Statistical signal processing (5CTA0)

Parametric spectral estimation

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Part 1: Random variables and Random Signals

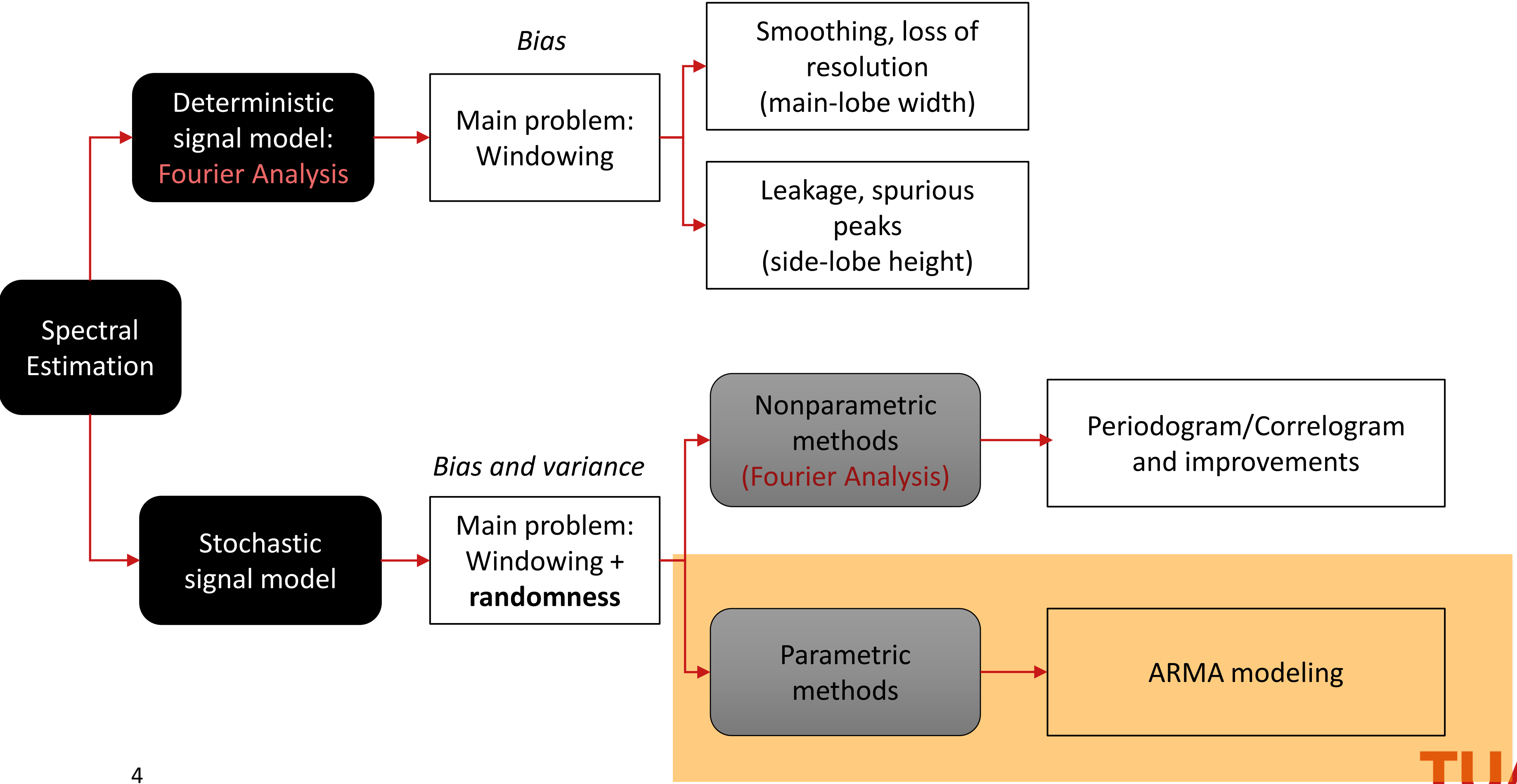
Part 3

Spectral estimation

3.1: Introduction to spectral estimation
3.2: Non-parametric spectral estimation
3.3: Parametric spectral estimation

Outline

- Introduction
- AR spectral estimation
- Model order selection
- MA spectral estimation
- ARMA spectral estimation


Spectral Estimation
Deterministic signal model: Fourier Analysis
Bias
Main problem: Windowing
Smoothing, loss of resolution (main-lobe width)
Leakage, spurious peaks (side-lobe height)
Stochastic signal model
Bias and variance
Main problem: Windowing + randomness
Nonparametric methods (Fourier Analysis)
Periodogram/Correlogram and improvements
Parametric methods
ARMA modeling

Spectral analysis: overview of methods

Non-parametric:

- Classic approach based on FTD + windowing
- Estimation from (finite) signal samples
- No prior assumption on mechanism that correlates the samples

Parametric:

- Based on signal model
- Exploiting knowledge (or guess) of correlation structure in signal
- Reduces to estimating parameters from model

Signal modeling

Choose model structure and order

STEP 1
Model selection

Focus on ARMA

Signal modeling

Choose model structure and order → Estimate model parameters

STEP 1
Model selection

STEP 2
Model estimation

Focus on ARMA

Focus on LS

Signal modeling

Choose model structure and order → Estimate model parameters → Check candidate model performance

STEP 1
Model selection

STEP 2
Model estimation

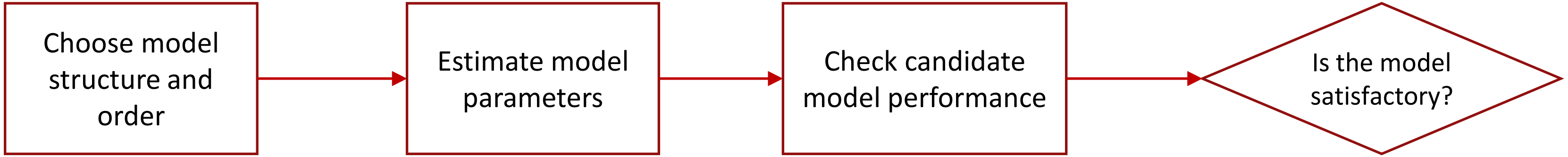
STEP 3
Model validation

Focus on ARMA

Focus on LS

R^2,FPE, AIC

Signal modeling



STEP 1
Model selection

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Model estimation

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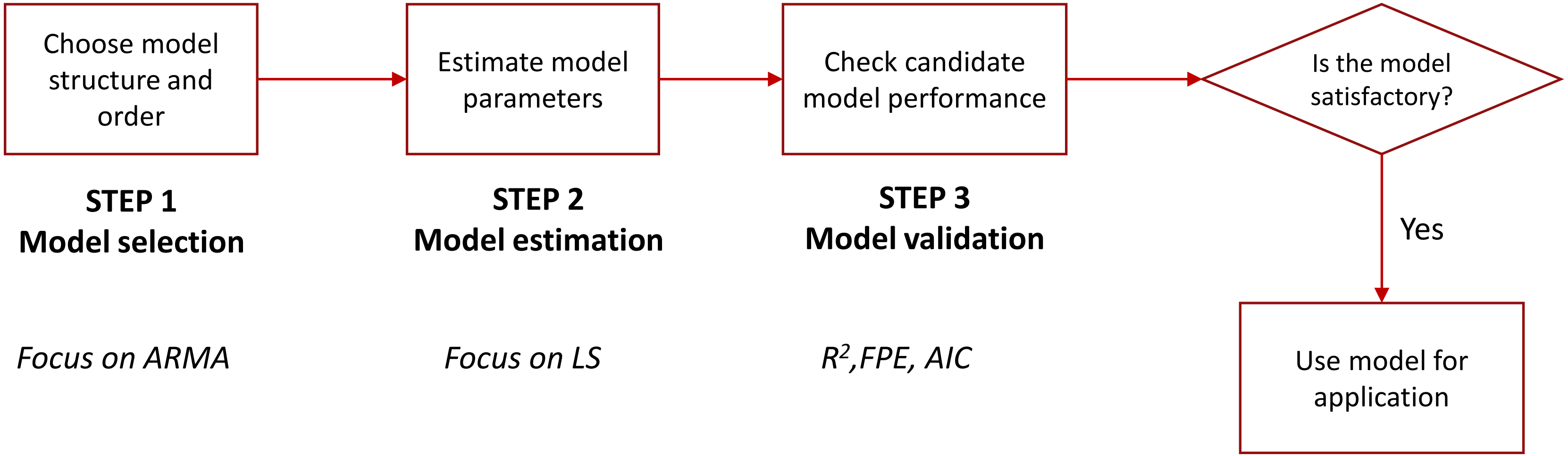
Model validation

Focus on ARMA

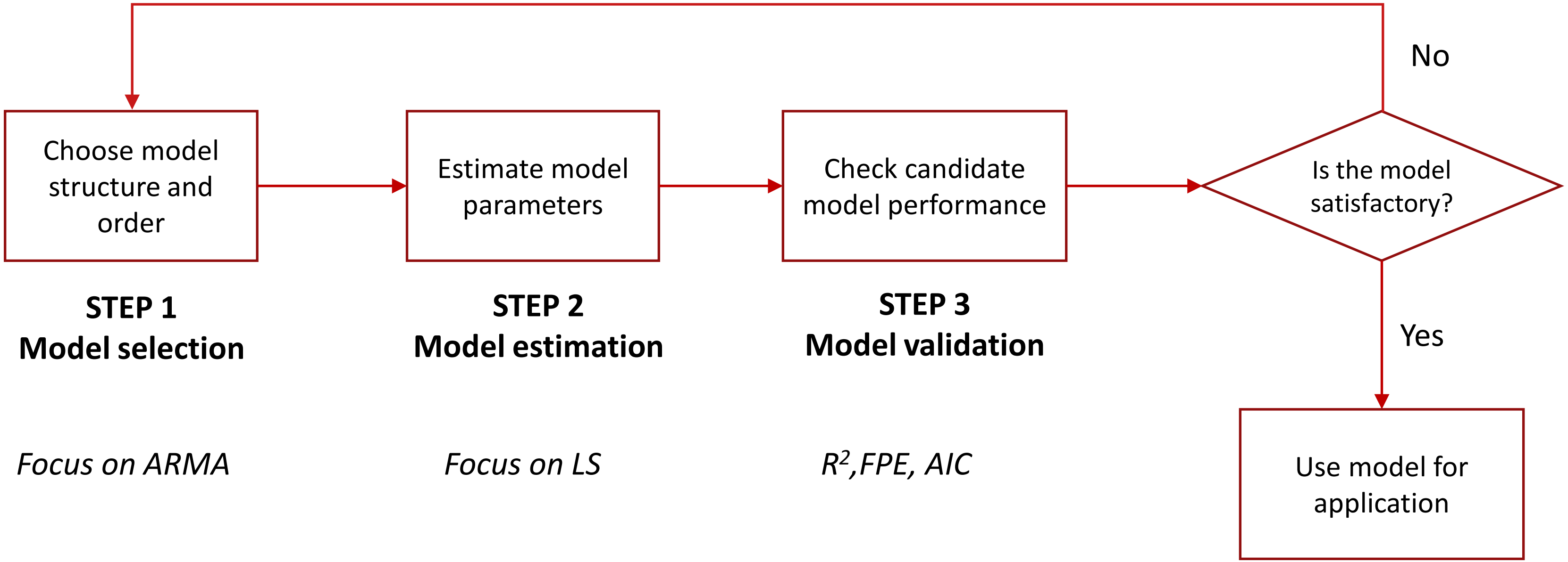
Focus on LS

R^2,FPE, AIC

Signal modeling



Signal modeling

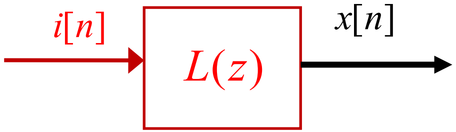


AR Spectral estimation

Parametric spectral estimation

AR spectral estimation

- Assume $x[n]$ is generated by driving a LTI system represented by P-th order AR model with WGN



AR spectral estimation

- Assume $x[n]$ is generated by driving a LTI system represented by P-th order AR model with WGN

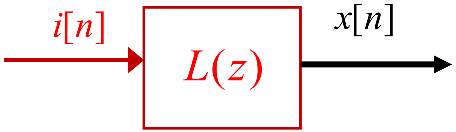
$$i[n] \longrightarrow \boxed{L(z)} \xrightarrow{x[n]}$$

$$x[n] = i[n] - \sum_{p=1}^{P} a_p x[n-p] = i[n] + \hat{x}[n]$$

$$= i[n] - a_1 x[n-1] - a_2 x[n-2] - \ldots - a_p x[n-p]$$

AR spectral estimation

- Assume $x[n]$ is generated by driving a LTI system represented by P-th order all-poles model with WGN



$$x[n] = i[n] - \sum_{p=1}^{P} a_p x[n-p] = i[n] + \hat{x}[n]$$

$$= i[n] - a_1 x[n-1] - a_2 x[n-2] - \ldots - a_p x[n-p]$$

$$L(z) = \frac{1}{1 + \sum_{p=1}^{P} a_p z^{-p}} \quad \Rightarrow \quad P(e^{j\theta}) = \frac{\sigma_i^2}{\left|1 + \sum_{k=1}^{P} a_p e^{-jp\theta}\right|^2}$$

- Spectral estimation reduces to finding P+1 parameters: $[a_1, \ldots, a_P]$ and ${\sigma_i}^2$

AR spectral estimate via Yule-Walker

$$E\{(x[n]-\hat{x}[n])^2\} = E\{(x[n]+\sum_{p=1}^{P} a_p x[n-p])^2\} = E\{i^2[n]\} = \hat{\sigma}_i^2$$

$$r_x[l] = \begin{cases} \sigma_w^2 - \sum_{k=1}^{p} a_k r_x[|l|-k] & \text{for } l=0 \\ -\sum_{k=1}^{p} a_k r_x[|l|-k] & \text{for } l>0 \end{cases}$$

Step 1
Calculate $r[0], r[1], \dots, r[P]$ from data
Step 2
Obtain P+1 equations from Yule-Walker

$$r[0] = \sigma_w^2 - a_1 r[-1] - a_2 r[-2] - \dots - a_P r[-P]$$

$$r[1] = 0 - a_1 r[0] - a_2 r[-1] - \dots - a_P r[-P+1]$$

$$\vdots$$

$$r[P] = 0 - a_1 r[P-1] - a_2 r[P-2] - \dots - a_P r[0]$$

Yule Walker for AR(P) model

$$r[-l] = r[l]$$

For AR(P) models, Yule-Walker equations can be rewritten as

$$\begin{pmatrix} \sigma_i^2 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r[0] & r[1] & \dots & r[P-1] & r[P] \\ r[1] & r[0] & \dots & \dots & r[P-1] \\ \dots & \dots & \dots & \dots & \dots \\ r[P-1] & r[0] & \dots & r[0] & r[1] \\ r[P] & r[P-1] & \dots & r[1] & r[0] \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ \dots \\ a_{P-1} \\ a_p \end{pmatrix}$$

$$\mathbf{r}_w = \mathbf{R}\mathbf{a} \quad \Rightarrow \quad \hat{\mathbf{a}} = \mathbf{R}^{-1}\mathbf{r}_w$$

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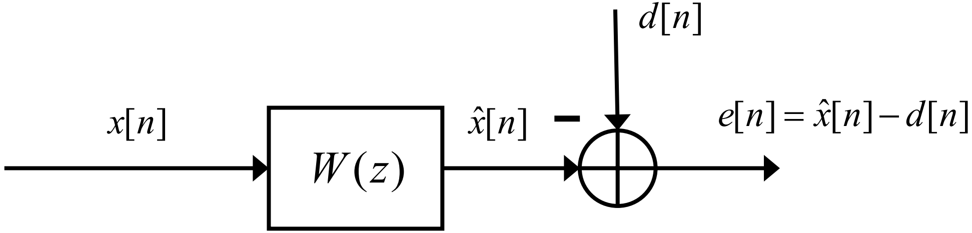
$$\mathbf{r}_w = \mathbf{R}\mathbf{a} \quad \Rightarrow \quad \hat{\mathbf{a}} = \mathbf{R}^{-1}\mathbf{r}_w$$

- AC matrix **R** has **Toeplitz** structure

Wiener filters (FIR)

- Wiener filters minimize minimum mean squared error (MMSE criterion): $\mathbf{w}_o = \arg\min_w (E\{e^2[n]\})$



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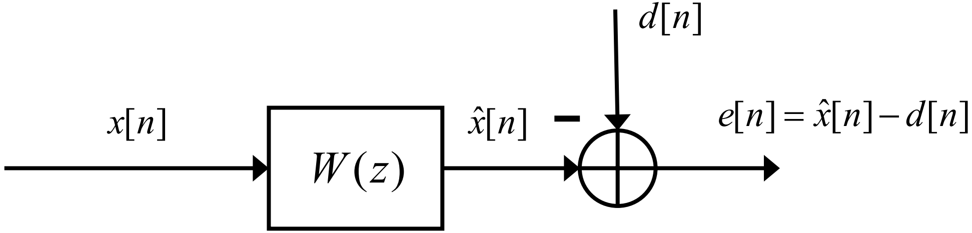

$$J = E\{e^2[n]\} = E\{(d[n] - \hat{x}[n])^2\} = E\{d^2[n]\} - \mathbf{w}^T \mathbf{r}_{dx} - \mathbf{r}_{dx}{}^T \mathbf{w} + \mathbf{w}^T \mathbf{R}_x \mathbf{w}$$

$$\frac{dJ}{d\mathbf{w}} = -2(\mathbf{r}_{dx} - \mathbf{R}_x \mathbf{w}) = \mathbf{0} \quad \Rightarrow \quad \mathbf{R}_x \mathbf{w} = \mathbf{r}_{dx}$$

Normal equations

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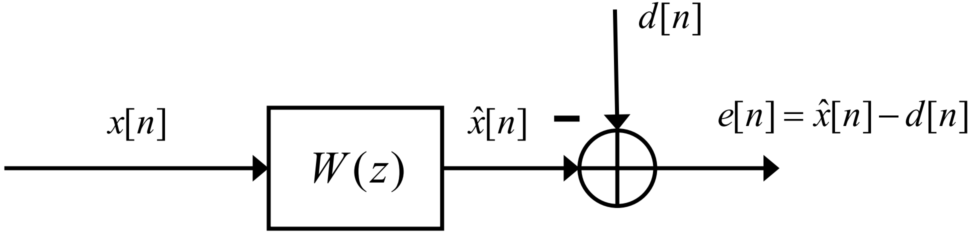

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Normal equations

Wiener FIR Filter

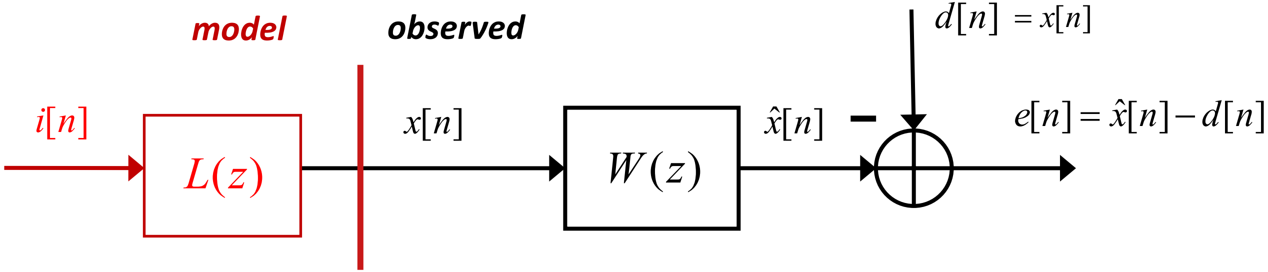
$$\boxed{\mathbf{w}_{opt} = \mathbf{R}_x^{-1} \mathbf{r}_{dx}}$$

Filter error

$$\boxed{J_{FIR} = r_d[0] - \sum_{i=0}^{N-1} w_{opt}[i] r_{dx}[i]}$$

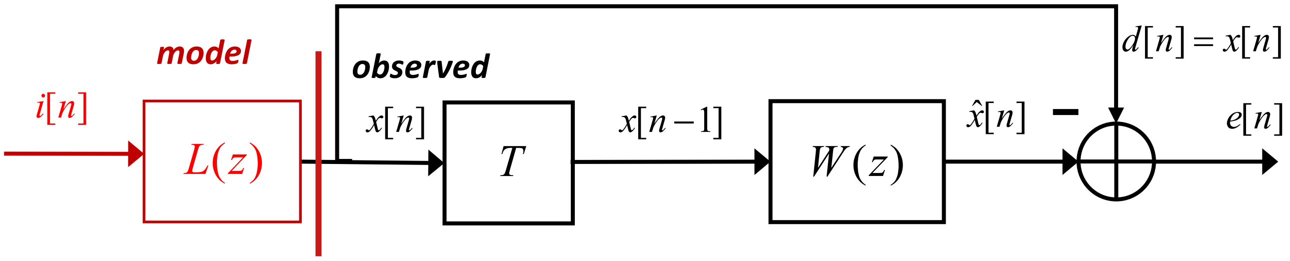
Wiener filter for linear prediction

- Wiener filters minimize minimum mean squared error (MMSE criterion): $\mathbf{w}_o = \arg\min_w (E\{e^2[n]\})$



1-step linear prediction for AR spectral estimate

- 1-step forward prediction: given an input signal $x[n]$, predict n^{th} sample based on previous $N-1$ samples



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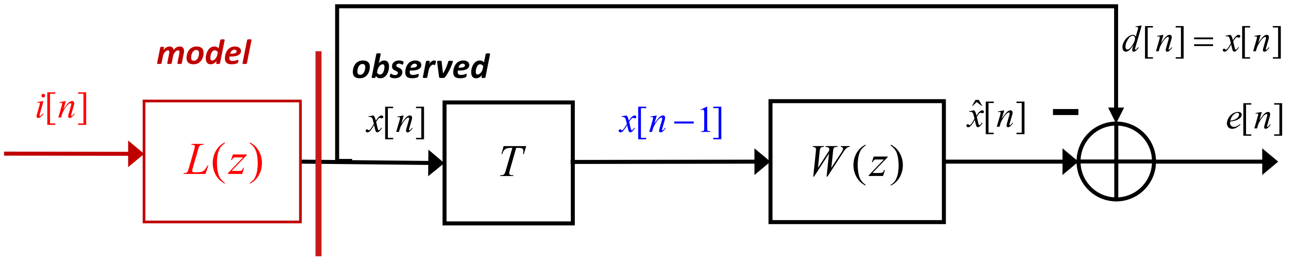

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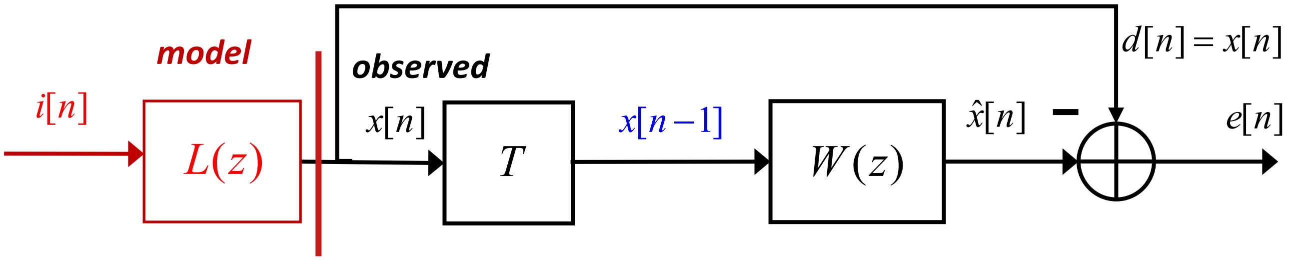

$$\mathbf{x}=[x[0],x[1],\ldots,x[N-1]];$$
$$\mathbf{d}=[x[1],x[2],\ldots,x[N-1]];$$
$$\mathbf{x}'=[x[0],x[1],\ldots,x[N-2]];$$

1-step linear prediction for AR spectral estimate

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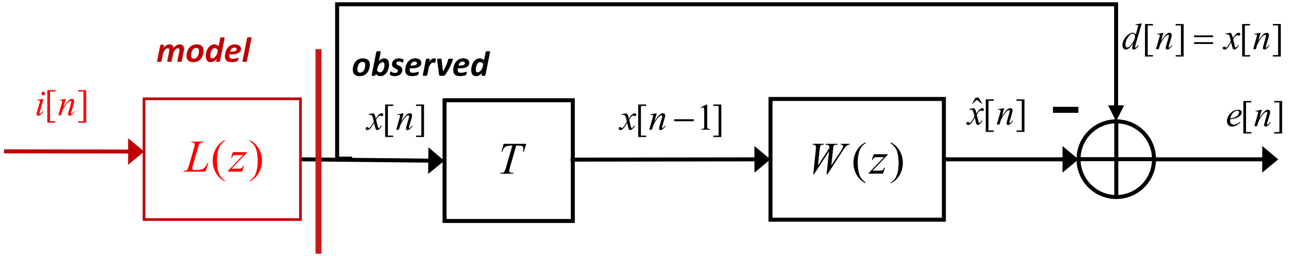

FIR: $\boxed{\mathbf{w}_{opt} = \mathbf{R}_x^{-1}\mathbf{r}}$ $\boxed{J_{\min} = r_x[0] - \mathbf{w}_{\mathbf{opt}}^{\mathbf{T}}\mathbf{r} = \hat{\sigma}_i^2}$

$r_x[l] = E\{x^2[n-1]\}$

$r_{dx}[l] = E\{x[n]x[n-1]\} = r_x[l-1]$

1-step linear prediction for AR spectral estimate

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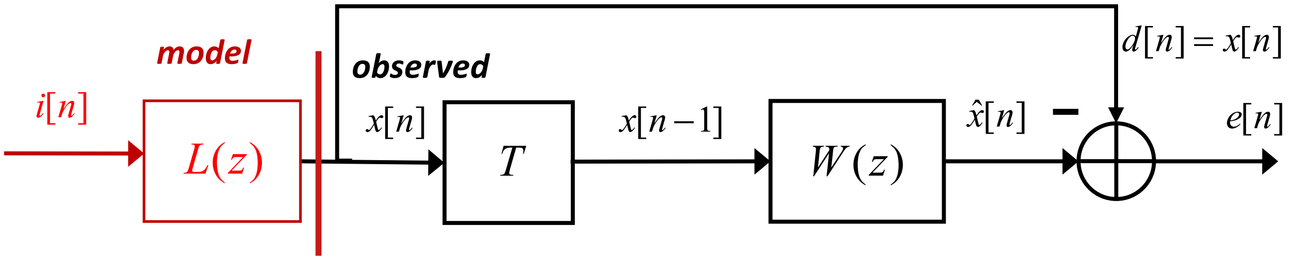

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$$r_{dx}[l] = E\{x[n]x[n-1]\} = r_x[l-1]$$

$$\mathbf{x} = [x[n-1],\dots,x[n-N+1]]^T$$

$$\mathbf{R}_{\mathbf{x}} = \mathrm{Toeplitz}\{r_x[0],\dots,r_x[N-2]\}$$

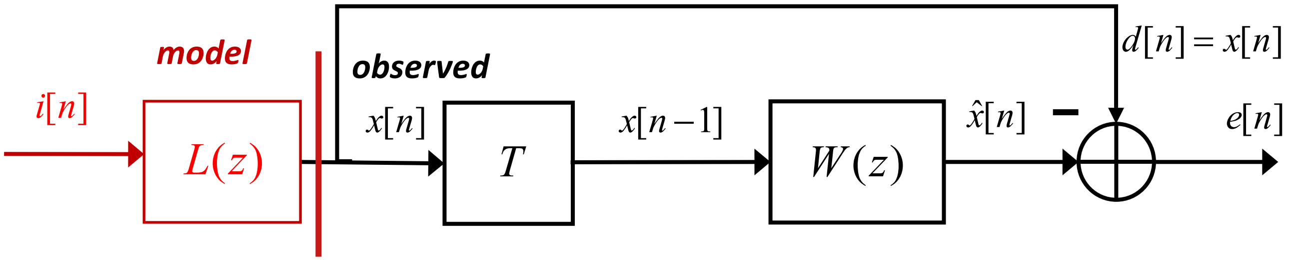
$$\mathbf{r} = [r_x[1],\dots,r_x[N-1]]^T$$

$$\mathbf{w}_{\mathbf{opt}} = [w_1, w_2, \dots, w_{N-1}]^T$$

FIR with N-1 coefficients

1-step linear prediction for AR spectral estimate

- 1-step forward prediction: given an input signal $x[n]$, predict n^{th} sample based on previous $N-1$ samples



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$$\mathbf{r} = [r_x[1],\dots,r_x[N-1]]^T$$

$$\mathbf{R}_\mathbf{x} = \text{Toeplitz}\{r_x[0],\dots,r_x[N-2]\}$$

$$\mathbf{w}_{\mathbf{opt}} = [w_1, w_2, \dots, w_{N-1}]^T = [-\hat{a}_1, -\hat{a}_2, \dots, -\hat{a}_p]^T$$

FIR with N-1 coefficients

AR spectral estimate

Estimate autocorrelation $\hat{r}[l]$ for $|l| \leq P$

AR spectral estimate

Estimate autocorrelation $\hat{r}[l]$ for $|l| \leq P$ → Calculate model parameters $\hat{a}_1, \ldots, \hat{a}_p$ and $\hat{\sigma}_i^2$ via Yule Walker (or via Wiener filter)

AR spectral estimate

Estimate autocorrelation $\hat{r}[l]$ for $|l| \leq P$

→

Calculate model parameters $\hat{a}_1, .., \hat{a}_p$ and $\hat{\sigma}_i^2$ via Yule Walker (or via Wiener filter)

→

1 Compute $\hat{P}\left(e^{j\theta}\right)$ as

$$\hat{P}_{AR}(e^{j\theta}) = \frac{\hat{\sigma}_i^{\,2}}{|1 + \hat{a}_1 e^{-j\theta} + \hat{a}_2 e^{-j2\theta} + \ldots + \hat{a}_P e^{-jP\theta}|^2}$$

AR spectral estimate

Estimate autocorrelation $\hat{r}[l]$ for $|l| \le P$

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Via DFT/FFT

Apply zero padding (L)

$$\hat{A}_L(e^{j\theta})|_{\theta=l\frac{2\pi}{L}} = \sum_{p=0}^{P} \hat{a}_p e^{-jpl\frac{2\pi}{L}}, \quad \hat{a}_0 = 1$$

2 Compute $\hat{P}\left(e^{j\theta}\right)$ as

$$\hat{P}_L(e^{j\theta}) = \hat{P}_{AR}(e^{j\theta})|_{\theta=l\frac{2\pi}{L}} = \frac{\hat{\sigma}_i^{2}}{|\hat{A}_L(e^{j\theta})|^2}$$

Model order selection

Parametric spectral estimation

Model order selection

Occam's razor: simplest solution is the best
When presented with competing hypotheses to solve a problem, one should select the solution with the fewest assumptions

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Reasons for parsimony

- Spectral estimate might degrade if model order is too high
- Computational complexity

Trade-offs:

- Residual error
- Overfitting (fitting noise instead of data)

Model order selection

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Trade-offs:

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Approaches

- Choose lowest model order such that the residual error is white
- Use criterion balancing between model order and goodness-of-fit

Model selection criteria

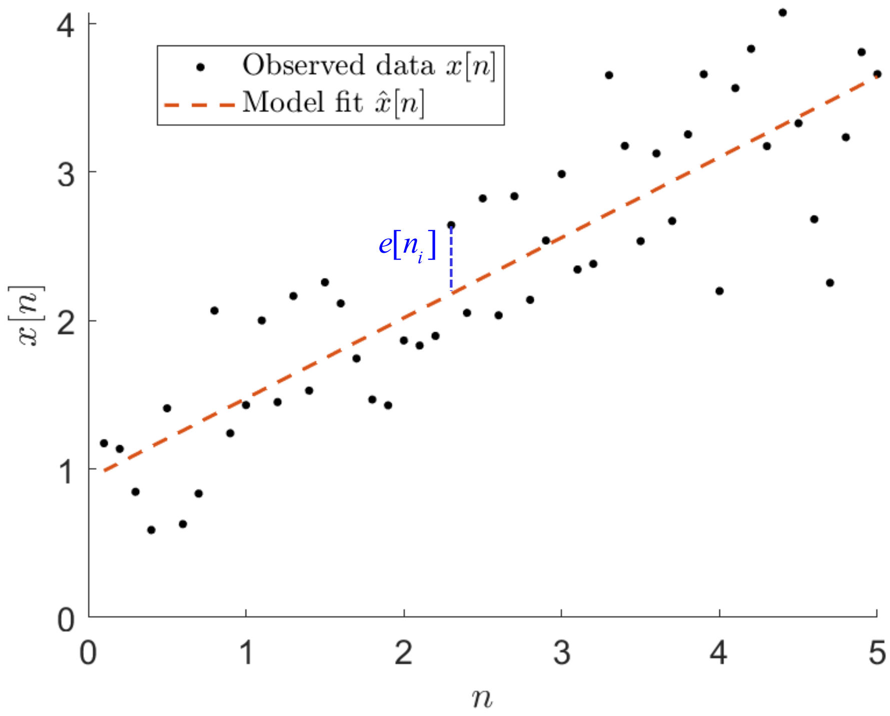
Coefficient of determination, R^2

Residuals: $e[n] = x[n] - \hat{x}[n]$

Residual variance: $\sigma_r^2 = \frac{1}{N}\sum_{n=0}^{N-1} e^2[n]$

Data variance: $\sigma_x^{\ 2} = \frac{1}{N}\sum_{n=0}^{N-1} (x[n] - \mu_x)^2$

$$\boxed{R^2 = 1 - \frac{\sigma_r^2}{\sigma_x^2}}$$



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Data variance: $\sigma_x^{\ 2} = \frac{1}{N}\sum_{n=0}^{N-1}(x[n]-\mu_x)^2$

$$\boxed{R^2 = 1 - \frac{\sigma_r^2}{\sigma_x^2}}$$

- Gives information about goodness of fit
- Statistical measure of how well the prediction approximate the observed data
- **Choose model with highest R^2**
- Cannot be used to compare models with different number of parameters

Model selection criteria

Final prediction error (FPE)

$$\boxed{\mathrm{FPE}(P) = \sigma_r^2 \frac{N + (P+1)}{N - (P+1)}}$$

- **Choose P that minimizes FPE**
- Reported to underestimate true model order

Model selection criteria

Final prediction error (FPE)

$$\boxed{\mathrm{FPE}(P) = \underbrace{\sigma_r^2}_{\text{Decreases with model order P}} \frac{N+(P+1)}{N-(P+1)}}$$

- **Choose P that minimizes FPE**
- Reported to underestimate true model order

Model selection criteria

Final prediction error (FPE)

$$\boxed{\mathrm{FPE}(P) = \underbrace{\sigma_r^2}_{\text{Decreases with model order P}} \underbrace{\frac{N+(P+1)}{N-(P+1)}}_{\text{Increases with model order P}}}$$

- **Choose P that minimizes FPE**
- Reported to underestimate true model order

Model selection criteria

Akaike's information criterion (AIC)

$$\boxed{\mathrm{AIC}(P) = N \cdot \ln(\hat{\sigma}_r^2) + 2P}$$

- **Choose P that minimizes AIC**
- Reported to overestimate true model order

TU/e

Model selection criteria

Akaike's information criterion (AIC)

$$\boxed{\mathrm{AIC}(P) = N \cdot \ln(\hat{\sigma}_r^2) + 2P}$$

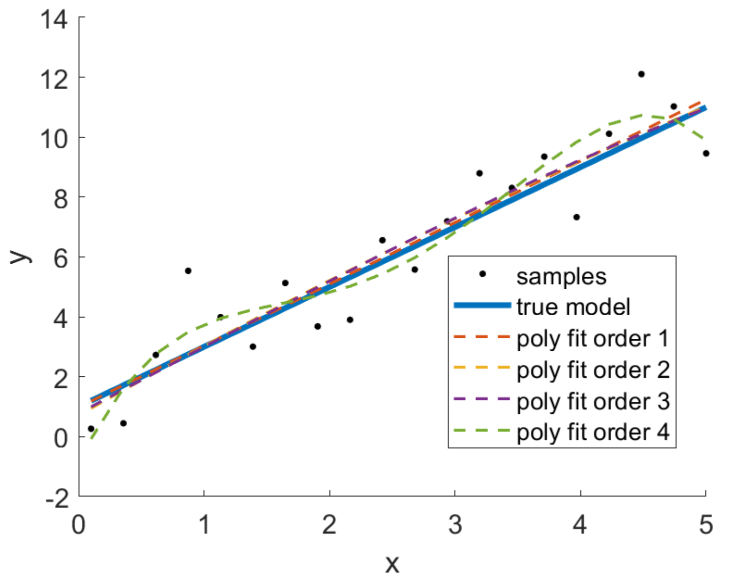
Correction for small samples

$$\boxed{\mathrm{AICc}(P) = N \cdot \ln(\hat{\sigma}_r^2) + 2P + \frac{2P(P+1)}{N-P-1}}$$

- **Choose p that minimizes AIC**
- Reported to overestimate true model order

Overfitting: example

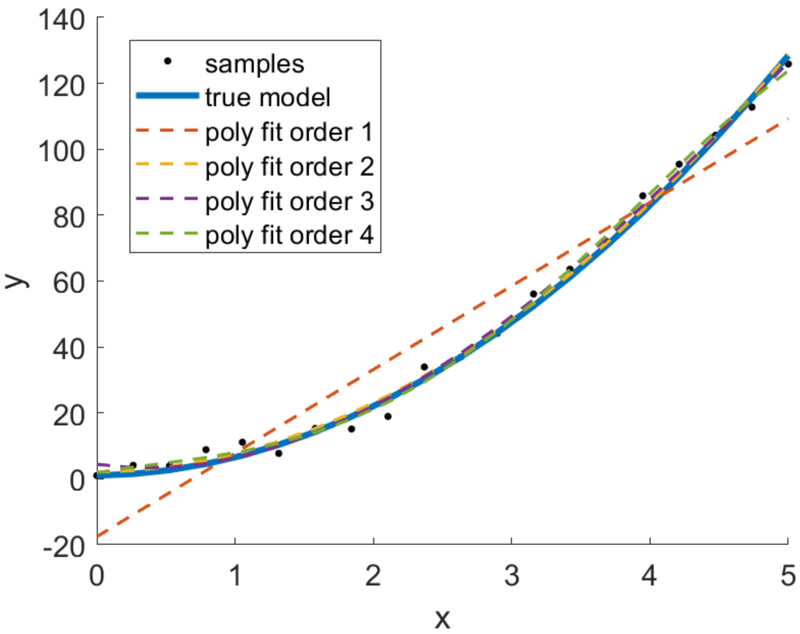
$$y[n] = ax[n] + b + w[n]$$



Model order	σ_r^2	R^2	FPE	AIC
1	0.8	1.00	0.98	-2.15
2	0.77	1.00	1.05	-0.43
3	0.74	0.99	1.12	1.57
4	0.73	0.99	1.21	4.28

Overfitting: example

$$y[n] = ax^2[n] + bx[n] + c + w[n]$$



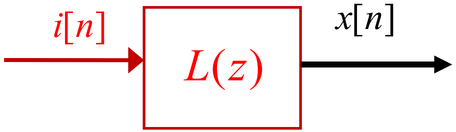
Model order	σ_r^2	R^2	FPE	AIC
1	125.3	0.93	153.14	98.84
2	12.55	1.00	16.98	55.3
3	11.53	1.00	17.3	56.4
4	11.37	1.00	18.95	59.29

MA Spectral estimation

Parametric spectral estimation

MA spectral estimate

- Assume $x[n]$ is generated by driving a LTI system represented by Q-th order MA model with WGN



$$x[n] = i[n] + \sum_{q=1}^{Q} b_q i[n-q] =$$

$$= i[n] + b_1 i[n-1] + b_2 i[n-2] + \ldots + b_Q i[n-q]$$

$$L(z) = 1 + \sum_{q=1}^{P} b_q z^{-q} \quad \Rightarrow \quad P(e^{j\theta}) = \sigma_i^2 \left| 1 + \sum_{q=1}^{Q} b_q z^{-jq\theta} \right|^2$$

MA spectral estimate via Yule-Walker

$$r[l] = \begin{cases} 0 & l > Q \\ \sigma_i^2 \sum_{k=|l|}^{Q} b_k b_{k-l} & 0 \le l \le Q \\ r[-l] & l < Q \end{cases}$$

Step 1
Calculate $r[0], r[1], \ldots, r[Q]$ from data
Step 2
Obtain Q+1 equations from Yule-Walker

$$r[l] = b_0 b_l \sigma_i^2 + b_1 b_{l+1} \sigma_i^2 + \ldots + b_{Q-l} b_Q \sigma_i^2$$

$$\begin{cases} \hat{r}[0] = \hat{b}_0 \hat{b}_0 \hat{\sigma}_i^2 + \hat{b}_1 \hat{b}_1 \hat{\sigma}_i^2 + \cdots + \hat{b}_Q \hat{b}_Q \hat{\sigma}_i^2 \\ \hat{r}[1] = \hat{b}_0 \hat{b}_1 \hat{\sigma}_i^2 + \hat{b}_1 \hat{b}_2 \hat{\sigma}_i^2 + \cdots + \hat{b}_{Q-1} \hat{b}_Q \hat{\sigma}_i^2 \\ \vdots \\ \hat{r}[Q] = \hat{b}_0 \hat{b}_Q \hat{\sigma}_i^2 \end{cases}$$

$\Rightarrow$ System of nonlinear equations

MA spectral estimate

Estimate autocorrelation $\hat{r}[l]$ for $|l| \leq Q$

MA spectral estimate

Estimate autocorrelation $\hat{r}[l]$ for $|l| \leq Q$

→

Calculate model parameters $\hat{b}_1, \ldots, \hat{b}_q$ and $\hat{\sigma}_i^2$ via nonlinear solution of Yule-walker equations

MA spectral estimate

Estimate autocorrelation $\hat{r}[l]$ for $|l| \leq Q$

Calculate model parameters $\hat{b}_1, \ldots, \hat{b}_q$ and $\hat{\sigma}_i^2$ via nonlinear solution of Yule-walker equations

1

Compute $\hat{P}\left(e^{j\theta}\right)$ as

$$\hat{P}_{MA}(e^{j\theta}) = \hat{\sigma}_i^2 \,|1 + \hat{b}_1 e^{-j\theta} + \hat{b}_2 e^{-j2\theta} + \ldots + \hat{b}_Q e^{-jQ\theta}|^2$$

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Via DFT/FFT

Apply zero padding (L)

$$\hat{B}_L(e^{j\theta})\Big|_{\theta = l\frac{2\pi}{L}} = \sum_{q=0}^{Q} \hat{b}_p e^{-jql\frac{2\pi}{L}}, \quad \hat{b}_0 = 1$$

2 Compute $\hat{P}\left(e^{j\theta}\right)$ as

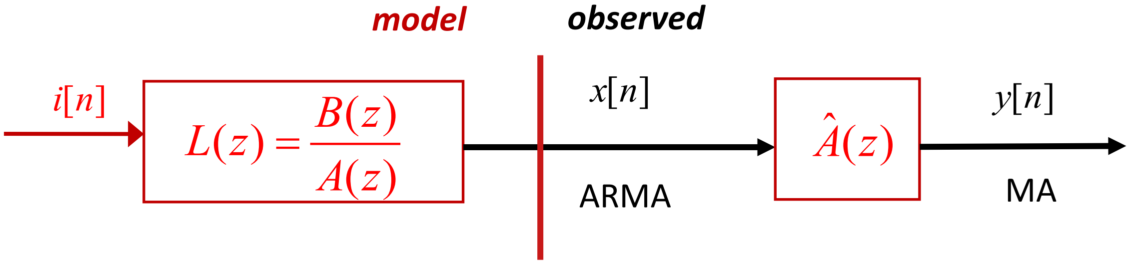
$$\hat{P}_L(e^{j\theta}) = \hat{P}_{MA}(e^{j\theta})\Big|_{\theta = l\frac{2\pi}{L}} = \hat{\sigma}_i^2 \, |\hat{B}_L(e^{j\theta})|^2$$

ARMA Spectral estimation

Parametric spectral estimation

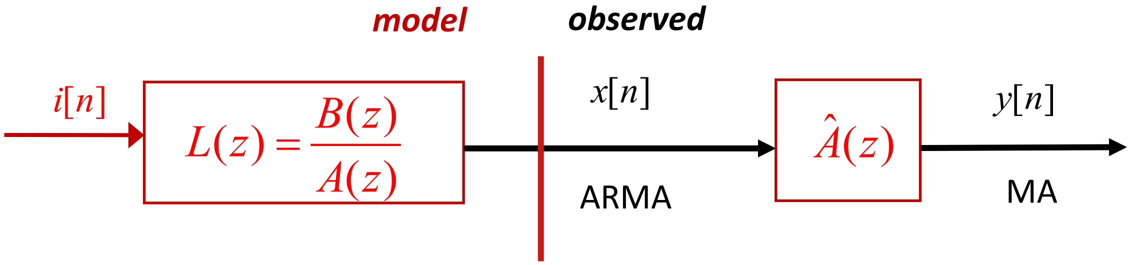
ARMA spectral estimation

- Idea: for $l > Q$ ARMA behaves exactly like AR



ARMA spectral estimation

- Idea: for $m > Q$ ARMA behaves exactly like AR
  - Use lags $|\tau| > Q$ to estimate $\hat{A}(e^{j\theta})$
  - Inverse filter ARMA sequence $x[n]$ by $\hat{A}(e^{j\theta})$
  - Use obtained sequence MA sequence $y[n]$ to calculate $r_y[\tau]$
  - Use $r_y[\tau]$ to calculate $\hat{B}(e^{j\theta})$ by solving nonlinear system of equation
  - Compute $\hat{P}_{ARMA}(e^{j\theta})$ as $\hat{P}_{ARMA}(e^{j\theta}) = \hat{\sigma}_i^{\,2} \dfrac{|\hat{B}_L(e^{j\theta})|^2}{|\hat{A}_L(e^{j\theta})|^2}$



ARMA spectral estimation

- In practice, any ARMA model can be approximated by an AR model by increasing model order
  - Use MA or ARMA estimation only when model structure is known

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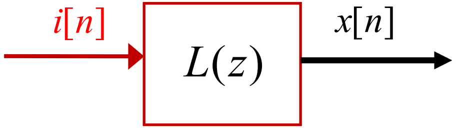
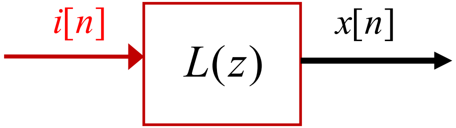
Example:



$$L(z) = \mathrm{ARMA}(4,3)$$

- Simulate random process $x[n]$ as LTI modeled as $L(z)$ and driven by white noise with $\sigma_i^2 = 5$
- Compare
  - True spectrum ARMA(4,3) model
  - Non-parametric estimation by periodogram with Hann window
  - Parametric estimation by AR(P) of increasing order

Example



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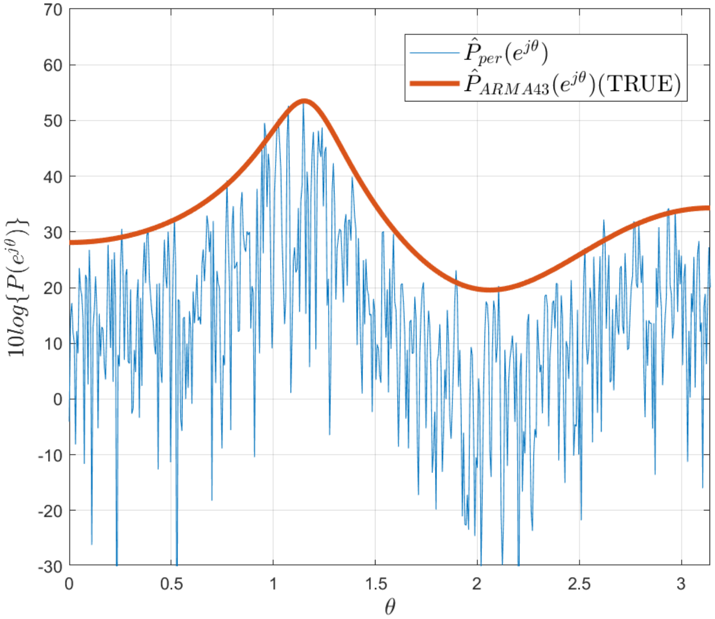

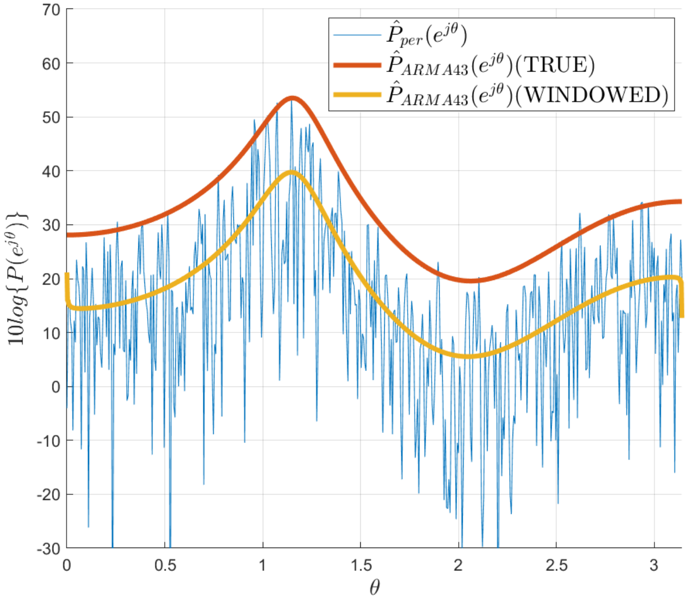

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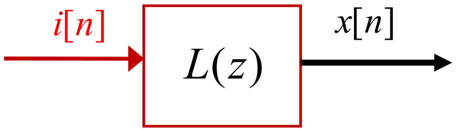

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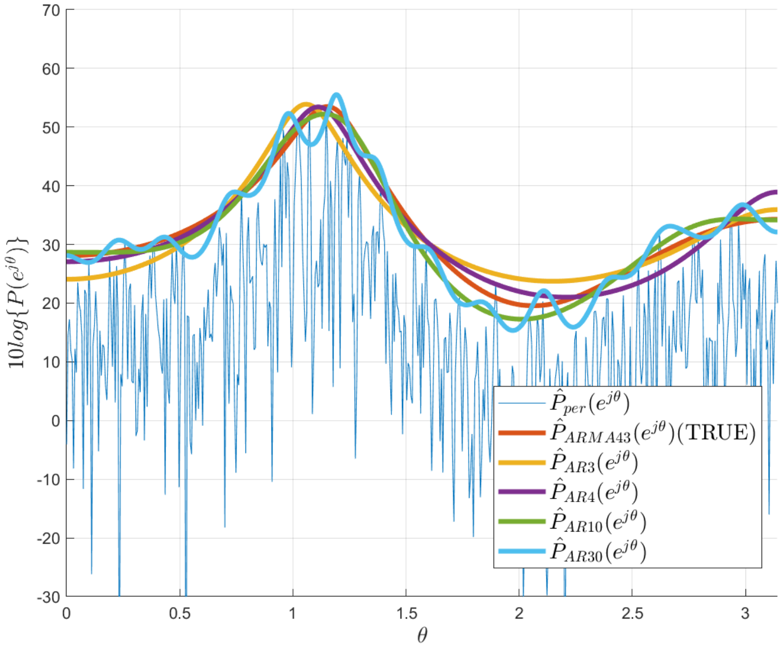
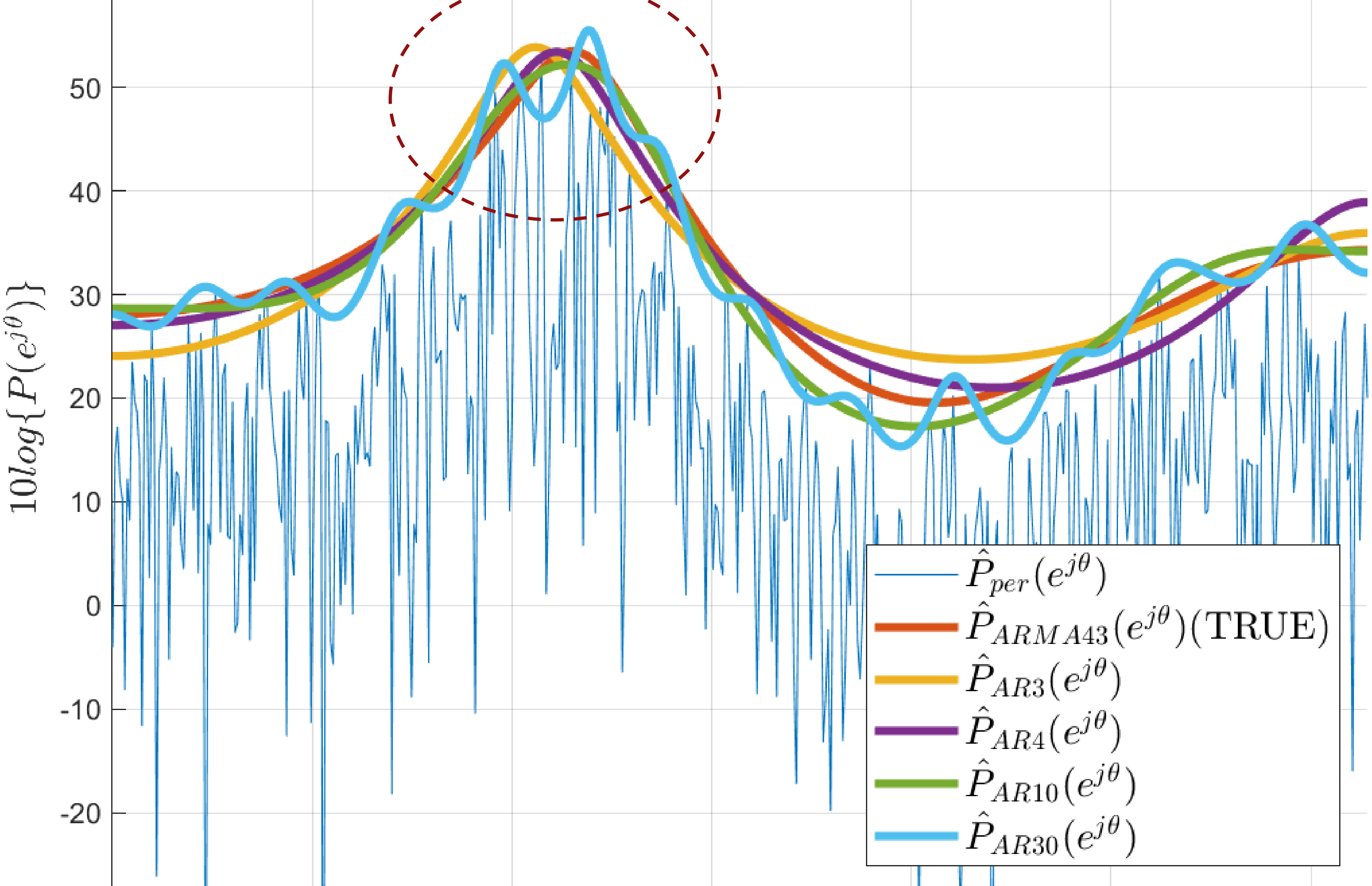
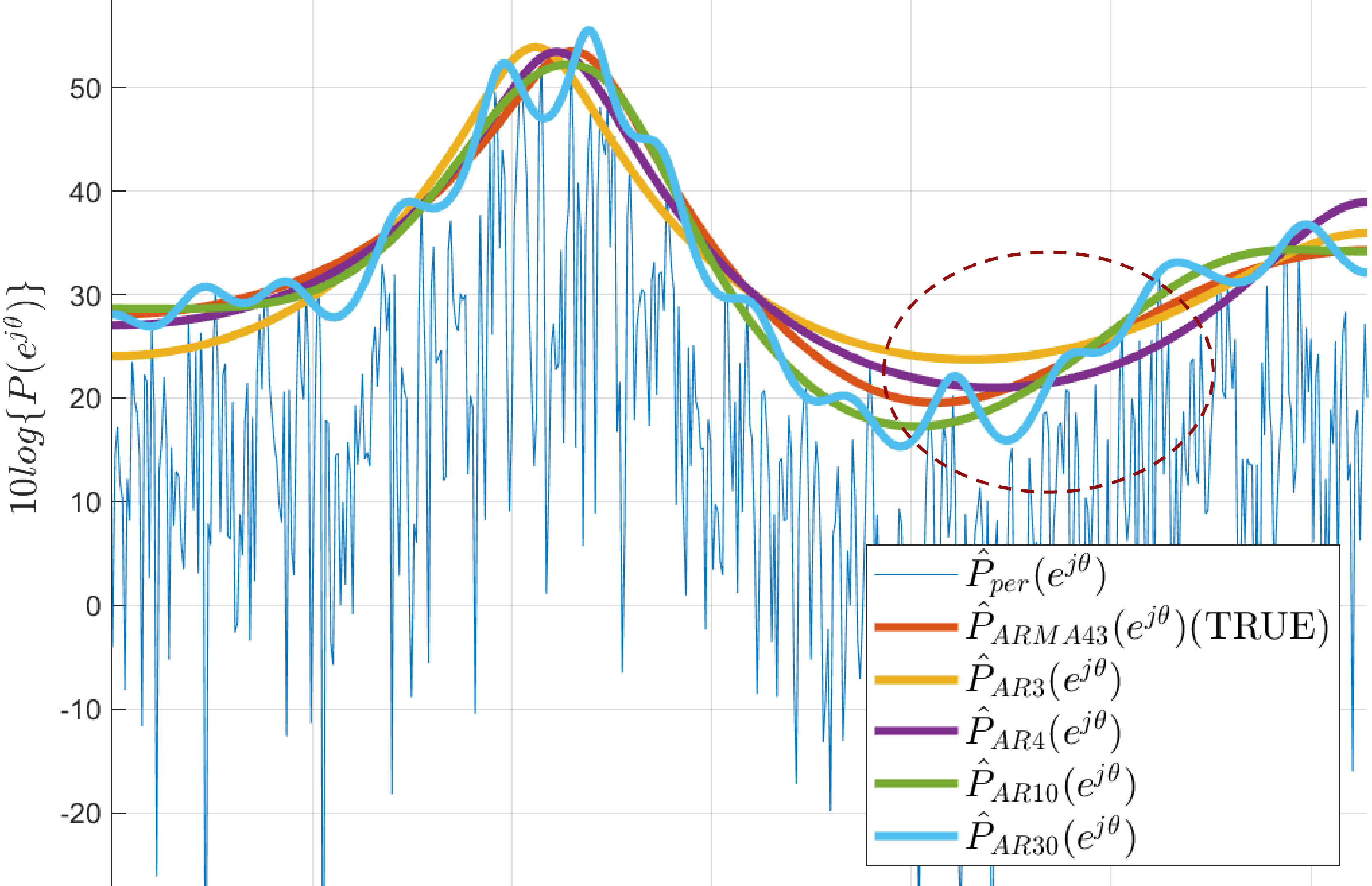


- Model order close to true P gives good estimation of peaks
- Need to increase model order for better estimation of valleys



$10log\{P(e^{j\theta})\}$
50
40
30
20
10
0
-10
-20
$\hat{P}_{per}(e^{j\theta})$
$\hat{P}_{ARMA43}(e^{j\theta})$(TRUE)
$\hat{P}_{AR3}(e^{j\theta})$
$\hat{P}_{AR4}(e^{j\theta})$
$\hat{P}_{AR10}(e^{j\theta})$
$\hat{P}_{AR30}(e^{j\theta})$



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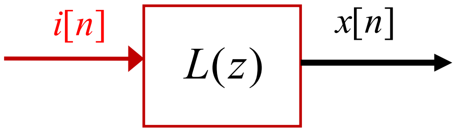

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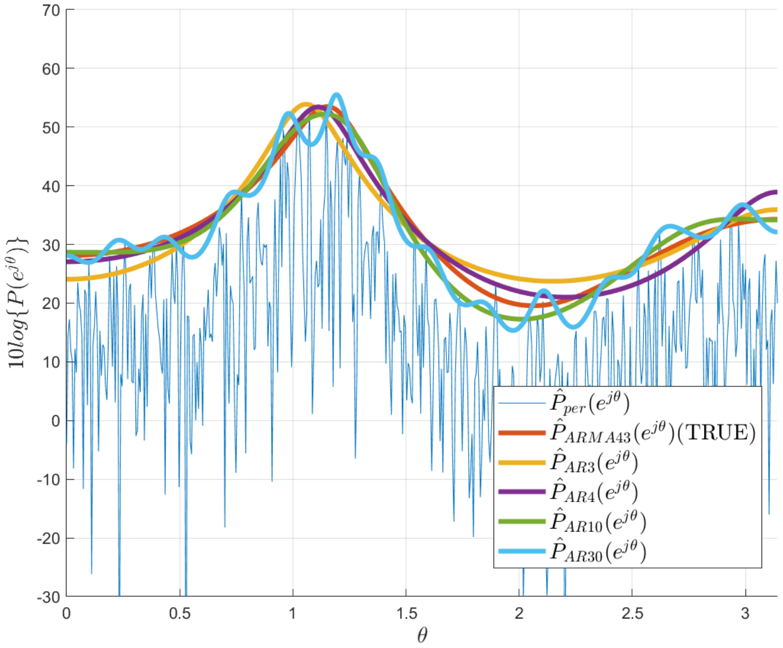

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Non-parametric approach

- Based on FTD of observed data
  - Periodogram/correlogram
- **Pros**: No prior knowledge required
- **Cons**: PSD derived from windowed data (explicitly or implicitly)
  - Assumption that data (or correlation) outside window is zero: unreasonable in most cases
  - Windows limit the resolution of the spectral estimate and causes spectral leakage

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Parametric approach

- Based on modeling the structure of the observed signal $x[n]$
- **Pros**: Overcome resolution limitations and spectral leakage
- **Cons**: Prior knowledge (or guess) required
  - Need to choose model structure and order
  - If chosen model is not appropriate PSD estimate is poor
  - MA and ARMA estimation requires resolution of non-linear equation: complex for high order models

Wrap up (I)

- **Parametric approaches** are based on a model of the random signal; the estimation of the power spectrum reduces to estimating the model parameters
- If we assume an **AR model** for the signal, the model parameters can be estimated by a linear system of equations (Yule-Walker)
- If the chosen **model order** is not appropriate, our estimate of the power spectral density will be inaccurate due to **under-** or **overfitting**.
- An appropriate model order can be chosen by a **selection criteria** that compromises between model error and number of parameters.

Wrap up (II)

- If we assume an **MA model** for the signal, the model parameters can be estimated by a non-linear system of equations
- **ARMA modeling** can be approached in two steps by estimating first the AR and then the MA model parameters; however, it is rarely performed.
- In practice, an ARMA model can be approximated by an AR model by increasing the model order
- Compared to non-parametric approaches, parametric approaches overcome the resolution and spectral leakage limitations, but might provide inaccurate estimate if the chosen model is not appropriate

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Statistical signal processing (5CTA0)

Parametric spectral estimation

Lecturer: Simona Turco

Electrical Engingeering, Signal Processing Systems group