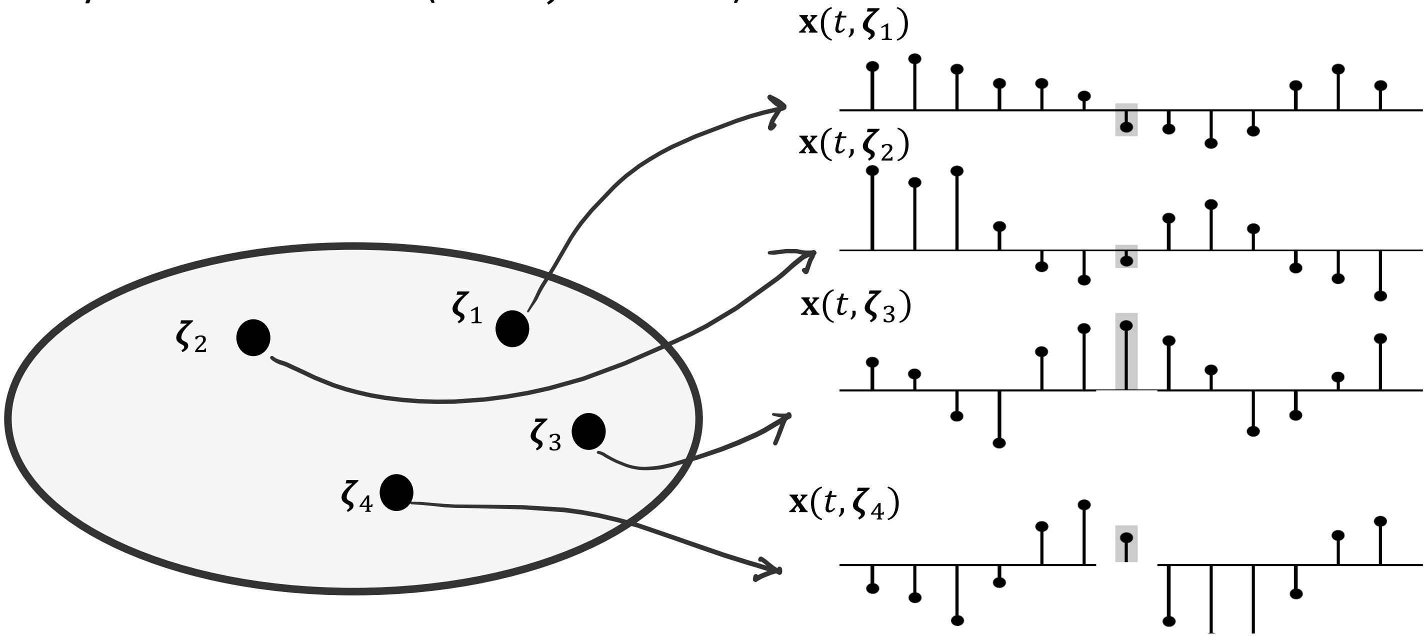
Random Processes, random variables, random signals

- Start with an experiment specified by its outcomes ζ forming the sample space $\mathcal{S}$.
- To every … each occurring with a di…
- The sam… ons constitute together a random process (or random signal)

The random process $\mathbf{X}(t, \boldsymbol{\zeta})$ maps each outcome of the sample space to a time function

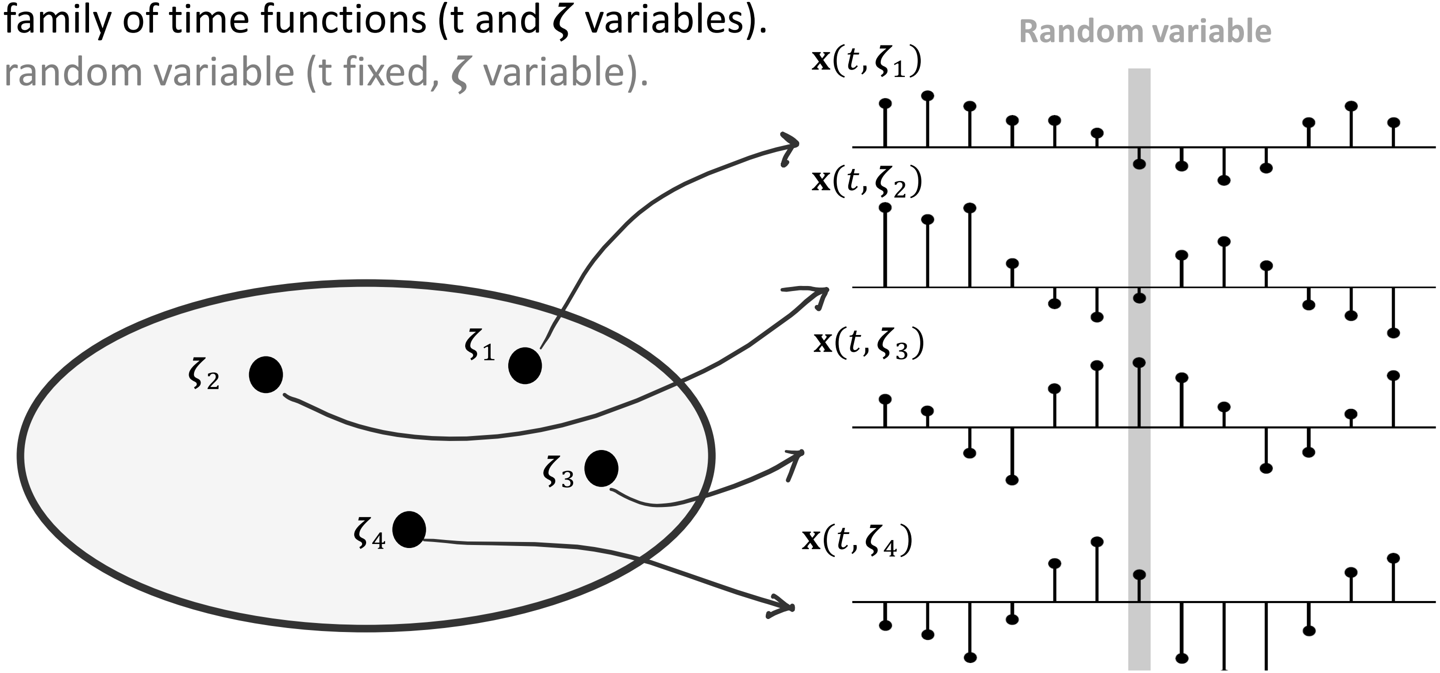
Random Processes and Random Variables

- Four ways to think about the function $\mathbf{X}(t, \zeta)$:
  - A family of time functions (t and ζ variables).



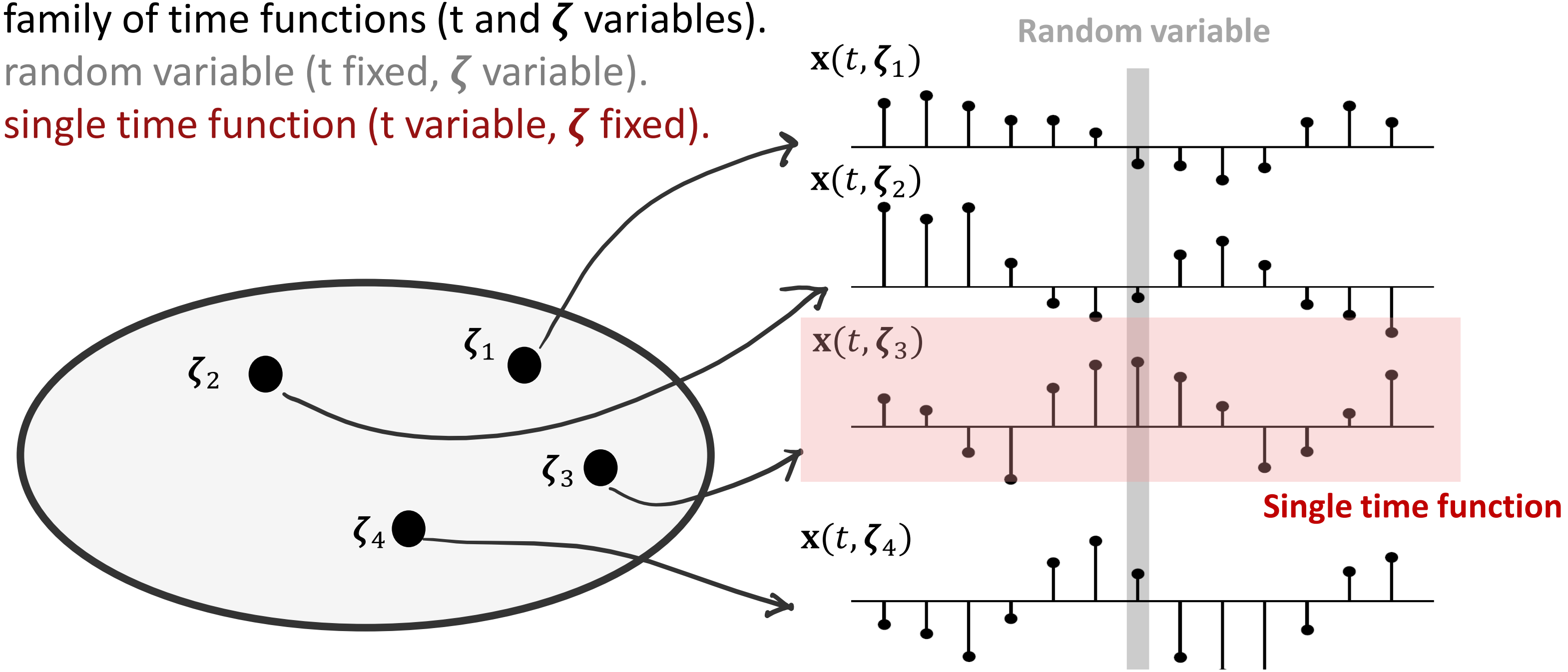
Random Processes and Random Variables

- Four ways to think about the function $\mathbf{X}(t, \zeta)$:
  - A family of time functions (t and ζ variables).
  - A random variable (t fixed, ζ variable).



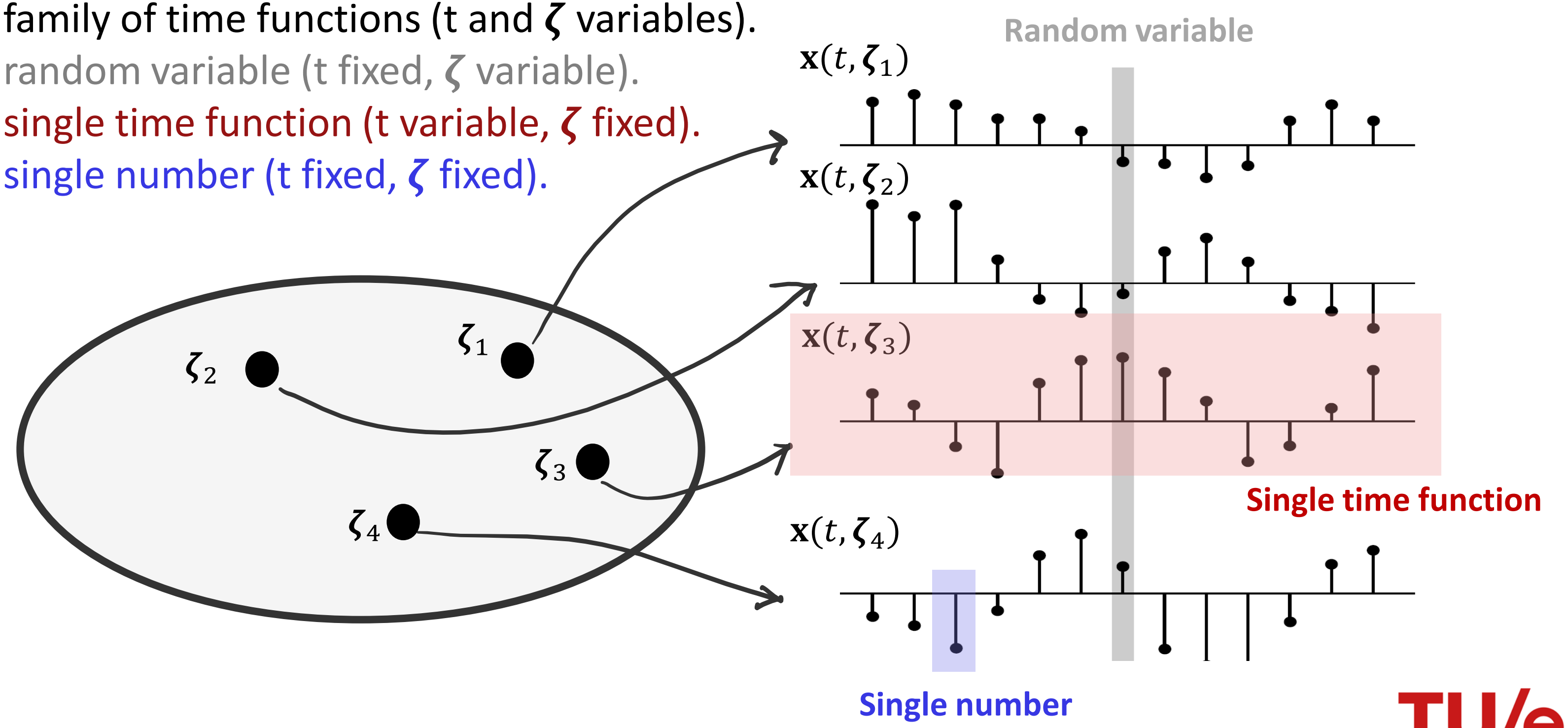
Random Processes and Random Variables

- Four ways to think about the function $\mathbf{X}(t,\zeta)$:
  - A family of time functions (t and ζ variables).
  - A random variable (t fixed, ζ variable).
  - A single time function (t variable, ζ fixed).



Random Processes and Random Variables

- Four ways to think about the function $X(t, \zeta)$:
  - A family of time functions (t and ζ variables).
  - A random variable (t fixed, ζ variable).
  - A single time function (t variable, ζ fixed).
  - A single number (t fixed, ζ fixed).



Statistics of Random Processes

- We are interested in describing the behavior of the random process.
- Each realization of $\mathbf{X}(t, \zeta)$ yields a different time function.

Statistics of Random Processes

- We are interested in describing the behavior of the random process.
- Each realization of $\mathbf{X}(t,\zeta)$ yields a different time function.
- Remember the statistics used for random vectors

Joint CDF $$P_{\mathbf{X}}(\mathbf{x}) = P_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = \Pr[X_1 \le x_1,\ldots,X_N \le x_N].$$

Joint PDF $$p_{\mathbf{X}}(\mathbf{x}) = p_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = \frac{\partial^N P_{X_1,\ldots,X_N}(x_1,\ldots,x_N)}{\partial x_1 \ldots \partial x_N}$$

Statistics of Random Processes

- For random processes $\mathbf{X}(t,\zeta)$, fixing the time results in a random variable. (Note: We dropped ζ and write $\mathbf{X}(t)$ instead of $\mathbf{X}(t,\zeta)$):

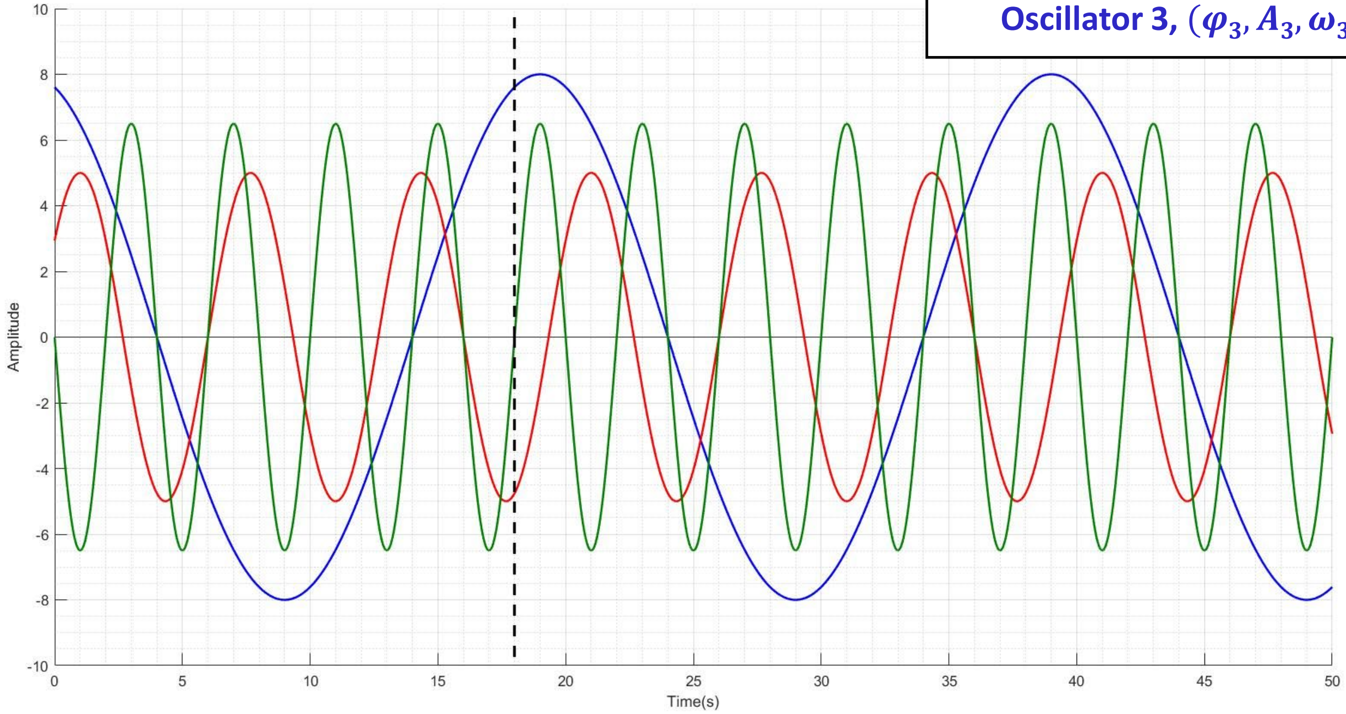
Joint CDF $$P_{\mathbf{X}(t)}(\mathbf{x};t) = P_{X(t_1),\ldots,X(t_N)}(x_1,\ldots,x_N) = \Pr[X(t_1) \le x_1,\ldots,X(t_N) \le x_N].$$

Joint PDF $$p_{\mathbf{X}(t)}(\mathbf{x};t) = p_{X(t_1),\ldots,X(t_N)}(x_1,\ldots,x_N) = \frac{P_{X(t_1),\ldots,X(t_N)}(x_1,\ldots,x_N)}{\partial x_1 \ldots \partial x_N}$$

Joint PMF $$p_{\mathbf{X}[n]}(\mathbf{x};n) = p_{X[n_1],\ldots,X[n_N]}(x_1,\ldots,x_N) = \Pr[X[n_1] = x_1,\ldots,X[n_N] = x_N].$$

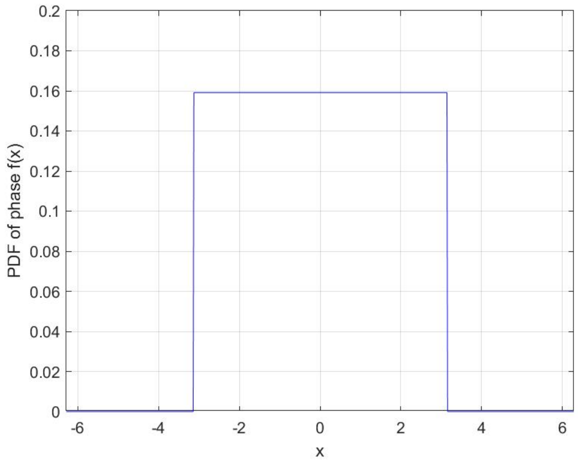
Random Processes: example

Oscillator 1, $(\varphi_1, A_1, \omega_1)$
Oscillator 2, $(\varphi_2, A_2, \omega_2)$
Oscillator 3, $(\varphi_3, A_3, \omega_3)$

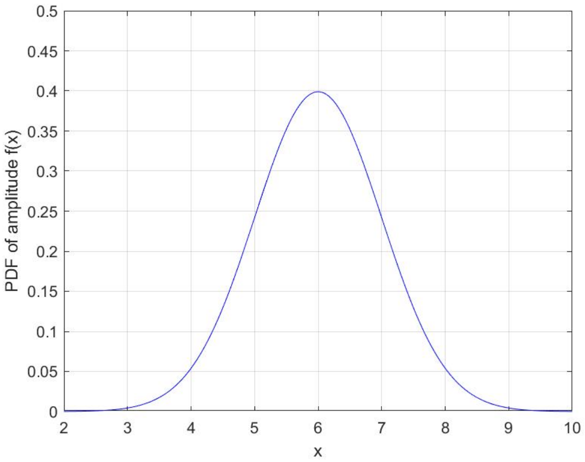


Random Processes: example

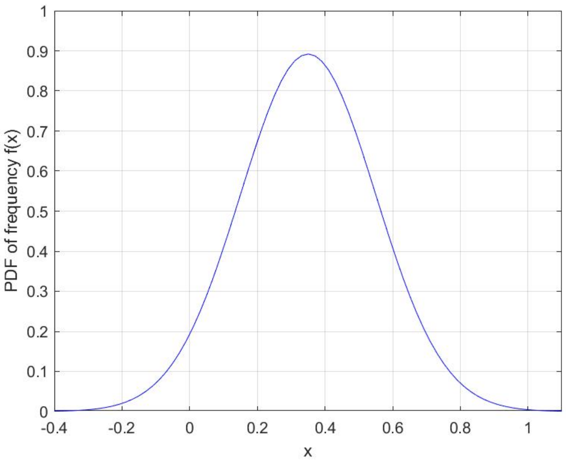
$p_\Phi(\varphi)$



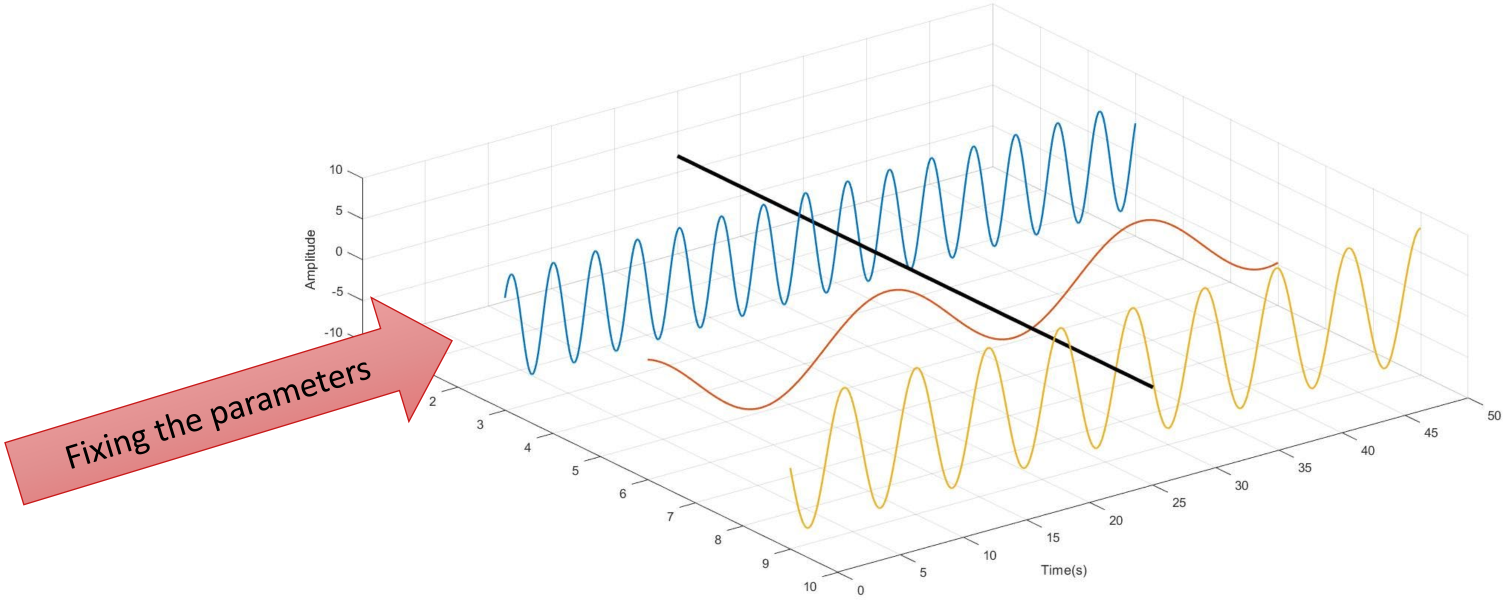
$p_A(a)$



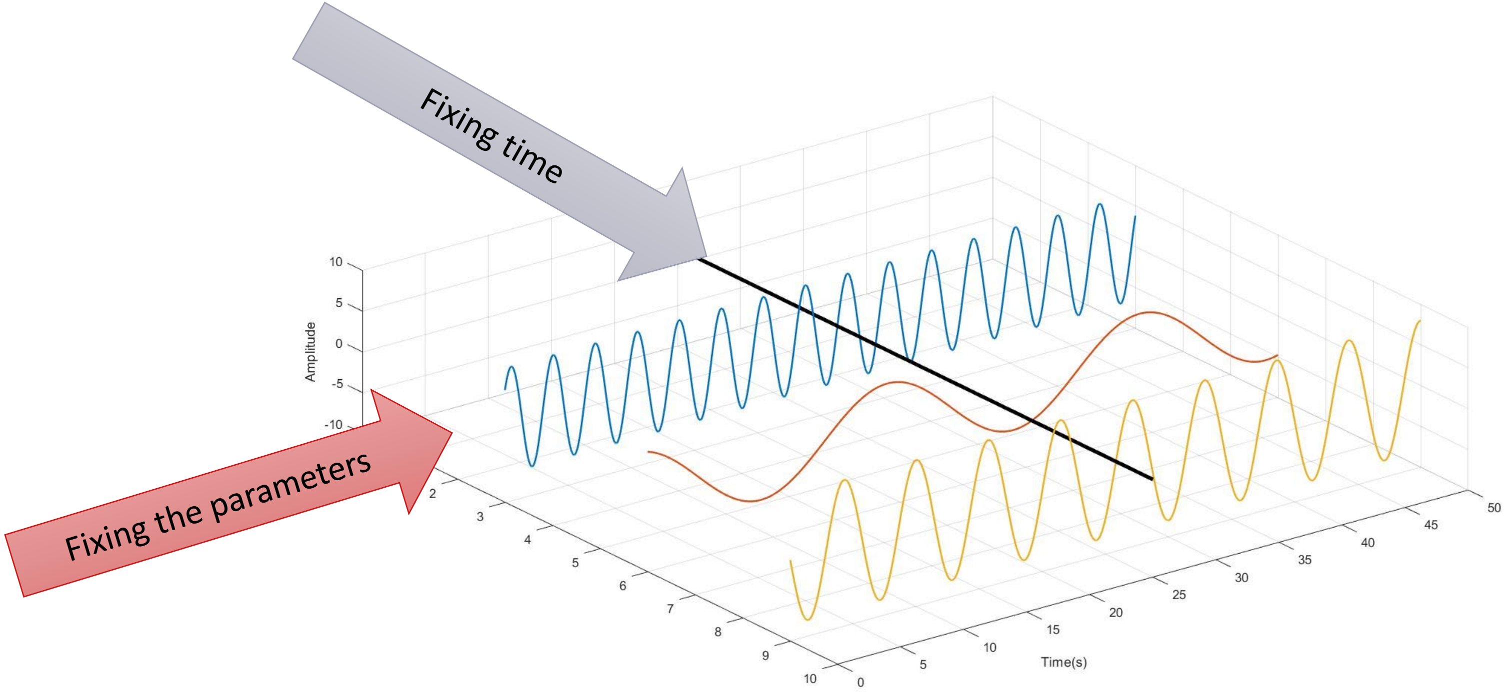
$p_\Omega(\omega)$



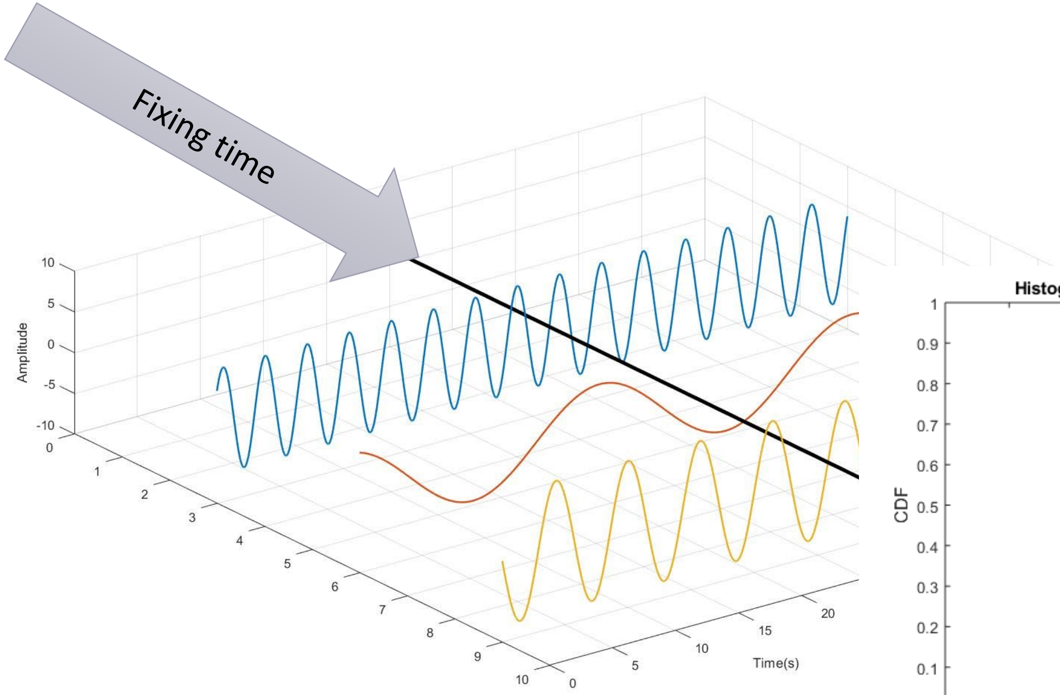
Random Processes: example

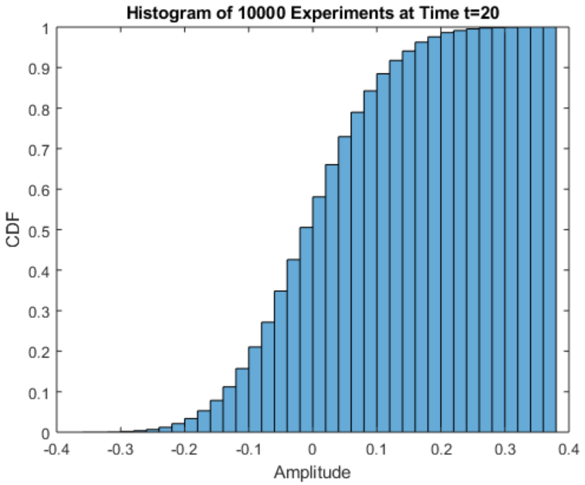


Random Processes: example



Random Processes: example





1st Order Statistics of Random Processes

- Mean of a Random Process:

DISCRETE:

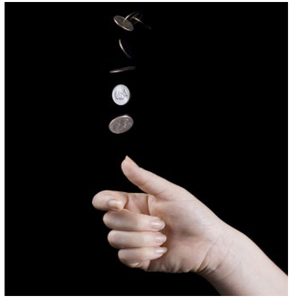
$$\mu_X[n] = E[\mathbf{X}[n]] = \sum_{x \epsilon \zeta} x p_X(x,t)$$

CONTINOUS:

$$\mu_X(t) = E[\mathbf{X}(t)] = \int_{-\infty}^{+\infty} x p_X(x,t) dx$$

1st Order Statistics of Random Processes

- Consider the example of tossing a coin:
  - Substitute 1 for heads, 0 for tails
  - At any repetition n, $p_{\mathrm{X}}(1;n) = p_{\mathrm{X}}(0;n) = 0.5$
  - Calculate the mean of the random process:



$$E\{\mathbf{x}[n]\} = \sum_{x=0}^{1} x \cdot p_{\mathrm{X}}(x;n) = 0.5$$

2^{nd} Order Distribution

Consider the random process $\mathbf{X}(t,\boldsymbol{\zeta})$, fixing the time at two different instances t_1 and t_2.

- The **joint cumulative distribution function** of two random variables $\mathrm{x}(t_1)$ and $\mathrm{x}(t_2)$ is called the second-order distribution of the process $\mathbf{X}(t,\boldsymbol{\zeta})$:

$$\mathrm{P}(x_1, x_2; t_1, t_2) = P[X(t_1) \leq x_1, X(t_2) \leq x_2]$$

- The **joint probability density function**:

$$p(x_1, x_2; t_1, t_2) = \frac{\partial^2 \mathrm{P}(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

2^{nd} Order Statistics

Consider the random process $\mathbf{X}(t,\zeta)$, fixing the time at two different instances t_1 and t_2.

- Why are we interested in the 2^{nd} order statistics?
  - Autocorrelation
  - Power Spectral Density

2^{nd} Order Statistics: Autocorrelation

- **Auto-correlation** of a random process:

Continuous

$$r(t_1, t_2) = E\{\mathrm{x}(t_1)\mathrm{x}(t_2)\} = \int_{-\infty}^{\infty} x_1 \cdot x_2 \cdot p(x_1, x_2; t_1, t_2) dx_1 dx_2$$

Discrete

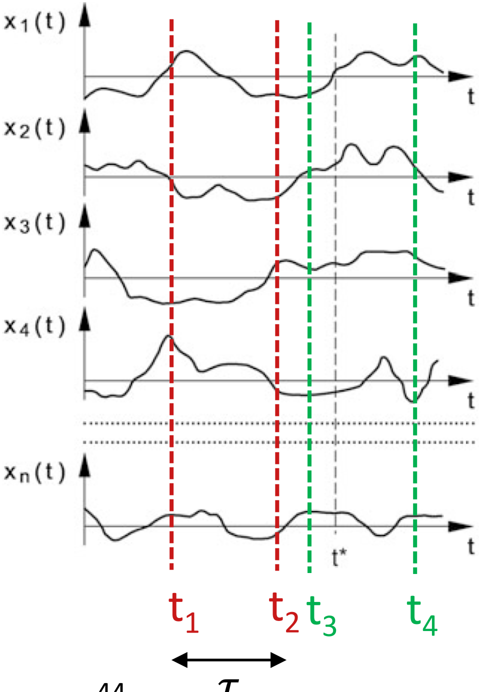
$$r[n_{1,} n_2] = E\{\mathrm{x}[n_1]\mathrm{x}[n_2]\} = \sum_{x \in \zeta} x_1 \cdot x_2 \cdot p(x_1, x_2; n_1, n_2)$$

- The autocorrelation gives an indication of how the two random variables (obtained by fixing the time at t_1 and t_2) change in relation to each other.

Note: here we focus on real signals, hence the complex conjugates are omitted

Stationarity

Random process



Statistical properties do not change over time

- 1^{st} order statistics:

$$E\{X(t_1)\} = E\{X(t_2)\} = \mu$$

$$E\{X(t)\} = E\{X(t+\tau)\} = \mu$$

- 2^{nd} order statistics:

$$r(t_1, t_2) = E\{X(t_1)\, X(t_2)\} = E\{\mathrm{X}(t_1+\tau)\, \mathrm{X}(t_2+\tau)\}$$

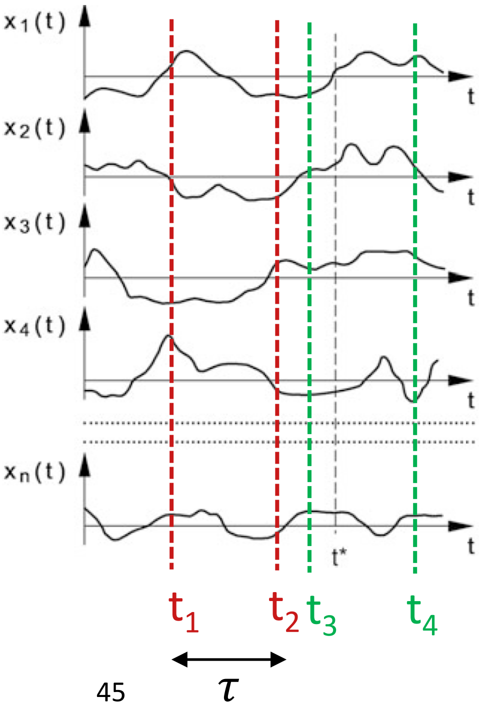
$$E\{\mathrm{X}(t_1)\, X(t_2)\} = E\{X(t_3)\, X(t_4)\} \text{ if } t_2 - t_1 = t_4 - t_3 = \tau$$

$$r(\tau) = E\{\mathrm{X}(t_1)\, \mathrm{X}(t_1+\tau)\}$$

2^{nd} order statistics depends only on the time lag

Stationarity

Random process



Statistical properties are constant

n-order Stationarity

Generalize over higher order density functions:

$$p(x_1, \cdots, x_n; t_1, \cdots, t_n) = p(x_1, \cdots, x_n; t_1 + \tau, \cdots, t_n + \tau)$$

Any order n → strict-sense stationarity

Up 2nd order → wide-sense stationarity

Strict sense stationary $\Longrightarrow$ Wide Sense stationary

Properties autocorrelation

- For **wide-sense stationary** signals the autocorrelation depends only on the shift

$$r_x[n_1, n_2] = r_x[l], \qquad \text{with} \quad l = n_1 - n_2$$

- The autocorrelation is maximum for $l = 0$, and it is equal to average power of the signal

$$r_x[0] = E\{|x[n]|^2\} = \sigma_x^2 + |\mu_x|^2 \geq 0$$

- The autocorrelation is a ***conjugate symmetric function***

$$r_x^*[-l] = r_x[l]$$

2nd Order Statistics: Power Spectral Density

- The **power spectral density** of a random process defined as the Fourier Transform of its autocorrelation (Wiener-Kintchine):

Continuous

$$P(e^{j\omega}) = \int_{-\infty}^{\infty} r(\tau) e^{-j\omega\tau} d\tau$$

Discrete

$$P(e^{j\omega}) = \sum_{n=-\infty}^{\infty} r[n] e^{-j\omega n}$$

The inverse relation is (inverse Fourier Transorm):

$$r(\tau) = \int_{-\infty}^{\infty} P(e^{j\omega}) e^{j\omega\tau} d\omega$$

$$r[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{j\omega}) e^{j\omega n} d\omega$$

Properties (auto)PSD

- The PSD is a real-valued periodic function of frequency with period 2π
  - $x[n]$ real-valued, then $P_x(e^{j\theta})$ is even: $P_x(e^{j\theta}) = P_x(e^{-j\theta})$

- The PSD is non-negative definite: $P_x(e^{j\theta}) \geq 0$

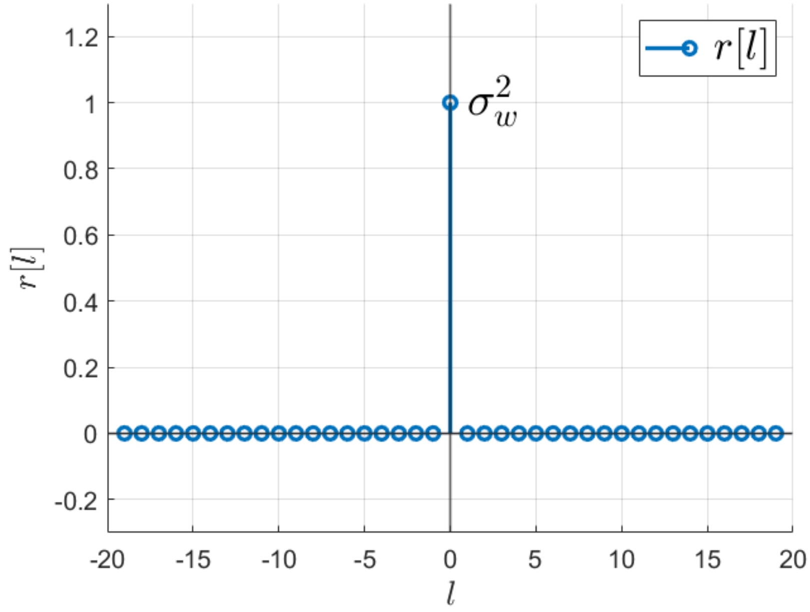
- The area under the PSD is non-negative and it equals the average power of $x[n]$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{j\theta})d\theta = r_x[0] = E\{|x[n]|^2\} \geq 0$$

Special-case: zero-mean white noise sequence

Given a zero-mean white noise sequence $w[n]$ with variance σ_w^2

$$r[l] = \sigma_w^2 \delta[l]$$



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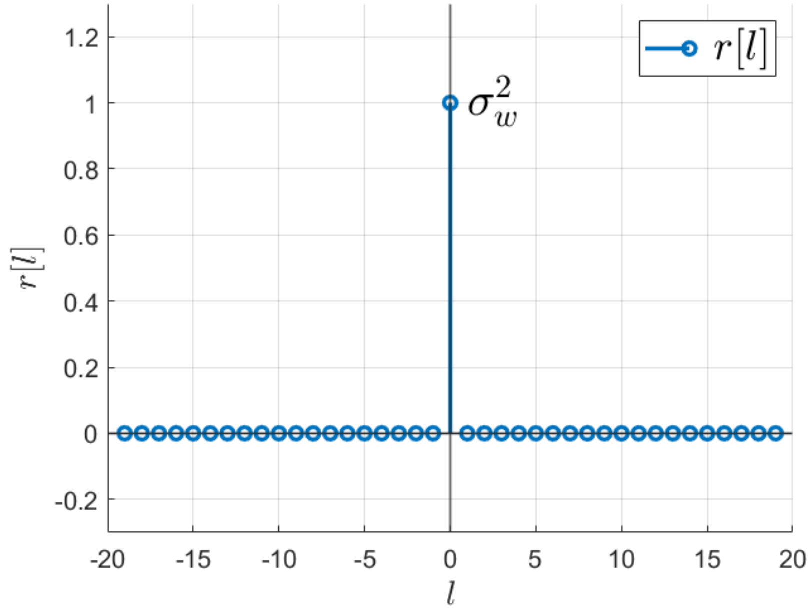


Correlation of each sample with itself

$$r[0] = E[w[n]w[n]] = E[w^2[n]]$$

Correlation of each sample with the next

$$r[1] = E[w[n]w[n-1]]$$

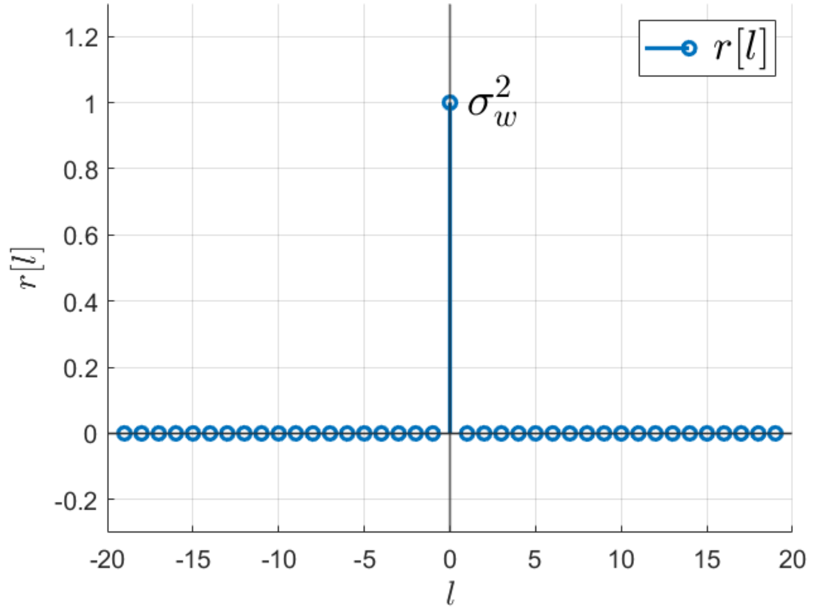
Correlation of each sample with 2 samples ahead

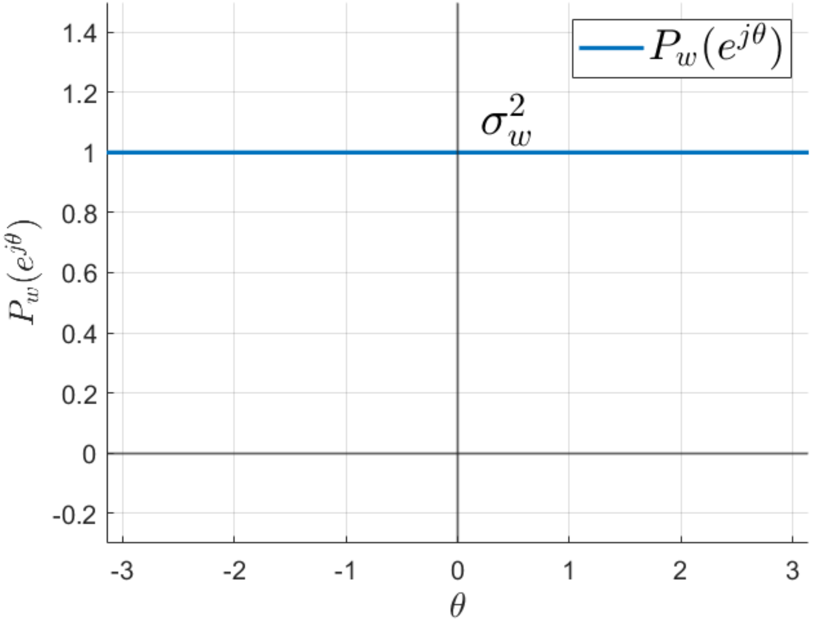
$$r[2] = E[w[n]w[n-2]]$$

Special-case: zero-mean white noise sequence

Given a zero-mean white noise sequence $w[n]$ with variance σ_w^2

$$r[l] = \sigma_w^2 \delta[l] \qquad\qquad P_w(e^{j\theta}) = \sigma_w^2 \quad \forall\, \theta$$

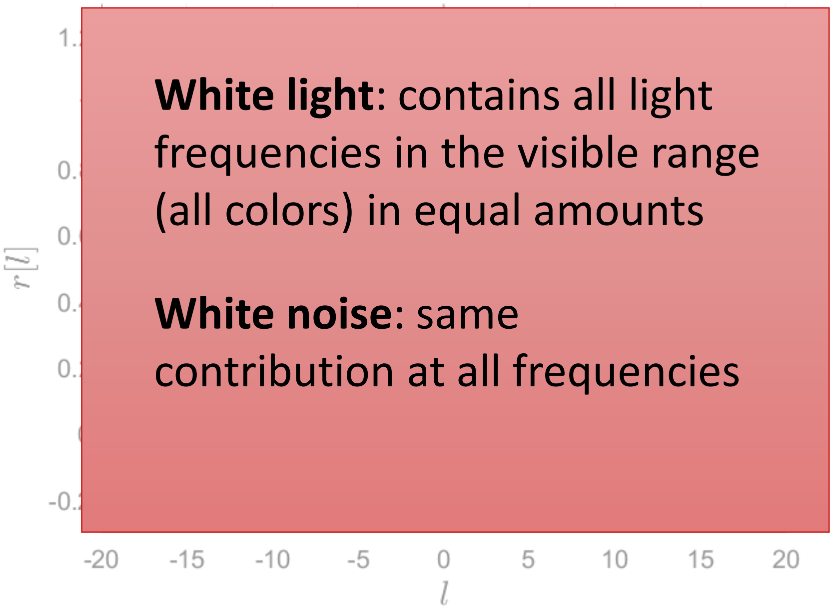




Special-case: zero-mean white noise sequence

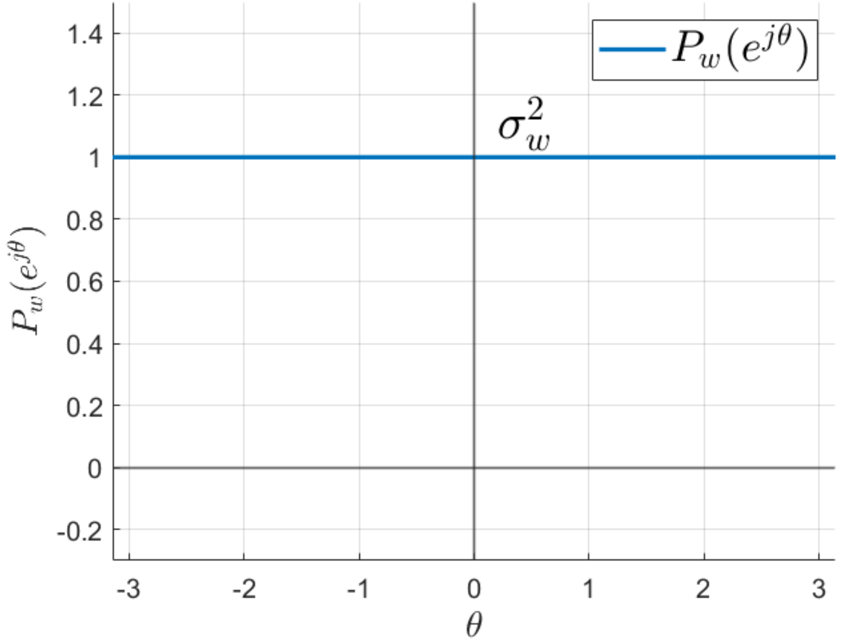
Given a zero-mean **white** noise sequence $w[n]$ with variance σ_w^2

$$r[l] = \sigma_w^2 \delta[l] \qquad\qquad P_w(e^{j\theta}) = \sigma_w^2 \quad \forall \theta$$



White light: contains all light frequencies in the visible range (all colors) in equal amounts

White noise: same contribution at all frequencies



Ideal signal statistics: discrete-time

Mean: $\mu_X[n] = \mathrm{E}\left[X[n]\right]$

Variance: $\sigma_X^2[n] = \mathrm{E}\left[\left|X[n] - \mu_X[n]\right|^2\right]$

Covariance: $c_X[n_1, n_2] = \mathrm{E}\left[\left(X[n_1] - \mu_X[n_1]\right)\left(X[n_2] - \mu_X[n_2]\right)^*\right]$

Correlation: $r_X[n_1, n_2] = \mathrm{E}\left[X[n_1] \cdot X^*[n_2]\right]$

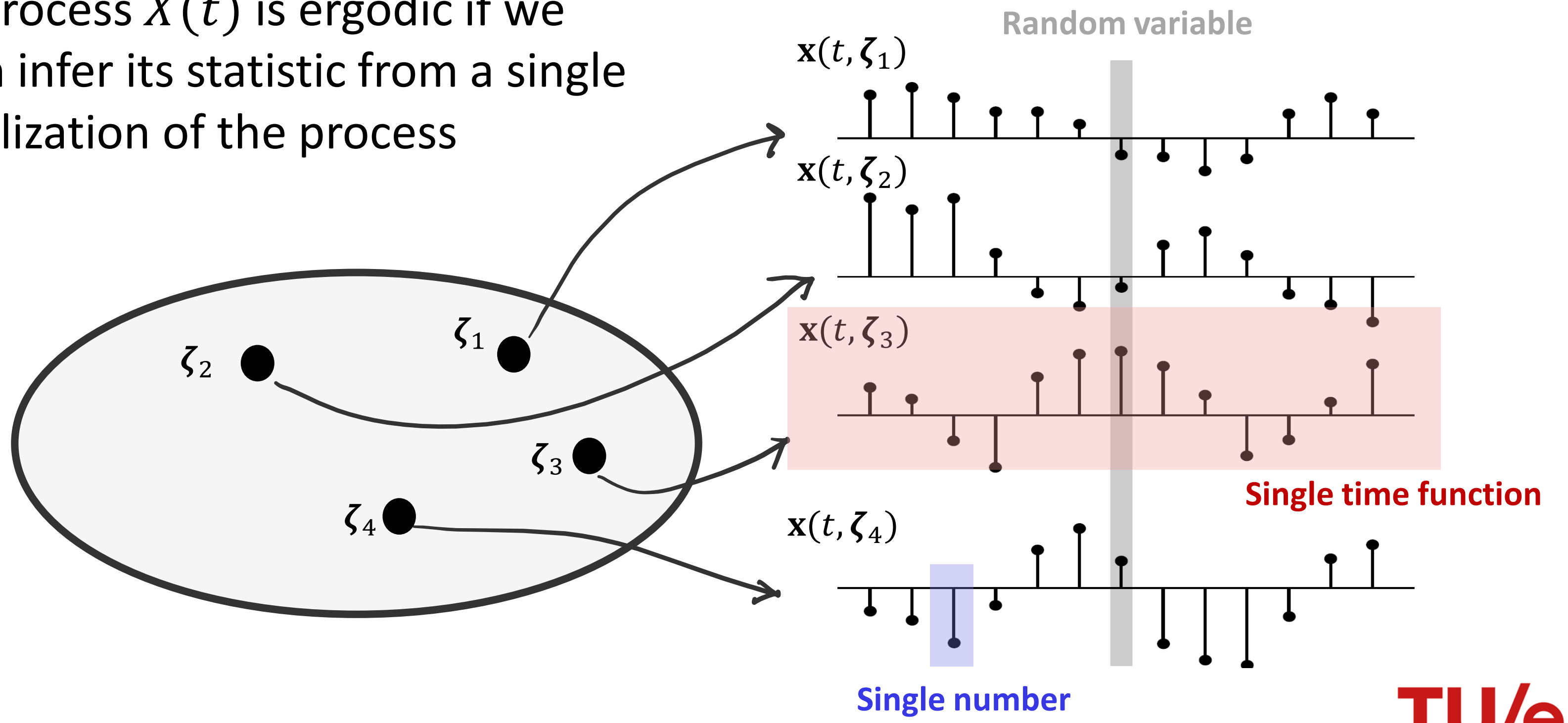
Cross-Covariance: $c_{XY}[n_1, n_2] = \mathrm{E}\left[\left(X[n_1] - \mu_X[n_1]\right)\left(Y[n_2] - \mu_Y[n_2]\right)^*\right]$

Cross-Correlation: $r_{XY}[n_1, n_2] = \mathrm{E}\left[X[n_1] \cdot Y^*[n_2]\right]$

Cross-Correlation coefficient: $\rho_{XY}[n_1, n_2] = \dfrac{c_{XY}[n_1, n_2]}{\sigma_X[n_1]\sigma_Y[n_2]}$

Ergodicity

- A process $X(t)$ is ergodic if we can infer its statistic from a single realization of the process



Ergodicity: general definition

- The formal definition of ergodicity states that a strict-sense stationary process $X(t)$ is strict-sense ergodic if **time average equals the ensemble average**.

- This means that any of the statistical properties of $X(t)$ of any order can be obtained by any of its single realizations $x(t)$, known during an infinite time interval.

Ergodicity

- A process is ergodic in the mean if

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} x(t) = \mu_x = E[X(t)] = \int_{-\infty}^{+\infty} x p_X(x;t) dx$$

equality only holds if the variance of the time-average tends to zero for $T \to \infty$

For discrete time signals in a window of length $N \to \infty$

$$\mathrm{E}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right\} = \mu_X \qquad \mathrm{Var}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right\} \xrightarrow{N \to \infty} 0$$

Ergodicity

- A process is ergodic in the autocorrelation if

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} x(t+\tau)x(t) = R_X(\tau) = E[X(t+\tau)X(t)]$$

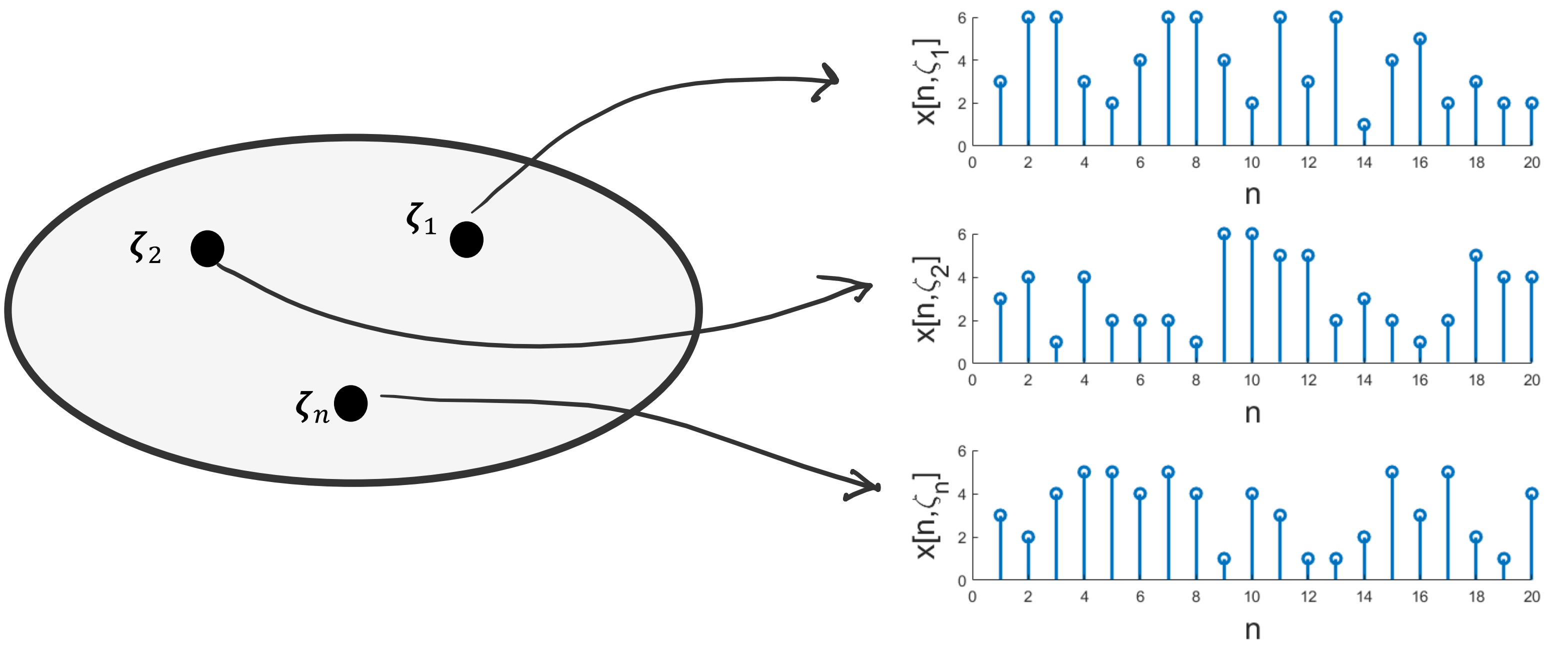
equality only holds if the variance of the time-average tends to zero for $T \to \infty$

For discrete time signals in a window of length $N \to \infty$

$$\mathrm{E}\left\{ \frac{1}{N} \sum_{n=0}^{N-1-|l|} x[n]x^*[n-l] \right\} = r_X[l] \qquad \mathrm{Var}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right\} \xrightarrow{N \to \infty} 0$$

Example 1

Random process: roll a dice infinitely many times



Example 1

Random process: roll a dice infinitely many times

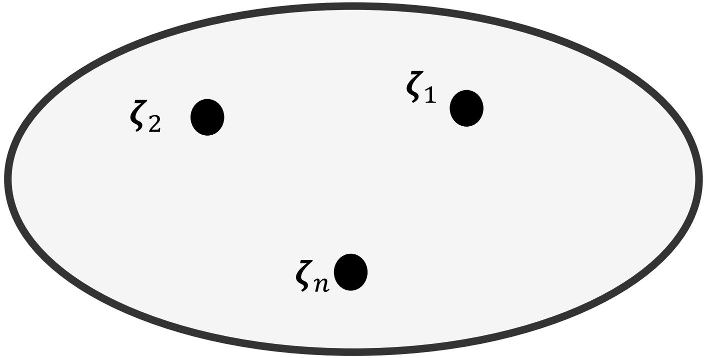
$$E\{x[n,\zeta_1]\} = E\{x[n,\zeta_2]\} = \ldots = E\{x[n,\zeta_n]\} = 3.5$$

$$E\{X[n]\} = 3.5$$

We can conclude that this process is **ergodic in the mean**

Example 1

Random process: roll a die infinitely many times



$$x[n,\zeta_1] = Uniform(1,6)$$

$$\vdots$$

$$x[n,\zeta_2] = Uniform(1,6)$$

$$\Downarrow$$

$$X[n] = Uniform(1,6)$$

Each ζ_i represents a realization of this random process.
If the dice are equal, the statistics of the whole process are the same as the ones of a single realization

Example 1

Random process: roll a die infinitely many times

$$x[n,\zeta_1] = Uniform(1,6)$$

ζ_2

We can conclude that this process is **strict-sense** ergodic

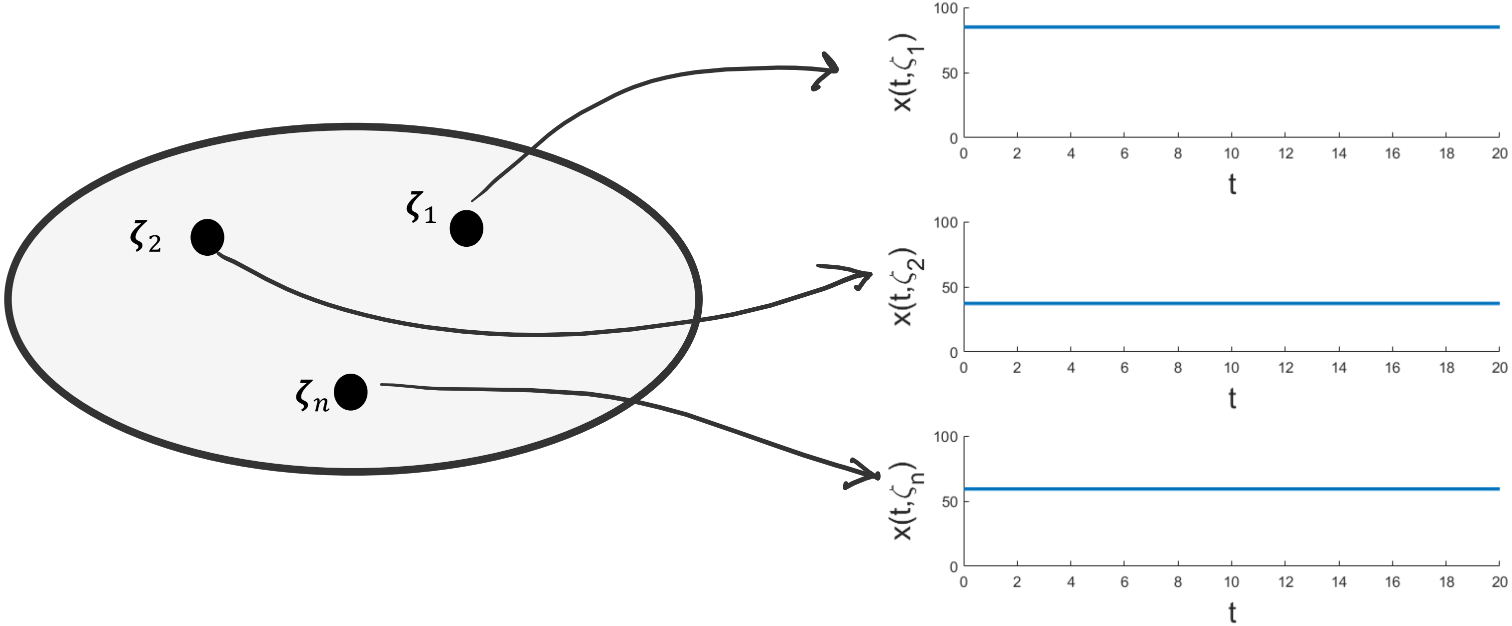
$$\ldots orm(1,6)$$

Each ζ_i represents a realization of this random process.
If the dice are equal, the statistics of the whole process are the same as the ones of a single realization

$$X[n] = Uniform(1,6)$$

Example 2

Random process: pick a resistor and measure a (constant) voltage over time.
Assumption: each measurement can give a random value distributed as $U(0{,}100)$



Example 2

Random process: pick a resistor and measure a (constant) voltage over time.
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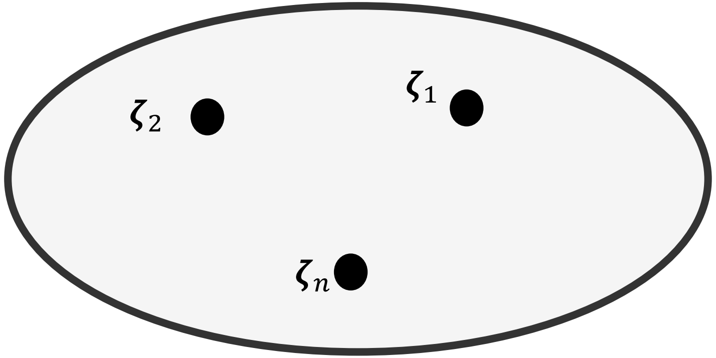


Each ζ_i represent picking a resistor. Once the resistor is picked, the value is constant.

$$x(t,\zeta_1) = a_1 \qquad a_1, a_2, \ldots, a_n \quad \text{Fixed over time}$$

$$x(t,\zeta_2) = a_2$$

$$\vdots$$

$$x(t,\zeta_n) = a_n \qquad X(t) \sim U(0{,}100)$$

Example 2

Random process: pick a resistor and measure a (constant) voltage over time.
Assumption: each measurement can give a random value distributed as $U(0,100)$

$$E\{x(t,\zeta_1)\} = a_1$$
$$E\{x(t,\zeta_2)\} = a_2$$
$$\vdots$$
$$E\{x(t,\zeta_n)\} = a_n$$

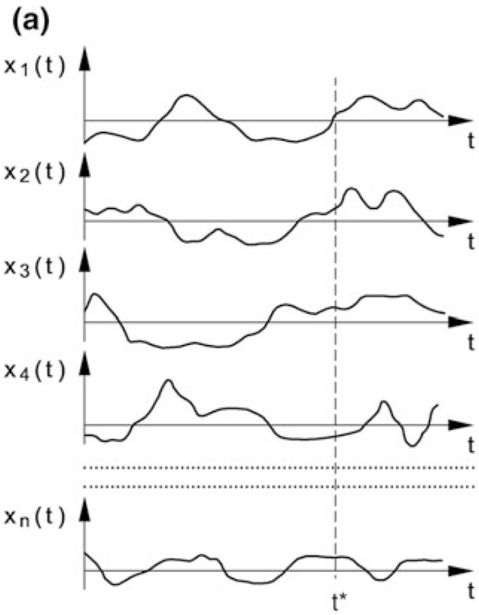
$$\neq$$

$$E\{X(t)\} = 50$$

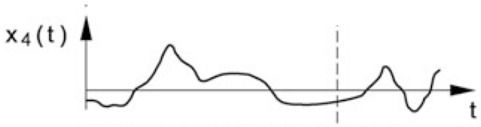
We can conclude that this process is **not** ergodic

Ergodicity

Random process



Ergodic process

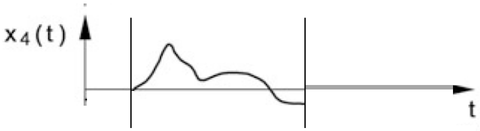


Statistics can be calculated by time-averaging over single representative members of the ensemble

Ergodicity of the mean

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbf{x}(t)dt = E\{\mathbf{x}(t)\}$$

Practice: Limited set of samples



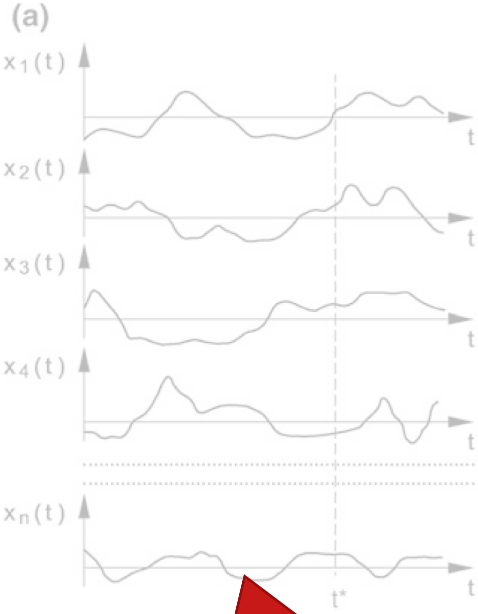
Segment of a single realization

$$E\{\cdot\} \Leftrightarrow \frac{1}{N}\sum_{n=0}^{N-1}\{\cdot\}$$

Ergodicity $\Longrightarrow$ Stationarity

Stationarity and Ergodicity

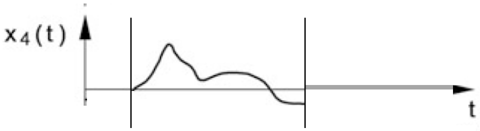
Random process



Ergodic process

Statistics can be calculated by time-averaging over single representative members of the ensemble

Practice:
limited set of samples

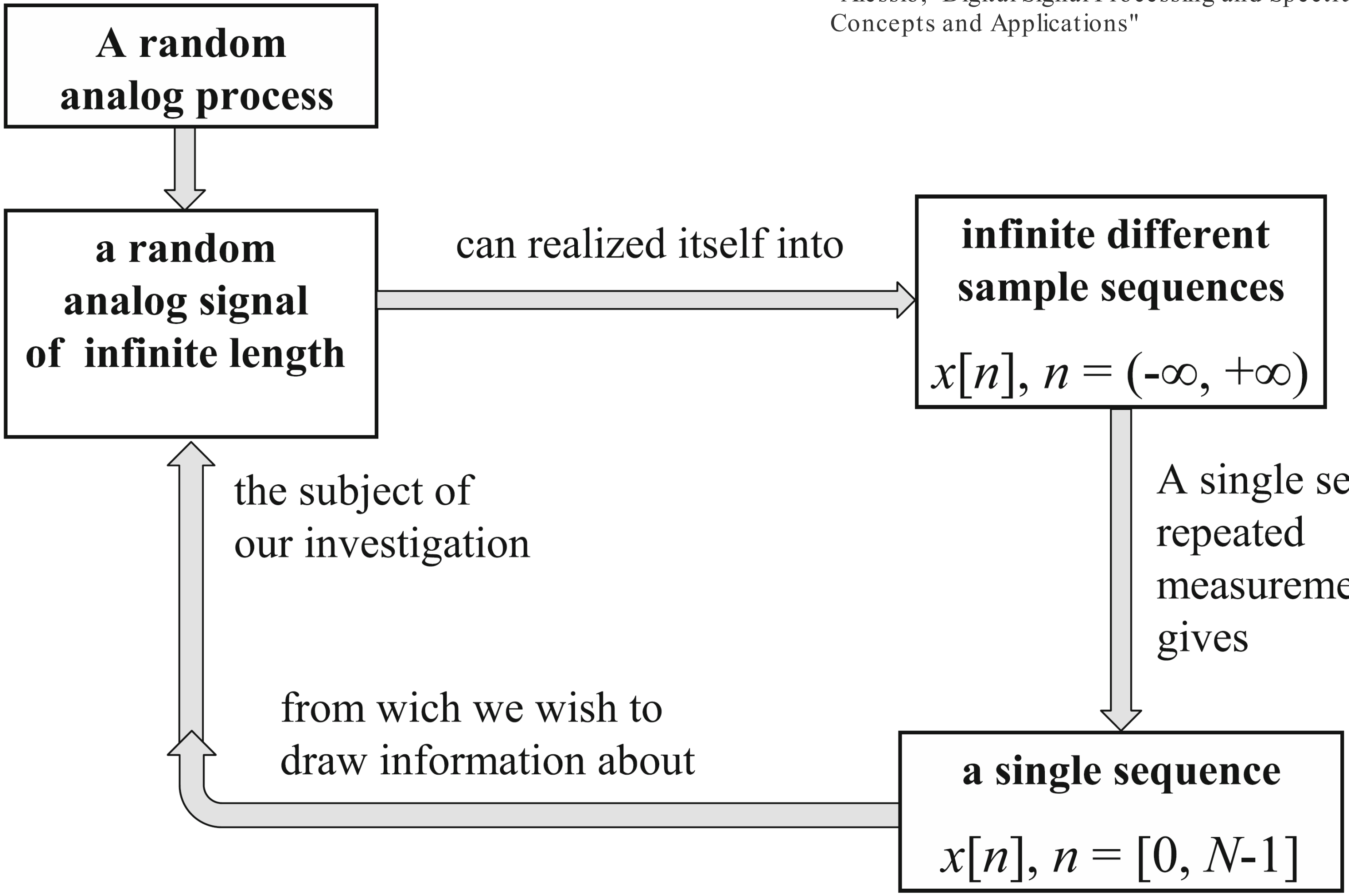


Segment of a single realization

$$E\{\cdot\} \Leftrightarrow \frac{1}{N}\sum_{n=0}^{N-1}\{\cdot\}$$

Draw information on the underlying random process

*Alessio, "Digital Signal Processing and Spectral Analysis for Scientists - Concepts and Applications"



Autocorrelation/Autocovariance

Discrete-time random process $X[n]$

Autocorrelation

$$r_X[n_1,n_2]=\mathrm{E}\left[X[n_1]\cdot X^*[n_2]\right]$$

WSS

$$r_X[l]=\mathrm{E}\left[X[n]X^*[n-l]\right]$$

Ergodic, limited time

$$\hat{r}_X[l]=\frac{1}{N}\sum_{n=0}^{N-1-|l|}x[n]x^*[n-l]=\hat{\gamma}_X[l]+|\hat{\mu}_X|^2$$

Autocovariance

$$c_X[n_1,n_2]=\mathrm{E}\left[\left(X[n_1]-\mu_X[n_1]\right)\left(X[n_2]-\mu_X[n_2]\right)^*\right]$$

WSS

$$c_X[l]=\mathrm{E}\left[\left(X[n]-\mu_X\right)\left(X[n-l]-\mu_X\right)^*\right]$$

Ergodic, limited time

$$\hat{c}_X[l]=\frac{1}{N}\sum_{n=0}^{N-1-|l|}(x[n]-\hat{\mu}_X)(x[n-l]-\hat{\mu}_X)^*$$

Approximate signal statistics: ergodic, limited-time signals

Mean: $$\hat{\mu}_X = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

Variance: $$\hat{\sigma}_X^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]-\hat{\mu}_X|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 - |\hat{\mu}_X|^2$$

Covariance: $$\hat{c}_X[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|} (x[n]-\hat{\mu}_X)(x[n-l]-\hat{\mu}_X)^* \qquad |l| \le L-1$$

Correlation: $$\hat{r}_X[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|} x[n]x^*[n-l] = \hat{\gamma}_X[l] + |\hat{\mu}_X|^2 \qquad |l| \le L-1$$

Approximate signal statistics: ergodic, limited-time signals

Cross-Covariance:

$$\hat{c}_{XY}[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|}(x[n]-\hat{\mu}_X)(y[n-l]-\hat{\mu}_Y)^*$$

Cross-Correlation:

$$\hat{r}_{XY}[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|}x[n]y^*[n-l] = \hat{\gamma}_{XY}[l]+\hat{\mu}_X\cdot\hat{\mu}_Y^*$$

Cross-Correlation coefficient:

$$\rho_{XY}[l] = \frac{c_{XY}[l]}{\sigma_X\sigma_Y}$$

Cross power spectral density

- The **cross-power spectrum** between two WSS processes can be calculated as

$$P_{XY}(e^{j\theta}) = \sum_{l=-\infty}^{\infty} r_{XY}[l]e^{-jl\theta} \quad \Leftrightarrow \quad r_{XY}[l] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{XY}(e^{j\theta})e^{jl\theta}\mathrm{d}\theta$$

- For the complex conjugate symmetry property of the correlation function:

$$r_{XY}[l] = r_{YX}^{*}[-l] \quad \Leftrightarrow \quad P_{XY}(e^{j\theta}) = P_{YX}^{*}(e^{j\theta})$$

Wrap up (I)

- Multiple random variables can be conveniently grouped as **random vectors**, of which we can define a joint probability model and calculate statistics
- A **random process** maps each outcome of the sample space to a time function. Each realization of the random process is a **time signal**.
- Random processes can be characterized by the joint probability model for each time instance
- Of practical interest are **1st order statistics** (mean, variance) and **2nd order statistics** (autocorrelation, power spectral density)

Wrap up (II)

- For **stationary processes**, the statistics do not change over time (1st order statistics are constant, 2nd order statistics only depends on time lags)
  - **Strict-sense stationarity**: statistics of any order n
  - **Wide-sense stationarity**: statistics up to order 2
- For **ergodic processes**, the statistical properties of any order can be obtained by any of its single realizations $x(t)$, known during an infinite time interval (strict-sense)
- In practice, signals are available only in a limited time window, and the we assume that the statistics can be **approximated in a finite-time interval**

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Statistical signal processing (5CTA0)

Lecture 2, part B

Lecturer: Simona Turco

Electrical Engingeering, Signal Processing Systems group