

## Lecture 2: Data Structures

American Computer Science League, February Contest

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## 1 Introduction

In the modern era where data is being produced at a rapidly increasing pace<sup>1</sup> and being used for ever-increasing applications<sup>2</sup>, we need fast ways of storing data. Ideally, we could develop one data structure to hold all types of data and perform all operations in the quickest time possible, however, such a structure is currently nonexistent. Instead, computer scientists need to be able to determine what data structures to use in their specific problem.

For the purposes of ACSL, we will be studying four data structures in the following lecture: stacks, queues, binary search trees, and heaps/priority queues. We will go through what each of them are, how they operate, operations you can do on them, and the runtime complexity for each operation. Hopefully, you will find the complexity analysis to be an interesting intellectual endeavor. If not, you will end up seeing this stuff a lot more in intermediate–advanced CS classes in college and in job interviews, so it’s a good idea to start learning them now.

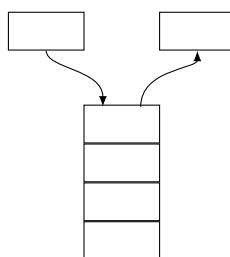
**Basics on runtime.** For the following analyses, we will be making use of Big(O) notation. If you don’t understand what expressions like  $\mathcal{O}(n \log n)$  mean, read [this Topoder article](#) for a brief introduction.

## 2 Stacks

A good mental model for a stack is to picture a stack of books. You start the school year by putting the Resnick and Halliday AP Physics textbook down, but you never touch it after. Eventually, after the AP is done and McClung or Bradford asks for the textbook, you go to search for it. However, since it is now under a pile of 5 other textbooks, it takes a lot of effort to pull out.

The book stack here works in exactly the same way as a stack data structure. It follows a “Last-In, First-Out” (LIFO) ordering, which means that the objects put in last come out first. In the textbook example, while it takes a lot of effort to pull out your physics textbook, your stats textbook (which you just put in yesterday) slides right off the top.

A visual representation of a stack follows. To sum up what we’ve been discussing, it is clear how incoming objects get placed on the top of the list, which is the same place where objects are removed.



For ACSL, we will be given one of two commands: `PUSH(element)` or `X = POP()`. `PUSH` pushes an element to the top of the stack, while `POP` removes the top element and assigns it to some variable. If the stack is empty, `POP` returns `NIL`.

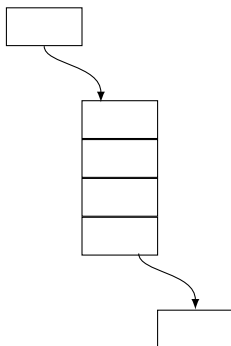
<sup>1</sup>With the Internet of Things, everything from a lightbulb to your car is generating data.

<sup>2</sup>With Machine Learning, data-driven services like Google Translate and FaceID reach billions of users each day

### 3 Queues

Similar to stacks, queues have the exact same list representation and the exact same operations, PUSH and POP. However, while stacks pop from the top of the list, queues pop from the bottom. In other words, queues have “First-In, First-Out” (FIFO) ordering.

A nice mental model for FIFO queues can be seen through the order line in a fast-food restaurant. As customers enter, they must wait till they reach the front until they can order. Once at the front, out of all the people in the queue, they are the first ones chronologically (First-In) who entered and the first ones out (First-Out). The following diagram sums everything up. Notice how we still add new items to the top. We just remove items from the bottom.



**Sample problem.** What are the values popped and the items remaining on the list if the following commands are applied to a stack and queue:

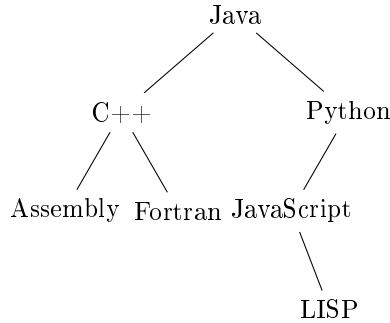
- |              |              |               |               |
|--------------|--------------|---------------|---------------|
| 1. PUSH("A") | 5. PUSH("R") | 9. X = POP()  | 13. PUSH("A") |
| 2. PUSH("M") | 6. X = POP() | 10. X = POP() | 14. PUSH("N") |
| 3. PUSH("E") | 7. PUSH("I") | 11. X = POP() |               |
| 4. X = POP() | 8. X = POP() | 12. PUSH("C") |               |

Answer: For a stack, the values popped are E, R, I, M, A, and NIL. The items remaining will be N and C, with N at the top. For a queue, the values popped are A, M, E, R, I, and NIL. The items remaining will be N, A, and C, with N at the top.

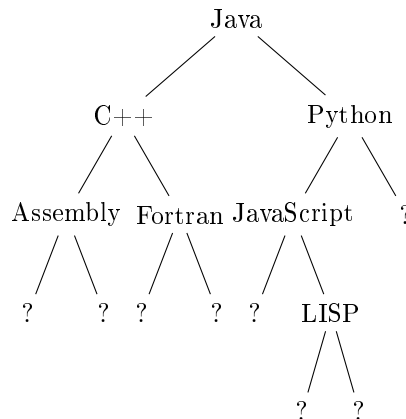
**A note on runtimes.** Both stacks and queues use Linked Lists as their underlying representations. For stacks, we store a reference to the top. For queues, we store references to both the top and bottom. Thus, when we want to push or pop, we just access the references directly, which is an  $\mathcal{O}(1)$  operation.

### 4 Binary search trees

This is where shit gets interesting... A binary search tree (BST) consists of several nodes arranged in a tree-like structure (i.e., with nodes branching off of other nodes). The following is a diagram for the BST of {Java, Python, C++, JavaScript, LISP, Assembly, Fortran} (in that order), which will help me explain some important information:



1. *Root node.* The root node forms the top of the tree. In this example, the root node is labeled “Java.”
2. *Child node.* Every node can have zero, one, or two children (nodes directly coming out of it). In the above diagram, the root node has “C++” and “Python” as its children.
3. *Ordering.* The BST is arranged in a particular structure. Namely, the left child of any node needs to have a value that is less than or equal to that node. The right child needs to have a value that is strictly greater than that node.
4. *Depth.* The root node is defined to have a depth of 0. Every subsequent node has a depth that is equal to how far away it is from the root node. For example, “C++” and “Python” have depth 1, while “LISP” has depth 3.
5. *Leaf node.* A leaf node is the most lonely node—it has no children. Here, “Assembly,” “Fortran,” and “LISP” are leaf nodes (which is quite inaccurate to reality since they pioneered several offspring languages).
6. *Internal nodes.* The set of nodes currently in the BST, {Java, Python, C++, JavaScript, LISP, Assembly, Fortran}.
7. *Internal path length.* The sum of the depths of the internal nodes,  $0 + 1 + 1 + 2 + 3 + 2 + 2 = 11$ .
8. *External nodes.* The set of nodes that could possibly be attached to the BST. I will demarcate these nodes with a “?” in the following diagram.



9. *External path length.* The sum of the depths of the hypothetical, external nodes,  $3 + 3 + 3 + 3 + 3 + 4 + 4 + 2 = 25$ . Note that the external path length can be calculated through the following formula:

$$(\text{internal path length}) + 2 * (\text{number of internal nodes}).$$

In this case, the formula would give  $11 + 2 * 7 = 25$ . We ran out of time to prove this in the 2018–19 season, so this remains an unsolved problem for future CPT members.

**BST traversal.** Traversal problems often ask for the order you would reach every single node given a certain traversal method. For the purposes of ACSL, there are three traversal methods for a BST: inorder, preorder, and postorder. Each refers to the order of traversal of the current node relative to the left and right nodes (e.g., inorder traverses left, current, right; preorder traverses current, left, right; and postorder traverses left, right, current). Given a specific traversal order, our method of determining which nodes come first will henceforth be called the *Jelinsky Method*.

The Jelinsky Method proceeds as follows:

- We write the traversal order of the Left child, the Right child, and the current node, as defined by the given traversal method.
- We cross off the next available letter and move in that corresponding direction.
- When we run out of letters, we move back up to that node's parent.
- When we cross off the C, we have “visited” the current node and add it to our list of visited nodes.

As a demonstration for the prior BST, let's assume we want to traverse in postorder. We write L, R, C next to every node and start from the root. We cross the L off of “Java” and move to its left child, “C++”. We cross off the L in “C++” and move to “Assembly”. We cross off the L and R in “Assembly” and try to move to its left and right children, but since no such children exist, we stay at “Assembly”. We cross off the C in “Assembly” and add it to our list of visited nodes. Thus, using postorder traversal, we arrive at “Assembly” first.

Next, since we have crossed off the L, R, and C in “Assembly,” we move back up to its parent, “C++”. We have already crossed off the L in “C++,” so we cross off the R and move into “Fortran.” As you can imagine, this process keeps on going until we have traversed the entire tree. Overall, our traversal order would be {Assembly, Fortran, C++, LISP, Javascript, Python, Java}.

**Searching for and inserting nodes into a BST.** The algorithms for searching and inserting into a BST are nearly identical: they both proceed down the tree comparing a given value to the nodes of the tree. Specifically, we start from the root node and do the following:

- If the given value is **less than or equal to** the current node, move to the left child.
- If the given value is greater than the current node, move to the right child.
- Keep on iterating until the value is either found (searching) or until you arrive at a place to add the node (inserting).

What's important about both of the above functions is that they use the BST's implicit ordering of the children relative to the parent node. This enables them to perform a maximum of  $d$  comparison operations, where  $d$  is the BST's depth. For a well-balanced tree (i.e., the nodes are not stacked on one side of the tree),  $d$  is approximately equal to the log base 2 of the number of nodes. This makes the running time  $\mathcal{O}(\log n)$ , where  $n$  is the number of nodes.

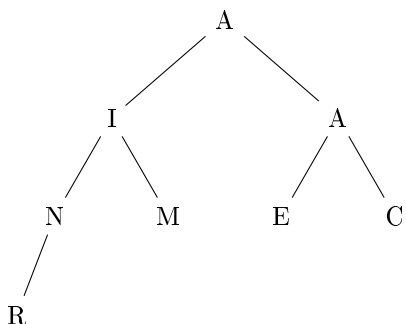
**Removing a node.** Removing is slightly more complex. To remove a node  $p$ , we first use the searching algorithm above to find it. Then, if  $p$  has no children, we simply remove it. If it has one child, we make  $p$ 's child become  $p$ 's parent's child.

The case where  $p$  has two children is the most complex: since both children could possibly have children of their own, we need to be careful. First, we make an arbitrary decision to make  $p$ 's left child become  $p$ 's parent's child. Then, we keep on moving the right branch of  $p$  down the left child's branch until we arrive at the highest node (the node furthest right) in the left branch. We know that every node in the original right branch is greater than this node, so we simply add the right branch as the right child of this node.

## 5 Heaps and priority queues

**Motivating discussion.** There are several applications where we want to find the smallest (or largest) node in a set. For example, let's say we are trying to drive from place A to place B in the shortest amount of time. We come up with estimates for the time it would take through different routes, and we constantly determine the minimum current estimate and perform updates to our estimates using new information<sup>3</sup>. In this case, both BSTs and Heaps provide  $\mathcal{O}(\log n)$  updates, but Heaps let you determine the minimum current estimate in  $\mathcal{O}(1)$  as opposed to  $\mathcal{O}(\log n)$ . To say it plainly, Heaps allow for really quick determination of the minimum element in your set.

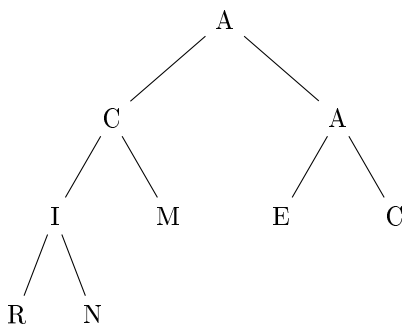
**Structure.** Heaps (also called Priority Queues) are fundamentally tree-like. Compared to BST's "Left-lesser, Right-greater" ordering, however, they enforce that every child (irrespective of left or right) has a value greater than that of the current node. This allows them to access the minimum node in  $\mathcal{O}(1)$  time. An example HEAP follows, built using the letters {A, M, E, R, I, C, A, N} (in that order).



Compared to BSTs, you may notice that the above Heap looks more completely filled. This is purposeful. Every level in the tree except the bottom-most level is completely filled. To add a new node, we go to the bottom-most level and fill from left to right. The completely filled nature of Heaps gets rid of some of the balancing problems of BSTs and ensures the depth  $d$  will always be the log base 2 of the number of nodes.

**Inserting a new node.** As mentioned above, we first insert a node in the deepest tree level from left to right. After insertion at the bottom, we “bubble” up, switching the node with its parent if it has a smaller value. Since the other child node is guaranteed to have a larger value than the parent (which in turn has a larger value than the inserted node), the Heap’s ordering is preserved under “bubbling”. Moreover, since our number of switches is at max the depth  $d$  of our tree, insertion is  $\mathcal{O}(\log n)$ !

This is what the above Heap would look like if we added the character C. Trace out the bubbling yourself to make sure you understand the process.

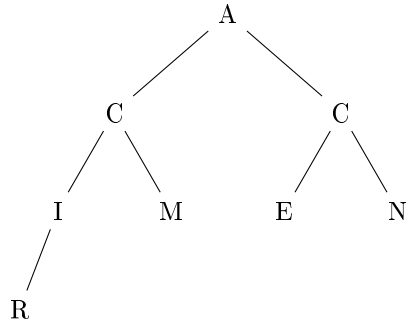


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<sup>3</sup>Such a process is used in Dijkstra's Algorithm

**Deleting the root node.** To delete the root node, we switch it with the bottom-most and right-most node and delete the old root. Then, instead of bubbling up, we bubble *down* from the new root node (previously one of the largest nodes), switching with the smaller of the two child nodes. This specific switch ensures that the other child node is still larger than its new parent node and that the Heap ordering is preserved. In addition, since we only have to perform  $d$  bubble downs, deleting the root node is  $\mathcal{O}(\log n)$ .

What follows is the result of deleting the root node from our updated Heap.



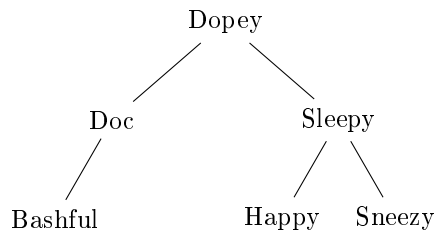
**Critical thinking.** In a min-heap (i.e., where the root node is the minimum element), what is the time complexity for finding the maximum element?

First, observe that since the children of a particular node will always be larger than it, we can find the node in the last level of the Heap. Moreover, compared to a BST where we would need to traverse all the way down the tree to get to the final level, we usually store Heaps in arrays due to their sequential and predictable algorithm for insertion and deletion. This enables us to look at the last few elements of the array to find the last level of the tree.

However, it isn't all well and good for Heaps. Since the number of nodes in the last layer is approximately equal to  $n/2$ , where  $n$  is the number of nodes, the time complexity would still be  $\mathcal{O}(n/2)$ , which reduces to  $\mathcal{O}(n)$ . This points to one of the disadvantages of Heaps. Compared to the more versatile BST which can find the minimum and maximum elements in  $\mathcal{O}(\log n)$ , Heaps are much more specialized. They perform some operations (e.g., finding the minimum value) in constant time, but take much more time for other operations compared to BSTs.

## 6 Exercises

1. When “Grumpy” is inserted into the following binary search tree, what is the new external path length of the tree?



2. Consider the following operations on an initially empty stack. If a POP is now performed, what value would be popped?
 

(a) PUSH(“B”)	(c) PUSH(“N”)	(e) PUSH(“D”)	(g) PUSH(“S”)
(b) PUSH(“T”)	(d) X = POP()	(f) X = POP()	(h) X = POP()

3. Consider an initially empty stack. After the following operations are performed, what is the value of Z?

(a) PUSH(3)

(d) Y = POP()

(g) Z = POP()

(b) PUSH(6)

(e) X = POP()

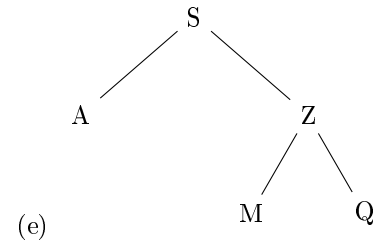
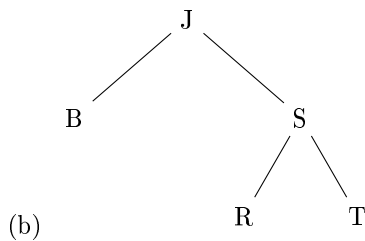
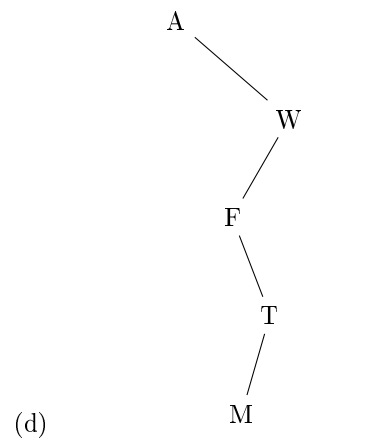
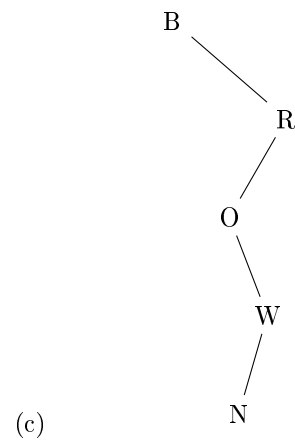
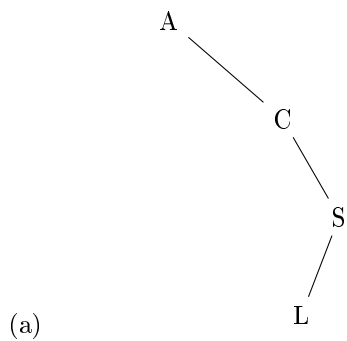
(c) PUSH(8)

(f) PUSH(X - Y)

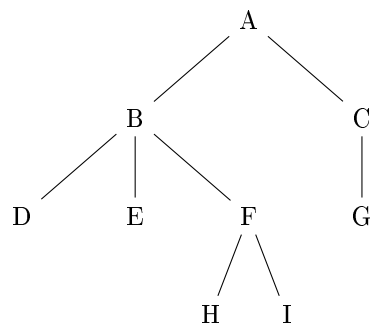
4. Create a min-heap with the letters in the word PROGRAMMING. What are the letters in the bottom-most row, from left to right?

5. Create a binary search tree from the letters in the word PROGRAM. What is the internal path length?

6. Which of the following binary trees are valid binary search trees?

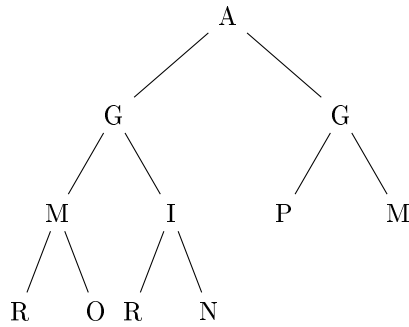


7. If one traversed the following tree in preorder, in what order would nodes be visited? Be sure your answer is neat and clear!

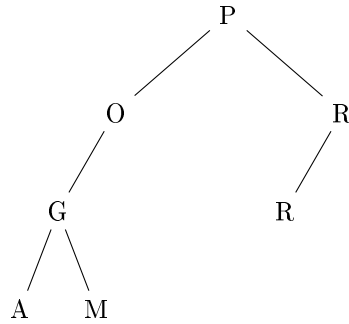


## 7 Solutions

1. 25
2. "T"
3. -2
4. Here is the min-heap. As can be seen, the bottom row contains the letters R, O, R, and N.



5. What follows is the BST. The internal path length is  $0 + 1 + 1 + 2 + 2 + 3 + 3 = 12$ .



6. Recall the "Left-lesser, Right-greater" ordering of BSTs. Using that, we get (a), (b), and (d).
7. Trees are not just limited to having 2 nodes! The answer here is A, B, D, E, F, H, I, C, G, A.