From (13.30) and (13.31) of the course reader, we have that

$$W_{\mathrm{M}}(e^{j\theta}) \geq 0 \iff \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} a_i^* w_{\mathrm{M}}[i-j]a_j \geq 0. \tag{1}$$

To prove this necessary and sufficient condition for a window to have a positive spectrum, we first prove that

$$\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} a_i^* w_{\mathrm{M}}[i-j]a_j \geq 0 \implies W_{\mathrm{M}}(e^{j\theta}) \geq 0. \tag{2}$$

Therefore, we express the window w_M by its IDFT, which yields

$$\sum_{n=0}^{M-1}\sum_{m=0}^{M-1} a_n^* a_m \frac{1}{2\pi}\int_{-\pi}^{\pi} W_{\mathrm{M}}(e^{j\theta})e^{j\theta(n-m)}d\theta. \tag{3}$$

After changing the order of integration and summations, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} W_{\mathrm{M}}(e^{j\theta}) \sum_{n=0}^{M-1} a_n^* e^{j\theta n} \sum_{m=0}^{M-1} a_m e^{-j\theta m} d\theta, \tag{4}$$

where we recognize the summations as the DTFT of the finite-length sequence a and its complex conjugate, which is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} W_{\mathrm{M}}(e^{j\theta})A^*(e^{j\theta})A(e^{j\theta})d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_{\mathrm{M}}(e^{j\theta})|A(e^{j\theta})|^2 d\theta. \tag{5}$$

From (2), we have that this is greater than 0 for all sequences a, and since $|A(e^{j\theta})|^2$ is real and positive, independent of the choice of a, it also follows that $W_{\mathrm{M}}(e^{j\theta})$ hast to be real and positive for all values of θ. Especially, we can chose a such that $|A(e^{j\theta})|^2$ is only nonzero for a small interval from $[\theta_a, \theta_b]$.

The implication

$$W_{\mathrm{M}}(e^{j\theta}) \geq 0 \implies \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} a_i^* w_{\mathrm{M}}[i-j]a_j \geq 0, \tag{6}$$

follows by showing that

$$\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} a_i^* w_{\mathrm{M}}[i-j]a_j < 0 \implies W_{\mathrm{M}}(e^{j\theta}) < 0 \tag{7}$$

for some sequence a and frequency θ_0. This can be shown in the same way as the necessity in the first part of the proof.

During the Q&A session, we talked about that the rectangular window violates this condition which is not true. A rectangular symmetric window of length $2M-1$ has spectrum

$$W_{\mathrm{M}}(e^{j\theta}) = \frac{\sin(\frac{\theta(2M-1)}{2})}{\sin(\frac{\theta}{2})}, \tag{8}$$

which is clearly negative for some values of θ. To show that also the right hand side of (1) is violated, we first notate that the double summation can be expressed as

$$\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} a_i^* w_{\mathrm{M}}[i-j]a_j = \mathbf{a}^H\mathbf{W}\mathbf{a}, \tag{9}$$

where the element W_{ij} of the matrix $\mathbf{W}$ is given by

$$\mathbf{W} = w_{\mathrm{M}[i-j]}. \tag{10}$$

Note that the matrix $\mathbf{W}$ is a Toeplitz matrix. Next, the condition

$$\sum_{i=0}^{M-1} \sum_{j=0}^{M-1} a_i^* w_{\mathrm{M}}[i-j] a_j = \mathbf{a}^H \mathbf{W} \mathbf{a} \geq 0 \tag{11}$$

implies that $\mathbf{W}$ is positive semidefinite.

Suppose now that we have a rectangular window of length 5, i.e, $M = 3$. Since (1) holds for any length, we can look, for example, at a 4×4 matrix which is

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \tag{12}$$

and has eigenvalues $\lambda_1 = -0.5616$, $\lambda_2 = 0$, $\lambda_3 = 1$, and $\lambda_4 = 3.5616$. Since one of the eigenvalue is negative, the matrix is not positive semidefinite.