# Integer Least Squares
# Search and Reduction Strategies

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## DEDICATION

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## ABSTRACT

In the worst case the integer least squares (ILS) problem is NP-Hard. Since its solution has many practical applications, there have been a number of algorithms proposed to solve it and some of its variations e.g., the box-constrained ILS problem (BILS). There are typically two stages to solving an ILS problem, the reduction and the search. Obviously we would like to solve instances of the ILS problem as quickly as possible, however most of the literature does not compare the run-time or FLOP counts of the algorithms, instead they use a more abstract metric (the number of nodes explored during the search). This metric does not always co-incide with the algorithms run-time. This thesis will review some of the most effective reduction and search strategies for both the ILS and BILS problems. By comparing the run-time performance of some search algorithms, we are able to see the advantages of each, which allows us to propose a new, more efficent search strategy that is a combination of two others. We also prove that two very effective BILS reduction strategies are theoretically equivalent and propose a new BILS reduction that is equivalent to the others but more efficeint.

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# CHAPTER 1
# Introduction

## 1.1 Least Squares Problem

Consider the following linear model for some observation vector $y$,

$$y = Ax + v. \tag{1.1}$$

Where $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ is called the "design matrix" and has full column rank, and $v \in \mathbb{R}^m$ is a noise vector which we assume is normally distributed with mean $0$ and covariance matrix $\sigma^2 I$. We would like to find the unique solution $x \in \mathbb{R}^m$ which minimizes the least squares residual,

$$\|Ax - y\|_2^2 . \tag{1.2}$$

This is called the least squares (LS) problem. If we expand (1.2) and set its gradient to $0$, we will arrive at the well known "normal equations" which can be written in matrix form as,

$$A^T A x = A^T y \tag{1.3}$$

$$x = (A^T A)^{-1} A^T y. \tag{1.4}$$

The solution of the least squares problem has numerous applications in almost every field of science and engineering.

## 1.2 Integer Least Squares Problems

The integer least squres (ILS) problem is a modification of the LS problem where the solution vector $x \in \mathbb{Z}^m$ is an unknown integer vector. We no longer have a closed-form solution for $x$ in this case, in fact, the problem is provably NP-Hard in the worst case.

A modification to the ILS problem is the box-constrained integer least squares problem (BILS). Here we have the following constraint on the solution vector,

$$x \in \mathcal{B} \tag{1.5}$$

$$\mathcal{B} = \{x \in \mathbb{Z}^n : l \leq x \leq u, l \in \mathbb{Z}^n, u \in \mathbb{Z}^n\}. \tag{1.6}$$

Some important applications such as MIMO wireless signal decoding depend on the solution of the BILS problem. MIMO stands for “multiple-input multiple-output”, it refers to the case where a wireless system has multiple input antennas transmitting a signal which is received by multiple output antennas. The signal received is our input vector $y$ from (1.1), it has undergone some linear transformation by the known “channel matrix” $A$ (design matrix) and some noise has been introduced during the transmission. Originally, we know that each element of $x$ came from some finite set of symbols that we may want to transmit or receive (we model this property with B). The purpose of such a system is to maximize throughput, however, the overall throughput of the system will depend on how quickly we can solve the BILS problem. Of course we need not solve the BILS problem exactly, but under the assumption that the noise has $0$ mean and is normally distributed, the BILS solution is more likely than any other possible solution to be the true solution $x$.

Other applications of BILS and ILS include global positioning systems, cryptography, lattice design, etc... Any application where the elements of $x$ are known to be integer, we should use ILS. If the elements of $x$ are drawn from some finite set, BILS is appropriate.

Even though the problem is NP-Hard, we still have some hope to get solutions quickly. In [1] the authors prove that under reasonable assumptions on the variance in the noise, the ILS problem can be solved in polynomial time using standard search algorithms.

The usual approach to solving an ILS or BILS problem consists of two phases, reduction and search. In the reduction phase, we transform the problem into an equivalent, but easier one. This involves manipulations such as permutations and possibly integer gauss transformations on the design matrix $A$. After reduction, we proceed to the search phase where we try to enumerate the possible solutions in an efficient manner.

# CHAPTER 2
# Search Algorithms

## References

[1] Babak Hassibi and Haris Vikalo. On the sphere-decoding algorithm i. expected complexity. In *IEEE Transactions on Signal Processing*, volume 53, August 2005.