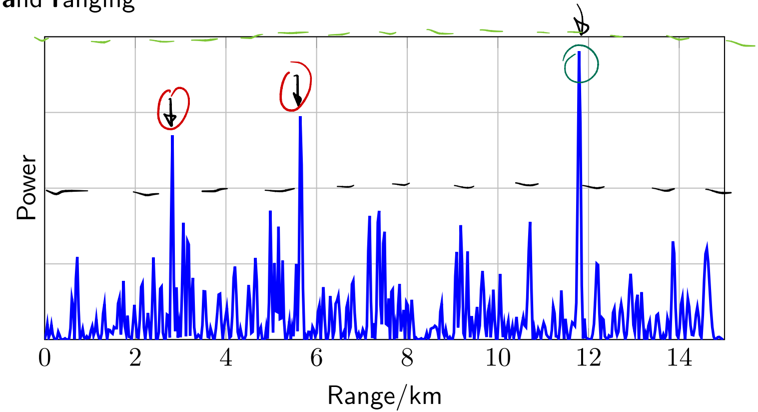
EINDHOVEN UNIVERSITY OF TECHNOLOGY

Statistical signal processing 5CTA0

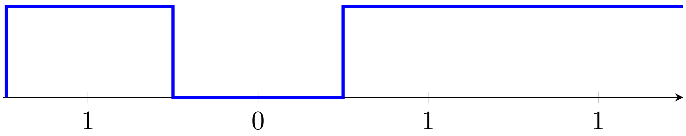
Detection theory

Example - Radar

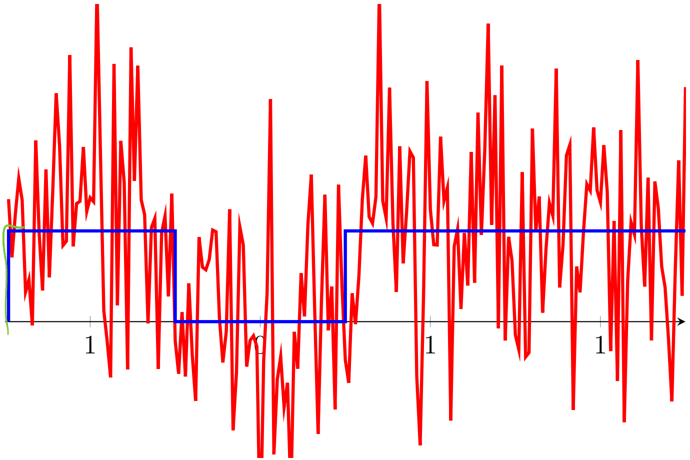
Radio **d**etection **a**nd **r**anging



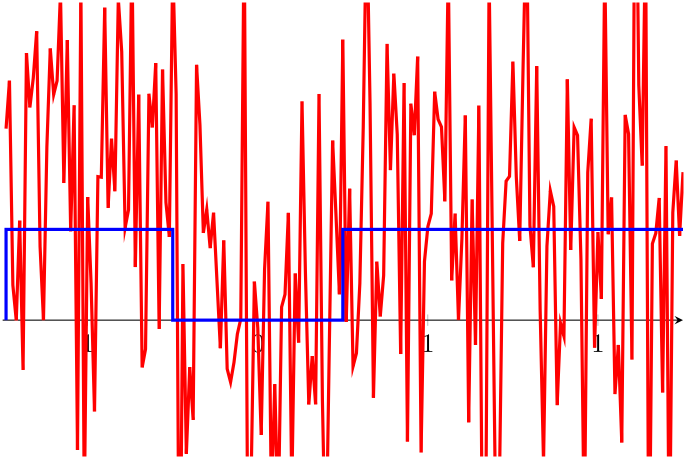
Example - Communication system



Example - Communication system



Example - Communication system

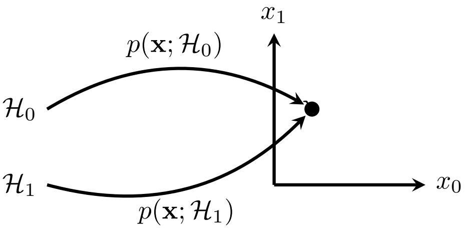


Binary hypothesis testing

- Discriminate between two hypothesis: null hypothesis $\mathcal{H}_0$ and alternative hypothesis $\mathcal{H}_1$
- Distribution of observation:

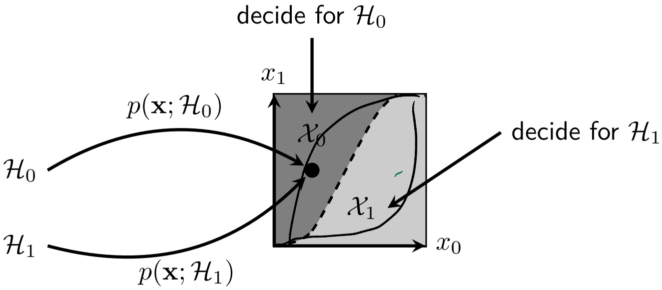
$$\begin{aligned} \mathcal{H}_0 &: \mathbf{x} \sim p(\mathbf{x}; \mathcal{H}_0) \\ \mathcal{H}_1 &: \mathbf{x} \sim p(\mathbf{x}; \mathcal{H}_1) \end{aligned}$$

Binary hypothesis testing



Observation space

Binary hypothesis testing



Observation space

- Assessing hypotheses: partition the observation space into two disjoint parts, $\mathcal{X}_0$ and $\mathcal{X}_1$, with $\mathcal{X}_0 \cup \mathcal{X}_1 = \mathbb{R}^N$ and $\mathcal{X}_0 \cap \mathcal{X}_1 = \emptyset$

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Binary hypothesis testing

- Scenarios:

	True hypothesis	
Decision	$\mathcal{H}_0$	$\mathcal{H}_1$
$\mathcal{H}_0$	true negative	false negative/miss/type II error
$\mathcal{H}_1$	false positive/false alarm/type I error	true positive/detection

$P_M = 1 - P_D$

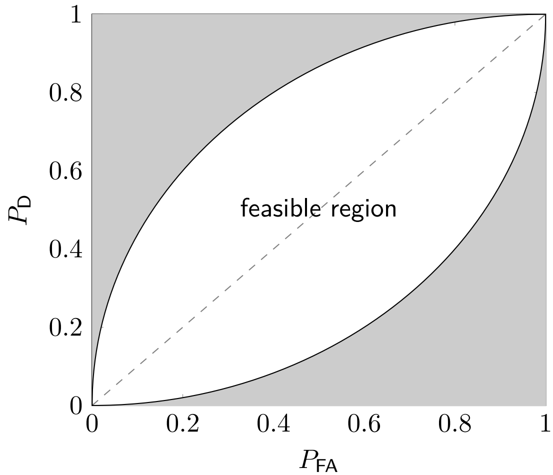
- Probability of detection:

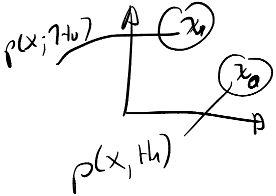
$$P_{\mathsf{D}} = \int_{\mathcal{X}_1} p(\mathbf{x}; \mathcal{H}_1) d\mathbf{x}$$

- Probability of false alarm:

$$P_{\mathsf{FA}} = \int_{\mathcal{X}_1} p(\mathbf{x}; \mathcal{H}_0) d\mathbf{x}$$

Binary hypothesis testing





Neyman-Pearson test

- Maximize P_{D} for $P_{\mathsf{FA}} \leq \alpha$
- Likelihood ratio test:

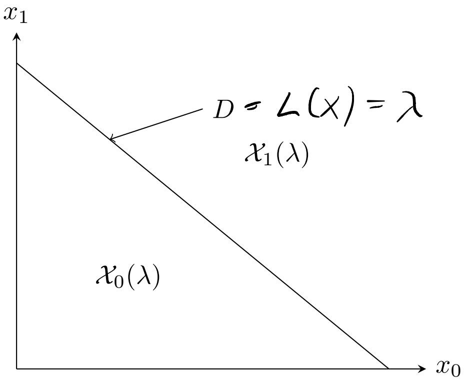
$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda$$

- Decision regions:

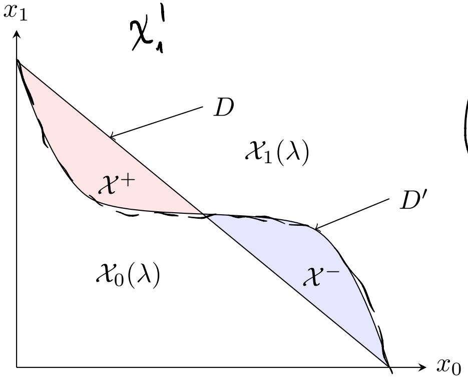
$$\mathcal{X}_1(\lambda) = \{\mathbf{x} : L(\mathbf{x}) \geq \lambda\}$$

$$\mathcal{X}_0(\lambda) = \{\mathbf{x} : L(\mathbf{x}) < \lambda\}$$

Optimality of the NP test



Optimality of the NP test



$$\int_{\mathcal{X}^+} p(\mathbf{x};\mathcal{H}_0)d\mathbf{x} = \int_{\mathcal{X}^-} p(\mathbf{x};\mathcal{H}_0)d\mathbf{x}$$

$$P_{FA}' = \int_{\mathcal{X}_1'} p(x,\mathcal{H}_0)\,dx$$

$$\mathcal{X}_1' = \mathcal{X}_1(\lambda) \cup \mathcal{X}^+ \setminus \mathcal{X}^-$$

$$= \int_{\mathcal{X}_1} p(x,\mathcal{H}_0)\,dx + \int_{\mathcal{X}^+} p(x,\mathcal{H}_0)\,dx - \int_{\mathcal{X}^-} p(x,\mathcal{H}_0)\,dx$$

$$= \int_{\mathcal{X}_1} p(x,\mathcal{H}_0)\,dx = P_{FA}$$

Optimality of the NP test

- Probability of detection for new decision region:

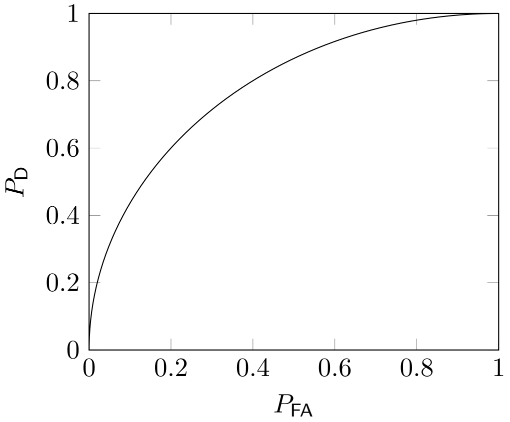
$$P'_D = \int_{\mathcal{X}'_1} p(\mathbf{x};\mathcal{H}_1)d\mathbf{x} = \int_{\mathcal{X}'_1} p(\mathbf{x};\mathcal{H}_1)\underbrace{\frac{p(\mathbf{x};\mathcal{H}_0)}{p(\mathbf{x};\mathcal{H}_0)}}_{1}d\mathbf{x} = \int_{\mathcal{X}'_1} L(\mathbf{x})p(\mathbf{x};\mathcal{H}_0)d\mathbf{x}$$

- $\mathcal{X}'_1 = \mathcal{X}_1(\lambda) \cup \mathcal{X}^+ \setminus \mathcal{X}^-$
-

$$P'_D = \int_{\mathcal{X}_1(\lambda)} L(\mathbf{x})p(\mathbf{x};\mathcal{H}_0)d\mathbf{x} + \int_{\mathcal{X}^+} L(\mathbf{x})p(\mathbf{x};\mathcal{H}_0)d\mathbf{x} - \int_{\mathcal{X}^-} L(\mathbf{x})p(\mathbf{x};\mathcal{H}_0)d\mathbf{x}$$

$$= P_D + \underbrace{\underbrace{\int_{\mathcal{X}^+} L(\mathbf{x})p(\mathbf{x};\mathcal{H}_0)d\mathbf{x}}_{< \lambda \int_{\mathcal{X}^+} p(\underline{x},\mathcal{H}_0)dx} - \underbrace{\int_{\mathcal{X}^-} L(\mathbf{x})p(\mathbf{x};\mathcal{H}_0)d\mathbf{x}}_{\geq \lambda \int_{\mathcal{X}^-} p(x,\mathcal{H}_0)dx}}_{}\ < 0$$

Receiver operating characteristic



Example

- Discriminate between two Gaussian distribution

$$p(\mathbf{x}; \mathcal{H}_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x_n^2\right)$$

$$p(\mathbf{x}; \mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} (x_n - A)^2\right)$$

Example

- Likelihood ratio test:

$$L(\mathbf{x}) = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x_n - A)^2\right)}{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{\sigma^2}\sum_{n=0}^{N-1} x_n^2\right)}$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(NA^2 - 2NA\bar{x}\right)\right)$$

$$\bar{x} = \frac{1}{N}\sum_{n=0}^{N-1} x_n$$

- Log-likelihood:

$$\ln L(\mathbf{x}) = -\frac{1}{2\sigma^2}\left(NA^2 - 2NA\bar{x}\right) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \ln\lambda$$

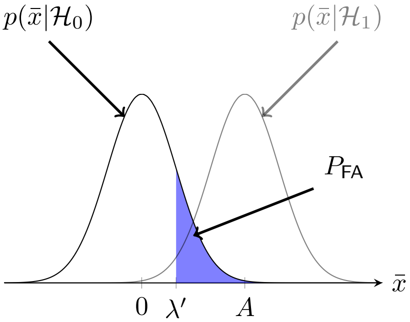
- Sufficient statistic:

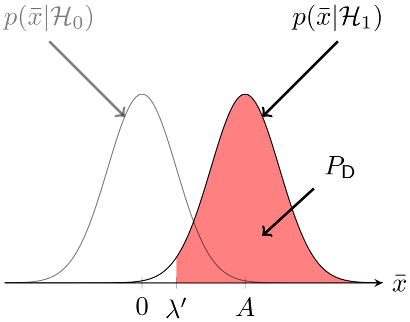
$$\bar{x} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \frac{\sigma^2}{NA}\ln\lambda + \frac{A}{2} = \lambda'$$

Example

- Distribution of sample mean:

$$p(\bar{x};\mathcal{H}_0) \sim \mathcal{N}(0,\sigma^2/N)$$
$$p(\bar{x};\mathcal{H}_1) \sim \mathcal{N}(A,\sigma^2/N)$$





Bayesian test

- Bayes risk:

$$\mathcal{R} = \int_{\mathcal{X}_0} C_{0,0} P_0 p(\mathbf{x}|\mathcal{H}_0) d\mathbf{x} + \int_{\mathcal{X}_1} C_{1,0} P_0 p(\mathbf{x}|\mathcal{H}_0) d\mathbf{x} + \int_{\mathcal{X}_0} C_{0,1} P_1 p(\mathbf{x}|\mathcal{H}_1) d\mathbf{x} + \int_{\mathcal{X}_1} C_{1,1} P_1 p(\mathbf{x}|\mathcal{H}_1) d\mathbf{x}$$

- $\mathcal{X}_0 \cup \mathcal{X}_1 = \mathbb{R}^N$ and $\mathcal{X}_0 \cap \mathcal{X}_1 = \emptyset$:

$$\int_{\mathcal{X}_0} p(\mathbf{x}|\mathcal{H}_i) d\mathbf{x} = 1 - \int_{\mathcal{X}_1} p(\mathbf{x}|\mathcal{H}_i) d\mathbf{x},$$

$i = 0, 1$

$$\mathcal{R} = C_{0,0} P_0 + C_{0,1} P_1 + \int_{\mathcal{X}_1} P_0 (C_{1,0} - C_{0,0}) p(\mathbf{x}|\mathcal{H}_0) + P_1 (C_{1,1} - C_{0,1}) p(\mathbf{x}|\mathcal{H}_1) d\mathbf{x}$$

Bayesian test

- Cost:

$$C_{0,1} > C_{1,1}$$
$$C_{1,0} > C_{0,0}$$

- $$\int_{\mathcal{X}_1} P_0(C_{1,0} - C_{0,0})p(\mathbf{x}|\mathcal{H}_0) + P_1(C_{1,1} - C_{0,1})p(\mathbf{x}|\mathcal{H}_1)d\mathbf{x}$$

- Decision region:

$$\mathcal{X}_1 = \{\mathbf{x} : P_0(C_{1,0} - C_{0,0})p(\mathbf{x}|\mathcal{H}_0) + P_1(C_{1,1} - C_{0,1})p(\mathbf{x}|\mathcal{H}_1) \leq 0\}$$

- Include $\mathbf{x}$ to $\mathcal{X}_1$ if

$$\frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}|\mathcal{H}_0)} \geq \frac{(C_{1,0} - C_{0,0})P_0}{(C_{0,1} - C_{1,1})P_1} = \lambda$$

- The left hand side is again the likelihood ratio test $L(\mathbf{x})$

Bayesian

- Typical cost assignment:

$$C_{1,1} = C_{0,0} = 0$$
$$C_{0,1} = C_{0,1} = 1$$

- Bayes risk:

$$\mathcal{R} = \underbrace{P_0 \int_{\mathcal{X}_1} p(\mathbf{x}|\mathcal{H}_0) d\mathbf{x}}_{P_{\mathsf{FA}}} + \underbrace{P_1 \int_{\mathcal{X}_0} p(\mathbf{x}|\mathcal{H}_1) d\mathbf{x}}_{P_{\mathsf{M}} = 1 - P_{\mathsf{D}}}$$

Total probability of error.

$$\frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}|\mathcal{H}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \frac{P_0}{P_1} = \lambda$$

Known signal in noise

$$\begin{aligned} \mathcal{H}_0 &: \quad x_n = w_n \\ \mathcal{H}_1 &: \quad x_n = s_n + w_n \end{aligned}$$

$$p(\mathbf{x}; \mathcal{H}_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x_n^2\right)$$

$$p(\mathbf{x}; \mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} (x_n - s_n)^2\right)$$

Known signal in noise - likelihood ratio test

- Likelihood ratio test:

$$L(\mathbf{x}) = \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}(x_n - s_n)^2 - \sum_{n=0}^{N-1} x_n^2\right)\right) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda$$

- Log-likelihood ratio test:

$$\ln L(\mathbf{x}) = \frac{1}{\sigma^2}\sum_{n=0}^{N-1} x_n s_n - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1} s_n^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \ln\lambda$$

$$\sum_{n=0}^{N-1} x_n s_n \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \sigma^2 \ln\lambda + \frac{1}{2}\sum_{n=0}^{N-1} s_n^2 = \lambda'$$

Known signal in noise - matched filter

- Convolution:

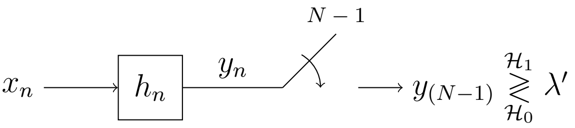
$$y_n = \sum_{l=0}^{N-1} x_l h_{n-l}.$$

- Filter:

$$h_n = s_{N-1-n}$$

- Sample at $N-1$:

$$y_{N-1} = \sum_{l=0}^{N-1} x_l s_{(\cancel{N-1}+\overset{l}{\cancel{n}}\cancel{-N+1})} = \sum_{l=0}^{N-1} x_l s_l$$



Matched filter - SNR

- Definition:

$$\text{SNR} = \frac{\mathrm{E}^2\left[y_{(N-1)}; \mathcal{H}_1\right]}{\mathrm{Var}\left[y_{(N-1)}; \mathcal{H}_0\right]}$$

- Sample $y_{(N-1)}$:

$$\mathcal{H}_1 : y_{(N-1)} = \sum_{l=0}^{N-1} (s_l + w_l) h_{N-1-l} = \mathbf{h}^T(\mathbf{s} + \mathbf{w})$$

$$\mathcal{H}_0 : y_{(N-1)} = \sum_{l=0}^{N-1} w_l h_{N-1-l} = \mathbf{h}^T \mathbf{w}$$

$\mathbf{s} = [s_0, s_1, \ldots, s_{(N-1)}]^T$, $\mathbf{w} = [w_0, w_1, \ldots, w_{N-1}]^T$, and $\mathbf{h} = [h_{(N-1)}, h_{(N-2)}, \ldots, h_0]^T$

Matched filter - SNR

- SNR:

$$\text{SNR} = \frac{\mathrm{E}^2\left[\mathbf{h}^T(\mathbf{s}+\mathbf{w})\right]}{\mathrm{Var}\left[\mathbf{h}^T\mathbf{w}\right]} = \frac{(\mathbf{h}^T\mathbf{s})^2}{\sigma^2\mathbf{h}^T\mathbf{h}}$$

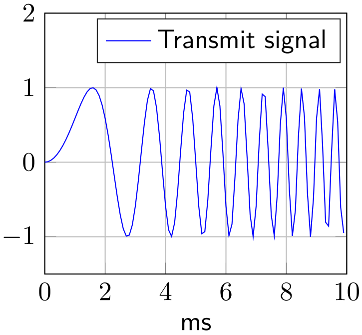
- Cauchy-Schwarz inequality:

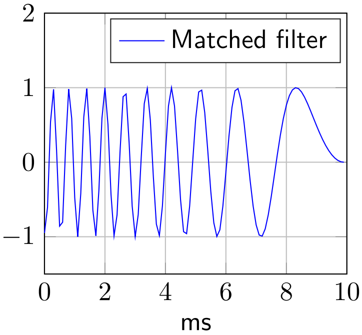
$$(\mathbf{h}^T\mathbf{s})^2 \leq (\mathbf{h}^T\mathbf{h})(\mathbf{s}^T\mathbf{s})$$

- Equality:

$$\mathbf{h} = c\mathbf{s}$$

Example - Radar





Example - Radar