Statistical signal processing (5CTA0)

Non-parametric spectral estimation

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Electrical Engingeering, Signal Processing Systems group

Part 1: Random variables and Random Signals

Part 3

Spectral estimation

3.1: Introduction to spectral estimation

3.2: Non-parametric spectral estimation

3.3: Parametric spectral estimation

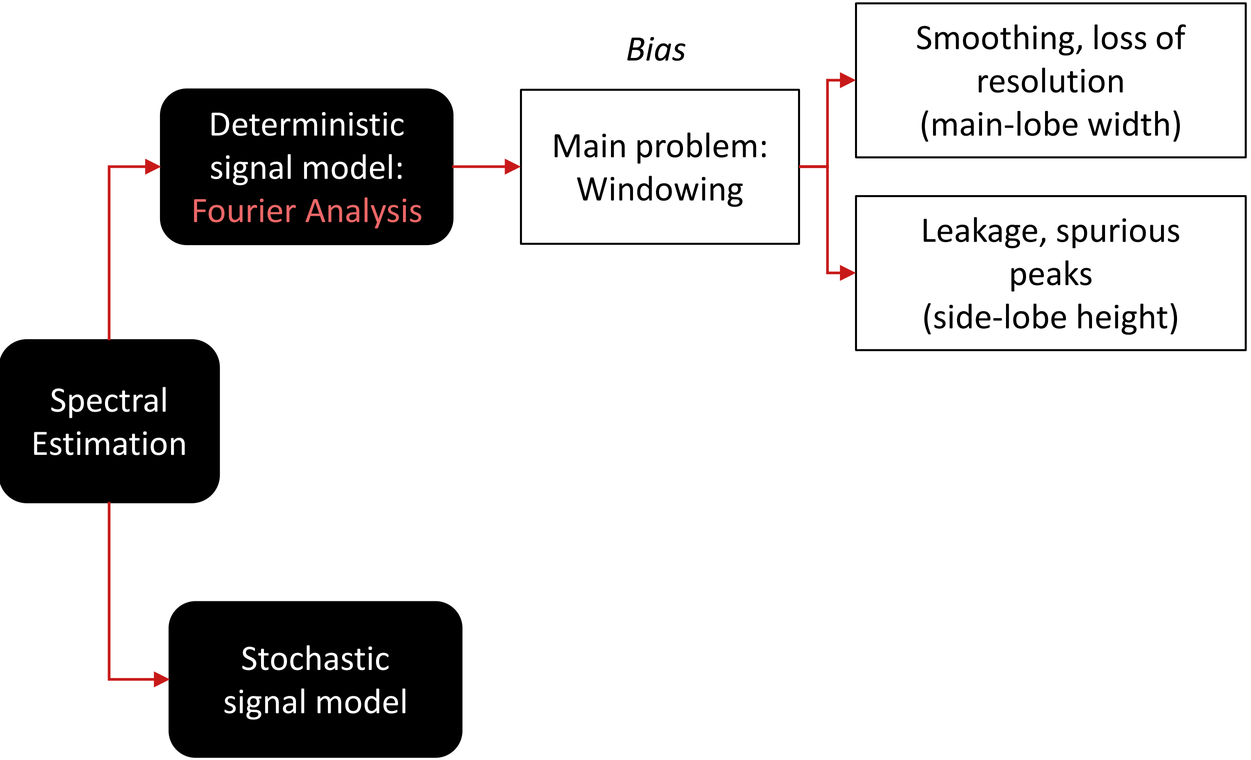
Outline

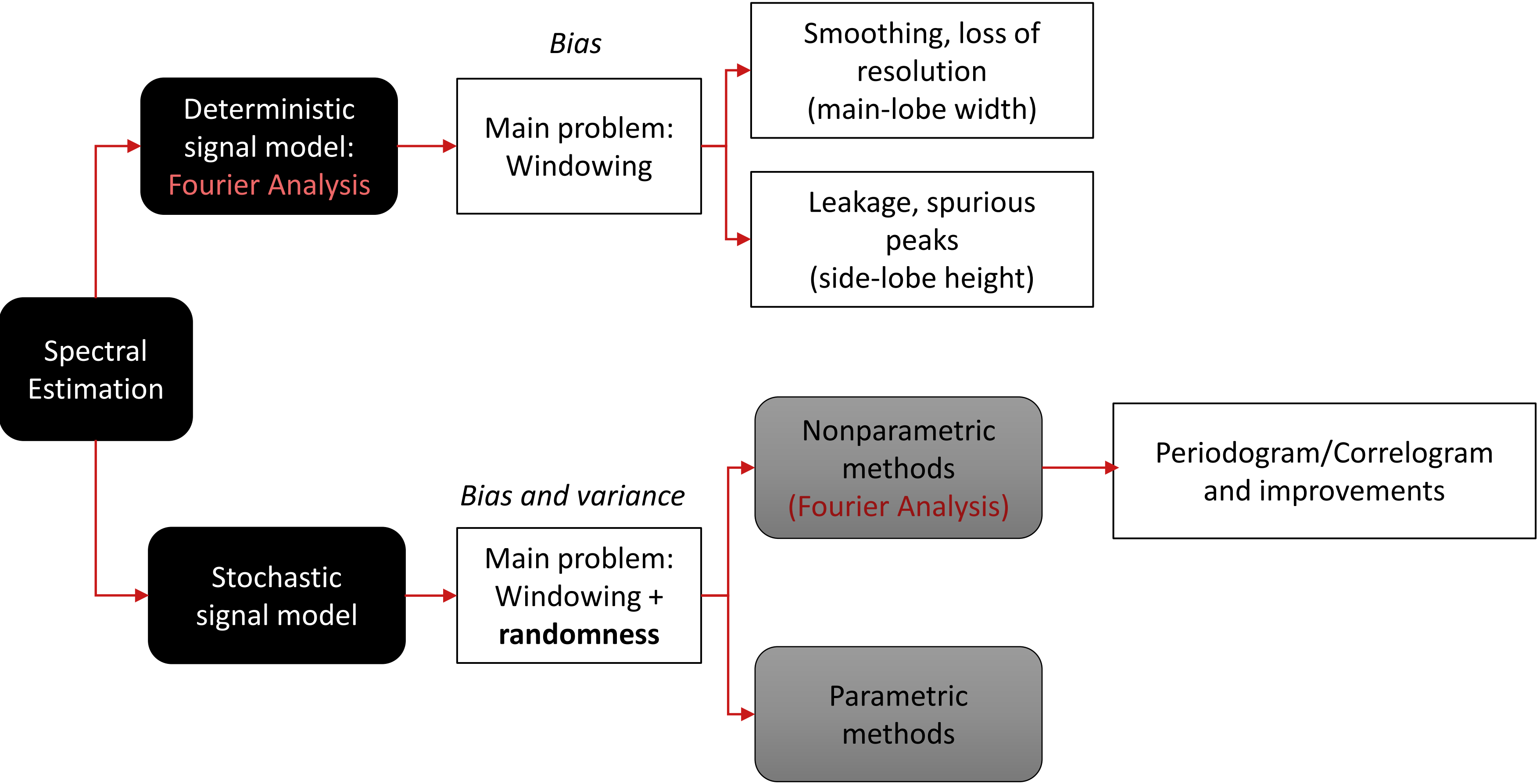
- Introduction
- Biased and unbiased estimators of AC
- Performance of periodogram/correlogram:
- Periodogram improvements:
  - Bartlett method
  - WOSA method
- Correlogram and improvements:
  - Blackman-Tukey method

Deterministic signal model: Fourier Analysis

Spectral Estimation

Stochastic signal model


Spectral Estimation
Deterministic signal model: Fourier Analysis
Stochastic signal model
Bias
Main problem: Windowing
Smoothing, loss of resolution (main-lobe width)
Leakage, spurious peaks (side-lobe height)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Bias and variance
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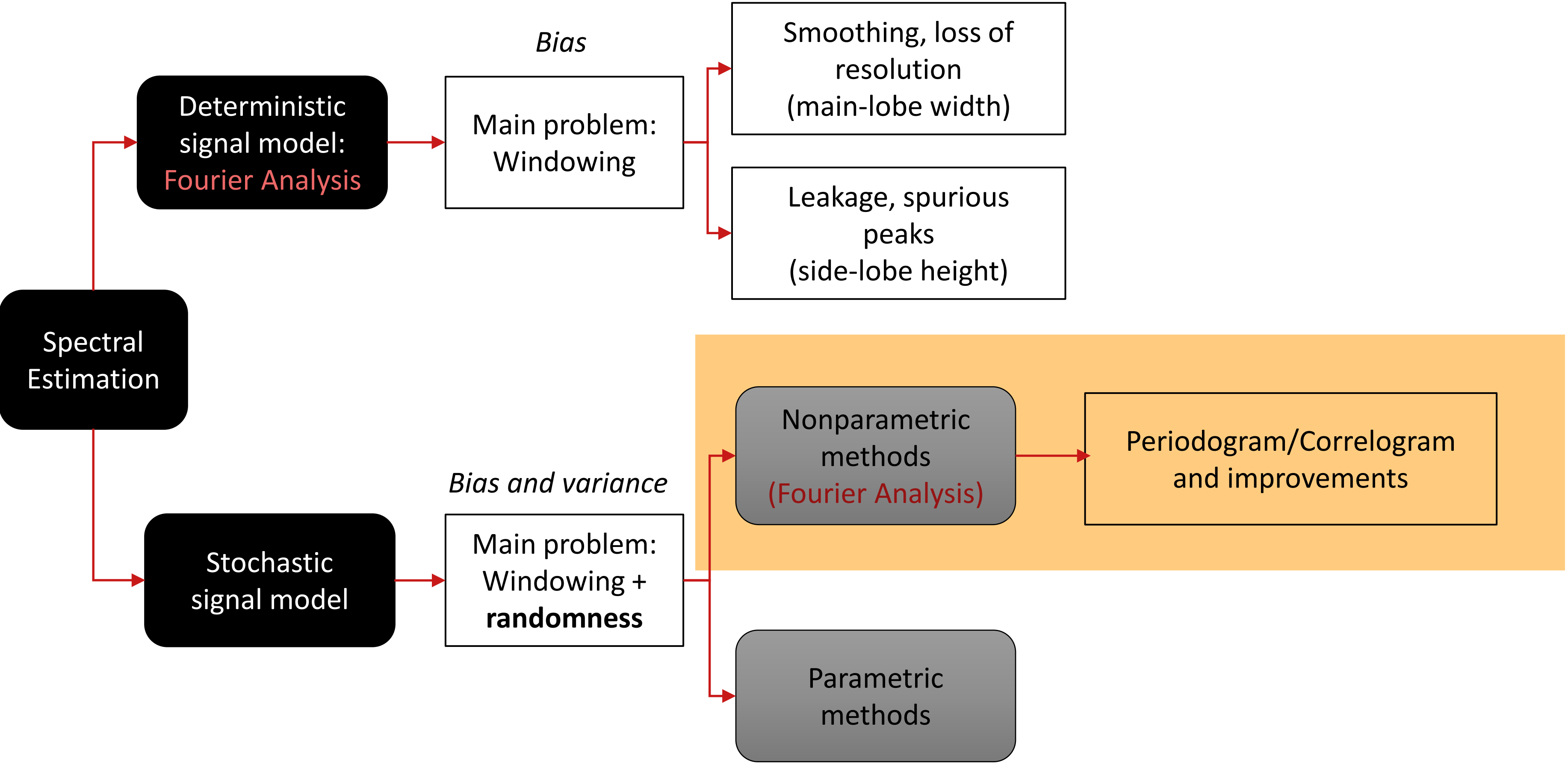
Nonparametric methods (Fourier Analysis)
Periodogram/Correlogram and improvements
Parametric methods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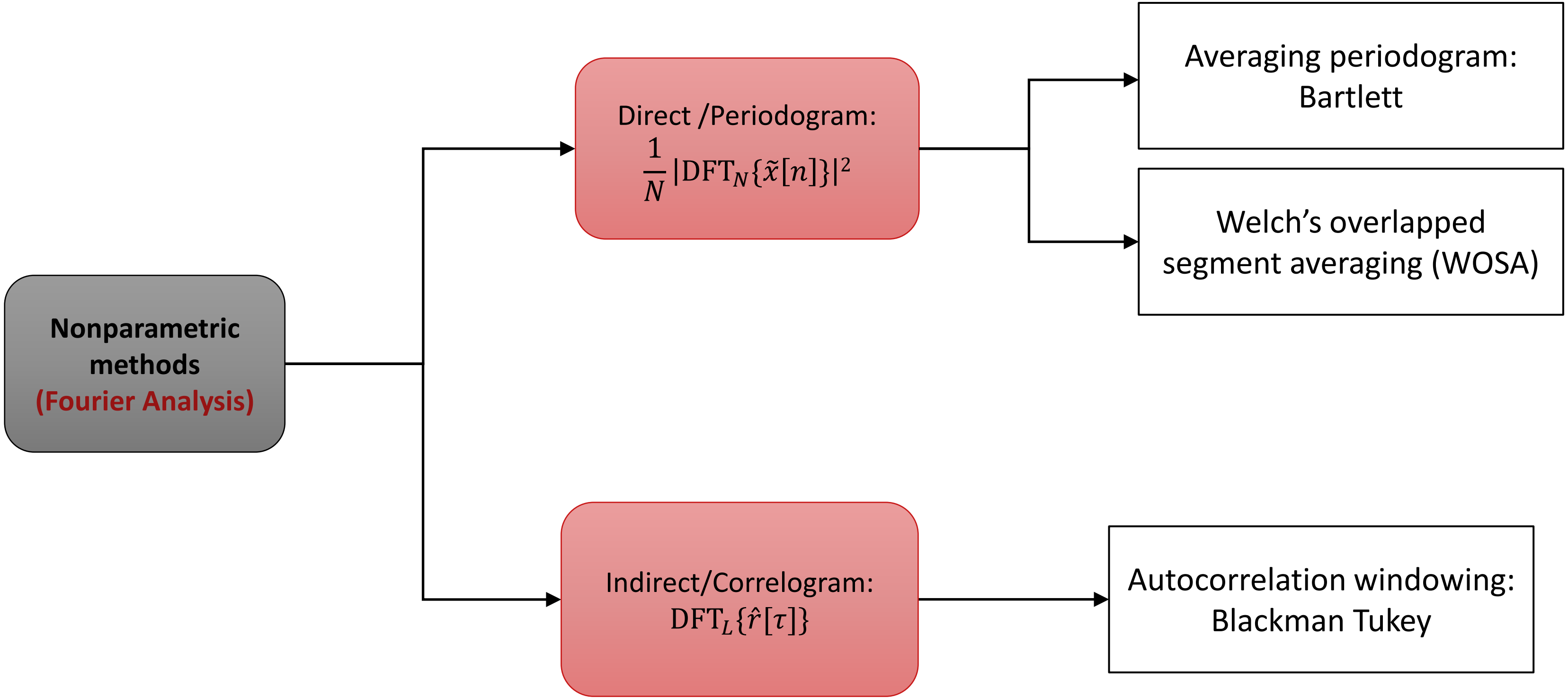
Non-parametric PSD estimators



Bias and variance of non-parametric PSD estimators

Non-parametric spectral estimation

Biased and unbiased AC estimators

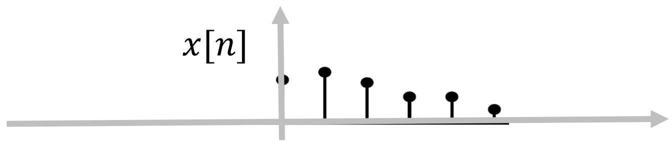
Unbiased

$$\hat{r}_{ub}[\tau] = \begin{cases} \frac{1}{N-|\tau|}\sum_{n=\tau}^{N-1} x[n]x^*[n-\tau] & 0 \le \tau \le N-1 \\ \hat{r}_{ub}[\tau] & -(N-1) \le \tau < 0 \\ 0 & else \end{cases}$$

Biased

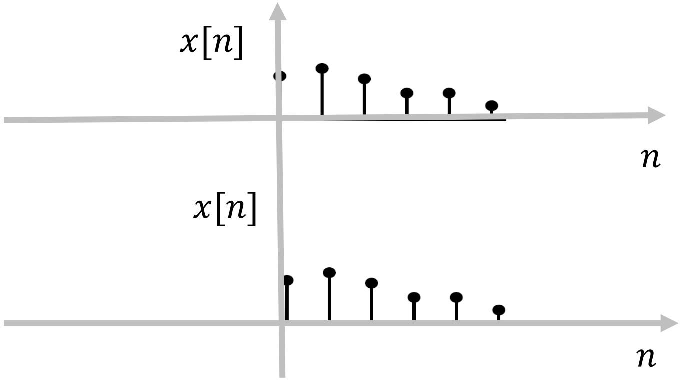
$$\hat{r}_{b}[\tau] = \begin{cases} \frac{1}{N}\sum_{n=\tau}^{N-1} x[n]x^*[n-\tau] & 0 \le \tau \le N-1 \\ \hat{r}_{b}[\tau] & -(N-1) \le \tau < 0 \\ 0 & else \end{cases}$$

Autocorrelation calculation



$$\sum_{n=\tau}^{N-1} x[n]x^*[n-\tau]$$

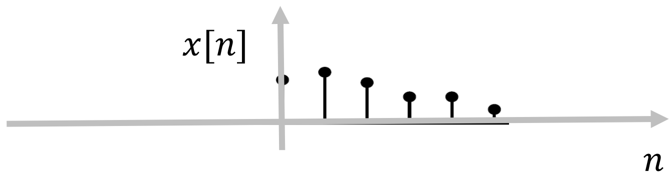
Autocorrelation calculation



$$\sum_{n=\tau}^{N-1} x[n]x^*[n-\tau]$$

$\tau = 0,$ average over N samples

Autocorrelation calculation



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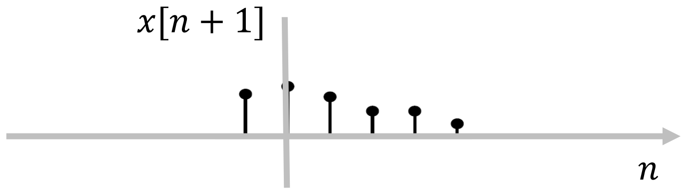
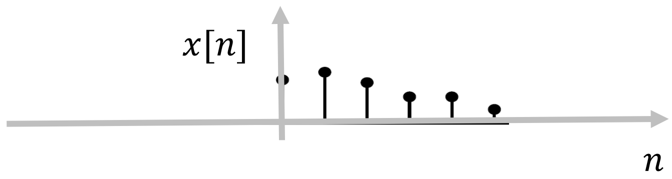


$\tau = 1,$ average over $N-1$ samples

Autocorrelation calculation



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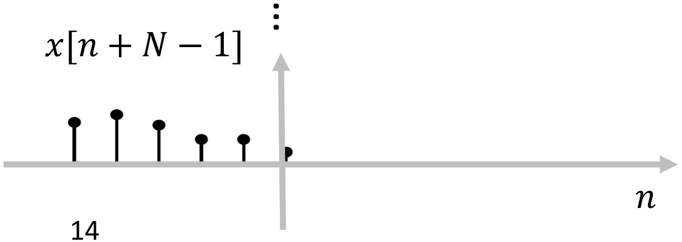
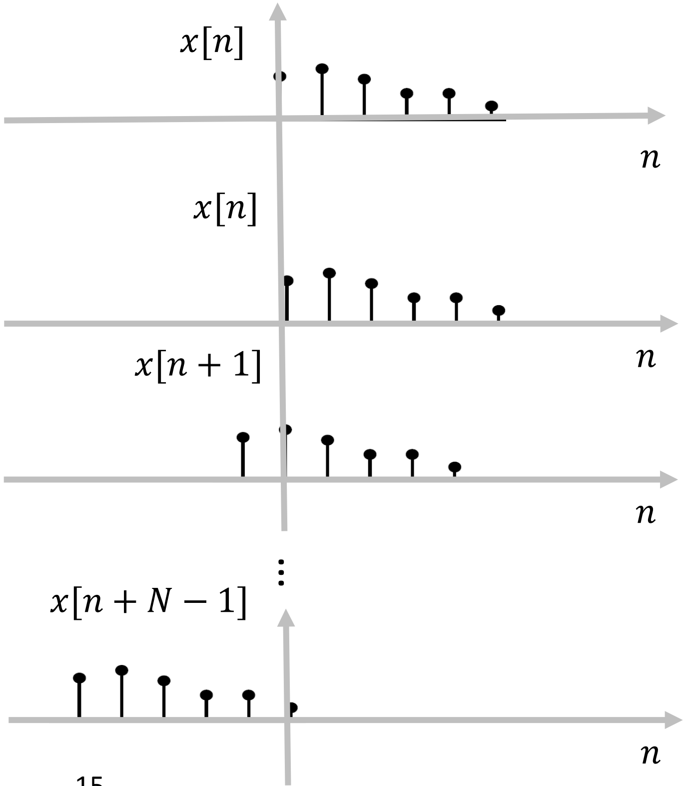


$\tau = N - 1,$ average over 1 sample

Autocorrelation calculation



$$\sum_{n=\tau}^{N-1} x[n]x^*[n-\tau]$$

$\tau = 0,$ average over N samples

$\tau = 1,$ average over $N - 1$ samples

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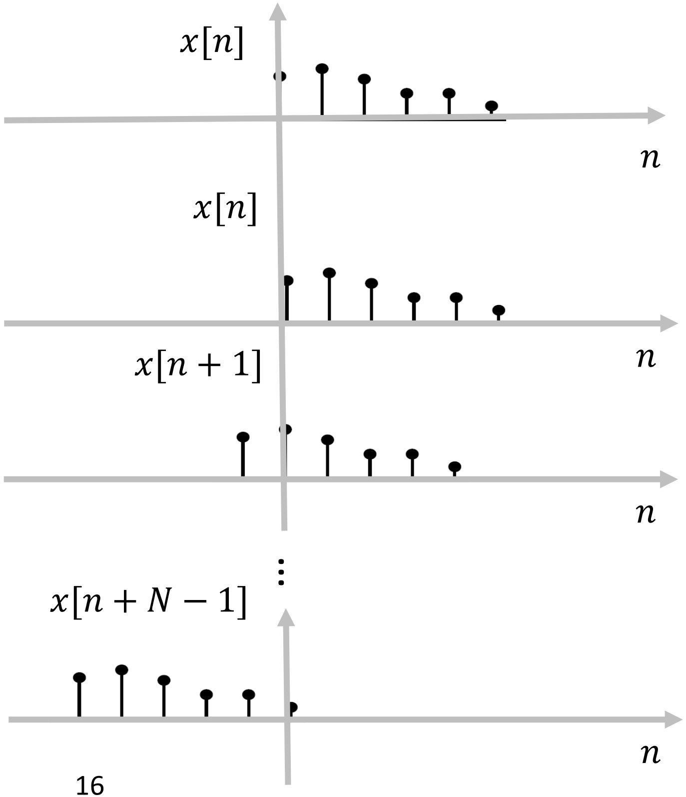
Autocorrelation calculation



Unbiased estimator:
For each τ, take average over $N - |\tau|$ samples

Unbiased estimators AC

$$E\{\hat{r}_{ub}[\tau]\} = E\left\{\frac{1}{N-|\tau|}\sum_{n=\tau}^{N-1} x[n]x^*[n-\tau]\right\} = \frac{1}{N-|\tau|}\sum_{n=\tau}^{N-1} E\left\{x[n]x^*[n-\tau]\right\}$$

$$E\{\hat{r}_{ub}[\tau]\} = \frac{1}{N-|\tau|}\sum_{n=\tau}^{N-1} r[\tau] = \cancel{\frac{1}{N-|\tau|}}\cancel{(N-|\tau|)}r[\tau] = r[\tau]$$

$$\boxed{E\{\hat{r}_{ub}[\tau]\} = r[\tau]}$$

Biased estimators AC

$$E\{\hat{r}_b[\tau]\} = E\left\{\frac{1}{N}\sum_{n=\tau}^{N-1} x[n]x^*[n-\tau]\right\} = \frac{1}{N}\sum_{n=\tau}^{N-1} E\left\{x[n]x^*[n-\tau]\right\}$$

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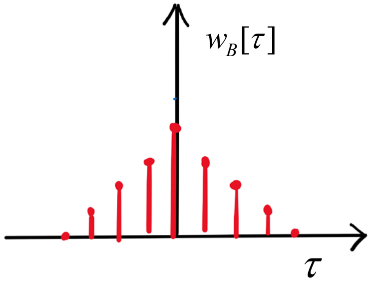
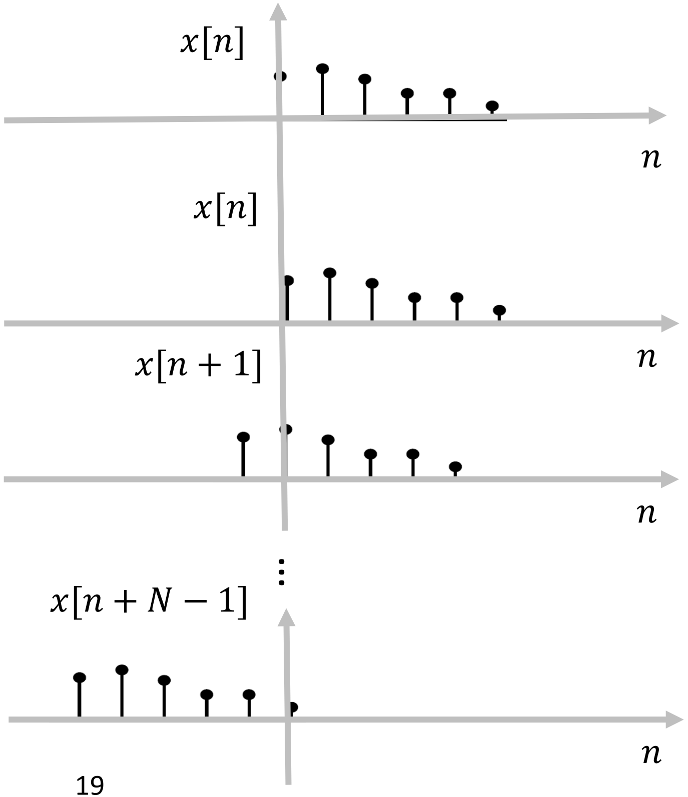


Triangular (Bartlett) window

$$w_B[n] = \begin{cases} \dfrac{N-|\tau|}{N} & |\tau| \leq N-1 \\ 0 & elsewhere \end{cases}$$

Autocorrelation calculation



Unbiased estimator:
For each τ, take average over $N - |\tau|$ samples

Variance:
For $\tau \approx N$, almost no averaging: large variance (unreliable estimate)

Rule of thumb: for signal of length N, calculate autocorrelation on L lags, with $L \sim \frac{N}{4}$

Biased and unbiased AC estimators

Unbiased

- Unbiased: $E\{\hat{r}_{ub}[\tau]\} = r[\tau]$

Biased

- Biased: $E\{\hat{r}_b[\tau]\} = r[\tau] \cdot r_w\,[\tau] \neq r[\tau]$
    - Asymptotically unbiased

Biased and unbiased AC estimators

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Biased and unbiased AC estimators

Unbiased

- Unbiased: $E\{\hat{r}_{ub}[\tau]\} = r[\tau]$
- Consistent: $Var\{\hat{r}_{ub}[\tau]\} \to 0, N \to \infty$
- For $\tau \approx N$, almost no averaging: large variance (unreliable estimate)
- Non-negativity not fulfilled: PSD can become negative

Biased

- Biased: $E\{\hat{r}_b[\tau]\} = r[\tau] \cdot r_w\,[\tau] \neq r[\tau]$
  - Asymptotically unbiased
- Consistent: $Var\{\hat{r}_b[\tau]\} \to 0, N \to \infty$
- For $\tau \approx N$, almost no averaging: large variance (unreliable estimate)
- **PSD non-negative fulfilled**

Periodogram/Correlogram: Bias

$$E\{P_{ind}(e^{j\theta})\} = E\left\{\sum_{\tau=-(N-1)}^{N-1} \hat{r}_b[\tau]e^{-j\tau\theta}\right\}$$

$$E\{\hat{r}_b[\tau]\} = \frac{N-|\tau|}{N}r[\tau] = r_w[\tau]r[\tau]$$

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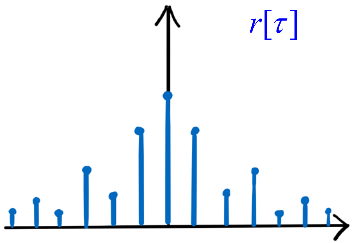
- The expected value of the periodogram can be interpreted as the convolution in the frequency domain of the true spectrum with the Fourier transform of the correlation window

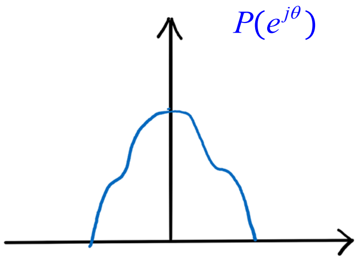
  -> **smoothed periodogram**

"Correlation" domain



$\Leftrightarrow$

Frequency domain



“Correlation” domain

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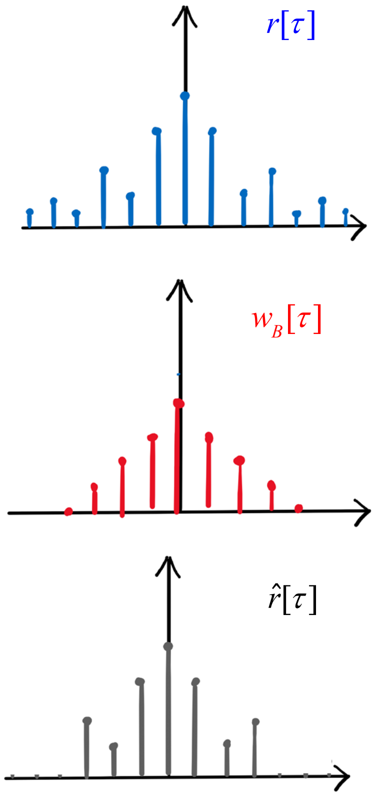

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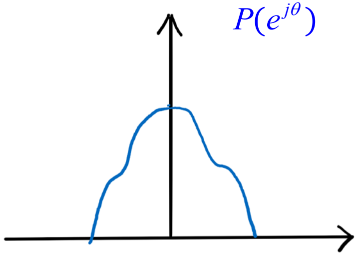
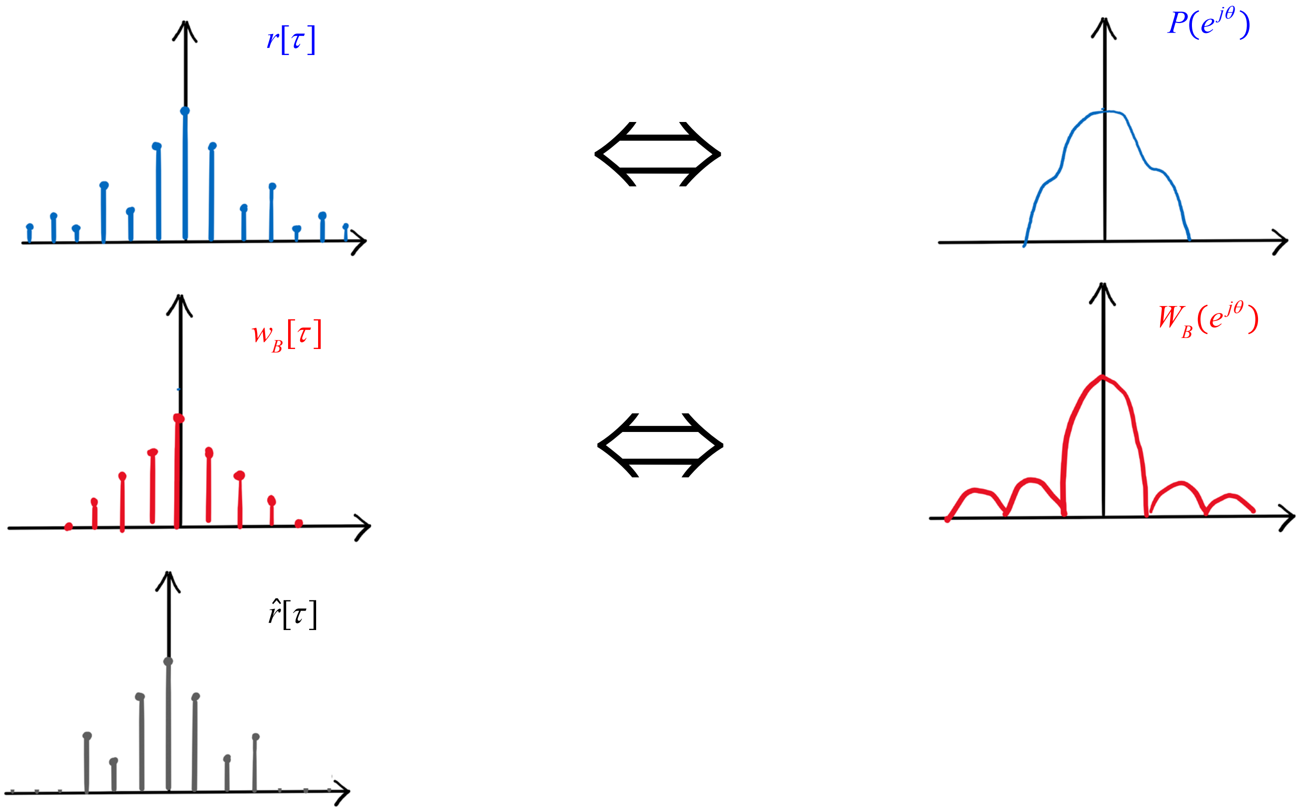
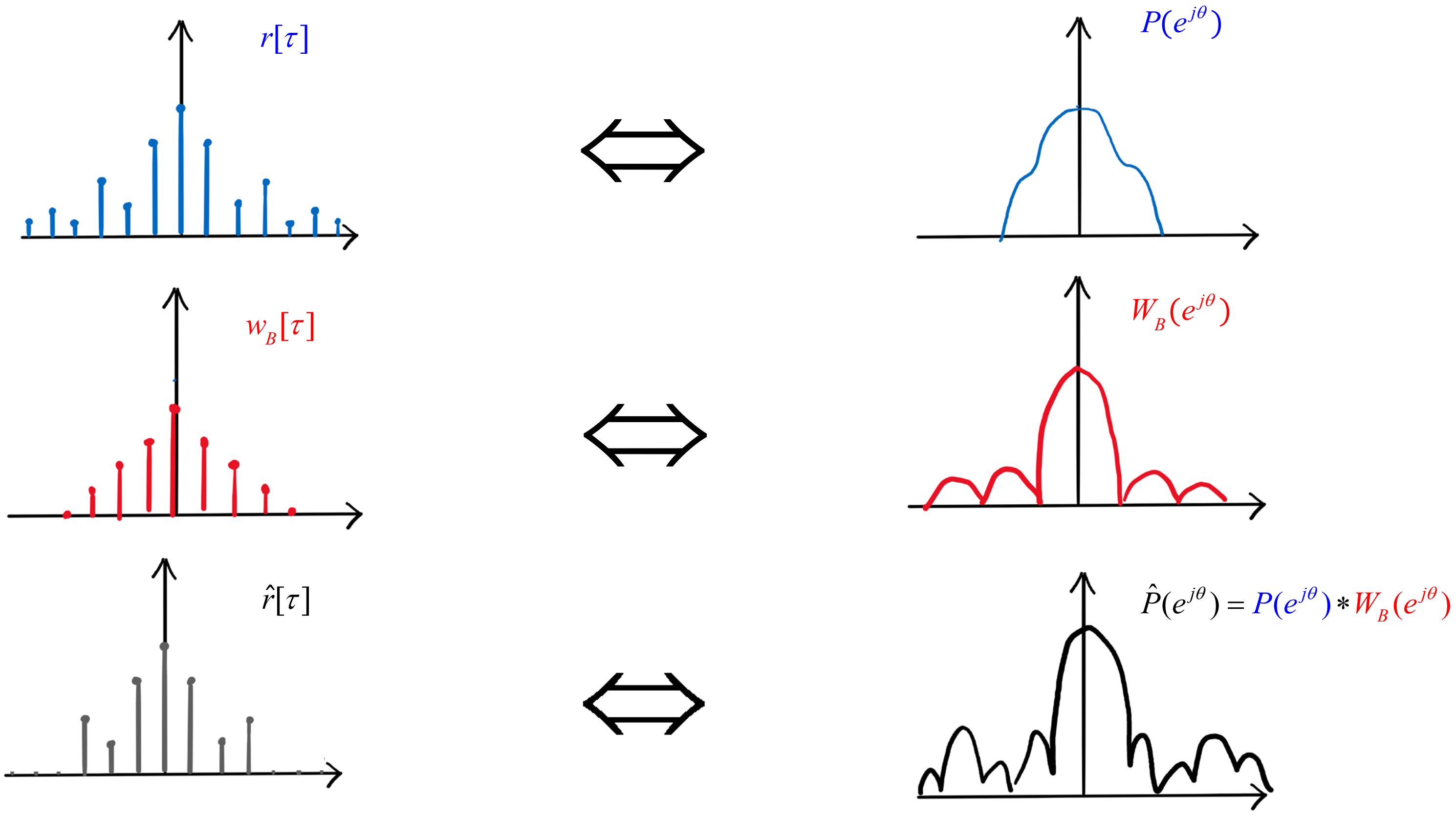


Correlation domain **Frequency domain**



“Correlation” domain **Frequency domain**



Periodogram/Correlogram: Bias

$$E\{P_{ind}(e^{j\theta})\} = E\left\{\sum_{\tau=-(N-1)}^{N-1} \hat{r}_{ub}[\tau]e^{-j\tau\theta}\right\}$$

$$E\{\hat{r}_{ub}[\tau]\} = r[\tau]$$

$$E\{P(e^{j\theta})\} = \mathrm{E}\left\{\sum_{\tau=-\infty}^{\infty} (r[\tau]w_R[\tau])e^{-j\tau\theta}\right\}$$

- $w_R[\tau]$, rectangular window defined as

$$w_R[\tau] = \begin{cases} 1 & |\tau| \le N-1 \\ 0 & elsewhere \end{cases}$$

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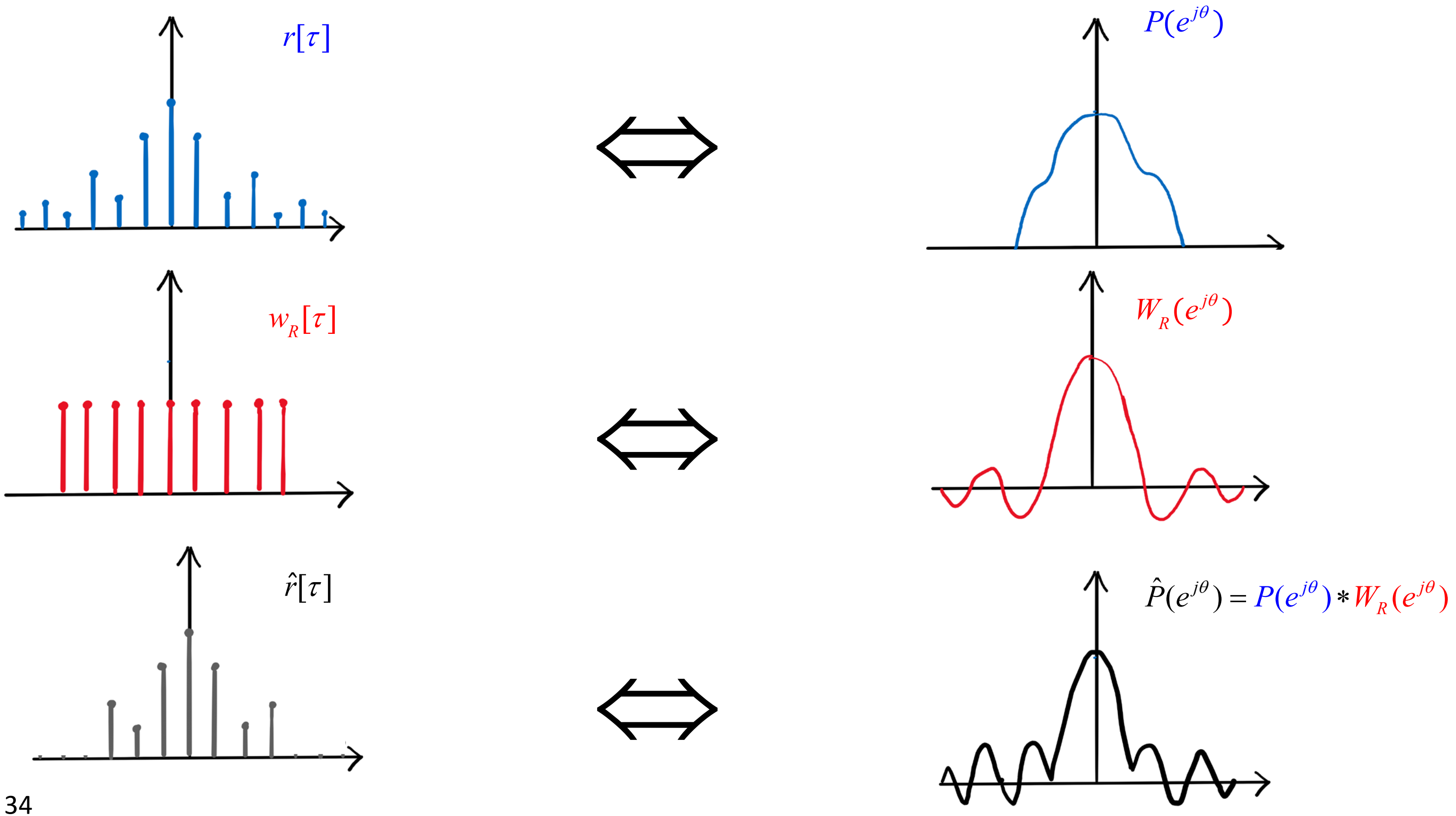
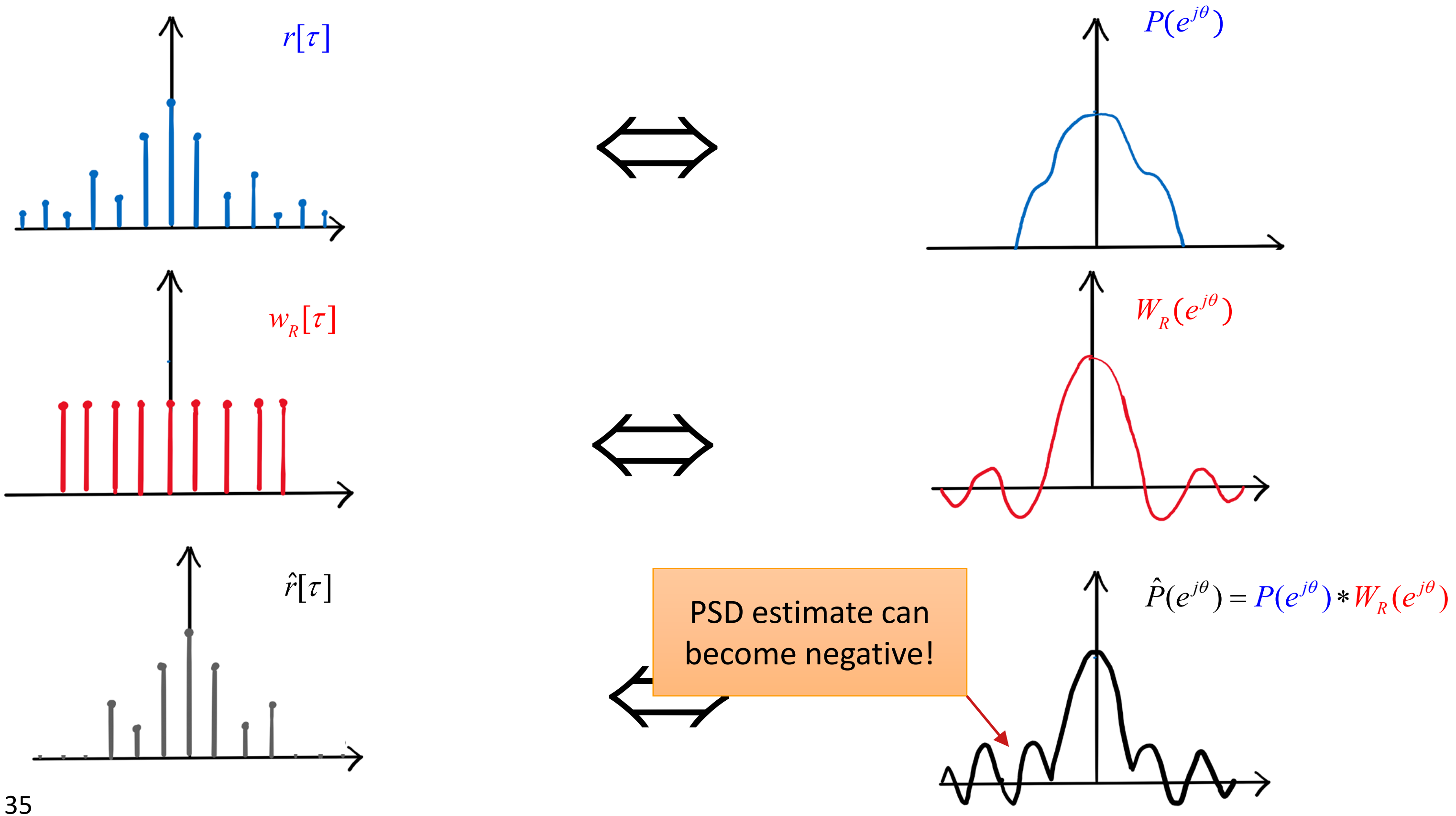
$$w_R[\tau] = \begin{cases} 1 & |\tau| \leq N-1 \\ 0 & elsewhere \end{cases} \quad \xleftrightarrow{\mathcal{F}} \quad W_R(e^{j\theta}) = \left(\frac{\sin(N\theta/2)}{\sin(\theta/2)}\right)e^{-j\theta(N-1)/2}$$

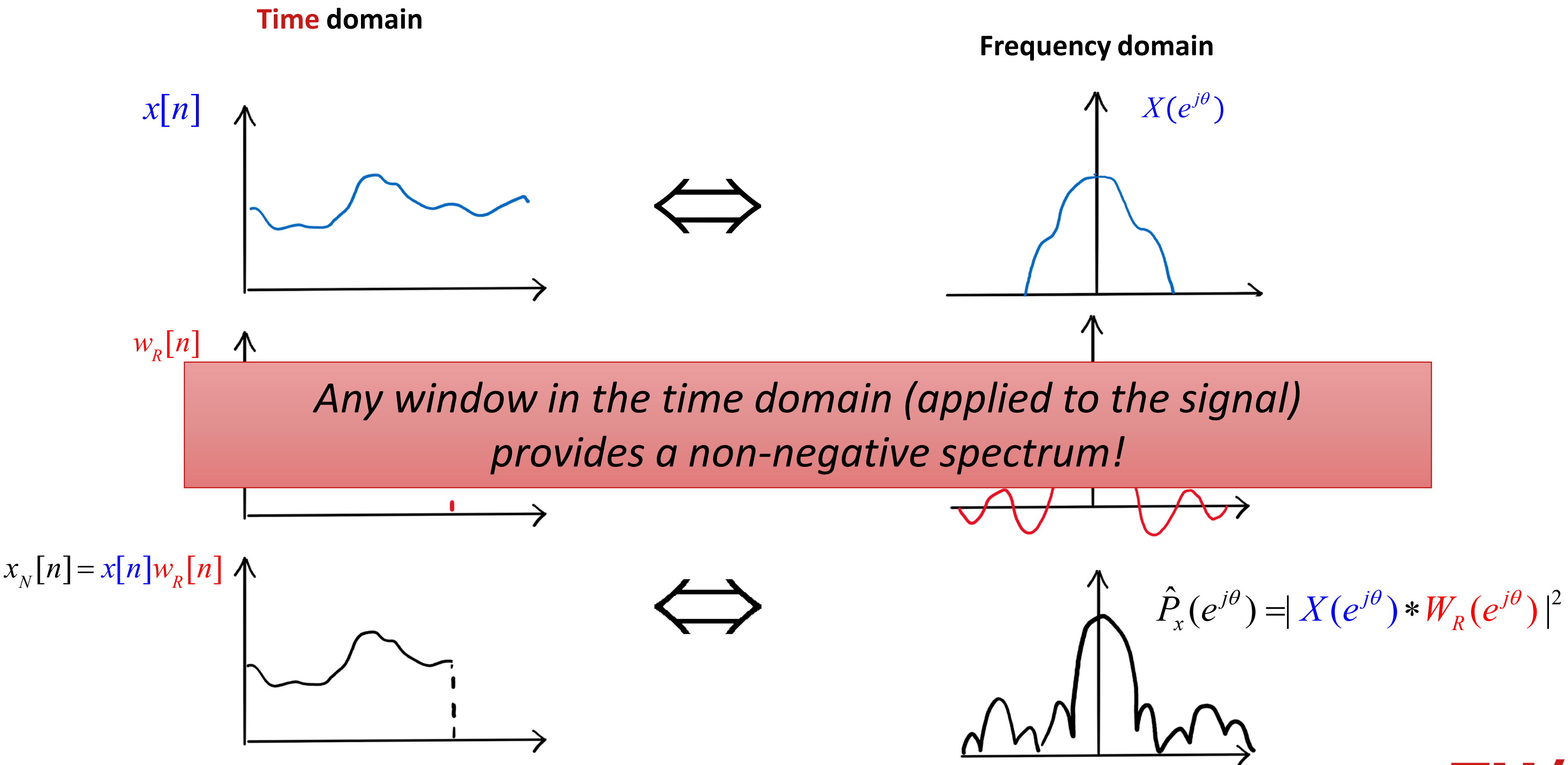
“Correlation” domain

Frequency domain



“Correlation” domain | **Frequency domain**





Any window in the time domain (applied to the signal) provides a non-negative spectrum!

Periodogram/Correlogram: Bias

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- The periodogram is the Fourier transform of the **biased estimated** of $r[\tau]$
- The periodogram is <u>asymptotically unbiased</u>

Periodogram/correlogram: Variance

- Approximate expression for AR(1) process $\quad \mathrm{var}\{P(e^{j\theta})\} = \sigma^4{}_i \left\{ 1 + \left(\frac{1}{N} \frac{\sin(\theta N)}{\sin\theta} \right)^2 \right\}$

$$N \to \infty \qquad \mathrm{var}\{P(e^{j\theta})\} \to \sigma^4{}_i$$

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- For white Gaussian sequence input to LTI $\quad \mathrm{var}\{P(e^{j\theta})\} = P^2(e^{j\theta})\left\{1 + \left(\frac{1}{N}\frac{\sin(\theta N)}{\sin\theta}\right)^2\right\}$

$$N \to \infty \qquad \mathrm{var}\{P(e^{j\theta})\} \to P^2(e^{j\theta})$$

Periodogram/correlogram: Variance

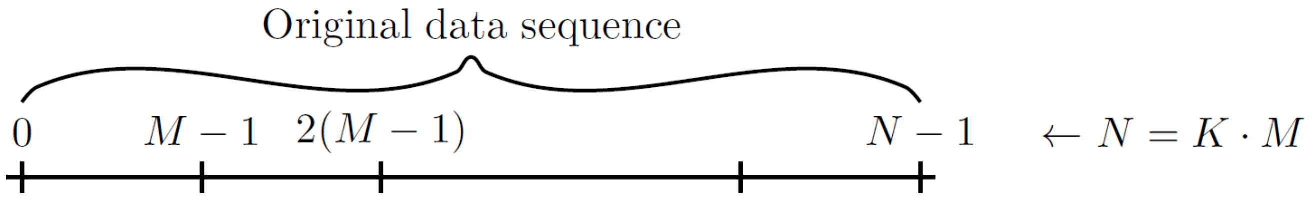
- Approximate expression for AR(1) process $$\operatorname{var}\{P(e^{j\theta})\} = \sigma^4{}_i \left\{1+\left(\frac{1}{N}\frac{\sin(\theta N)}{\sin\theta}\right)^2\right\}$$
- For white Gaussian sequence input to LTI $$\operatorname{var}\{P(e^{j\theta})\} = P_x^2(e^{j\theta})\left\{1+\left(\frac{1}{N}\frac{\sin(\theta N)}{\sin\theta}\right)^2\right\}$$

- The periodogram is not a consistent estimator, i.e., the **variance does not decrease** as $N \to \infty$
- The variance of the periodogram is **proportional to the square of the true spectrum**

Periodogram improvements

Non-parametric spectral estimation

Averaged periodogram: Bartlett's method



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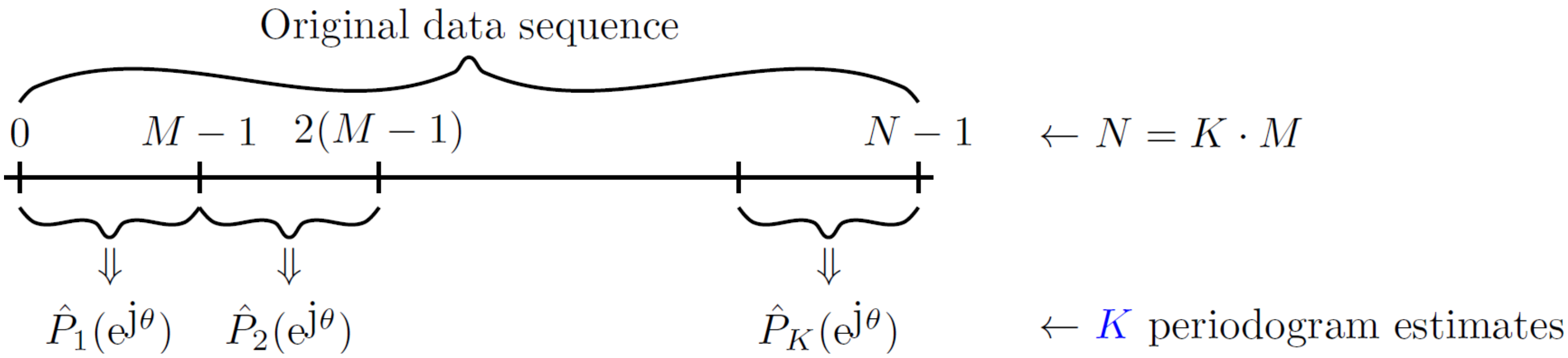

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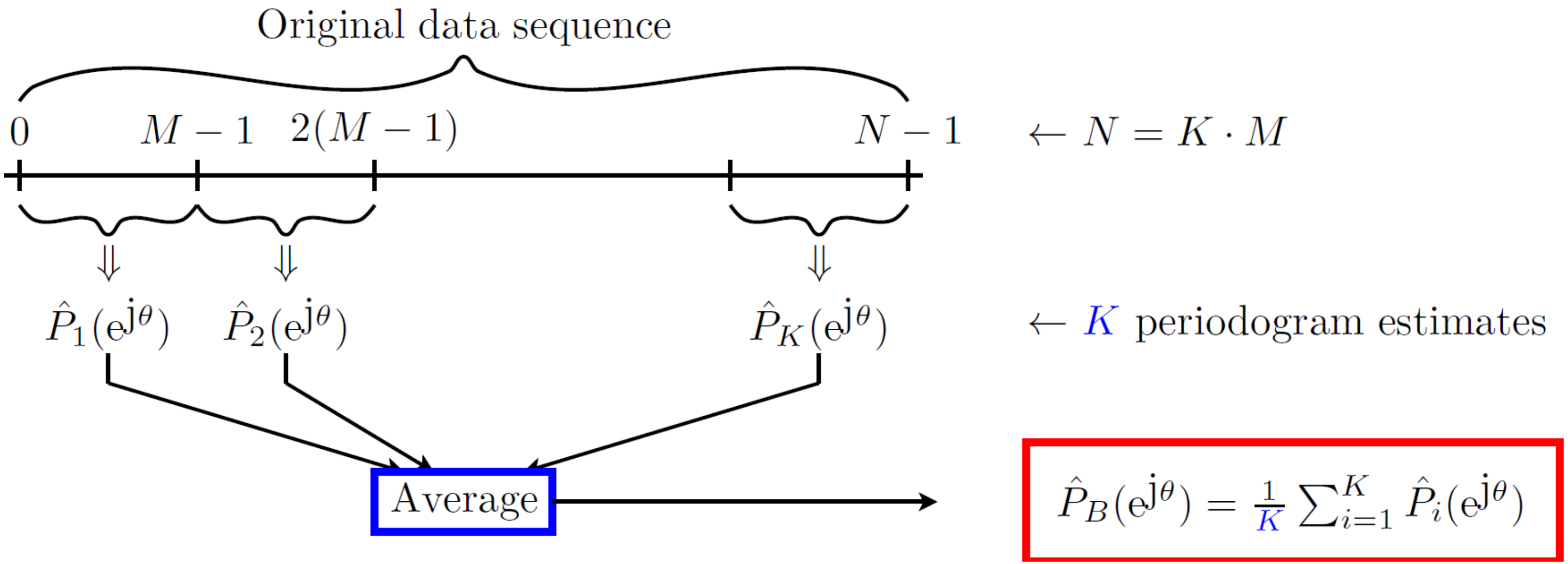
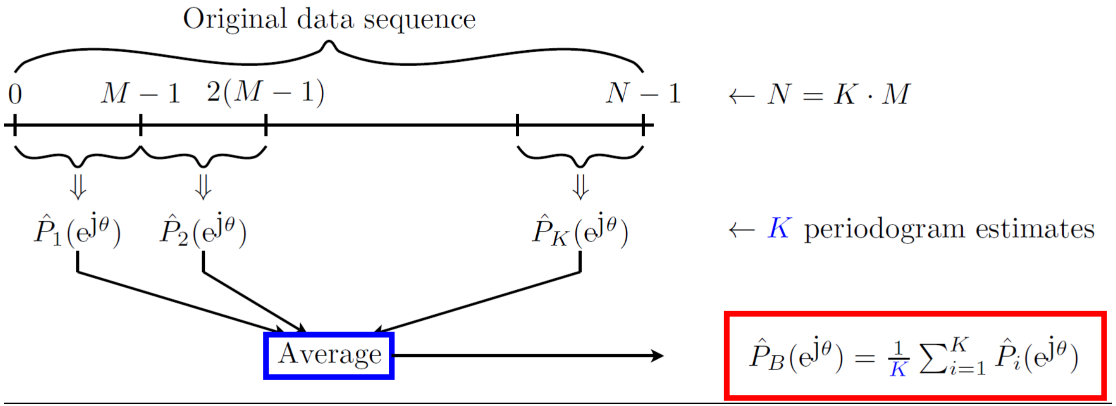


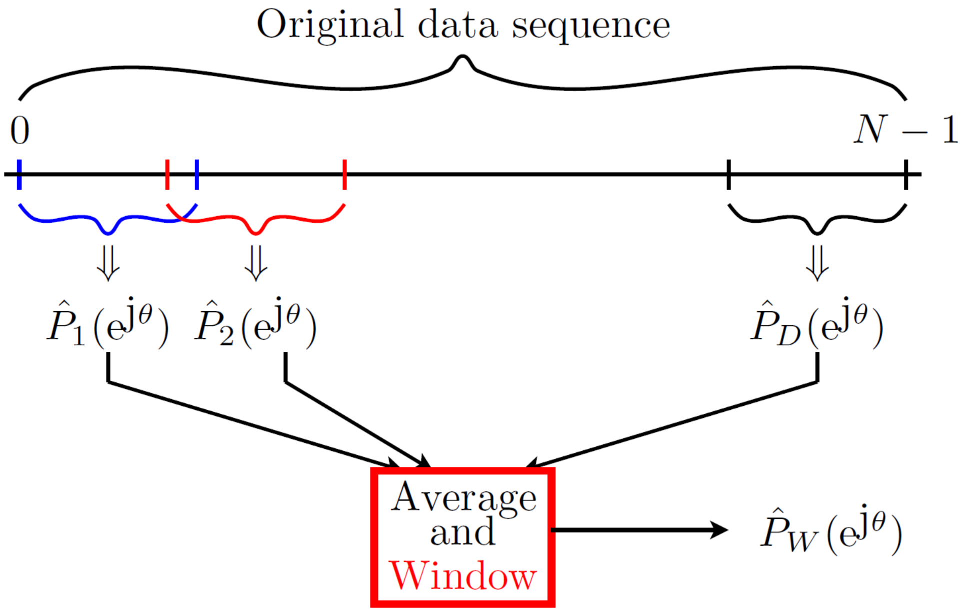
Averaged periodogram: bias and variance



- Bias: $E\{P_B(e^{j\theta})\} = \frac{M-|\tau|}{M} \sum_{\tau=-(M-1)}^{M-1} r[\tau] e^{-j\tau\theta}$ Increased!
- Variance: $\mathrm{Var}\{P_B(e^{j\theta})\} \approx \frac{1}{K} \mathrm{var}\{P_{dir}(e^{j\theta})\}$ Decreased!

TU/e

WOSA: Welch's overlapped segment averaging



- Allowing overlap, longer segments
- Even when signal is white, different **raw periodograms are not independent**: variance will not reduce as $1/K$
- Typical overlaps: 50% and 75%
- Matlab function `pwelch`

Correlogram improvements

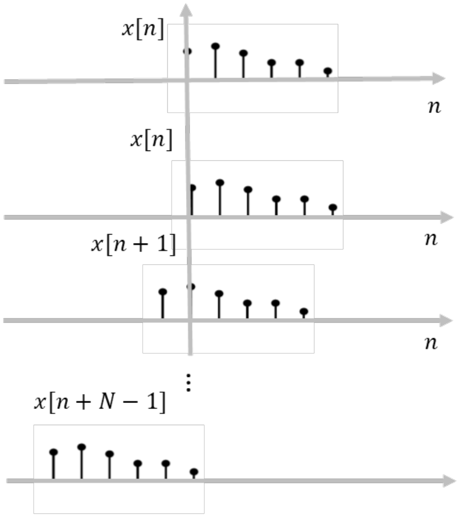
Non-parametric spectral estimation

Correlogram: Blackman-Tukey method

- Idea: apply window to the raw autocorrelation estimate prior to transforming

$$P_{BT}(e^{j\theta}) = \sum_{\tau=-(L-1)}^{L-1} w_L[\tau]\hat{r}_b[\tau]e^{-j\tau\theta}$$

$w_L[\tau]$ symmetric, length $2L-1$, with typically $L \ll N$



$\tau = 0,$ average over N samples

$\tau = 1,$ average over $N-1$ samples

$\tau = N-1,$ average over 1 sample

Correlogram: Blackman-Tukey method

- Idea: apply window to the raw correlation estimate prior to transforming

$$P_{BT}(e^{j\theta}) = \sum_{\tau=-(L-1)}^{L-1} w_L[\tau]\hat{r}_b[\tau]e^{-j\tau\theta}$$

$w_L[\tau]$ symmetric, length $2L-1$, with typically $L \ll N$

- For $w_L[\tau]$ triangular and $L = N$ the BT correlogram is equivalent to the raw correlogram

- The Blackman-Tukey estimator can be interpreted as the convolution of the true spectrum by the transform of the window function

$$P_{BT}(e^{j\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{j\theta})W_L(e^{j(\theta-\phi)})d\phi$$

Correlogram: valid PSD

- Not all windows can be used, e.g., Rectangular, Hamming and Hann not allowed since FTD can become negative for some values of θ.

- Sufficient conditions for $P_{BT}(e^{j\theta})$ to be non-negative

$$W_L(e^{j\theta}) \geq 0; \; -\pi < \theta \leq \pi \qquad \text{or} \qquad \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} a_i w_L[i-j] a_j \geq 0$$

Blackman-Tuckey: Bias and Variance

- Bias: $$E\{P_{BT}(e^{j\theta})\} \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{j\theta})W_L(e^{j(\theta-\phi)})d\phi \qquad \lim_{N\to\infty} W_L(e^{j\theta}) = A\delta(\theta)$$

Blackman-Tuckey: Bias and Variance

- Bias: $E\{P_{BT}(e^{j\theta})\} \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{j\theta}) W_L(e^{j(\theta-\phi)}) d\phi$ $\qquad \lim_{N\to\infty} W_L(e^{j\theta}) = A\delta(\theta)$

$P_{BT}(e^{j\theta})$ is **asymptotically unbiased** if $A = 1$. This occurs if

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} W_L(e^{j\theta}) d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} A\delta(\theta) d\theta = 1 \Rightarrow w_L[0] = 1$$

Blackman-Tuckey: Bias and Variance

- Variance: $\mathrm{var}\{P_{BT}(e^{j\theta})\} \approx \dfrac{P^2(e^{j\theta})}{N}\left(\sum_{\tau=-(L-1)}^{L-1} w_L^2[\tau]\right)$ (for N sufficiently large)

$P_{BT}(e^{j\theta})$ is **consistent**: for $N \to \infty$, $\mathrm{var}\{P_{BT}(e^{j\theta})\} \to 0$:

Example

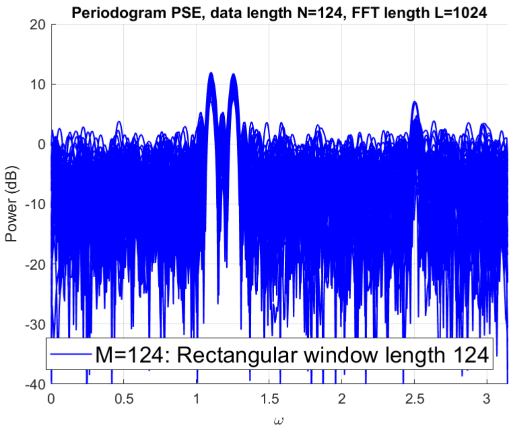
Random process modeled by

$$x[n] = \cos(0.35\pi n) + \cos(0.4\pi n) + 0.25\cos(0.8\pi n) + g[n] \qquad n = 0,1,\ldots,N-1$$

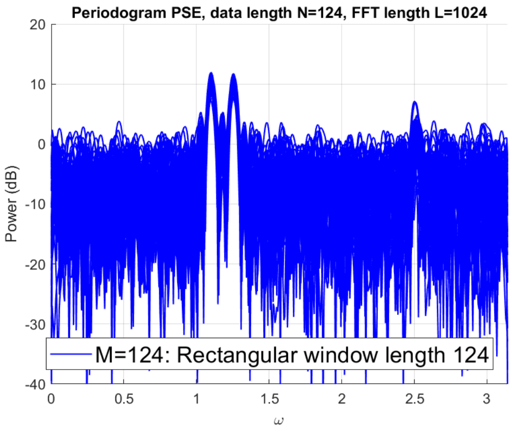
- $g[n]$ WGN, zero-mean, unit variance
- 50 realizations of the random process
- Simulations with varying N

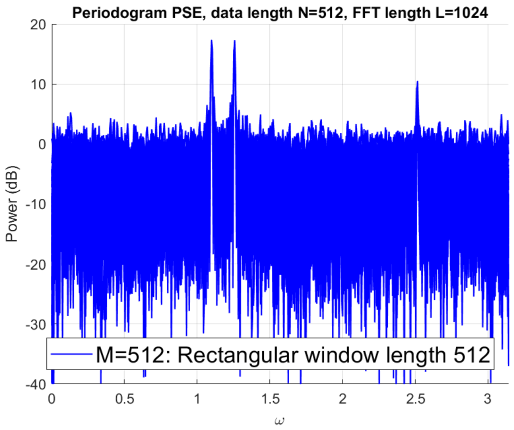
See MATLAB function `periodogram`

“Raw” periodogram



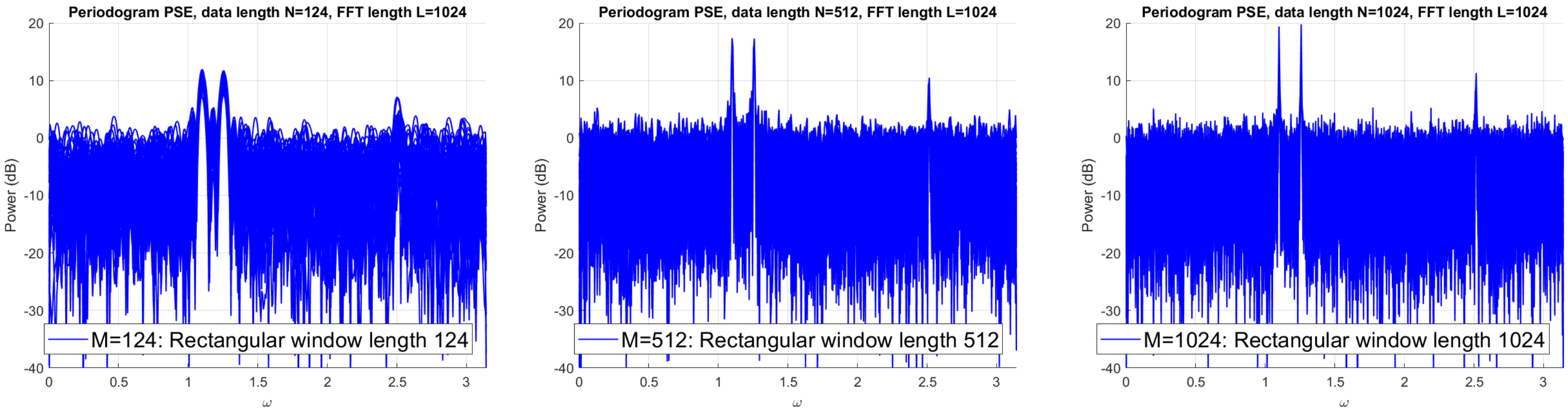
"Raw" periodogram





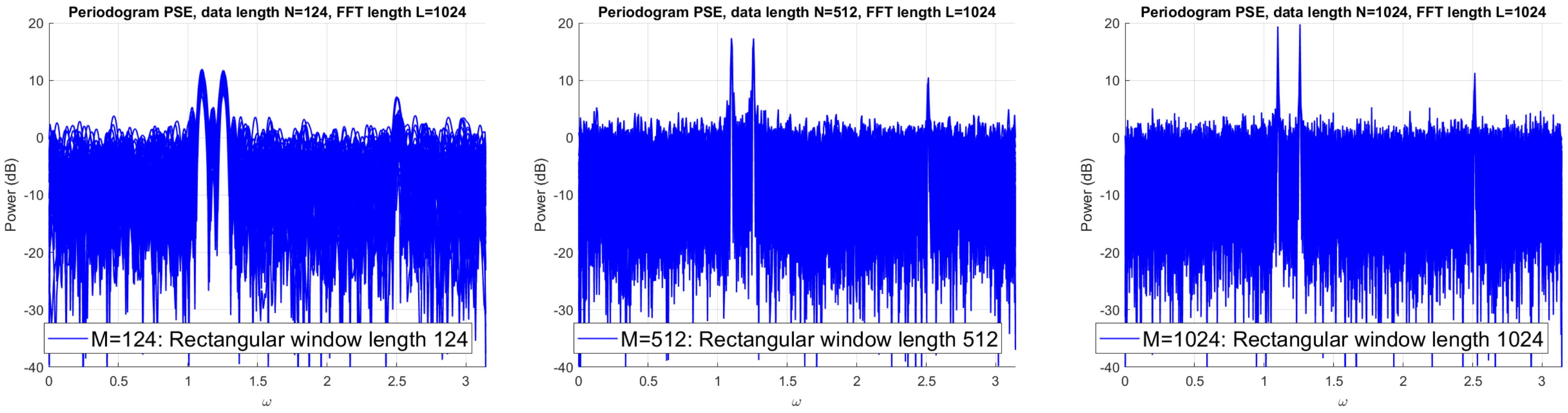
See MATLAB function `periodogram`

“Raw” periodogram



See MATLAB function `periodogram`

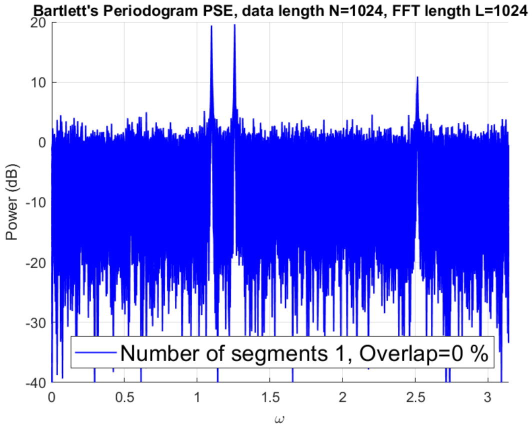
"Raw" periodogram



- Resolution improves with N
- Variance does not decrease with N
  - The periodogram is not a consistent estimator

See MATLAB function `pwelch`

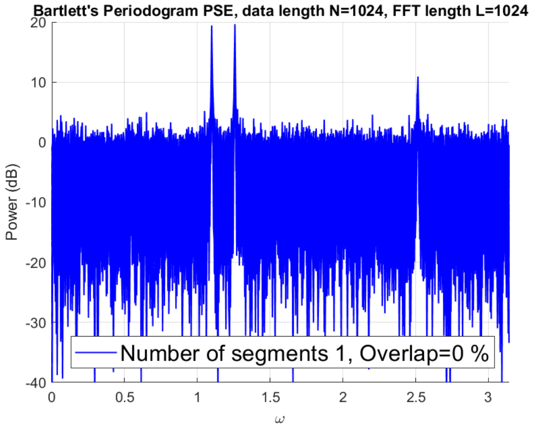
Bartlett's periodogram

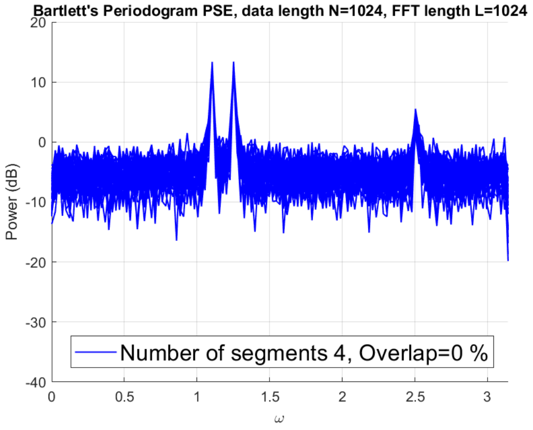


- $K = 1$ equivalent to raw periodogram
- Equivalent to WOSA for 0% overlap

See MATLAB function `pwelch`

Bartlett's periodogram





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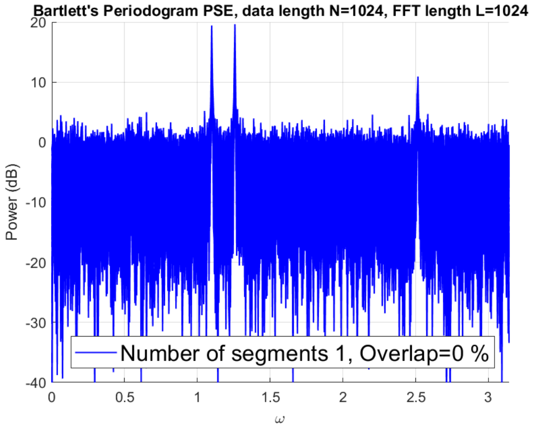
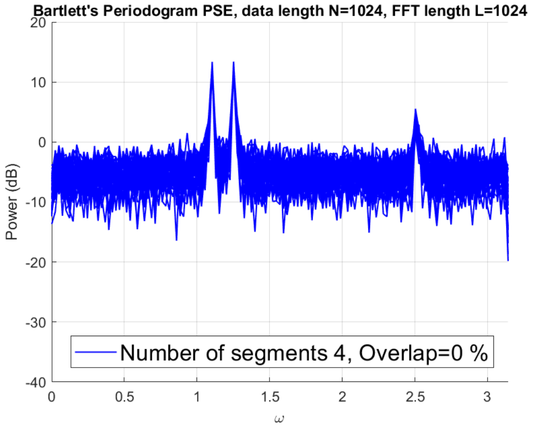
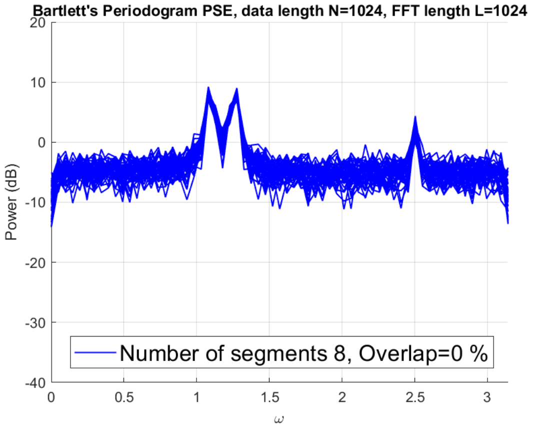
Bartlett's periodogram







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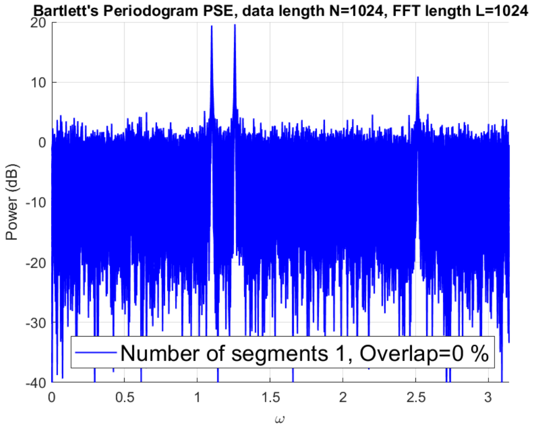
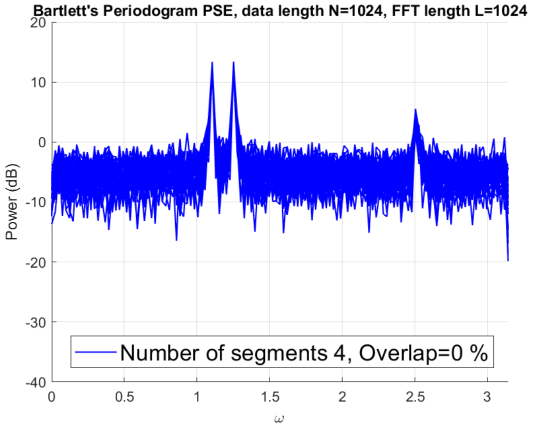
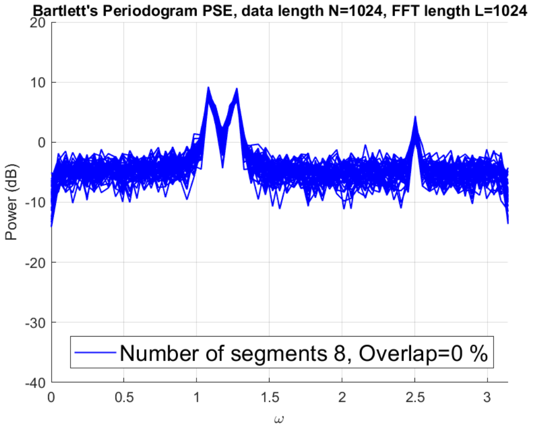
See MATLAB function `pwelch`

Bartlett's periodogram



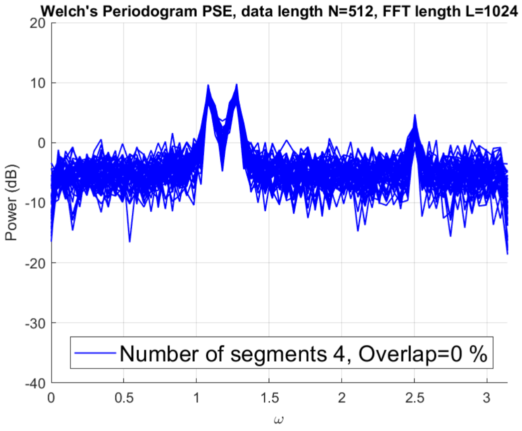




- $K = 1$ equivalent to raw periodogram
- Equivalent to WOSA for 0% overlap
- Resolution (bias) deteriorates with K
- Variance decreases with K

See MATLAB function `pwelch`

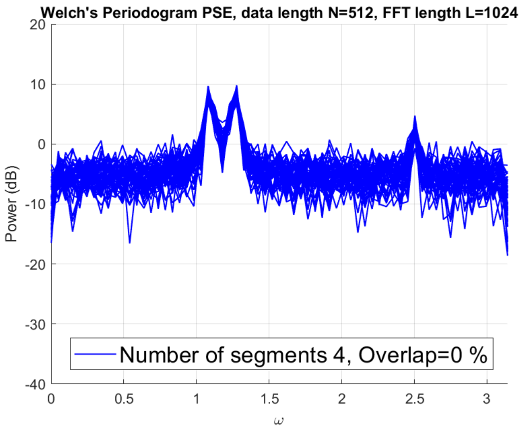
WOSA periodogram

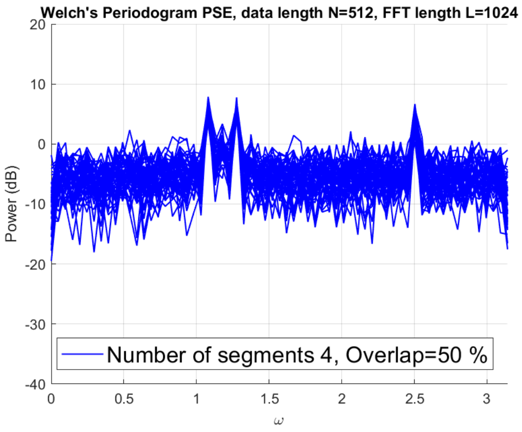


- Equivalent to Bartlett for 0% overlap

See MATLAB function `pwelch`

WOSA periodogram

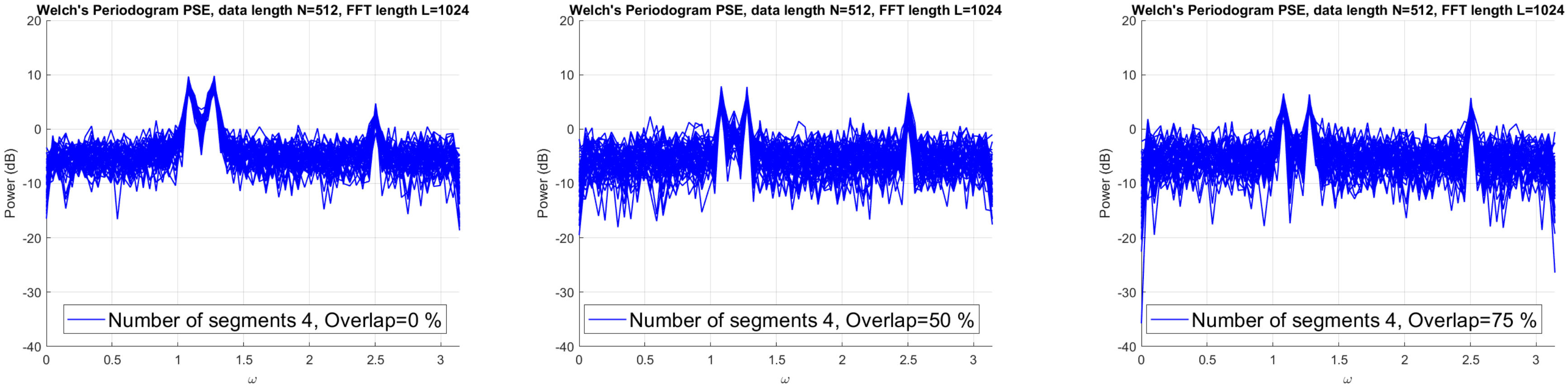




- Equivalent to Bartlett for 0% overlap

See MATLAB function `pwelch`

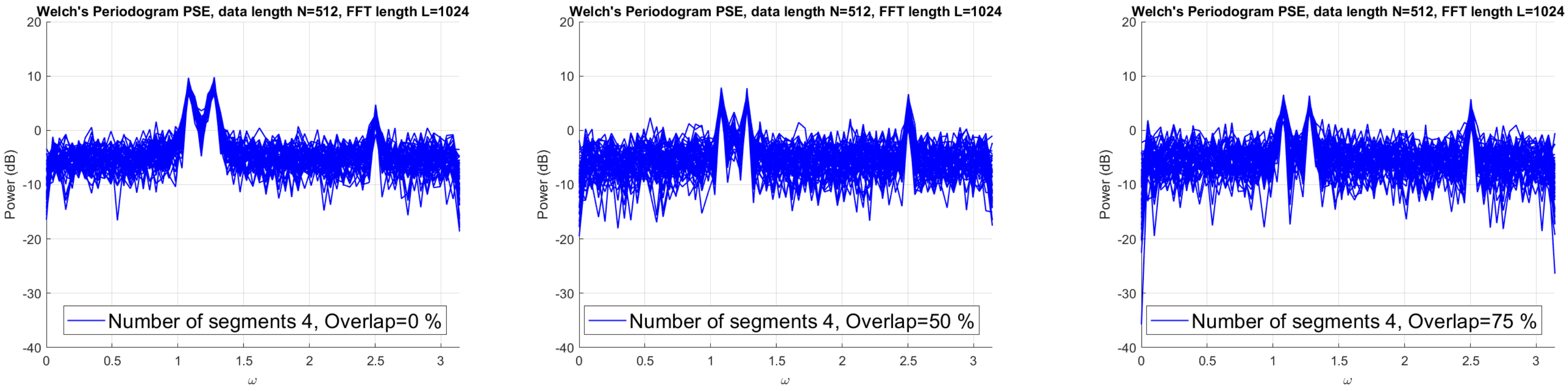
WOSA periodogram



- Equivalent to Bartlett for 0% overlap

See MATLAB function `pwelch`

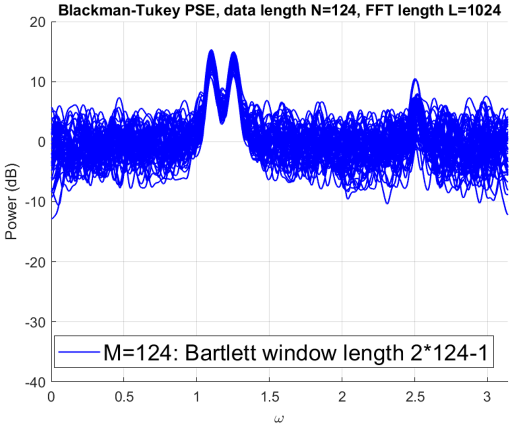
WOSA periodogram



- Equivalent to Bartlett for 0% overlap
- Resolution (bias) improves with overlap
- Variance does not decrease as $1/K$

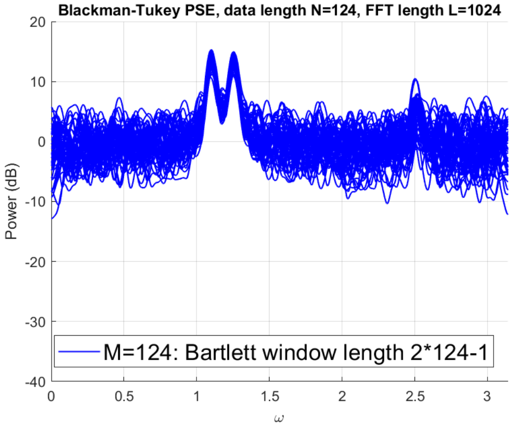
***Correlation window** length 2*M -1 fixed*
***Data length** N varying*

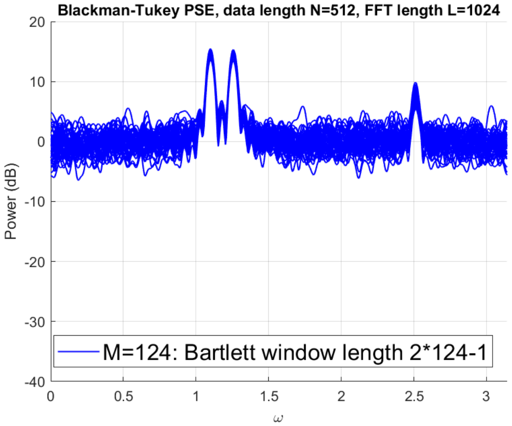
Blackman-Tuckey correlogram



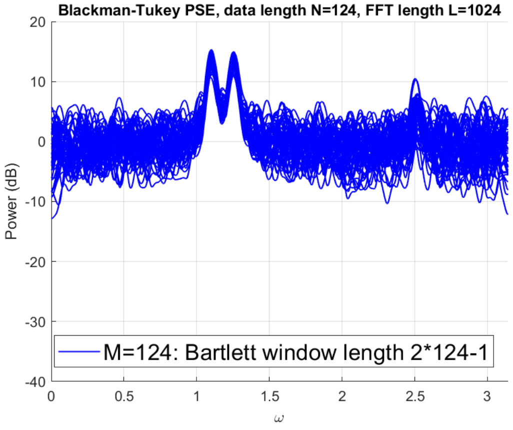
Blackman-Tuckey correlogram

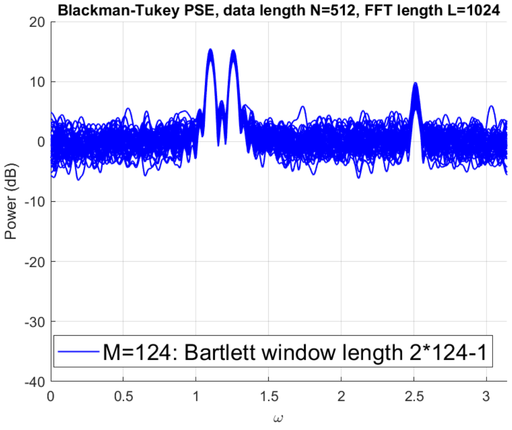
***Correlation window** length 2*M -1 fixed*
***Data length** N varying*

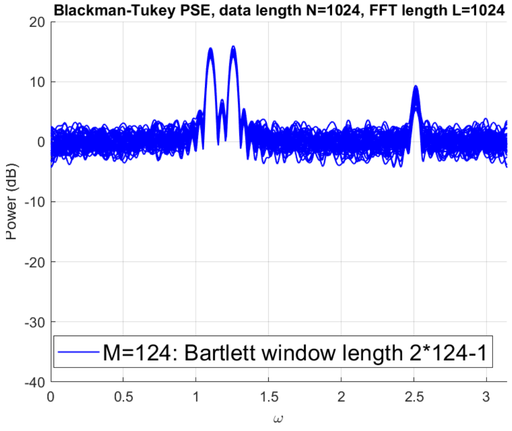




Blackman-Tuckey correlogram



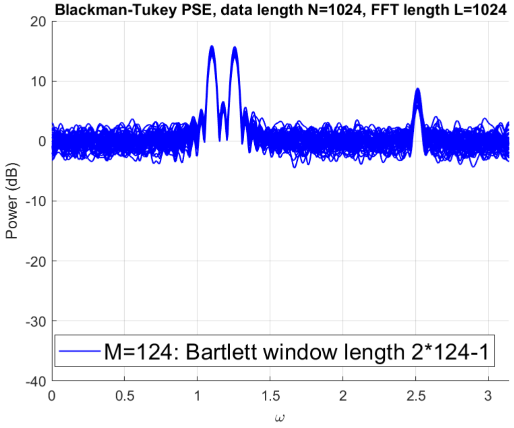




- For fixed correlation window length, increasing N
  - Resolution ~ unchanged
  - Variance decreases

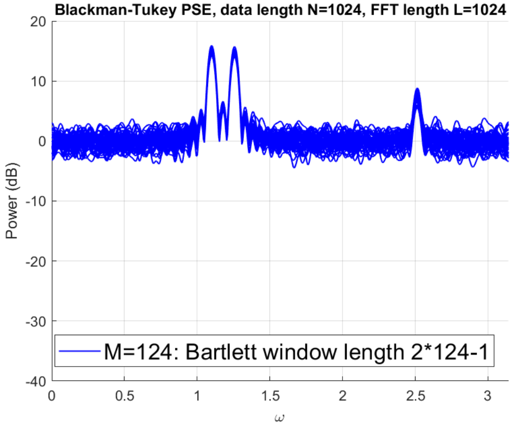
*__Correlation window__ length 2*M -1 varying*
__Data length__ N fixed

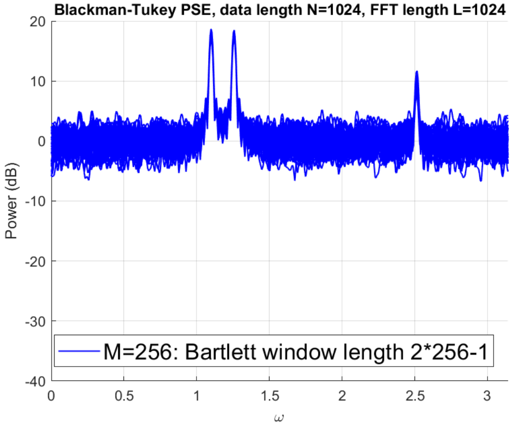
Blackman-Tuckey correlogram



***Correlation window** length 2*M -1 varying*
***Data length** N fixed*

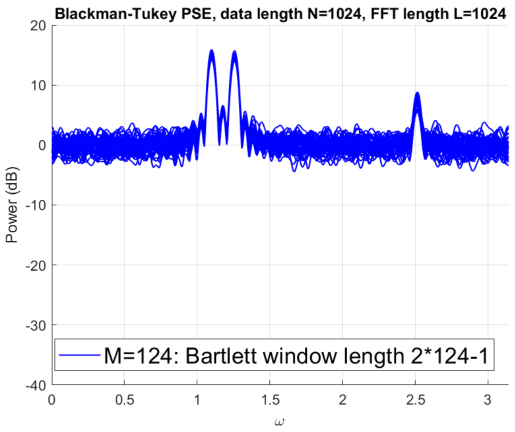
Blackman-Tuckey correlogram

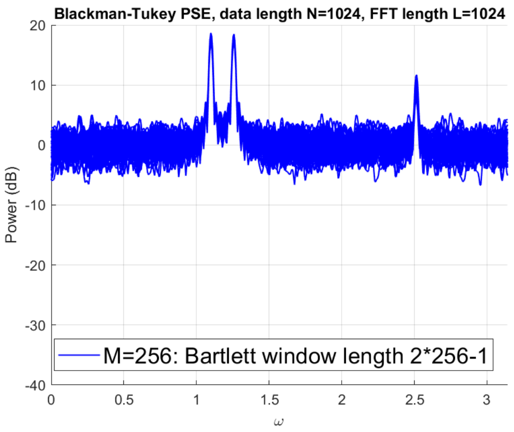


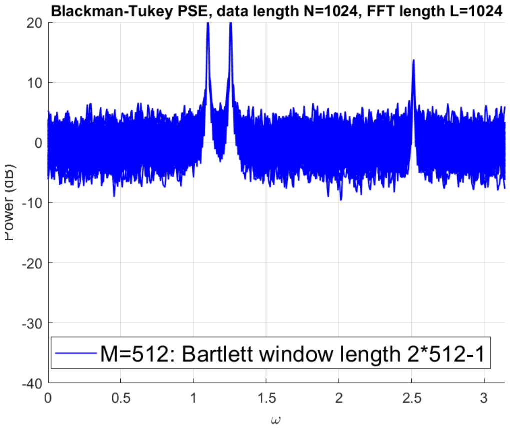


*__Correlation window__ length 2*M -1 varying*
__Data length__ N fixed

Blackman-Tuckey correlogram







- For fixed correlation window length, increasing N
  - Resolution ~ unchanged
  - Variance decreases
- For fixed N, increasing correlation window length:
  - Resolution improves
  - Variance slightly increases

Wrap up (I)

- **Non-parametric approaches** are based on the Fourier transform of the random signal or of an estimate of its autocorrelation function
- The **biased estimate** of the autocorrelation function is preferred as it is asymptotically unbiased, and it ensures **non-negativity** of the power spectral estimate
- The **expected value** of the **periodogram/correlogram** can be interpreted as the convolution in the frequency domain of the true spectrum with the Fourier transform of the correlation window
- The periodogram/correlogram is the Fourier transform of the biased estimator of the AC, but it is **asymptotically unbiased**

Wrap up (II)

- The periodogram/correlogram is **not a consistent estimator**, as the variance does not decrease as $N \to \infty$
- The **variance** of the periodogram/correlogram is approx. proportional to the square of the true spectrum.
- The variance of the periodogram can be decreased by the **averaging (overlapping) periodogram** methods (Bartlett and WOSA); however, the bias increases.
- The **Blackman-Tukey** method improves the correlogram by applying a suitable window to the raw autocorrelation (prior to DFT); it provides an **asymptotically unbiased** and **consistent** estimator of the PSD.

Statistical signal processing (5CTA0)

Non-parametric spectral estimation

Lecturer: Simona Turco

Electrical Engingeering, Signal Processing Systems group