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# State-Space Model of Spatially Distributed Flow Heating with Time-Varying Parameters

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***Abstract*—The paper proposes a methodology for creating a dynamic model of a flow heater with time-varying flow rate and heat transfer coefficient. The first-order hyperbolic equation is used as the mathematical description of the heating process. The numerical model of the heating process is obtained using the spectral method of analysis and synthesis of distributed systems. The orthonormal function basis is used for the transition to the state space. A method for the consideration of non-zero boundary conditions is proposed. The state-space model of the flow heater is obtained and its implementation in Simulink using the MATLAB S-Function block is proposed. The results of the study of the obtained model are presented in the article. The dynamic behavior of the model is compared with the analytical solution of the hyperbolic equation. The dependence of the model accuracy on the dimensionality of the state vector is investigated.**

***Keywords—state-space model, flow heating, variable flow rate, distributed system, spectral method, hyperbolic equation***

## I. INTRODUCTION

A heating fluid or gas flow model can be used as the basis for simulating more complex technological processes. These are mainly heat exchangers of various types. Dynamic models with the accuracy required to solve automatic control problems are based on a one-dimensional first-order hyperbolic equation. Such an equation has an analytical solution. But only if the model parameters do not change over time. However, in most practical cases, the control action of the heater outlet temperature control system is to change the flow rate and/or heat transfer coefficient. This leads to the use of numerical methods to create dynamic models.

There are different ways of implementing numerical models. The use of finite element models provides the necessary accuracy, but such models are resource-intensive. As a result, when solving problems in the design of automatic control systems, such models are used at best to verify the performance of the system.

The most popular modelling approaches focus on implementing models in software for computer simulation of dynamic systems (MATLAB/Simulink, VisSim, SimInTech, etc.). Such an approach allows the behavior of control systems to be studied in a single software package, without the additional task of linking different software packages.

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## II. LITERATURE REVIEW

In [1] an overview of methods for solving fluid flow and heat transfer problems is given. The author classifies the methods used to solve computational fluid dynamics problems. The most popular are mesh-based methods. These are characterized by high computational costs. These methods are typically used to obtain accurate results, from which simplified models are then identified and validated.

In [2, 3, 4, 5] the Finite Volume Method (FVM) is used in combination with other approaches to model heat exchangers of different designs to improve accuracy. In [2] a comparison is made between the behavior of heat exchanger models obtained with the FVM approach and the Moving Bound approach. In [3] a thermodynamic investigation of a cascaded latent heat storage system is carried out. The FVM is used for modelling. The resulting model is then used to optimize the parameters of the system. In [4], the FVM of a plate heat exchanger is used for transient simulations in MATLAB/Simulink. It is then used to improve energy efficiency and control heat pump systems. In [5], the authors use a sequence of lumped model slices to simulate a water reservoir. The temperature and pressure are assumed to be uniform in each slice. The slices are connected in series, creating a spatial temperature variation. The model is used to design predictive controllers for tankless gas-fired boilers with time-varying delay [6]. Control-oriented modelling approaches for distributed parameter systems are presented in [7]. The authors present the heat transfer process between the heater tube and the gas flow as a system of two models: a piecewise homogeneous distribution of finite volume element temperatures of the air canal and the finite element representation of the tube temperature.

In [6,7,8,9] the obtained dynamic models of the heat transfer process are presented in the form of state-space models. This allows the application of the mathematical apparatus of state space for the synthesis of automatic control systems. Linearized models are used to solve temperature stabilization problems [10]. Temperature behavior is considered in terms of deviations from steady state. This provides an analytical solution that is suitable for analysis and parametric identification of the model. In [11], an approach focuses on the implementation of a variable flow rate model based on a variable transport delay block, which is implemented in software for computer simulation of dynamic systems with lumped parameters (MATLAB/Simulink, VisSim, SimInTech, etc.).

The application of the state space model can be based on the spectral method of distributed systems [12,13,14,15]. An expansion of the partial differential equation in an orthonormal function basis, the solution appears as an infinite-dimensional system of ordinary differential equations. The transition to a state space model is made by limiting the number of orthonormal basis functions used.

The paper discusses the use of the spectral method to implement a Simulink model of a flow heater using an MATLAB S-function block.

## III. Mathematic model

The temperature field $Q(z,t)$ of a flow with a variable flow-rate shows the dynamic behavior described by a one-dimensional hyperbolic equation of the first order:

$$\frac{\partial Q(z,t)}{\partial t} + v(t)\cdot\frac{\partial Q(z,t)}{\partial z} = \beta(t)\cdot\left(T(t) - Q(z,t)\right), \quad (1)$$
$$0 \le z \le L, \quad t > 0,$$

with the initial and boundary conditions

$$Q(z,0) = Q_0(z), \quad Q(0,t) = g(t), \quad (2)$$

where $v(t)$ – the flow speed; $\beta(t)$ – the heat transfer coefficient; $T(t)$ – the heater temperature, assumed to be the same along the heater length; $L$ – the length of the heater; $Q_0(z)$ – the initial temperature distribution; $g(t)$ – the function that describes the flow temperature changes at the heater inlet.

Due to the variable parameters of the equation, the flow velocity $v(t)$ and the heat transfer coefficient $\beta(t)$, equation (1) has no analytical solution. It is solved using a spectral method for the analysis and synthesis of distributed systems [12].

Using an orthonormal function basis, a spectral representation of the temperature distribution $Q(z,t)$ can be obtained:

$$Q(z,t) = \sum_{h=1}^{\infty} P(h,z)\varphi_Q(h,t) = \mathbf{P}_0^{\mathrm{T}}(z)\mathbf{\Phi}_Q, \quad (3)$$

where $\mathbf{P}_0(z) = \left[P(h,z)\right]_{\infty\times 1}$ is a vector of orthonormal functions,

$$P(h,z): \quad \int_0^L P(\bar{h},\xi)P(h,\xi)d\xi = \begin{cases} 1, & if \quad \bar{h} = h; \\ 0, & if \quad \bar{h} \ne h, \end{cases} \quad (4)$$
$$h = 1,2,...$$

$\mathbf{\Phi}_Q = \left[\varphi_Q(h,t)\right]_{\infty\times 1}$ – vector of time modes of temperature distribution,

$$\varphi_Q(h,t) = \int_0^L Q(\xi,t)P(h,\xi)d\xi, \quad h = 1,2,\dots . \quad (5)$$

In [12] it is shown that for the $\frac{\partial Q(z,t)}{\partial z}$ term of equation (1) it is also possible to obtain a spectral representation of the form

$$\frac{\partial Q(z,t)}{\partial z} = \mathbf{P}_1\mathbf{\Phi}_Q + \mathbf{\Gamma}_0 \quad (6)$$

where $\mathbf{P}_1 = \left[P_1(h,\bar{h})\right]_{\infty\times\infty}$ is first-order operational differentiation matrix, the components of which are defined as

$$P_1(h,\bar{h}) = \int_0^L P(\bar{h},\xi)\left(\frac{\partial P(h,\xi)}{\partial \xi}\right)d\xi, \ h = 1,2,\dots, \ \bar{h} = 1,2,\dots; (7)$$

The $\mathbf{\Gamma}_0 = \left[\varphi_g(h,t)\right]_{\infty\times 1}$ vector takes into account the non-zero boundary conditions, $Q(0,t)$, and its components are defined by the expression

$$\varphi_g(h,t) = \int_0^L g(t)\delta(\xi)P(h,\xi)d\xi, \quad h = 1,2,\dots . \quad (8)$$

Equation (1) in spectral form is

$$\frac{\partial \mathbf{\Phi}_Q}{\partial t} = \mathbf{K}_1\left(\mathbf{P}_1\mathbf{\Phi}_Q + \mathbf{\Gamma}_0\right) + \mathbf{K}_2\mathbf{\Phi}_Q + \mathbf{K}_3\cdot\mathbf{\Phi}_T, \quad (9)$$

where $\mathbf{K}_1 = -v(t)\mathbf{E}$, $\mathbf{K}_2 = -\beta(t)\mathbf{E}$, $\mathbf{K}_3 = \beta(t)\mathbf{E}$, $\mathbf{E}$ is the identity matrix. The components of the matrix $\mathbf{\Phi}_T = \left[\varphi_T(h,t)\right]_{\infty\times 1}$ are calculated for $T(t)$ in the same way as (5):

$$\varphi_T(h,t) = T(t)\int_0^L P(h,\xi)d\xi, \quad h = 1,2,... \quad (10)$$

The system of linear equations (9) can be used to implement a state space model of the outlet flow temperature.

In [14,15] the basis

$$P(h,z)=\sqrt{2}\sin\left(\frac{(2h-1)\pi z}{2L}\right),\quad h=1,2,... \tag{11}$$

is used to simulate a flow heater. It provides high model accuracy when restricted to a small number of countable terms of the infinite series (11).

However, the basis (11) has the disadvantage that all functions of the basis are equal to zero at the point $z$=0. Therefore, it is not possible to account for non-zero initial conditions with (8). In [14,15] the authors consider the temperature distribution of the flow in deviations from the inlet temperature. This approach is justified if the inlet temperature is assumed to be quasi-stationary. Its slow fluctuations do not affect the quality of the control.

In cases where the heater inlet temperature variation needs to be accounted for in the model, it is possible to use a technique where the inlet temperature effect is replaced by an equivalent heat flux at the start part of the heater $[0,d]$.

In steady-state mode ($\beta(t)=\beta_c=const$, $v(t)=v_c=const$, $T(t)=T_c=const$, $g(t)=g_c=const$), the temperature field at the point $z$ is determined by the expression:

$$Q(z,\infty)=T_c\cdot\left(1-\exp\left(-\beta_c z/v_c\right)\right)+g_c\exp\left(-\beta_c z/v_c\right). \tag{12}$$

The effect of the heater inlet temperature $g_c$ on the flow temperature value is determined by the multiplier $\exp\left(-\beta_c z/v_c\right)$. It can be used to calculate the flow temperature value due to the inlet temperature $g_c$ at the point $d$. From (12) it is possible to obtain the heating temperature $T_g$ at the start of the heater $[0,d]$, which gives an equivalent flow temperature at the heater outlet

$$T_g=g_c\frac{\exp\left(-\beta_c d/v_c\right)}{1-\exp\left(-\beta_c d/v_c\right)} \tag{13}$$

Expression (13) allows us to use the basis (11), replacing the expression for calculating the boundary conditions $\mathbf{\Gamma}_0$:

$$\varphi_g(h,t)=T_g\int_0^d P(h,\xi)d\xi,\quad h=1,2,\ldots. \tag{14}$$

Expression (13) is obtained for steady state heat transfer. By substituting the variables $v(t)$, $\beta(t)$ in (13), we observe a model error of limited duration $L/v(t)$. An estimate of the error is given in the Simulink model study section.

## IV. Implementation and Study of the Simulink Model

To implement the Simulink model, a state-space representation of the solution is used

$$\begin{cases}\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{Bu}\\ \mathbf{y}=\mathbf{Cx}\end{cases} \tag{15}$$

The dimensionality of the state space N is determined by the required accuracy of the model and the available computational resources. In (15), the $\mathbf{\Phi}_Q$ vector (5), truncated to the first $N$ components, is used as a component of the state vector $\mathbf{x}$. The sum of the truncated vectors $\mathbf{\Phi}_T$ (10) and $\mathbf{\Gamma}_0$ (14) is used as the input vector $\mathbf{u}$. The output vector of the model is the heater outlet temperature $\mathbf{y}=\left[Q(L,t)\right]_{1\times 1}$.

The coefficient matrices of the state-space model are defined by the following expressions, according to (9), (7), (11):

$$\mathbf{A}=-v(t)\mathbf{P}_1-\beta(t)\mathbf{E},\quad \mathbf{B}=\beta(t)\mathbf{E},\quad \mathbf{C}=\mathbf{P}_0^{\mathrm{T}}(L). \tag{16}$$

The state-space model with variable coefficients is implemented in MATLAB/Simulink using the S-function block [16]. The calculation of matrix components can be implemented in MATLAB using the Symbolic Math Toolbox. Figure 1 shows the script to generate the auxiliary variables P0d, P0L, Pout, P1 to calculate $\mathbf{\Gamma}_0$, $\mathbf{\Phi}_T$, $\mathbf{P}_0(L)$, $\mathbf{P}_1$ respectively.

If you are using the S-Function block, you must describe the structure of the block, the block methods that are called when the Simulink model calculation is initialized, when the state vector is calculated, and when the model output is calculated. For a compact presentation of the running source code in this paper, the code uses global variables that are created in the MATLAB workspace by calling the script in Fig. 1.

```
clear;
global N P0d P0L Pout P1 d
N=20; % state-space dimension
P0d=zeros(N,1);
P0L=zeros(N,1);
Pout=zeros(N,1);
P1=zeros(N,N);
d=0.1;
syms x
for h=1:N
    h1 = sym(h);
    P  = sqrt(2)*sin((2*h1-1)*pi/2*x);
    P0d(h) = double(int(P, x, 0, sym(d) ));
    P0L(h) = double(int(P, x, 0, 1 ));
    Pout(h)= double(subs(P, x, 1));
    dP = diff(P,x);
    for h_=1:N
        h2 = sym(h_);
        P_ = sqrt(2)*sin((2*h2-1)*pi/2*x);
        P1(h_,h)=double(int(dP*P_, x, 0, 1));
    end
end
```

Fig. 1. Script to calculate auxiliary variables using Simbolic Math Toolbox.

The definition of the S-Function is shown in Fig. 2. The sfFlowHeating function defines the number of inputs and outputs of the block, the dimensionality of the state vector and the names of the methods called during initialization, calculation of state variables and outputs.

The initial state of the state vector is set during initialization (Fig. 3). The Outputs function implements the output calculation from the second equation of the system (15). The Derivatives function (Fig. 3) calculates the first equation of the system (15). The variable Tg is calculated by equation (13) from the flow rate and heat transfer coefficient values entered into the S-Function block during the model simulation.

```
function sfFlowHeating(block)
  setup(block);
function setup(block)
  block.NumInputPorts =4; % Tinp, Tenv, vFlow, betta
  block.NumOutputPorts = 1; % Toutput
  block.SetPreCompInpPortInfoToDynamic;
  block.SetPreCompOutPortInfoToDynamic;
  global N
  block.NumContStates = N; % State-space dimension
  block.SampleTimes = [0 0]; % Cont. sample time
  block.SetAccelRunOnTLC(false);
  block.SimStateCompliance = 'DefaultSimState';
  block.RegBlockMethod('InitializeConditions', @Init);
  block.RegBlockMethod('Outputs', @Outputs);
  block.RegBlockMethod('Derivatives', @Derivatives);
```

Fig. 2. Description of the S-Function.

```
function Init(block)
  block.ContStates.Data(:) = 0;

function Outputs(block)
  global Pout
  x = block.ContStates.Data;
  block.OutputPort(1).Data = Pout'*x;

function Derivatives(block)
  global P0d P0L P1 d
  g = block.InputPort(1).Data;
  T = block.InputPort(2).Data;
  v = block.InputPort(3).Data;
  betta =  block.InputPort(4).Data;
  x = block.ContStates.Data;
  Tg=g*exp(-betta*d/v)/(1-exp(-betta*d/v));
  block.Derivatives.Data = ...
    -betta*diag(ones(block.NumContStates,1))*x ...
    -v*(P1*x) + betta*(P0L*T + P0d*Tg);
```

Fig. 3. S-function methods.

The implemented S-Function was used to carry out a computational experiment, which allowed us to evaluate its accuracy depending on the dimensionality of the state vector and the length of the start part of the heater to account for the initial flow temperature *d*. The model used for the study is a flow heater with the following characteristics: *L*=1 m, *d*=0.1 m, $\beta(t) = 0.1\ s^{-1}$, *T*(*t*) = 10 °C, *v*(*t*)=0.1 m/s, *g*(*t*)=2 °C.

Figs. 4-8 show plots of the heater outlet temperature during a step change in parameters as a function of the dimensionality of the state vector. The plots are compared with the ideal case, for which the transients are calculated by the analytical method. Fig. 4 shows the transient for a step change in flow rate from 0.1 m/s to 0.2 m/s at time t=2 s. Fig. 5 shows the transient when the heat transfer coefficient changes in steps from 0.1 $s^{-1}$ to 0.2 $s^{-1}$ at time t=2 s. Fig. 6 shows the transient response to a step change in heating temperature at time t=2 s from 10 °C to 15 °C.

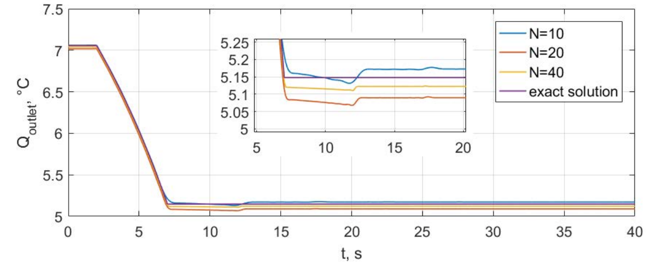


Fig. 4. Transient process with step change in flow rate.

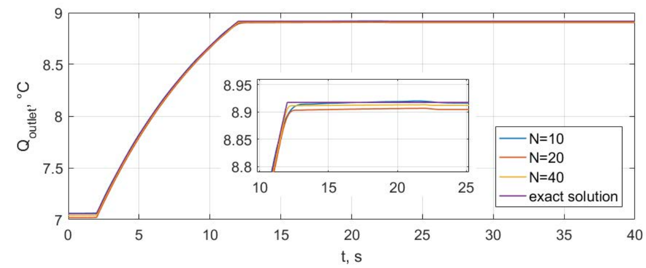


Fig. 5. Transient process when changing the heat transfer coefficient.

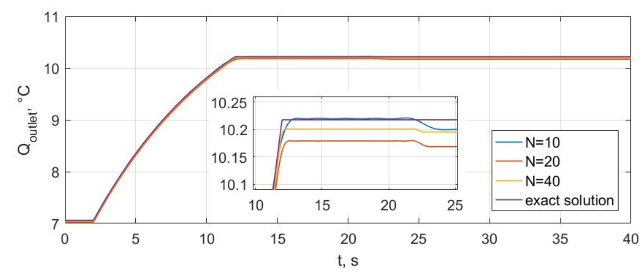


Fig. 6. Transient process with step change in heater temperature.

Figs. 7, 8 show the transients during a step change in the flow temperature at the heater inlet. The vertical red line indicates the moment (t=2 s) when the inlet temperature changes from 2 °C to 6 °C. In Fig. 7 the transient has been calculated with a constant value of *d*=0.1 for different values of the dimensionality of the state vector. Fig. 8 shows the transient process for a state vector of dimension *N*=20 for different values of the length of the start part *d*.

## V. Results and Discussion

Analysis of Figs. 5-8 shows that the behaviour of the state-space model is adequate to the exact analytical solution with stepwise influence on all model inputs. The accuracy of the model increases as the dimensionality of the state space increases. In contrast to the exact solution, where the transient ends in a time equal to the duration of the flow along the heater, $\tau = L/v(t)$, the transient in the state-space model continues for an interval in $3\tau$. The state space model shows the highest accuracy in response to changes in heat transfer coefficient and heating temperature. The lowest accuracy is observed in response to changes in input temperature. Fig. 7

clearly shows that as the inlet temperature changes, false dynamics (oscillations) occur due to the limited number of terms in the decomposition series. In addition, the characteristic outliers occur at times multiples of $\tau$ time from the moment the signal changes at the input.

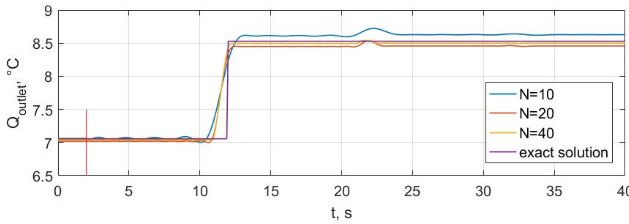


Fig. 7. Transient process with step changes in inlet temperature, with different state vector dimensions.

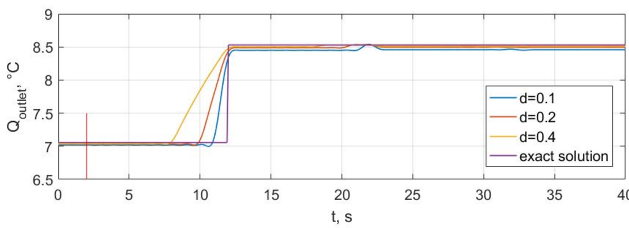


Fig. 8. Transient response to step changes in inlet temperature with different values for the length of the start part of the heater.

In Figs. 7, 8 it can be seen that the start of the temperature change at the outlet occurs earlier than in the exact solution. This is due to the chosen method of accounting for the influence of the inlet temperature through the equivalent flux in the start part of the heater. Fig. 8 clearly shows that as the start part $d$ increases, the time for the outlet temperature to change from the initial value to the final value increases. However, increasing the start part d leads to a reduction in the amplitude of the false dynamics. This state-space model behavior can be used for specific modelling problems. In a real situation, a step change in the heater inlet temperature is difficult to implement and is rare. Transient analysis has shown that the state space model has a static error when the heater inlet temperature is not zero. This error can be reduced or even eliminated in any application of the model by reducing the value of parameter $d$ or by introducing a correction factor into expression (13).

## VI. Conclusions

The proposed approach to flow heater modelling can be used to build models of arbitrary systems involving the process of heating fluid or gas flow. The proposed methodology can be used to implement a model of a cross-flow heater with interacting flows. For modelling direct-flow or counter-flow heaters, additional refinement of the model is required to specify the spatially dependent temperature distribution of the heater.

The model can be implemented in any dynamic systems simulation package that supports state space block description with the ability to change coefficient matrices during simulation.

## References


[1] M. L. Hosain, Fluid Flow and Heat Transfer Simulations for Complex Industrial Applications : From Reynolds Averaged Navier-Stokes towards Smoothed Particle Hydrodynamics: PhD Dissertation. Västerås: Mälardalen University, 2018.

[2] Y. Vaupel, W. R. Huster, F. Holtorf, A. Mhamdi, and A. Mitsos, “Analysis and improvement of dynamic heat exchanger models for nominal and start-up operation,” Energy, vol. 169, pp. 1191–1201, 2019.

[3] C. Zhang, X. Zhang, L. Qiu, and Y. Zhao, “Thermodynamic investigation of cascaded latent heat storage system based on a dynamic heat transfer model and DE algorithm,” Energy, vol. 211, 118578, 2020.

[4] E. Salazar-Herran, K. Martin-Escudero, L. A. del Portillo-Valdes, I. Flores-Abascal, and N. Romero-Anton, “Flexible dynamic model of PHEX for transient simulations in Matlab/Simulink using finite control volume method,” International Journal of Refrigeration, vol. 110, pp. 83–94, 2020.

[5] A. F. Quintã, J. A. F. Ferreira, A. Ramos, N. A. D. Martins, and V. A. F. Costa, “Simulation models for tankless gas water heaters,” Applied Thermal Engineering, vol. 148, pp. 944–952, 2019.

[6] A. F. Quintã, I. Ehtiwesh, N. Martins, J. A. F. Ferreira, “Gain scheduling model predictive controller design for tankless gas water heaters with time-varying delay,” Applied Thermal Engineering, vol. 213, 118669, 2022.

[7] A. Rauh, L. Senkel, H. Aschemann, V. V. Saurin, and G. V. Kostin, “An integrodifferential approach to modeling, control, state estimation and optimization for heat transfer systems,” Int. J. Appl. Math. Comput. Sci., vol. 26, no. 1, pp. 15–30, 2016.

[8] Q. Miao, S. You, W. Zheng, X. Zheng, H. Zhang, and Y. Wang, “A grey-box dynamic model of plate heat exchangers used in an urban heating system,” Energies, vol. 10, no. 9, 1398, 2017.

[9] Y. Wang, S. You, W. Zheng, H. Zhang, X. Zheng, and Q. Miao, “State space model and robust control of plate heat exchanger for dynamic performance improvement,” Applied Thermal Engineering, vol. 128, pp. 1588–1604, 2018.

[10] A. I. Danilushkin and I. A. Danilushkin, “Numerical analytical model of gas transport on the gas-main pipeline,” Bulletin of the Tomsk Polytechnic University, Geo Assets Engineering. vol. 326, pp. 96–103, 2015.

[11] I. Danilushkin, “A numerical-analytical model of heat transfer for a variable rate flow,” 2019 XXI International Conference Complex Systems: Control and Modeling Problems (CSCMP), Samara, Russia, 2019, pp. 416–419.

[12] V. A. Koval’, Spectral Method of Analysis and Synthesis of Distributed Systems: Tutorial. Saratov: STUS, 2010, 145 p.

[13] V. A. Koval’, “Solving the distributed control problem on the basis of the spectral method,” in Vestnik of State Technical University of Saratov, vol. 4, no. 3(61), pp. 96–105, 2011.

[14] V. A. Koval', V. N. Osenin, S. I. Suyatinov, and O. Y. Torgashova, “Synthesis of discrete controller for construction of a distributed controller of temperature conditions of steam oil heater,” Journal of Computer and Systems Sciences International, vol. 50, no. 4, pp. 638–653, 2011.

[15] I. A. Danilushkin and K. V. Kavkaev, “Simulation of the temperature field of flow with variable velocity in Simulink,” in Vestnik of Samara State Technical University (Technical Sciences Series), vol. 1, no. 53, pp. 174–178, 2017.

[16] S-Function Concepts defined by The MathWorks, Inc. Available: https://www.mathworks.com/help/simulink/sfg/s-function-concepts.html