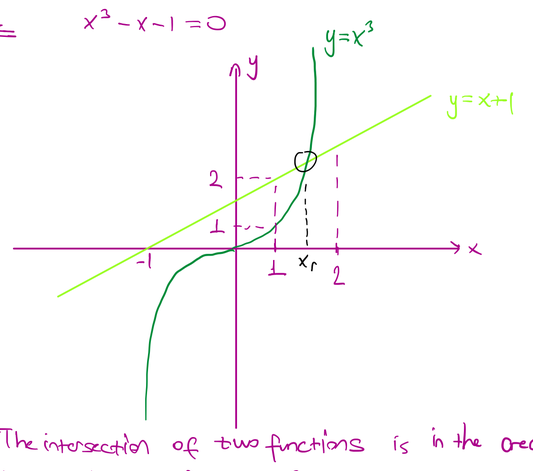
SLT-5

07 October 2022 13:20

5.1 $x^3 - x - 1 = 0$



The intersection of two functions is in the area of $x \in [1,2]$

in our solution $f(x) = x^3 - x - 1 = 0$

Therefore, $\boxed{x \approx a - \frac{f(a)}{f'(a)}}$

$a = 1$ is first initial guess,

$$x_1 = 1 - \frac{1^3 - 1 - 1}{3(1)^2 - 1} = 1.5$$

$$x_2 = 1.5 - \frac{(1.5)^3 - 1.5 - 1}{3(1.5)^2 - 1} = 1.3478260870$$

$$x_3 = 1.3478260870 - \frac{(1.3478260870)^3 - (1.3478260870) - 1}{(1.3478260870)^2 - 1}$$

$$x_3 = 1.3252003990$$

Sanity check $x_3^3 - x_3 - 1 = 0.002058 \approx 0$

5.2 a) $x_n = r^n + w_n$, $w_n \sim N(0, \sigma^2)$, $r > 0$

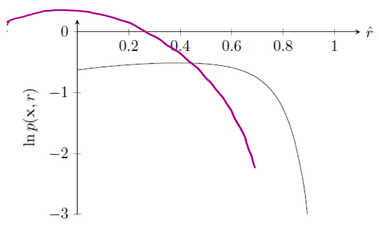


Figure 1: Log-likelihood function after observing x.

$$p(x|r) = \frac{1}{(2\pi\sigma)^{N/2}} \cdot \exp\left(-\frac{1}{2\sigma} \sum_{n=0}^{N-1} (x_n - r^n)^2\right)$$

$$\ln p(x|r) = \ln\left(\frac{1}{(2\pi\sigma)^{N/2}} \cdot \exp\left(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x_n - r^n)^2\right)\right)$$

$$\ln p(x|r) = -\frac{N}{2} \ln(2\pi\sigma) - \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x_n - r^n)^2$$

$$\frac{\partial \ln p(x|r)}{\partial r} = -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} 2.(x_n - r^n)(-n.r^{n-1})$$

$$\left\{ \frac{\partial \ln p(x|r)}{\partial r} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left(x_n.n.r^{n-1} - n.r^{2n-1}\right) \right.$$

$$\left\{ \frac{\partial^2 \ln p(x|r)}{\partial r^2} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left(x_n.n.(n-1)r^{n-2} - n.(2n-1).r^{2n-2}\right) \right.$$

$$r_{m+1} = r_m - \frac{\cancel{\frac{1}{\sigma^2}} \sum_{n=0}^{N-1} \left(x_n.n.r^{n-1} - n.r^{2n-1}\right)}{\cancel{\frac{1}{\sigma^2}} \sum_{n=0}^{N-1} \left(x_n.n(n-1)r^{n-2} - n.(2n-1)r^{2n-2}\right)}$$

$$r_{m+1} = r_m - \frac{\sum_{n=0}^{N-1} \left(x_n.n.r^{n-1} - n.r^{2n-1}\right)}{\sum_{n=0}^{N-1} \left(x_n.n.(n-1)r^{n-2} - (2n-1).n.r^{2n-2}\right)}$$

c) Initial r should be chosen from the left side $0 < r < 0.5$ since if it is selected as $r > 0.5$ (right side of the root, calculated gradient can explode or vanish and it can diverge

$$r_{is} > 0.5 \longrightarrow f(x) \rightsquigarrow -\infty$$

5.3 $w_R[i] = \begin{cases} 1 & \text{for } i = 0, 1, \dots, N-1 \\ 0 & \text{elsewhere} \end{cases}$

FTD:

$$W_R(e^{j\theta}) = \sum_{i=-\infty}^{\infty} w_R[n] e^{-jn\theta} = \sum_{i=0}^{N-1} e^{-jn\theta}$$

from exponential series sum formula

$$= \frac{1 - e^{-jN\theta}}{1 - e^{-j\theta}}$$

$$= \frac{\left(e^{jN/2\theta} - e^{-jN/2\theta}\right) e^{-j\frac{N}{2}\theta}}{\left(e^{j1/2\theta} - e^{-j1/2\theta}\right) e^{-j1/2\theta}}$$

$$= \frac{\frac{1}{2j}\left(e^{jN/2\theta} - e^{-jN/2\theta}\right) e^{-j\frac{N}{2}\theta}}{\frac{1}{2j}\left(e^{j1/2\theta} - e^{-j1/2\theta}\right) e^{-j1/2\theta}}$$

$$= \frac{\sin(N\theta/2)}{\sin(\theta/2)} \cdot e^{-j\frac{N-1}{2}\theta} \quad \text{--- Eq (1)}$$

FTD is continuous function of frequency, and in practice continuous signals cannot be directly recorded on computers.

DFT & Eq (1) is a discrete function of frequency. We are sampling in N Data points.

DFT:

$$W_R[k] = \sum_{n=0}^{N-1} w_R[n] . e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1 - e^{-j.2\pi k}}{1 - e^{-j\frac{2\pi k}{N}}}$$

$$= \frac{\left(e^{j\pi k} - e^{-j\pi k}\right) e^{-j\pi k}}{\left(e^{\frac{j\pi k}{N}} - e^{-\frac{j\pi k}{N}}\right) e^{-\frac{j\pi k}{N}}}$$

$$= \frac{\frac{1}{2j}\left(e^{j\pi k} - e^{-j\pi k}\right) e^{-j\pi k}}{\frac{1}{2j}\left(e^{\frac{j\pi k}{N}} - e^{-\frac{j\pi k}{N}}\right) e^{-\frac{j\pi k}{N}}}$$

$$= \frac{\sin(j\pi k)}{\sin(j\pi k/N)} \cdot e^{\frac{(1-N)(j\pi k)}{N}}$$

$, k = 0, 1, \dots, N-1$ --- (2)

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b) L: zero padded length

Zero padded DFT can be expressed as:

$$W_R[k] = \sum_{n=0}^{N-1} w_R[k] e^{-j\frac{2\pi}{L}kn}$$

$k = 2N$ ..

$$\Rightarrow \frac{1 - e^{-j\pi k}}{1 - e^{-j\frac{\pi}{N}k}}$$

$$= \frac{\sin(j\pi k/2)}{\sin(j\pi k/2N)} \cdot e^{-\frac{(N-1)(j\pi k/2)}{N}} \quad \text{--- Eq (3)}$$

By using zero padding we decrease the step size in N (window length) true in continuous spectrum. Thus we increased the FSD sampling

It leads to better approximation.

5.4 5.4

$$x[n] = \sin\left(\frac{f_1}{f_s} 2\pi n\right) + \sin\left(\frac{f_2}{f_s} 2\pi n\right) + w[n]$$

$f_s = 100$ Hz

a) $N = 30$

$$\frac{1.81.\pi}{N-1} < \left| 2\pi \frac{f_1}{100} - 2\pi \frac{f_2}{100} \right|$$

the difference between frequency peaks in the frequency axis should be at least 0.20 Hz

$$\frac{1.81.\pi}{29} < \left| 2\pi \frac{f_1}{100} - 2\pi . \frac{f_2}{100} \right|$$

$$|f_1 - f_2| > 3.12 \text{ Hz}$$

b) If triangular window is used then,

$$\frac{5.01.\pi}{N-1} < \left| 2\pi . \frac{f_1}{100} - \frac{2\pi f_2}{100} \right|$$

$$|f_1 - f_2| > 8.63 \text{ Hz}$$

In order to distinguish f_1 and f_2 peaks in $N=30$ triangular windows $|f_1 - f_2|$ minimum distance should be at least 8.63 Hz

Therefore we can conclude that rectangular window is better in terms of resolution in the same window length.

$$p(x|\lambda) = -\sum_{n=0}^{N-1} \frac{(x_n - \exp(\lambda n))^2}{2\sigma^2}$$

$$\frac{\partial p(x|\lambda)}{\partial \lambda} = -\sum_{n=0}^{N-1} 2.(x_n - \exp(\lambda n)).(-n\exp(\lambda n))$$

$$\frac{\partial p(x|\lambda)}{\partial \lambda} = \sum_{n=0}^{N-1} 2x_n.n.\exp(\lambda n) - \exp(2\lambda n).n$$

$$\frac{\partial^2 p(x|\lambda)}{\partial \lambda^2} = \sum_{n=0}^{N-1} 2x_n.n^2.\exp(\lambda n) - 2n^2\exp(2\lambda n)$$

$$\lambda_{m+1} = \lambda_m = \frac{\sum_{n=0}^{N-1} 2x_n.n.\exp(\lambda n) - n.\exp(2\lambda n)}{\sum_{n=0}^{N-1} 2x_n.n^2.\exp(\lambda n) - 2n^2\exp(2\lambda n)}$$