Solution Suggestion
Exam - TTK4115 Linear System Theory
December 18, 2013

Problem 1

a) The observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where the matrices $\mathbf{A}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Because the observability matrix has full column rank, i.e. $\text{rank}(\mathcal{O}) = 2 = n$, we conclude that the system is observable.

b) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2.$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalue $\lambda = 0$ with multiplicity 2. The corresponding eigenvectors can be obtained from the kernel of the matrix $(\mathbf{A} - \lambda\mathbf{I})$:

$$\ker(\mathbf{A} - \lambda\mathbf{I}) = \ker\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \qquad \Longrightarrow \qquad \mathbf{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $\mathbf{q}$ is the corresponding eigenvector. Note that $\mathbf{A}$ has only one eigenvector associated with λ. Because the eigenvalue λ of $\mathbf{A}$ has multiplicity 2, and $\mathbf{A}$ has only one eigenvector associated with λ, the eigenvalues of $\mathbf{A}$ are not (all) distinct. Therefore, the system cannot be transformed into a diagonal form using a similarity transformation.

c) The system is (marginally) stable in the sense of Lyapunov, if and only if all the eigenvalues of $\mathbf{A}$ have negative or zero real parts and all the Jordan blocks corresponding to eigenvalues with zero real parts are 1×1. We observe that $\mathbf{A}$ is a Jordan block of size 2×2 with eigenvalue $\lambda = 0$ (which implies a zero real part). Therefore, the system is not stable in the sense of Lyapunov.

Problem 2

a) Before we find the step response of the system from u_2 to y, we first determine its transfer function matrix. The system can be written in the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned}$$

with

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

and where the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

In the Laplace domain, the input-output relation of the system is given by

$$y(s) = \mathbf{H}(s)\mathbf{u}(s).$$

where the transfer function matrix is given by

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+5 & -2 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+5} & \frac{2}{(s+5)(s-1)} \\ 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{(s+5)(s-1)} & \frac{2}{s+5} \end{bmatrix}. \end{aligned}$$

Hence, we have

$$y(s) = \frac{2}{(s+5)(s-1)} u_1(s) + \frac{2}{s+5} u_2(s).$$

For the step response from u_2 to y, we have

$$u_1(s) = 0 \qquad \text{and} \qquad u_2(s) = \frac{1}{s}.$$

Therefore, $y(s)$ is given by

$$y(s) = \frac{2}{s(s+5)} = \frac{2}{5s} - \frac{2}{5(s+5)}.$$

By transforming this expression to the time domain, we obtain the step response

$$y(t) = \mathcal{L}^{-1}\{y(s)\} = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = \frac{2}{5} - \frac{2}{5}e^{-5t}$$

for all $t \geq 0$.

b) The controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & -10 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Because the controllability matrix has full row rank, i.e. $\operatorname{rank}(\mathcal{C}) = 2 = n$, we conclude that the system is controllable.

c) The control input for the system with LQ controller and feed-forward is given by

$$\mathbf{u}(t) = \underbrace{-\mathbf{K}\mathbf{x}(t)}_{\text{LQ controller}} + \underbrace{\mathbf{P}r(t)}_{\text{feed-forward}},$$

where $\mathbf{P}$ is a matrix that should be defined. With this control input, the system equations are given by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{BP}r(t)\\ y(t) &= \mathbf{Cx}(t).\end{aligned}$$

In the Laplace domain, the relation between the reference r and the output y is given by

$$y(s) = G(s)r(s),$$

with

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{BP}.$$

If the stationary error between r and y is zero, then y converges to r as time goes to infinity for every constant reference r. Assuming that r is constant, i.e., $r(t) = r_c$ for all t, where r_c is a constant, we have

$$r(s) = \mathcal{L}\{r(t)\} = r_c\mathcal{L}\{1\} = \frac{r_c}{s}.$$

Using the finite value theorem, we obtain

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sG(s)r(s) = \lim_{s\to 0} \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{BP}r_c = -\mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{BP}r_c.$$

Let the matrix $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

We get

$$\begin{aligned}\lim_{t\to\infty} y(t) &= -\mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{BP}r_c\\ &= -\begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} -5 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.0884 & 2.5239 \\ 0.7003 & 0.1768 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} r_c\\ &= -\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -6.4006 & 1.6464 \\ -0.0884 & -1.5239 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} r_c\\ &= -\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -0.1539 & -0.1663 \\ 0.0089 & -0.6466 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} r_c\\ &= -\begin{bmatrix} -0.1663 & -0.3079 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} r_c\\ &= (0.1663p_1 + 0.3079p_2)r_c.\end{aligned}$$

If the stationary error between r and y is zero, we have

$$\lim_{t\to\infty} y(t) = r_c.$$

Therefore, we require that

$$0.1663p_1 + 0.3079p_2 = 1.$$

Any values of p_1 and p_2 that fulfill this condition ensure that the stationary error between r and y is zero. For instance, if we choose $p_1 = 0$, then it follows that $p_2 = 3.2481$ satisfies this condition. The corresponding feed-forward is given by

$$\mathbf{P}r(t) = \begin{bmatrix} 0 \\ 3.2481 \end{bmatrix} r(t).$$

A block diagram of the system with LQ controller and feed-forward is given in Fig. 1.

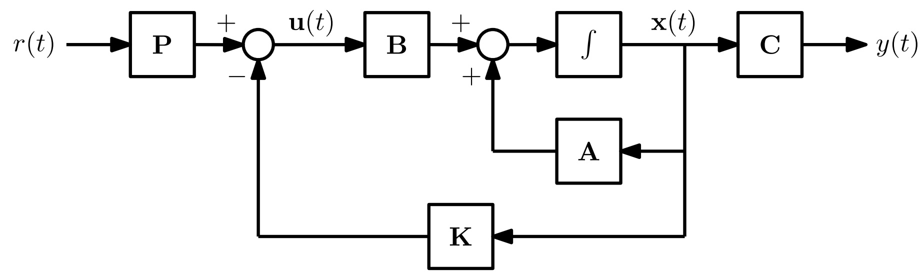

Fig. 1: Block diagram of the system with LQ controller and feed-forward.

d) The eigenvalues of the system with LQ controller are the eigenvalues of the closed-loop system matrix $\mathbf{A} - \mathbf{BK}$. The eigenvalues of $\mathbf{A} - \mathbf{BK}$ can be calculated from the characteristic polynomial of $\mathbf{A} - \mathbf{BK}$, which is given by

$$\det(\mathbf{A} - \mathbf{BK} - \lambda\mathbf{I}) = \begin{vmatrix} -6.4006 - \lambda & 1.6464 \\ -0.0884 & -1.5239 - \lambda \end{vmatrix} = \lambda^2 + 7.9245\lambda + 9.8994.$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues

$$\lambda_{1,2} = \frac{-7.9245 \pm \sqrt{7.9245^2 - 4 \cdot 9.8994}}{2} = -3.9622 \pm 2.4083,$$

which can alternatively be written as

$$\lambda_1 = -1.5539 \qquad \text{and} \qquad \lambda_2 = -6.3706.$$

e) We note that, from the output equation of the system, it follows that $y = x_1$. The cost function that is minimized to obtain the LQ controller is given by

$$J_{LQR} = \int_0^t \mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)dt.$$

To obtain a faster response in y, we can put more weight on the cost of deviation of the state x_1. We can do this by increasing the first element on the diagonal of $\mathbf{Q}$. Moreover, by looking at the equation for the dynamics of the system, we see that the input u_1 mainly influences x_2, while the input u_2 only influences x_1. Therefore, we can also get a faster response in y if we put less weight on the cost of the input u_2 (allow for a larger input u_2). We can do this by decreasing the second element on the diagonal of $\mathbf{R}$.

f) We consider an observer of the following form:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t)).$$

The control law is now given by

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + \mathbf{P}r(t),$$

A block diagram of the system with LQ controller and observer is given in Fig. 2. The poles of the observer are the eigenvalues of the matrix $\mathbf{A} - \mathbf{LC}$. As a rule of thumb, the poles of the observer should be chosen 5 to 10 faster than the fastest pole of the closed-loop system. The fastest pole of the closed-loop system is $\lambda_2 = -6.3706$. This implies that the real part of the poles of the observer should be chosen

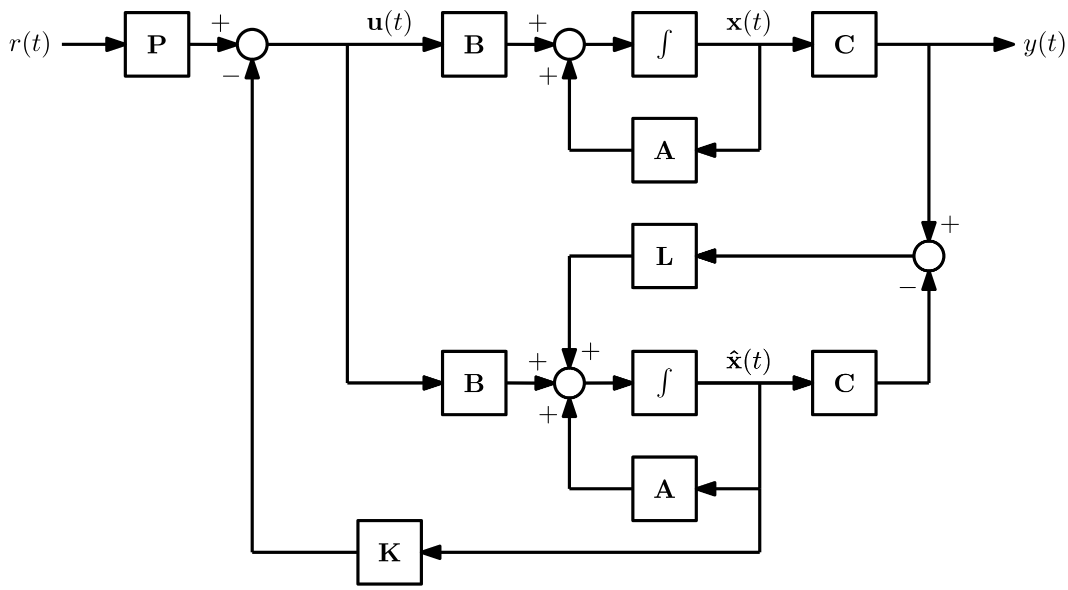

Fig. 2: Block diagram of the system with LQ controller and observer.

between approximately -64 and -32. By introducing the estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, the system with LQ controller and observer can be written as

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{BP} \\ \mathbf{0} \end{bmatrix} r(t)$$
$$y(t) = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}.$$

Because the system matrix has a block-triangular form, the eigenvalues of the system are equal to the eigenvalues of blocks on the diagonal of the matrix, i.e., the eigenvalues of the system are equal to the eigenvalues of $\mathbf{A} - \mathbf{BK}$ and the eigenvalues of $\mathbf{A} - \mathbf{LC}$. Therefore, the poles of the LQ controller (i.e. the eigenvalues of $\mathbf{A} - \mathbf{BK}$) are <u>not</u> affected by the observer. (The observer only adds two new poles to the system (i.e. the eigenvalues of $\mathbf{A} - \mathbf{LC}$) without affecting two "old" poles that correspond to the controller.) In the literature, this is known as the separation property/theorem.

Problem 3

a) The autocorrelation function $R_X(\tau)$ can be obtained from the inverse Fourier transform of the spectral density function $S_X(\omega)$:

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(\omega)\} = 0.1\mathcal{F}^{-1}\{1\} = 0.1\delta(\tau),$$

where δ is the Dirac function.

b) The process $X(t)$ is a white-noise process. It is characterized by a constant spectral density function. Moreover, its mean is zero (i.e. $\mu_X = 0$) and its variance is infinite (i.e. $\sigma_X^2 = \infty$).

c) The impulse-response function of the filter can be obtained from its transfer function via the inverse Laplace transform:

$$G(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1,$$

for $t \geq 0$. The output $Y(t)$ of the filter is given by

$$Y(t) = \int_0^t G(\tau)X(t-\tau)d\tau.$$

The mean $\mu_Y(t)$ of the process $Y(t)$ is obtained as follows:

$$\mu_Y(t) = E\left[Y(t)\right] = E\left[\int_0^t G(\tau)X(t-\tau)d\tau\right] = \int_0^t G(\tau)E[X(t-\tau)]d\tau = \int_0^t \mu_X d\tau = \mu_X t.$$

Because the process $X(t)$ has zero mean (i.e. $\mu_X = 0$), we have

$$\mu_Y(t) = 0.$$

Hence, $Y(t)$ is a zero-mean process. The variance $\sigma_Y^2(t)$ of the process $Y(t)$ is given by

$$\begin{aligned}
\sigma_Y(t) = E\left[Y^2(t)\right] &= E\left[\int_0^t G(\tau_1)X(t-\tau_1)d\tau_1 \int_0^t G(\tau_2)X(t-\tau_2)d\tau_2\right] \\
&= E\left[\int_0^t G(\tau_2)\int_0^t G(\tau_1)X(t-\tau_1)X(t-\tau_2)d\tau_1 d\tau_2\right] \\
&= \int_0^t G(\tau_2)\int_0^t G(\tau_1)E\left[X(t-\tau_1)X(t-\tau_2)\right]d\tau_1 d\tau_2 \\
&= \int_0^t G(\tau_2)\int_0^t G(\tau_1)R_X(\tau_2-\tau_1)d\tau_1 d\tau_2 \\
&= 0.1\int_0^t\int_0^t \delta(\tau_2-\tau_1)d\tau_1 d\tau_2 \\
&= 0.1\int_0^t 1 d\tau_2 \\
&= 0.1t.
\end{aligned}$$

A random process is said to be stationary if the density functions describing the process are invariant under a translation of time. Consequently, properties such as the mean and the variance of the process do not change over time if the process is stationary. Because the variance $\sigma_Y(t)$ changes over time, the process $Y(t)$ is not stationary. We note that the filter is an integrator. An integrated white-noise process is called a Wiener (or Brownian-motion) process. Because $X(t)$ is a white-noise process, $Y(t)$ is a Wiener process.

Problem 4

a) The Kalman filter estimates $\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2$ are obtained by repeating the following steps:

Step 1: Compute the Kalman gain

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

Step 2: Update the estimate

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(z_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-)$$

Step 3: Update the error covariance

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)^T + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^T$$

Step 4: Project ahead to $k+1$

$$\hat{\mathbf{x}}_{k+1}^- = \mathbf{\Phi}_k \hat{\mathbf{x}}_k$$
$$\mathbf{P}_{k+1}^- = \mathbf{\Phi}_k \mathbf{P}_k \mathbf{\Phi}_k^T + \mathbf{Q}_k$$

For $k = 0$, we obtain:

$$\begin{aligned}
\mathbf{K}_0 &= \mathbf{P}_0^- \mathbf{H}_0^T (\mathbf{H}_0 \mathbf{P}_0^- \mathbf{H}_0^T + \mathbf{R}_0)^{-1} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.2 \right)^{-1} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \\
\hat{\mathbf{x}}_0 &= \hat{\mathbf{x}}_0^- + \mathbf{K}_0 (z_0 - \mathbf{H}_0 \hat{\mathbf{x}}_0^-) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \left(0.5 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} \right) = \begin{bmatrix} 0.35 \\ 0.1 \end{bmatrix} \\
\mathbf{P}_0 &= (\mathbf{I} - \mathbf{K}_0 \mathbf{H}_0) \mathbf{P}_0^- (\mathbf{I} - \mathbf{K}_0 \mathbf{H}_0)^T + \mathbf{K}_0 \mathbf{R}_0 \mathbf{K}_0^T \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right)^T + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} 0.2 \begin{bmatrix} 0.5 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.4 \end{bmatrix} \\
\hat{\mathbf{x}}_1^- &= \mathbf{\Phi}_0 \hat{\mathbf{x}}_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.35 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.1 \end{bmatrix} \\
\mathbf{P}_1^- &= \mathbf{\Phi}_0 \mathbf{P}_0 \mathbf{\Phi}_0^T + \mathbf{Q}_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 4.4 \end{bmatrix}
\end{aligned}$$

Subsequently, for $k = 1$, we get:

$$\begin{aligned}
\mathbf{K}_1 &= \mathbf{P}_1^- \mathbf{H}_1^T (\mathbf{H}_1 \mathbf{P}_1^- \mathbf{H}_1^T + \mathbf{R}_1)^{-1} = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 4.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 4.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.2 \right)^{-1} = \begin{bmatrix} 0.7143 \\ 0.5714 \end{bmatrix} \\
\hat{\mathbf{x}}_1 &= \hat{\mathbf{x}}_1^- + \mathbf{K}_1 (z_1 - \mathbf{H}_1 \hat{\mathbf{x}}_1^-) = \begin{bmatrix} 0.45 \\ 0.1 \end{bmatrix} + \begin{bmatrix} 0.7143 \\ 0.5714 \end{bmatrix} \left(1 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.45 \\ 0.1 \end{bmatrix} \right) = \begin{bmatrix} 0.8429 \\ 0.4143 \end{bmatrix} \\
\mathbf{P}_1 &= (\mathbf{I} - \mathbf{K}_1 \mathbf{H}_1) \mathbf{P}_1^- (\mathbf{I} - \mathbf{K}_1 \mathbf{H}_1)^T + \mathbf{K}_1 \mathbf{R}_1 \mathbf{K}_1^T \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8429 \\ 0.4143 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 4.4 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8429 \\ 0.4143 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right)^T \\
&\quad + \begin{bmatrix} 0.8429 \\ 0.4143 \end{bmatrix} 0.2 \begin{bmatrix} 0.8429 & 0.4143 \end{bmatrix} = \begin{bmatrix} 0.1429 & 0.1143 \\ 0.1143 & 4.1714 \end{bmatrix} \\
\hat{\mathbf{x}}_2^- &= \mathbf{\Phi}_1 \hat{\mathbf{x}}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8429 \\ 0.4143 \end{bmatrix} = \begin{bmatrix} 1.2571 \\ 0.4143 \end{bmatrix} \\
\mathbf{P}_2^- &= \mathbf{\Phi}_1 \mathbf{P}_1 \mathbf{\Phi}_1^T + \mathbf{Q}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1429 & 0.1143 \\ 0.1143 & 4.1714 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4.5429 & 4.2857 \\ 4.2857 & 8.1714 \end{bmatrix}
\end{aligned}$$

Finally, for $k = 2$, we obtain:

$$\begin{aligned}
\mathbf{K}_2 &= \mathbf{P}_2^- \mathbf{H}_2^T (\mathbf{H}_2 \mathbf{P}_2^- \mathbf{H}_2^T + \mathbf{R}_2)^{-1} = \begin{bmatrix} 4.5429 & 4.2857 \\ 4.2857 & 8.1714 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4.5429 & 4.2857 \\ 4.2857 & 8.1714 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.2 \right)^{-1} \\
&= \begin{bmatrix} 0.9578 \\ 0.9036 \end{bmatrix} \\
\hat{\mathbf{x}}_2 &= \hat{\mathbf{x}}_2^- + \mathbf{K}_2 (z_2 - \mathbf{H}_2 \hat{\mathbf{x}}_2^-) = \begin{bmatrix} 1.2571 \\ 0.4143 \end{bmatrix} + \begin{bmatrix} 0.9578 \\ 0.9036 \end{bmatrix} \left(1.3 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1.2571 \\ 0.4143 \end{bmatrix} \right) = \begin{bmatrix} 1.2982 \\ 0.4530 \end{bmatrix}
\end{aligned}$$

To summarize, we have obtained:

$$\hat{\mathbf{x}}_0 = \begin{bmatrix} 0.35 \\ 0.1 \end{bmatrix}, \qquad \hat{\mathbf{x}}_1 = \begin{bmatrix} 0.8429 \\ 0.4143 \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{x}}_2 = \begin{bmatrix} 1.2982 \\ 0.4530 \end{bmatrix}.$$

b) The system can be written in the following form:

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \mathbf{w}_k$$
$$z_k = h(\mathbf{x}_k) + v_k$$

with

$$\mathbf{f}(\mathbf{x}_k) = \mathbf{\Phi}_k \mathbf{x}_k \qquad \text{and} \qquad h(\mathbf{x}_k) = x_{1,k} + x_{2,k}^3.$$

The extended Kalman filter estimates $\hat{x}_0, \hat{x}_1, \hat{x}_2$ are obtained by repeating the following steps:

Step 1: Compute the Kalman gain

$$\mathbf{H}_k = \left.\frac{dh}{d\mathbf{x}_k}\right|_{\mathbf{x}_k=\hat{\mathbf{x}}_k^-} = \begin{bmatrix}1 & 3\hat{x}_{2,k}^2\end{bmatrix}$$
$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

Step 2: Update the estimate

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k (z_k - h(\hat{\mathbf{x}}_k^-))$$

Step 3: Update the error covariance

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T$$

Step 4: Project ahead to $k+1$

$$\hat{\mathbf{x}}_{k+1}^- = \mathbf{\Phi}_k \hat{\mathbf{x}}_k$$
$$\mathbf{P}_{k+1}^- = \mathbf{\Phi}_k \mathbf{P}_k \mathbf{\Phi}_k^T + \mathbf{Q}_k$$

For $k = 0$, we obtain:

$$\mathbf{H}_0 = \begin{bmatrix}1 & 0.03\end{bmatrix}$$

$$\mathbf{K}_0 = \mathbf{P}_0^- \mathbf{H}_0^T (\mathbf{H}_0 \mathbf{P}_0^- \mathbf{H}_0^T + \mathbf{R}_0)^{-1} = \begin{bmatrix}0.2 & 0\\ 0 & 0.4\end{bmatrix} \begin{bmatrix}1\\ 0.03\end{bmatrix} \left(\begin{bmatrix}1 & 0.03\end{bmatrix} \begin{bmatrix}0.2 & 0\\ 0 & 0.4\end{bmatrix} \begin{bmatrix}1\\ 0.03\end{bmatrix} + 0.2 \right)^{-1} = \begin{bmatrix}0.4996\\ 0.0300\end{bmatrix}$$

$$\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}_0^- + \mathbf{K}_0 (z_0 - h(\hat{\mathbf{x}}_0^-)) = \begin{bmatrix}0.2\\ 0.1\end{bmatrix} + \begin{bmatrix}0.4996\\ 0.0300\end{bmatrix} (0.5 - 0.2010) = \begin{bmatrix}0.3494\\ 0.1090\end{bmatrix}$$

$$\begin{aligned}
\mathbf{P}_0 &= (\mathbf{I} - \mathbf{K}_0 \mathbf{H}_0) \mathbf{P}_0^- (\mathbf{I} - \mathbf{K}_0 \mathbf{H}_0)^T + \mathbf{K}_0 \mathbf{R}_0 \mathbf{K}_0^T \\
&= \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}0.4996\\ 0.0300\end{bmatrix} \begin{bmatrix}1 & 0.03\end{bmatrix} \right) \begin{bmatrix}0.2 & 0\\ 0 & 0.4\end{bmatrix} \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}0.4996\\ 0.0300\end{bmatrix} \begin{bmatrix}1 & 0.03\end{bmatrix} \right)^T \\
&\quad + \begin{bmatrix}0.4996\\ 0.0300\end{bmatrix} 0.2 \begin{bmatrix}0.4996 & 0.0300\end{bmatrix} = \begin{bmatrix}0.1001 & -0.0060\\ -0.0060 & 0.3996\end{bmatrix}
\end{aligned}$$

$$\hat{\mathbf{x}}_1^- = \mathbf{\Phi}_0 \hat{\mathbf{x}}_0 = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \begin{bmatrix}0.3494\\ 0.1090\end{bmatrix} = \begin{bmatrix}0.4583\\ 0.1090\end{bmatrix}$$

$$\mathbf{P}_1^- = \mathbf{\Phi}_0 \mathbf{P}_0 \mathbf{\Phi}_0^T + \mathbf{Q}_0 = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \begin{bmatrix}0.1001 & -0.0060\\ -0.0060 & 0.3996\end{bmatrix} \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 4\end{bmatrix} = \begin{bmatrix}0.4877 & 0.3936\\ 0.3936 & 4.3996\end{bmatrix}$$

Subsequently, for $k = 1$, we get:

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0.0356 \end{bmatrix}$$

$$\mathbf{K}_1 = \mathbf{P}_1^- \mathbf{H}_1^T (\mathbf{H}_1 \mathbf{P}_1^- \mathbf{H}_1^T + \mathbf{R}_1)^{-1} = \begin{bmatrix} 0.4877 & 0.3936 \\ 0.3936 & 4.3996 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0356 \end{bmatrix} \left(\begin{bmatrix} 1 & 0.0356 \end{bmatrix} \begin{bmatrix} 0.4877 & 0.3936 \\ 0.3936 & 4.3996 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0356 \end{bmatrix} + 0.2 \right)^{-1}$$

$$= \begin{bmatrix} 0.6956 \\ 0.7629 \end{bmatrix}$$

$$\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_1^- + \mathbf{K}_1(z_1 - h(\hat{\mathbf{x}}_1^-)) = \begin{bmatrix} 0.4583 \\ 0.1090 \end{bmatrix} + \begin{bmatrix} 0.6956 \\ 0.7629 \end{bmatrix} (1 - 0.4596) = \begin{bmatrix} 0.8342 \\ 0.5212 \end{bmatrix}$$

$$\mathbf{P}_1 = (\mathbf{I} - \mathbf{K}_1 \mathbf{H}_1) \mathbf{P}_1^- (\mathbf{I} - \mathbf{K}_1 \mathbf{H}_1)^T + \mathbf{K}_1 \mathbf{R}_1 \mathbf{K}_1^T$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6956 \\ 0.7629 \end{bmatrix} \begin{bmatrix} 1 & 0.0356 \end{bmatrix} \right) \begin{bmatrix} 0.4877 & 0.3936 \\ 0.3936 & 4.3996 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6956 \\ 0.7629 \end{bmatrix} \begin{bmatrix} 1 & 0.0356 \end{bmatrix} \right)^T$$

$$+ \begin{bmatrix} 0.6956 \\ 0.7629 \end{bmatrix} 0.2 \begin{bmatrix} 0.6956 & 0.7629 \end{bmatrix} = \begin{bmatrix} 0.1387 & 0.0108 \\ 0.0108 & 3.9798 \end{bmatrix}$$

$$\hat{\mathbf{x}}_2^- = \mathbf{\Phi}_1 \hat{\mathbf{x}}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8342 \\ 0.5212 \end{bmatrix} = \begin{bmatrix} 1.3554 \\ 0.5212 \end{bmatrix}$$

$$\mathbf{P}_2^- = \mathbf{\Phi}_1 \mathbf{P}_1 \mathbf{\Phi}_1^T + \mathbf{Q}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1387 & 0.0108 \\ 0.0108 & 3.9798 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4.1402 & 3.9906 \\ 3.9906 & 7.9798 \end{bmatrix}$$

Finally, for $k = 2$, we obtain:

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 0.8151 \end{bmatrix}$$

$$\mathbf{K}_2 = \mathbf{P}_2^- \mathbf{H}_2^T (\mathbf{H}_2 \mathbf{P}_2^- \mathbf{H}_2^T + \mathbf{R}_2)^{-1} = \begin{bmatrix} 4.1402 & 3.9906 \\ 3.9906 & 7.9798 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8151 \end{bmatrix}$$

$$\cdot \left(\begin{bmatrix} 1 & 0.8151 \end{bmatrix} \begin{bmatrix} 4.1402 & 3.9906 \\ 3.9906 & 7.9798 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8151 \end{bmatrix} + 0.2 \right)^{-1} = \begin{bmatrix} 0.4579 \\ 0.6500 \end{bmatrix}$$

$$\hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_2^- + \mathbf{K}_2(z_2 - h(\hat{\mathbf{x}}_2^-)) = \begin{bmatrix} 1.3554 \\ 0.5212 \end{bmatrix} + \begin{bmatrix} 0.4579 \\ 0.6500 \end{bmatrix} (1.3 - 1.4970) = \begin{bmatrix} 1.2652 \\ 0.3932 \end{bmatrix}$$

To summarize, we have obtained:

$$\hat{\mathbf{x}}_0 = \begin{bmatrix} 0.3494 \\ 0.1090 \end{bmatrix}, \qquad \hat{\mathbf{x}}_1 = \begin{bmatrix} 0.8342 \\ 0.5212 \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{x}}_2 = \begin{bmatrix} 1.2652 \\ 0.3932 \end{bmatrix}.$$