Long term behaviour of the Elo scores of time-homogeneous players

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Abstract

In this project, we will investigate the Elo rating system. We will look at how accurately the rating system rates players and the time taken for the ratings to converge to a player's true strength. We look to build on the work that David Aldous has published in his papers on the subject. In our simulations, we see errors between the players' ratings and their strengths even when a considerable amount of games have been played. We look to quantify these errors and we find out that the ratings are consistently off, although by a dilation. The significance of the errors being a dilation is then considered. Although the Elo ratings do not perfectly reflect the players' strengths, they still can be used to extremely accurately calculate the win probabilities, which is one ultimate purpose of the Elo rating system. Another of the primary uses is to rank a set of players based on their ability. We show that if there is a dilation, then the ranking based on rating should still, for the most part, reflect the ranking based on strengths. As well as making general conclusions about the Elo rating system, in this report we focus on the case when our functions are from the logistic function family. These are the predominantly used functions in this rating system and we make specific conclusions related to changing particular constants in these functions.

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1 Introduction

1.1 Motivations

One of the purposes of the Elo scores is to represent the true ability of a player. The Elo ratings can then be used to compare two players and determine the probability of each player winning the game. First we look at the standard simplified version of how the ratings are updated, which is often found online when researching the subject.

1.2 Description of Elo scores

Consider a set of players $\mathscr{P}$, where $\mathscr{P} = \{1, ..., N\}$. Let $A, B \in \mathscr{P}$ and y_A and y_B be the rating/score of players A and B respectively. Consider a game where there are only two outcomes: either Player A wins or Player B wins. The set of possible game scores for Player A is $S_A = \{0, 1\}$ whereas for Player B their game score is $S_B = 1 - S_A$.

The expected win probabilities for players A and B are claimed to be

$$E_A = \frac{1}{1 + 10^{(y_B - y_A)/400}}, \qquad E_B = \frac{1}{1 + 10^{(y_A - y_B)/400}},$$

respectively. Due to the nature of the game, it follows that $E_A + E_B = 1$.

After a game has taken place, each player's rating gets updated, the formulas for updating Player A's and B's rating are

$$y'_A := y_A + K(S_A - E_A); \qquad y'_B := y_B + K(S_B - E_B),$$

respectively, where y'_A and y'_B are the updated ratings, y_A and y_B are the old ratings, K is a constant called K-factor, S_A and S_B are the game score and E_A and E_B are the expected win probabilities [11].

The Elo rating assumes that a player's performance is a normally distributed random variable (bell-shaped). Each curve is identical for each player apart from where the centre of the curve is. The centre of the curve is the average performance level for a player and is also the individual's Elo rating. The Elo rating of a player goes up or down depending on the result of a game. The size of the change of rating after a game depends upon the difference between the two players' Elo ratings before the game took place.

1.3 Claims of how Elo scores work

In the long run, Player A's rating, y_A, and Player B's rating, y_B, should reflect their true strength, x_A and x_B respectively, up to translation of the whole set of scores. It is often claimed that it takes approximately 30 games for this to occur.

One of the Elo rating system's main positive features is that the updates can be implemented immediately after a game. The reason for this is only the previous rating and the game score are needed to calculate the new rating.

1.4 History

1.4.1 Arpad Elo

Arpad Elo was born in Egyházaskesző, Hungary on 25^{th} August 1903 [7]. In 1913, he and his family emigrated to the United States where he then completed a Physics degree at the University of Chicago. After completing his degree, he taught various courses in Physics and Astronomy at university level at Marquette University and the University of Wisconsin.

Figure 1: Arpad Elo

At the age of 10, after he moved to the USA, Arpad Elo learnt chess by reading the Encyclopedia Britannia in his high school library. He was the Wisconsin State

Champion eight times between the ages of 32 and 58, won over 40 tournaments and drew twice with one of the best chess players in US history, Rueben Fine [13].

During the early 1930s, Arpad Elo helped increase chess interest across America by running a programme offering evening classes in chess at park social centres in many cities [9]. He became President of the American Chess Federation in 1935. He was one of the United States Chess Federation's founders in 1939, a merger between the American Chess Federation and the National Chess Federation of the United States (USCF).

In 1959, Arpad Elo was asked by the USCF to improve the chess rating system used in the US chess community. Previously the USCF was primarily using the Harkness Rating to rate chess players.

Elo designed a rating system that was more objective and mathematical. The system that he designed, named the Elo rating system, was adopted by the USCF in 1960 and then ten years later by FIDE - the international chess competition's governing body. Although the rating system was more advanced than the Harkness Rating, it was still simple enough to do the calculations with pencil and paper which Elo did himself until the mid-1980s [12].

1.4.2 Similar methods and alternatives

The Harkness rating was created by Kenneth Harkness who was a Scottish chess organiser. The system is relatively simple, when a player competes in a tournament, the average rating of their competition is calculated. If the player scores 50% they receive the average competition rating as their performance rating. If they score more than 50% their new rating is the competition average plus 10 points for each percentage point above 50. If they score less than 50% their new rating is the competition average minus 10 points for each percentage point below 50 [6].

However, the Harkness rating was not very accurate and had serious problems. For example, players could gain points despite losing every game in a tournament and lose points after winning them all [5]. This is why the USCF asked Arpad Elo to create a new rating system.

1.4.3 Original use of Elo scores in chess

Table 1 shows how the FIDE used an interval scale to classify players based on their Elo rating. The numbers assigned to any given level of chess proficiency remain entirely arbitrary. Both the class subdivision into 200 points and the choice of 2000 as the reference point were already steeped in tradition by the time Arpad Elo came up with his rating system. So although other numbers could have been used, he kept these features due to their general acceptance by players [4].

Table 1: Elo rating scale categories

Elo Rating	**Chess Proficiencies**
>2599	World Championship Contenders
2400-2599	Most Grandmasters Most International Masters
2200-2399	Most National Masters
2000-2199	Candidate Masters, Experts
1800-1999	Amateurs - Class A Category 1
1600-1799	Amateurs - Class B Category 2
1400-1599	Amateurs - Class C Category 3
1200-1399	Amateurs - Class D Category 4
<1200	Novices

1.4.4 Spread of Elo scores

Nowadays, the Elo rating system is not just used for chess. The rating system has been adapted for many modern-day applications. Several online video games use a variation of the Elo rating system to assess their players' quality such as Counter-Strike: Global Offensive [3]. These ratings are then used for matchmaking so that players of similar ability play each other to try to make their games as competitive as possible.

Until recently, Elo ratings were a part of the dating application Tinder's algorithm, which assessed how others interacted with someone's profile. It assigned a numerical value based on the number of likes and nopes a person received which was then used to show someone their potential matches based on similarities in the way others would engage with profiles [10].

1.5 Literature review

One of the existing pieces of literature available about Elo scores is 'Elo Ratings and the Sports Model: A Neglected Topic in Applied Probability?' by David Aldous [1]. In this paper, one of the questions Aldous raises is: after infinite games how close is a player's rating to their true strength? In real-world practical terms, we do not know a player's true strength, however, through simulations we can get an insight into the quality of this limit. As, in the real-world, we cannot measure the difference between ratings and strength, we can instead look at the errors in predicted win probabilities. We can view the prediction error by comparing the implied win-probability function with the model's true win probability.

Another question the paper raises is: how well can Elo rating models incorporate players' changing strengths? The original Elo rating system assumed that a player's true strength was a fixed number. However, it is improbable that a player's true strength will remain constant in practical terms. For changing strengths to be implemented all strength estimates would need to be updated after every single match. This would counteract one of the main positives of the Elo rating system - that it is quick and easy to use. In this report we have assumed that a player's strength is fixed.

David Aldous also questions the claim that ratings tend to converge to a team's true strength relative to its competitors after about 30 matches. This claim is often stated online when viewing Elo related subjects, however, there seems to be minimal foundation to this claim, with the origin unknown. One theory is that this claim may have arisen from traditional Statistics textbooks asserting that sample size 30 suffices for Normal approximation.

1.6 This project

In this project, we will explore what is the limit of the Elo scores process. We will investigate what the limit looks like, if it is any good and how quickly and accurately it can be used to measure a player's strength. Then we will quantify the errors in measuring strengths and determining win probabilities and see what contributes to these errors.

2 Setting, theoretical framework and main result

2.1 Assumptions

Before introducing the Simple Game Model, we will make some assumptions about the game and the players:

- The set of players, $\mathscr{P} = \{1, ..., N\}$, is fixed - no players will arrive or depart. In this project, we will use teams and players interchangeably.

- For $A, B \in \mathscr{P}$, either Player A wins and Player B loses, or vice versa. There are no draws. The game score for Player A, S_A, is 1 if they win and 0 if they lose, whereas for Player B their game score is $S_B = 1 - S_A$.
- The game is symmetric. There is no difference between saying Player A plays Player B and Player B plays Player A. In other words, neither player has any advantage such as home-field advantage in sport or first go in board games.
- The strengths of the players do not change with time.

2.2 The Simple Game Model

The Simple Game Model is used to model the outcome of a game and is used to calculate the probability of a player winning a game. When Player A with strength, x_A, competes against Player B with strength, x_B, we have

$$\mathbb{P}(A \text{ beats } B) = W(x_A - x_B),$$

where W is a win probability function which satisfies the following conditions:

$$\begin{aligned} &W : \mathbb{R} \to (0,1) \text{ is continuous, strictly increasing,} \\ &\lim_{x \to \infty} W(x) = 1, \\ &W(-x) + W(x) = 1, \quad \forall x \in \mathbb{R}. \end{aligned} \tag{1}$$

The first condition in (1) is so that the bigger the difference in players' strengths, the higher the probability of the player with the larger strength winning. The second condition means that if a player's strength is infinitely better than whom they are playing they will definitely win. The final condition in (1) means that the probability of Player A beating Player B and the probability of Player B beating Player A must equal 1 as they are the only two possibilities (since there are no draws).

It is clear to see that the only parameters that the Simple Game Model uses are W, the win probability function, and x, the player's strength. One example for W that satisfies the above conditions is the logistic function,

$$L(u) = \frac{1}{1 + \exp(-u)}, \quad \forall u \in \mathbb{R}.$$

In this report we will just be using variations of the logistic function in our simulations. The logistic function is regarded as the standard win probability function.

Next we will look at where we centre the players' strengths, however the choice is somewhat arbitrary due to the following remark.

Remark. If $x' = x + k$, for $k \in \mathbb{R}$, i.e. $\forall A \in \mathscr{P}$, $x'_A = x_A + k$, we have the same $\mathbb{P}(A \text{ beats } B)$. This means the strengths are invariant by translation and so only the differences in strengths matters.

In this project we will centre them at 0, consequently we have

$$\sum_{A \in \mathscr{P}} x_A = 0.$$

Claim 1. When $\sum x_A = 0$, for $x \in \mathbb{R}^n$, then knowing the strengths of all players is equivalent to knowing the differences between all player's strengths.

Proof. To prove this claim first we define the function f as follows:

$f : \mathbb{R}^n \to \mathbb{R}^n \; x \mapsto (x_2 - x_1, ..., x_N - x_{N-1}, \sum x_A)$.

If the mapping defined by f is an isomorphism then the claim holds. To prove it is isomorphic we have to show that f is linear and that f is a bijection.

First, we will prove f is linear: To prove this we need to check that it is compatible with addition and scalar multiplication.

Take any $(x_1, x_2, ..., x_N), (x'_1, x'_2, ..., x'_N) \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} f((x_1, x_2, ..., x_N) + (x'_1, x'_2, ..., x'_N)) &= f(x_1 + x'_1, x_2 + x'_2, ..., x_N + x'_N) \\ &= ((x_2 + x'_2) - (x_1 + x'_1), ..., (x_N + x'_N) \\ &\quad - (x_{N-1} + x'_{N-1}), \sum x_A + x'_A) \\ &= (x_2 - x_1) + (x'_2 - x'_1), ..., \\ &\quad + (x_N - x_{N-1}) + (x'_N - x'_{N-1}), \\ &\quad \sum x_A + \sum x'_A) \\ &= (x_2 - x_1, ..., x_N - x_{N-1}, \sum x_A) \\ &\quad + (x'_2 - x'_1, ..., x'_N - x'_{N-1}, \sum x'_A) \\ &= f(x_1, x_2, ..., x_N) + f(x'_1, x'_2, ..., x'_N). \end{aligned}$$

We have proved that f is closed under addition, next we will check to see if it is closed under scalar multiplication. Hence we have

$$\begin{aligned}
f(c((x_1, x_2, ..., x_N)) &= f(cx_1, cx_2, ..., cx_N)) \\
&= (cx_2 - cx_1, ..., cx_N - cx_{N-1}, \sum cx_A) \\
&= (c(x_2 - x_1), ..., c(x_N - x_{N-1}), c\sum x_A) \\
&= c((x_2 - x_1), ..., (x_N - x_{N-1}), \sum x_A)) \\
&= cf(x_1, x_2, ..., x_N, \sum x_A).
\end{aligned}$$

Therefore we have proved that f is closed under scalar multiplication. As we have previously proved f is closed under addition, we have now proved that f is a linear mapping.

Now we will prove f is bijective. To help us prove this we recall the following theorem:

Theorem 2.1. *If f is linear and the dimension of the domain is equal to the dimension of the codomain then f is bijective if and only if f is injective.*

We have just previously proved that f is linear so the first assumption holds. The second assumption also holds as the dimension of the domain and the dimension of the codomain are both equal to N. This means that f is bijective if and only if it is also injective.

To prove that f is injective we must prove that the kernel of f is equal to $\{0\}$.

Hence we need to prove that if $f(x_1, x_2, ..., x_N) = 0$ then $(x_1, x_2, ..., x_N) = 0$.

If $f(x_1, x_2, ..., x_N) = 0$ then this means $(x_2 - x_1), ..., (x_N - x_{N-1}), \sum x_A = 0$. Using the first $N-1$ equations we see that $x_1 = x_2, x_2 = x_3, ..., x_{N-1} = x_N$, thus $x_A = x_B$ for all $A, B \in \mathscr{P}$. The final equation tells us that $\sum x_A = 0$, as all the x values are equal to each other the only way this is possible is if $x_A = 0, \quad \forall A \in \mathscr{P}$. Thus we have proved that if $f(x) = 0$ then $x = 0$. This is equivalent to saying that $\ker(f) \subset 0$, which means f is injective.

As we have proved f is injective, this means that it is also bijective from the previously stated theorem. As f is bijective and linear this means the mapping is an isomorphism and hence we have proved the claim.

$\square$

2.3 Elo-type ratings

We have introduced the Simple Game Model in the previous subsection, now we need to show how a player's score is updated. Each Player A is given some initial rating, a real number, y_A. We will use the terms ratings and scores interchangeably throughout the report. When Player A plays Player B, the ratings of each player are updated after the game using an update function U.

$$\begin{aligned} &\text{If Player } A \text{ beats Player } B \text{ then:} \\ &Y_A := y_A + U(y_A - y_B); \qquad Y_B := y_B - U(y_A - y_B). \\ &\text{If player } A \text{ loses to Player } B \text{ then:} \\ &Y_A := y_A - U(y_B - y_A); \qquad Y_B := y_B + U(y_B - y_A). \end{aligned} \tag{2}$$

Here, Y_A and Y_B are the updated ratings of Player A and B respectively. Note that the sum of the ratings remains constant. We also require the function U to satisfy the qualitative conditions:

$$\begin{aligned} &U : \mathbb{R} \to (0, \infty) \text{ is } C^1, \text{ strictly decreasing,} \\ &\lim_{u \to \infty} U(u) = 0. \end{aligned} \tag{3}$$

The purpose of condition (3) is so the transfer of points is in line with the result's significance. For example, if Player A has an initial Elo score much bigger than Player B, then if we assume that ratings are a reasonable estimation of strength, it is expected that Player A will win the game. Because of this, if Player A wins, the transfer of score from Player B to Player A will be small as the result was expected. However, in this situation, if Player B beats Player A it could be for two main reasons. Firstly, it could be that the initial Elo ratings were not accurate and so this condition will mean that there will be a large transfer of points helping to fix this problem. Whereas, the ratings may have been correct and Player B may have fairly beaten Player A: not always does the better player win, due to luck and other factors.

We will also impose another condition on U

$$k_U := \sup_u |U'(u)| < 1. \tag{4}$$

Claim 2. If condition (4) holds then for any $y \in \mathbb{R}$ the functions

$$f : x \mapsto x + U(x - y); \qquad g : x \mapsto x - U(y - x), \tag{5}$$

are increasing.

That is, the rating updates when a player with (variable) strength x plays a player of fixed strength y is an increasing function of the starting strength x.

Proof. As U is a C^1 function this means that the functions f and g are differentiable.

Differentiating the function f first we get:

$$f'(x) = 1 + U'(x - y) \quad \forall x \in \mathbb{R}.$$

From (4) as $\sup_u |U'(u)| < 1$ we have

$$\begin{aligned} f'(x) &> 1 + (-1) \\ f'(x) &> 0. \end{aligned}$$

Hence as $f'(x) > 0 \quad \forall x \in \mathbb{R}$, f is an increasing function. Likewise differentiating function g we get:

$$g'(x) = 1 + U'(x - y) \quad \forall x \in \mathbb{R}.$$

From (4) as $\sup_u |U'(u)| < 1$ we have

$$\begin{aligned} g'(x) &> 1 - (1) \\ g'(x) &> 0. \end{aligned}$$

Thus as $g'(x) > 0$ for all $x \in \mathbb{R}$, g is an increasing functions.

Therefore if (4) holds both f and g are increasing functions. $\square$

Like with the players' strengths, as the ratings updates function, $U(y_A - y_B)$, only use the differences of ratings and not the actual ratings the choice of where to centre the Elo ratings is arbitrary. In claim (1), we mentioned that knowing the strengths of all players was equivalent to knowing the differences between all player's strengths. This is the same with the ratings: knowing the ratings of all the players is equivalent to knowing the differences between all players' ratings. We will centre the ratings at 0, hence we have

$$\sum_{A \in \mathscr{P}} y_A = 0.$$

Additionally, before any players have played any games, we will assign all the players an initial Elo rating of 0. This proposes the following question which we will investigate later.

Question. Is it better to start all players' initial ratings at 0 or to give a spread of ratings for all players?

2.4 Schedule of matches

We need to specify how the matches are scheduled to make sense of what happens after numerous games have been played.

Given a match schedule (where we specify what matches occur between t and $t+1$), the Simple Game model for match results, and using the Elo rating system as seen in (2) we get the process

$$Y(t) = (Y_A(t))_{A \in \mathscr{P}},$$

where $Y_A(t)$ is the rating of Player A after a total of t matches has been played (called the scores process).

At $t = 0$, when no matches have been played, we set every players' Elo rating to 0 so at $t = 0$ we have

$$y_A(0) = 0, \quad \forall A \in \mathscr{P}.$$

As the outcome of matches is random, for $t \geq 1$ the scores that are obtained are also random. This means that $Y(t)$ for $t \geq 1$ are random variables. Hence, Y is a random process.

The following are three examples of how games are scheduled in the real-world:

- League format such as the English Premier League in football.
- Single-elimination tournaments such as a tennis Grand Slam tournament.
- No central scheduling such as online games.

The league format is typically used in team games, matches where there is no central scheduling usually involves individual players and single elimination tournaments can be either. For a league format, there are n teams where each pair of teams plays exactly once during a season (or twice such as in football). Whereas in single-elimination tournaments, when players lose a game they are out of the tournament and consequently play no further games.

A league format guarantees that every team will play all of the other teams of differing strength in a season. If we are assuming that the more games a team plays the more likely it is that the Elo scores will be closer to the team's true strength, then for the sake of accuracy this schedule is preferred to a tournament. As in a single-elimination tournament, half the set of teams will only play one match, meaning that it is unlikely that their Elo score will reflect their true strength after their loss.

Mathematically, if we want the ratings to reflect the players' strengths correctly, we would want every pair to play infinite times. It is natural to consider the following two schedules: random matching schedule and round robin schedule.

The random matching process has some property called continuous-state Markov chain. Having this property is useful as it allows theorems related to Elo scores and the Simple Game Model to be proven (beyond this project's scope). Between t and $t+1$, we say that either one pair of players chosen at random play a game (random matching schedule), or all possible pairs of players play each other (round robin schedule). Out of the three real-world examples we named earlier, random matching associates most closely with no central scheduling. The main difference between these two is that when there is no central scheduling, games are often organised so that players of a similar ability play each other, especially in the online competitive gaming scene. Whereas in random matching the choice of opponent for a player is entirely random.

2.5 Link between the Simple Game Model and the Elo scores

The Elo scores rating system is not directly connected to probability. The scores process only uses player ratings, $Y(t)$ and the function U but does not use the parameters (W, x) of the Simple Game Model. Despite this, there is a heuristic connection between the Simple Game Model and the rating algorithm which we will now describe.

Consider Player A, with initial rating y_A and rating after the game of Y_A plays against Player B, with initial rating y_B and rating after the game of Y_B. Then the expectation of rating change for Player A and Player B respectively are

$$\begin{aligned} \mathbb{E}[Y_A - y_A] &= U(y_A - y_B)W(x_A - x_B) - U(y_B - y_A)W(x_B - x_A), \\ \mathbb{E}[Y_B - y_B] &= U(y_B - y_A)W(x_B - x_A) - U(y_A - y_B)W(x_A - x_B). \end{aligned} \tag{6}$$

Proof. For Player A we know if they beat Player B their score is increased by $U(y_A - y_B)$ and if they lose their score is decreased by $U(y_A - y_B)$. The probability that Player A wins is $W(x_A - x_B)$ and the probability that they lose is $W(x_B - x_A)$. Combining these bits of information together we get

$$\mathbb{E}[Y_A - y_A] = U(y_A - y_B)W(x_A - x_B) - U(y_B - y_A)W(x_B - x_A).$$

Same proof for Player B. □

Now, consider the case where the functions U and W are related by

$$\frac{U(u)}{U(-u)} = \frac{W(-u)}{W(u)}, \quad \forall u \in \mathbb{R}. \tag{7}$$

David Aldous calls (7) the balance relation.

Consider a match where Player A plays Player B. Assume that the balance relation (7) holds. Then we get the following claim:

Claim 3. (a) If $(y_A - y_B) = (x_A - x_B)$ then $\mathbb{E}[Y_A - Y_B] = (y_A - y_B)$.

(b) If $(y_A - y_B) > (x_A - x_B)$ then $\mathbb{E}[Y_A - Y_B] < (y_A - y_B)$.

(c) If $(x_A - x_B) > (y_A - y_B)$ then $\mathbb{E}[Y_A - Y_B] > (y_A - y_B)$.

Proof. (a) As $(y_A - y_B) = (x_A - x_B)$ then let $u = (y_A - y_B) = (x_A - x_B)$ and $-u = (y_B - y_A) = (x_B - x_u)$. From Equation (7), we have

$$\begin{aligned} \frac{U(u)}{U(-u)} &= \frac{W(-u)}{W(u)} \\ U(u)W(u) &= W(-u)U(-u). \end{aligned}$$

Then for Player A we have

$$\begin{aligned} \mathbb{E}[Y_A - y_A] &= U(y_A - y_B)W(x_A - x_B) - U(y_B - y_A)W(x_B - x_A) \\ &= U(u)W(u) - U(-u)W(-u) \\ &= 0. \end{aligned} \tag{$\sim$}$$

And then for Player B we have

$$\begin{aligned} \mathbb{E}[Y_B - y_B] &= U(y_B - y_A)W(x_B - x_A) - U(y_A - y_B)W(x_A - x_B) \\ &= U(-u)W(-u) - U(u)W(u) \\ &= 0. \end{aligned} \tag{$\square$}$$

Hence

$$\begin{aligned} \mathbb{E}[Y_A - Y_B] - (y_A - y_B) &= \mathbb{E}[Y_A - Y_B] - \mathbb{E}[y_A - y_B] \\ &= \mathbb{E}[Y_A - y_a] - \mathbb{E}[Y_B - y_B] \\ &= 0 \quad \text{combining } (\sim) \text{ and } (\square). \end{aligned}$$

Therefore $\mathbb{E}[Y_A - Y_B] = (y_A - y_B)$.

(b) We have that $(y_A - y_B) > (x_A - x_B)$, it follows that
$(y_B - y_A) < (x_B - x_A)$.
As U is a strictly decreasing function and we have that

$$\begin{aligned}\frac{U(y_A - y_B)}{U(y_B - y_A)} &< \frac{U(x_A - x_B)}{U(x_B - x_A)}\\ &= \frac{W(x_A - x_B)}{W(x_B - x_A)}.\end{aligned}$$

Hence we have

$$U(y_A - y_B)W(y_B - y_A) < U(x_B - x_A)W(x_A - x_B). \tag{$\star$}$$

Then for Player A we have

$$\begin{aligned}\mathbb{E}[Y_A - y_A] &= U(y_A - y_B)W(x_A - x_B) - U(y_B - y_A)W(x_B - x_A)\\ &< 0 \quad \text{(from } (\star)).\end{aligned} \tag{$\triangle$}$$

And then for Player B we have

$$\begin{aligned}\mathbb{E}[Y_B - y_B] &= U(y_B - y_A)W(x_B - x_A) - U(y_A - y_B)W(x_A - x_B)\\ &> 0 \quad \text{(from } (\star)).\end{aligned} \tag{$\triangledown$}$$

Thus

$$\begin{aligned}\mathbb{E}[Y_A - Y_B] - (y_A - y_B) &= \mathbb{E}[Y_A - Y_B] - \mathbb{E}[y_A - y_B]\\ &= \mathbb{E}[Y_A - y_a] - \mathbb{E}[Y_B - y_B]\\ &< 0 \quad \text{combining } (\triangle) \text{ and } (\triangledown).\end{aligned}$$

Therefore $\mathbb{E}[Y_A - Y_B] < (y_A - y_B)$.

(c) As $(x_A - x_B) > (y_A - y_B)$, it follows that
$(x_B - x_A) < (y_B - y_A)$.
Because U is a strictly decreasing function we have that

$$\begin{aligned}\frac{U(y_A - y_B)}{U(y_B - y_A)} &> \frac{U(x_A - x_B)}{U(x_B - x_A)}\\ &= \frac{W(x_A - x_B)}{W(x_B - x_A)}.\end{aligned}$$

Therefore

$$U(y_A - y_B)W(y_B - y_A) > U(x_B - x_A)W(x_A - x_B). \qquad (\dagger)$$

Then for Player A we have

$$\begin{aligned}\mathbb{E}[Y_A - y_A] &= U(y_A - y_B)W(x_A - x_B) - U(y_B - y_A)W(x_B - x_A) \\ &> 0 \quad \text{(from } (\dagger)). \end{aligned} \qquad (\triangleright)$$

And then for Player B we have

$$\begin{aligned}\mathbb{E}[Y_B - y_B] &= U(y_B - y_A)W(x_B - x_A) - U(y_A - y_B)W(x_A - x_B) \\ &< 0 \quad \text{(from } (\dagger)). \end{aligned} \qquad (\triangleleft)$$

Hence

$$\begin{aligned}\mathbb{E}[Y_A - Y_B] - (y_A - y_B) &= \mathbb{E}[Y_A - Y_B] - \mathbb{E}[y_A - y_B] \\ &= \mathbb{E}[Y_A - y_a] - \mathbb{E}[Y_B - y_B] \\ &> 0 \quad \text{combining } (\triangleright) \text{ and } (\triangleleft).\end{aligned}$$

Therefore $\mathbb{E}[Y_A - Y_B] > (y_A - y_B)$.

□

The previous observations suggest that if the scores differences are greater than the strengths difference then the updates process will in expectation reduce this difference. However, the changes in scores needed for the scores difference to equal the strengths difference is unknown. Due to this, the updates process may over or underestimate the changes required.

Ideally, due to the previous claim we would hope that, for any $A, B \in \mathscr{P}$, $(Y_A - Y_B)$ will converge in some way to $(x_A - x_B)$. Due to the previous claims and remarks, we would also expect Y_A to converge in some way to x_A, $\forall A \in \mathscr{P}$.

If we know W or U, we can find U or W that satisfy (7).

Lemma 2.2. *(a) Given U satisfying (3) and (4) the unique solution W to (1) and (7) is*

$$W(u) := \frac{U(-u)}{U(u) + U(-u)}. \qquad (8)$$

(b) Given W satisfying (1):

(i) If U satisfies (3), (4) and (7) then U is of the form

$$U(u) = W(-u)S(u), \tag{9}$$

for some symmetric function S, (i.e. $S(u) = S(-u) \quad \forall u \in \mathbb{R}$).

(ii) If U is given by $U(u) = W(-u)S(u)$, for some strictly positive, symmetric function S then

(–) (7) holds.

(–) (3) holds if S is also of class C^1 and $\frac{S'(u)}{S(u)} < \frac{W'(-u)}{W(-u)}$.

(–) (4) holds if S is also of class C^1 for some $\varepsilon \in (0,1)$

$$-\varepsilon < W(-u)S'(u) - S(u)W'(-u) < 0. \tag{10}$$

Proof. (a) First, assume that $W(u)$ satisfies (1) and (7). Recall the balance relation

$$\frac{U(u)}{U(-u)} = \frac{W(-u)}{W(u)}.$$

From (1) we have that $W(-u) = 1 - W(u)$, hence

$$\begin{aligned}
\frac{U(u)}{U(-u)} &= \frac{1 - W(u)}{W(u)} \\
\frac{U(u)}{U(-u)} &= \frac{1}{W(u)} - 1 \\
\frac{U(u)}{U(-u)} + 1 &= \frac{1}{W(u)} \\
\frac{U(u)}{U(-u)} + \frac{U(-u)}{U(-u)} &= \frac{1}{W(u)} \\
\frac{U(u) + U(-u)}{U(-u)} &= \frac{1}{W(u)} \\
\frac{U(-u)}{U(u) + U(-u)} &= W(u).
\end{aligned}$$

Therefore we have proved, given that (1) and (7) hold, then $W(u) = \frac{U(-u)}{U(u)+U(-u)}$. Now we need to prove that given $W(u) = \frac{U(-u)}{U(u)+U(-u)}$, W satisfies (1) and (7). First we will take that $W(u) = \frac{U(-u)}{U(u)+U(-u)}$ then prove that (1) holds and after that prove that (7) holds.

One of the criteria of (1) is that $W(-u) + W(u) = 1$.

$$\begin{aligned} W(-u) &= \frac{U(-(-u))}{U(-u) + U(-(-u))} \\ &= \frac{U(u)}{U(-u) + U(u)}. \end{aligned}$$

Hence

$$\begin{aligned} W(-u) + W(u) &= \frac{U(u)}{U(-u) + U(u)} + \frac{U(-u)}{U(u) + U(-u)} \\ &= \frac{U(-u) + U(u)}{U(-u) + U(u)} \\ &= 1. \end{aligned}$$

Next, we will prove that $W(u)$ is a continuous, strictly increasing function.

From (3), we have $U : \mathbb{R} \to (0, \infty)$, it follows that $U(-u) + U(u) \neq 0$ for all $u \in \mathbb{R}$. Consequently as $W(u)$ is well defined and the numerator and denominator are continuous then W is also continuous.

Differentiating $W(u)$ we get

$$\begin{aligned} W'(u) &= \frac{-U'(-u)(U(u) + U(-u)) - U(-u)(U'(u) - U'(-u))}{(U(u) + U(-u))^2} \\ &= \frac{-U'(-u)U(u) - U'(-u)U(-u)) - U(-u)U'(u) + U(-u)U'(-u)}{(U(u) + U(-u))^2} \\ &= \frac{-U'(-u)U(u) - U(-u)U'(u)}{(U(u) + U(-u))^2} \\ &= -\left[\frac{(U'(-u)U(u) + U(-u)U'(u)}{(U(u) + U(-u))^2}\right]. \end{aligned}$$

From (3), $U(u) > 0$ for all $u \in \mathbb{R}$ and $U'(u) < 0$ for all $u \in \mathbb{R}$ as U is a strictly decreasing function. This means the numerator is negative and thus $W'(u) > 0$ for all $u \in \mathbb{R}$ and therefore W is a strictly increasing function.

The final part of (1) that we need to prove is that $\lim_{u\to\infty} W(u) = 1$. We know from (3) that $\lim_{u\to\infty} U(u) = 0$. Also from (3) we know U is valued in $(0, +\infty)$, hence

$$\lim_{u\to-\infty} U(u) = l \in (0, +\infty).$$

Next we can manipulate $W(u)$ to get

$$\begin{aligned}
W(u) &= \frac{U(-u)}{U(u)+U(-u)} \\
&= \frac{\frac{U(-u)}{U(-u)}}{\frac{U(u)}{U(-u)} + \frac{U(-u)}{U(-u)}} \\
&= \frac{1}{\frac{U(u)}{U(-u)} + 1}.
\end{aligned}$$

To work out the $\lim_{u\to\infty} W(u)$ we can take limits as u tends to infinity of each component of the fraction to get

$$\lim_{u\to\infty} W(u) = \frac{\lim_{u\to\infty} 1}{\lim_{u\to\infty} \frac{U(u)}{U(-u)} + \lim_{u\to\infty} 1}.$$

Clearly $\lim_{u\to\infty} 1 = 1$. Also

$$\begin{aligned}
\lim_{u\to\infty} \frac{U(u)}{U(-u)} &= \frac{\lim_{u\to\infty} U(u)}{\lim_{u\to\infty} U(-u)} \\
&= \frac{0}{l} \\
&= 0.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\lim_{u\to\infty} W(u) &= \frac{\lim_{u\to\infty} 1}{\lim_{u\to\infty} \frac{U(u)}{U(-u)} + \lim_{u\to\infty} 1} \\
&= \frac{1}{0+1} \\
&= 1.
\end{aligned}$$

This means we have proved that (2.2a) implies (1).

Now we will prove (2.2a) also implies that (7) holds.

$$\begin{aligned}
W(u) &= \frac{U(-u)}{U(u)+U(-u)} \\
\frac{1}{W(u)} &= \frac{U(u)+U(-u)}{U(-u)} \\
\frac{1}{W(u)} &= \frac{U(u)+U(-u)}{U(-u)} \\
\frac{1}{W(u)} &= \frac{U(u)}{U(-u)} + \frac{U(-u)}{U(-u)} \\
\frac{1}{W(u)} &= \frac{U(u)}{U(-u)} + 1 \\
\frac{1}{W(u)} &= \frac{U(u)}{U(-u)} + 1 \\
\frac{1}{W(u)} - 1 &= \frac{U(u)}{U(-u)} \\
\frac{1}{W(u)} - \frac{W(u)}{W(u)} &= \frac{W(u)}{W(u)}\frac{U(u)}{U(-u)} \\
\frac{1-W(u)}{W(u)} &= \frac{U(u)}{U(-u)}.
\end{aligned}$$

From earlier in the proof we know that $1 - W(u) = W(-u)$, thus

$$\frac{W(-u)}{W(u)} = \frac{U(u)}{U(-u)},$$

and so if (2.2a) holds then (7) also holds.

(b) (i) First we will prove that if (3), (4) and (7) hold then U is of the form described by (2.2b). Let S be defined by

$$S(u) = \frac{U(u)}{W(-u)}.$$

Then, by construction, we have $U(u) = S(u)W(-u)$. From (7) we have

$$\begin{aligned}
\frac{U(u)}{U(-u)} &= \frac{W(-u)}{W(u)} \\
\frac{U(u)}{W(-u)} &= \frac{U(-u)}{W(u)}.
\end{aligned}$$

Hence

$$\begin{aligned} S(u) &= U(u)W(-u) \\ &= U(-u)W(u) \\ &= U(-u)W-(-u) \\ &= S(-u), \end{aligned}$$

and so $S(u) = S(-u)$.

And so we have that if (3), (4) and (7) hold then (2.2b) holds.

(ii) Next we will prove that if $U(u) = W(-u)S(u)$ where $S(u)$ is a symmetric, strictly positive function then (3), (4) and (7) hold.

(–) Firstly we will check that (7) holds.
We can rearrange $U(u) = W(-u)S(u)$ to get

$$S(u) = \frac{U(u)}{W(-u)},$$

and so

$$\begin{aligned} S(-u) &= \frac{U(-u)}{W(-(-u))} \\ &= \frac{U(-u)}{W(u)}. \end{aligned}$$

Therefore as $S(u) = S(-u)$,

$$\frac{U(u)}{W(-u)} = \frac{U(-u)}{W(u)},$$

which rearranges to

$$\frac{U(u)}{U(-u)} = \frac{W(-u)}{W(u)},$$

and so (7) holds.

(–) Next we will check that (3) holds. Recall we make additional assumptions on S that it is also of class C^1 and $\frac{S'(u)}{S(u)} < \frac{W'(-u)}{W(-u)}$. Firstly W is continuous and we have defined S so that it is also continuous, therefore U is continuous. To see if U is a decreasing function we differentiate it (which we can as U is C^1) and obtain

$$U'(u) = W(-u)S'(u) - S(u)W'(-u).$$

For U to be a strictly decreasing function we need $U'(u) < 0$ therefore

$$\begin{aligned} 0 &> W(-u)S'(u) - S(u)W'(-u) \\ S(u)W'(-u) &> W(-u)S'(u). \end{aligned}$$

So provided that we define S so that that

$$\frac{S(u)}{S'(u)} > \frac{W(-u)}{W'(-u)},$$

which is equivalent to

$$\frac{S'(u)}{S(u)} < \frac{W'(-u)}{W(-u)},$$

then (3) holds.

(–) Finally we will check that (4) holds. Recall we make additional assumptions on S that it is of class C^1 for some $\varepsilon \in (0, 1)$. We note that

$$|U'(u)| = |W(-u)S'(u) - S(u)W'(-u)|.$$

For (4) to hold, we require

$$k_U := \sup_u |W(-u)S'(u) - S(u)W'(-u)| < 1.$$

Hence there exists some $\varepsilon \in (-1, 0)$ so that

$$-\varepsilon < W(-u)S'(u) - S(u)W'(-u)| < 0,$$

thus (4) holds with some assumptions including (2.2b) for some ε. So we have proved that if U is defined as in (2.2b) then (3), (4) and (7) hold provided we stipulate additional requirements on S.

□

2.6 Convergence of Elo scores

One of the main points of interest in this report is to investigate the convergence of Elo scores. David Aldous proposes the following theorem:

Theorem 2.3. *Under our standing assumptions (1, 3, 4) on W and U, for each x and for any initial ratings $y(0)$ we have $Y(t) \xrightarrow{d} Y(\infty)$ as $t \to \infty$. In addition, the limit does not depend on $y(0)$.*

Note that this theorem does not make any other assumptions on U and W this includes not assuming that the balance relation (7) between functions W and U holds.

Before we delve into what the theorem means, we will define what it means for something to converge in distribution.

Definition 2.4. *A sequence of random variables $X_1, X_2, X_3...$ converges in distribution to a random variable X, shown by $X_n \xrightarrow{d} X$, if*

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

for all x at which $F_X(x)$ is continuous. F_n and F are the cumulative distribution functions of random variables X_n and X respectively.

We saw at the end of Section (2.6) that when the balance relation (7) holds there is a tendency for a player's rating to move towards their true strength. Due to this, it is no surprise that this part of the theorem holds when the balance relation occurs.

However, this theorem does not assume that the balance relation holds. It is quite surprising that when we do not specify a relation between U and W, and only use our initial standing assumptions (1, 3, 4), there is a limit. This theorem does not tell us about the quality of $Y(\infty)$ and if it is close or not to x (the players' true strength). Later in this project, we will investigate the effects of when the balance relation holds including how it affects the convergence in distribution.

2.7 Questions to investigate

In the following few sections, we will use programming techniques to investigate various aspects of the Elo scores update process and the win probability function.

In the next section, we will first show how we will set up the simulations in statistical programming software R. We will then analyse the simulation, looking at if convergence "has happened" and if so how good the limit is and how quickly it is reached.

In future simulations, we will then change a variable to see what effect this has, and see if the same conclusions we make in the first simulation still hold. We will answer some of the questions over the following few sections: What is the limit of the scores and how good is it? How quickly do the scores converge to their limits? Does the balance relation (7) holding affect the limit? Do the initial ratings affect the limit?

3 Simulation 1

3.1 Setup

3.1.1 Standard setup

The first step of setting up the Elo update process is to select a Simple Game Model. The two variables needed for the Simple Game Model are the win probability function, W, and the player's strengths, x.

But first, we need to specify the number of players taking part in the games. Recall from the assumptions we made in Subsection 2.1 that the set of players, $\mathscr{P} = \{1, ..., N\}$, is fixed. So no players will arrive or depart hence $|\mathscr{P}| = N$ will remain the same throughout a simulation.

For the win probability function, W, recall that the following conditions need to be met:

$$
\begin{aligned}
&W : \mathbb{R} \to (0, 1) \text{ is continuous, strictly increasing,}\\
&\lim_{x \to \infty} W(x) = 1,\\
&W(-x) + W(x) = 1, \quad \forall x \in \mathbb{R}.
\end{aligned}
$$

For the players' strengths, x, we will design them so that they are uniformly spread. In other words, we will specify a value h such that $|x_A - x_B| = h$ when $|A - B| = 1, \quad \forall A, B \in \mathscr{P}$. We have also mentioned earlier that we want $\sum_{A \in \mathscr{P}} x_A = 0$, hence we will make sure that the strengths are centred at 0 and so $x_A = h(A - \frac{N+1}{2})$.

The next step is to specify the items needed for the Elo-type ratings: the update function, U, and the initial ratings of the players, $y(0)$.

For the update function, U recall that the following conditions need to be met:

$$
\begin{aligned}
&U : \mathbb{R} \to (0, \infty) \text{ is } C^1, \text{ strictly decreasing.}\\
&\lim_{u \to \infty} U(u) = 0.\\
&k_U := \sup_u |U'(u)| < 1.
\end{aligned}
$$

For the initial scores of these players, we stated earlier in the report that $y_A(0) = 0, \quad \forall A \in \mathscr{P}$, hence all players' initial rating will be 0.

In our simulations we will be determining $(Y(t))_{t \in \mathbb{N}}$ where $Y(0) = 0$ and $Y(t + 1)$

is calculated using $Y(t)$, who plays who (the schedule), the outcomes of the games and the update function.

We also need to specify the simulations' schedule, i.e. what games occur between t and $t+1$. We will use the random matching schedule which we detailed earlier in Section 2.4. We will choose one pair of players uniformly at random to play between t and $t+1$, simulate the outcome and then update the two players' ratings accordingly. We will repeat this up to a time horizon $T \in \mathbb{N}^*$.

Later, as Theorem 2.3 talks about convergence in distribution, we will need to look at multiple samples to investigate this convergence. We will let $\mu \in \mathbb{N}^*$ be the number of samples.

3.1.2 Specifying parameters for the first simulation

For the first simulation, we first need to specify each part of the previous subsection's setup. Firstly, for the win probability function, we will set

$$W(u) = \frac{1}{1 + \exp(-\alpha_1 u)}, \quad \forall u \in \mathbb{R},$$

and the update function we will set

$$U(-u) = W(u), \quad \forall u \in \mathbb{R}.$$

We will specify the value of α once we have chosen the set of players. Here we have set up W and U in such a way that the balance relation (7) holds as we have $U(u) = cW(-u)$ with $c = 1$.

In this simulation, we will have $|\mathscr{P}| = 5$ and $h = 5$. This means there are five players where $|x_A - x_B| = 5$ when $|A - B| = 1, \quad \forall A, B \in \mathscr{P}$. We choose Player 1 to have the lowest strength of -10, Player 3 the middle strength of 0 and Player 5 the highest strength of 10. Each player starts with an initial score of 0 and so we have the following setup:

Table 2: Simulation 1 players' strengths and initial scores

Player ($A \in \mathscr{P}$)	**Strength** (x_A)	**Initial Score** ($y_A(0)$)
1	-10	0
2	-5	0
3	0	0
4	5	0
5	10	0

Now that we have set up the players playing in the games, we will now choose the value of α in $W(u)$. The value of alpha will determine the probabilities of winning a game for each player. The bigger the absolute value of alpha, the more skill-based rather than luck-based the game is. For example, a snooker game should have a higher value of alpha in the win probability function than snakes and ladders. This is because the player's strength plays a far more prominent role than luck in the former compared to the latter. We will assume that the five players in our games are of reasonable strength so each player will have more than a negligible chance of beating any other player. Due to this, we will choose $\alpha_1 = \frac{\ln(1/4)}{20} \approx 0.069$ (3 d.p.) meaning the best player (Player 5) has an 80 percent chance of beating the worse player (Player 1). This also means that when the players' strengths differ by five there is roughly a 59 percent chance that the stronger player wins.

Hence we have

$$W(u) = \frac{1}{1 + \exp(\frac{-\ln(1/4)}{20}u)}, \quad \forall u \in \mathbb{R}.$$

Note that this win probability function meets the requirements of (1) and the update function meet the requirements of (3) and (4) as $k_U < 1$.

For the simulations, we will set $T = 7,500$ meaning 7,500 games will take place with each player playing approximately 3,000 times.

3.2 Looking at a sample trajectory of the scores

First we can see how the rating of the five players varies over the time period $\{1, ..., T\}$ from one of the samples. As we are only looking at a sample trajectory, we will fix $m \in \{1, 2, ..., \mu\}$ to be 1 so that we are just looking at the first sample.

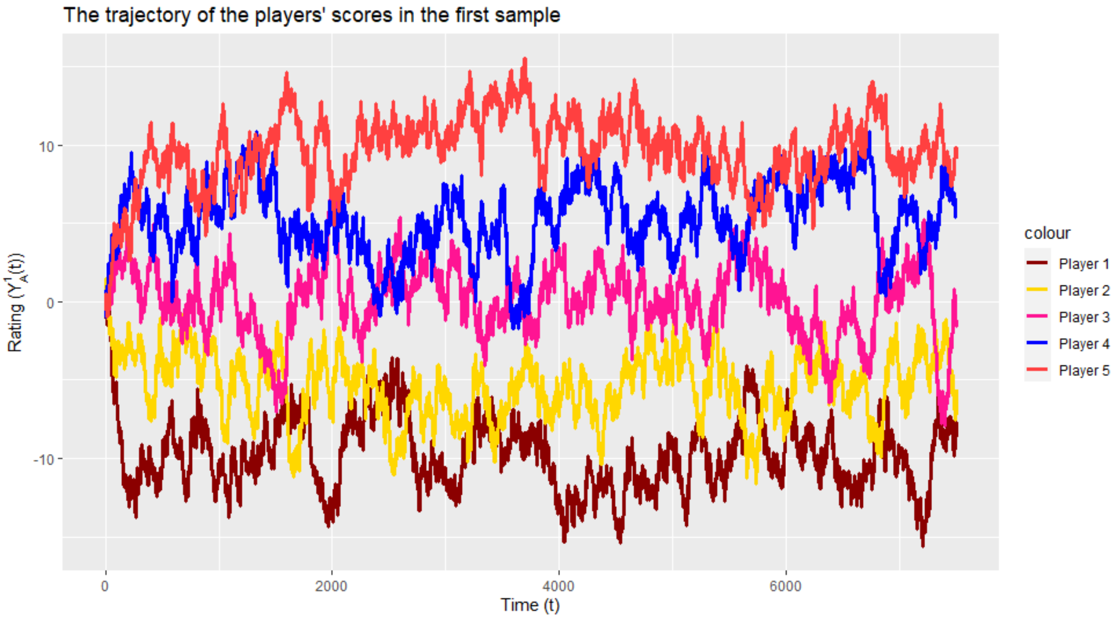

Figure 2: Players' rating over the time period $\{1, ..., T = 7,500\}$

From Figure 2, we can see that despite all players starting with an initial score of 0, by $T = 7,500$ all players' scores seem to be around the same as their strength. It appears that for most of the time period convergence "has happened". We also see a reasonably symmetric pattern for the trajectories of players with the same absolute value for their strength (e.g. Player 1 with strength -10 and Player 5 with strength 10). This again is not too surprising considering they both had an initial score of 0 and their strength is the same distance away from 0 just in the opposite direction. Due to this instead of looking in detail at Player 1 (weakest player), Player 3 (middle strength) and Player 5 (strongest player) we will just look at just Players 1 and 3 and assume this pattern holds for all further simulations.

Another noteworthy observation from Figure 2 is that if we were to rank our players based on their score at t then this would mostly be the same ranking as using their true strength. However, there are multiple times during the time period when the curves cross over each other which would lead to the rankings not being the same.

Between time t and $t + 1$, only two of the players' scores will change as only one game occurs consisting of two players. Due to this, between this time period three of the curves will be horizontal while the other two curves change. For these two curves, the change of rating will be the same just in opposite directions, with the winner's score increasing and the loser's score decreasing by the same amount. This is hard to see on Figure 2 due to T being so large. So as an aside, we can look at the trajectories from $t = 1$ to $t = 20$. Hence in Figure 3, we can see the times where certain players do not play illustrated by a horizontal line.

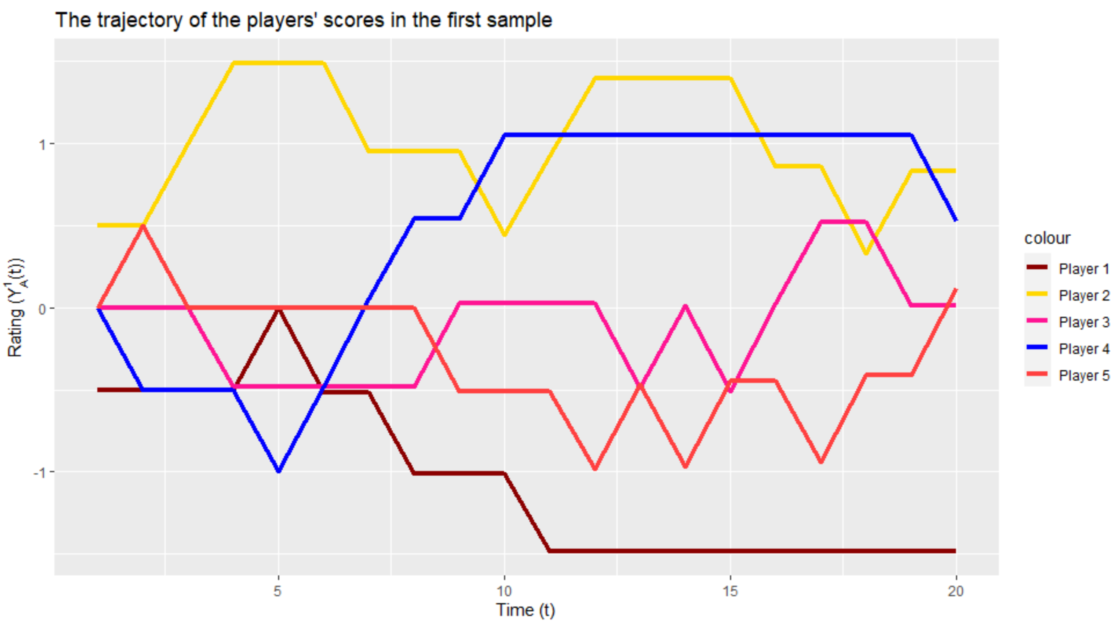

Figure 3: Players' rating up to t=20

For example, from Figure 3, we can see that between $t = 10$ and $t = 11$ Player 2 beat Player 1 and so the other three players did not play a match during this time.

3.3 Looking at the distribution of the score at T

Another way we can view our results is by looking at certain players' score at $t = T$ for each $m \in \{1, 2, ..., \mu = 500\}$. This means we will look at $(Y_A^m(T))_{m\in\{1,...,\mu\}}$. We will look at the scores for Players 1 and 3. We can see the distribution of their scores by plotting the empirical probability distribution function (EPDF) for the players [8]. The EPDF tells us the probability of observing a value within a specific range. We compute the EPDF in R using the function 'stat_density' from the 'ggplot2' package. The plot for Player 1 can be seen in Figure 4.

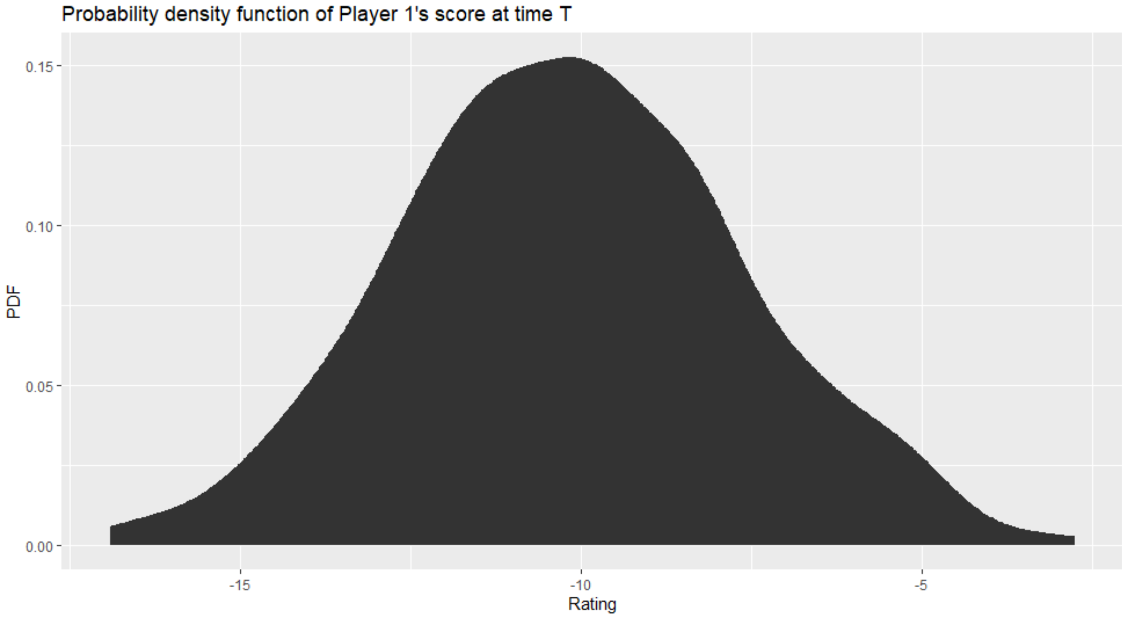

Figure 4: The empirical probability distribution of Player 1's score at time T

Figure 4 shows that in the vast majority of samples, Player 1's score at T is between -5 and -15. We also see by eye that the scores at time T are centred close but seemingly slightly below the player's true strength of -10. We can calculate the empirical mean for each player at time T using the following formula

$$\overline{Y}_A(T) = \frac{1}{\mu}\sum_{m=1}^{\mu} Y_A^m(T).$$

For Player 1, we calculate that $\overline{Y}_1(T) = -10.16$ (2 d.p.), which confirms our initial thought that the scores at T are centred slightly below -10. We can also calculate the empirical standard deviation for each player at time T using the following formula

$$\overline{\sigma}_1(T) = \overline{\sigma}(Y_1(T)) = \left[\frac{1}{\mu-1}\sum_{m=1}^{\mu}\Big(Y_1^m(T) - \overline{Y}_1(T)\Big)^2\right]^{1/2}.$$

For Player 1 we calculate that $\overline{\sigma}_1(T) = 2.51$ (2 d.p.). This means that letting $I_1(T) = [\overline{Y}_1(T) \pm \delta\overline{\sigma}_1(T)] = [-10.16 \pm \delta 2.51]$, for $\delta \in \mathbb{R}_+$, then $x_1 \in I_1(T)$ iff $\delta \geq 0.07$ (2 d.p.). This suggests $\overline{Y}_1(T)$ is a good estimate of Player 1's true strength as the player's strength is less than one standard deviations away from their empirical mean rating at T. In further simulations, it will be fascinating to compare the standard deviations to see the different variation between the values at T across the samples when changing certain variables.

We will also plot the empirical probability distribution function for Player 3, as shown in Figure 5.

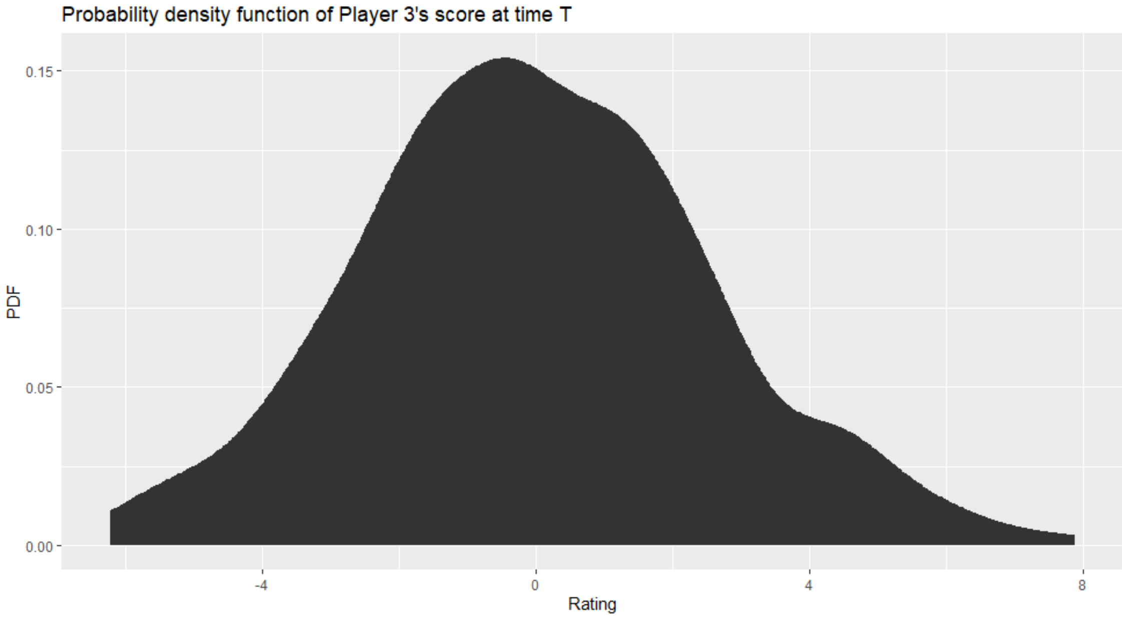

Figure 5: The empirical probability distribution of Player 3's score at time T

From Figure 5 we see by eye that the samples' score at T is centred extremely close to 0, which is the player's strength. This is expected as the initial scores were 0 for all players and Player 3 had the central strength. More accurately, we can calculate the empirical mean $\overline{Y}_3(T) = 0.00$ (2 d.p.) which is the same value as the strength to two decimal places. The empirical standard deviation for Player 3, $\overline{\sigma}_3(T)$, is 2.56 (2 d.p.). This is fairly similar to Player 1's scores standard deviation at time T of 2.51.

3.4 Looking at the evolution of the distribution of the score

The next way we will look at our results from the first simulation is to look at the evolution of the distribution of a player's score over the time period. We want to do this because Aldous' Theorem (Theorem 2.3) is about convergence in distribution. Hence to look at the convergence in distribution we need to look at the evolution of the distribution of the score. Often when convergence in distribution occurs we have that $E[Y(t)] \to E[Y(\infty)]$ and $\text{StdDev}[Y(t)] \to \text{StdDev}[Y(\infty)]$ as $t \to \infty$ [2]. Because of this, we will look at how the empirical mean and empirical standard deviation of the players' score changes over the time period represented by $\overline{Y}_A(t)$

and $\overline{\sigma}_A(t)$ respectively. The empirical mean for Player A is given by

$$\overline{Y}_A(t) = \frac{1}{\mu}\sum_{m=1}^{\mu} Y_A^m(t) \tag{11}$$

and the empirical standard deviation for Player A is given by

$$\overline{\sigma}_A(t) = \left[\frac{1}{\mu-1}\sum_{m=1}^{\mu}(Y_A^m(t) - \overline{Y}_A(t))^2\right]^{1/2}. \tag{12}$$

.

Hence, we will plot the empirical mean for all players across the time period.

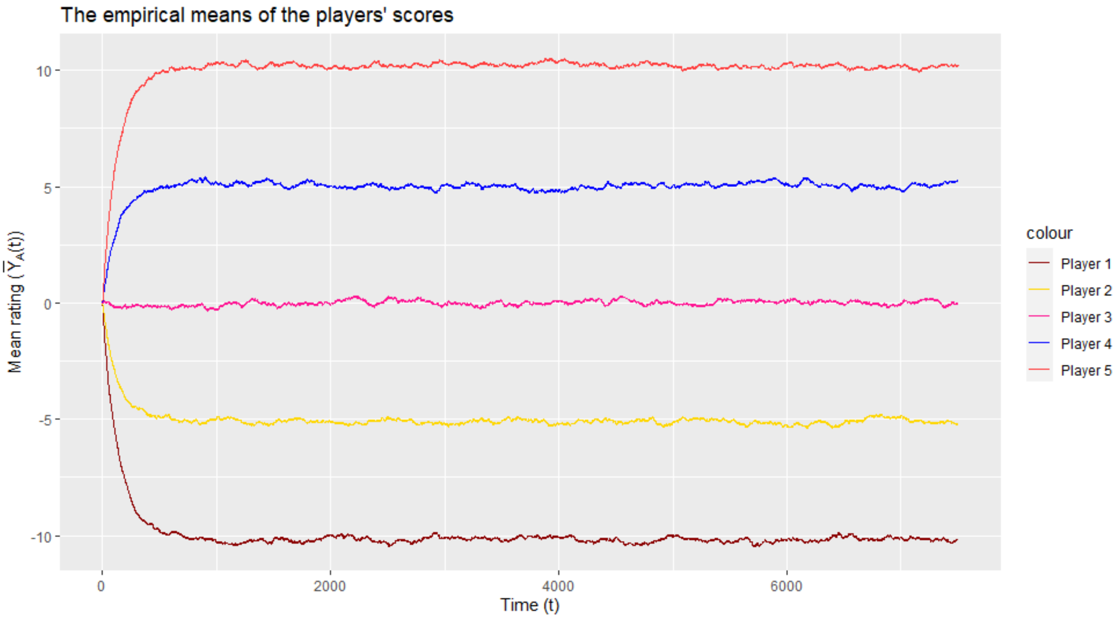

Figure 6: The empirical mean rating of all players at time t

Looking at the empirical mean in Figure 6 we see a similar pattern to the trajectory of the players across one sample (seen in Figure 2) with far fewer fluctuations (less noise). This is expected as it is the empirical mean across all samples rather than just the trajectory of one sample, where a series of unlikely wins or losses causes more of an effect. It appears to take until approximately $\bar{t} = 750$ for convergence in distribution "to happen".

Next, we can look at the empirical standard deviation across the time period for the players given by $\overline{\sigma}_A(t)$.

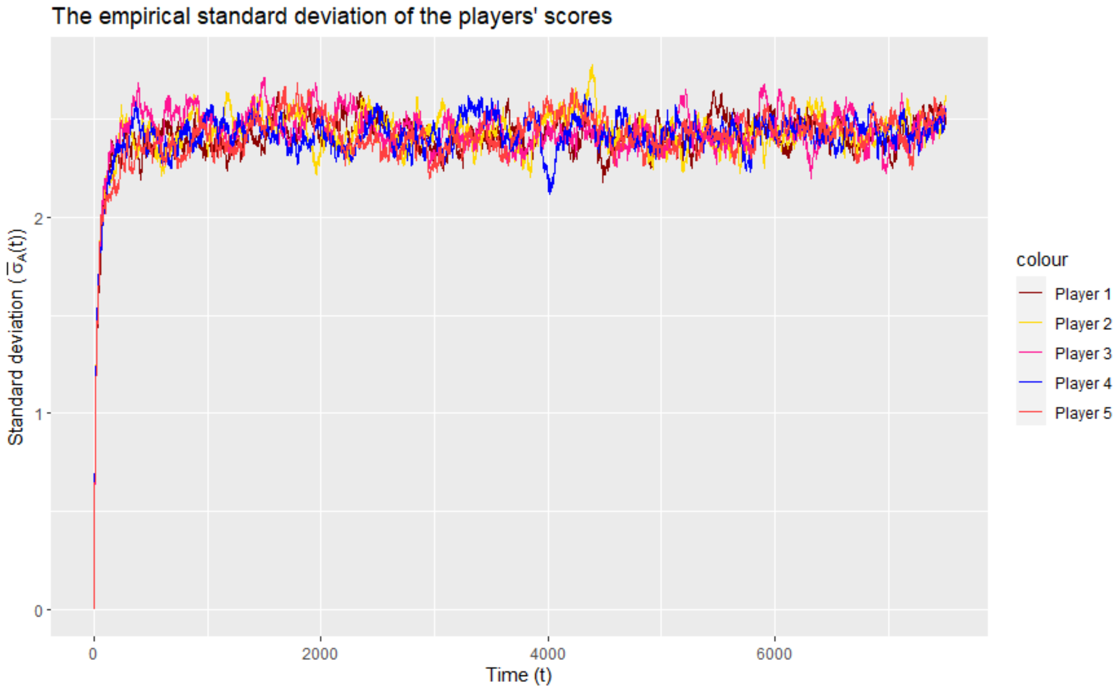

Figure 7: The empirical standard deviation of all players' scores at time t

Figure 7 shows the relationship between the empirical standard deviation and time. We see that the standard deviation sharply increases across the samples and then starts to fluctuate around 2.4 for most of the time period for all players. The standard deviation of each player's score is very similar for all t up to our time horizon. The empirical standard deviation starts at 0 for all players, as initially there is only one observation so there is no difference between the results. Then there is a significant jump when there is a second observation, due to there being a difference between them. As more scores get observed, it takes a little time for the empirical standard deviation to adjust and eventually converge around a value. This seems to occur at around a similar time to when we saw all the players' empirical means appear to converge which was $\bar{t} = 750$. Hence from Figure 7 we can conclude that the standard deviation does not depend on the player. It will be interesting to see if this is the case for other simulations where we change certain variables and also if there is a difference in the standard deviation value along the time period.

3.5 Looking at the evolution of the time average

Finally we can look at the historical mean of the players' rating in the sample $m = 1$ given by $\widetilde{Y}_A^1(t)$. We will use the terms historical mean and rolling mean interchangeably. Instead of going to time $T = 7,500$ we will go to $T^* = 500,000$. We will do this so the initial rating has less of an effect on the later historical means. Also, as the historical mean only looks at one sample, this does not drastically increase computing time. The historical mean of Player A's rating at time t in sample 1 is given by

$$\widetilde{Y}_A^1(t) = \frac{1}{t+1} \sum_{s=0}^{t} Y_A^1(s).$$

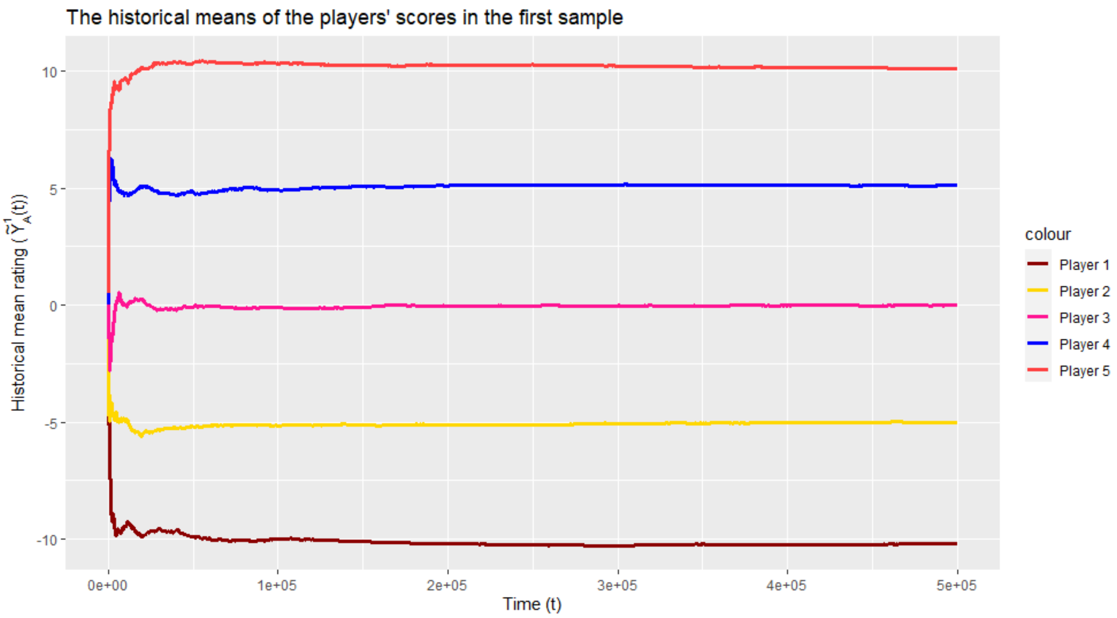

Figure 8: The historical mean of the players' ratings

Figure 8 shows a very similar pattern to Figure 6 with smoother curves compared to a sample trajectory in Figure 2. Both Figure 8 and Figure 2 use results from just one sample. However, historical means in the former lead to much smoother curves as a series of unlikely wins/losses have less of a significant effect on the overall time-average. Aldous' Theorem (Theorem 2.3) does not say that the historical means must converge, however this often occurs with Markov chains when there is convergence in distribution.

Looking specifically at Player 1 we can calculate that the historical mean at time T^* in this sample is $\widetilde{Y}_1^1(T^*) = -10.18$. This value is much closer to the player's strength than the value of Player 1's score at T, $Y_1^1(T)$, which is -10.50. This shows that it is likely in this sample that Player 1 unexpectedly lost some games close to the time horizon reducing their score away from their true strength. This shows that the historical mean gives us a much better estimate of the player's strength as it is affected far less by a series of unlikely wins/losses (which we call noise).

The rolling mean at t in one sample can also estimate the empirical mean at t across all samples and thus the expected value at t. This is because $\widetilde{Y}_A^m(t) \approx \overline{Y}_A(t) \approx E[Y_A(t)]$. In this case we have $\widetilde{Y}_1^1(T^*) = -10.18$ and $\overline{Y}_1(T) = -10.16$. To assess how good the estimate $\widetilde{Y}_1^1(T^*)$ is of $\overline{Y}_1(T)$ we can compare the difference to the standard deviation. The standard deviation of the empirical rating of Player 1 at T is 2.51. Calculating $\left|\widetilde{Y}_1^1(T^*) - \overline{Y}_1(T)\right| = 0.02 \leq 2.51\widetilde{\delta}$ when $\widetilde{\delta} \geq 0.008$ (3 d.p.). Hence this indicates $\widetilde{Y}_1^1(T^*)$ is an excellent estimate of $\overline{Y}_1(T)$. The rolling mean at T^* in one sample takes far less computing time than the empirical mean at T.

3.6 Summary of simulation 1

To summarise simulation 1, we can try to assess the quality of the limit. Looking at our players' trajectories across one sample we see that convergence "has happened" within the time period $[0, T]$. As the limit is 'reached' before T we can treat T as ∞ as we would not expect the distribution to change much past T. However, from the empirical probability distributions, we see that the mean of the scores at T is not equal to the players' strength with $|\overline{Y}_A(T)| > |x_A|$ but the difference is "not too much", which we will quantify. This indicates that there could be a small amount of systematic bias in the estimator which we will be investigating in the following subsection.

3.7 Errors

From David Aldous' papers, we learn of the three sources of error when using Elo-ratings: Mismatch error, lag error and noise.

The mismatch error is caused when the update function is not adapted to the win-probability function. This occurs when the balance relation does not hold. In this simulation, we had that $U(u) = W(-u)$ and so the balance relation held. This means any errors in the scores in this simulation are not down to the mismatch error.

The lag error is caused by the data coming from past results affected by past and not current strength. In our Simple Game Model subsection (Subsection 2.2) one of the assumptions we make is that the players' strength does not change over the time period. This means the win probabilities are the same throughout the time period so this will not cause errors. However, when calculating the historical mean we see that because of the initial scores being 0, they can be weighted down by earlier scores. When calculating the historical means we go to the time horizon of $T^* = 500,000$. This should result in the earlier scores having a minimal effect on the time horizons for later times.

The final error Aldous mentions is noise which is caused by the randomness of recent match results. This error is evident in our plot of the players' rating trajectories (Figure 2). For example, if we look at Player 3, despite for most of the time period their rating is around 0, there are points where the player's rating is below -5 which a series of unlikely losses would have caused. The noise error is less evident when using the empirical means (Figure 6) however it is not a real-life case as we have repeated the simulation 500 times to achieve these results. In the real-world using the historical mean (Figure 8) is the best way to minimise noise as a series of unlikely wins/losses affects the historical mean less than a sample trajectory.

3.7.1 Measuring the error in terms of strengths

Although the empirical mean of the players' score at T is close to the players' true strength it does not seem to be exact. For us to compare simulations, we want to quantify the quality of the limit for each simulation. We can compare the limit's quality by evaluating the distribution of $Y(T)$. To do this we can look at a distributional measure of the error using the formula

$$\overline{D}(x, Y(T)) = \left[\frac{1}{N}\sum_{A\in\mathscr{P}}\left|\overline{Y}_A(T) - x_A\right|^2\right]^{1/2}. \tag{13}$$

We can calculate the distributional measure of the error for our first simulation using this formula and we get $\overline{D}(x, Y(T)) = 0.18$ (2 d.p.). This will be an interesting value to compare later simulations with.

We can also use another formula to measure the error which is

$$\overline{\mathscr{D}}(x, Y(T)) = \overline{\mathbb{E}}\left[\frac{1}{N}\sum_{A\in\mathscr{P}}\left|Y_A(T) - x_A\right|^2\right]^{1/2}, \tag{14}$$

where

$$\overline{\mathbb{E}}[X] = \frac{1}{\mu}\sum_{m=1}^{\mu} X^m.$$

Using this formula we get $\overline{\mathscr{D}} = 0.11$ (2 d.p.), again this will be an fascinating value to compare later simulations with.

3.7.2 Looking at the dilation

In this simulation (as well as previous ones not included in this report), it appears that the empirical mean rating of each player at T may be off from their strength by a similar multiple. If this is the case then it would mean that we have a dilation. If it is a dilation although the scores at T may not perfectly reflect the player's strength they may still be able to be used to accurately determine the win probability. We will explore this further in the next subsection.

To see if the empirical mean score at T is away from the player's strength by a similar multiple we can calculate the λ_A for each player such that $\overline{Y}_A(T) = \lambda_A x_A$. We can also calculate $\lambda_{A,B}$ where

$$\lambda_{A,B} = \frac{\overline{Y}_A(T) - \overline{Y}_B(T)}{x_A - x_B}.$$

We will calculate all λ_A in Table 3 and all possible $\lambda_{A,B}$ values in Table 4.

Table 3: Calculating λ_A values

Player	**Empirical mean at T** ($\overline{Y}_A(T)$)	**Strength** (x_A)	**Lambda** (λ_A)
1	-10.15	-10	1.025
2	-5.25	-5	1.05
3	0.00	0	n/a
4	5.23	5	1.046
5	10.17	10	1.017

Table 4: Calculating $\lambda_{A,B}$ values

Players	$\overline{Y}_A(T) - \overline{Y}_B(T)$	$x_A - x_B$	$\lambda_{A,B}$
1 and 2	-4.9	-5	0.98
1 and 3	-10.15	-10	1.015
1 and 4	-15.38	-15	1.025
1 and 5	-20.32	-20	1.016
2 and 3	-5.25	-5	1.05
2 and 4	-10.48	-10	1.048
2 and 5	-15.42	-15	1.028
3 and 4	-5.23	-5	1.046
3 and 5	-10.17	-10	1.017
4 and 5	-4.94	-5	0.988

We can calculate the average λ value and the standard deviation across the 14 values calculated for this simulation. Doing this we get a mean lambda ($\overline{\lambda}$) of 1.025 (3 d.p.)

and s.d. (σ_λ) of 0.022 (3 d.p.). As the lambda values are close together (as seen by the small standard deviation), for large times we likely have $\overline{Y}(t) \approx \lambda x$. As we only have completed 500 samples going up to the time horizon of $T = 7,500$ then we see that not all the lambda values are exactly the same due to noise/randomness. If we could do infinite samples up to an infinite time period, we would expect lambda values to be identical. However based on our results, we have that $\overline{Y}(t) \approx 1.025x$. Also as $1 \notin [\overline{\lambda} - \sigma_\lambda, \overline{\lambda} + \sigma_\lambda] = [1.003, 1.047]$ this suggests that we can say with some confidence that there is a dilation.

In the real world, we obviously would not be able to calculate the value of λ_A or $\lambda_{A,B}$ as we require the players' true strength value which is unknown. We have seen empirically that through the Elo scores model the absolute value of the empirical mean at T is greater than the player's strength. Can we see this analytically? We can investigate by looking at a more straightforward two-player situation which we will do in the following simulation. Before that another way we can measure the errors is by looking at win probabilities.

3.7.3 Measuring the error in terms of win probabilities

We can also look at the error by comparing the win probability using the player's strength with the win probability using the player's Elo score at T. The Elo-scores process only needs to know which player wins and not either of the win probability function, W, or the players' strengths, x_A. So it would not matter if the players' rating at T did not perfectly reflect their strengths if the win probabilities were still accurate. As we have seen in the previous subsection it appears that there is a dilation. Because of this we would expect the win probabilities to still be accurate as $W(x_A - x_B) = W'(x'_A - x'_B)$ for some dilation. To do this we need a formula to calculate the win probability using the Elo scores which is

$$W_U(u) = \frac{U(-u)}{U(u) + U(-u)}, \tag{15}$$

this is called the Elo-induced win probability function (from Lemma 2.2a).

We can calculate the error between the win probability using the strengths and the win probability using the historic means in the Elo-induced formula at t by

$$\widetilde{E}(t) = \frac{1}{\binom{N}{2}} \left[\sum_{A=1}^{N-1} \sum_{B>A}^{N} \left| W_U\left(\widetilde{Y}_A^1(t) - \widetilde{Y}_B^1(t)\right) - W(x_A - x_B) \right| \right]. \tag{16}$$

We choose to use the historical means as this is a figure we can obtain in the real-world and are less affected by noise than just the scores at t. We can plot this up to $T^* = 500,000$ which can be seen in Figure 9.

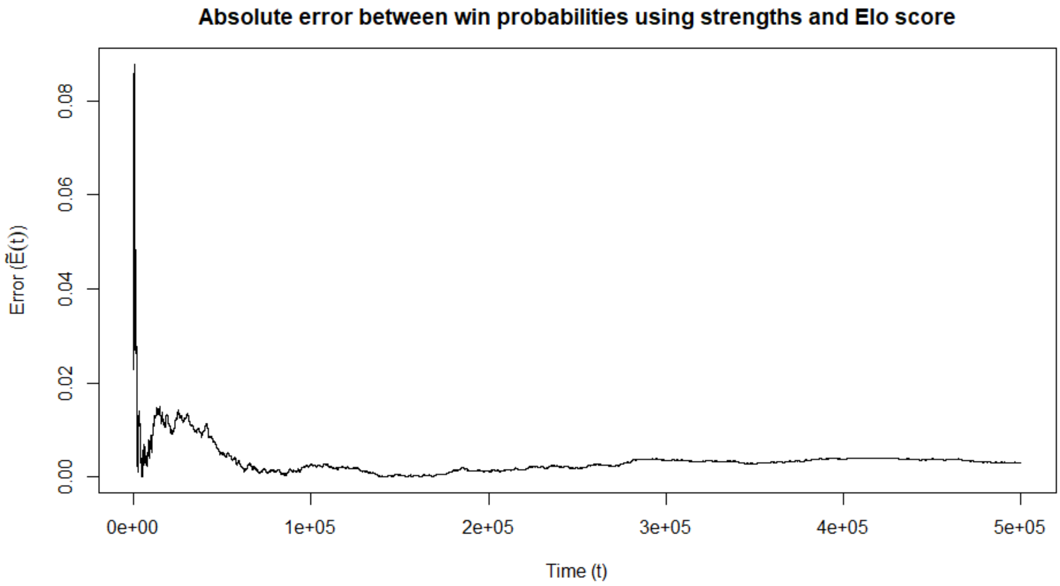

Figure 9: Error between win probability using strength and win probability using Elo-induced win probability with rolling mean at T^*

The error using the historical means at T^* is $\widetilde{E}(t) = 0.003$. This is a very small figure considering that the probability of a stronger player beating a weaker player ranges between 59% and 80%. So this shows that we can use the historical means in Formula (15) instead of the strengths to accurately determine the probability of a player winning a game. We will measure this error for all other simulations to see how changing certain parameters affects the value.

We will also estimate the time taken for the error to converge which we will call τ. The way we will calculate τ for all our simulations is given by

$$\tau = \inf\{t : \forall t' \geq t, \left|\widetilde{E}(t)\right| \leq \epsilon_0\}. \tag{17}$$

Using 17 we then have a way of quantifying when convergence 'has happened'. In this and future simulations we will set the value of $\epsilon_0 = 0.015$. The value of tau is of interest as we will see from what time the error between the win probabilities does not change much from. For this simulation we calculate that $\tau = 1,626$, which we will compare with future simulations.

Instead of using the historical means, we could use the empirical means to measure the error (we will look at doing this in our two player simulation). However, the advantage of using the historical mean is that this value is obtainable in the real-world whereas the empirical is not.

Next we will conduct a simulation with two players instead of five to see if the dilation can be explained analytically.

4 Simulation 2 - Two players

4.1 Setup

This two-player model will have the same update function and win probability function as in the first simulation. We will also keep $h = 5$. First, we will do a simulation to see if we get the same empirically as we did in the first simulation but this time we will set the players' initial scores equal to their strengths to make sure the initial scores do not affect the dilation. The setup can be seen in Table 5.

Table 5: Two player simulation player's strengths and initial score

Player ($A \in \mathscr{P}$)	**Strength** (x_A)	**Initial Score** ($y_A(0)$)
1	-2.5	-2.5
2	2.5	2.5

4.2 Looking at the distribution of the score at T

First, we will look at the empirical mean score of the players at t across the 500 samples which we show in Figure 10.

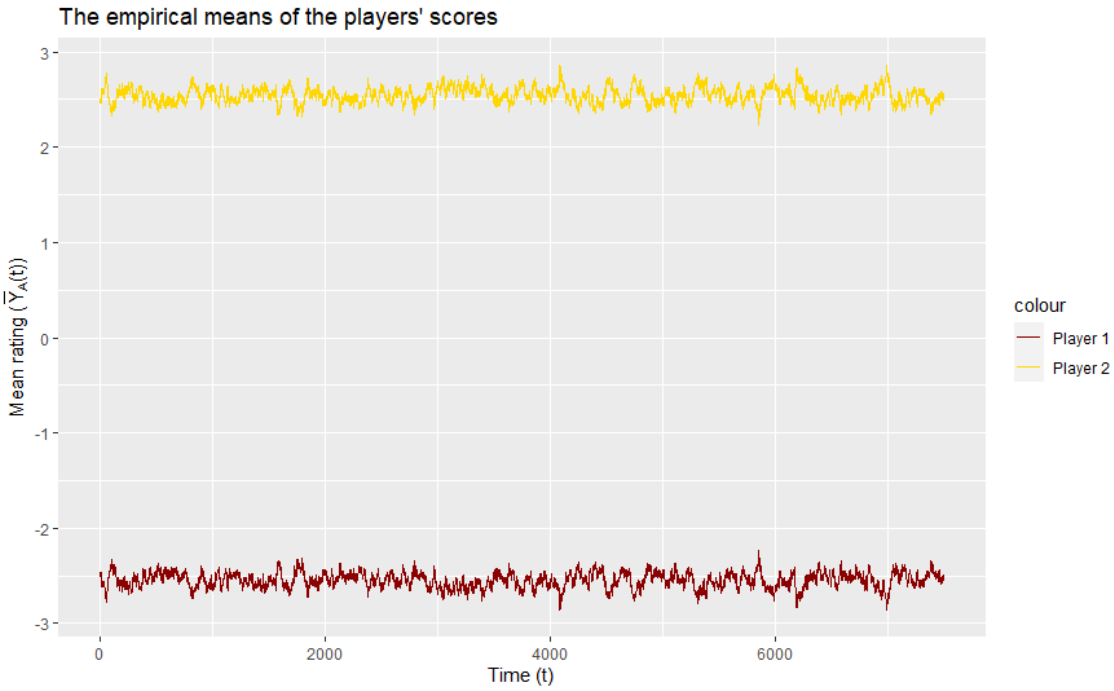

Figure 10: The empirical mean rating of both players at time t

Again we see that the scores seem to converge close to the players' true strength but not preciously. This can be seen with the empirical means at the time horizon, T, for both players which are $\overline{Y}_1(T) = -2.558$ and $\overline{Y}_2(T) = 2.558$. We also for these players see the same pattern as we did in the first simulation that $|\overline{Y}_A(T)| > |x_A|$. This answers one of our earlier questions as we see that the initial ratings do not seem to affect the quality of the limit. Using the same formula as we did earlier for λ_A we get $\lambda_A = 1.023$ (3 d.p.) for both players. As λ_A is the same for both Players 1 and 2, the value of $\lambda_{1,2}$ will also be 1.023. We can again use formulas (13) and (14) to measure the size of the error. In this case we get $\overline{D}(x, Y(T)) = 0.06$ (2 d.p.) and $\overline{\mathscr{D}}(x, Y(T)) = 0.09$ (2 d.p.). These values are slightly smaller than the ones we obtained in simulation 1.

4.3 Analytical analysis

As we have seen this result empirically from both simulations, next we will see if we see this same pattern analytically. We will do this by first calculating the possible differences in scores between Player 1 and Player 2 up to $t = 3$. The possible rating differences can be seen in the following tree where an upwards move will be when Player 1 beats Player 2, whereas a downwards move will be Player 2 beating Player

1.

Table 6: A tree demonstrating the difference in scores between Player 1 and Player 2, $\Delta Y(t) = Y_2(t) - Y_1(t)$

t=0	**t=1**	**t=2**	**t=3**
			1.6033
		2.6965	
			3.6033
	3.8284		
			3.5352
		4.6965	
			5.5352
5			
			3.4701
		4.6291	
			5.4701
	5.8284		
			5.4033
		6.6291	
			7.4033

Using the Simple Game Model we can calculate the probability of the results and in turn the probability of the change in scores. The probability of Player 1 beating Player 2 is $p = W(x_1 - x_2) = (1 + \exp{(-\frac{\ln(1/4)}{20}(x_1 - x_2))})^{-1} = (1 + \exp{(\frac{ln(1/4)}{-4})})^{-1} = 0.414$ (3 d.p.) and so the probability of Player 2 beating Player 1 is $1 - p = 1 - 0.414 = 0.586$ (3 d.p.). These are the settings we had in simulation 1. Using these probabilities and the scores changes in the tree of the evolution of $\Delta Y(t)$, we can calculate the expectation of the change in rating between time t and $t + 1$:

$$\begin{aligned}
(\mathbb{E}[\Delta Y(1)]) &= (3.8284)p + (5.8284)(1-p) \\
&= 5. \\
(\mathbb{E}[\Delta Y(2)]) &= (2.6965)p^2 + (4.6965)p(1-p) \\
&+ (4.6291)(1-p)p + (6.6291)(1-p)^2 \\
&= 5.00019. \\
\mathbb{E}[\Delta Y(3)]) &= (1.6033)p^3 + (3.6033)p^2(1-p) \\
&+ (3.5352)(1-p)^2p + (5.5352)p(1-p)^2 \\
&+ (3.4701)(1-p)p^2 + (5.4701)(1-p)^2p \\
&+ (5.4033)(1-p)^2p + (7.40433)(1-p)^2p \\
&= 5.00055.
\end{aligned}$$

Here we have calculated the $(\mathbb{E}[\Delta Y(t)])$ for $1 \leq t \leq 3$. For t=1 we get $\mathbb{E}[\Delta Y(1)] = 5$ which is equal to $(x_2 - x_1)$. This is to be expected as because the balance relation holds and $(y_2 - y_1) = (x_2 - x_1) = 5$ from Claim 3a this means $(\mathbb{E}[\Delta Y(1)]) = 5$. We can continue to calculate $(\mathbb{E}[\Delta Y(t)]$ for further time periods using statistical software 'R'. We can also calculate $\lambda(t) = \frac{\mathbb{E}[\Delta Y(t)]}{\Delta x}$ and see how it varies. We do this up to $t = 25$ as beyond this time takes too much computing time. The reason for doing this is that this is the way we calculate λ so we will see if we get analytically what we saw empirically in both simulations.

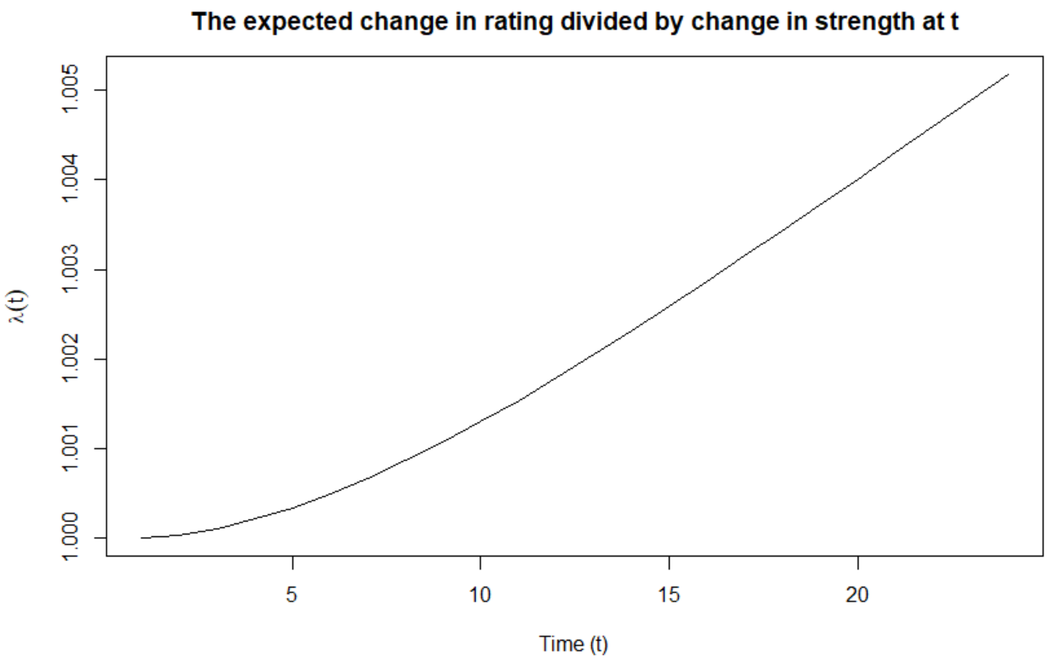

Figure 11: The difference between the expected change in rating and the change in strength at time t

In Figure 11 we see that the value of $\lambda(t)$ increases throughout the time period. This analytically confirms what we saw empirically in both our simulations. In our two-player model at the time horizon we had, $\overline{Y}_1(T) = -2.558$ and $\overline{Y}_2(T) = 2.558$ meaning that $\Delta Y(T) = \overline{Y}_2(T) - \overline{Y}_1(T) = 5.115$. Ideally, if the Elo scores update process perfectly calculated their true strength (with $Y_A(t) = x_A(t)$) then we would have $\Delta Y(T) = \Delta x$ but we do not have this as $\Delta x = 5$. However, the graph shows us that in expectation this does not occur either. As time increases past the times we have plotted in Figure 11 we would expect the values to converge at around 1.023 (3 d.p.) as this is the value we calculated earlier for $\lambda_{1,2}$ from our two player simulation.

4.4 Win probability

As we did in simulation 1, we can look at the error between using the win probabilities with the players' strengths and the win probabilities using the scores.

First, we can then calculate the error between the win probability using strengths and win probability using the Elo-induced win probability formula by

$$\overline{\mathcal{E}}(t) = \mathbb{E}\left[\left|W_U\Big(Y_1(t) - Y_2(t)\Big) - W(x_1 - x_2)\right|\right].$$

We can also calculate the error using the empirical mean values at t using the formula

$$\overline{E}(t) = \left|W_U\Big(\overline{Y}_1(t) - \overline{Y}_2(t)\Big) - W(x_1 - x_2)\right|.$$

We can plot both these results to see the absolute difference between the probability of Player 1 beating Player 2 using W_U instead of W over time as seen in Figure 12. Error 1 (red line) is $\overline{\mathcal{E}}(t)$ and error 2 (lime line) is $\overline{E}(t)$.

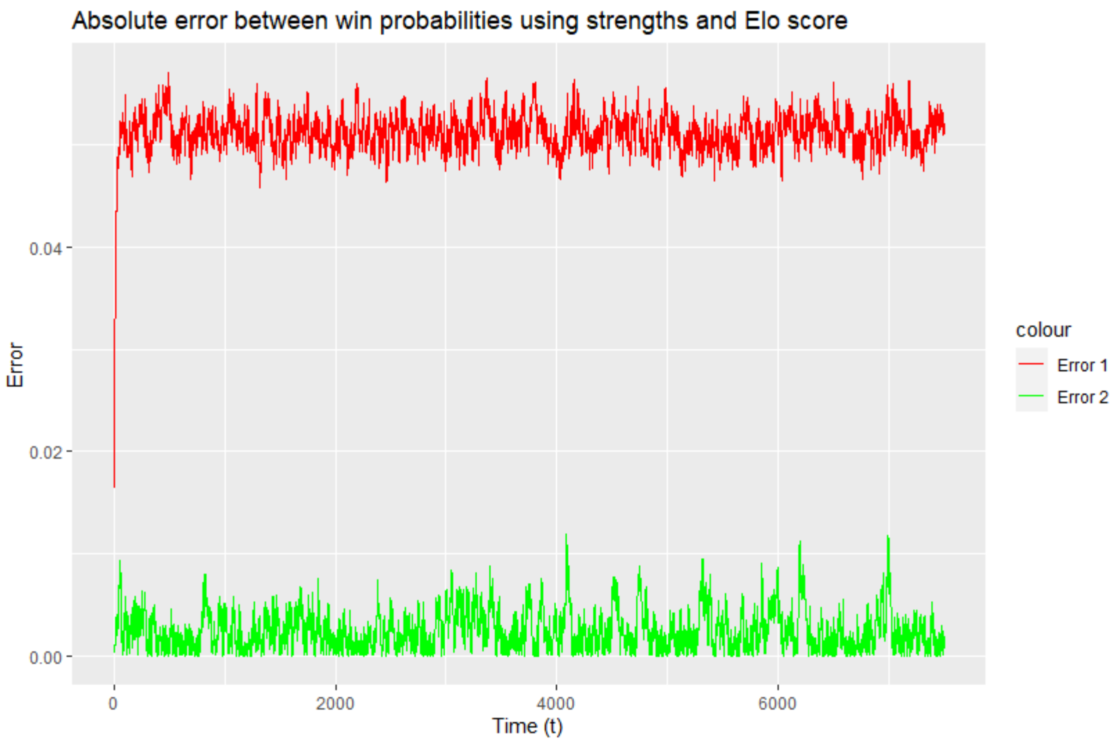

Figure 12: Error between win probability using strength and win probability using Elo-induced win probability with empirical mean at T

Firstly looking at error 1 we can calculate that $\overline{\mathcal{E}}(T) = 0.052$ (3 d.p.). Considering that the probability of Player A beating Player B is 0.414, the mean error is 12.34% of the value, which is quite big. Whereas, we get $\overline{E}(T) = 0.002$ (3 d.p.), this is much smaller error than $\overline{\mathcal{E}}(T)$. However in the real-world we cannot obtain empirical means so instead we will look at using the historical mean in one sample.

We do this by using the formula

$$\widetilde{E}(t) = \left| W_U\left(\widetilde{Y}_1^1(t) - \widetilde{Y}_2^1(t)\right) - W(x_1 - x_2) \right|.$$

Hence we plot this in Figure 13.

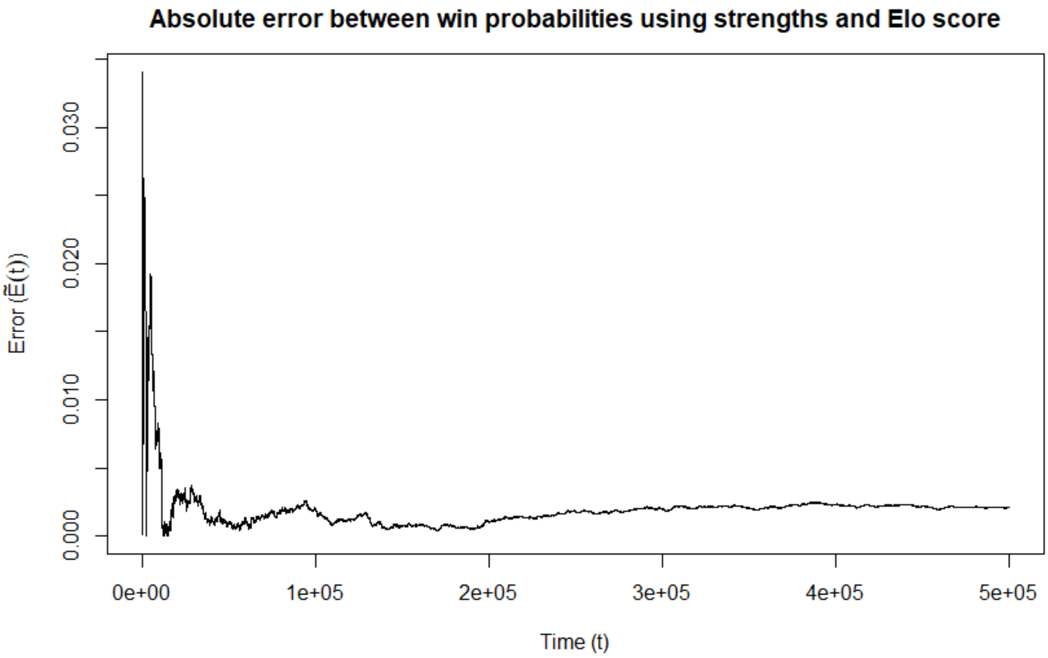

Figure 13: Error between win probability using strength and win probability using Elo-induced win probability with rolling mean at T^*

The mean error using the historical means at T^* is $\widetilde{E}(T^*) = 0.002$ which is the same as the corresponding value using the empirical means at T. This is to be expected as we mentioned earlier the historical mean over a long period is a excellent estimate of the empirical mean. Nevertheless, this is good confirmation that a value we can obtain in the real-world can give us a reasonable estimate of the win probabilities.

Also this value is around the same as we obtained simulation 1, $\widetilde{E}(T^*) = 0.003$. This shows that the number of players has a minimal effect on this value as the only change we made between the two simulations was reducing the players from five to two. Although it does make sense for the two-player model to have a slightly smaller value. This is because each player has played 500,000 games when $t = T^*$ whereas in the five-player case the average number of games for each player is $200,000$. Hence these historical mean values are likely to be a more accurate measure of the strengths and so are better at determining the win probabilities.

We can see that many of our findings from simulation 1 can be backed up analytically through the more straightforward two-player model. We will now look at more simulations so that we can compare the simulations when we change specific parameters.

5 Simulation 3 - Balance relation not holding

5.1 Setup

One of the main points of interest of the report is to look at the effect of when the balance relation (7) holds compared to when it does not. Due to this we will conduct another simulation in a similar level of detail to simulation 1 to look at the effect it has on our measurements.

Therefore in this simulation, we will change the update function so that the requirements (3) and (4) are still met but the balance relation is not. It will be interesting to see if any convergence still occurs and how "good" it is. Aldous' Theorem (Theorem 2.3) does not say that the balance relation (7) has to hold so we would still expect the scores to converge to some number. Nevertheless, there is no guarantee that the player's score converges close to the player's strength from the theorem.

We will keep the win probability function the same as simulation 1 so we have

$$W(u) = \frac{1}{1 + \exp\left(-\alpha_1 u\right)}, \quad \forall u \in \mathbb{R},$$

where $\alpha_1 = (\frac{-\ln(1/4)}{20}) \approx 0.069$ (3 d.p.) but we will change the update function to

$$U(u) = \frac{1}{1 + \exp(\alpha_2 u)}, \quad \forall u \in \mathbb{R}$$

where $\alpha_2 = \frac{-\ln(1/19)}{20} \approx 0.147$ (3 d.p.) and so $U(u) \neq W(-u), \quad \forall u \in \mathbb{R}$.

This update function still meets the requirements of (3) and (4). However, we can show that the balance relation does not hold as, from Lemma 2.2b, the balance relation holds iff $\frac{W(u)}{U(-u)}$ is a symmetric function. Here we have

$$G(u) = \frac{W(u)}{U(-u)} = \frac{1 + \exp(\frac{\ln(1/19)}{20} u)}{1 + \exp(\frac{\ln(1/4)}{20} u)}$$

and it is clear that $G(u) \neq G(-u)$ and so G is not symmetric. This means that the balance relation does not hold.

If we had $W(u) = U(-u)$, with this update function then $W = logistic(\alpha_2)$ and so Player 5 would have a 95% chance of beating Player 1 and if the player's strengths differ by five then the stronger player would have roughly a 68% chance of winning. However, with this win probability function (same as we used in the first simulation), Player 5 has an 80% chance of beating Player 1 and when the strengths differ by five the stronger player has roughly a 59% chance of winning. Because of this we would expect to get a considerable mismatch error.

5.2 Looking at a sample trajectory of the scores

We will start off this simulation the same way as we did with simulation 1 by plotting a sample trajectory. We will again look at how the scores of the players vary over the first sample, this can be seen in Figure 14.

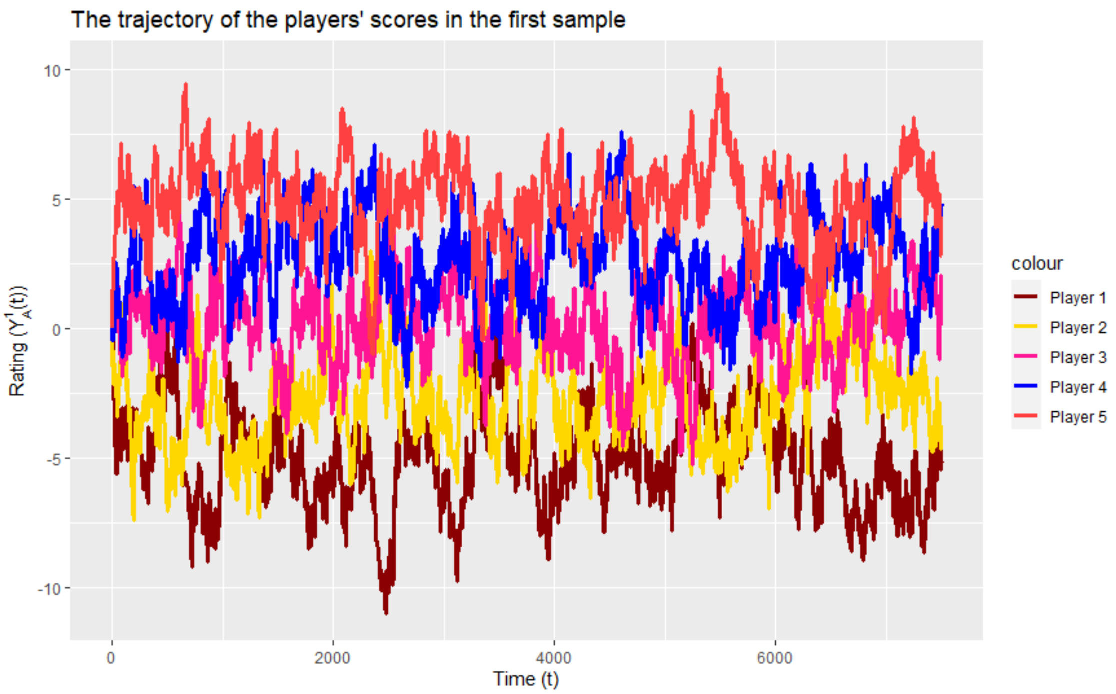

Figure 14: Players' rating over the time period $\{1, ..., T = 7,500\}$

Comparing Figure 14 with the corresponding figure from simulation 1 (Figure 2) we see that the curves cross over far more in this simulation (far more noise). If we were to rank our players at time t using this sample there is less chance that the ranking would be the same as the strength ranking. We also see that in this sample the players' scores do not seem to be around their strength (apart from Player 3).

For example, Player 1's strength is -10 but in this trajectory, there are very few times where their score is at or below their strength and we have $Y_1^1(T) = -5.32$. We will see if this pattern is the same across all of 500 trajectories, first by looking at the distribution of Player 1's scores at T.

5.3 Looking at the distribution of the score at T

First, we will look at the probability distribution function for Player 1's scores at T.

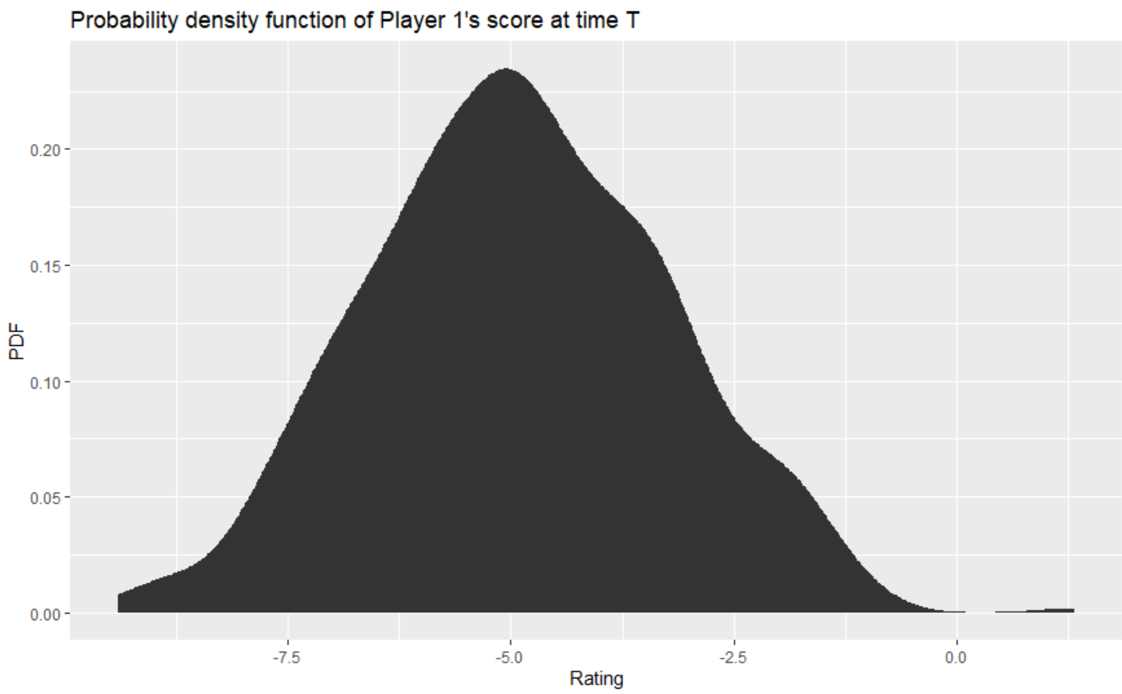

Figure 15: The empirical probability distribution of Player 1's score at time T

It is immediately clear from Figure 15 that Player 1's scores at T are far further away from their strength of -10 in this simulation compared to the last. We can calculate that $\overline{Y}_1(T) = -4.91$ (2 d.p.) and $\overline{\sigma}_1(T) = 1.67$ (2 d.p.). This means that letting $I_1(T) = [\overline{Y}_1(T) \pm \delta\overline{\sigma}_1(T)] = [-4.91 \pm \delta 1.67]$, for $\delta \in \mathbb{R}_+$, then $x_1 \in I_1(T)$ iff $\delta \geq 3.05$ (2 d.p.). This suggests $\overline{Y}_1(T)$ is a very bad estimate for x_1 as it is over 3 standard deviations away.

Next, we will look at the probability distribution function for Player 3's scores at T.

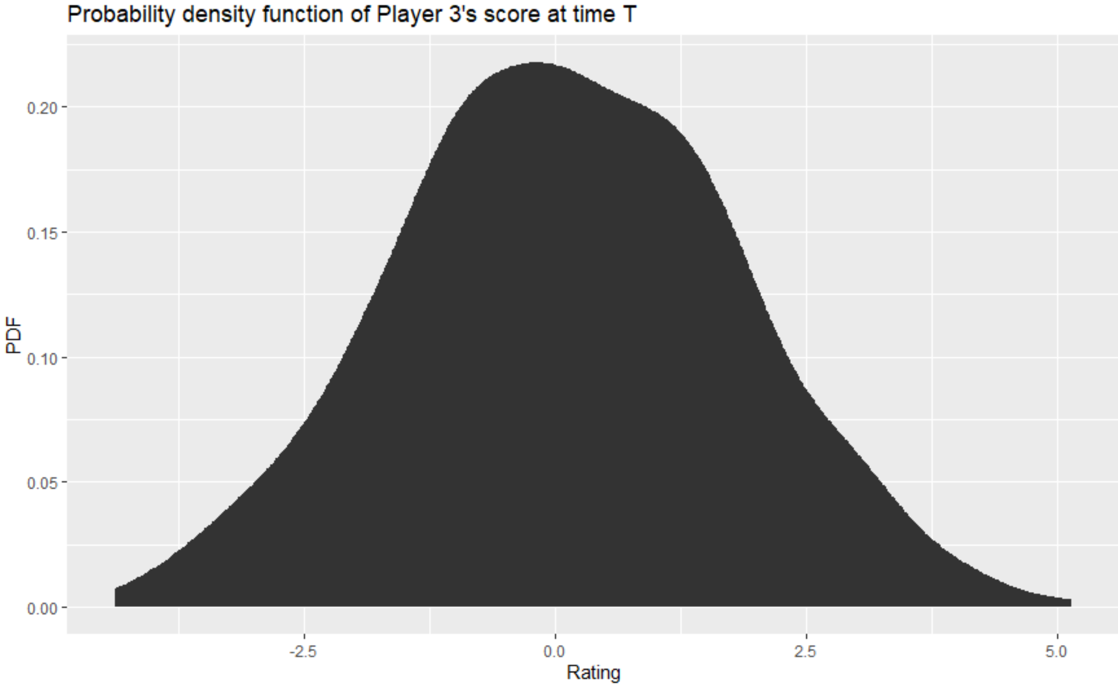

Figure 16: The empirical probability distribution of Player 3's score at time T

Unlike with Player 1, we see by eye from Figure 16 that Player 3's rating at T looks reasonable close to their strength of 0 despite the balance relation not holding. We calculate that $\overline{Y}_3(T) = 0.11$ (2 d.p.) and $\overline{\sigma}_3(T) = 1.68$ (2 d.p.). This means that letting $I_3(T) = [\overline{Y}_3(T) \pm \delta\overline{\sigma}_3(T)] = [0.11 \pm \delta 1.68]$, for $\delta \in \mathbb{R}_+$, then $x_3 \in I_3(T)$ iff $\delta \geq 0.07$ (2 d.p.). This suggests $\overline{Y}_3(T)$ is a superb estimate for x_3 as it is far less than one standard deviation away. It makes sense that the balance relation not holding has less of an effect on Player 3 than the other players. This is due to Player 3's initial score being equal to their strength and that they are the middle player when ranked due to strength.

5.4 Looking at the evolution of the distribution of the score

Now we will look at the empirical rating at t for each of the players.

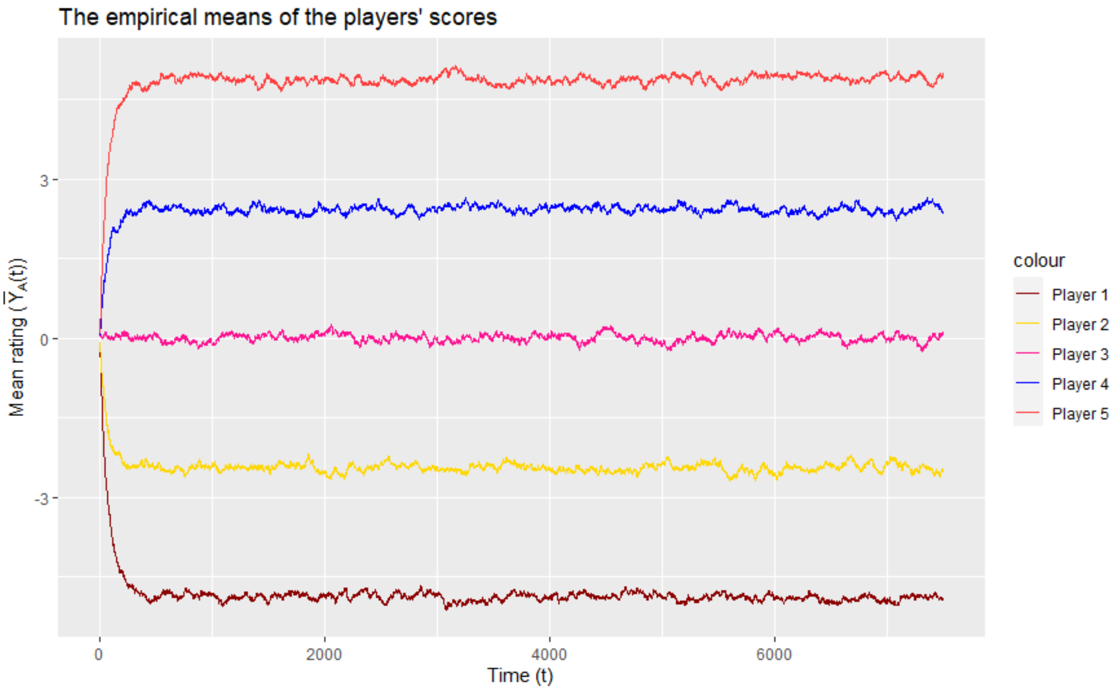

Figure 17: The empirical mean rating of all players at time t

From Figure 17, we can see by eye that the empirical mean score at T for each player is far away from their strength for all players apart from our medium strength player, Player 3. However, if we rank them based on their empirical means at t then we would get the same ranking as from their strength. The convergence in distribution seems "to occur" quicker in this simulation than the first at around $\bar{t} = 500$ compared to approximately $\bar{t} = 750$.

We can use the formulas

$$\overline{D}(x, Y(T)) = \left[\frac{1}{N}\sum_{A\in\mathscr{P}}\left|\overline{Y}_A(T) - x_A\right|^2\right]^{1/2}$$

and

$$\overline{\mathscr{D}}(x, Y(T)) = \mathbb{E}\left[\frac{1}{N}\sum_{A\in\mathscr{P}}\left|Y_A(T) - x_A\right|^2\right]^{1/2},$$

which we used in the first simulation to compare the size of error in measuring strengths between the two simulations. We get $\overline{D}(x, Y(T)) = 3.59$ (2 d.p.) and $\overline{\mathscr{D}}(x, Y(T)) = 0.18$ (2 d.p.). In the first simulation we obtained $\overline{D}(x, Y(T)) = 0.18$ (2 d.p.) and $\overline{\mathscr{D}}(x, Y(T)) = 0.11$ (2 d.p.) hence this shows the size of error is much larger in this simulation compared to the first. This is expected as in the first simulation the balance relation held whereas in this simulation it did not. This means we get additional mismatch error in this simulation which did not exist in the first.

We can also calculate λ_A and $\lambda_{A,B}$ as we did in simulation 1 where $\lambda_A = \frac{\overline{Y}_A(T)}{x_A}$ and $\lambda_{A,B} = \frac{\overline{Y}_A(T) - \overline{Y}_B(T)}{x_A - x_B}$ which are displayed in the following tables.

Table 7: Calculating λ_A values

Player	**Empirical mean at T** ($\overline{Y}_A(T)$)	**Strength** (x_A)	**Lambda** (λ_A)
1	-4.91	-10	0.491
2	-2.51	-5	0.502
3	0.11	0	n/a
4	2.32	5	0.464
5	4.98	10	0.498

Table 8: Calculating $\lambda_{A,B}$ values

Players	$\overline{Y}_A(T) - \overline{Y}_B(T)$	$x_A - x_B$	$\lambda_{A,B}$
1 and 2	-2.4	-5	0.48
1 and 3	-5.02	-10	0.502
1 and 4	-7.23	-15	0.482
1 and 5	-9.89	-20	0.495
2 and 3	-2.62	-5	0.524
2 and 4	-4.83	-10	0.483
2 and 5	-7.49	-15	0.499
3 and 4	-2.21	-5	0.442
3 and 5	-4.87	-10	0.487
4 and 5	-2.66	-5	0.532

Calculating the mean and standard deviation we get $\overline{\lambda} = 0.491$ and $\sigma_\lambda = 0.022$. Hence like in the first simulation because σ_λ is very small we can say $\overline{Y}(t) \approx \lambda x$ but in this simulation with $\lambda = 0.491$ instead of 1.025. It is clear that $1 \notin [\overline{\lambda} - \sigma_\lambda, \overline{\lambda} + \sigma_\lambda] = [0.469, 0.513]$ so we can say with high confidence that there is again a dilation.

Next, we can look at the empirical standard deviation for each player at t.

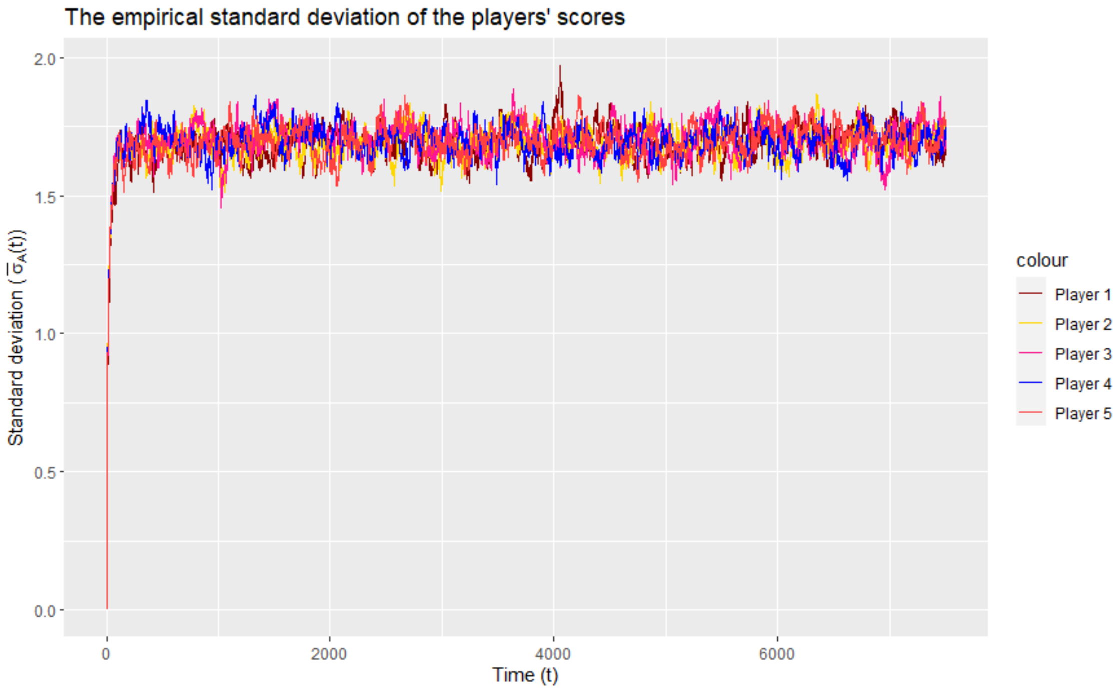

Figure 18: The empirical standard deviation of the rating of all players at time t

From Figure 18 we see that the standard deviation value does not seem to depend on the player - which is the same as in simulation 1. The standard deviation of the players' scores seems to converge at around 1.7, which is lower than in the first simulation where it was 2.4. However the difference is quite small. This explains why in Figure 14 our trajectories cross over each other much more than in the first simulation. As the trajectories are closer together due to lambda decreasing from 1.025 to 0.491 and the standard deviation only decreasing from 2.4 to 1.7.

5.5 Win probabilities

Like we did in the first two simulations, we will now look at the error in win probabilities between using strengths and the Elo-scores induced win probabilities using historical means. We again calculate the historical means up to $T^* = 500,000$ and then use Formula (16). We then plot the results in Figure 19.

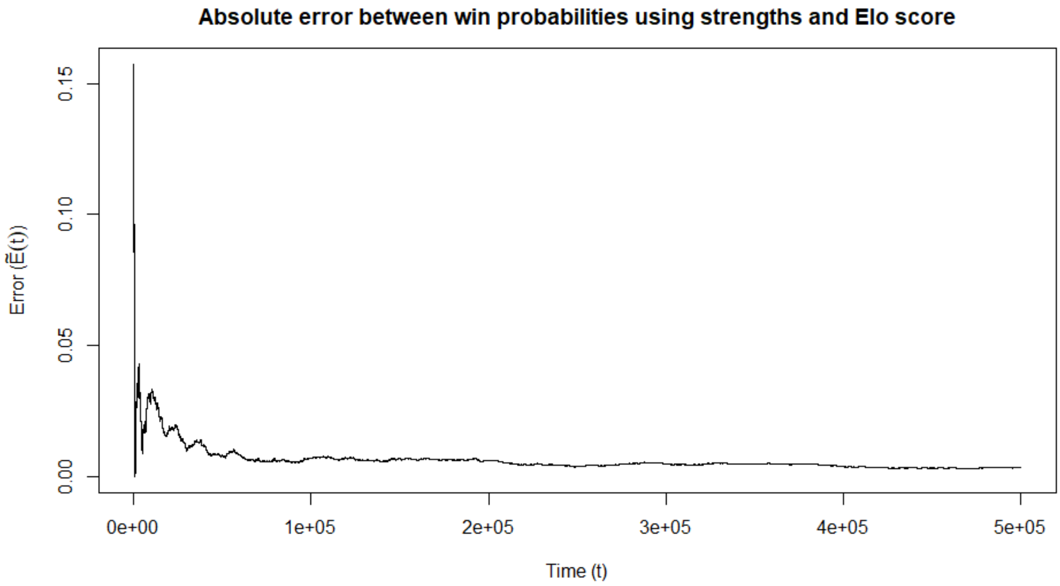

Figure 19: Error between win probability using strength and win probability using Elo-induced win probability with rolling mean at T^*

The mean error using the historical means at T^* is $\widetilde{E}(T^*) = 0.004$ which is still reasonably small. This is quite surprising as the players' scores at T are very far off their strengths. But this shows that using these historical mean at T^* to calculate the win probabilities through the Elo-induced win probability function we still get relatively accurate win probabilities. Also we try quantify the time where convergence 'has occurred' using Formula (17) and we get $\tau = 26,286$. Tau is much larger than the corresponding figure from simulation 1 which was $\tau = 1,626$. When we conduct more simulation later we will see if we spot a reason for this being the case.

6 Further simulations

In this section we will conduct further simulations. These simulations will primarily focus upon investigating the errors in the strengths and win probabilities. Due to this, we can discount some of the plots that we have looked at in previous simulations such as the sample trajectories. Instead we will focus on looking at the empirical means across the samples and the historical mean in one sample over a more extended time period. Using the empirical mean at T we will be able to investigate the dilation. Whereas using the historical mean at T^* we will be able to see how the win probabilities using these scores compare to the win probabilities using strengths. This is an intriguing value to look at as the win probability using historical means is a number that we can obtain in the real-world. Whereas we cannot get the win probability using strengths as we do not know the players' true strengths in the real world.

In these simulations the only changes we will make will be the alpha values in the win probability function, W, and in the update function, U. We will investigate when the value of α is the same in both simulations meaning the balance relation (7) holds and some where it does not and what effect this had. We will also change the magnitude of both these values to see what the impact is. As the only variables we are changing are the alpha values this means in all simulations we will still have five players with $h = 5$. This means $|x_A - x_B| = 5$ when $|A - B| = 1$.

6.1 Simulation 4

In this simulation, we will keep $U(u)$ the same as the previous simulation but we change $W(u)$ so that the balance relation (7) holds. Consequently we have

$$W(u) = \frac{1}{1 + \exp\left(-\alpha_2 u\right)}, \quad \forall u \in \mathbb{R},$$

and

$$U(u) = \frac{1}{1 + \exp(\alpha_2 u)}, \quad \forall u \in \mathbb{R}$$

where $\alpha_2 = \frac{-\ln(1/19)}{20} \approx 0.147$ (3 d.p.) and so $U(u) = W(-u), \quad \forall u \in \mathbb{R}$.

As we are now more interested in investigated how the dilation compares across the simulations, we will go straight to looking at the player's empirical mean scores at t.

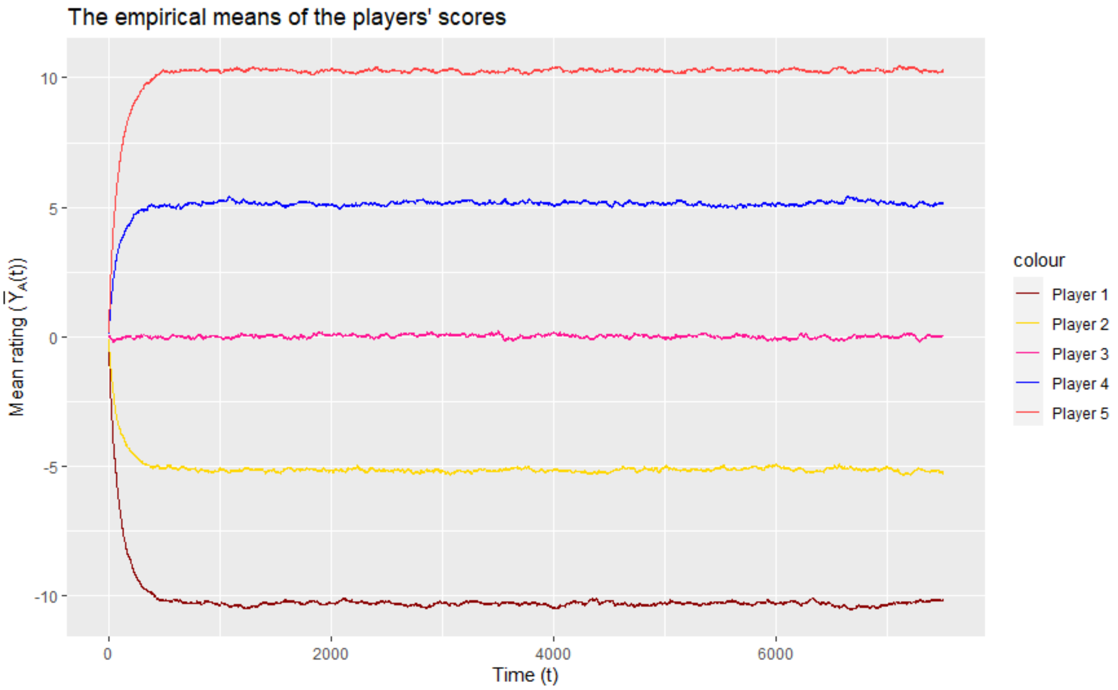

Figure 20: The empirical mean rating of all players at time t

Looking at Figure 20, we can see the empirical means look much more similar to simulation 1 (where the balance relation held) than simulation 3 (where it did not). However like in the first simulation it still does not seem that $Y_A(T) = x_A$ for all players.

We will now use Formulas (13) and (14) to measure the error in strengths. We get $\overline{D}(x, Y(T)) = 0.21$ (2 d.p.) and $\overline{\mathscr{D}}(x, Y(T)) = 0.08$ (2 d.p.). In the first simulation we obtained $\overline{D}(x, Y(T)) = 0.18$ (2 d.p.) and $\overline{\mathscr{D}}(x, Y(T)) = 0.11$ (2 d.p.) and in the second we obtained $\overline{D}(x, Y(T)) = 3.59$ (2 d.p.) and $\overline{\mathscr{D}}(x, Y(T)) = 0.18$ (2 d.p.). This shows that in the two simulations where the balance relation holds both the error distance measurements are very similar.

Again we will calculate λ_A and $\lambda_{A,B}$ with $\lambda_A = \frac{\overline{Y}_A(T)}{x_A}$ and $\lambda_{A,B} = \frac{\overline{Y}_A(T) - \overline{Y}_B(T)}{x_A - x_B}$ to look at the dilation. We will exclude the tables showing our calculations and go straight to computing the mean and standard deviation which are $\overline{\lambda} = 1.027$ and $\sigma_\lambda = 0.011$ respectively. Again as σ_λ is small we can say $\overline{Y}(t) \approx \lambda x$ with $\lambda = 1.027$. This is very close to the value of λ that we calculated in the first simulation when the balance relation also held which was 1.025. Again it is clear that $1 \notin [\overline{\lambda} - \sigma_\lambda, \overline{\lambda} + \sigma_\lambda] = [1.016, 1.038]$ so we can say with some confidence that there is a dilation.

We will now look at the error in win probability as we have in previous simulations. We will again use Formula (16) to plot how the error in win probability changes over time which can be seen in Figure 21.

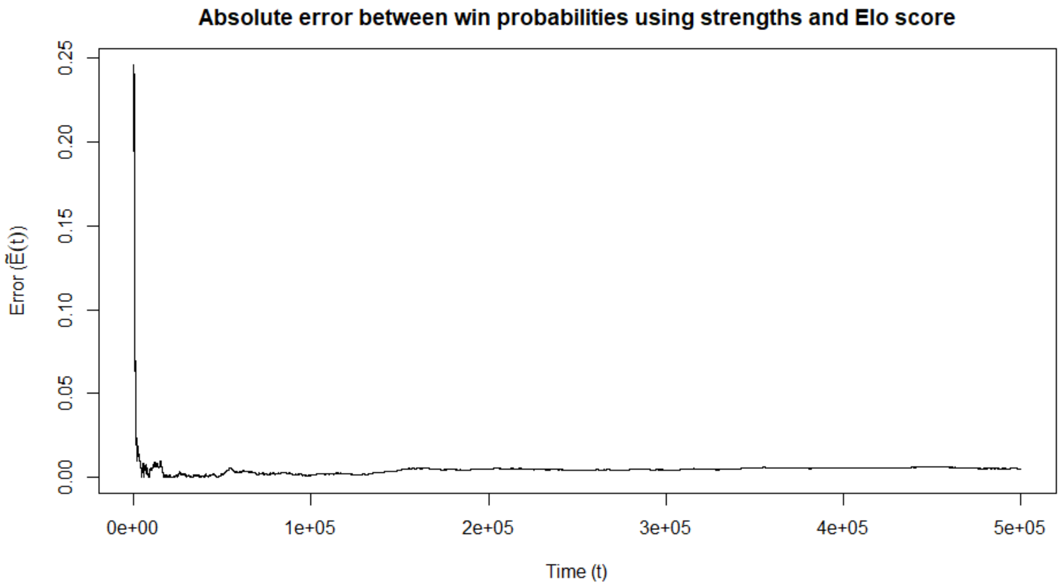

Figure 21: Error between win probability using strength and win probability using Elo-induced win probability with rolling mean at T^*

We calculate that the win probability error using the historical means at T^* is $\widetilde{E}(T^*) = 0.005$. We can also use Formula (17) to try quantify when convergence 'has happened' and we get $\tau = 2,442$. This value is very similar to the corresponding tau values in simulation 1, $\tau = 1,626$, and much smaller than the value from simulation 3, $\tau = 26,286$.

6.2 Simulation 5

In this simulation we will let the balance relation hold again but use a completely different α value. This way we will be able to see how it compares to Simulation 1 and 4 where the balance relation also held. Hence we set

$$W(u) = \frac{1}{1 + \exp\left(-\alpha_3 u\right)}, \quad \forall u \in \mathbb{R},$$

and

$$U(u) = \frac{1}{1 + \exp(\alpha_3 u)}, \quad \forall u \in \mathbb{R}$$

where $\alpha_3 = \frac{-\ln(1/3)}{5} \approx 0.220$ (3 d.p.) and so $U(u) = W(-u), \quad \forall u \in \mathbb{R}$.

Again as we are only interested in the errors we will just look at the empirical means across the samples and the historical mean in one sample.

First we will look at the empirical means in Figure 22.

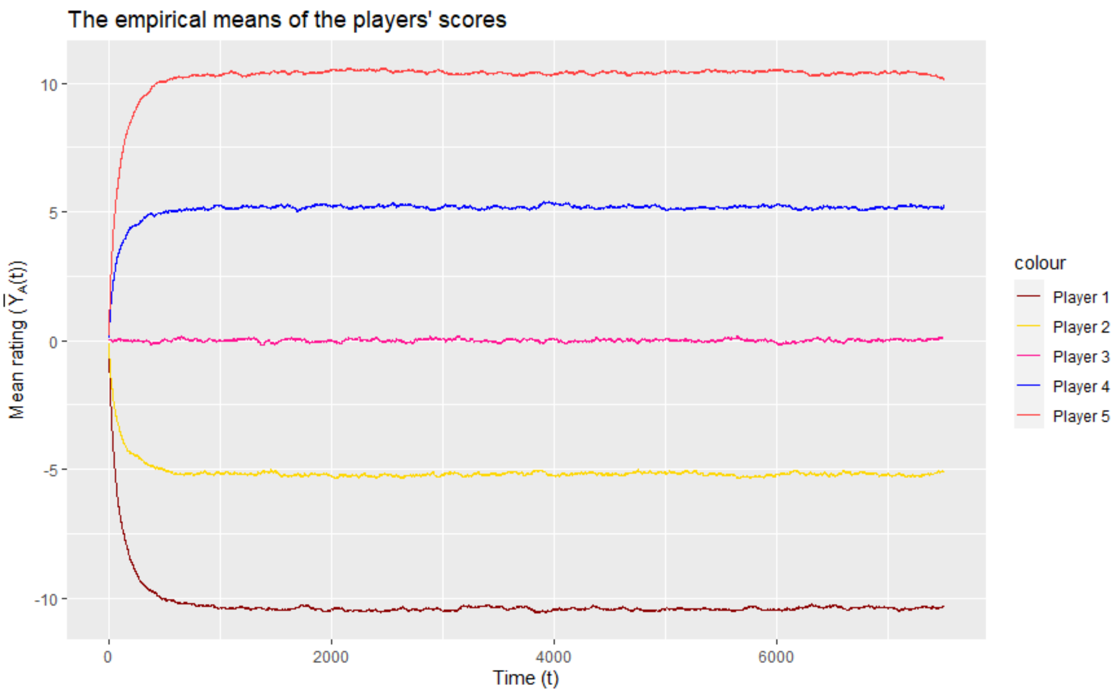

Figure 22: The empirical mean rating of all players at time t

By eye we can see that the empirical mean scores at T are not exactly equal to the players' strengths. This is confirmed when we calculate the measures of strength errors and we get $\overline{D}(x, Y(T)) = 0.23$ (2 d.p.) and $\overline{\mathscr{D}}(x, Y(T)) = 0.06$ (2 d.p.).

To measure the dilation, we again calculate the values of lambda and compute the mean and standard deviation which are $\overline{\lambda} = 1.031$ and $\sigma_\lambda = 0.016$ respectively. Hence we can say $\overline{Y}(t) \approx \lambda x$ with $\lambda = 1.031$ due to σ_λ being small. This is very close to the value of λ that we calculated in the first simulation when the balance relation also held which was 1.027. Again it is clear that $1 \notin [\overline{\lambda} - \sigma_\lambda, \overline{\lambda} + \sigma_\lambda] = [1.016, 1.038]$ so we can say with some confidence that there is a dilation.

We can now, like we have in the previous simulations, measure the error in win probabilities at t up to $T^* = 500,000$ which we plot in Figure 23.

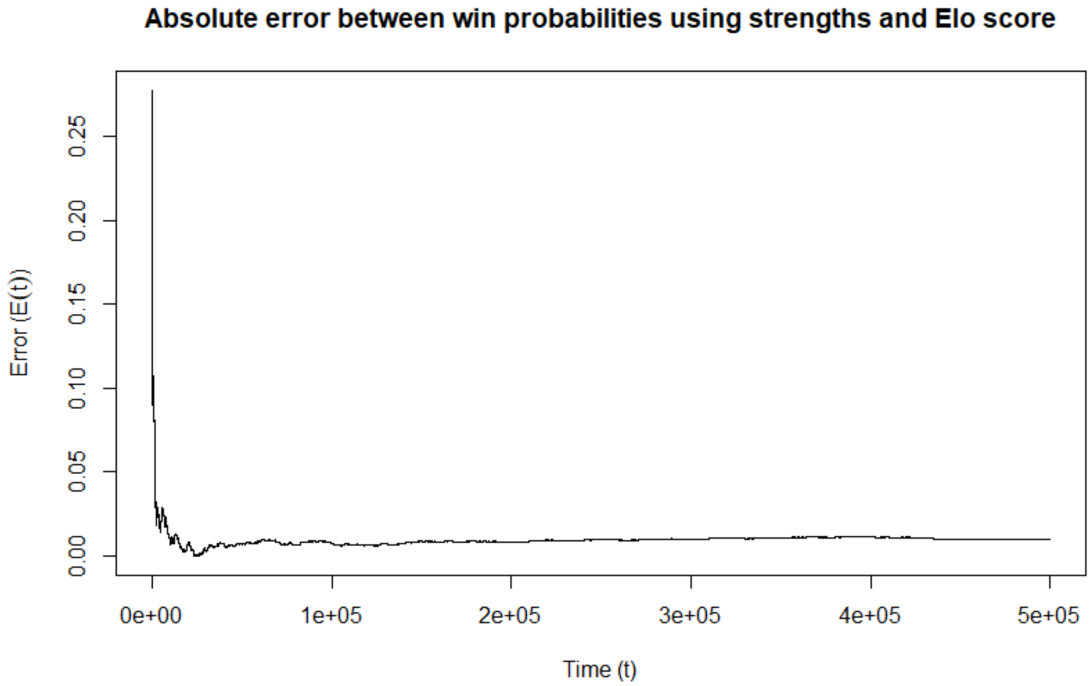

Figure 23: Error between win probability using strength and win probability using Elo-induced win probability with rolling mean at T^*

The mean error using the historical means at T^* is $\widetilde{E}(T^*) = 0.010$. This is the biggest value across all five simulations that we have completed so far but is still fairly small. Also using Formula (17) we can calculate that $\tau = 8,300$.

6.3 Extra simulations

We have now looked in great detail at two 5-players simulations (Simulation 1 and 3). We then conducted two others (Simulation 4 and 5) to investigate the errors in strengths and win probabilities. We will now produce twelve more simulations and collate the results in Table 9 in the following subsection. This will result in a sample size large enough so we can have some confidence in our conclusions.

As we seem to get the same relationship from $\overline{D}$ and $\overline{\mathscr{D}}$ we will just focus on just $\overline{D}$. In these simulations we will continue to change the values of α_W and α_U to see what effect this has on the errors. We choose to look at seven different alpha values. This means we have a possible total of forty-nine different couples when choosing which value to have as α_W and α_U, seven where the balance relation holds and forty-two where it does not. We will look at all seven cases where the balance relation (7) holds and nine where it does not. However we will choose these nine to explore a range of different conditions such as when $\alpha_W > \alpha_U$ or the effect of just changing α_U.

Before we look at all the simulations in the next subsection, we will draw attention to two in particular where the balance relation holds:

In simulation 9, we will choose $\alpha_5 = \frac{-\ln(1/99)}{20} \approx 0.230$ which is the biggest alpha value across the simulations. It means that the strength of the players is a big factor in determining who wins the game. This would be a game where there is a high skill dependence and randomness plays a minimal role such as chess. Due to this we will have it so our strongest player has a 99% chance of beating the weakest player (strength difference of 20). This also means that when the scores vary by five the stronger player has a 76% chance of winning.

Whereas in simulation 17, we will choose alpha to be $\alpha_7 = \frac{-\ln(19/21)}{5} \approx 0.020$ which is the smallest alpha value across the simulations. Hence this alpha value has the opposite effect to the previously mentioned simulation. This means that the difference in rating has a smaller effect on which player wins than in the previous simulations. This would be when the game does not require much skill and so randomness plays a huge role. This alpha value means that when the players' strengths differ by five, then there is a 52.5% chance of the stronger player winning. This, in turn, means that when the strengths vary by twenty, the stronger player has a 59.88% chance of winning.

6.4 Discussions of results

In this subsection we will summarise the analysis we have conducted on the Elo scores process over the previous few sections. In Table 9 we collate the results from the simulations and show each of the measures we have calculated.

Table 9: Comparing the measures across the simulations

Sim	α_W	α_U	λ	$\overline{D}$	$\widetilde{E}(T^*)$	τ
1	α_1	α_1	1.025	0.18	0.003	1,626
3	α_1	α_2	0.491	3.59	0.004	26,286
4	α_2	α_2	1.027	0.21	0.005	2,442
5	α_3	α_3	1.031	0.23	0.010	8,300
6	α_1	α_3	0.335	4.71	0.007	3,560
7	α_2	α_3	0.704	2.10	0.006	11,861
8	α_4	α_4	1.013	0.12	0.002	28,629
9	α_5	α_5	1.039	0.28	0.009	4,500
10	α_5	α_4	5.772	33.74	0.004	6,000
11	α_2	α_1	2.152	8.15	0.003	4,398
12	α_3	α_2	1.527	3.73	0.005	3,184
13	α_4	α_5	0.189	5.73	0.004	1,409
14	α_3	α_1	3.218	15.67	0.002	6,557
15	α_3	α_5	0.996	0.08	0.007	11,926
16	α_6	α_6	1.017	0.14	0.002	7,379
17	α_7	α_7	1.021	0.20	0.002	1,068

Where $\alpha_1 = -\frac{\ln(1/4)}{20} \approx 0.069$, $\alpha_2 = -\frac{\ln(1/19)}{20} \approx 0.147$, $\alpha_3 = -\frac{\ln(1/3)}{5} \approx 0.219$, $\alpha_4 = -\frac{\ln(9/11)}{5} \approx 0.040$, $\alpha_5 = -\frac{\ln(1/99)}{20} \approx 0.230$, $\alpha_6 = -\frac{\ln(4/6)}{5} \approx 0.081$ and $\alpha_7 = -\frac{\ln(19/21)}{5} \approx 0.020$.

Firstly, as seen by Table 9, we have discounted Simulation 2. In this case we had two players whereas in all the others we had five players. If we were not to take this into account, then we would be making unfair comparisons. This is because at time $T = 7,500$, it is guaranteed that both players would have played 7,500 games in the two-player case. Whereas in the five-player case, each player would have played an average of 3,000 games. Hence in the two-player case they play on average more than double the number of games so we will not consider it. Ultimately we used Simulation 2 mainly to see if the empirical results from Simulation 1 could be backed up analytically, which they could.

Our simulations have specifically looked at two cases: when the balance relation holds (7) and when it does not. When looking at how well the Elo rating system determines the player's strength, it is clear that the balance relation holding makes a huge difference.

First we will look at the difference between our lambda values and their distance from 1, to measure how good the cases are at measuring the strength. We get a mean difference when the balance relation holds of 0.025, with the lambda values varying from 1.013 to 1.039. This is compared to a mean difference from 1 of 1.217, with the lambda values varying from 0.189 to 5.772, when the balance relation does not hold. This is a huge difference and shows the importance of the balance relation holding to measure the strength accurately. On top of this we can compare the measures of error, $\overline{D}$, in each case. When the balance relation holds we get a mean value of $\overline{D} = 0.19$. When it does not, we get a much higher mean of $\overline{D} = 8.43$. This again emphasises the importance of the balance relation holding when trying to determine the players' strengths accurately.

In Figure 24, we look at when the balance relation holds and what effect the magnitude of α has on lambda.

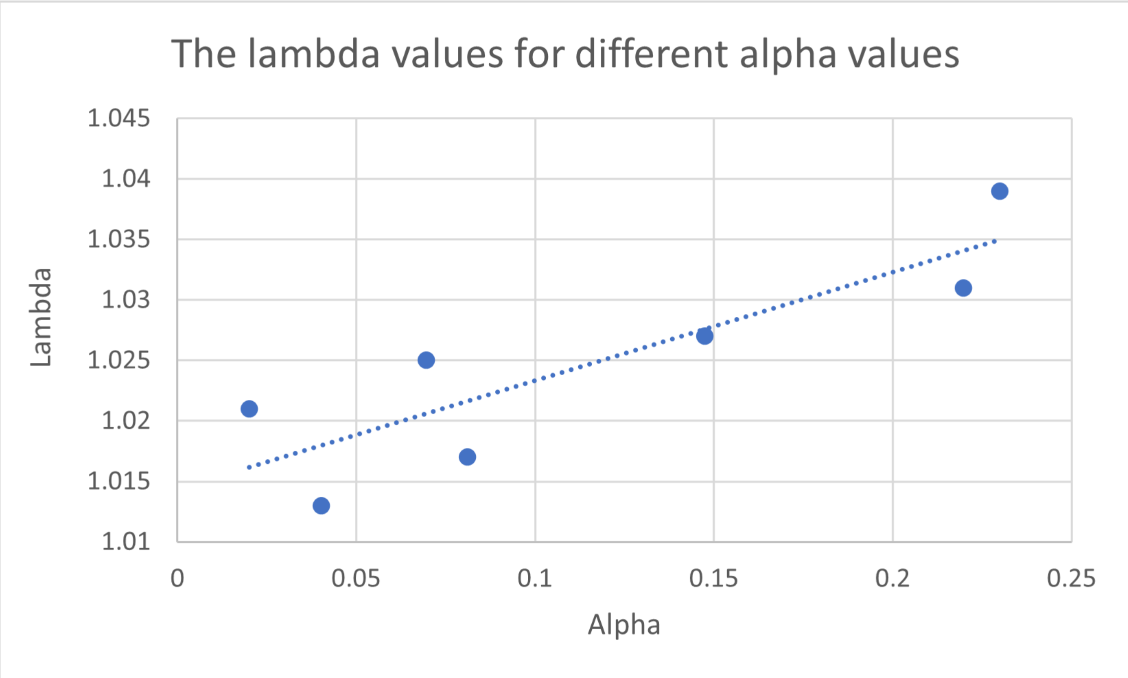

Figure 24: The value of lambda for various alphas when the balance relation (7) holds

Based on displaying our results on a scatter plot and adding a linear trend line, it is clear that there is a positive correlation between lambda and alpha. As we have previously stated, the bigger the value of lambda the larger the dilation, which is seen as a negative thing. This can be demonstrated with us plotting $\overline{D}$ at the various alpha values as seen in Figure 25.

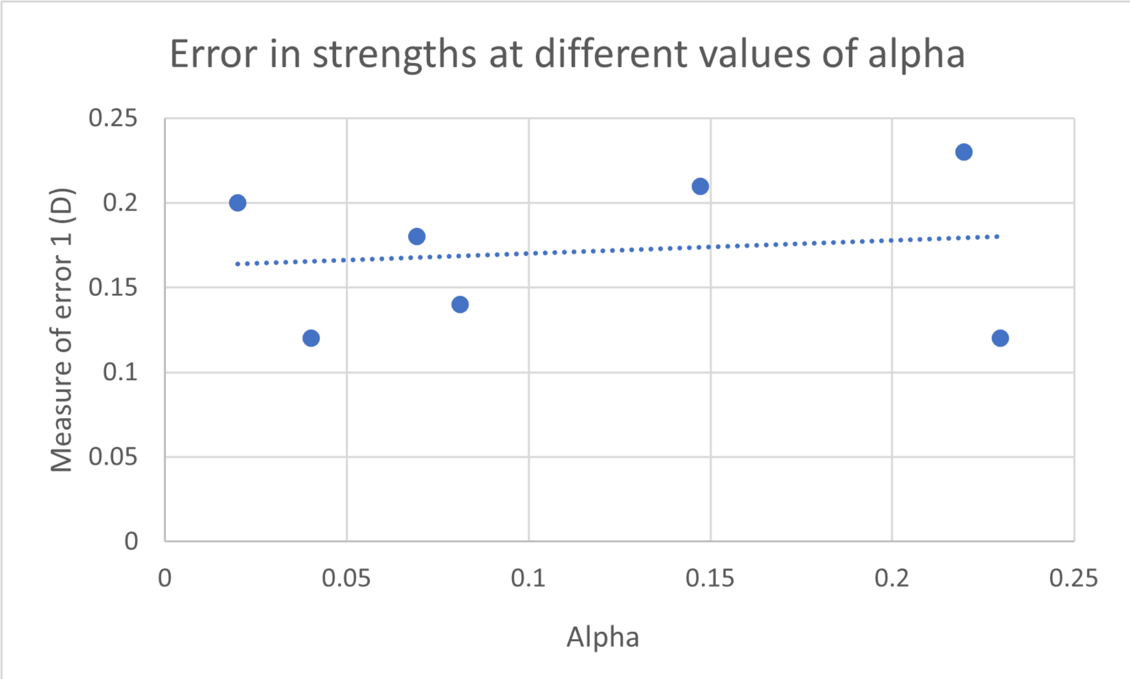

Figure 25: $\overline{D}$ at different values of alpha when the balance relation holds

We can see that there is also a positive correlation between the value of alpha and $\overline{D}$. However we see that the gradient of the linear trend line is very shallow, showing the correlation is weak. This means that as long as the balance relation holds, the error in strengths does not massively change. It makes sense for this to be a weak positive correlation by looking at the way we calculate $\overline{D}$ which is

$$\overline{D}(x, Y(T)) = \left[\frac{1}{N} \sum_{A \in \mathscr{P}} \left| \overline{Y}_A(T) - x_A \right|^2 \right]^{1/2}.$$

As $\overline{Y}_A(T) \approx \lambda x_A$ we can rewrite the formula in the following way

$$\begin{aligned}
\overline{D}(x, Y(T)) &= \left[\frac{1}{N}\sum_{A\in\mathscr{P}} \left|\overline{Y}_A(T) - x_A\right|^2\right]^{1/2} \\
&= \left[\frac{1}{N}\sum_{A\in\mathscr{P}} \left|\lambda x_A - x_A\right|^2\right]^{1/2} \\
&= \left[\frac{1}{N}\sum_{A\in\mathscr{P}} \left|x_A(\lambda - 1)\right|^2\right]^{1/2} \\
&= \left[|\lambda - 1|^2 \frac{1}{N}\sum_{A\in\mathscr{P}} |x_A|^2\right]^{1/2} \\
&= |\lambda - 1| \left[\frac{1}{N}\sum_{A\in\mathscr{P}} |x_A|^2\right]^{1/2}.
\end{aligned}$$

Hence we can see that as lambda increases then so should $\overline{D}$. However when the balance relation holds the values of lambda only range from 1.013 to 1.039 and so the effect is small, which explains why the correlation is weak.

Furthermore the value of lambda also dictates how far apart each players' scores are from each other. With a bigger lambda there is a higher chance of the ranking based on scores being the same as the ranking based on strengths, which is a benefit. This is because it is less likely for players' score trajectories to cross over each other. A bigger alpha values means the more skill-based a game is. Hence we can conclude it is more likely that the ranking based on players' scores will be the same as the ranking based on strength in a game where strength is a bigger factor in who wins, than randomness.

Next, we will look at when the balance relation does not hold. The main observation we can gain, is that when the value of alpha in the win probability function (α_W) is greater than the value of alpha in the update function (α_U), then the dilation is always above one. Whereas if the relationship is the other way round, then the dilation is sometimes below one. This can be seen in Figure 26 where we plot lambda for various values of $(\alpha_W - \alpha_U)$.

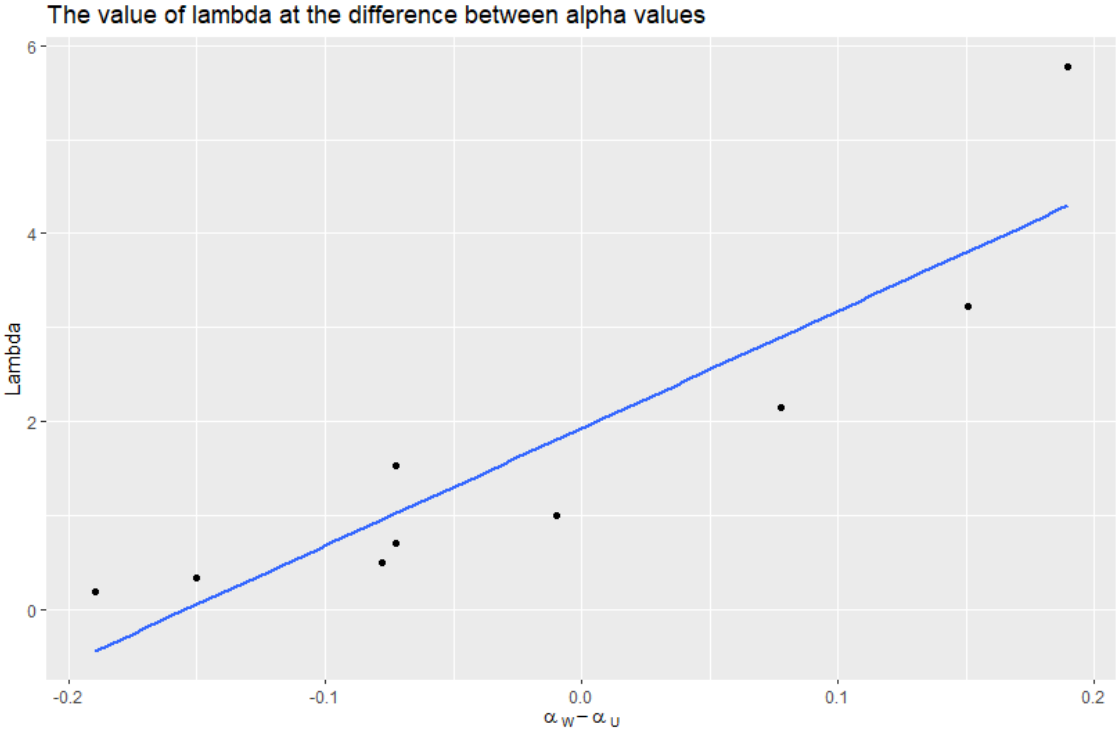

Figure 26: The value of lambda for various differences of alphas

We can see from Figure 26 that there is a positive correlation between the lambda and $\alpha_W - \alpha_U$. Hence, in the majority of cases, the bigger the absolute value of the difference, the larger the dilation. We can also see from the plot that when the difference is 0, the lambda value is between 1 and 2. When the difference is zero this means the balance relation holds. Hence this matches with Figure 24 when we were getting lambda values between 1 and 2. However the values are much closer to 1 than 2 in those simulations compared to what this trend line suggests.

This shows us the importance of the balance relation holding if we want to determine the true strengths of the players. However in reality, it is tough to make sure that the balance relation holds. As assuming that W follows a Simple Game Model, we do not know what W is. Therefore it is very unlikely that the U that we choose will be the conjugate of W, which is needed for the balance relation to hold.

We can also plot the measure of the errors of strengths ($\overline{D}$) against the difference in alpha ($\alpha_W - \alpha_U$) as seen in Figure 27.

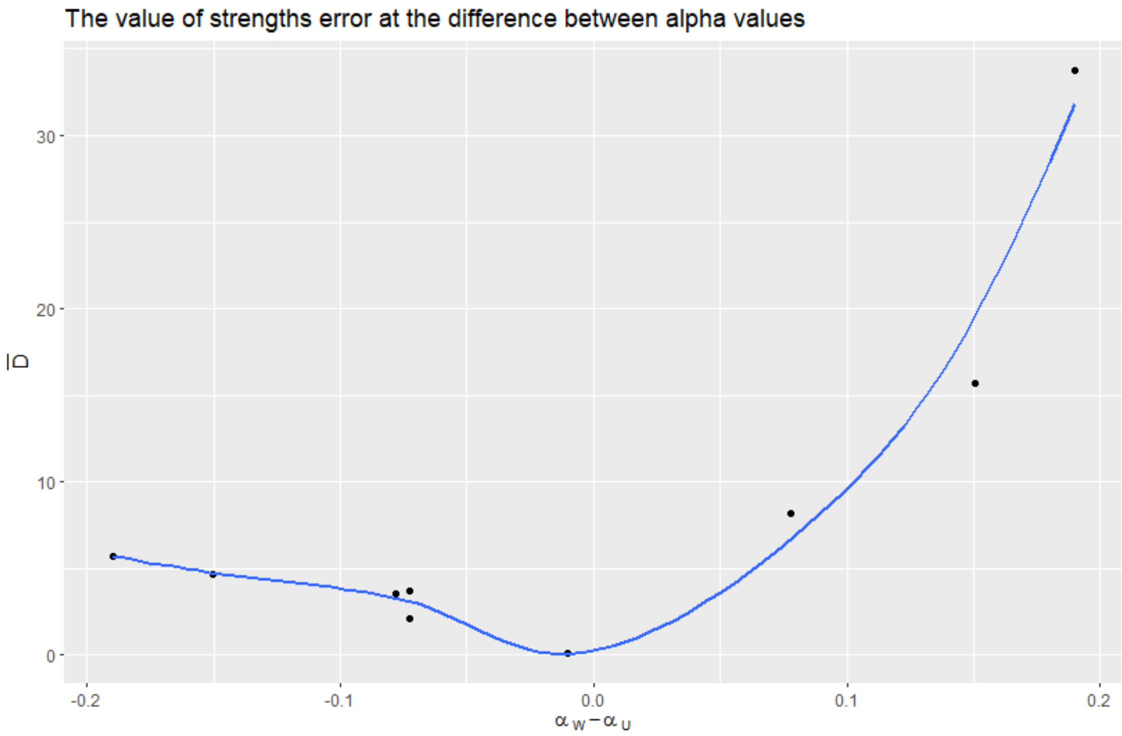

Figure 27: $\overline{D}$ at the difference of alpha values when the balance relation does not hold

We have added an exponential trend line to the plot, rather than linear, due to it being the absolute value of the difference that effects $\overline{D}$. It could be strongly argued that it is better to choose alpha too big in the update function rather than too small. One of the reasons for this is because the value of $\overline{D}$ increases far more rapidly for positive values of $(\alpha_W - \alpha_U)$ rather than negative. Also, under-specifying the alpha value in the update function compared to the win probability function can lead to a lambda being below one. This is detrimental because it leads to the scores of the players being close to each other. This means it is more likely for their score trajectories to cross over each other and so the rankings based on their scores may not reflect the ranking based on strengths.

As long as the win probabilities using these scores are still accurate then it could be strongly argued that the scores not perfectly reflecting their strengths is not very important. This is due to the fact that the Simple Game Model uses W and x together to produce a win probability and not separately. Due to this, instead of looking at the error associated with strengths, we will instead look at the error in win probabilities. The value of error in win probabilities is calculated by comparing the win probabilities using strengths, with those using the historical means at $T^* = 500,000$ in Formula (15). By looking at the values of $E(T^*)$ we see that the values are extremely low for all simulations. We will now compare the mean win probabilities errors in the simulations when the balance relation holds with when it does not, as we did with the strengths. We calculate $E(T^*) = 0.00471$ when the balance relation holds whereas it is $E(T^*) = 0.00466$ when it does not. This shows that the error associated with win probabilities is not affected by the balance relation based on our results. This is not too surprising, as we have shown in previous sections, that we seem to get a dilation in both cases between the scores and their strengths. Since the win probability is determined using W and x together, we can have $W(x_A - x_B) = W'(x'_A - x'_B)$ meaning that a dilation can lead to the same results.

Another interesting thing to look at is the effect of swapping over α_W and α_U. We have conducted four pairs of simulations where the alpha value in the win probability function is swapped with the alpha in the update function. These simulations are 3 with 11, 6 with 14, 7 with 12 and 10 with 13. It may be expected that we would see the same sized dilation in the opposite direction between the two. However this is not the case. Another assumption we may have made is that the error in win probabilities would be the same however this is also not always the case. We get the same value for $E(T^*)$ in simulation 10 and 13 but not in any of the other three.

The final thing we can investigate is the speed of convergence. We have calculated tau for each simulation, which is the time taken for the error in win probabilities to be below 0.015. To see if there's is a connection between the alpha values (α_U or α_W) and τ we look at a series of plots. Outside the report we have plotted tau against each α_U value, each α_W value and $\alpha_W - \alpha_U$ and we see no clear correlations. Hence we can conclude that based on our results, the values of alpha in the logistic functions do not effect the speed of convergence. A claim often made about Elo scores that we mentioned earlier in the report is that it takes 30 games for a score to reflect a player's strength. In David Aldous' papers he questions how accurate this claim is. Based on our results we also have reason to doubt the statement. For all our simulations, it takes far more matches in order for the win probabilities using the historical means to accurately represent the win probabilities using strengths.

7 Conclusion

This report has managed to investigate the quality of the Elo scores rating process through running simulations where we changed the update and win probability functions. We immediately noticed in our first simulation that the empirical mean rating at the time horizon did not perfectly reflect the players' strengths. We were then able to use the two-player simulation to back up our empirical results analytically.

We showed empirically that convergence in distribution "occurred". We found that the scores of each player at a time horizon had a bell-shaped distribution that was not always centred at their strength. However we showed that this was a dilation and there existed a unique λ for each simulation such that $x_A = \lambda\mathbb{E}[Y(T)]$. Showing that there was convergence in distribution was one of the report's primary objectives as we wanted to investigate the theorem that David Aldous proposed (Theorem 2.3). The intriguing part of the theorem there was no requirement for the balance relation (7) to hold. We were able to show that this was indeed the case. However, we learnt that if (7) holds, then the strengths' dilation is far smaller, leading to more minor strength errors.

Not only were we able to show that convergence in distribution occurred, we also showed that there was convergence in the time averages. This was possibly to be expected, seeing as when there is convergence in distribution for a Markov chain, there is often convergence of the time averages.

We also showed that these time averages could be used to determine the win probabilities accurately. This was the case regardless of whether the balance relation held or not. We can say that it is likely that these results would hold for all types of U and W that meet the requirements set out earlier in the report. Hence these are general conclusions for the Elo rating update process.

However in this report we exclusively kept W and U in the family of the logistic function. Due to this we were able to make specific conclusions about the Elo rating system when W and U are both variations of the logistic function. We looked at the effect of changing the alphas values in both functions. We learnt that changing these values affected the magnitude of the dilation of the players' scores away from their strengths. However we saw no connection between these alpha values and the win probabilities in terms of the size of the error or the speed of convergence.

Although we only looked at the logistic function in this report, the functions are not restricted to being of this type. As long as W meets the requirements of (1) and U complies with (3) and (4) then they are suitable functions. In future work, we could look at alternatives such as when W and U are variations of the Gaussian CDF. In this family of functions, we could look at the effect of changing the value of the standard deviation. There is no guarantee that we would get the same conclusions that we obtained changing the alpha value in the logistic functions.

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