Gauss' Circle Problem

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The Gauss' circle problem investigates how many integer lattice points there are in or on a circle which is centred at the origin with radius r [3]. We define a lattice point as follows:

Definition 0.1. *A point (x,y) in R^2 is said to be a* ***lattice point*** *if $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$.*

It is named the Gauss' circle problem as German mathematician and physicist Carl Friedrich Gauss (1777-1855) made the first progress on a solution.

We illustrate the problem shown in Figure 1 which shows the number of lattice points in circles of radius 1,2 and 3 with each lattice point circled in black.

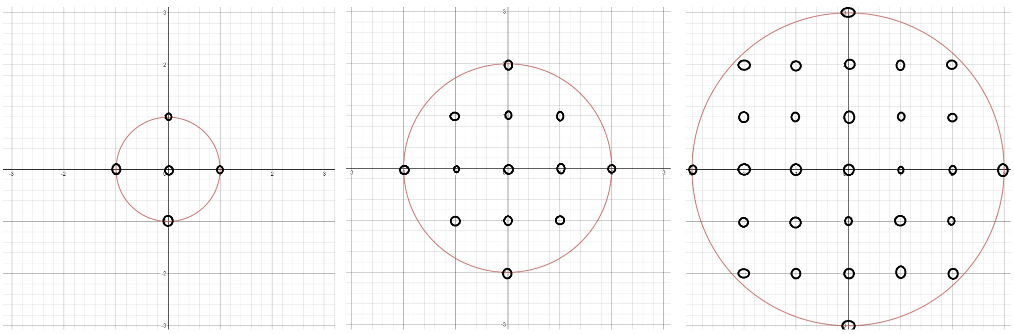

Figure 1: Lattice points in a circle of radius r=1,2,3

As the equation of the circle centred at the origin can be given in Cartesian coordinates by $x^2 + y^2 = r^2$, the problem is equivalent to calculating how many ordered pairs (m, n) exist such that $m^2 + n^2 \leq r^2$ where

$m \geq 0, n \geq$ and $r \geq 0$. We denote $N(r)$ as the number of solutions to the inequality.

Example 1. Using Figure 1, we can calculate the number of integer lattice points there are in a circle of radius 1,2 and 3.

Table 1: Example of ordered pairs for different radius

Radius	**Ordered Pairs**	$N(r)$
1	{(-1,0),(0,1),(0,0),(0,-1),(1,0)}	5
2	{(-2,0),(-1,1),(-1,0),(-1,-1), (0,2),(0,1), (0,0),(0,-1),(0,-2),(1,1),(1,0),(1,-1),(2,0)}	13
3	{(-3,0),(-2,2),(-2,1),(-2,0), (-2,-1),(-2,-2), (-1,2),(-1,1),(-1,0),(-1,-1),(-1,-2),(0,3),(0,2), (0,1),(0,0),(0,-1),(0,-2), (0,-3),(1,2),(1,1), (1,0),(1,-1),(1,-2),(2,2),(2,1)(2,0),(2,-1), (2,-2),(3,0)}	29

Note in our example we have used the first three positive integer numbers for r however we could have chosen non-integer values for r.

Intuitively, the number of lattice points in a circle of radius r should be roughly the area of the circle (πr^2) [2]. One of the possible ways to visualise this is to associate a unit square with each lattice point, for example for each lattice point Q within the circle we will associate a unit square where Q is the bottom left corner as seen in Figure 2.

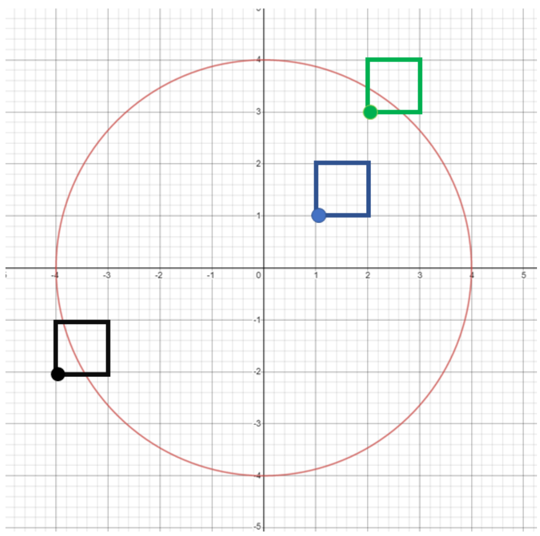

Figure 2: Unit squares associated with selected lattice points

For a circle with $r = 4$, we see that for many of the lattice points within the circle the associated unit square will also be fully in the circle, such as the blue unit square associated with the lattice point (1,1). However, for the green lattice point (2,3) which is in the circle, much of the associated unit square is outside the circle. Whereas the black lattice point (-4,-2) isn't in the circle but much of the associated unit square is within the circle. This indicates that the area of the circle is not a perfect measure of how many lattice points are in a circle of radius r centred at the origin. Due to this, we have

$$N(r) = \pi r^2 + E(r), \tag{1}$$

where $E(r)$ is a relatively small absolute value called the error term. Gauss' circle problem is now related to finding bounds for $E(r)$. Gauss proved that $|E(r)| \leq 2\sqrt{2}\pi r$ and other mathematicians have taken the problem further. We will now show how an initial upper bound can be found.

As the diagonal of the unit square is $\sqrt{2}$, each square to the upper right of a lattice point in the circle will be fully contained in a circle of radius $r + \sqrt{2}$. Hence the total area of all unit squares, which is equal to N(r), is at most the area of a circle with radius $r + \sqrt{2}$, i.e.

$$N(r) \leq \pi(r + \sqrt{2})^2 = \pi r^2 + 2\pi\sqrt{2}r + 2\pi. \tag{2}$$

Likewise, the entire area of the circle with radius $r - \sqrt{2}$ is covered by unit squares to the upper right of integer points within the circle of $r - \sqrt{2}$. Therefore the total area of all unit squares, which is equal to N(r), is at least the area of a circle with radius $r - \sqrt{2}$, i.e.

$$N(r) \geq \pi(r - \sqrt{2})^2 = \pi r^2 - 2\pi\sqrt{2}r + 2\pi. \tag{3}$$

Hence combining (2) and (3) we get

$$\begin{aligned}
\pi r^2 - 2\pi\sqrt{2}r + 2\pi \leq N(r) &\leq \pi r^2 + 2\pi\sqrt{2}r + 2\pi \\
-2\pi\sqrt{2}r \leq N(r) - \pi r^2 - 2\pi &\leq 2\pi\sqrt{2}r \\
|N(r) - \pi r^2 - 2\pi| &\leq 2\pi\sqrt{2}r \\
|N(r) - \pi r^2| &\leq 2\pi\sqrt{2}r + 2\pi.
\end{aligned}$$

Substituting a rearranged version of equation (1) we get

$$|E(r)| \leq 2\pi\sqrt{2}r + 2\pi.$$

Therefore we have obtained an upper bound for the error term. We can then use the big O notation which ignores constants to get

$$|E(r)| \leq O(r)$$

and so we finally get

$$N(r) = \pi r^2 + O(r). \tag{4}$$

Since Gauss proved equation (4) in circa 1800, the bounds have been improved. Writing $|E(r)| \leq Cr^\theta$, the best bounds on θ are currently

$$\frac{1}{2} < \theta \leq \frac{131}{208} \approx 0.62981,$$

with the lower limit proved independently by Godfrey Harold Hardy and Edmund Landau in 1915 and the upper limit by Edmund Landau in 2003 [1].

References

[1] Hardy, G. H. [1991], *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*, AMS Chelsea Pub.

[2] Takloo-Bighash, R. [2018], *A Pythagorean Introduction to Number Theory: Right Triangles, Sums of Squares, and Arithmetic*, Springer.

[3] Wikipedia [no date], 'Gauss circle problem', Available at: `https://en.wikipedia.org/wiki/Gauss_circle_problem`. (Last accessed on 08 November 2020).