MSci Project Handbook
2020–2021

PER ARDUA AD ALTA

School of Mathematics
University of Birmingham

Contents

Regulations for MSci projects

1. Introduction

Every final year student on the MSci programme must undertake a project worth 40 credits. The project module gives the students an opportunity to study in depth, under supervision, some area of mathematics or statistics that particularly interests them. It is intended to give students an idea of mathematical research and teaches key mathematical skills, including writing a properly referenced project dissertation, oral presentation of advanced material and mathematical typesetting.

2. Conduct of the Project

2.1 Members of staff (as potential supervisors) will submit proposals for possible projects to be distributed to students on the MSci programme (including those studying abroad) in time for projects to be assigned before the end of the session preceding the student's final year of study.

2.2 Each project will be overseen by a supervisor and a co-assessor. Joint supervisors and/or co-assessors may be deemed appropriate.

2.3 All MSci projects will run over both terms of the session. With the agreement of the supervisor and the director, students may choose to do either 20 credits worth of work in each term or 10 credits in the Autumn term and 30 in the Spring term.

2.4 Supervisors may wish to suggest some preparatory reading or investigation over the summer vacation before the official start of the project.

2.5 Co-assessors will be appointed no later than the start of the new session and a project plan will be agreed between the supervisor, co-assessor and student at a meeting with the student held during the first 2 weeks of the Autumn term.

2.6 The co-assessor should be present for at least one additional meeting in each term. The timing of such meetings should enable the co-assessor to contribute to the progress review process in the School.

2.7 The student shall have regular meetings with the supervisor. In the Autumn term, weekly meetings are recommended.

2.8 Save in exceptional circumstances and only with the permission of the Head of School, no change of supervisor or project allocation will be permitted after the end of the second week of the final year.

2.9 Introductory training in LaTeX will be provided early in the Autumn term for those students who have not been exposed to LaTeX in their third year.

2.10 Students will produce a substantial dissertation based on the mathematical or statistical work they have carried out over the two terms.

2.11 Students will give an oral presentation based on the project work. This will be scheduled as soon as possible after the end of the Main Examination Period.

2.12 Members of staff will not normally be available for consultation relating to their project after the end of Spring Term. Students who wish to consult their supervisor or other member of staff about their project after the end of Spring Term must obtain the permission of the Director of the MSci Programme.

3. Assessment

3.1 The following elements will contribute to the assessment of the project: two interim reports, a dissertation and an oral presentation.

3.2 The two interim reports will outline research undertaken to date. The first will be produced by the end of the sixth week of Semester 1, the second by the end of the third week of the Semester 2. Each will provide a summary of the work done so far and an outline plan of future work, possibly including library search, other background research or bibliography.

3.3 The interim reports will be assessed by the co-assessor.

3.4 The dissertation should include any significant work undertaken in the course of the project, though preliminary exercises may be omitted or relegated to an appendix. The length of the dissertation should not exceed 100 pages, excluding references and appendices. The font size must be 12pt.

3.5 The dissertation will be assessed by the supervisor and co-assessor.

3.6 The dissertation will be written for specialists in the field. They will be prepared using LaTeX (or other version of TeX) unless there are compelling reasons to the contrary and the permission of the Head of School has been obtained.

3.7 You should produce a signed and dated copy of the following declaration:'I warrant that the content of this dissertation is the direct result of my own work and that any use made in it of published or unpublished material is fully and correctly referenced.' A copy of this page should be submitted to the Undergraduate Office by 12 noon on the Wednesday of the last week of Semester 2.

3.8 The dissertation should be uploaded on Canvas by noon on the Thursday of the first week of the Easter vacation following the commencement of the project.

3.9 Penalties will be applied for late submission. Projects submitted after the deadline will lose 1 mark. Projects submitted after 4.15 p.m. on the day of the deadline will lose an additional 4 marks. Projects submitted after noon the following Monday will lose a further additional 1 mark. For each subsequent day thereafter, an additional 1 mark will be deducted for a project not submitted before noon. No project will be accepted later than noon on the Friday two weeks after the deadline.

3.10 The oral presentation should last for about 20 minutes, with a further 10 minutes for questions and discussion about the work contained in the project. This presentation will be assessed by a small committee appointed by the Head of School and chaired by the Director of the MSci Programme. It will contain representatives from the research groups in the School.

3.11 The oral presentations should be accessible to a general mathematical audience and should be assessed as presentations of technical material. Supervisors are encouraged to offer advice in their preparation.

3.12 The presentation slides should be submitted online the day before the presentation is scheduled.

3.13 In exceptional circumstances and with the permission of the Head of School the oral presentation may be waived. In such cases, detailed assessment arrangements will be agreed by the supervisor, the Director of the MSci Programme and the Head of School. Some additional assessed element, such as a critique of an important paper in an area related to the project, may be required.

3.14 Attendance at all oral presentations is required. Penalties for non-attendance will be assessed by the Director of the MSci Programme.

4. Marking Procedure

4.1 The project will be marked according to the following categories:

- technical content of the dissertation (42 marks);
- development and execution of the dissertation (20 marks);
- presentation of the dissertation (13 marks);
- conclusions of the dissertation (9 marks);
- two interim reports (3 marks each);
- oral presentation (10 marks)

4.2 The supervisor and co-assessor should produce marks for the dissertation independently of one another according to an agreed marking scheme. After discussion as necessary, a final mark will then be agreed. If no agreement is reached between the supervisor and co-assessor, the Head of School will institute an appropriate procedure to arrive at the final mark.

4.3 The marking scheme for the dissertation should reflect the nature of the project. Suggestions of possible categories for assessment are listed below. A mark of zero will be returned for presentation if the dissertation has not been typeset using (La)TeX, unless specific permission has been given by the Head of School.

• **Technical content of dissertation**: mastery of the subject matter; historical perspective; relation to undergraduate courses; conceptual and methodological difficulty; mathematical accuracy and clarity of exposition; innovative aspects; examples cited (relevance of); originality, independently derived results and proofs, new results, proofs, simple generalizations, significant insight, original research, examples constructed.

• **Development and execution of project**: plan outline, aims and objectives; sensible development of aims; appropriate approach and methodology; evidence of work done, computer programs, experiments, field work, etc.; library research, use of references; new source materials; level of independence and of assistance; analysis and appreciation of results, comparison with experimental data or known examples.

• **Presentation of dissertation**: aims, structure and conclusions clearly laid out; suitable length; quality of exposition and organisation; diagrams; bibliography complete, appropriate, properly referenced, other source material properly referenced.

• **Conclusions of dissertation**: Conclusion and suggestions for further work; stated aims met or reasons why not; placement of work in wider context.

4.4 The supervisor and co-assessor will jointly provide a written report on the dissertation, which includes the mark scheme, and information on how their marks were arrived at. This report shall be made available to the external examiner together with a copy of the project dissertation at least two weeks before the School Board of Examiners meets to discuss examination results.

4.5 The interim reports will be marked taking into account evidence of reasonable progress and accuracy of summary of work to date, viability of proposed plan of study and other appropriate criteria. A mark of zero will be returned if a report has not been typeset using (La)TeX unless specific permission has been given by the Head of School.

4.6 Marks for the oral presentations will be awarded according to the following criteria:

- delivery (audibility, use of slides, blackboard etc; style, interaction with audience);
- structure (organization, clarity of explanation, comprehensibility);
- content (level, amount of material covered, success of presentation in relation to difficulty of material);
- questions (mastery of topic and handling questions).

4.7 Approximately two weeks prior to the oral presentations, there will be a meeting of project supervisors and co-assessors to discuss and moderate the marks of the written work. This meeting will be chaired by the Director of the MSci Programme.

5. Plagiarism

5.1 The School abides by the University's guidelines on plagiarism. The University regards plagiarism as a very serious form of cheating and will impose the severest penalties in all cases of cheating.

5.2 Supervision and scrutiny of project work will be sufficiently closely arranged to ensure that:

• signs of plagiarism (whether intentional or unintentional) in early drafts or pieces of work may be detected in good time and drawn to the attention of the student and then followed up by a written warning if necessary;

• students are not allowed to get so far behind with their work that they may be tempted to turn to plagiarism in an effort to catch up.

5.3 Students must make appropriate use of references and footnotes when using material from published or unpublished sources to avoid any suspicion of plagiarism. If in any doubt the student should discuss the matter with their project supervisor.

5.4 Students must include in their dissertation a signed declaration that unreferenced material is their own work as described in 3.7.

6. Health and Safety

6.1 The School abides by the University Health and Safety policy. Where a project is anything other than low risk, it is the responsibility of the project supervisor and ultimately the Head of School to ensure that appropriate Health and Safety procedures are in place. If a supervisor deems that a project is not low risk, or is in any doubt, the matter should be referred to the Director of the MSci Programme. Desk based work, standard use of proprietary electrical equipment, use of Class 1 lasers and on-campus data collection are examples of low risk activity: off-campus data collection, experiments with electrical equipment, use of robots, testing materials to breaking point are examples of possible higher risk activities.

Projects in Pure Mathematics

1 Topics in Graph Theory

Supervisor: **Deryk Osthus**

Co-assessor: **Daniela Kühn**

Description: The aim of the project is to study one of the topics of last years Graph Theory course in more depth. There will be scope to study recent research developments. Possible directions include:

- *Hamilton cycles in graphs and directed graphs:* A Hamilton cycle in a graph G is a cycle which contains all vertices of G. Unfortunately, nobody knows how to decide efficiently whether a graph has a Hamilton cycle. Probably this is not possible, as otherwise P = NP. So researchers have been looking for natural and simple sufficient conditions which guarantee that a (directed) graph has a Hamilton cycle.
- *Graph colouring problems:* Here the question is how many colours are needed to colour the vertices of a graph in such a way that endvertices of an edge have different colours. This has connections to the 4-colour-theorem which states that every planar map can be coloured with at most 4 colours such that adjacent regions have different colours.
- *Extremal Graph Theory:* Here the general question is which conditions force the existence of a certain substructure in a graph. So this is more general than the Hamilton cycle problem mentioned above.

Prerequisites: LH Graph Theory

References:

[1] R. Diestel, Graph Theory, Springer 1997.

[2] D. West, Introduction to Graph Theory, Prentice Hall 2001.

2 The mathematics of card shuffling

Supervisor: **Nikolaos Fountoulakis**

Co-assessor: **Matthew Jenssen**

Description: How many times one should shuffle a deck of cards so that it is random? We will put this question in the context of Markov chains and random walks on the symmetric group. This project will consider a number of card shuffling methods which will be analysed as random walks on permutations. If run for sufficiently long time, they provably shuffle a deck of cards almost perfectly, that is, an almost random permutation is created. We will utilise a mix of probabilistic and algebraic techniques in order to analyse the number of steps needed to reach an almost perfect shuffle.

Prerequisites: 2AC, 2S

References:

[1] D. Levin, Y. Peres and E. Wilmer, *Markov chains and mixing times*, AMS, 2017.

[2] P. Diaconis, *Group representations in probability and statistics*, Inst. of Mathematical Statistics,1988.

3 Rumour spreading and epidemic processes on graphs

Supervisor: **Nikolaos Fountoulakis**

Co-assessor: **Matthew Jenssen**

Description: This project will explore a variety of mathematical models for the spread of rumours or diseases on a network. The main questions we will address are the following. How long does it take until a rumour is spread over the entire network? How long will a disease will stay active for in certain classes of networks? The models we will consider are probabilistic. They exploit randomness in order to spread a rumour as quickly as possible or describe the evolution of an infection as a sequence of random infections between neighbouring nodes.

Prerequisites: 2S

References:

[1] A. Frieze and G. R. Grimmett, Shortest-path problem for graphs with random arc-lengths, *Disc. App. Math.* **10** (1985), 57–77.

[2] T. E. Harris, Contact interactions on a lattice, *Annals of Probability* **2** (1974), 969–988.

[3] R. Pemantle, The contact process on trees, *Annals of Probability* **20** (1992), 2089–2116.

4 Topics in Combinatorics

Supervisor: **Andrew Treglown**
Co-assessor: **Richard Montgomery**
Description: This project will involve focusing on a major recent theme in Combinatorics, with the potential to look at start-of-the-art research papers. The nature of the project is flexible; possible directions include:

- *Embedding problems for graphs:* A wealth of research has gone into establishing sufficient conditions that ensure a graph contains a given subgraph. For example, Mantel's theorem tells us a graph with more than half of all possible edges must contain a triangle. Many powerful techniques have been developed for attacking questions of this type such as the *absorbing method*. The project could explore some of the key recent results in the area, as well as the methods used in their proofs.
- *Posets:* A poset is a set equipped with a relationship $\leq$ that satisfies reflexive, transitive and antisymmetric properties. One famous poset is the so-called Boolean lattice $\mathcal{P}(n)$; this set simply consists of all subsets of $\{1,\ldots,n\}$ ordered by inclusion. So for example $\{1,2\} \leq \{1,2,3\}$ since $\{1,2\} \subseteq \{1,2,3\}$. One can ask extremal questions about the Boolean lattice $\mathcal{P}(n)$. For example, what is the size of the largest collection of sets from $\mathcal{P}(n)$ so that no two of the sets is comparable (i.e. one isn't the subset of the other)? The project could explore some of the fundamental topics in this area.

Prerequisites: 3Com, 3GphT
References: [1] P.J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press, 1994.

5 Arithmetic Ramsey theory

Supervisor: **Andrew Treglown**
Co-assessor: **Matthew Jenssen**
Description: Ramsey theory is a central branch of combinatorics, which has connections to other branches of mathematics such as number theory. A typical question in this field asks how large a mathematical structure must be in order to guarantee a particular type of pattern emerges. For example, in 1930 Ramsey proved that no matter how one r-colours the edges of the complete graph K_n on n vertices, provided n is sufficiently large (compared to r and t), K_n must contain a monochromatic copy of K_t (that is, a copy of K_t whose edges are all coloured the same). Although Ramsey's theorem is often seen as the foundation stone of the area, Ramsey-type results in the setting of the integers were proved before this by Hilbert (1892), Schur (1916) and van der Waerden (1927). Indeed, Schur's theorem asserts that however one r-colours the first n natural numbers, provided n is sufficiently large, there is a monochromatic solution to $x + y = z$.
This project will focus on such Ramsey properties of the integers. The project has the potential to study a range of different topics including: (i) Folkman's theorem, a wide generalisation of Schur's theorem; (ii) Ramsey properties of *random* sets of integers; and (iii) bounds on so-called van der Waerden numbers.
Prerequisites: 3Com
References: [1] B.M. Landman and A. Robertson, *Ramsey Theory on the Integers*, American Mathematical Society, 2004.

6 The Random Graph Process

Supervisor: **Dr Richard Montgomery**
Co-assessor: **Dr Allan Lo**
Description: The study of random graphs brings together combinatorial and probabilistic techniques to create a rich area. One of the most elegant ways to study random graphs is to consider the *random graph process*. Starting with a large number of vertices, but no edges, we iteratively choose uniformly at random a pair of vertices without an edge between them, and add this missing edge. The key question is: how does a typical random graph process develop? This project will look at different aspects of the random graph process, considering questions like the following. How long do we expect the process to run before a cycle appears? How long before other chosen subgraphs, like Hamilton cycles, are likely to appear? For each k, how long before there is typically a subgraph with minimum degree k? When such a minimum degree k subgraphs appears, what will it typically look like?

Prerequisites: 3GphT and 2S
References: [1] Bollobás, B. (2001). Random graphs. Cambridge University Press.
[2] Janson, S., Luczak, T. and Rucinski, A. (2011). Random graphs. John Wiley & Sons.

7 Algebraic techniques in combinatorics

Supervisor: **Dr Richard Montgomery**

Co-assessor: **Dr Andrew Treglown**

Description: Algebraic techniques provide many of the most surprising and elegant proofs in combinatorics. With appropriate chosen settings, notions of linear independence and the study of the roots of polynomials may be used to infer strong results in extremal combinatorics. This project will look at the influence of algebraic techniques in Combinatorics through the exposition of the proofs of several important results, and the common methods they employ.

Suitable results would include, for example, Lovász's determination of the Shannon capacity of the 5-cycle, the Ray-Chaudhuri-Wilson Theorem, and recent results on the cap-set problem using the polynomial method.

Prerequisites: 3Com, 3GphT and 2AC

References:

[1] Babai, L. and Frankl, P. (1992) Linear Algebra Methods in Combinatorics: With Applications to Geometry and Computer Science.

[2] Jukna, S. (2011) Extremal combinatorics: with applications in computer science. Springer.

8 Infinitary combinatorics

Supervisor: **Robert Leek**

Co-assessor: **Richard Kaye**

Description: Infinite sets (especially uncountable) behave very differently from finite sets. A simple example is that a function between finite sets of the same cardinality is injective if and only if it is surjective; the corresponding statement for infinite sets does not hold.

After developing the basic theory of ordinal and cardinal numbers, including transfinite induction / recursion, the student will have a choice of topics to investigate, such as:

- *Aronszajn*, *Suslin*, and *Kurepa* trees and lines.
- The *(Generalised) Continuum Hypothesis* ((G)CH) as well as other cardinal-arithmetic statements, such as *Easton's theorem* and the *Singular Cardinals Hypothesis* (SCH).
- Clubs and stationary sets, and club-guessing principles such as $\diamondsuit$, $\clubsuit$, and $\square$.
- Combinatorial Cardinal Characteristics of the Continuum: *Cichón's diagram* connecting the measureable and topological properties of subsets of the real line, *Martin's axiom* and 'long' diagonalisations.
- Large cardinals and elementary embeddings.

The project can either focus solely on the combinatorics or look into applications of them to other areas of mathematics.

Prerequisites: Mathematical maturity and rigour. No finitary combinatorics knowledge will be required. If the student wants to investigate applications, they should have some familiarity in that broad area.

References:

[1] Kenneth Kunen, *Set Theory*. Studies in Logic, 34. College Publications, London., 2011. pp. viii+401. ISBN: 978-1-84890-050-9.

9 Extremal set theory

Supervisor: **Eoin Long**

Co-assessor: **Allan Lo**

Description: Extremal set theory is an active area of Combinatorics in which the main objects of interest are collections of finite sets. Though the structures are simple, they offer much flexibility, and can be used to represent many interesting problems. The project will investigate this rich area and present some recent developments. The project is flexible, but two possible directions include:

- *Intersection theorems.* Here one is given a collection $\mathcal{F}$ of finite sets satisfying certain restrictions and aims to describe the intersections which appear among the sets in $\mathcal{F}$ (e.g. the values of $|F_1 \cap F_2|$ where $F_1, F_2 \in \mathcal{F}$). Results here are widely applicable, and have some surprising consequences in geometry and number theory.
- *Isoperimetric inequalities.* Here one is interested in the 'growth' of a collection $\mathcal{F}$ of finite sets. Formally, given a collection $\mathcal{F}$ of finite sets, with $F \subset X$ for all $F \in \mathcal{F}$, let $\mathcal{G}$ denote those subsets of X which either lie in $\mathcal{F}$ or differ on a single element from a set in $\mathcal{F}$. (You could think of $\mathcal{G}$ as a 'neighbourhood' of $\mathcal{F}$.) How large must $|\mathcal{G}|$ be, given $|\mathcal{F}|$? This topic has close connections in Analysis and to the 'concentration of measure' phenomenon.

Prerequisites: 3Com, 3GphT.

References:

[1] B. Bollobás. *Combinatorics: Set Systems, Hypergraphs, Families of Vectors, and Combinatorial Probability.* 1986, Cambridge University Press.

[2] P. Frankl and N. Tokushige. *Invitation to intersection problems for finite sets*, Journal of Combinatorial Theory, Series A, Volume 144, 157–211.

10 Topics in combinatorial geometry

Supervisor: **Eoin Long**

Co-assessor: **Andrew Treglown**

Description: Many compelling combinatorial problems arise from geometric objects. This project will explore central topics in the field of combinatorial geometry, focusing on core results and proof techniques. There are many possible directions here, which are best illustrated by examples:

- (*Metric structure*) How many unit distances can exist between n points in the plane?
- (*Ramsey-type behaviour*) Given a set of n points in the plane, no three on a line, how many points from the set can you find in convex position?
- (*Discrepancy*) How should you select a set $S \subset [0,1]^d$ with $|S| = n$ in order to minimize $\left|\text{area}(R) - \frac{|R \cap S|}{n}\right|$ over all axis-parallel boxes $R \subset [0,1]^d$?
- (*Intersection patterns*) You are given a collection of n balls in the plane and told that among any 100 balls from the collection, some 3 must share a point. Can you find a constant number of points (i.e. independent of n) which intersect all the balls?

Prerequisites: 2LALP, 3Com, 3GphT.

References:

[1] P. Agarwal and J. Pach, *Combinatorial geometry*, Volume 37 (2011), John Wiley & Sons.

[2] J. Matoušek. *Lectures on discrete geometry*, Volume 212, (2013) Springer.

11 Hypergraph Turán Problem

Supervisor: **Allan Lo**

Co-assessor: **Richard Montgomery**

Description: The Turán number $\mathrm{ex}(n, F)$ is the maximum number of edges in a graph on n vertices without a copy of F. A celebrated result of Turán determined the exact value of $\mathrm{ex}(n, K_t)$ for all $t \in \mathbb{N}$. It is a long-standing problem in Extremal Combinatorics to develop some understanding of these numbers. On the other hand, the approximate value of $\mathrm{ex}(n, F)$ is well understood. Erdős and Stone showed that if the chromatic number $\chi(F) = t$, then $\mathrm{ex}(n, F) = \mathrm{ex}(n, K_t) + o(n^2)$. However much less is known about Turán numbers of hypergraphs. Here, an r-uniform hypergraph is a generalisation of a graphs whose edges consist of r vertices instead of 2. The hypergraph analogous result of Erdś̋s and Stone does not hold. Furthermore, Frankl and Rödl proved that there does not exists a countable set $\Omega \subseteq [0, 1]$ such that $\mathrm{ex}(n, F) = (t + o(1))\binom{n}{r}$ for some $t \in \Omega$. Even determining the asymptotic value of Turán number of complete r-uniform hypergraphs remains one of the most important open questions in Extremal Combinatorics. Despite the lack of progress on the hypergraph Turán problem, different ideas and powerful approaches has been developed. For instance, hypergaphs Lagrangian, links graphs, stability, counting, Flag algebra, Delta-systems, intersection semilattice lemma. The project explores some of these techniques as well as recent developments.

Prerequisites: Graph Theory (3GphT)

References:

[1] A. Sidorenko. *What we know and what we do not know about Turn numbers*. Graph and Combin. 11 (1995) 179–199.

[2] P. Keevash. *Hypergraph Turán problems*. Surveys in combinatorics 392 (2011) 83–140.

[3] D. Mubayi and J. Verstraëte. *A survey of Turn problems for expansions*. Recent Trends in Combinatorics. Springer, Cham, 2016. 117–143.

12 Ramsey theory: Complete disorder is impossible

Supervisor: **Allan Lo**

Co-assessor: **Johannes Carmesin**

Description: In a party of six people there is always a group of three people who all know each other or are all strangers. However, how large do you need a party to be to ensure there is a group of six people who all know each other or are all strangers? (This question is notoriously difficult. If you solve it exactly then you would become a famous mathematician!)

Ramsey theory concerns questions of this type. Namely, how large do you require a structure to be to guarantee some desired substructure. Many problems in Ramsey theory can be stated in a graph theoretical way. However, Ramsey theory has developed into a broad area with questions arising, for example, from Number theory. The project has scope to focus on a number of topics in Ramsey theory. In particular, there will be chance to explore recent breakthroughs in the subject. Here are just two possible topics to investigate:

Bounding Ramsey number: In general, determine the (asymptotic) values of $R(s, t)$ is an open problem. Ajtai, Komlos and Szemerédi showed that when fixed s and t tends to infinity, $R(s, t) \leq O(\frac{t^{s-1}}{(\log t)^{s-2}})$. This bound is sharp when $s = 3$. However, less seems to be known for the multicolour or hypergraph generalisations.

Partitioning into monochromatic components: A classical result of Gerencsér and Gyárfás from 1967 states that any 2-edge-coloured complete graph K_n can be partitioned into a red and blue path. Can we partition into a red and blue cycle instead? A recent elegant proof of Bessy and Thomassé answers this question affirmatively. Is similar statement holds for r-edge-colouring with $r \geq 3$? How about replacing the host graphs with graphs with small independence number; large minimum degree; or bipartite graphs. See [5] for a survey on the recent developments along this direction.

Prerequisites: Graph Theory (3GphT) and Combinatorics & Communication Theory (3Com)

References:

[1] R. Diestel, *Graph Theory*, Springer, 1997.

[2] R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*, John Wiley & Sons, 1990.

[3] S. Radziszowski, *Small Ramsey numbers*, Electron. J. Combin, DS1.

[4] D. Conlon, J. Fox, and B. Sudakov. *Recent developments in graph Ramsey theory*. Surveys in combinatorics 424 (2015) 49–118.

[5] A. Gyárfás, *Vertex covers by monochromatic pieces - a survey of results and problems*, Discrete Math. 339 (2016) 1970–1977.

13 Combinatorics in 3-dimensions

Supervisor: **Johannes Carmesin**

Co-assessor: **TBA**

Description: Graph Minor Theory sits at the interface between Topology and Graph Theory, with many algorithmic applications. The area started with Kuratowski's characterisation of planarity in terms of forbidden minors as well as the famous Hadwiger Conjecture, which would be a far reaching generalisation of the famous Four-Colour-Theorem. An important aspect of Graph Minor Theory is the connection between embeddings of graphs in 2-dimensional surfaces and the Minor Relation. Here a minor of a graph is obtained by deleting edges and contracting connected edge sets to single vertices. In the context of plane graphs the minor relation is particularly natural; it is equivalent to the subgraph relation combined with planar duality. The discovery of this connection began with Kuratowski's theorem in the 1930s and led to the proof of the Robertson-Seymour Theorem, which is often regarded as the deepest theorem of Combinatorics today.

The Graph Minor Theory of Robertson and Seymour has had a transformative impact on Combinatorics as a whole with deep implications to Computer Science. In very rough terms, their structure theorem establishes a connection between the minor relation on graphs and embeddings of graphs in 2-dimensional surfaces. The simplest example of this connection is Kuratowski's planarity criterion from 1930, which characterises the class of planar graphs in terms of forbidden minors. Recently, I have been able to prove a 3-dimensional analogue of Kuratowski's theorem, answering questions of Lovasz, Pardon and Wagner. Opening the door to many related exciting research questions. Indeed, one can try to extend other fundamental theorems from Graph Theory to this new 3-dimensional setting.

The first step in the project would be to acquire the necessary background. The second step would be to formulate a conjecture of how a certain theorem from Graph Theory might extend to 3D. The ultimate goal would be to prove this conjecture.

This project would be an ideal chance to try how doing a PhD would be like, and to demonstrate that you are a top student in the school (with the possiblity of obtaining a research paper at the end of this project). It would be greatly appreciated if students got in contact with myself (email: j.carmesin@bham.ac.uk) when they are interested in this topic in order to discuss further details.

Prerequisites: background in Topology and Combinatorics, or a strong background in one of these areas.

References:
[1] J. Carmesin, *Embedding simply connected 2-complexes in 3-space – I. A Kuratowski-type characterisation*, available at "https://arxiv.org/pdf/1709.04642.pdf"

[2] R. Diestel, *Graph Theory* (5th edition), Springer-Verlag, 2016.

[3] J. Oxley, *Matroid Theory* (2nd edition). Oxford University Press, 2011.

[4] M. A. Armstrong, *Basic Topology.* Springer-Verlag, 1983.

14 Geometrisation of 3-manifolds.

Supervisor: **Johannes Carmesin**

Co-assessor: **TBA**

Description: Proving the Poincare Conjecture, Perelman proved that every simply connected compact 3-manifold is isomorphic to the 3-sphere. A far reaching generalisation of this is the Geometrisation Theorem, which provides a decomposition theorem of compact 3-manifolds into 3-manifolds from ”basic classes”.
The goal of this project is to understand the Geometrisation Theorem, and to give a clear presentation of that theorem understandable to a student. The first step would be to read the relevant aspects in the literature. The second step would be to become familiar with the basic classes of 3-manifolds appearing in the Geometrisation Theorem and to give appropriate examples. A focus in this project would be on explaining relevant concepts by examples and in simple language.
One of the difficulties in this project (or motivation for this project) is that the topic requires a diverse background, and that some parts of the literature might be hard to understand for a student.
It would be greatly appreciated if students got in contact with myself (email: j.carmesin@bham.ac.uk) when they are interested in this topic in order to discuss further details.

Prerequisites: background in Differential Geometry and Topology, a little bit of Algebraic Topology might also be helpful

References:
[1] G. Perelman, *The entropy formula for Ricci flow and its geometric applications*, 2002, available at ”https://arxiv.org/abs/math.DG/0211159”

[2] A. Hatcher, *Algebraic Topology.*, Cambridge Univ. Press, 2002.

[3] A. Hatcher, *Notes on basic 3-manifold topology.*, available at ”http://www.math.cornell.edu/ hatcher/3M/3Mfds.pdf”

[4] J. Morgan and G. Tian, *Ricci Flow and the Poincare Conjecture.*, available at ”https://arxiv.org/abs/math/0607607v2”

15 Heat-flow monotonicity in mathematical analysis

Supervisor: **Jon Bennett**

Co-assessor: **Alessio Martini**

Description: One form of the classical Cauchy–Schwarz inequality states that

$$\int f(x)g(x)dx \leq \left(\int f(x)^2dx\right)^{1/2}\left(\int g(x)^2dx\right)^{1/2},$$

where f and g are suitable nonnegative real functions. It was discovered relatively recently that this fundamental inequality has a rather striking proof based on *heat flow*.

Theorem. *Suppose that $u(x,t)$ and $v(x,t)$ are the solutions to the heat equation with initial temperature distributions $f(x)^2$ and $g(x)^2$ respectively. Then, the function*

$$Q(t) := \int u(x,t)^{1/2}v(x,t)^{1/2}dx$$

is increasing,

$$\lim_{t\to 0} Q(t) = \int fg$$

and

$$\lim_{t\to\infty} Q(t) = \left(\int f^2\right)^{1/2}\left(\int g^2\right)^{1/2}.$$

As you may be able to spot, the Cauchy–Schwarz inequality is an immediate corollary of this.
This project will seek analytic, algebraic and geometric explanations for such heat-flow monotonicity phenomena in the context of a variety of inequalities in mathematical analysis.

Prerequisites: 2RCA, MVA and 3FFA.

References:
[1] J. Bennett, “Heat-flow monotonicity related to some inequalities in euclidean analysis”, Harmonic and Partial Differential Equations, Contemporary Mathematics 505, American Mathematical Society, (2010), 85–96.

16 Analysis of Harmonic Measure

Supervisor: **Andrew Morris**

Co-assessor: **Susana Gutierrez**

Description: This project in mathematical analysis is motivated by the behaviour of waves at rough interfaces. The initial aim is to rigorously define a measure ω, known as the **harmonic measure**, on $\mathbb{R}^n$. We will then prove that $u := \int_{\partial\Omega} f \,\mathrm{d}\omega$ is harmonic (i.e. it solves Laplace's equation $\Delta u = 0$) on a Lipschitz domain Ω and that u equals f on the boundary $\partial\Omega$ of the domain. These are domains with corners and edges which can dramatically effect wave propagation. Many questions remain unanswered in the area but recent progress has been made using harmonic measure (see, for instance, [2]). The method is based on constructing and analysing a measure on the domain boundary to generate solutions. A detailed development of the method can be found in, for instance, [1] and [3].

Prerequisites: 3FFA or 3LAn or 3Top.

References:

[1] John Garnett and Donald Marshall, *Harmonic Measure*, New Mathematical Monographs, Vol. 2, Cambridge University Press, Cambridge, 2005.

[2] Steve Hofmann, Carlos Kenig, Svitlana Mayboroda, and Jill Pipher, *Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators*, J. Amer. Math. Soc. 28 (2015) 483–529.

[3] Carlos Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, No. 83, AMS, Providence, RI, 1994.

17 Analysis of Layer Potentials

Supervisor: **Andrew Morris**

Co-assessor: **Diogo Oliveira e Silva**

Description: This project in mathematical analysis is motivated by the behaviour of waves at rough interfaces. The initial aim is to rigorously define fundamental solutions $E(x, y)$ for use in integral operators $Sf(x) := \int_{\mathbb{R}^n} E(x, y)f(y) \,\mathrm{d}y$, known as **layer potentials**, on Lipschitz domains Ω. We will see that $u := Sf$ is harmonic (i.e. it solves Laplace's equation $\Delta u = 0$) in Ω, whilst the relationship between u and f on the boundary $\partial\Omega$ of the domain involves the theory of singular integrals, which we will develop. Lipschitz domains have corners and edges which can dramatically effect wave propagation. Many questions remain unanswered in the area but recent progress has been made using layer potentials (see, for instance, [2]). The method is based on constructing and analysing a fundamental solution to generate layer potentials which behave as singular integrals at the domain boundary. A detailed development of the method can be found in, for instance, [1] and [3].

Prerequisites: 3FFA or 3LAn or 3Top.

References:

[1] E. B. Fabes, M. Jodeit and N. M. Rivière, *Potential techniques for boundary value problems on C^1-domains*, Acta Math. 141 (1978), 165–186.

[2] Steve Hofmann, Marius Mitrea and Andrew J. Morris, *The method of layer potentials in L^p and endpoint spaces for elliptic operators with L^∞ coefficients*, Proc. London Math. Soc. 111 (2015), no. 3, 681–716.

[3] Carlos E. Kenig, *Elliptic boundary value problems on Lipschitz domains*, Beijing Lectures in Harmonic Analysis, (AM-112), edited by E. M. Stein, Princeton University Press, 1986, pp. 131–184.

18 Functional Calculus for Linear Operators

Supervisor: **Andrew Morris**

Co-assessor: **Alessio Martini**

Description: This is a project in functional analysis that aims to develop a meaningful definition of e^A when A is a matrix. It will examine the spectral properties of A that make such possible and how this can be extended to define $f(A)$ for a more general class of functions f. This is known as the functional calculus of A and the collection $(e^{tA})_{t>0}$ is an example of a strongly continuous semigroup. These are important objects to study because they can provide solutions to Cauchy problems for differential equations that arise in a myriad of applications. In particular, the functional calculus can be used to justify formal manipulations, such as $\frac{d}{dt}(e^{tA}) = Ae^{tA}$, which resemble the properties of the ordinary exponential function. The techniques have also been used recently in [3] to provide a new solution to the longstanding Kato Conjecture. The project will examine these topics in detail, following treatments such as in [1] and [2].

Prerequisites: 3FFA or 3LAn or 3Top.

References:

[1] David Albrecht, Xuan Duong and Alan McIntosh, *Operator theory and harmonic analysis*, Instructional Workshop on Analysis and Geometry, Part III (Canberra, 1995), Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 34, Austral. Nat. Univ., Canberra, 1996, pp. 77–136.

[2] Wolfgang Arendt, Charles J.K. Batty, Matthias Hieber and Frank Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, vol. 96, Birkhäuser Verlag, Basel, 2001.

[3] Andreas Axelsson, Stephen Keith and Alan McIntosh, *Quadratic estimates and functional calculi of perturbed Dirac operators*, Invent. Math. 163 (2006), no. 3, 455–497.

19 Chaos in Discrete Dynamical Systems

Supervisor: **Chris Good**

Co-assessor: **Rob Leek**

Description: Suppose that $f : [0,1] \to [0,1]$ is a continuous function, then for any $x \in [0,1]$, $f^2(x) = f\big(f(x)\big)$,$f^3(x) = f\big(f^2(x)\big)$, and so on. Discrete dynamics studies the behaviour of these iterates of f. For example,a point x is periodic if $f^n(x) = x$ for some $n \geq 1$. Then Sharkovskii's Theorem tells us that if we have a point of period 3, then we have points of all other periods, andif we have a point of period 7 then we have points of all other periods except possibly 3 and 5. Li and Yorke coined the term chaos in their seminal paper 'Period three implies chaos.'It turns out that even very simple functions such as quadratic maps can exhibit chaotic behaviour and, perhaps surprisingly, one can make a detailed study of chaos using only elementary analysis of the real line.In this project you could look at the proof of Sharkovskii's Theorm or the Li-Yorke Theorem or at other definitions of chaos such as Devaney's. An alternative would be to investigate the period doubling route to chaos.The topics covered in this project must be different from those covered in Rob Leek's 4th Year module on the same subject.

Prerequisites: 2RCA 'Real and Complex Variable Theory' is essential.'Metric Spaces and Topology' would be useful.'Chaos' would be a nice extra.

References: Search for 'Chaos' and 'Sharkovskii' on Web.

20 Ergodic Theory

Supervisor: **Chris Good**

Co-assessor: **Tony Samuel**

Description: A dynamical system consists of a continuous map f on a compact metric space X. X is often called the phase or state space and f describes how points (or states) in X evolve with time. Ergodic theory studies the average, long term behaviour of such systems. For example, a preliminary result in this direction is Poincar's recurrence theorem which says that almost all points in any subset of the phase space eventually revisit the set. It turns out that Ergodic theory is a powerful tool that can be used to prove results in a number of other areas, such as number theory, fractal geometry and statistical physics. In this project you will prove Birkhoff's Ergodic Theorem and then go on to look at some applications, such as a proof of Van der Waerden's Theorem (if you split the integers into a finite number of pieces, at least one of them will contain arithmetic progressions of arbitrary length) or Szemeredi's Theorem (which says that any sufficiently large subset of the integers contains arithmetic progressions of arbitrary length.

Prerequisites: e.g. Measure Theory (3LAn), Metric Spaces and Topology (3Top) would be useful, 3ANDS would be a nice extra.

References: Search for 'Ergodic Theory' on the Web

21 Badly approximable numbers by dyadic rationals

Supervisor: **Dr. Tony Samuel**

Co-assessor: **Sabrina Kombrink**

Description: Every irrational number x has an approximation constant $c(x)$ defined by

$$c(x) = \liminf_{q \to \infty} q|qx - p|,$$

where p is the nearest integer to qx. The quantity $q|qx - p|$ measures how well x is approximated by the rational number p/q. A number x is said to be *badly approximable* if $c(x) > 0$. The set of Badly approximable numbers has been well studied and is as small as it can be in terms of Lebesgue measure (namely having Lebesgue measure zero), but as large as it can be in terms of Hausdorff dimension (that is, having dimension one). In this project we will consider properties of the set of numbers which are not only badly approxiable by rationals, but also by dyadic rationals.

Prerequisites: Real & Complex Analysis; Number Theory (Metric Spaces and Topology would be beneficial, but not necessary)

References:

[1] A. Y., Khinchin, *Continued fractions*, Dover Publications (1997)

[2] J., Nilsson, On numbers bady approximable by dyadic rationals, *Israel J. Math.* **171** (2009)

[3] K., Dajani and C., Kraaikamp, *Ergodic theory of numbers*, Carus Mathematical Monographs (2002)

22 Space filling curves

Supervisor: **Dr. Tony Samuel**

Co-assessor: **Dr. Diogo Oliveira e Silva**

Description: A curve in the unit square is called space-filling if it covers the whole unit square. The existence of space filling curves caused some disturbance in the 1920's when mathematicians were investigating how to tell the difference between Euclidean spaces of various dimensions topologically, and examples of space filling curves made people wonder whether the problem might be harder than it appeared. On the one hand it is easy to show that the line and the plane are not homeomorphic, for example removing a point disconnects the line. On the other hand it was not so clear why, say, the plane and three-dimensional space could not be homeomorphic. In this project we will learn the basic definitions and facts about space filling curves and then move onto a more specialised topic.

Prerequisites: Real & Complex Analysis (Metric Spaces and Topology would be beneficial, but not necessary)

References:

[1] H., Sagan, *Space-Fillling Curves*, New York, Springer-Verlag (1994)

[2] M., McClure, Self-similar Structure in Hilbert's Space Filling Curve, *Mathematics Magazine*, **76** (2003)

[3] M., McClure, The Hausdorff Dimension of Hilbert's Co-ordinate functions, *Real Analysis Exchange*, **24** (1998)

[4] F., Dreher and T., Samuel. Continuous images of Cantor's ternary set, *Amer. Math. Monthly* **121** (2014)

23 Sphere Packings

Supervisor: **Diogo Oliveira e Silva**

Co-assessor: **Simon Goodwin**

Description: The sphere packing problem asks for a densest packing of congruent solid spheres in d-dimensional Euclidean space $\mathbb{R}^d$. Until very recently, optimal sphere packings were only known in dimensions 1, 2 and 3. On March 14, 2016, Maryna Viazovska announced a proof in dimension $d = 8$ which stirred the world of mathematics.
The aim of this project is to study the sphere packing problem, and to investigate some of the analytic tools that have led to progress in the field. In particular, there will be a focus on Viazovska's proof, which shows that the E_8 root lattice is the densest sphere packing in eight dimensions via a deep, beautiful and conceptually simple argument.

Prerequisites: 2RCA, 3LAn

References:

[1] H. Cohn and N. Elkies, *New upper bounds on sphere packings. I.* Ann. of Math. (2) **157** (2003), no. 2, 689–714.

[2] M. Viazovska, *The sphere packing problem in dimension 8.* Ann. of Math. (2) **185** (2017), no. 3, 991–1015.

[3] D. Zagier, *Elliptic modular forms and their applications. The 1-2-3 of modular forms.* 1–103, Universitext, Springer, Berlin, 2008.

24 Fourier meets Gauss.

Supervisor: **Diogo Oliveira e Silva**

Co-assessor: **Christopher Good**

Description: *In how many ways can a given integer be written as the sum of two squares?* A cursory examination of the size of this function reveals a high degree of irregularity, so that it is not possible to capture by a simple analytic expression the essential behaviour of this function for large values of the argument.
An inspired idea of C. F. Gauss was to inquire instead about the *average* behaviour of this arithmetic function. One aim of this project is to understand how Fourier Analysis leads to progress on this and other related questions.

Prerequisites: 2RCA, 3LAn

References:
[1] E. M. Stein and R. Shakarchi, *Functional Analysis.* Princeton Univ. Press, 2011.

[2] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton Univ. Press, 1993.

25 Nonlinear Schrödinger equations

Supervisor: **Susana Gutierrez**

Co-assessor: **Andrew Morris**

Description: The 1-dimensional free Schrödinger equation from quantum mechanics is the partial differential equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0,$$

where $x \in \mathbb{R}$ and t represents time. For a suitably well-behaved function $f : \mathbb{R} \longrightarrow \mathbb{C}$, one may use elementary properties of the Fourier transform to explicitly find the solution of this (linear) equation subject to the initial condition $u(0, x) = f(x)$. However, for important nonlinear variants of this differential equation, such as the cubic nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0,$$

this explicit analysis breaks down completely. The answers to fundamental questions concerning the existence and uniqueness of solutions to such equations (under appropriate initial conditions) are far from evident, and will form the focus of this project. The extent to which this analysis applies in higher dimensions (and to more general nonlinear partial differential equations) is of significant current interest, and will also be investigated.
The prerequisites here are a reasonable grasp of real and complex analysis and metric spaces theory.

Prerequisites: 2RCA-Real and Complex Analysis and 3Top-Metric Spaces and Topology. 3FFA is also recommended but not neccessary.

References:
[1] Books on the Contraction Mapping Theorem and Picard's Theorem about the existence and uniqueness of solutions to ODE's; e.g. Metric Spaces by Victor Bryant, Cambridge University Press, 1985.

[2] E. M. Stein and R. Shakarchi, Fourier Analysis - An Introduction, Princeton, 2003 (§2.2 and 3.3).

[3] J. Duoandikoetxea Fourier Analysis, translated by D. Cruz-Uribe, AMS 2001.

[4] F. Linares, G. Ponce Introduction to Nonlinear Dispersive Equations, Springer-Verlag, 2015

26 Sharp weighted estimates for singular integrals

Supervisor: **Maria Carmen Reguera**

Co-assessor: **Diogo Oliveira e Silva**

Description: Given the singular integral operator, known as the Hilbert transform,

$$Hf(x) := \int_{\mathbb{R}} \frac{f(y)}{x-y} dy$$

we want to study the boundedness of such operator in the weighted Lebesgue space $L^2(w)$, where w is a weight, i.e. a positive locally integrable function. This question arises very naturally when studying certain PDEs. It is a classical result in Harmonic Analysis that such operator is bounded in $L^2(w)$ if and only if the weight belongs to the so called Muckenhoupt A_2 class. We will embark in studying such classical results as well as the new advances in the area related to sharp estimates. We will prove sharp weighted inequalities of the operator norm using a very powerful inequality provided by A. Lerner [3]. The mathematics of this project are fresh and remarkably powerful, the inequality provided by A. Lerner have shaken up the field of Harmonic Analysis and the study of singular integrals.

Prerequisites: 2RCA Real and Complex analysis, 3FFA Functional and Fourier Analysis.

References:

[1] Duoandikoetxea, J.; Fourier analysis. Translated and revised from the 1995 Spanish original by David Cruz-Uribe. Graduate Studies in Mathematics, 29. American Mathematical Society, Providence, RI, 2001. xviii+222 pp

[2] Lerner, Andrei K.; A simple proof of the A_2 Conjecture. Int. Math. Res. Not. IMRN 2013, no. 14, 3159–3170.

[3] Lerner, Andrei K.; On an estimate of Calderón-Zygmund operators by dyadic positive operators. J. Anal. Math. 121 (2013), 141 161.

27 Sparse domination for oscillatory integrals

Supervisor: **Maria Carmen Reguera**

Co-assessor: **Susana Gutiérrez**

Description: We are interested in studying **oscillatory operators**. Such classical operators include the Bochner-Riesz operator, whose boundedness is a very famous open problem in the area of Harmonic Analysis. In order to understand the behavior of such operator, in this project we aim to control it by so called **sparse operators** ([3]). This domination by sparse operators has proven to be a very powerful tool in the study of non-oscillatory operators and its weighted estimates, see work of A. Lerner ([2]) among others. The mathematics of this project are at the forefront of the latest advances in the area and present an excellent opportunity to learn these advances in a control way.

Prerequisites: 2RCA Real and Complex analysis, 3FFA Functional and Fourier Analysis.

References:

[1] Duoandikoetxea, J.; Fourier analysis. Translated and revised from the 1995 Spanish original by David Cruz-Uribe. Graduate Studies in Mathematics, 29. American Mathematical Society, Providence, RI, 2001. xviii+222 pp

[2] Lerner, Andrei K.; On an estimate of Calderón-Zygmund operators by dyadic positive operators. J. Anal. Math. 121 (2013), 141 161.

[3] Lacey, Michael T.; Mena, Dario; Reguera, Maria Carmen; Sparse Bounds for Bochner Riesz Multipliers. J. Fourier Anal. Appl. 25 (2019), no. 2, 523 537.

28 Central limit theorem for stochastic fractional heat equation

Supervisor: **Jingyu Huang**

Co-assessor: **Hong Duong**

Description: Consider the stochastic fractional heat equation on the real line:

$$\frac{\partial u}{\partial t} = -(-\Delta)^{\alpha/2}u + \sigma(u)\dot{W} \quad t > 0\,, x \in R$$

with a constant initial condition. $\dot{W}$ is a space time white noise. The fundamental solution for the fractional heat equation is localized, thus, $u(t,x)$ and $u(t,x')$ are almost i.i.d. random variables when x and x' are far apart. Thus, the integral

$$\int_{-R}^{R} u(t,x)dx$$

is almost a sum of independent random variables. Question: does the above integral converge in distribution to a Gaussian random variable?

Prerequisites: 2nd year Statistics, 3rd year Methods in Partial Differential Equations

References: A Central Limit Theorem for the stochastic heat equation. Jingyu Huang, David Nualart, Lauri Viitasaari. Preprint.

29 Solution to stochastic fractional heat equation in $L^1(R)$

Supervisor: **Jingyu Huang**

Co-assessor: **Hong Duong**

Description: Consider the stochastic fractional heat equation on the real line:

$$\frac{\partial u}{\partial t} = -(-\Delta)^{\alpha/2}u + \sigma(u)\dot{W} \quad t > 0\,, x \in R$$

with initial condition $u_0(x)$. $\dot{W}$ is a space time white noise. Is it true that $u_0 \in L^1(R)$ is equivalent to that $u(t,\cdot) \in L^1(R)$ a.s. for any $t > 0$?

Prerequisites: 2nd year Statistics, 3rd year Methods in Partial Differential Equations

References: Decorrelation of total mass via energy. By Le Chen, Davar Khoshnevisan, Kunwoo Kim. Potential Analysis, 45 157-166.

30 The Heisenberg Group

Supervisor: **Alessio Martini**

Co-assessor: **Olga Maleva**

Description: The Heisenberg uncertainty principle of quantum mechanics tells us that position q and momentum p of a particle cannot be simultaneously measured with arbitrary precision, i.e.,

$$\Delta p\, \Delta q \geq \hbar/2,$$

where $\hbar$ is the reduced Plack constant. In order to structurally encode the uncertainty principle, one can model position and momentum as linear operators (think of matrices) satisfying the "canonical commutation relations"

$$[q,p] = qp - pq = i\hbar I,$$

where I is the identity. These relations are intimately connected with the structure of the group of 3×3 matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ for $a, b, c \in \mathbb{R}$, known as the *Heisenberg group*. Besides its role in the foundation of quantum mechanics, the Heisenberg group has applications in many different parts of mathematics. The aim of this project is to explore some of these applications.

Prerequisites: 2LALP (Linear Algebra and Linear Programming), 2MVA (Multivariable & Vector Analysis) and 2RCA (Real & Complex Analysis) are essential. 2AC (Algebra and Combinatorics 2), 3FFA (Functional and Fourier Analysis) and 3Top (Metric Spaces and Topology) would be advantageous.

References:

[1] G. B. Folland, *Harmonic analysis in phase space*, Princeton University Press, 1989.

[2] S. Thangavelu, *Harmonic analysis on the Heisenberg group*, Birkhäuser, 1998.

31 Geodesics in sub-Riemannian Geometry

Supervisor: **Alessio Martini**

Co-assessor: **Andrew Morris**

Description: The distance between two points can be defined as the infimum of the lengths of the paths joining the two points. This definition makes sense on $\mathbb{R}^n$ as well as on more general (e.g., Riemannian) manifolds. Suppose now that some of the paths are forbidden, because, at each point of the manifold, moving in certain directions is not allowed. Think, e.g., of the configuration space of a skate moving on a plane without sliding laterally: the direction in which the skate can move depends on its orientation. This kind of constraint gives rise to a particular geometry on the underlying manifold, called *sub-Riemannian geometry*. One of the main problems in sub-Riemannian geometry is characterising length-minimising curves, also known as *geodesics*, and, in particular, understanding their smoothness properties. The aims of this project are to review the basic definitions and results of sub-Riemannian geometry, to illustrate them in particular examples, and to investigate sufficient conditions under which length-minimising curves can be proved to be smooth.

Prerequisites: 2LALP (Linear Algebra and Linear Programming), 2MVA (Multivariable & Vector Analysis) and 2RCA (Real & Complex Analysis) are essential. 2DE (Differential Equations), 3FFA (Functional and Fourier Analysis) and especially 3Top (Metric Spaces and Topology) would be advantageous.

References:
[1] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, American Mathematical Society, 2002.

[2] E. Le Donne, G. P. Leonardi, R. Monti, and D. Vittone, Extremal curves in nilpotent Lie groups, *Geometric and Functional Analysis* 23 (2013), 1371–1401.

32 Baire's category theorem and Banach-Mazur game

Supervisor: **Olga Maleva**

Co-assessor: **Alessio Martini**

Description: In 1935 the Polish mathematician Stanislaw Mazur proposed the following game.
There are two players called Player I and Player II. A subset S of the interval $[0,1]$ is fixed beforehand, and the players alternately choose subintervals $I_n \subset [0,1]$ so that $I_{n+1} \subseteq I_n$ for each $n \geq 1$. Player I wins if the intersection of all I_n intersects S, and player II wins if he can force this intersection to be disjoint from S.
Mazur observed that if S can be covered by a countable union of sets whose closure has empty interior (S is of first category) then the second player wins while if the complement of S is of first category then the first player wins. Later Banach proved that these conditions are not only necessary for the existing winning strategies but are also sufficient.
This is a very powerful method in Analysis. For example, it implies the Baire theorem which says that the intersection of any collection of open dense subsets of $\mathbb{R}$ is dense.
The game can be generalised to an arbitrary topological space X. Then in order to decide whether a certain property describes a *typical* object of X it is enough to show that there is a winning strategy for Player I with respect to the set of objects satisfying the given property. Remarkably, one can show in this way that a 'typical' continuous function is differentiable at no point!
The project will cover a range of topics connected with the Banach-Mazur game and Baire category theorem.

Prerequisites: 3Top Metric Spaces and Topology 3FFA Fourier and Functional Analysis

References:
[1] B. Bolobas, Linear Analysis. Cambridge University Press, Cambridge, 1999.

[2] J. Oxtoby, Measure and category. A survey of the analogies between topological and measure spaces. Second edition. Graduate Texts in Mathematics, 2. Springer-Verlag, New York-Berlin, 1980.

33 Partial Cubes

Supervisor: **Prof S Shpectorov**
Co-assessor: **Dr C Hoffman**
Description: The vertex and edge graph of the n-dimensional cube is known as the Hamming cube graph Q_n. Partial cubes are isometric subgraphs of Hamming cubes, that is, a graph G is a partial cube if $G \subseteq Q_n$ for some n and for any two vertices of G the distance between them is the same when measured in G and in Q_n.
The theory of partial cubes is a lively area of Combinatorics, where new research can start literally minutes after one learns the basic definitions. Partial cubes have many applications in, as well as outside, mathematics.
Prerequisites: 3GphT Graph Theory.
References: [1] "Graphs and Cubes", by Sergei Ovchinnikov.

34 Clifford Algebras and Spinors

Supervisor: **Prof S Shpectorov**
Co-assessor: **Prof CW Parker**
Description: This might be an interesting project in particular for those who are interested in application to Physics. The focus of the project will be on the definition of the Clifford algebras over the real numbers and their action on the corresponding space of spinors. On the way there, you can learn chapters from Linear Algebra that are rarely covered in the basic university courses, but which are very useful for applications. This includes the concepts of the dual space and tensors, classification of bilinear forms, etc. The famous quaternion algebra will appear as a particular case of Clifford algebra.
Prerequisites: 3GrpT Group Theory useful, but not strictly necessary.
References: [1] "Clifford Algebras: An Introduction", by D.J.H. Garling.

35 Groups and geometries

Supervisor: **Prof S Shpectorov**
Co-assessor: **Dr C Hoffman**
Description: The concept of diagram geometries was introduced by J. Tits as a source of natural objects on which finite simple groups act. One interesting class of geometries is called buildings, and among buildings one finds such famous examples as the n-dimensional projective space and the polar spaces of different types. The project includes the basics of Coxeter groups, the axioms and elementary properties of buildings, and examples of buildings.
Prerequisites: 3GrpT Group Theory useful, but not strictly necessary.
References:

36 The Omnibus Project

Supervisor: **Prof S Shpectorov**
Co-assessor: **Prof CW Parker**
Description: I would be happy to consider a project with a motivated student on any topic of her/his liking in any area of pure mathematics including, but not restricted to Algebra, Combinatorics, and Geometry.
Prerequisites: Motivation and mathematical maturity.
References: Bring your own.

37 Topics in Group Theory

Supervisor: **Chris Parker**
Co-assessor: **Sergey Shpectorov**
Description: The start point of this project is the end point of Sergey's group theory course. The objective is to understand more advanced group theory. The project can focus on numerous different directions and is suitable for more than one student.

1. use group theory to determine of the structure of the stabiliser of a vertex in a graph which admits a transitive automorphism group.
2. understand how the structure of certain subgroups of a group influences the structure of the group (local to global results).
3. the finite simple groups (the building blocks of groups).
4. the structure of certain classes of p-groups, p a prime (maximal class p-groups).
5. the subgroup structure of soluble groups (Hall's Theorems, Carter subgroups).
6. the O'Nan-Scott Theorem and generalizations (permutation group theory).
7. algorithms in group theory (Todd Coxeter, Low Index Subgroups).
8. anything that interests you that is associated with group theory.

Prerequisites: 3Grp
Corequisites: 4TAl

38 Algebraic number theory

Supervisor: **Simon Goodwin**
Co-assessor: **Chris Parker**
Description: Algebraic number theory can be viewed as the study of generalizations of the ring of integers $\mathbb{Z}$ obtained by adjoining roots of monic polynomials over $\mathbb{Z}$. This theme has been developed in the third year module "Number Theory", and this project gives you the opportunity to delve further. The first aim of the project will be to cover material in [1], which consolidates and generalizes the theory you have already covered. This moves on to cover prime factorizations of ideals in rings of algebraic integers, and subsequently to outline ideas behind Andrew Wiles' proof of Fermat last theorem. Further topics and different approaches can be investigated with [2] being one of many possible resources.
Prerequisites: 2AC Algebra 2, 3Num Number Theory
References:

[1] I. Stewart and D. Tall, *Algebraic Number Theory and Fermat's Last Theorem*, Fourth Edition, CRC Press, Taylor and Francis Group, 2016.

[2] A. Fröhlich and M. Taylor, *Algebraic Number Theory*, Cambridge Advanced Studies in Mathematics, Cambridge University Press, 1991.

39 Galois theory

Supervisor: **Simon Goodwin**
Co-assessor: **Chris Parker**
Description: Galois theory concerns roots of polynomials and their symmetries. A highlight of this project will be the proof of the insolubility of the quintic equation. The first aim of the project will be to cover material in [1] or [2] on the theory of field extensions and Galois groups building up to the fundamental theorem of Galois theory. Along the way you will see how Galois theory can be used to answer fundamental questions about whether it is possible to "square a circle" or to "construct a regular 17-gon". Further topics can be investigated with possibilities including the theory of finite fields, or the inverse Galois problem.
Prerequisites: 2AC Algebra 2, 3GrpT Group Theory (desirable but not essential)
References:

[1] D. J. H. Garling, *A Course in Galois Theory*, Cambridge University Press, 1986.

[2] I. Stewart, *Galois Theory*, Fourth Edition, CRC Press, Taylor and Francis Group, 2015.

40 Homology theory for Riemann surfaces

Supervisor: **Prof. M. Mazzocco**

Co-assessor: **Dr T. Kelly**

Description: The basic idea of homology theory is to classify spaces in terms of their holes. A typical example is ta circle and a disk, namely a circle is not a disk because the circle has a hole through it while the disk does not. The aim of the theory is to associate a sequence of abelian groups or modules to topological spaces.Such groups, called homology groups, are in some sense refinements of the Euler characteristic. In this project the student will familiarise themselves with simplicity homology and work out the example of Riemann surfaces of any genus.

Prerequisites: Metric and Topological spaces.

41 Analysing a combinatorial game

Supervisor: **Richard Kaye**

Co-assessor: **David Leppinen**

Description: Certain kinds of combinatorial games (finite games of perfect information) have interesting mathematical theory. Many of these games are too complicated to analyse directly. See for example the books by Conway et al called ‘Winning Ways’.

This project would look at some game or family of games and attempt to analyse it (to work out the winning strategies for example) by a variety of methods. Key amongst these would be computer methods, such as computer search, and building a tree of all possible positions and moves in the game. Thus this project may well involve programming skills (java or C++ might be suitable languages, but the use of classes as well as numerical programming will be important) and will also extend the student’s knowledge of one of these programming languages.

Prerequisites: 2NP, interest in programming

References: [1] ‘Winning Ways’, Conway, Berlekamp and Guy (any edition)

42 Mathematics built on different foundations

Supervisor: **Richard Kaye**

Co-assessor: **Corneliu Hoffman**

Description: Pure Mathematics as normally done requires so-called ‘classical logic’ with ‘the law of excluded middle’ under foundational assumptions (axioms) given by the axioms of the set theory ZFC (or similar).

Occasionally other logics or other foundations are suggested. Thus for example, classical logic may be replaced by intuitionistic logic and the resulting theory is something rather more ‘constructive’ where the law of excluded middle is not assumed at all times, and mathematical objects must be constructed directly rather than have their existence proved by some proof by contradiction. Analysis is interesting when studied in this way.

A recent suggestion on these lines is Homotopy Type Theory (HoTT) which as well as being an intended foundation for computer science is also proposed as an alternative framework for mathematics. This project would be to look at this (or a related foundational framework) to see how different mathematics might look under such foundations. HoTT is also used in areas of computability and automatic theorem proving, so there may be scope for exploration of it in these contexts too.

The main reference for HoTT is a free download at http://homotopytypetheory.org/book/ It is expected that students will have taken computability and logic. This project description is just a suggestion at present and there is scope to vary it after discussion with the supervisor.

Prerequisites: 3CLog

References: [1] The HoTT book, http://homotopytypetheory.org/book/

43 A project in nonstandard mathematics

Supervisor: **Richard Kaye**

Co-assessor: **Chris Good**

Description: Nonstandard analysis (NSA) was invented by Abraham Robinson in the 1960s and puts the infinitesimal calculus on a rigorous basis. A taste of this topic was given in the third year computability and logic module. One of its strengths is its ability to construct interesting analytic or topological structures from finite discrete structures.

This project would look at NSA in general and/or one or more of its applications, either in the context of analysis or in the context of number theory. There is plenty of scope to choose the exact area of application according to the student’s interests.

Prerequisites: 3CLog

References: [1] The Mathematics of Logic, C.U.P., chapter 12.

44 Public-key cryptography

Supervisor: **Nat Queen**

Co-assessor: **Simon Goodwin**

Description: Users of a public-key cryptosystem can communicate securely over insecure electronic channels such as e-mail without previously exchanging secret keys. Public-key cryptography can also be used for secure digital signatures. Various public-key cryptosystems exist, such as RSA, Diffie–Hellman, ElGamal and elliptic-curve cryptography. This project will analyse such cryptosystems, study their security and compare their relative advantages. Computer code will be developed and tested for implementing such cryptosystems.

Prerequisites: Ability to write computer programs in any language, and 3Num Number Theory

References:

[1] Douglas R. Stinson, *Cryptography: Theory and Practice*

[2] Neal Koblitz, *A Course in Number Theory and Cryptography*

Projects in Applied Mathematics

45 Mathematical modelling of interactions between microbes, the host and the environment

Supervisor: **Dr Sara Jabbari**

Co-assessor: **Dr Rosemary Dyson**

Description: This project will apply mathematical modelling techniques to mechanisms of microbe survival. Bacteria and fungi are extremely resilient organisms, capable of adapting to a wide variety of environments, be that in a host during infection or for survival outside the host. An ordinary differential equation model will be developed to represent a particular microbe and its environment. The student will be able to choose from a variety of applications, for example, bacterial interactions with the host immune system, or the response of bacteria to exposure to novel antimicrobials. Both numerical (using Matlab) and asymptotic analyses will be required to investigate the resulting systems.

Prerequisites: 3Bio (Mathematical Biology)/4AMM (Advanced Mathematical Modelling), 4AMB (Advanced Mathematical Biology), 3PTA (Perturbation Theory and Asymptotics)/ 4AMA (Applied Mathematical Analysis)

References:
[1] G Karlebach and R Shamir. *Nature Reviews Molecular Cell Biology* (2008) 9:770-780.

[2] PA Roberts, RM Huebinger, E Keen, AM Krachler, S Jabbari. Predictive modelling of a novel anti-adhesion therapy to combat bacterial colonisation of burn wounds. *PLoS computational biology* (2018) 14:5.

46 Population dynamics of a biological waste-water treatment technology

Supervisors: **Dr Sara Jabbari, Dr Alexandra Tzella**

Co-assessor: **Dr Sam Johnson**

Description: The Orsini group in the School of Biosciences are developing an exciting new waste-water treatment technology that uses tiny waterfleas called Daphnia to consume waste products and filter the water thus eliminating the need for costly and complex filters. One might assume that increasing the number of Daphnia in the treatment plant would increase the efficiency of the consumption or filtering process. However, there is effectively a threshold on the number of Daphnia that can be supported by the local environment. The optimal number of Daphnia (that will vary depending on conditions, e.g. heavy rainfall will dilute the system) can be estimated by developing mathematical models of population dynamics and these are usually based on differential equations. Due to the time difference between Daphnia being born and themselves being able to reproduce, delay differential equations (DDEs) are required to model the time it takes for Daphnia to mature. DDE models are very interesting because they are capable of generating rich and complex dynamics that can be surprisingly different to models based on ordinary differential equations. This project will develop a class of DDE models that mirror the experimental data available from the Orsini group. The behaviour of the models will be explored numerically using Runge-Kutta methods and analytically using a dynamical systems approach. The student will be required to use Matlab (or similar software) to undertake this project. Relevant codes (in Matlab) will be provided at the outset; however some code development will be required as part of the project tasks.

Prerequisites: 3Bio (Mathematical Biology)/4AMM (Advanced Mathematical Modelling), 4AMB (Advanced Mathematical Biology), 3ANDS (Applied Nonlinear Dynamical Systems)/ 4AMA (Applied Mathematical Analysis)

References:
[1] Daphne water solutions for the removal of emerging pollutants from wastewater: market assessment and technical feasibility, UKRI https://gtr.ukri.org/projects?ref=NE%2FS015515%2F1

[2] Delay differential equations with applications in population dynamics, Yang Kuang, eBook ISBN: 9780080960029 (available on FindIt@Bham)

47 Modelling the collective motion of birds

Supervisor: **Dr Galane J. Luo**

Co-assessor: **Dr Samuel Johnson**

Description: Understanding collective animal motion is a topic of active scientific research, with far-reaching implications in diverse fields including biomimetics and sociology. Mathematical models provide unique insights into the underlying mechanisms for synchronous behaviour in groups. This project will look at two phenomena relating to the movement of large groups of birds: *flocking*, defined as flight with a common velocity; and *murmuration*, meaning flight with swooping and swirling patterns. The student will investigate the Cucker-Dong model for flocking, and extend the model in order to incorporate flocking and murmuration under a unified theoretical framework. If possible, the phase transition between flocking and murmuration will be explored.

Prerequisites: 2DE or 2DE3 (Differential Equations) essential; 3Bio (Mathematical Biology) and 3ANDS (Applied Nonlinear Dynamical Systems) useful.

References:

[1] Wang, X. and Lu, J., 2019. *Collective Behaviors Through Social Interactions in Bird Flocks*. IEEE Circuits and Systems Magazine, 19(3), pp.6-22.

[2] Vicsek, T. and Zafeiris, A., 2012. *Collective motion*. Physics reports, 517(3-4), pp.71-140.

[3] Cucker, F. and Dong, J.G., 2010. *Avoiding collisions in flocks*. IEEE Transactions on Automatic Control, 55(5), pp.1238-1243.

48 The vitamin C clock reaction

Supervisor: **Prof. Dave Smith**

Co-assessor: **Dr Sara Jabbari**

Description: Clock reactions occur in some important industrial applications, such as cement hydration, and are an enduring demonstration experiment used in the teaching of chemical kinetics. They are usually characterised by a prolonged *induction period*, during which little appears to be happening, followed by a sudden change or *switchover*, often revealed by a change of colour in the solution. A true clock reaction is characterised by a predictable and repeatable induction time. This project will build on earlier work by MSci and MSc students in Birmingham which led to a publication [1]. We will focus on the vitamin C clock [2], which can be performed with safe household chemicals. The main challenge will be to extend the analysis of ref. [1] to a modified situation in which hydrogen peroxide is at a comparable concentration to the other chemicals, which is likely to change the form of the kinetics. The mathematical techniques will build on the phase plane analysis in 2DE, law of mass action in 3Bio and also provide the opportunity to carry out numerical simulations and/or approximate solutions via matched asymptotic expansions [3]. There will also be the opportunity to carry out experiments and use these to estimate model parameters. The project will therefore provide the opportunity to acquire many of the core skills in interdisciplinary research.

Prerequisites: 2DE, 3Bio

References:

[1] Kerr R, Thomson WM and Smith DJ. 2019. Mathematical modelling of the vitamin C clock reaction. R. Soc. open sci. 6: 181367

[2] Wright SW. 2002. The vitamin C clock reaction. J. Chem. Edu. 79, 41

[3] Billingham J, Needham DJ. 1992. Mathematical modelling of chemical clock reactions. I Induction, inhibition and the iodate-arsenous-acid reaction. Phil. Trans. R. Soc. Lond. A340, 569–591

49 Helping sperm reach their destination

Supervisor: **Prof. Dave Smith**

Co-assessor: **Dr Rosemary Dyson**

Description: The sperm of internally-fertilising species such as us have a difficult job to do – they have to swim many times their body lengths and locate the egg in the female reproductive tract so that fertilisation can occur. The transport mechanism by which tens or hundreds of millions of sperm deposited at the cervix lead to a few tens or hundreds near the egg is not well-characterised. It is not even known whether this process can be approximated as diffusion. Through soft lithography microchannel experiments, our collaborators in the Centre for Human Reproductive Science and Birmingham Women's Fertility Centre are now able to collect detailed data on how sperm migrate, and how the microarchitecture (such as walls and grooves) changes the migration rate. This project will consider models of how a population of cells spreads out, starting with simple Fickian diffusion, then progressing to take into account persistent random walks. The mathematical techniques will involve both partial differential equations such as the heat equation and telegraph equation, and/or individual-based modelling of discrete cells. There will be the opportunity to parameterise and test models against experimental results.

Prerequisites: 2DE, 3Bio, 3MoPD

References:

[1] EA Gaffney, H Gadêlha, DJ Smith, JR Blake, JC Kirkman-Brown. 2011. Annual Review of Fluid Mechanics 43, 501–528

[2] EA Codling, MJ Plank and S Benhamou. 2008. Random walk models in biology. J. R. Soc. Interface 5, 813–834

50 Models of plant root growth

Supervisor: **Dr Rosemary Dyson**

Co-assessor: **Dr Galane Luo**

Description: Understanding plant root growth is essential to promote healthy plant growth in normal and stressed (e.g. during a drought) environments. A root grows through the elongation of some of its cells, pushing the root forward through the soil. These cells differ from animal cells through the presence of a tough cell wall surrounding the cell, which maintains a high internal turgor pressure whilst allowing significant growth. It is the (varying) mechanical properties of this cell wall which control growth, whilst the driving force for expansion is provided by the turgor pressure.

Potential projects include investigating how bending of the root, for example as displayed under a gravity stimulus, is generated via differential wall properties across the root tissue, or developing models to determine the mechanical properties of the cell wall via atomic force microscopy.

Prerequisites: 3CM Continuum mechanics, 3Bio Mathematical Biology, 3MoPDE Modelling with PDEs likely, depending on actual project.

51 Modelling in mathematical medicine

Supervisor: **Dr Rosemary Dyson**

Co-assessor: **Dr David Smith**

Description: The use of mathematical modelling within medical research is widespread, covering topics such as cancer, inflammation, neuroscience, physiological fluid mechanics and many others. This project will involve working closely with relevant colleagues in other Schools to tackle a specific biological question through formulating, analysing and solving (using analytical and/or numerical techniques) mathematical models. It is likely to involve models of how cells interact with each other or their environment as this is a common feature of both cancer and inflammation.

Prerequisites: 3CM Continuum mechanics, 3Bio Mathematical Biology, 3MoPDE Modelling with PDEs likely, depending on actual project.

52 The influence of flows on the propagation of travelling fronts

Supervisors: **Dr Alexandra Tzella**

Co-assessor: **Dr John Meyer**

Description: A vast number of phenomena in chemistry and biology in which a single reactant or population spreads are described in the form of a travelling front. The simplest model that captures this spreading is the Fisher-Kolmogorov equation which combines the effects of spatial diffusion, growth and saturation [1]. However, in many environmental situations, e.g. in marine ecology, the population is carried by an ambient flow. A challenge is then to determine how the flow may influence the characteristics of the travelling front, such as its speed and shape. This is particularly important for understanding the evolution of reactants in the environment, a phenomenon that is directly related to the growth and maintenance of oceanic plankton populations [2]. This visually dramatic example can also have a large-scale impact on the climate of our planet. The main objective of this project is to investigate the influence of periodic flows on the propagation of travelling fronts. We will focus on the sharp front arising in Fisher–Komogorov type models in the limit of small molecular diffusivity and fast reaction and on its heuristic approximation by an alternative model, the so-called G equation [3]. The investigation will require a numerical implementation.The preferred computational package is MATLAB, although familiarity with a programming environment should be sufficient. Most relevant code will be provided at the outset; however some code development may be required as part of the project tasks.

Prerequisites: 3PTA (Perturbation Theory and Asymptotics), 3ANDS (Applied Nonlinear Dynamical Systems).

References:
[1] J. D. Murray, *Mathematical Biology I: An Introduction*, Springer, 2002.

[2] E. R. Abraham, *The generation of plankton patchiness by turbulent stirring*, Nature, 391, 577, 1998.

[3] A. Tzella and J. Vanneste, *FKPP fronts in cellular flows: the large-Peclet regime*, SIAM J. Appl. Math. , 79(1), 131-152, 2019.

53 Diffusion in porous media: the large deviation regime

Supervisors: **Dr Alexandra Tzella**

Co-assessor: **Dr Hong Duong**

Description: The complexity of a porous environment such as groundwater is a major obstacle for obtaining a unified description of the generation and movement of chemicals over a range of scales and porous configurations. As a result, multiple modelling approaches have been adopted: from pore-scale to coarse-grained models. Although pore-scale models have a solid physical foundation, computational domains are still too small to be of any use for predictions at the observation scale. Coarse-grained models on the other hand treat a porous medium as a continuum and as a result are very efficient and cost-effective. However, they are largely empirical, lacking understanding from first principles and consequently unable to incorporate crucial microstructural information e.g. a particular geometrical configurations. Multiscale approaches such as homogenisation theory overcome the above challenges by combining the efficiency of coarse-grained models with the accuracy of pore-scale models. This is particularly well-developed for periodic porous media, enabling [1] to demonstrate that the evolution of a passive scalar (e.g. a pollutant) is ultimately represented by an effective diffusion equation. The effective diffusion equation has been recently re-examined as an appropriate coarse-grained model following the realisation that it does not provide accurate predictions of low scalar concentrations. This limitation motivated the development of a novel multiscale approach that uses the theory of large deviations describing rare events to capture the entire distribution of a passive scalar released suddenly inside a periodic flow [2]. However, the approach has not been exploited for diffusion inside a random porous medium. The main objective of this project is to use the theory of large deviations to investigate the influence of random geometry on the scalar concentration at long times. The investigation will require a numerical implementation. The preferred computational package is MATLAB, although familiarity with a programming environment should be sufficient.

Prerequisites: 3PTA (Perturbation Theory and Asymptotics), 3MoPD (Methods of Partial Differential Equations

References:
[1] Auriault and Adler, *Taylor dispersion in porous media: Analysis by multiple scale expansions*, Adv. Water Resour., 18:217-226, 1995.

[2] P. H. Haynes and J. Vanneste, *Dispersion in the large-deviation regime. Part I: shear flows and periodic flows*, J. Fluid Mech., 745, 321-350, 2014.

54 Multiscale analysis of partial differential equations

Supervisor: **Dr Hong Duong**

Co-assessor: **Jingyu Huang**

Description: Complex systems in nature and in applied sciences often exhibit multiple spatial and temporal scales. The associated structures and phenomena give rise to many interesting mathematical problems. Examples include continuum limits of atomistic models for material defects, hydrodynamic limit of interacting particle systems and collective behaviour in biological and social systems. The aim of this project is to study the asymptotic behaviour of some partial differential equations arising from statistical physics and molecular dynamics such as the Fokker Planck equation and the kinetic Fokker Planck equation.

Prerequisites: 2DE3 (Differential Equations); 3PTA4 (Perturbation Theory and Asymptotics); 3MePD (Methods in Partial Differential Equations).

References: [1] G. A. Pavliotis and A. Stuart. *Multiscale Methods: Averaging and Homogenization*. Springer (Berlin), 2008.

[2] Weinan E. *Principles of Multiscale Modeling*. Cambridge University Press, 2011.

55 Evolutionary game theory and population dynamics

Supervisor: **Dr Hong Duong**

Co-assessor: **Dr Fabian Spill**

Description: Evolutionary Game Theory is the application of game theory to evolving populations in biology defining a framework of contests and strategies into which Darwinian competition can be modelled. It has proven to be a powerful and versatile mathematical framework for the modelling and analysis of complex biological (and economical) systems whenever there is frequency dependent selection. At the core of evolutionary game theory and population dynamics is the replicator-mutator equation, which is a set of differential equations that describe the evolution of the frequencies of the different types in the population.
The aim of this project is to study the replicator-mutator equation and related models. To capture the rapidly change (uncertainty) of the environment, some randomness is possibly incorporated. In these scenarios, an important problem is to understand the statistics of equilibria points of the models.

Prerequisites: 2DE3 (Differential Equations).

References: [1] J. Maynard Smith and G. R Price. *The logic of animal conflicts*. Nature, (246):15-18, 1973.

[2] J. Hofbauer and K. Sigmund. *Evolutionary Games and Population Dynamics*. Cambridge: Cambridge University Press, 1998.

56 Powering flexible microscale robots with autophoretic reactions

Supervisor: **Tom Montenegro-Johnson**

Co-assessor: **Alexandra Tzella**

Description: Artificial phoretic microswimmers, such as Janus particles, use chemical reactions of a solute fuel catalysed at their surface in order to self-propel. Such propulsion entails no moving parts, and is highly suited to the miniaturisation required for micro-scale robots for a range of industrial or medical applications. These phoretic swimmers exhibit a wealth of interesting individual and group dynamics, such as phase-separation, crystallisation, and spontaneous collective motion for isotropic particles. In this project, the student will examine some fundamental problems in the phoretic motion of a new type of swimmer based on flexible filaments, for example spontaneous symmetry-breaking leading to swimming. The student can choose to either focus on analytical techniques, numerical modelling, or a combination of the two, and so should have a keen interest in mathematics and/or computational techniques.

Prerequisites: none

References: [1] Microtransformers: Controlled microscale navigation with flexible robots.Thomas D. Montenegro-Johnson. Phys. Rev. Fluids 3, 062201(R).

57 Modelling shape-changing composite materials in microscale flows

Supervisor: **Tom Montenegro-Johnson**
Co-assessor: **Rosemary Dyson**
Description: Thermoresponsive polymers are soft materials that expand when heated. These materials can be micro-3D printed onto scaffolds of standard polymers to create composite materials with fascinating properties. The mathematical study of these materials promises to be a new and interesting subfield of continuum mechanics. In this project, the student will develop theoretical and/or computational models of these materials, and examine the dynamics of composite filaments, sheets, and structured manifolds in microscale fluid flows.
Prerequisites: none
References: [1] Controlling the shape of 3D microstructures by temperature and light. Hippler et al. Nature Communications, volume 10, Article number: 232 (2019)

58 Mathematical modelling to predict the total removal of tumour cells or bacteria

Supervisor: **Dr Fabian Spill**
Co-assessor: **Dr Sara Jabbari**
Description: Cancer and bacteria alike can be difficult to treat since initial therapy may appear to have eradicated the respective diseases, yet a relapse may occur. Often, this is because the bulk of the tumour or bacteria is removed, but a handful or tumour cells or bacteria survive treatment. These few cells initially cause no symptoms and may not be detectable, yet in both cases a few cells or bacteria may regrow. Moreover, in both cases, the regrown tumour or bacterial population may now have increased resistance to treatment. In this project, we will develop mathematical models and study techniques to predict when total extinction of tumour or bacterial populations occurs, and what may be done to increase the extinction probability. We will develop stochastic models that may predict the survival of every single cancer or bacterial cell, and then employ analytic and/or numerical techniques to solve the resulting equations. While for this project no particular knowledge of stochastic processes is required, an eagerness to learn about this important mathematical field is required.
Prerequisites: 3Bio (Mathematical Biology))
References: [1] F. Spill et.al., Hybrid approaches for multiple-species stochastic reactiondiffusion models, *Journal of Computational Physics* (2015).

[2] R. de la Cruz et.al., The effects of intrinsic noise on the behaviour of bistable cell regulatory systems under quasi-steady state conditions *Journal of Chemical Physics* (2015)

59 The impact of cell geometry on cell behaviour

Supervisor: **Dr Fabian Spill**
Co-assessor: **Dr Thomas Montenegro-Johnson**
Description: Cells in the human body occur in vastly different shapes, despite having the same genetic makeup. Recent evidence also suggests that the geometry may affect how the cell behaves. This is due to the fact that intracellular signalling molecules that dictate cell behaviour localise to specific compartments of the cell. These compartments include the cell membrane, i.e. the boundary of the cell. Accurate mathematical descriptions taking cell geometry into account thus require the study of bulk-boundary problems on potentially complicated geometry. In this project, we will investigate simple networks of reacting intracellular signalling molecules, and determine their output in dependence on cell geometry. That way, we aim to classify how cellular deformations act as switches that may dictate cell behaviour. Depending on skills and interests, this project may be successfully accomplished through numerical solution of the derived equations, or through analytic derivations that simplify the geometries to simplify the resulting equations . Moreover, this project may be performed by closely working with our experimental collaborators by analysing real biological data of cells with varying shapes [2].
Prerequisites: 2DE (Differential Equations) 3Bio (Mathematical Biology))
References: [1]F . Spill et.al., Effects of 3D Geometries on Cellular Gradient Sensing and Polarization, *Physical Biology* (2016)

[2] JE Sero and C. Bakal, Multiparametric analysis of cell shape demonstrates that -PIX directly couples YAP activation to extracellular matrix adhesion, *Cell Systems* (2017)

60 Trophic coherence and ecosystem stability

Supervisor: **Samuel Johnson**

Co-assessor: **Fabian Spill**

Description: Rainforests, coral reefs and other very large ecosystems seem to be the most stable in nature, in the sense that fluctuations in species density do not lead to their sponteneous collapse. However, this has been regarded as paradoxical ever since Robert May showed that, all else being equal, diversity should lead to instability [1,2]. We have recently proposed a solution to this puzzle: food webs (the networks of who eats who in an ecosystem) display a topological property called "trophic coherence" which suppresses feedback and, under cetain assumptions, makes the systems more stable [3,4]. Furthermore, if networks are sufficiently coherent, the diversity-stability relationship could be inverted. However, some of the assumptions behind these predictions may make them unrealistic, and other preliminary work suggests that in some situations trophic coherence can reduce stability. The aim of this project is to study the effects of network topology – and, in particular, of trophic coherence – on more realistic ecosystem models and measures of stability. This will likely require a combination of theoretical and numerical analysis.

Prerequisites: 3Bio and 2NP are not essential but would be an advantage.

References:

[1] R.M. May, "Will a large complex system be stable?", *Nature* 238:413414 (1972).

[2] K.S. McCann, "The diversity-stability debate", *Nature* 405:228233 (2000).

[3] S. Johnson, V. Domínguez-García, L. Donetti, and M.A. Muñoz, "Trophic coherence determines food-web stability", *PNAS* 111, 17923 (2014).

[4] S. Johnson and N.S. Jones, "Looplessness in networks is linked to trophic coherence", *PNAS* 114, 5618 (2017).

61 The role of trophic coherence in complex systems

Supervisor: **Samuel Johnson**

Co-assessor: **Fabian Spill**

Description: The simplest useful description of a complex system is often as a graph, or network. For instance, brains, ecosystems or financial networks are often regarded as graphs whose vertices are neurons, species or banks, while edges represent interactions such as synaptic signalling, predation or financial exposure. Although much of the work on such systems assumes interactions to be symmetric (undirected networks), in many cases one should take into account that this is not the case (directed networks). We have recently shown that directed networks can exhibit a degree of order called *trophic coherence*, and that this feature has many important topological and dynamical effects [1–3]. But these results suggest many questions: How can we measure coherence when there are no *basal* (or *source*) nodes? Can we extend the definition to take account of networks with positive and negative interactions? What mechanisms could produce high coherence? What can trophic levels and coherence tell us about the functions of different nodes? Is there are general relationship between coherence and stability? This is quite an open project, in that the student can follow any of these leads, or indeed others related to this research.

Prerequisites: Some knowledge of graph theory and/or dynamical systems would be useful, but not essential.

References:

[1] S. Johnson, V. Domínguez-García, L. Donetti, and M.A. Muñoz, "Trophic coherence determines food-web stability", *PNAS* 111, 17923 (2014).

[2] J. Klaise and S. Johnson, "From neurons to epidemics: How trophic coherence affects spreading processes", *Chaos* 26, 065310 (2016).

[3] S. Johnson and N.S. Jones, "Looplessness in networks is linked to trophic coherence", *PNAS* 114, 5618 (2017).

62 Biocapsule formation for cell therapy

Supervisor: **Dr. Jamal Uddin**

Co-assessor: **Dr. Q. Wang**

Description: Encapsulation of a biological active substance like a cell or a protein is a growing field of biomedical research. Advances in this field are unlocking the potential for refined treatment of many diseases including cancer and diabetes. Cell encapsulation involves encasing cells in a protective outer membrane and forms a fundamental part of modern day medical and pharmaceutical research. Encapsulated cells, which belong to a wider class of biocapsules, offer a promising route in avoiding hostile *in vivo* responses of the bodies natural immune system. In the treatment of cancer, engineered stem cells are protected by a hydrogel membrane which offers a physical barrier from hostile factors. The hydrogel can be designed to biodegrade in a controlled way, and consequently provides a vehicle for the stem cell to migrate to the location of a tumour [1]. In this project we seek to develop a mathematical model for biocapsule formation through atomisation of coaxial liquid jets. The model incorporates modelling of the rheology of a suspension of cells and will involve an analysis of the stability of such coaxial jets [2] and [3].

Prerequisites: 3MePD, 3PTA, 3MoPD.

References:
[1] T. M. Kauer et al., (2012), Nature Neuro. **15**, 197-204.

[2] C. McIlroy and O. G. Harlen, (2014), Phys. Fluids, **26**, 033101.

[3] M. Afzaal, J. Uddin et al., (2015), Phys. of Fluids, **27**, 4.

63 Polymer Nanofibers in Biomedicine

Supervisor: **Dr. Jamal Uddin**

Co-assessor: **Dr. Q. Wang**

Description: Nanofibers, which are fibers typically less than one micrometer in diameter, are slowly being introduced into the market as technologies to successfully manufacture them in large volumes become available [1]. The properties of nanofibers, and in particular their relative large surface area to volume ratio, have led to applications in numerous fields including tissue scaffolding, drug delivery, lithium ion batteries, filtration and optics. As an example, researchers from North Carolina State University are using nanofibers to delivery therapeutic drugs. They have developed an elastic material that is embedded with needle like carbon nanofibers. The material is intended to be used as balloons which are inserted next to diseased tissue, and then inflated. When the balloon is inflated the carbon nanofibers penetrate diseased cells and delivery therapeutic drugs [2]. In this project we will examine the Rotary Jet Spinning (RJS) method for fiber production and look at mathematical models to determine relevant stability of such fibers.

Prerequisites: 3MePD, 3PTA, 3MoPD.

References:
[1] J. J. Rogalski et al., (2017)., Nanocomposites Review, 3, 4.

[2] R. C. Pearce et al., (2013), ACS Applied Materials & Interfaces.

[3] P. Mellado et al., (2011), App. Phys. Lett., 99, 203107.

64 Distensible tube wave energy converter

Supervisor: **Dr W. R. Smith**

Co-assessor: **Prof Y. D. Shikhmurzaev**

Description: The distensible tube wave energy converter is a recognized representative of the next generation of marine energy devices, it being sufficiently different from current prototypes to achieve a substantial reduction in the cost of energy. It is a submerged tube, full of sea water, located just below the surface of the sea. The tube undergoes a complex interaction with the sea waves which run along its length. The result is a bulge wave in the tube which, providing certain criteria are met, grows in amplitude and captures the wave energy. Engineering models have been successful in guiding the development of the first prototypes of the distensible tube; however, significant open problems hinder further progress. A comprehensive mathematical model will be formulated using fundamental principles from mechanics and systematic asymptotic analysis. The open problem of the interaction between the distensible tube and the waves will be investigated.

Prerequisites: Continuum Mechanics (3CM), Methods in Partial Differential Equations (3MePD), Perturbation Theory (3PTA)

References:
[1] W. R. Smith. Wave-structure interactions for the distensible tube wave energy converter. Proc. R. Soc. A472, 20160160, 2016.

65 The pitfalls of investigating laminar rotational flows with the Euler equations

Supervisor: **Dr W. R. Smith**

Co-assessor: **Dr Q. X. Wang**

Description: The amplitude of laminar high-Reynolds-number rotational flows is determined by viscous effects. Therefore, the high-Reynolds-number limit is singular. These facts are not well understood in fluid mechanics and, as a result, the Euler equations are used to simulate laminar rotational flows at large Reynolds number. A classical exact inviscid vortex solution will be investigated in this limit using the most recently developed techniques from singular perturbation theory.

Prerequisites: Continuum Mechanics (3CM), Methods in Partial Differential Equations (3MePD), Perturbation Theory (3PTA)

References:

[1] W. R. Smith and J. G. Wissink. Travelling waves in two-dimensional plane Poiseuille flow. SIAM J. Appl. Math., 75:2147-2169, 2015

[2] W. R. Smith and J. G. Wissink. Asymptotic analysis of the attractors in two-dimensional Kolmogorov flow. Eur. J. Appl. Math., 29: 393-416, 2018.

66 Non-local reaction-diffusion equations

Supervisor: **J. C. Meyer**

Co-assessor: **D. J. Needham**

Description: Models of biological and chemical processes often depend on the concentration of species locally in space but also on the bulk property of that species in some subset of its domain [2]. Consequently we will consider the following boundary value problem for $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ given by:

$$u_t - Du_{xx} = u\left(1 - \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy\right) \quad \text{in } \mathbb{R} \times (0,\infty) \tag{66.1}$$

with $D > 0$ constant and kernel $\phi : \mathbb{R} \to [0, \infty)$ such that $||\phi||_1 = 1$. Moreover, we have regularity of solutions given by

$$u \in C(\mathbb{R} \times [0,\infty)) \cap L^\infty(\mathbb{R} \times [0,\infty)) \cap C^{2,1}(\mathbb{R} \times (0,\infty)) \tag{66.2}$$

and initial data,

$$u(\cdot, 0) = u_0 \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}). \tag{66.3}$$

The aim of the project is to establish a qualitative theory for travelling waves in various asymptotic limits depending on D, c and ϕ, specifically in relation to whether travelling wave solutions are oscillatory or non-oscillatory. If time permits we will consider well-posedness results for the full boundary value problem.

Pre/Co-requisites: Ideally, perturbation theory and asymptotics, applied nonlinear dynamical systems, methods in partial differential equations, reaction diffusion theory, numerical methods II and functional and fourier analysis, but not necessarily all are required.

References:

[1] G. Nadin, B. Perthame and M. Tang, "Can a travelling wave connect two unstable states? The case of the nonlocal Fisher equation. *C. R. Acad. Sci. Paris, Ser. I*, **349**, (2011), 553-557.

[2] V. Volpert, *Elliptic Partial Differential Equations, Volume 2: Reaction-Diffusion Equations.* (Birkhäuser, Basel, 2011).

67 On statics or dynamics of a spherical bubble

Supervisor: **Q Wang**
Co-assessor: **D Leppinen**
Description: This project is concerned with radial oscillation and translation of microbubbles in Newtonian and non-Newtonian fluids subject to ultrasound. This phenomenon is found in sonochemistry, ultrasound cleaning and ultrasound medical applications. The dynamics will be modelled by nonlinear ordinary differential equations, which can be solved numerically using the Runge-Kutta method. The project is suitable for students who are interested in computations and physical analysis of phenomena.

Prerequisites: 2DE, 3MePD, 3CM, 3MoPD. Familiarity with MATLAB
Corequisites: 4TAM, 4Wav, 4RDT or 4TAM, 4Wav for TPAM students
References:
[1]QX Wang 2016 Local energy of a bubble system and its loss due to acoustic radiation. Journal of Fluid Mechanics, 797, 201230. DOI: 10.1017/jfm.2016.281

[2] QX Wang 2014 Multi-oscillations of a bubble in a compressible liquid near a rigid boundary. Journal of Fluid Mechanics, 745, 509-536. doi: 10.1017/jfm.2014.105

[3] QX Wang, JR Blake 2011 Non-spherical bubble dynamics in a compressible liquid. Part 2. Standing acoustic wave. Journal of Fluid Mechanics, 679, 559-581.

[4] QX Wang, JR Blake 2010 Non-spherical bubble dynamics in a compressible liquid. Part 1. Travelling acoustic wave. Journal of Fluid Mechanics, 659, 191-224.

68 Dynamics of liquid bridges

Supervisor: **Yulii Shikhmurzaev**
Co-assessor: **TBC**
Description: Liquid bridges connecting two solid bodies appear in a number of applications, from crystal growth to nanopens. In this project, you will consider the shape and parameters of liquid bridges depending on the wettability of the solid bodies, their motion and other factors. The project develops both mathematical and numerical skills.

69 Dynamics of a pulsating bubble

Supervisor: **Yulii Shikhmurzaev**
Co-assessor: **TBC**
Description: This project considers spherically-symmetric pulsations of a gas bubble in an incompressible liquid, from the Rayleigh solution to the influence on the process of insoluble and soluble surfactants. This problem will allow you to learn about such phenomena as adsorption-desorption dynamics and the modelling of chemical reactions in the framework of a highly nonlinear hydrodynamic problem.

70 Nonlinear oscillations of droplets

Supervisor: **Dr David Leppinen**
Co-assessor: **Dr Jama Uddin**
Description: We consider the oscillations of a droplet in an otherwise quiescent environment due to the action of surface tension. For small initial deformations, an inviscid droplet will oscillate indefinitely. For larger initial deformations, it is possible that under the action of surface tension, the droplet will break up into two or more smaller droplets. A combination of nonlinear asymptotic analysis and numerical simulations (using an existing boundary integral code) will be used to examine the transition from droplets which oscillate due to surface tension and droplets which break up.
Prerequisites: 3CM, 2DE, 3ANDS (or 3ANDS4 as a co-requisite)

71 Numerical methods in mathematical finance

Supervisor: **Dr David Leppinen**

Co-assessor: **Prof Chris Parker**

Description: This project will involve the student implementing and comparing different numerical methods that can used to determine the value of options. It is expected that the student will write their own computer codes to value Euorpean and American style options. Binomial (and related higher order methods) will be compared with finite difference approaches. Strategies for valuing path dependent options will be explored.

Prerequisites: 3FFin, 2NP

72 The Large Time Structure in a Classical Evolution problem for a Spatially Inhomogeneous Generalized Burgers Equation.

Supervisor: **Prof. D. J. Needham**

Co-assessor: **TBC**

Description: The unidirectional propagation of finite amplitude acoustic waves in a spatially inhomogeneous background leads to the study of a classical ervolution problem for a generalized inhomogeneous Burgers equation. This project uses the method of matched asymptotic coordinate expansions to examine the large time structure of the solution to this evolution problem.

73 The Development of a Wax Layer on the Interior Wall of a Pipe Transporting Heated Oil.

Supervisor: **Prof. D. J. Needham**

Co-assessor: **TBC**

Description:

74 How did the tiger get its stripes

Supervisor: **D. Loghin**

Co-assessor: **A. Bespalov**

Description: In his 1952 seminal paper, Alan Turing proposed and analysed a mechanism for pattern formation in the form of two coupled reaction-diffusion equations posed on a ring – a choice which enabled analysis. While he managed to identify stationary wave patterns for this simple system, he admitted that for general domains in 2D/3D and for more than two morphogens (i.e., reaction-diffusion equations), 'The difficulties are, however, such that one cannot hope to have any very embracing theory of such processes, beyond the statement of the equations. It might be possible, however, to treat a few particular cases in detail with the aid of a digital computer'. He concluded that 'the computational treatment of a particular case was most illuminating'

Reaction-diffusion systems are now standard nonlinear models in Applied Mathematics, with applications ranging from biology, biochemistry, morphogenesis to population dynamics and epidemic modeling. In general, only limited analysis is available, with most approaches being a mix of mathematical analysis, numerical analysis and computation, as Turing envisaged. The approach on this project will reflect this. Specifically, it will involve a review of RD theory, a detailed description and related analysis of the set of methods required for the numerical treatment of nonlinear PDE systems and computational investigations for two-dimensional domains of pattern selection (spots or stripes), sensitivity to initial and boundary conditions, growing domains, models with more than two morphogens etc. Familiarity with, or willingness to learn, programming and numerical methods is a must.

Prerequisites: Essential: LH Numerical Methods II. Desirable: LH Methods in Partial Differential Equations, LH Applied Nonlinear Dynamical Systems

Corequisites: LM Reaction-Diffusion Theory, LM Computational Methods for Scientific and Engineering Applications

References:

[1] A. M. Turing, *The Chemical Basis of Morphogenesis*, Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences, Vol. 237, No. 641. (Aug. 14, 1952), pp. 37-72.

[2] J. D. Murray, *Mathematical Biology II: Spatial Models and Biomedical Applications*, Springer, 2001.

[3] A. Quarteroni, R. Sacco, F. Saleri, *Numerical Mathematics*, Springer, 2000

75 Variational inequalities: the obstacle problem

Supervisor: **D. Loghin**

Co-assessor: **A. Bespalov**

Description: Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with boundary $\partial\Omega$. Consider the following partial differential inequality:

$$\text{PDI}: \begin{cases} \mathcal{L}u & \geq f \quad \text{in } \Omega \\ u & \geq \psi \quad \text{in } \Omega \\ (\mathcal{L}u - f)(u - \psi) & = 0 \quad \text{in } \Omega \\ u & = 0 \quad \text{on } \partial\Omega \end{cases}$$

When $\mathcal{L} = -\Delta$, the PDI is known as the obstacle problem and is used to model the shape of a membrane fixed at the boundary and lying over an obstacle function ψ (think circus tent). It is also related to the minimal surface problems (think soap films) and free boundary problems; finally, in finance, it is equivalent to the American option pricing problem. PDI can also be regarded as a Linear Complementarity Problem (LCP).

Questions of existence and uniqueness are harder for PDI than for PDE. Closed-form solutions are rare in general; for this reason, the only possible approach to finding a solution is numerical. A typical discretization method involves reformulating PDI as a variational inequality which is then discretized by finite difference or finite element methods. The resulting problem is a discrete LCP which can be solved in various ways. The aims of the project include a review of the current theory for variational inequalities, methods of discretization and LCP solution methods.

Prerequisites: Essential: LH Numerical Methods II., LH Methods in Partial Differential Equations. Desirable: Linear Analysis

Corequisites: LM Numerical Linear Algebra and Applications, LM Computational Methods for Scientific and Engineering Applications

References:

[1] D. Kinderlehrer, G. Stampacchia *An Introduction to Variational Inequalities and Their Applications*, SIAM, 2000.

[2] C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Dover 2009.

76 Domain decomposition methods

Supervisor: **D. Loghin**

Co-assessor: **Natalia Petrovskaya**

Description: Domain decomposition methods are standard solution methods for large scale calculations arising in the area of numerical solution of PDE. The approach is to divide up the computational domain into smaller domains leading to a set of smaller, independent problems. The advantages are obvious: (i) the problems are smaller and thus easier to solve numerically and (ii) they can be solved in parallel. The downside is that this re-formulation of the problem is only possible through knowledge and suitable discretisation of integro-differential operators defined on the interfaces generated through the subdivision of the domain. This is a hard problem which may not have a known solution in general. The aim of the project is to review existing domain decomposition techniques for elliptic equations and to generalise and analyse some standard choices of interface operators to the case of elliptic problems with variable coefficients.

Prerequisites: LH Numerical Methods II., LH Methods in Partial Differential Equations

Corequisites: LM Numerical Linear Algebra and Applications, LM Computational Methods for Scientific and Engineering Applications

References:

[1] A. Quarteroni and A. Valli, Domain Decomposition Methods for Partial Differential Equations, Oxford University Press, 1999.

[2] T. F. Chan, Domain Decomposition Algorithms, Acta Numerica, 1994, pp. 61-143.

77 Solving differential equations using deep neural networks

Supervisor: **Dr Alex Bespalov**

Co-assessor: **Dr Daniel Loghin**

Description: Artificial neural networks were inspired by their biological prototypes (that constitute, for example, the human brain) and developed into the computational framework for many machine learning algorithms. In particular, deep neural networks (DNNs) have recently demonstrated spectacular results for a wide range of data-intensive applications from language processing to computer vision and image classification. On the other hand, DNNs can be seen as a powerful approximation technique and, as such, they can be used for numerical solution of 'purely' mathematical problems, e.g., the reconstruction of a multivariable function given by a (very large) set of data points, solving a large system of (non-)linear equations, or solving a differential equation on a complicated domain.

In the first part of this project, the student will learn main ingredients of artificial neural networks and the mathematical techniques behind them [1, 2]. The project will then proceed to the application of this methodology to a selected mathematical problem (e.g., the numerical solution of a differential equation [3, 4]). Depending on the student's interests, this may involve a theoretical study of approximation properties of DNNs for the problem at hand or development and implementation of concrete algorithms for this problem.

The project could be the first step towards future PhD studies in the area of numerical analysis and scientific computing.

Prerequisites: Differential Equations (2DE), Numerical Methods II (3NM).

Corequisites: Topics in Applied Mathematics (Advanced Numerical Methods, 4TAM-B), Methods in Partial Differential Equations (3MePD).

References:

[1] I. Goodfellow, Y. Bengio and A. Courville, *Deep Learning*, MIT Press, Boston, 2016 (see also `http://www.deeplearningbook.org`).

[2] C. F. Higham and D. J. Higham, *Deep learning: an introduction for applied mathematicians*, 2018, `https://arxiv.org/abs/1801.05894`.

[3] J. Berg and K. Nyström, *A unified deep artificial neural network approach to partial differential equations in complex geometries*, Neurocomputing, 317 (2018), pp. 28–41.

[4] W. E and B. Yu, *The Deep Ritz method: A deep learning-based numerical algorithm for solving variational problems*, Commun. Math. Stat., 6 (2018), pp. 1–12.

78 Stochastic finite element methods

Supervisor: **Dr Alex Bespalov**

Co-assessor: **Dr Natalia Petrovskaya**

Description: Stochastic finite element methods (SFEMs) are modern techniques for numerical solution of partial differential equations with random data (often called stochastic PDEs). The key idea underlying these methods consists in using parametric representations of both the random data (e.g., coefficients in PDEs, boundary data, etc.) and the random solution to PDE in terms of a large (possibly infinite) number of stochastic parameters. The original stochastic PDE problem is then reformulated in a parametric deterministic form, that can be discretised with finite elements. Two main variants of the SFEMs are stochastic collocation and stochastic Galerkin finite element methods. Approximate solutions in these methods are sought in the tensor product space $X_h \otimes \mathcal{P}$, where X_h is a conventional finite element space associated with physical domain and $\mathcal{P}$ is a set of multivariate polynomials in stochastic parameters. When computing approximate solutions, stochastic collocation methods rely on samples in the parameter space to generate solutions to decoupled finite element problems, whereas stochastic Galerkin methods compute solution coefficients in the basis of $X_h \otimes \mathcal{P}$ directly via single (but very large) calculation.

Depending on the interests of the student, this project may pursue various aspects of one or both variants of the SFEM. These aspects will range from looking into advantages and drawbacks of different approaches to theoretical analysis and MATLAB-implementation of the algorithms. Interested students are encouraged to see the supervisor to discuss the details. The project could be the first step towards future PhD studies in the area of numerical analysis and scientific computing.

Prerequisites: Numerical Methods II (3NM), Methods in Partial Differential Equations (3MePD).
Interest and ability to program in MATLAB (or strong enthusiasm to learn MATLAB) are desirable.

Corequisites: Topics in Applied Mathematics (Advanced Numerical Methods, 4TAM-B).

References:

[1] R. G. Ghanem and P. D. Spanos, *Stochastic finite elements: a spectral approach*, Springer-Verlag, New York, 1991.

[2] I. Babuška, R. Tempone, and G. E. Zouraris, *Galerkin finite element approximations of stochastic elliptic partial differential equations*, SIAM J. Numer. Anal., 42 (2004), pp. 800–825.

[3] I. Babuška, F. Nobile, and R. Tempone, *A stochastic collocation method for elliptic partial differential equations with random input data*, SIAM J. Numer. Anal., 45 (2007), pp. 1005–1034.

[4] C. Schwab and C. J. Gittelson, *Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs*, Acta Numer., 20 (2011), pp. 291–467.

[5] M. D. Gunzburger, C. G. Webster, and G. Zhang, *Stochastic finite element methods for partial differential equations with random input data*, Acta Numer., 23 (2014), pp. 521–650.

79 Numerical algorithms for stochastic differential equations

Supervisor: **Xiaocheng Shang**

Co-assessor: **Alex Bespalov**

Description: Stochastic differential equations (SDEs) are widely used in science and engineering in order to model systems where random effects play a significant role; applications include molecular dynamics [1], complex hydrodynamic behaviour of DNA and blood flow, and Bayesian sampling techniques used in emerging machine learning applications. However, the design of efficient and accurate numerical algorithms for such systems is highly nontrivial, since they are often in high dimensional spaces (or large-scale datasets in data science applications) and the largest stepsize usable is often limited in order to maintain numerical stability. In this project, the student will first learn how to optimally design and analyze numerical algorithms for the popular Langevin dynamics, based on the idea of splitting the vector field of the stochastic system in such a way that each subsystem can be solved exactly [2]. The student will then explore how to apply the splitting techniques to more challenging settings, e.g., the generalized Langevin dynamics that incorporates memory effects [3,4]. The project will prepare the student well if the student wishes to pursue PhD studies in the area of numerical analysis and scientific computing.

Prerequisites: The project requires basic knowledge of numerical methods for ODEs (exposure to SDEs will be helpful) and programming in MATLAB.

References:

[1] Leimkuhler, B. and Matthews, C. (2015) *Molecular Dynamics: With Deterministic and Stochastic Numerical Methods*. Springer.

[2] Leimkuhler, B. and Matthews, C. (2013) ‘Rational construction of stochastic numerical methods for molecular sampling’, *Appl. Math. Res. Express*, 2013(1), pp. 34–56.

[3] Baczewski, A. D. and Bond, S. D. (2013) ‘Numerical integration of the extended variable generalized Langevin equation with a positive Prony representable memory kernel’, *J. Chem. Phys.*, 139(4), pp. 044107.

[4] Ceriotti, M., Bussi, G. and Parrinello, M. (2010) ‘Colored-noise thermostats a la carte’, *J. Chem. Theory Comput.*, 6(4), pp. 1170–1180.

80 Estimating pest insect population density from trap counts

Supervisor: **Natalia Petrovskaya**

Co-assessor: **Alex Bespalov**

Description: In ecological studies, populations are usually described in terms of the population density or population size. Having these values known over a period of time, conclusions can be made about a given species, community, or ecosystem as a whole. In particular, in pest management, the information gained about pest abundance in a given field or area is then used to make a decision about pesticide application. To avoid unjustified decisions and unnecessary losses, the quality of the information about the population density is therefore a matter of primary importance. However, the population density is rarely measured straightforwardly, e.g. by direct counting of the individuals. In the case of insects, their density is often estimated based on trap counts. The problem is that, once the trap counts are collected, it is not always clear how to use them in order to obtain an estimate of the population density in the field. The aim of this project is to overcome current limitations of trapping methods used in ecological studies through developing a theoretical and computational framework that enables a direct estimate of populations from trap counts. A mean-field diffusion model will be considered to investigate if it is capable of revealing the generic relationship between trap catches and population density.

Prerequisites: LI Numerical Methods and Programming (good programming skills are essential for the project!), LI Differential Equations, LH Methods in Partial Differential Equations.

References:

[1] [1] Crank, J., 1975. The mathematics of diffusion (2nd edition). Oxford University press, Oxford.

[2] Kot, M., 2001. Elements of Mathematical Ecology. Cambridge University Press, Cambridge.

[3] Okubo, A., Levin SA, 2001. Diffusion and Ecological Problems: Modern Perspecives. Springer, Berlin.

81 Numerical study of patchy invasion

Supervisor: **Natalia Petrovskaya**

Co-assessor: **Alex Bespalov**

Description: Gypsy moth is regarded as one of the top most harmful invasive species. Its invasion in the northeastern USA has led to widespread forest defoliation, wildlife disruption and even a change in biogeochemical conditions over the area of 10^6 km^2. Spread of gypsy moth has a few distinct features such as a patchy spatial distribution of the gypsy moth population, which is largely uncorrelated to the environmental heterogeneity, and a high variability (almost over an order of magnitude) in the spread rates. The aim of this project is to understand whether the patchy structure can result from the interplay between two natural factors such as wind dispersal and viral infection. In order to check this hypothesis, we describe the gypsy moth spread with a diffusive SI model, investigate the stability of its non-spatial counterpart, and study the properties of the non-spatial system by means of extensive computer simulations to predict the parameter range where patchy paterns can appear. The predictions made for the non-spatial model will then be validated in numerical solution of the original spatio-temporal model.

Prerequisites: LI Numerical Methods and Programming, LI Differential Equations, LH Methods in Partial Differential Equations.

References:

[1] Lewis, M.A., Kareiva, P., 1993. Allee dynamics and the spread of invading organisms. Theoretical Population Biology 43, 141158.

[2] Gerardi, M.H., Grimm, J.K., 1979. The history, biology, damage and control of the gypsy moth Porthetria dispar (L.) Fairleigh Dickinson University Press, Rutherford, NJ.

[3] Jankovic, M. Petrovskii, S. 2013. Gypsy moth invasion in North America: A simulation study of the spatial pattern and the rate of spread. Ecological Complexity 14, 132–144.

Projects in Optimization

82 Multicriteria decision making models for usability of websites

Supervisor: **Dr Sándor Zoltán Németh**

Co-assessor: **Prof Jinglai Li**

Description: By browsing the World Wide Web, everyone comes across many badly designed, hard-to-use websites. There are usability criteria, which should be respected by web designers in order for the users to easily find what they need, without getting frustrated. The project consists in designing a multicriteria decision method for rating websites based on usability criteria. For designing such a method, genetic programming might also be considered.

Prerequisites: Basic linear algebra and vector calculus. If genetic programming will be considered, then basic programming skills (Matlab and/or C++) are needed.

References:
[1] Thomas L. Saaty. *The analytic hierarchy process. Planning, priority setting, resource allocation.* McGraw-Hill, 1980.

[2] J. P. Brans and Ph. Vincke. A preference ranking organisation method (the PROMETHEE method for multiple criteria decision-making). *Management Science*, 31:647–656, 1985.

[3] J. P. Brans, Ph. Vincke, and B. Mareshal. How to select and how to rank projects: The PROMETHEE method. *European Journal of Operational Research*, 24:228–238, 1986.

[4] B. Roy. The outranking approach and the foundations of ELECTRE methods. *Theory and Decision*, 31(1):49–73, 1991.

[5] P. Csáki, F. Fölsz, T. Rapcsák and Z. Sági. On tender evaluations *Journal of Decision Systems*, 7:179–194, 1998.

[6] Anikó Ekárt and S. Z. Németh. Stability of tree structured decision functions. *European Journal of Operational Research*, 160:676–695, 2005.

83 A^* for solving Rubik's cube

Supervisor: **Dr Sándor Zoltán Németh**

Co-assessor: **Prof Michal Kočvara**

Description: Rubik's cube can be solved in at most 20 moves from any state. Algorithms exist which always lead to the solution, but in much more steps. A^* is an informed search method that can be used to find the optimal solution. The goal of the project is to find a suitable heuristic function and a representation of the cube.

Prerequisites: Some basic Matlab skills would be desirable for the examples.

References:
[1] Zbigniew Michalewicz and David B. Fogel, How to Solve It: Modern Heuristics, Springer (March 1, 2004)

[2] Ruhul A. Sarker, Hussein A. Abbass and Charles S. Newton, Heuristic and Optimization for Knowledge Discovery, Idea Group Publishing (February 7, 2002)

[3] S. J. Russell and P. Norviq, Artificial intelligence: a modern approach, Englewood Cliffs, N.J.: Prentice Hall, 1995.

[4] Nils J. Nilsson, Principles of Artificial Intelligence, Morgan Kaufmann Publishers; Reprint edition (June, 1986)

[5] http://www.edenwaith.com/products/pige/tutorials/a-star.php

[6] http://en.wikipedia.org/wiki/A%2A_algorithm

84 Duality in tropical linear programming

Supervisor: **Sergey Sergeev**

Co-assessor: **Sándor Zoltán Németh**

Description: Tropical linear programming [1] is a new and rapidly evolving area of idempotent mathematics, discrete optimisation and linear algebra with a wide range of applications. One of the key questions is duality of tropical linear programs. It is well known that an analogue of duality of tropical linear programs holds [2]. However, nothing is known about the duality gap for tropical integer programming. You will investigate this question and derive bounds or exact formulae for the gap.

Prerequisites: 2LALP, 3IPCO

References:
[1] Butkovic, P (2010) Max-linear Systems: Theory and Algorithms, Springer Monographs in Mathematics, Springer-Verlag, London

[2] Hoffman, A J (1963) On abstract dual linear programs. Naval Res. Logist. Quart. 10 (369 - 373).

85 Image compression via PDE

Supervisor: **Professor M Kocvara**

Co-assessor: **Dr D Loghin**

Description: Image compression has been one of the success stories of Computer Science; every student knows compression formats like jpeg, jpeg2000 and mpeg. Their goal is to preserve as much information about the original image with as few data. The jpeg format typically uses just about 10 per cent of the original size of the image.
In this project, the student will approach the image compression problem from the point of view of numerical analysis and partial differential equations. It has been shown that if we keep just ten percent of pixels of the original image, selected randomly or in a more clever way, we can solve a simple PDE on the full domain using a mask of the remaining pixels to obtain a very good approximation of the original image. The student will study the basic technique and program it using MATLAB, firstly with a random selection of the remaining pixels and Laplacian operator for the PDE. If time permits, the student will try to improve the reconstructed image by optimizing the position of the remaining pixels or by choosing a more suitable operator. The resulting large-scale system of linear equations will be solved by an iterative method, namely by the so-called multigrid method.

Prerequisites: The project requires basic knowledge of numerical methods for PDE and linear algebra and programming in Matlab.

References: `http://www.mia.uni-saarland.de/Research/IP_Compress.shtml` and the references therein

86 Alternating direction method of multipliers in semidefinite optimization

Supervisor: **Professor M Kocvara**

Co-assessor: **Prof Jinglai Li**

Description: The alternating direction method of multipliers (ADMM) is an algorithm that solves convex optimization problems by breaking them into smaller pieces, each of which are then easier to handle. It has recently found wide application in a number of areas. Semidefinite optimization (SDO) is a branch of constrained optimization. Unlike in standard inequality constraint, when a vector is required to be nonnegative, in SDO a matrix is required to be positive semidefinite. SDO has many applications in various areas, through finances to chemical engineering, to image processing.
The goal of the project is to examine applications of ADMM in semidefinite optimization. The student will learn basics of ADMM and SDO and will apply various versions of ADMM to solve large-scale SDO problems.

Prerequisites: The project requires basic knowledge of linear algebra, linear programming and, if available, conic optimization.

References:

[1] Z. Wen, D. Goldfarb, and W. Yin. Alternating direction augmented Lagrangian methods for semidefinite programming, Mathematical Programming Computation 2.3-4 (2010): 203-230.

[2] N. Parikh and S. Boyd. Block splitting for distributed optimization, Mathematical Programming Computation 6.1 (2014): 77-102.

[3] R. Madani, A. Kalbat, and J. Lavaei. ADMM for sparse semidefinite programming with applications to optimal power flow problem. Decision and Control (CDC), 2015 IEEE 54th Annual Conference on. IEEE, 2015.

87 Tropical optimization

Supervisor: **Sergey Sergeev**

Co-assessor: **Yun-Bin Zhao**

Description: We will consider optimization problems, where the objective function and the constraints are defined in terms of the max-plus arithmetics $\oplus = \max$ and $\otimes = +$ [1]. A general theory of tropical linear and fractional linear optimization has been developed [1,2], and it also involves an interesting connection with parametric mean-payoff games [2]. The goal of the project will be to review the existing tropical optimization problems and to develop a new idea or direction, for example the bi-level tropical optimization or optimization over symmetrization.

Prerequisites: Optimization modules such as Game Theory, Heuristic Optimization or Nonlinear Programming are relevant, but none of them necessary. Some knowledge of Algebra, Convex Geometry and/or Combinatorics is also welcome.

References:

[1] P. Butkovič. Max-linear systems: Theory and Algorithms. Springer, London, 2010.

[2] S. Gaubert, R.D. Katz and S. Sergeev. Tropical linear programming and parametric mean-payoff games. *J. of Symbolic Computation*, 47 (12) 2012, 1447-1478.

88 Analysing demand preferences by means of tropical geometry

Supervisor: **Sergey Sergeev**

Co-assessor: **Timothy Magee**

Description: There are various situations in economics where agents bid for indivisible bundles of goods and the profit from these bundles is described by certain utility function. When these agents compete on the same market, the question of existence of competitive equilibrium arises. Baldwin and Klemperer [1]recently developed an approach that allows to describe the conditions for existence of such equilibrium in terms of tropical geometry, which had been used before only in the context of pure mathematics, helping in particular to enumerate algebraic curves and describe their asymptotic properties. It is proposed that you will learn and understand some of the theory described in [1], and one of the possible research directions is to describe in detail how it applies to some of the known models of economics and management, such as [2].

Prerequisites: Some knowledge of Convex Geometry, Combinatorics, Game Theory and/or Algebraic Geometry is welcome, but not absolutely necessary.

References:

[1] E. Baldwin and P. Klemperer. Understanding preferences: “Demand types”, and the existence of equilibrium with indivisibilities. Preprint, available online: www.nuff.ox.ac.uk/users/klemperer/demandtypes.pdf

[2] A.S. Kelso and V.P. Crawford. Job matching, coalition formation, and gross substitutes. Econometrica: Journal of the Econometric Society, 121 (5), 966-1005.

89 Optimization methods for portfolio selection

Supervisor: **Sergey Sergeev**

Co-assessor: **Michal Kočvara**

Description: Some portfolio selection models, such as LAMAD model, are formulated as mixed-integer linear programming problems, that is, optimization problems where some variables are constrained to be integer [1,2]. We are going to study some optimization techniques developed for such problems. The project will include development of new software and actual application to portfolio optimization problems, in order to test the existing algorithms, apply them and possibly improve them.

Prerequisites: Some knowledge of Non-Linear Programming, Integer Programming and Heuristic Optimization will be very useful, but not absolutely necessary.

References:

[1] R. Cornuejols and R. Tutuncu. Optimization in Finance. Cambridge University Press, 2007.

[2] F. Cesarone, A. Scozzari, F. Tardella. A new method for mean-variance portfolio optimization with cardinality constraints. *Annals of Operations Research* 205 (1), 2013, 213-234.

Projects in Statistics

90 Long term behaviour of the Elo scores of time-homogeneous players

Supervisor: **Dr Arnaud Lionnet**

Co-assessor: **Dr Fabian Spill**

Description: In many games, winning requires both skill and chance. The *games* we consider can be board games or sports, so long as they are 2-player games. Likewise, a *player* can in practice be a team, as is often the case in sports. So if the results of matches are partly random, how can one know how good the players are? One approach, proposed by Arpad Elo for Chess and that has since then been adopted in a wide range of sports and online gaming sites, is to attribute a score y_i to each player $i \in \{1, \ldots, N\}$, called the Elo score, and supposed to reflect their strengths x_i. When two players meet, the Elo score of the winner is increased and the score of the loser is decreased by as much. While this scoring/ranking method is ubiquitous, little is known about its properties. Given a probabilistic model for the outcomes of the games, which draws random outcomes for matches between players i and j, and given a schedule for matches between players, we can look at the evolutions of the individual scores. More precisely, we look at the vector $Y(t) = \big(Y_i(t)\big)_{i=1\ldots N}$, where $Y_i(t)$ is the Elo score of player i after t matches have been played, for $t \in \mathbb{N}$. The main question to investigate in this project is the behaviour of $Y(t)$ as the number of matches $t \to +\infty$. D. Aldous obtained that there exists a probability distribution $\mathcal{Y}$ such that $Y(t)$ converges to $\mathcal{Y}$ in probability. But this is only an existence result and does not give information on *what* the limit distribution looks like. Do the scores of the players converge to their true strengths? Or at least, to a distribution that separates the strengths well enough so that one can rank players with a reasonable degree of confidence? The goal of the project is to study this distribution $\mathcal{Y}$ using computer simulations.

Prerequisites: Good background in probability and statistics. Good programming skills.

References:

[1] D. Aldous, Elo ratings and the sports model: A neglected topic in applied probability?, *Statistical Science* (2017).

[2] D. Aldous, Mathematical foundations of dynamic sports ratings. Overview and open problems. *Working paper* (2017+)

91 Statistical Process Control

Supervisor: **Biman Chakraborty**

Co-assessor: **Hui Li**

Description: Controlling process parameters in any industrial process is an important aspect. It not only reduces cost and efficiency, it is a requirement in many industries to maintain the quality of their product and service. The most popular visual tool to monitor a process variable is a control chart. There are several control charting techniques available in the literature, e.g. Shewert's control chart, CUSUM and EWMA charts. Almost all of these tools depend heavily on the assumption of underlying normal distribution for the process parameter. In this project, we will investigate the performance of some of these proposed methods when the underlying distribution deviates from normality, using simulations as well as theoretical derivations. We will also consider some techniques, which does not depend on the normal distribution and hence can be applied to a larger number of situations without worrying about normality. But they may lack in efficiency when the true distribution is indeed normal. This project will involve understanding of the basic ideas and extensive simulations using either R or MATLAB. For theoretical derivations, basic knowledge of statistics and probability is good enough.

Prerequisites: 2S/2S3: Statistics

References: [1] Montgomery, D.C. (2012) *Statistical Quality Control: A Modern Introduction*, 7th Edition, Wiley

92 Nonparametric Classification

Supervisor: **Biman Chakraborty**

Co-assessor: **Hui Li**

Description: Statistical classification is the research area that studies the design and operation of systems that recognize and classify patterns in data. Important application domains are image analysis, computer vision, character recognition, speech analysis, man and machine diagnostics, person identification, industrial inspection, financial data analysis and forecast, genetics. The project starts with a review some basic concepts from parametric classification techniques. These techniques aim at classify data by constructing analytical functions which estimate the statistical distribution of data samples. However, it is not always possible to describe this distribution analytically. In such cases, more general, non-parametric techniques have to be applied. We will review some popular nonparametric classification techniques and if possible, some improvements on them. This project will involve understanding of the basic ideas and extensive comparison studies using simulations with either R or MATLAB. For theoretical derivations, basic knowledge of statistics and probability is good enough.

Prerequisites: 2S/2S3: Statistics

References: [1]R.O. Duda, P.E. Hart, D. Stork (2001), *Pattern Classification*, 2nd edition, Wiley

93 Behind the scenes: Modelling the outcome of top international piano competitions

Supervisor: **Hui Li**

Co-assessor: **Biman Chakraborty**

Description: In general, talent has been linked to peak performances (high achievement) in musical performance. However, empirical research on the probability of winning in music performance is limited and it could offer novel insights into the musical activities. In this project, we focused on individual achievements, i.e., the places in the winner list in top piano international competitions which takes place in recent years and model their outcomes considering individual's achievements, repertoire structure and the composition of the jury team. Students are expected to use multiple regression and logistic regression type of models to explore the project.

Prerequisites: e.g. 2S, 3SMFE, 3AS etc

References:
[1] Evidence of bias in the Eurovision song contest: modelling the votes using Bayesian hierarchical models Marta Blangiardo & Gianluca Baio

[2] Getting into the musical zone: trait emotional intelligence and amount of practice predict flow in pianists.

[3] Modeling and prediction of competitive performance in swimming upon neural networks Jurgen Edelmann-nusser , Andreas Hohmann & Bernd Henneberg.

[4] Rater fairness in music performance assessment: Evaluating model-data fit and differential rater functioning Brian C. Wesolowski, Stefanie A. Wind, George Engelhard, Jr.

Projects in Education

94 Developing skills required to understand proof

Supervisor: **J. C. Meyer**
Co-assessor: **R. Leek**
Description: We will investigate types of mathematics questions that can be given to students in computer based assessments that are focussed on students development of a deep understanding of proof. The overall aim of the project is motivate and implement the creation of content that encourages students to understand proofs, and to what extent this content can be assessed automatically using a computer. The output(s) of this project will be created using Möbius and potentially be used as teaching materials in the future academic years. This project may be of particular interest to students with an interest in mathematics education and programming.
Prerequisites: Familiarity with Maple and at least one high level programming language.
References:
[1] M. Greene and P. Shorter, “Conceptual understanding weighting system: a targeted assessment tool”. *Teaching Mathematics and its applications: An international journal of the IMA*, **36**(1), p.1-17. https://doi.org/10.1093/teamat/hrw003

[2] Möbius Courseware, https://www.digitaled.com/products/courseware/.

[3] C. Sangwin. *Computer aided assessment of mathematics.* (OUP, Oxford, 2013).

95 Why, and how, does the specialist and more-able mathematics student use mathematics support?

Supervisor: **Michael Grove**
Co-assessor: **Chris Good**
Description: Mathematics underpins all Science, Technology, Engineering, and Mathematics (STEM) disciplines and is important in many others too, yet for various reasons, including personal, educational and social, many students arrive at university with negative attitudes about mathematics and a perception that they are ‘not very good at maths’. Further, many students also experience issues in their learning of mathematics, again for a range of reasons, as they progress through their undergraduate programmes or seek to make the transition into employment. To respond to such issues, many universities now offer ‘mathematics support provision’, principally through mathematics support centres, which form an offering that is additional students’ regular teaching in lectures and tutorials, and which acts as an important mechanism for helping students to achieve their full potential. Although such support was originally conceived as a way of offering support to struggling non-specialist students, there is now an increasing body of evidence that it has an important role in supporting specialist mathematics students, many of whom are actually doing rather well, with their mathematical learning.

Following-on from an MSci project on this theme in 2017/18, the results of which have just been accepted for publication in an international mathematics education journal, this project will seek to better understand the reasons as to why an increasing number of specialist and more-able mathematics students choose to access mathematics support. In particular it will seek to understand how, and why, such students are using the mathematics support centre here in Birmingham, the relationship this has with the support provided for their mathematical learning by the School of Mathematics, and how this differs from the use made of mathematics support by non-specialist mathematics learners. This project will build upon the results of a survey undertaken in 2017/18, and will collect new data through the use of qualitative methodologies, and as such, you should be prepared to collect and analyse data via interviews, focus groups and through observational studies. Whilst dependent upon the outcomes from the research, this project represents a unique opportunity to undertake a project with the potential of publishable outcomes.
Prerequisites: None
References:
[1] Croft, A.C. and Grove, M.J. (2015) ‘Progression within mathematics degree programmes’, in Grove, M.J., Croft, A.C., Kyle, J. and Lawson, D.A. (Editors) *Transitions in undergraduate Mathematics Education*. Higher Education Academy.

[2] Grove, M.J. and Overton, T.L. (Editors) (2013) *Getting Started in Pedagogic Research within the STEM Disciplines*. University of Birmingham and The Higher Education Academy. Available at: http://www.birmingham.ac.uk/Documents/college-eps/college/stem/getting-started-in-stem-pedagogic-optimised.pdf

[3] Grove, M.J., Guiry, S. and Croft, A.C. (Accepted) ‘Specialist and more-able mathematics students: understanding their engagement with mathematics support’. *Manuscript accepted for publication*.

96 How to write good questions in Undergraduate Mathematics

Supervisor: **Dr Dirk F M Hermans**

Co-assessor: **Dr M Grove or Dr D Leppinen**

Description: In this project, we will study what makes a good assessment in Mathematics. In particular, we will expplore the impact of Computer Aided Assessment (CAA) in Mathematics, as well as the pedagogical and mathematical issues it might raise. This will include the use of CAA in both formative and summative assessment and how traditional assessment methodologies can be amended or partially replaced by CAA. Following on from this pedagogical research, this projects then aims to further develop a framework for describing questions, with all their attributes, which is then transferable to the various CAA systems. These question ”factsheets” go beyond stating the question and correct solution, but look at how randomization may take place, how the questions should be classified, the potential detailed feedback streams and the assessment context in which these questions can be used. The project has access to CAA systems for testing the implementation of such new questions, and a possible outcome, besides the descriptors of the pseudo-questions, could be a new resource for students.

Prerequisites: None

References: [1] Sangwin, C.J., Computer Aided Assessmenmt of Mathematics, 2013.

97 How to develop valid competence tests in Undergraduate Mathematics

Supervisor: **Dr Dirk F M Hermans**

Co-assessor: **Dr M Grove or Dr D Leppinen**

Description: In this project, we will study what makes a good competence test in Mathematics. In some countries, it is required to provide a solid argument of why a certain mark is taken as the threshold of an award, be it entry into a course or a pass at an examination. This project will study the theory behind competence tests and sketch in more detail, probably on the basis of a specific example, how such tests are developed and maintained. It will build on Classical Test Theory and Individual Response Theory and will include exploring how non-mathematical subjects are making use of these.

Prerequisites: None

References: [1] Sangwin, C.J., Computer Aided Assessmenmt of Mathematics, 2013.