SLT 6

Thursday, 13 October 2022 13:41

Exercise 6.1 biased estimate $\hat{\rho}[\tau]$ for the auto correlation

$$\hat{\rho}[\tau] = \begin{cases} \frac{1}{N}\sum_{k=\tau}^{N-1} x[k]x[k-\tau] & 0 \le \tau \le N-1 \\ \hat{\rho} & -(N-1) \le \tau < 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\sum_{-\infty}^{\infty} \hat{\rho}[\tau]e^{-j\tau\theta} = \sum_{\tau=-(N-1)}^{N-1}\left(\frac{1}{N}\sum_{k=|\tau|}^{N-1} x[k]x[k-\tau]\cdot e^{-j\tau\theta}\right)$$

$\tau = k-n$

$\boxed{k-|\tau| = n}$
$$= \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1}\sum_{k=|\tau|}^{N-1} x[k]x[k-\tau]\cdot e^{-j\tau\theta}$$
$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x[k]x[n]e^{-j(k-n)\theta}$$
$$= \frac{1}{N}\left(\sum_{k=0}^{N-1} x[k]\cdot e^{-jk\theta}\right)\left(\sum_{n=0}^{N-1} x[k]e^{-jn\theta}\right)$$
$$= \frac{1}{N}\left|\sum_{k=0}^{N-1} x[k]e^{-jk\theta}\right|^2 = \hat{P}(e^{j\theta})$$

Second one: $n=k$, $k=k-n$

$\sum_{\tau=-(N-1)}^{N-1}$, $k-n=-(N-1)$ → something

$n = k-\tau$, $\tau = k-n$, $\sum_{n=0}^{N-1}$

Exercise 6.2

$2M-1 = 2N+1$

$2M = 2N+2$

$\boxed{M = N+1}$

a) Window length $2M-1 = 2N+1$

$M = N+1$, Therefore

$$\hat{P}_{BT}(e^{j\theta}) = \sum_{\tau=-N}^{N} w[\tau]\hat{r}[\tau]e^{-j\theta\tau}$$
$$\hat{P}_{BT}(e^{j\theta}) = \frac{1}{2\pi}\hat{P}(e^{j\theta}) * W_M(e^{j\theta})$$
$$E[\hat{P}_{BT}(e^{j\theta})] = \frac{1}{2\pi}E[\hat{P}(e^{j\theta})] * W_M(e^{j\theta})$$
$$E[\hat{P}_{BT}(e^{j\theta})] = \frac{1}{2\pi}P(e^{j\theta}) * W_B(e^{j\theta}) * W_M(e^{j\theta})$$

Since the BT window has significantly smaller window length than spectrum window length function, we can neglect it

$$E[\hat{P}_{BT}(e^{j\theta})] \approx P(e^{j\theta}) *_{2\pi} W_M(e^{j\theta})$$

it can be written as:

$$E[\hat{P}_{BT}(e^{j\theta})] \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{j\theta})W_M(e^{j(\theta-\phi)})d\phi$$

$W_M(e^{j\theta}) \ge 0.$

We know that our symmetric correlation lag window is Bartlett window

$$W_M(e^{j\theta}) = \frac{1}{N+1}\left(\frac{\sin((N+1)\theta/2)}{\sin(\theta/2)}\right)^2, \quad -\pi < \theta \le \pi$$

which is positive definite.

b) If it is rectangular window,

$$W_M(e^{j\theta}) = \left(\frac{\sin((N+1)\theta/2)}{\sin(\theta/2)}\right), \quad -\pi < \theta \le \pi$$

which might be negative in the given interval.

13.3.1
- Define window
- Must be checked $W(e^{j\theta}) \ge 0$
  ↳ $\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i W_m[i-j] a_j \ge 0$
- $\lim_{m\to\infty} W_m(e^{j\theta})$ must be equal to $A\delta(\theta)$

6.3

$x[k] = i[k] - \frac{3}{4}x[k-1] \quad i[k] \sim N(0, \sigma_i^2)$

a)

$\rho[\tau] = E\{x[k]\cdot x[k-\tau]\}$

$\rho[\tau] = E\{(i[k] - \frac{3}{4}x[k-1])(i[k-\tau] - \frac{3}{4}x[k-\tau-1])^*\}$

$\rho[\tau] = E\{(i[k]i^*[-\tau] - \frac{3}{4}i[k]x[k-\tau-1]^* - \frac{3}{4}x[k-1]i^*[k-\tau] + \frac{9}{16}x[k-1]\cdot x^*[k-\tau-1]\}$

$= E\{i[k]i^*[-\tau]\} - \frac{3}{4}E\{i[k]\cdot x^*[k-\tau-1]\} - \frac{3}{4}E\{x[k-1]i^*[k-\tau]\} + \frac{9}{16}\{x[k-1]\cdot x^*[k-\tau-1]\}$

→ $\sigma_i^2\delta[\tau]$; 0; 0; $\rho_x[k-1]$

$\rho[\tau] = \sigma_i^2 - \frac{3}{4}\cdot\rho[k-1] \quad \tau = 0$

$\boxed{\rho[\tau] = -\frac{3}{4}x[\tau-k]} \quad |\tau| > 0$

$\rho[0] = \frac{7}{16} - \frac{3}{4}\rho[1]$

$\rho[1] = -\frac{3}{4}\rho[0]$

$\rho[0] = \frac{7}{16} + \frac{9}{16}\rho[0]$

$\frac{7}{16}\rho[0] = \frac{7}{16} \to \rho[0] = 1 \Rightarrow \rho[1] = -3/4$

$\rho[2] = -\frac{3}{4}\rho[1] = (-\frac{3}{4})^2 \Rightarrow \rho[\tau] = (-\frac{3}{4})^{|\tau|}$

b)

$$\hat{\rho}[\tau] = \begin{cases} (-\frac{3}{4})^{|\tau|} & \tau = -1, 0, 1 \\ 0 & \text{elsewhere} \end{cases}$$

$\hat{P}(e^{j\theta}) = \sum_{\tau=-1}^{-1}\hat{\rho}[1]e^{+j\theta} + \sum_{\tau=0}^{0}\hat{\rho}[0]\cdot e^{0} + \sum_{\tau=1}^{1}\hat{\rho}[1]e^{-j\theta}$

$\hat{P}(e^{j\theta}) = -\frac{3}{4}e^{j\theta} + 1 - \frac{3}{4}\cos e^{-j\theta}$

$\hat{P}(e^{j\theta})\ 1 - \frac{3}{4}(\cos\theta + j\sin\theta) - \frac{3}{4}(\cos\theta - j\sin\theta)$

$\hat{P}(e^{j\theta}) = 1 - \frac{3}{2}\cos\theta$ for instance $\theta = 0 \to \hat{P}(e^{j\theta}) = -0.5$

with this correlogram, our PSD estimation can be negative, variance cannot be negative.

$\hat{P}(e^{j\theta}) = \frac{1}{N-|\tau|}\sum_{\tau=-(N-1)}^{N-1}[x[k]x[k-|\tau|]]e^{-j\theta|\tau|}$

$E\{\hat{P}(e^{j\theta})\} = \frac{1}{N-|\tau|}\sum_{\tau=-(N-1)}^{N-1}E\{x[k]x[k-|\tau|]\}e^{-j\theta|\tau|}$

$E\{\hat{P}(e^{j\theta})\} = \frac{N-|\tau|}{N-|\tau|}\sum_{\tau=-(N-1)}^{N-1}\hat{\rho}_x\cdot e^{-j\theta|\tau|}$

$E\{\hat{P}(e^{j\theta})\} = \sum_{\tau=-(N-1)}^{N-1} -\frac{3}{4}^{|\tau|}\cdot e^{-j\theta|\tau|}$

$E\{\hat{P}(e^{j\theta})\} = \sum_{\tau=-(N-1)}^{0}\left((-\frac{3}{4})^{-\tau}e^{+j\theta\tau}\right) + \sum_{\tau=0}^{N-1}\left((-\frac{3}{4})^{\tau}e^{-j\tau\theta}\right) - 1$

$\sum_{\tau=0}^{N-1}(-\frac{3}{4})^{\tau}\cdot e^{-j\theta\tau} - \sum_{\tau=0}^{N-1}\left((-\frac{3}{4})^{\tau}e^{-j\theta\tau}\right) - 1$

$E\{\hat{P}(e^{j\theta})\} = 2\sum_{\tau=0}^{N-1}(-\frac{3}{4})^{\tau}e^{-j\theta\tau + j\theta\tau}$

$2\cdot\sum_{0}^{N-1}(-\frac{3}{4})^{\tau}(\cos(\tau\theta)) - 1$

$N=2 \to 2\cdot(1 - \frac{3}{4}\cos\theta) - 1$

if $\theta = 0 \to$ it will be negative

6.4

$\sigma_i^2 = 1$, $H(z) = \frac{1 - \frac{2}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$ $\quad b_1 = -2/3$, $a_1 = -1/2$

$$H(z) = \frac{1 + \sum_{q=1}^{Q} b_q\cdot z^{-q}}{1 + \sum_{p=1}^{P} a_p z^{-p}}$$

$$\rho_x[l] = \begin{cases} \sigma_i^2\sum_{k=|l|}^{q} b_k h[k-|l|] - \sum_{k=1}^{p} a_k r_x[l] & \text{for } 0 \le |l| \le q \\ -\sum_{k=1}^{p} a_k r_x[|l|-k] \end{cases}$$

$p = 1, q = 1, \sigma_i^2 = 1$

$$\rho_x[k] = \begin{cases} 1\cdot\sum_{k=l}^{1} b_k h[k-|l|] - \sum_{k=1}^{1} a_1 r_x[|l|-k] & \text{for } 0 \le |l| \le q \\ -\sum_{k=l}^{1} a_k\cdot r_x[|l|-k] & \text{for } |l| > t \end{cases}$$

$\rho[0] = -\sum_{k=0}^{1} h[k] - \sum_{k=1}^{1}(-\frac{1}{2}r_x[-k])$

$r[-k] = r[k]$ → signal is real

$\rho[l=0] = b_0\cdot h[0] + b_1 h[1] - a_1\rho[-1]$ ——(1)

$\rho[l=1] = b_1 h[0] - a_1\rho[0]$ ——(2)

$\frac{b_1 h[0] - \rho[1]}{a_1} = b_0\cdot h[0] + b_1 h[1] - \rho[-1]$

$H(z) = \frac{1 - \frac{2}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}} = \frac{2}{3}\frac{(3z-2)}{(2z-1)}$

by polynomial division

$H(z) = 1 - \frac{z^{-1}}{6} + \ldots$

z^{-1}: $h[k] = \delta[k] - \frac{\delta[k-1]}{6}$

$h[0] = 1 - \frac{1}{6}\cdot 0 = 1$

$h[1] = -1/6$

$2z-1\ \overline{)\,3z-2}$, quotient $\frac{3}{2} - \frac{z^{-1}}{4}$; $-(3z - \frac{3}{2})$; $-\frac{1}{2}$; $-(-\frac{1}{2} + \frac{z^{-1}}{4})$; $-\frac{z^{-1}}{4}$

1.23 exercise booklet

$X(z) = H(z)I(z)$

$x[k] = i[k] - \frac{2}{3}i[k-1] + \frac{1}{2}x[k-1]$

$\rho[\tau] = E\{x[k]x[k-\tau]\}|_{\tau=0} = E\{x[k]x[k]\}$

$= \sigma_i^2 + \frac{4}{9} - \frac{2}{6}\sigma^2 = \frac{1}{2}E\{x[k]x[k-1]\}$

$E\{x[k]x[k-1]\}$

$= -\frac{2}{3} + \frac{1}{2}E\{x[k]x[k]\}$

$E\{x[k]x[k]\} = \frac{10}{9} + \frac{1}{2}E\{x[k]x[k-1]\}$

$E\{x[k]x[k]\} = \frac{28}{27}$

$\rho[l=0] = b_0\cdot h[0] + b_1 h[1] - a_1\rho[-1]$ ——(1)

$\rho[l=1] = b_1 h[0] - a_1\rho[0]$ ——(2)

$\rho[0] = 1 + \frac{1}{9} + \frac{1}{2}\rho[1]$

$\rho[1] = -\frac{2}{3} + \frac{1}{2}\rho[0]$ $\quad$ ARMA(1,1)

$\rho[0] = \frac{28}{27}$, $\rho[1] = \frac{-4}{27}$

$\rho[2] = -\sum_{k=1}^{1} a_k\cdot r_x[|l|-k] = \frac{1}{2}r_x[1]$

$\rho[3] = \frac{1}{2}\cdot r_x[2] = \frac{1}{2}\cdot\frac{1}{2}\cdot r_x[1]$

$$r_x[l] = \begin{cases} \frac{28}{27} & \text{for } l = 0 \\ \frac{-4}{27}\cdot(\frac{1}{2})^{|l|-1} & \text{for } |l| \ge 1 \end{cases}$$

b) AR(1) $\Rightarrow x[k] = i[k] - a_1 x[k-1]$

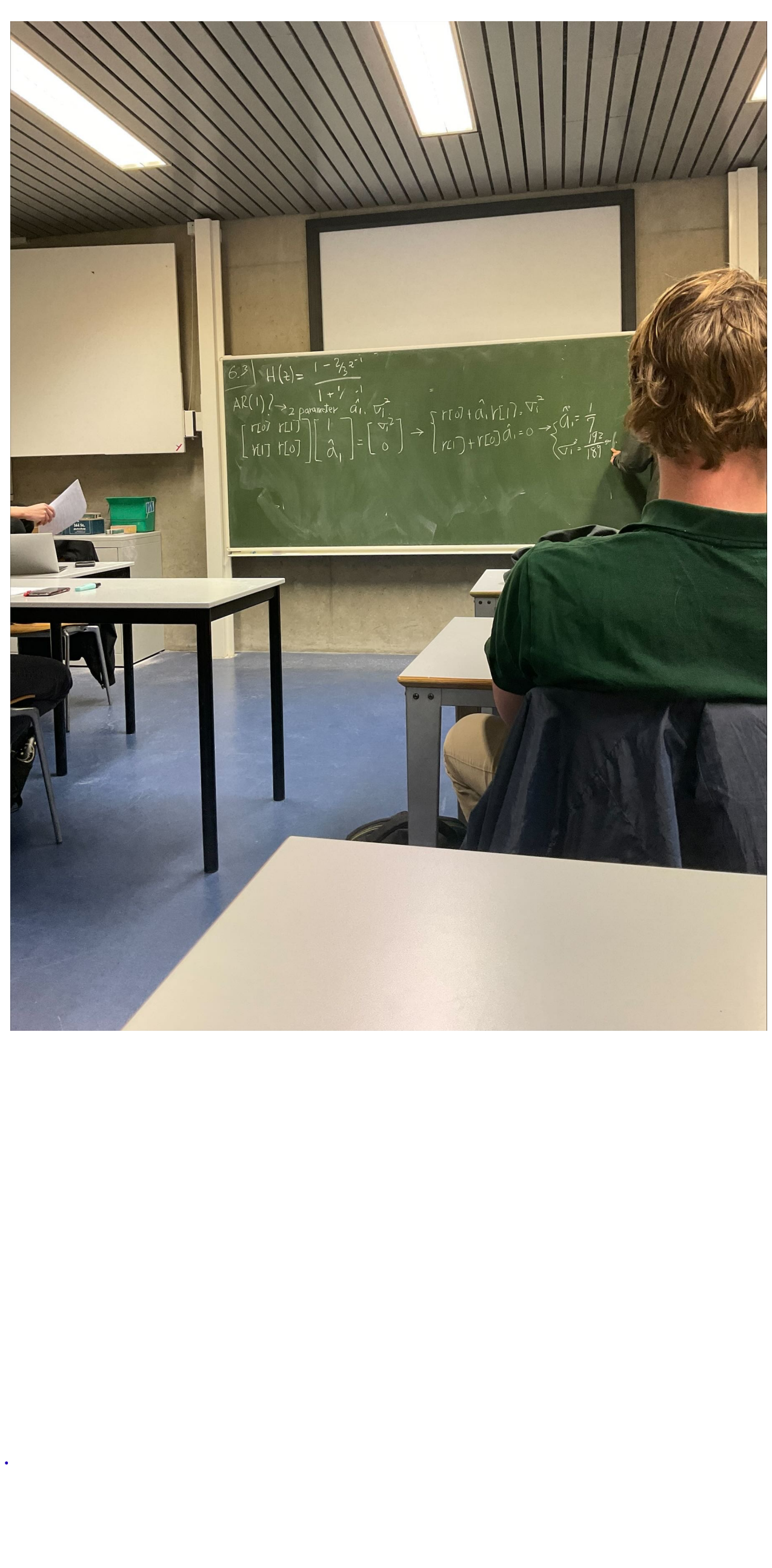
$r_x[\tau] = E[x[k]\cdot x^*[k-\tau]]$

$r_x[\tau] = E[x[k]\cdot(i[k-\tau] - a_1 x[k-1-\tau])]$

$r_x[\tau] = E[x[k]i[k-\tau] - a_1 x[k]\cdot x[k-1-\tau]]$

$r_x[\tau] = r_{i,x}[\tau] - a_1 r_x[1-\tau]$

We know that $r_{i,x}[k] = \sigma_i^2\delta[k]$



$r_x[\tau] = \sigma_i^2\delta[k] - a_1\cdot r_x[1-\tau]$

$r_x[0] = \sigma_i^2 - a_1\cdot r_x[1]$

$r_x[1] = -a_1\cdot r_x[0]$

$r_x[0] = \sigma_i^2 + a_1^2\cdot r_x[0]$

$r_x[0](1-a_1^2) = \sigma_i^2$

$\boxed{r_x[0] = \frac{\sigma_i^2}{1-a_1^2}}$

$\boxed{r_x[1] = -a_1\frac{\sigma_i^2}{1-a_1^2}}$

$r_x[0] = \frac{28}{27} = \frac{\sigma_i^2}{1-a_1^2}$

$28\cdot(1-a_1^2) = 27\sigma_i^2$ ——(1)

$-\frac{4}{27} = -a_1\cdot\frac{\sigma_i^2}{1-a_1^2}$ ——(2)

$\sigma_i^2 = 1$

$a_1 = 0.142$

c)

$P_{AR(1)} = \frac{\sigma_i^2}{|1 + a_1 e^{-j\theta}|^2} = \frac{1}{|1 + 0.142(\cos\theta - j\sin\theta)|^2}$

$P_{AR(1)} = \frac{1}{|1 + 0.142\cdot e^{-j\theta}|^2} = \frac{1}{|(1 + 0.142\cos\theta) - 0.142 j\sin\theta)^2}$

$= \frac{1}{|(1+0.142\cos\theta)^2 + (0.142)^2\sin^2\theta)}$

$= \frac{1}{|1 + 2\times 0.142\cos\theta + (0.142)^2\cdot\cos\theta + (0.142)^2\cdot\sin^2\theta|}$

$= \frac{1}{|1 + 0.284\cdot\cos\theta + (0.142)^2|}$

$P_{AR(1)} = \frac{1}{|1.02016 + 0.284\cdot\cos\theta|}$