EINDHOVEN UNIVERSITY OF TECHNOLOGY

Statistical signal processing (5CTA0)

Introduction to spectral estimation

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Electrical Engingeering, Signal Processing Systems group

Content overview

Part 1
Random Variables and Random Signals

Part 2
Estimation Theory

Part 3
Spectral Estimation

Part 4
Detection Theory

Part 1: Random variables and Random Signals

Part 3

Spectral estimation

3.1: Introduction to spectral estimation
3.2: Non-parametric spectral estimation
3.3: Parametric spectral estimation

Outline

- Introduction
- Energy and power signals
- Direct and indirect method
- Recap: Fourier Transform and zero-padding
- Windowing:
  - Resolution loss
  - Spectral leakage

Spectral estimation

- **Goal**: determine the spectral content, i.e. distribution of power over frequencies, of a random signal

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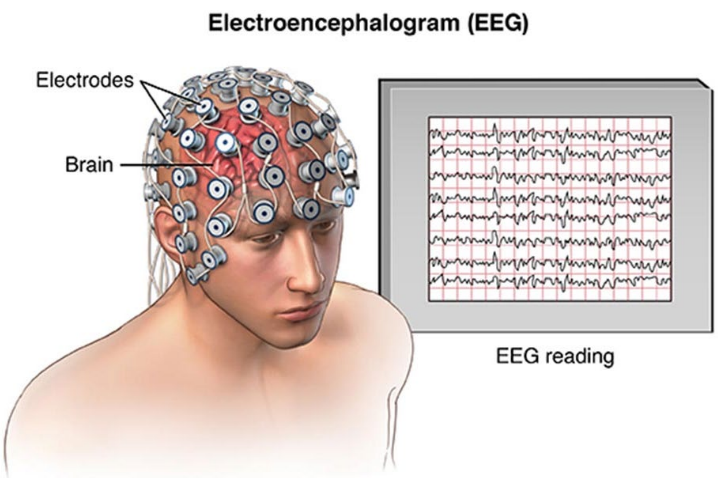
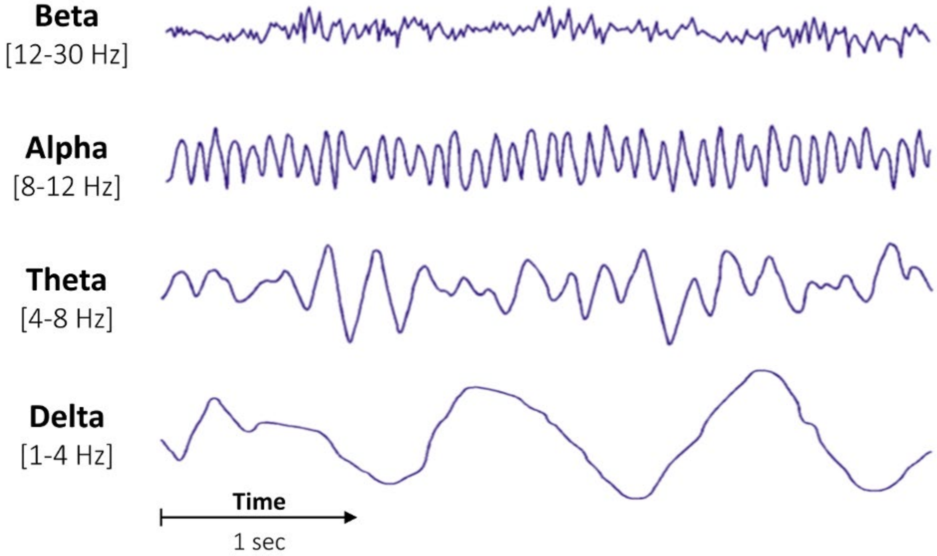
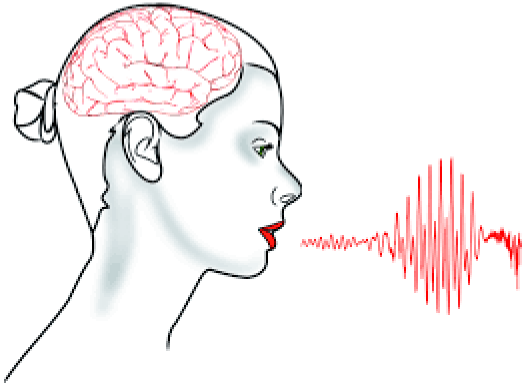
- **Applications**: medical diagnosis, speech analysis, seismology and geophysics, radar and sonar, nondestructive fault detection, prediction of economic trends…

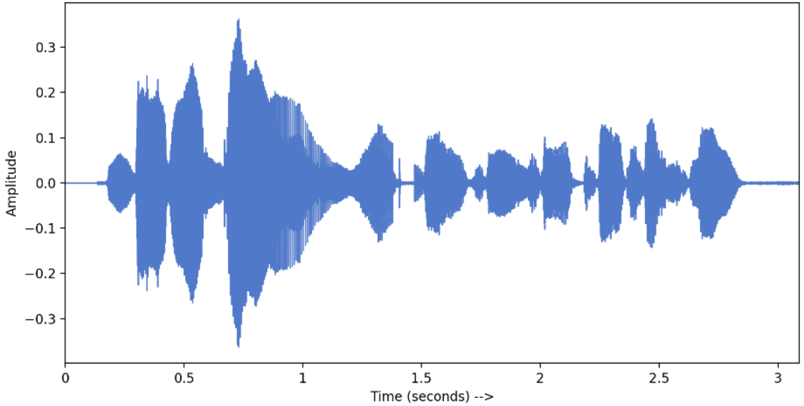
Application example: Sleep analysis

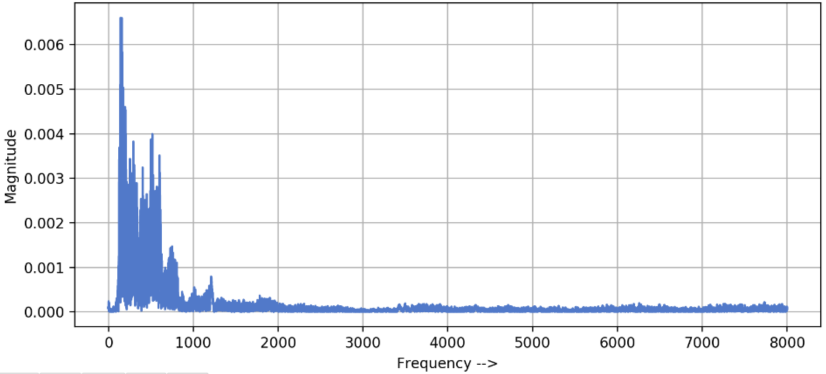




Application example: Speech analysis







Spectral estimation

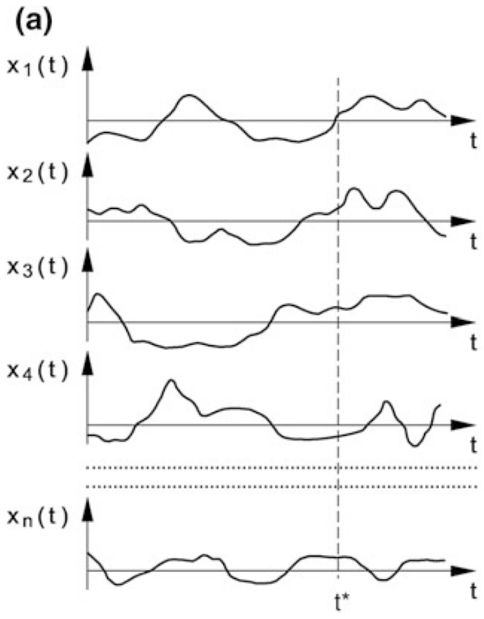
- **Goal**: determine the spectral content, i.e. distribution of power over frequencies, of a random signal

- **Applications:** medical diagnosis, speech analysis, seismology and geophysics, radar and sonar, nondestructive fault detection prediction of economic trends…

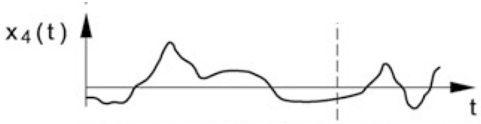
- **Problem:** in practice, only a limited set of observations is available

Ergodicity

Random process

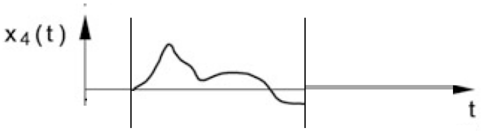


Ergodic process



Statistics can be calculated by time-averaging over single representative members of the ensemble

Practice: Limited set of samples



Segment of a single realization

$$E\{\cdot\} \Leftrightarrow \frac{1}{N}\sum_{n=0}^{N-1}\{\cdot\}$$

Ergodicity

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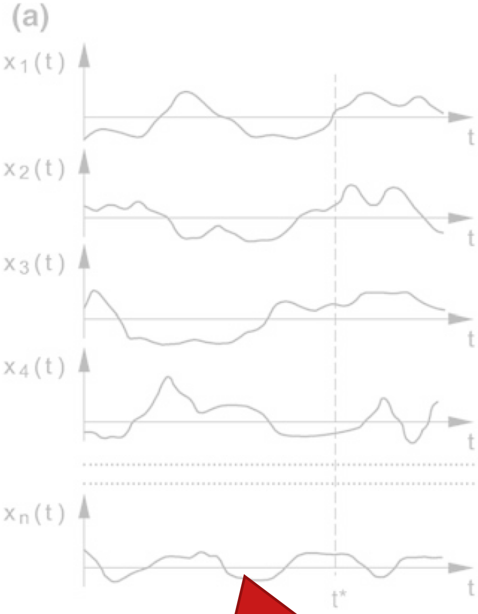

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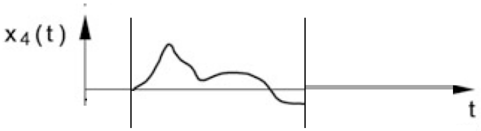

Segment of a single realization

$$E\{\cdot\} \Leftrightarrow \frac{1}{N}\sum_{n=0}^{N-1}\{\cdot\}$$

Draw information on the underlying random process

Wiener-Khintchine theorem

- The power spectral density (PSD) spectrum of a stationary random signal $x[n]$ is the Fourier transform of its autocorrelation (AC) function $r[l]$
- The PSD and the AC of a random signal are Fourier pairs

$$P_x(e^{j\theta}) = \sum_{l=-\infty}^{\infty} r_x[l] e^{-j\theta l} \qquad \Leftrightarrow \qquad r_x[l] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{j\theta}) e^{j\theta l} d\theta$$

Spectral analysis: problem

PROBLEM: limited set of *N* samples (observations)

- $x[n]$ available only in window: not $x[n]$ but $\tilde{x}[n]$ available
- Not autocorrelation $r[l]$ but $\hat{r}[l]$ available
- Window has effects on resolution and "leakage"
- The FDT exists only for signals with finite energy

We look for spectral **estimator** $\hat{P}(e^{j\theta}) = P(e^{j\theta}|\tilde{x}[n])$

- Bias, variance, consistency

TU/e

Spectral analysis: overview of methods

Non-parametric:

- Classic approach based on FTD + windowing
- Estimation from (finite) signal samples
- No prior assumption on mechanism that correlates the samples

Parametric:

- Based on signal model
- Exploiting knowledge (or guess) of correlation structure in signal
- Reduces to estimating parameters from model

Energy and power signals

Introduction to spectral estimation

Energy and power signals

- The energy (total power) of a signal $x[n]$ is given by

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \geq 0$$

The energy is zero iff $x[n] = 0$ for all n

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- The average power of a signal $x[n]$ is given by

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \geq 0$$

Energy and power signals

- A signal with finite energy ($0 < E_x < \infty$) is called energy signal
- A signal with finite average power ($0 < P_x < \infty$) is called power signal

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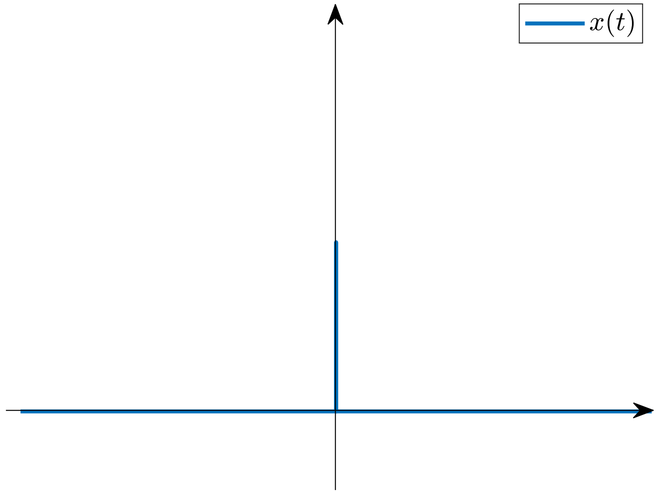
- A signal with finite energy ($0 < E_x < \infty$) is called energy signal
- A signal with finite average power ($0 < P_x < \infty$) is called power signal
    - Average power of an energy signal is zero!

Energy signal: example

Delta pulse

$$x(t) = A\,\delta(t)$$



Energy signal: example

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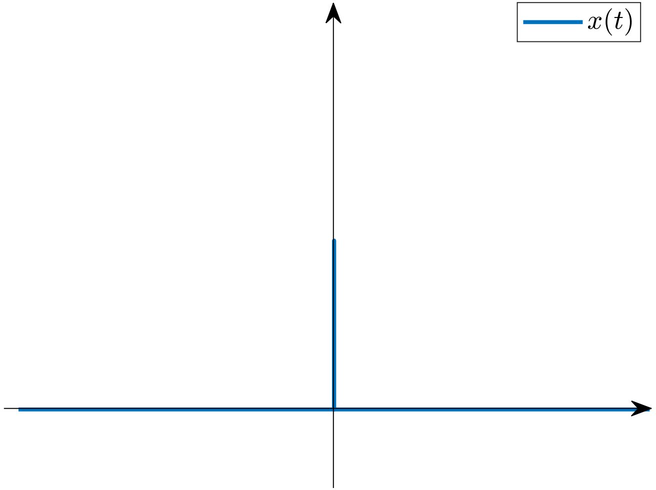

$$x(t) = A\,\delta(t)$$

$$E = \int_{-\infty}^{\infty} |\,x(t)\,|^2\, dt$$

$$E = A^2 \int_{-\infty}^{\infty} \delta(t)dt = A^2 < \infty$$

Energy signal

Energy signals: direct method

- FTD exists $X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n}$

- Energy is finite $E = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

Energy signals: direct method

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- Energy is finite $$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$
- Total energy can be rewritten in frequency domain as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\theta})|^2 \, d\theta$$

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- **Direct method** to calculate the Energy Spectral Distribution (ESD) function from observed signal $x[n]$

$$\boxed{E(e^{j\theta}) = |X(e^{j\theta})|^2 = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta} \right|^2}$$

Energy signals: indirect method

$$E(e^{j\theta}) = |X(e^{j\theta})|^2 = X(e^{j\theta})X^*(e^{j\theta}) = \left(\sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta}\right)\cdot\left(\sum_{l=-\infty}^{\infty} x^*[l]e^{jl\theta}\right) =$$

$$= \sum_{\tau=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} x[n]x^*[n-\tau]e^{-j\tau\theta}\right) = \sum_{\tau=-\infty}^{\infty} r[\tau]e^{-j\tau\theta}$$

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$\boxed{\tau = n - l}$

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- **Indirect method** to calculate the Energy Spectral Distribution (ESD) function from AC $r[\tau]$

$$\boxed{E(e^{j\theta}) = \sum_{\tau=-\infty}^{\infty} r[\tau]e^{-j\tau\theta}}$$

Energy and power signals

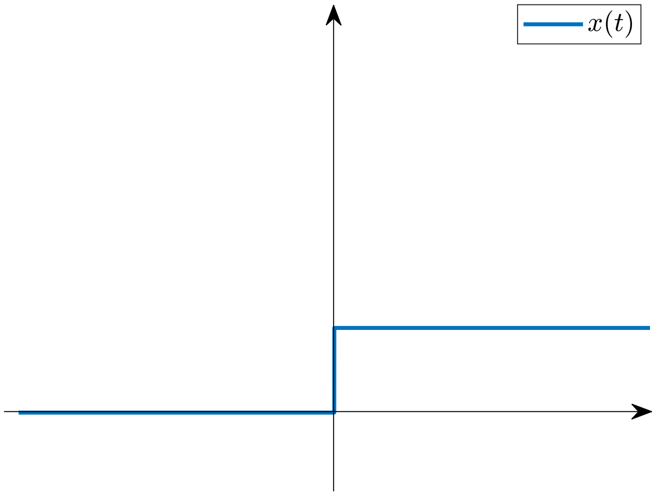
- A signal with finite energy ($0 < E_x < \infty$) is called energy signal
- A signal with finite average power ($0 < P_x < \infty$) is called power signal

In practice, we deal with segments of infinite-length signals:
Typically these are **aperiodic power signals**

Power signal: example

Step function

$$x(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



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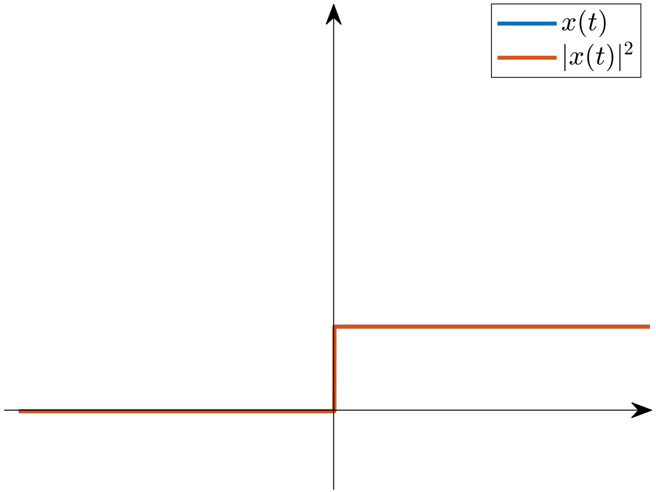

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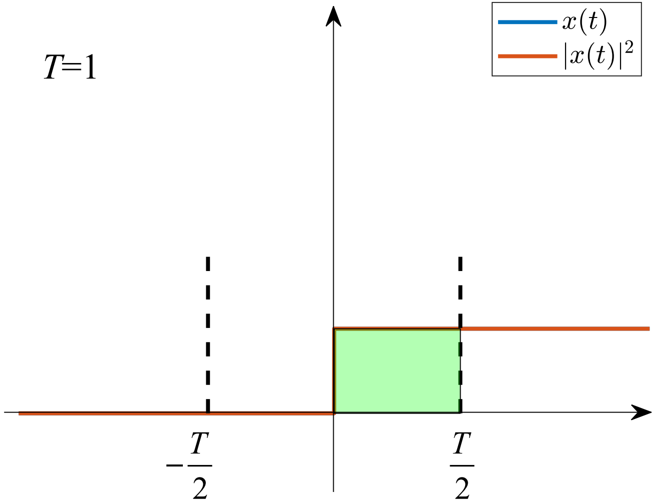
$$E = \int_{-\infty}^{\infty} |\, x(t) \,|^2 \; dt$$

$$E = \int_{0}^{\infty} 1 \; dt = t \Big|_0^{\infty} \rightarrow \infty$$

Power signal: example

Step function



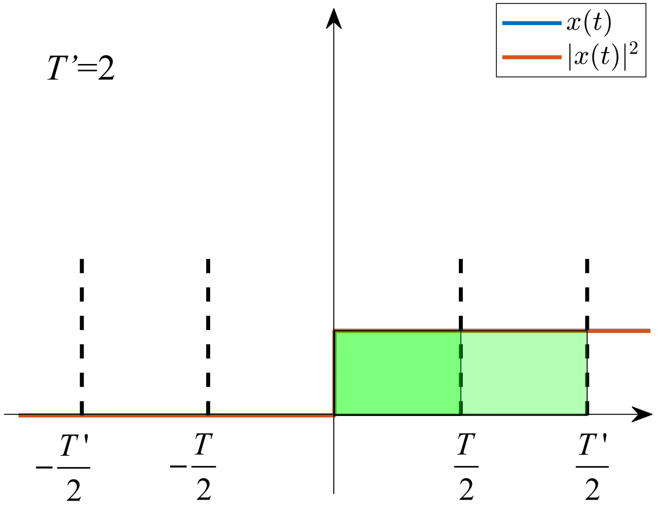
$$x(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |\, x(t) \,|^2 \, dt$$

$$\frac{1}{T} \int_{-T/2}^{T/2} |\, x(t) \,|^2 \, dt = 1 \int_{0}^{1/2} 1 \, dt = \frac{1}{2}$$

Power signal: example

Step function



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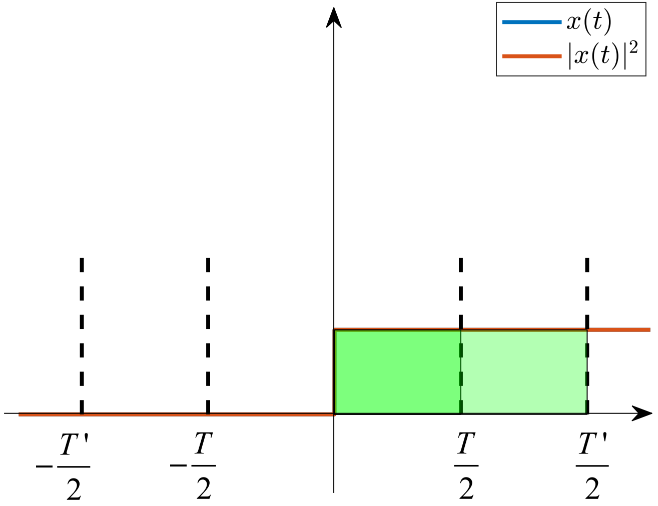

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$$P = \frac{1}{2} < \infty$$

Power signals: direct method

- Power signals: energy → ∞, power is finite
- Average Power: $$P = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

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- Power spectral density

$$P = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X(e^{j\theta})\right|^2 d\theta = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{-j\theta}) d\theta$$

- **Direct method** to calculate the Power Spectral Density (PSD) function from observed signal $x[n]$

$$\boxed{P(e^{j\theta}) = \frac{1}{N} |X_N(e^{j\theta})|^2 = \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] e^{-jn\theta} \right|^2}$$

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Power signals: indirect method

$$P(e^{j\theta}) = \frac{1}{N} |X(e^{j\theta})|^2 = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{p=0}^{N-1} x[n]x^*[p]e^{-j(n-p)\theta} = \sum_{\tau=-(N-1)}^{N-1} \left(\frac{1}{N} \sum_{n=|\tau|}^{N-1} x[n]x^*[n-|\tau|] \right) e^{-j\tau\theta}$$

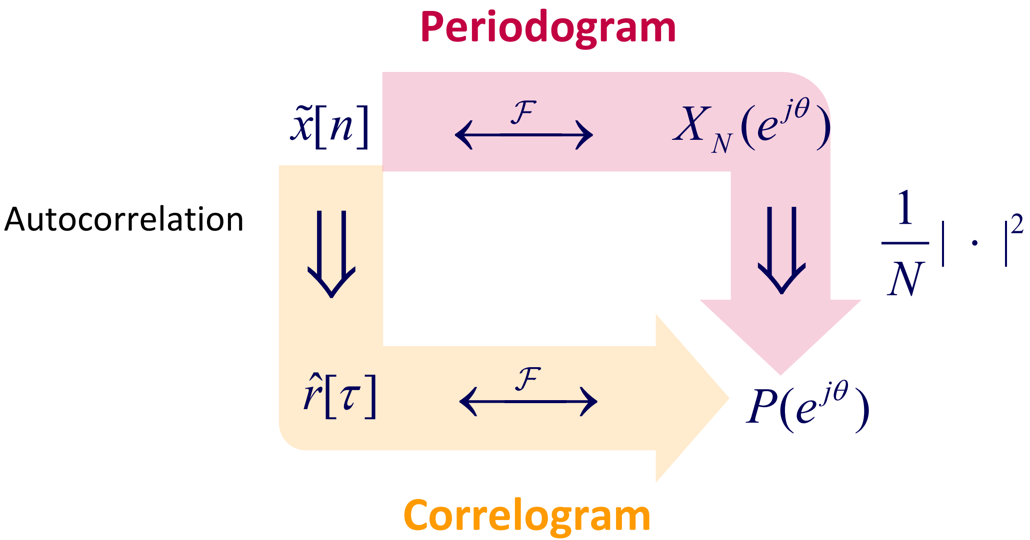
- With $r[\tau]$, estimate of AC of $x[n]$ defined as

$$r[\tau] = \frac{1}{N} \sum_{n=|\tau|}^{N-1} x[n]x^*[n-|\tau|] = r[-\tau]$$

- **Indirect method** to calculate the Power Spectral Distribution (PSD) function from AC $r[\tau]$

$$\boxed{P(e^{j\theta}) = \sum_{\tau=-(N-1)}^{N-1} r[\tau]e^{-j\tau\theta}}$$

Power signals: direct vs indirect



- **In practice:**
  - Stochastic signal→ power signal
  - Limited set of N samples available
  - AC can only be estimated over a limited number of lags

$$P_{dir}(e^{j\theta}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-jn\theta}\right|^2$$

$$P_{ind}(e^{j\theta}) = \sum_{\tau=-(N-1)}^{N-1} \hat{r}[\tau]e^{-j\tau\theta}$$

Recap: Fourier transform and zero padding

Introduction to spectral estimation

Fourier transform for discrete-time signals (FTD)

- Fourier transform for discrete time signals (**FTD**)

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta} \leftrightarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{jn\theta}\mathrm{d}\theta$$

FTD is a continuous function of frequency!

Condition for existence:

$$|X(e^{j\theta})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta}\right| < \infty$$

$$\left|\sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta}\right| \le \sum_{n=-\infty}^{\infty} \left|x[n]e^{-jn\theta}\right|$$

Magnitude of sum ≤ sum of magnitudes

$$\sum_{n=-\infty}^{\infty} \left|x[n]e^{-jn\theta}\right| < \infty$$

$$\left|x[n]e^{-jn\theta}\right| = \left|x[n]\right|\left|e^{-jn\theta}\right|$$

Magnitude of product = product of magnitudes

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty \quad \Rightarrow \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

FTD in principle exists only for energy signals!

Fourier transform for discrete-time signals (FTD)

- Fourier transform for discrete time signals (**FTD**)

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta} \leftrightarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{jn\theta}\mathrm{d}\theta$$

- N-point **FTD**: equivalent to windowing by a rectangular window

$$X_N(e^{j\theta}) = \sum_{n=0}^{N-1} x[n]e^{-jn\theta} = \sum_{n=-\infty}^{\infty} x[n]w[n]e^{-jn\theta} \qquad w[n] = \begin{cases} 1 & n = 0,\ldots,N-1 \\ 0 & else \end{cases}$$

$$\updownarrow$$

$$X(e^{j\theta}) * W(e^{j\theta})$$

FTD vs DFT

- Discrete-time Fourier transform (**DFT**)

$$X_p[k] = \sum_{n=0}^{N-1} x_p[n] e^{-j\frac{2\pi}{N}kn} \leftrightarrow x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_p[k] e^{j\frac{2\pi}{N}kn}$$

DFT is a discrete function of frequency!

$$X_N(e^{j\theta}) = \sum_{n=0}^{N-1} x[n] e^{-jn\theta} \qquad \to X_N(e^{j\theta})\Big|_{\theta = k\frac{2\pi}{N}}$$

DFT is equivalent to N-point FTD sampled at $\theta = k\dfrac{2\pi}{N}$

Zero padding

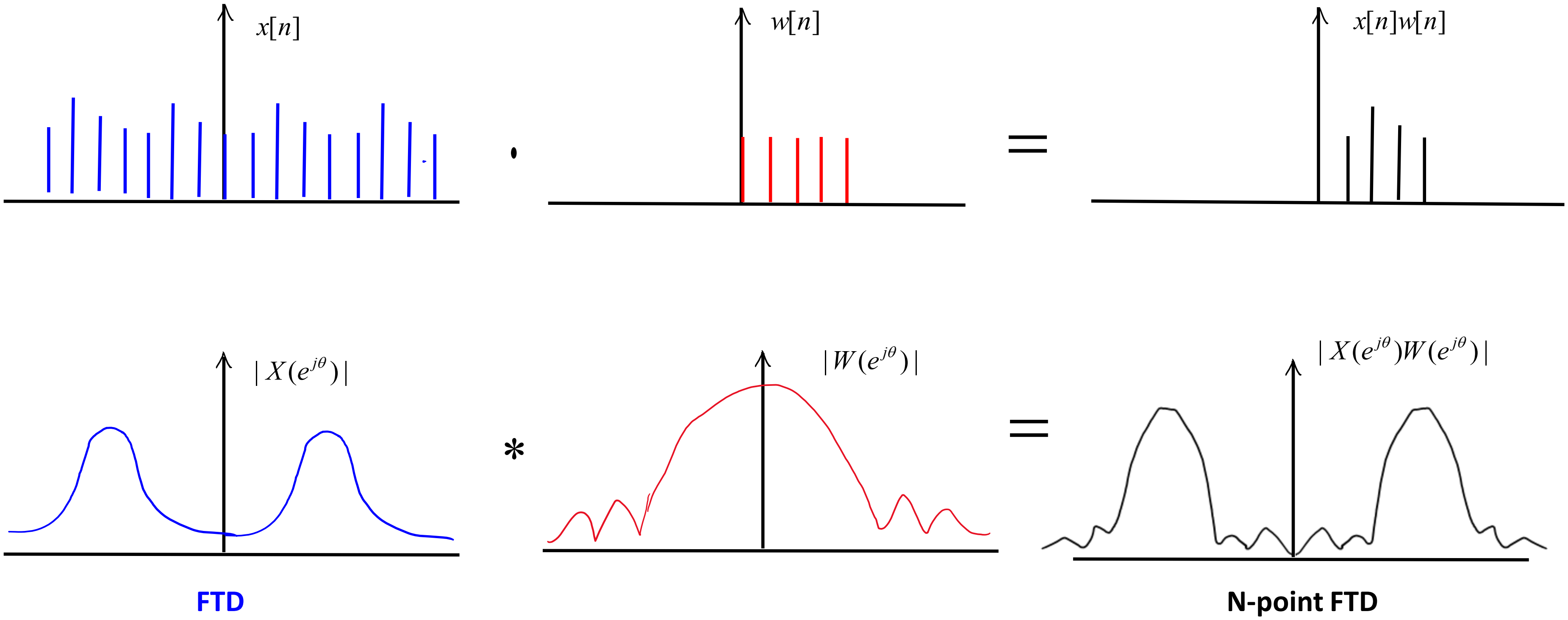
- N-point DFT calculates spectrum at specific frequency bins

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-njk\frac{2\pi}{N}} \triangleq X_N(e^{j\theta})\big|_{\theta=k\frac{2\pi}{N}} \qquad k=0,1,\ldots,N-1$$

- Zero-padding: extend data by $L-N$ zeros, and use L-point DFT ($L \geq N$)

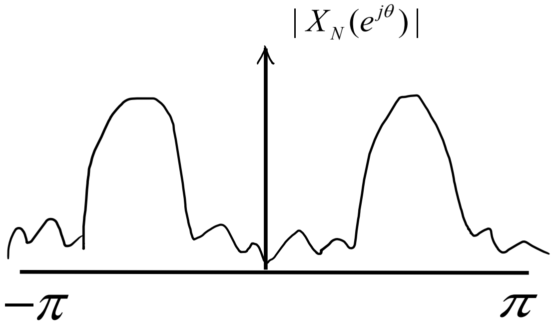
$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-njk\frac{2\pi}{L}} \triangleq X_N(e^{j\theta})\big|_{\theta=k\frac{2\pi}{L}} \qquad k=0,1,\ldots,L-1$$

N-point FDT: intuition

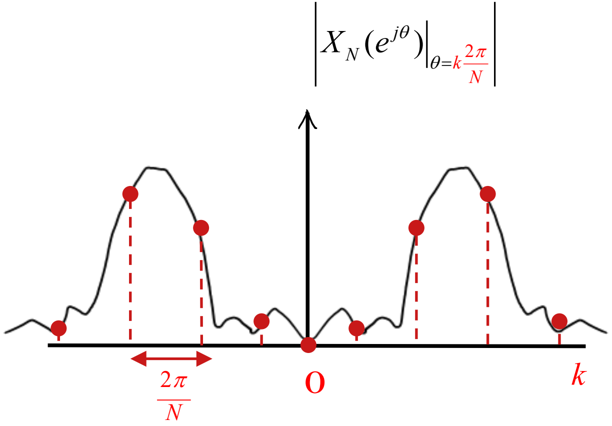


N-point FTD, DFT and zero-padding: intuition

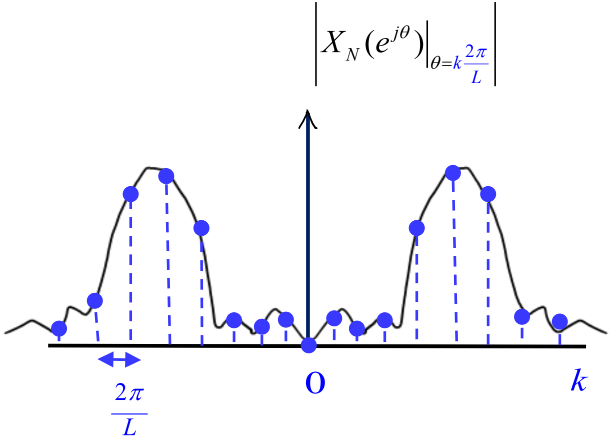
N-point FTD



DFT



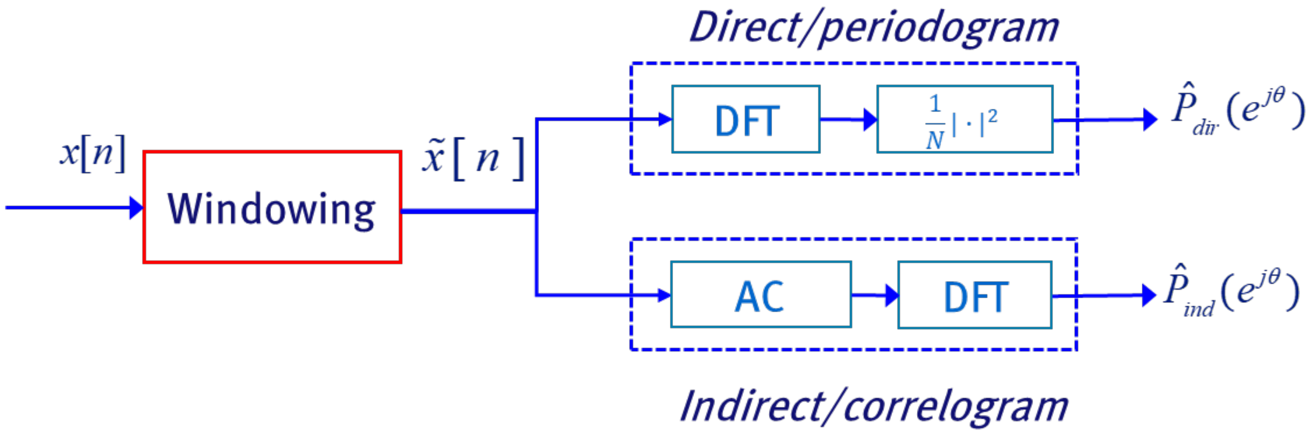
Zero-padded DFT



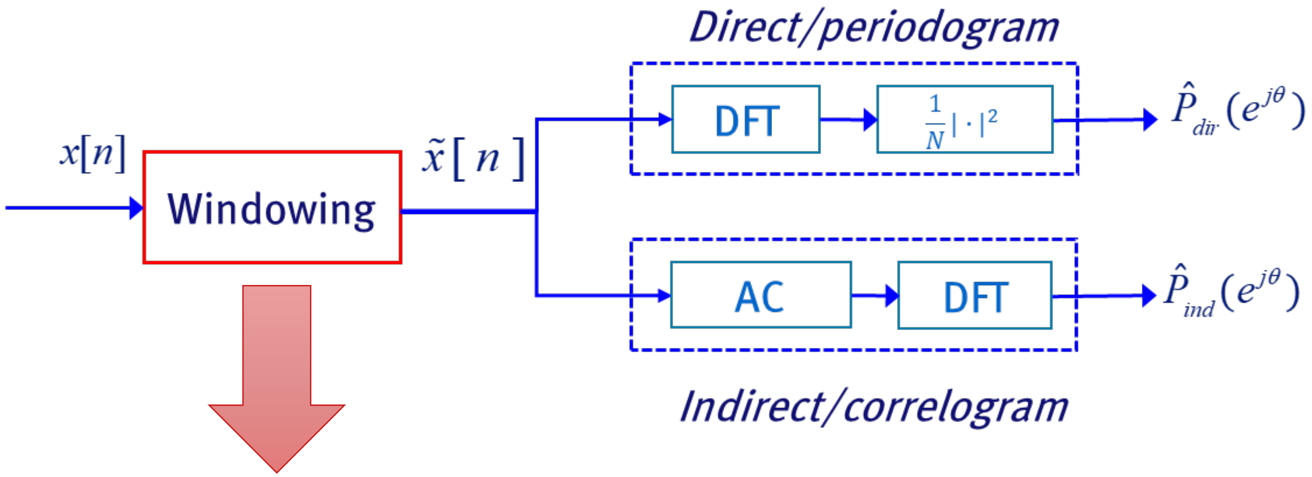
Windowing

Introduction to spectral estimation

Non-parametric spectral estimation: practice

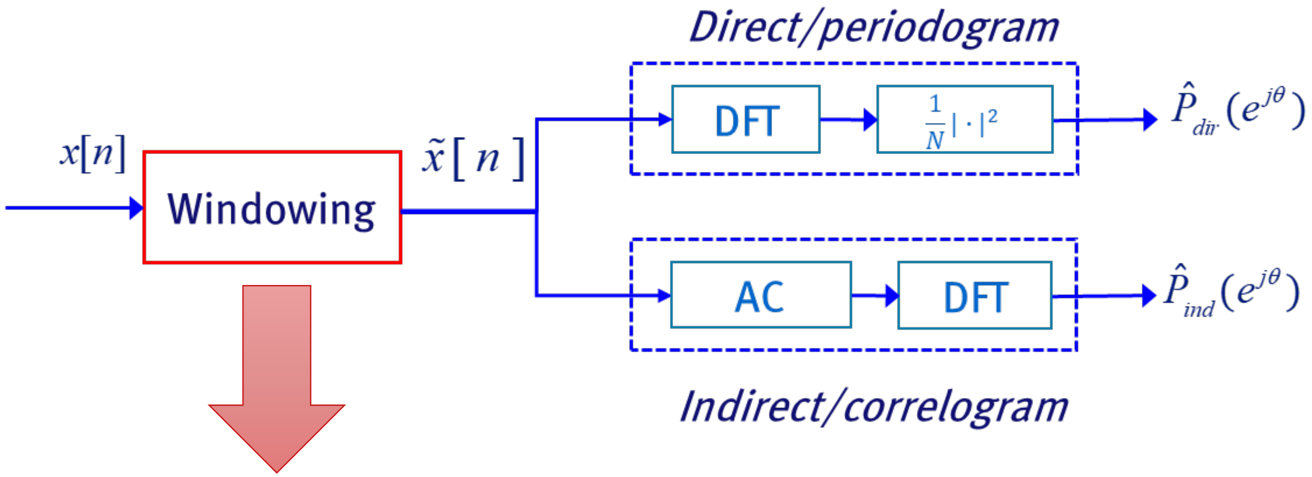


Non-parametric spectral estimation: practice



- Finite number of samples available
- Signal is stationary only in a window
- Need to reduce the computational complexity

Non-parametric spectral estimation: practice



- Finite number of samples available
- Signal is stationary only in a window
- Need to reduce the computational complexity

- Spectrum calculated by N-point DFT
- Spectral leakage
- Loss of resolution

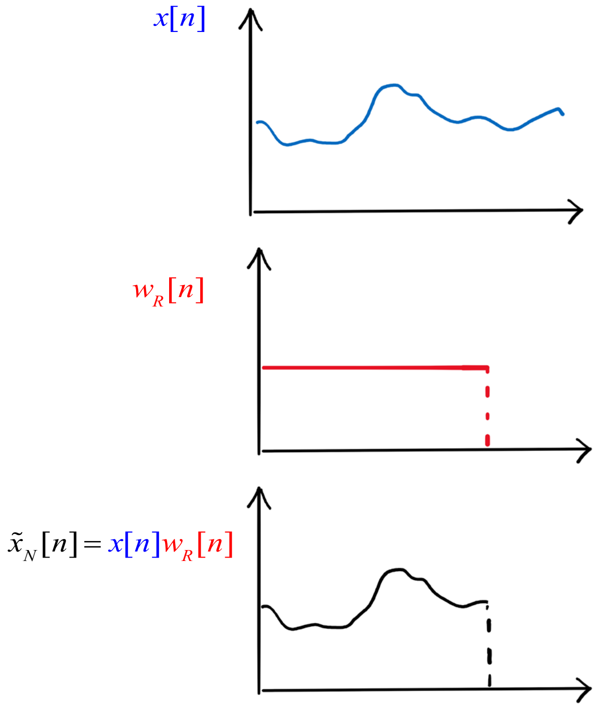
Rectangular window

- The finite length sequence $x_N[n]$ can be expressed as

$$\tilde{x}_N[n] = x[n]w_R[n]$$

with w_R, rectangular window of length N defined as

$$w_R[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases}$$



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$$\tilde{X}_N(e^{j\theta}) = X(e^{j\theta}) * W_R(e^{j\theta}) = \boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) W_R(e^{j(\theta-\phi)}) d\phi}$$

Periodic convolution

Rectangular window

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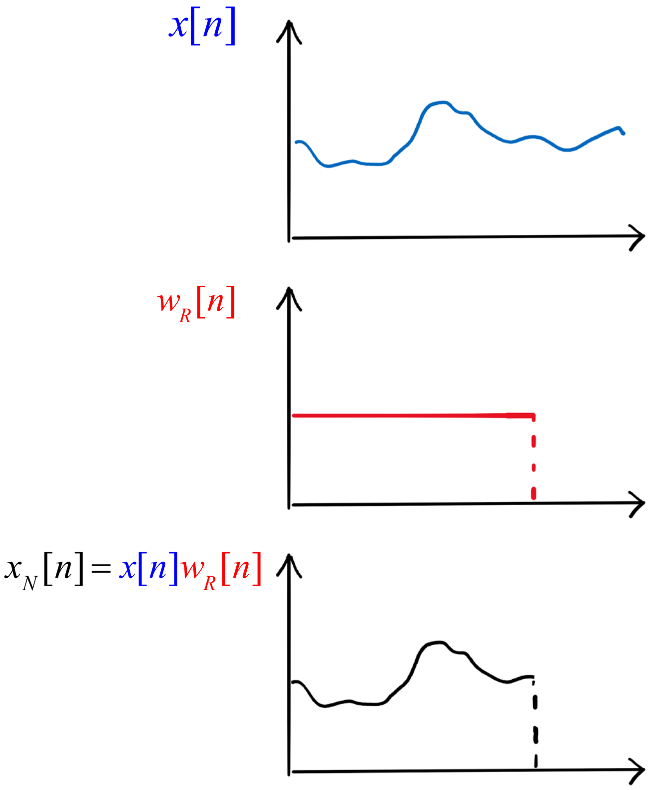
Periodic convolution

- And the DFT of $w_N[n]$ is given by

$$\boxed{W_R(e^{j\theta}) = \frac{\sin(N\theta/2)}{\sin(\theta/2)} e^{j\frac{N-1}{2}\theta}}$$

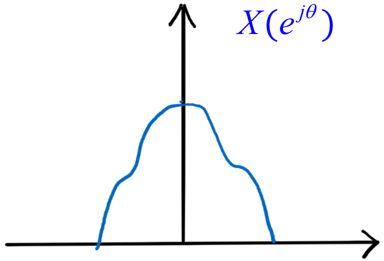
Periodic sinc function

Time domain



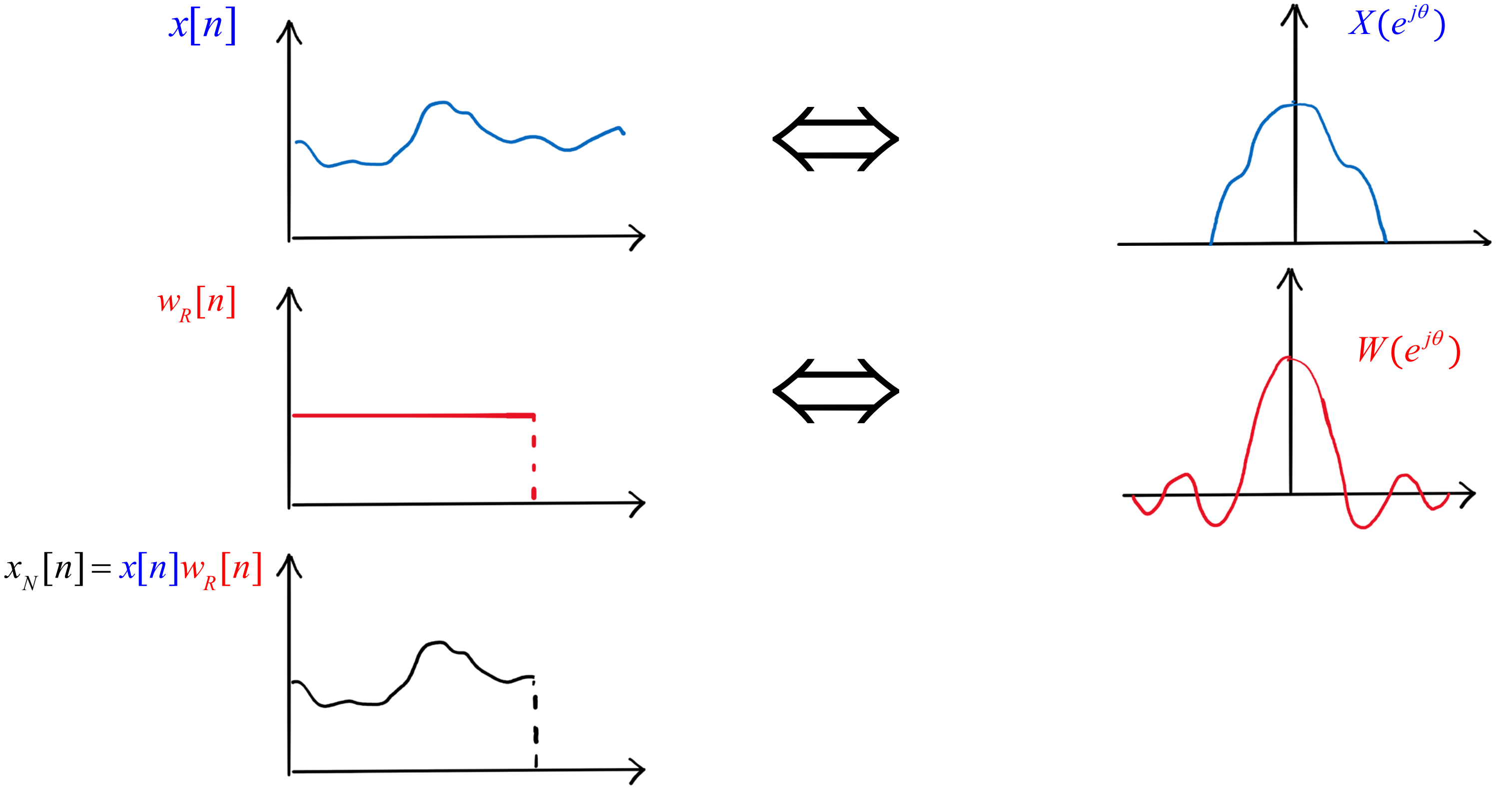
$\Leftrightarrow$

Frequency domain

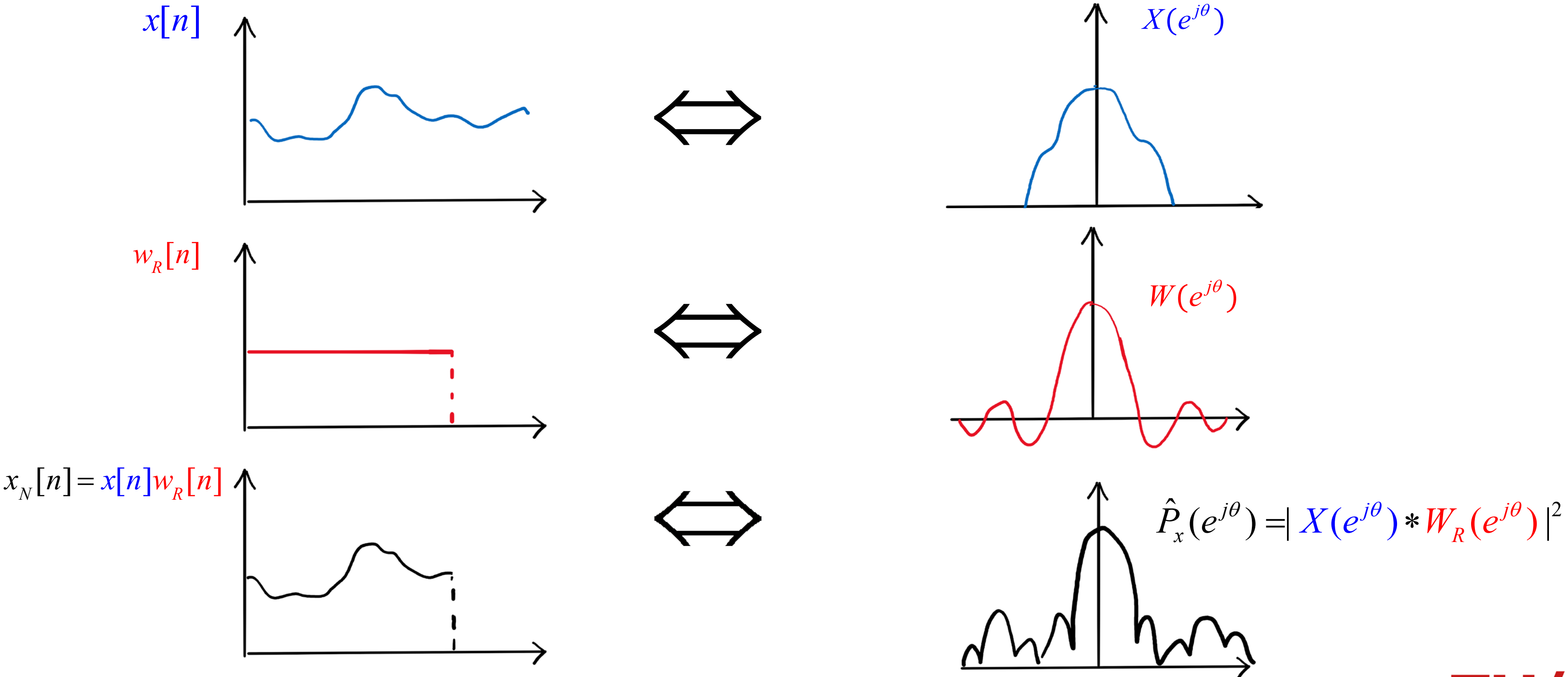


Time domain

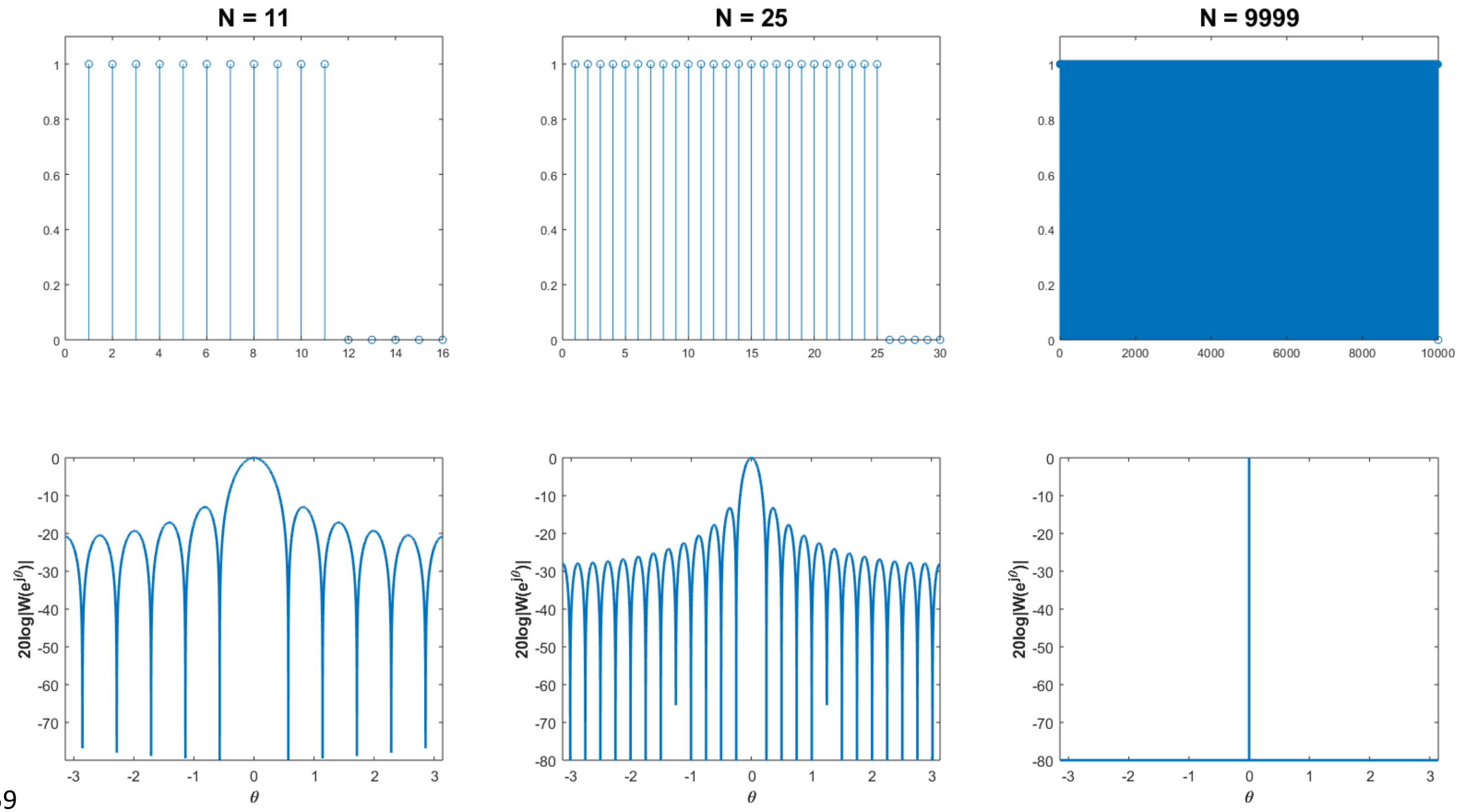
Frequency domain



Time domain **Frequency domain**



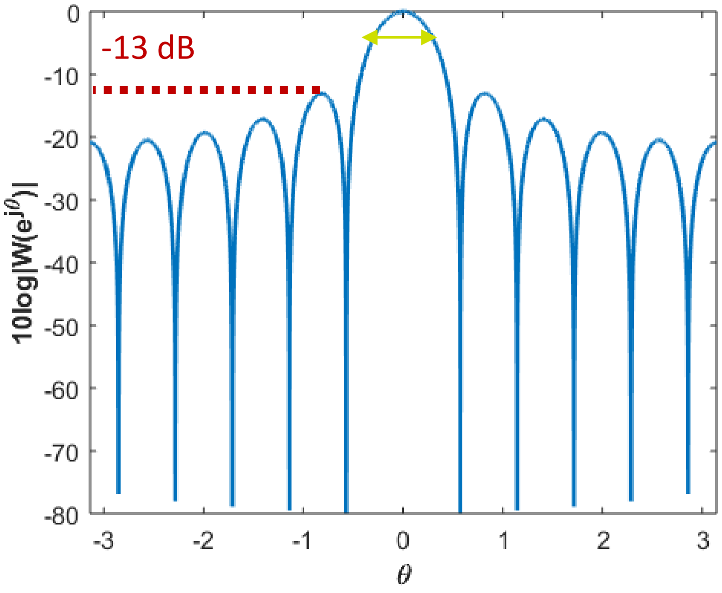
DFT of rectangular window



DFT of rectangular window

N=11

$BW_{3dB} = 0.51$



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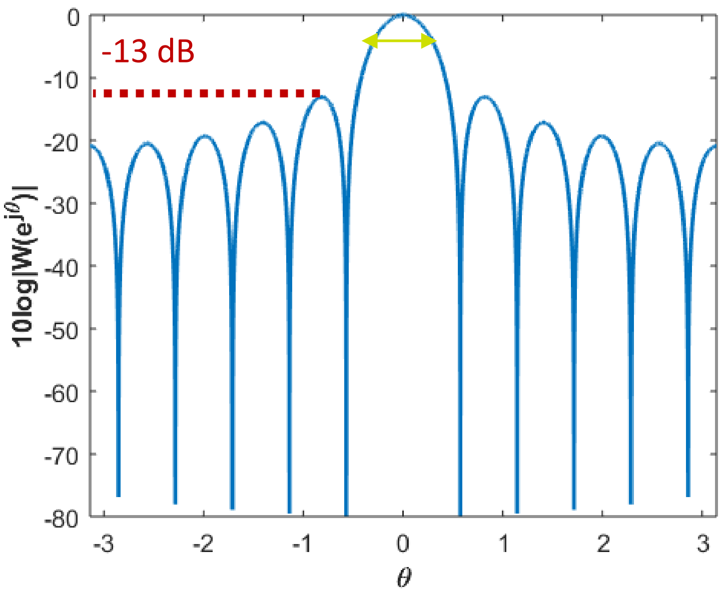
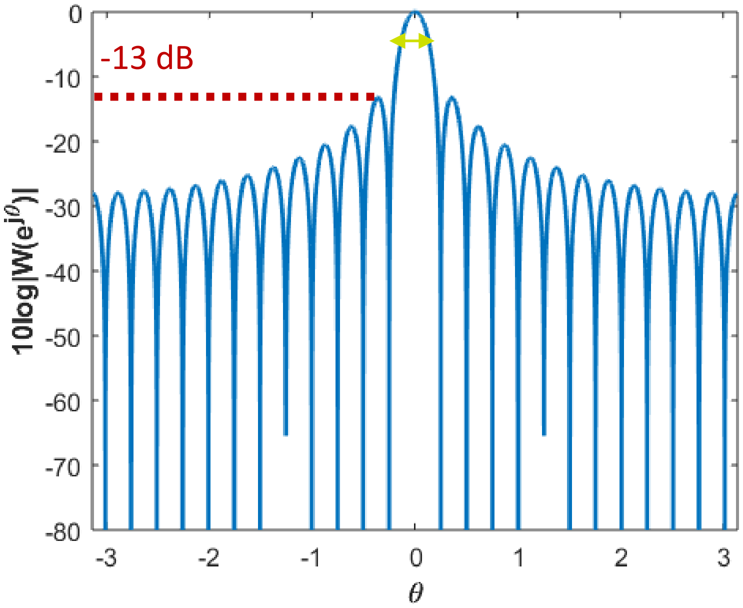


N=25 $BW_{3dB} = 0.22$



- **Main lobe**: $|\theta|<2\pi/N$, 3dB bandwidth (BW_{3dB})=1.81 $\pi/(N-1)$
- **First side lobe**: height of peak independent of N

Resolution and spectral leakage

$$E[\hat{P}_x(e^{j\theta})] = E[|W_R(e^{j\theta}) * X(e^{j\theta})|^2]$$

Window is deterministic

Expectation of $|X(e^{j\theta})|^2$ *is true power spectrum*

Resolution and spectral leakage

$$E[\hat{P}_x(e^{j\theta})] = E[|W_R(e^{j\theta}) * X(e^{j\theta})|^2] = \left(\frac{\sin(N\theta/2)}{\sin(\theta/2)}\right)^2 *_{2\pi} P_x(e^{j\theta})\Bigg|_{\theta=\frac{2\pi}{N}k}$$

Window transform *True spectrum* *Calculated at N points*

Resolution and spectral leakage

$$E[\hat{P}_x(e^{j\theta})] = E[|W_R(e^{j\theta}) * X(e^{j\theta})|^2] = \left(\frac{\sin(N\theta/2)}{\sin(\theta/2)}\right)^2 *_{2\pi} P_x(e^{j\theta})\Bigg|_{\theta=\frac{2\pi}{N}k}$$

$$= P_x(e^{j\theta}) *_{2\pi} W_{ML}(e^{j\theta}) + P_x(e^{j\theta}) *_{2\pi} W_{SL}(e^{j\theta})$$

Resolution and spectral leakage

$$E[\hat{P}_x(e^{j\theta})] = E[|W_R(e^{j\theta}) * X(e^{j\theta})|^2] = \left.\left(\frac{\sin(N\theta/2)}{\sin(\theta/2)}\right)^2 *_{2\pi} P_x(e^{j\theta})\right|_{\theta=\frac{2\pi}{N}k}$$

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$$W_{ML}(e^{j\theta}) = \begin{cases} W_R(e^{j\theta}) & |\theta| < \dfrac{2\pi}{N} \\ 0 & elsewhere \end{cases}$$

Resolution

Resolution and spectral leakage

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Resolution

$$W_{SL}(e^{j\theta}) = W_R(e^{j\theta}) - W_{ML}(e^{j\theta})$$

Spectral leakage

Resolution and spectral leakage

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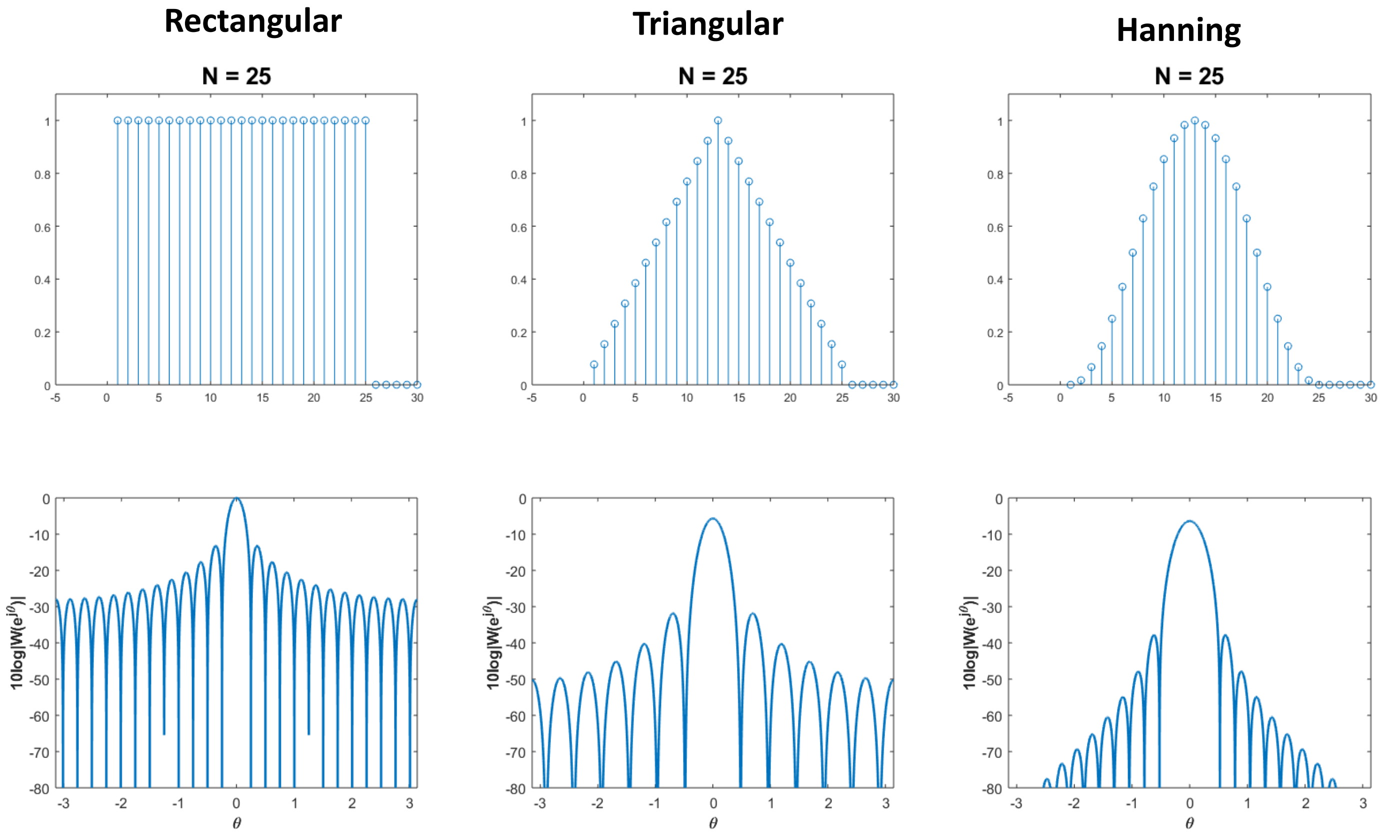
Spectral leakage

- $W_{ML}\ (ej^{\theta})$ smooths rapid variations and suppresses narrow peaks -> **ability to distinguish peaks**
- $W_{SL}\ (ej^{\theta})$ introduces ripples in smoothed regions of $X_N\ (ej^{\theta})$ and can create false peaks
- Options for improvement:
  - Increase N
  - Change window

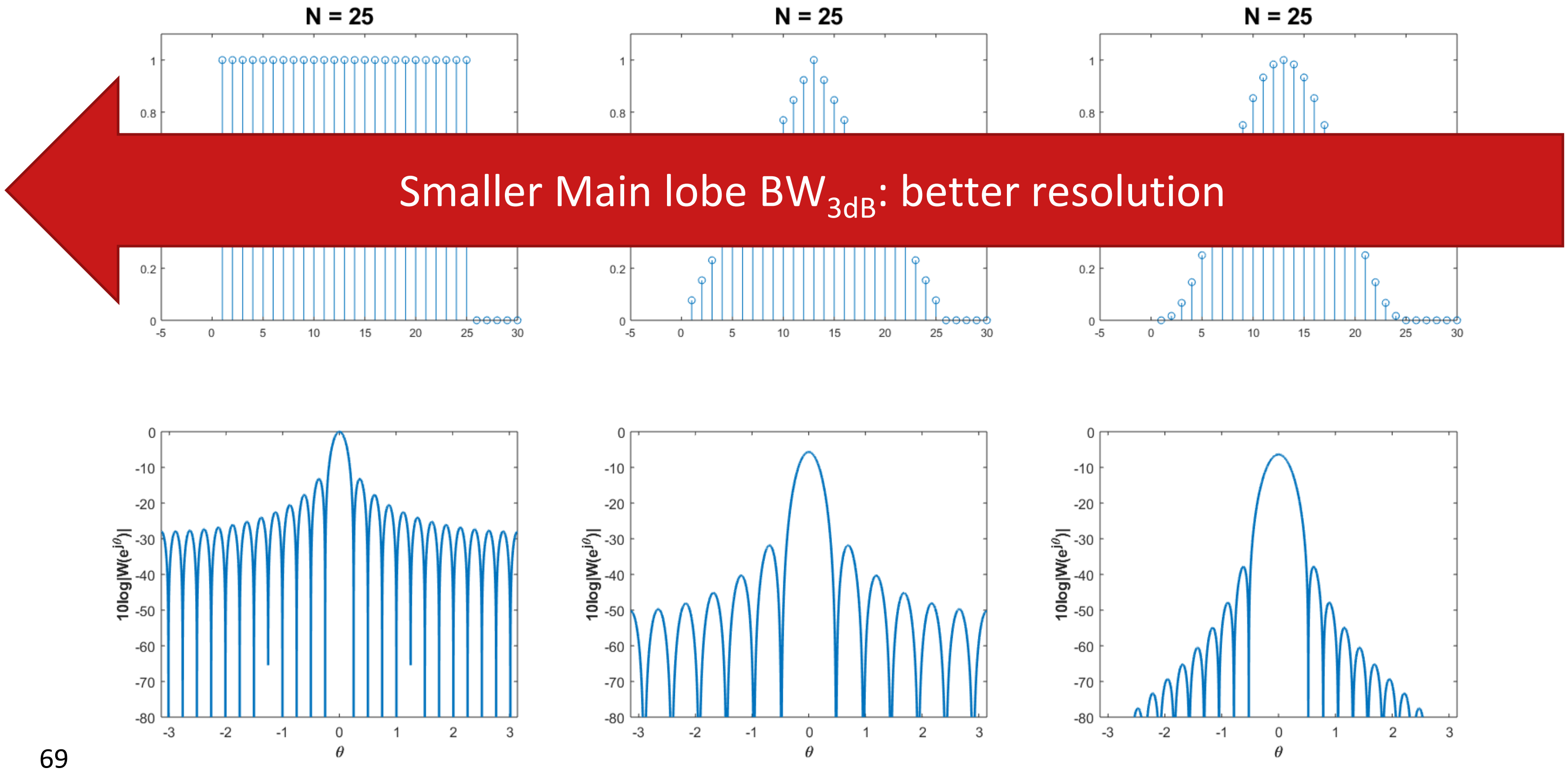
$$w[n] = \begin{cases} g[n] & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases}$$

Different windows

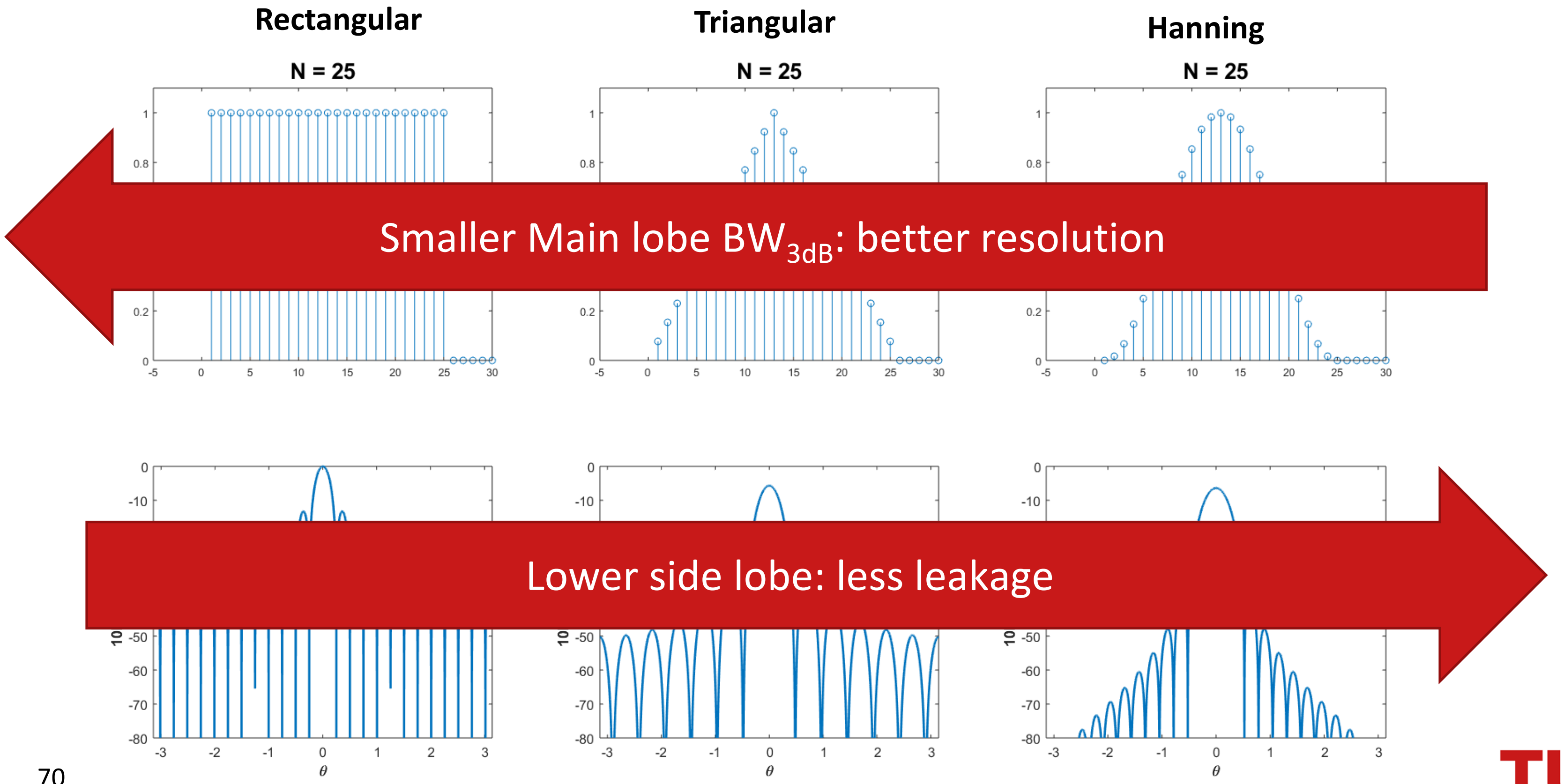


Different windows

Rectangular **Triangular** **Hanning**



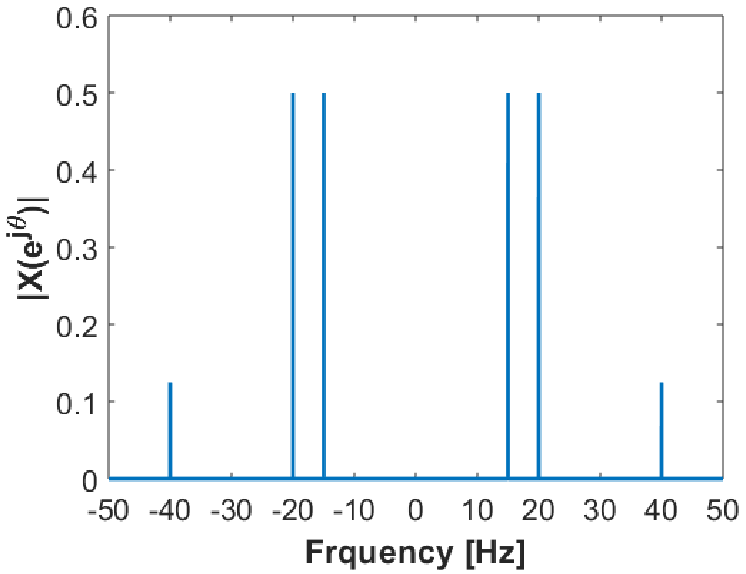
Different windows



Resolution loss: example

$$x[n] = \cos(f_1 / f_s \cdot 2\pi n) + \cos(f_2 / f_s \cdot 2\pi n) + 0.25\cos(f_3 / f_s \cdot 2\pi n)$$

$N \to \infty$



$f_1 = 15\,\text{Hz}$

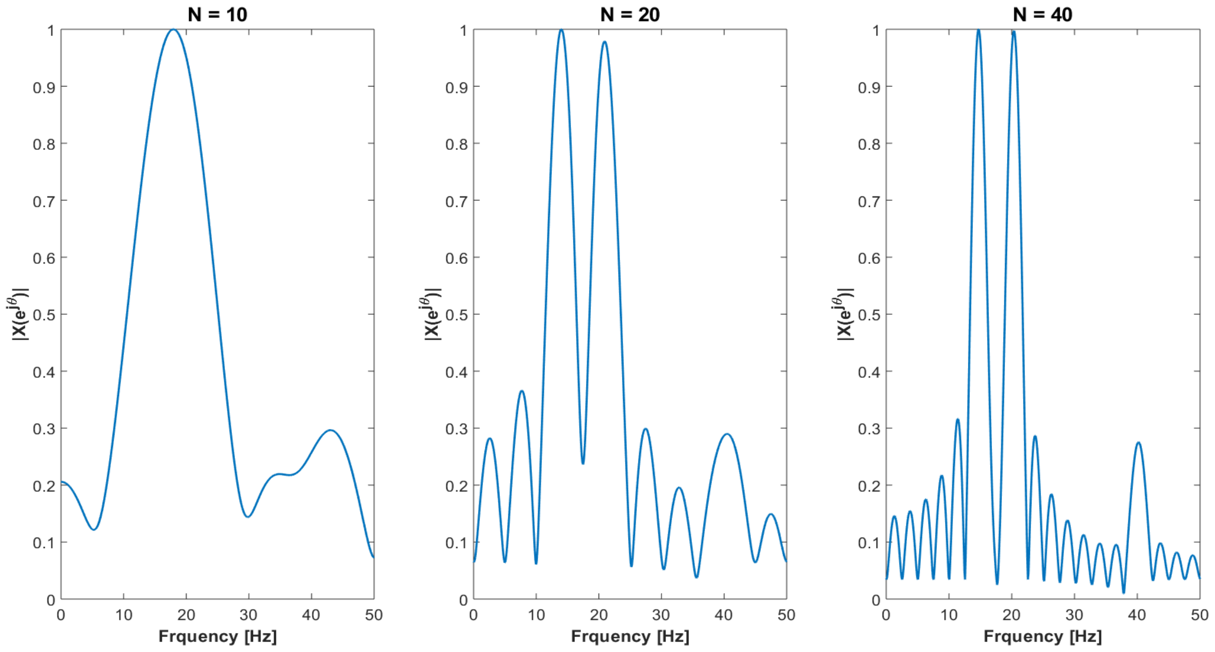
$f_2 = 20\,\text{Hz}$

$f_3 = 40\,\text{Hz}$

$f_s = 100\,\text{Hz}$

Resolution loss: example

Rectangular window



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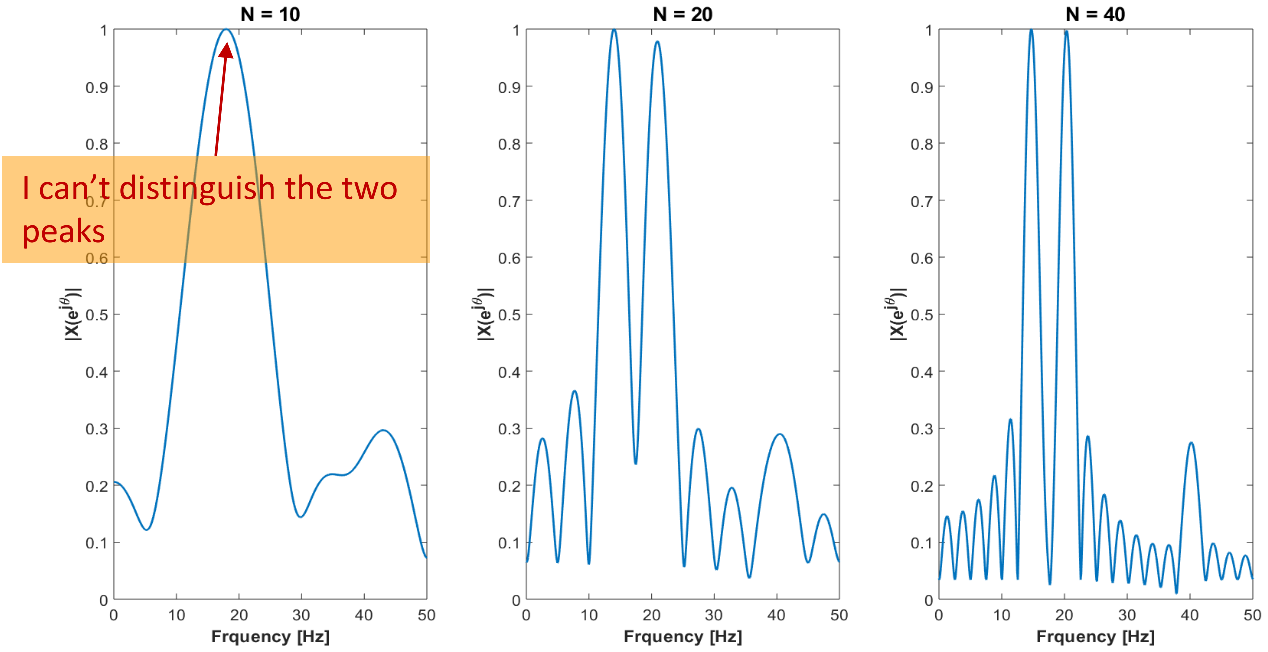

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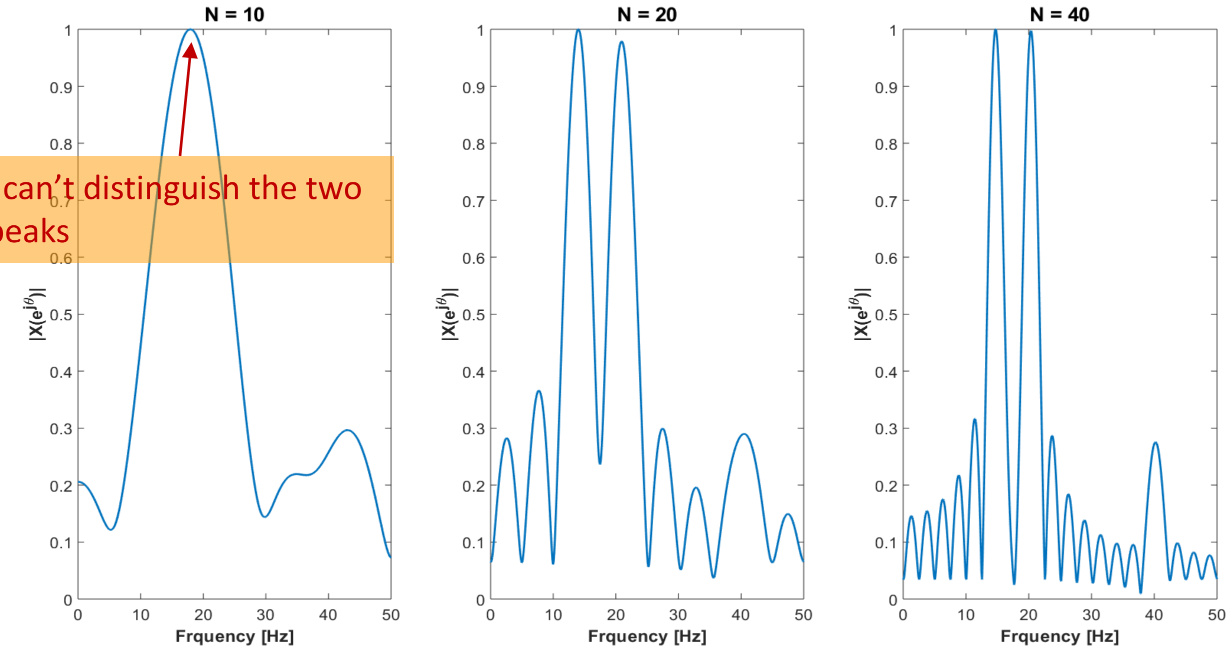


I can't distinguish the two peaks

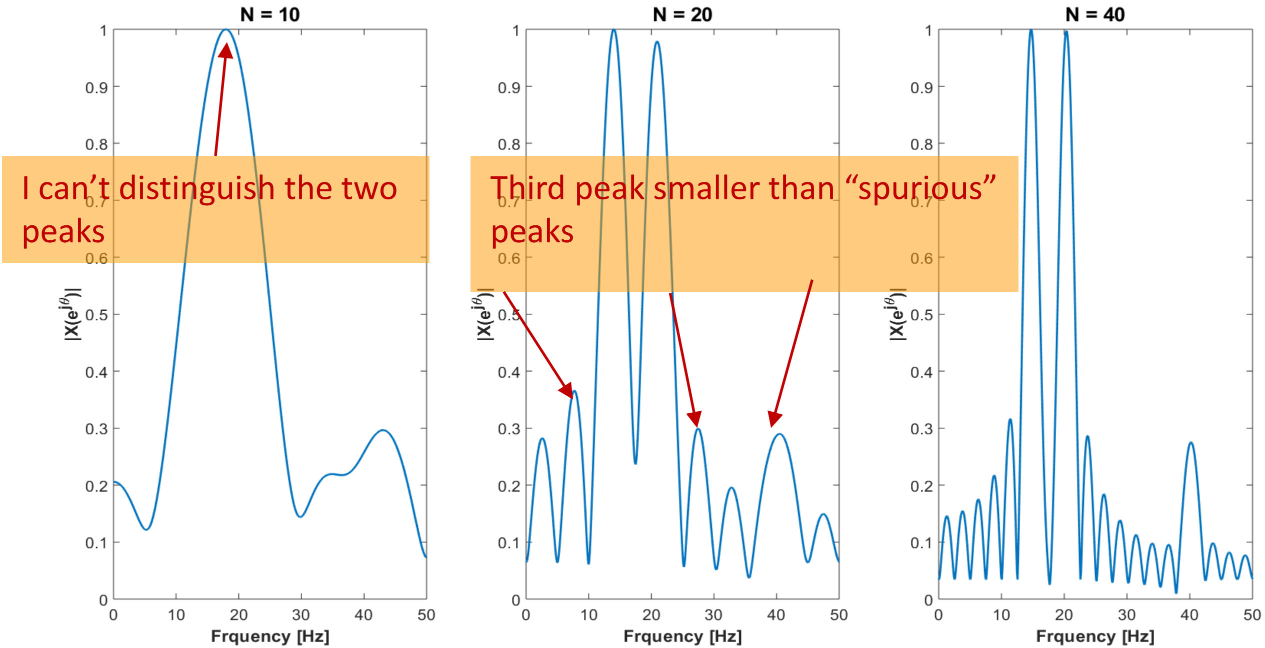
$$f_1 = 15\,\mathrm{Hz}$$
$$f_2 = 20\,\mathrm{Hz}$$
$$f_3 = 40\,\mathrm{Hz}$$

$$f_s = 100\,\mathrm{Hz}$$

$$\frac{1.81\pi}{N-1} < \left| 2\pi\frac{f_1}{f_s} - 2\pi\frac{f_2}{f_s} \right| \quad \Rightarrow N > 19$$

Resolution loss: example

Rectangular window



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Time-Bandwidth trade off

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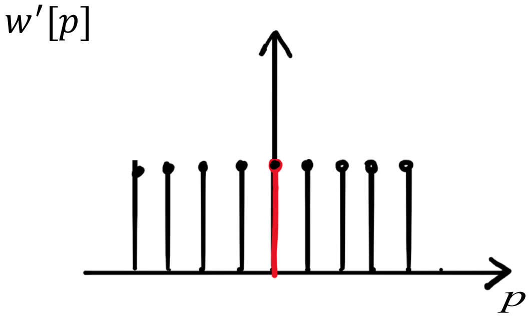

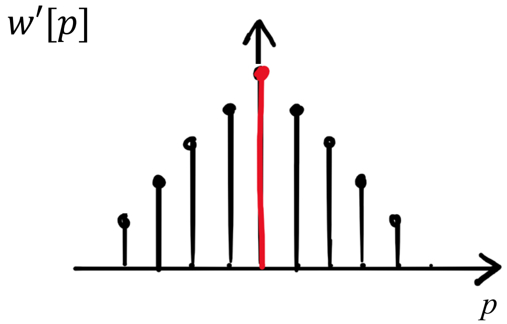

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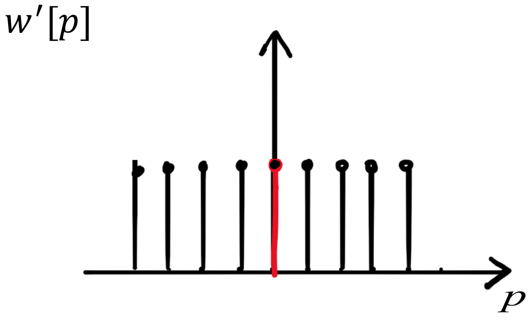

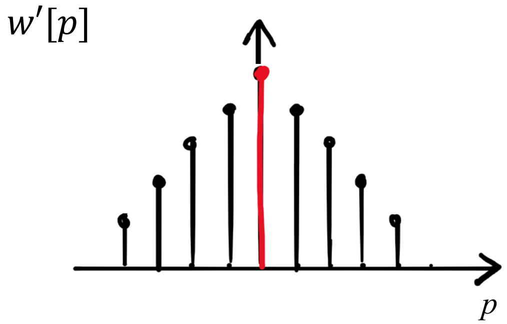

$$N_{eq} = \frac{\sum_{p=-M}^{M} w'[p]}{w'[0]}$$

$$\eta_{eq} = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} W'(e^{j\theta})d\theta}{W'(0)}$$

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Fundamental product

$$\boxed{N_{eq}\eta_{eq} = 1}$$

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	Rectangular	**Triangular**	**Hanning**
Peak side lobe	-13 dB	-27 dB	-32 dB
Main lobe BW_{3dB}	$1.81\frac{\pi}{N-1}$	$5.01\frac{\pi}{N-1}$	$6.27\frac{\pi}{N-1}$

Manolakis, “Statistical and adaptive signal processing”

Windowing: conclusions

- For $N \to \infty, W_R \to \delta$, then $\tilde{X}(e^{j\theta}) \to X(e^{j\theta})$, and thus the spectral content can be reconstructed exactly

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  - Resolution is limited by the window length N
  - Different windows attain different trade-offs between resolution and leakage
- Best practice:
  - Given N, choose window which gives the best compromise between resolution and leakage

Zero padding and resolution

- N-point DFT calculates spectrum at specific frequency bins

$$P[k] = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n] e^{-njk\frac{2\pi}{N}}\right|^2 \triangleq \hat{P}_N(e^{j\theta})\Big|_{\theta = k\frac{2\pi}{N}} \qquad k = 0,1,\ldots,N-1$$

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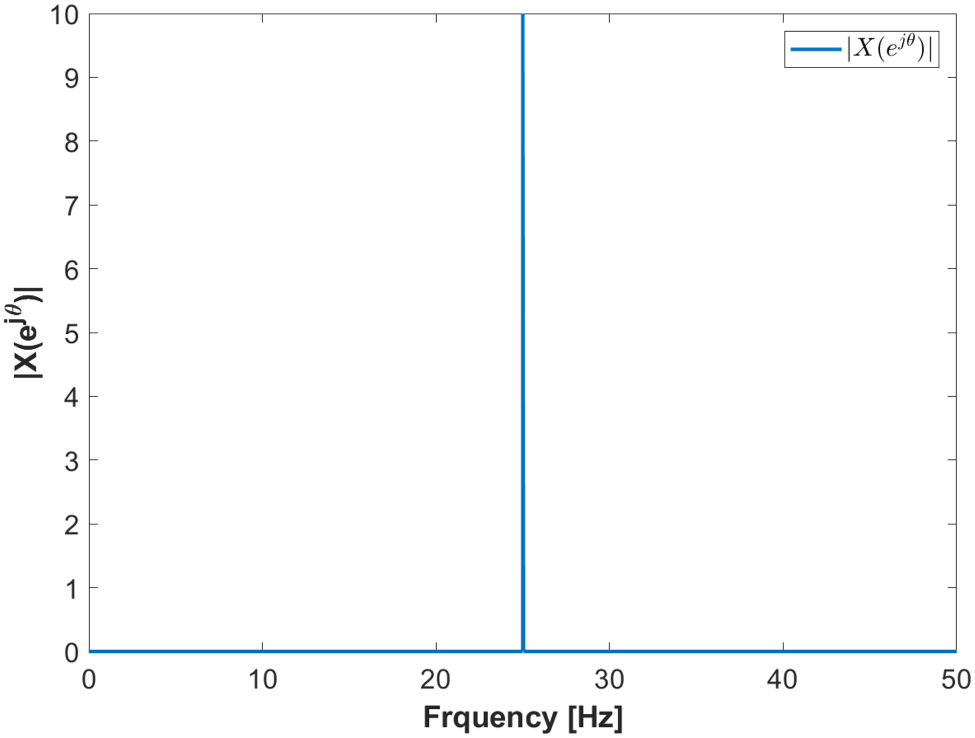
- Zero padding does **NOT** increase spectral resolution, but only provides interpolating values at more frequencies
- Resolution is only determined by N and window shape

Zero padding example

$$x[n] = 20\cos\left(\frac{f_1}{f_s}\cdot 2\pi n\right), f_1 = 25\,\text{Hz}$$



FTD (theoretical)

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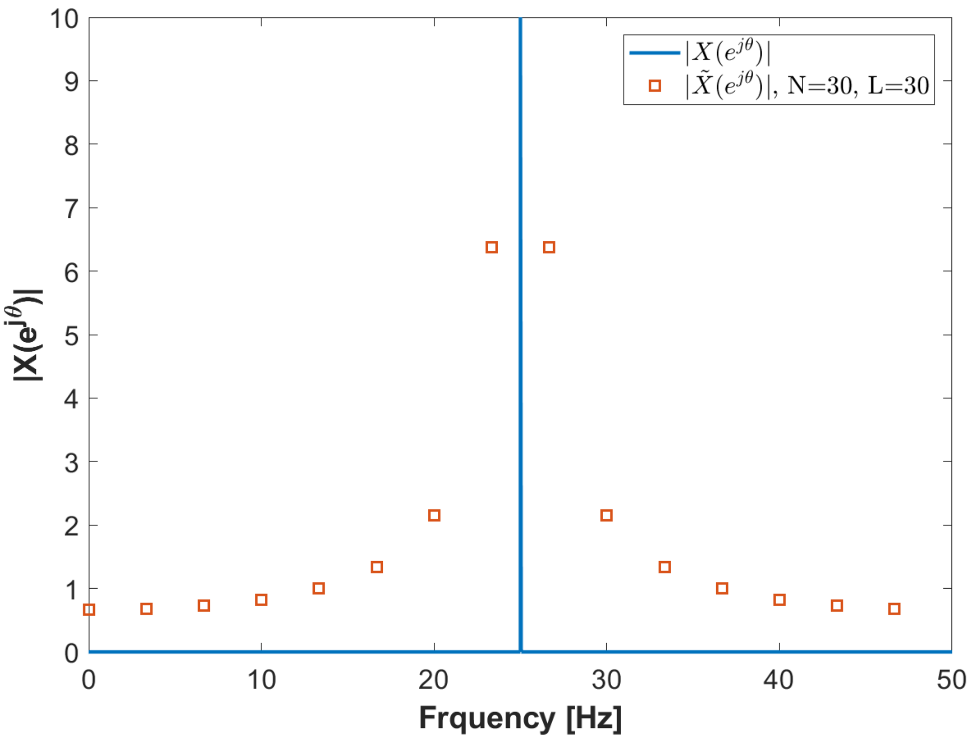

FTD (theoretical)

N-point DFT

TU/e

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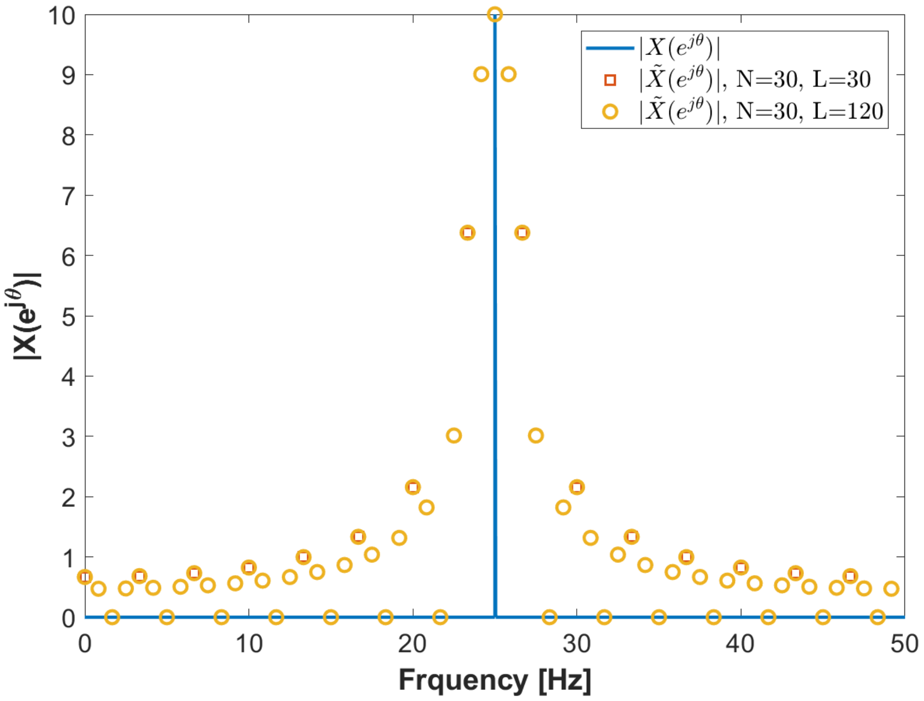

FTD (theoretical)

N-point DFT

L-point DFT

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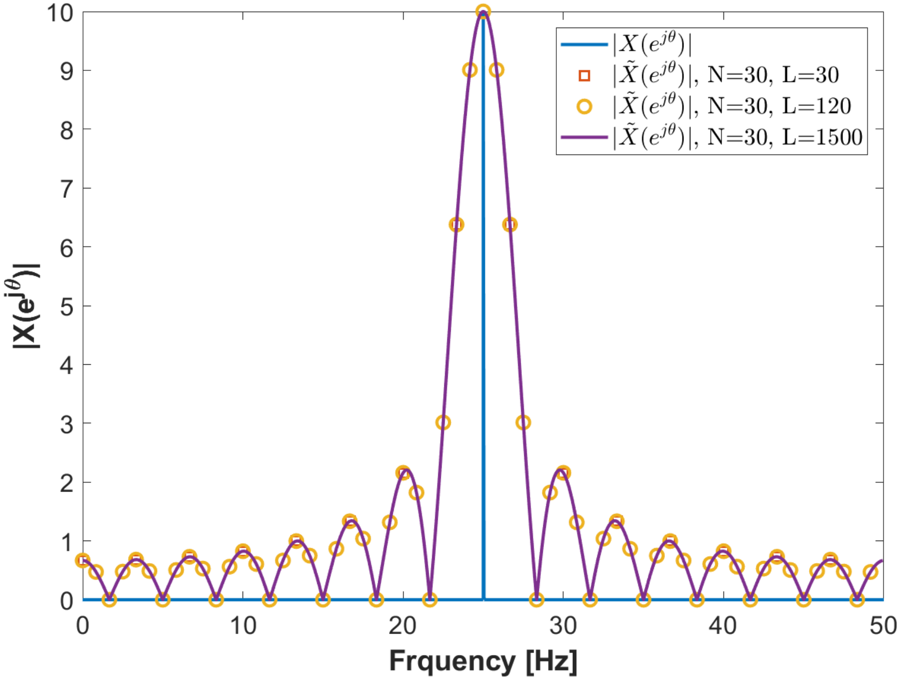

FTD (theoretical)

N-point DFT

L-point DFT

~N-point FTD (approx)

Wrap up (I)

- The goal of **spectral estimation** is to determine the distribution of the signal power over frequencies and has many applications in the engineering, economic and medical fields.
- In practice, we deal with **windowed** aperiodic power signals. As a result, we can obtain only an **estimate** of the power spectral density that is affected by **loss of resolution** and **spectral leakage**.
- The **estimate** of the power spectrum is given by the true spectrum convoluted by the transform of the window function

Wrap up (II)

- The **main lobe width** (3dB bandwidth) of the window transform affects the **resolution** of the spectral estimate and is inversely proportional to the length of the window (for any window)
- The height of the **side lobes** causes **spectral leakage**, that is the appearance of ripple and possibly spurious spectral peaks; this generally does not change with the length of the window
- Different windows attain different **trade-offs** in terms of resolution and leakage
- **Zero padding** does not improve spectral resolution, but allows to sample the spectral estimate at a finer frequency step

Statistical signal processing (5CTA0)

Introduction to spectral estimation

Lecturer: Simona Turco

Electrical Engingeering, Signal Processing Systems group