When a Straight-A Student isn't the Best: Fuzzy Ranking and Optimization from a Probabilistic Perspective

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***Abstract*—Ranking systems are typically straightforward when a deterministic order exists, but real-world scenarios often lack such clarity. In many applications, rankings are derived from aggregating diverse factors, leading to outcomes that can be controversial or subjective. To address this challenge, we propose a novel fuzzy ranking framework grounded in probability theory. Our approach represents ranks probabilistically, allowing for a more nuanced and flexible interpretation of rankings. We combine the search space constraints, regularization, initial guess and gradient descent technique to find the optimal solution. A case study demonstrates the practical utility and advantages of our method, showcasing how probabilistic fuzzy rankings can provide more robust and transparent decision-making support compared to conventional deterministic systems. This approach is particularly valuable in scenarios where traditional rankings fall short due to inherent uncertainties and variabilities in the data.**

***Index Terms*—Probability Density, Cumulative Probability Distribution, Maximum Likelihood Estimation**

I. Introduction

Ranking systems play a crucial role in various domains, including sports, academia, online platforms, and decision-making processes. These systems aim to order entities—such as teams, products, or institutions—based on aggregated scores derived from multiple factors. In ideal cases, a clear and deterministic order simplifies the task, providing an unambiguous ranking. However, in many real-world scenarios, such deterministic orderings are either impractical or misleading due to the inherent uncertainty and complexity of the factors involved. Factors contributing to a ranking often vary in significance, exhibit interdependencies, and may not be perfectly measurable, leading to controversies and debates surrounding the validity and fairness of the resulting rankings.

Traditional deterministic ranking methods, which rely on fixed score aggregations, often fail to capture the nuanced probabilistic nature of real-world data. This limitation has sparked interest in developing more flexible and accurate ranking models that reflect uncertainty and variability more effectively. Fuzzy ranking systems, which assign probabilistic weights or scores rather than absolute values, offer a promising alternative. By incorporating probability theory, such systems can better represent the inherent uncertainty in data, providing rankings that reflect a range of possible outcomes rather than a single, potentially controversial result.

In this paper, we propose a novel fuzzy ranking algorithm based on probability theory. Our method generates probabilistic rankings by considering uncertainties and variabilities within the aggregated factors. To optimize the objective function, we combine several techniques and evaluate the approach using a real-world case study. This demonstration underscores the practical relevance and potential advantages of our approach, particularly in scenarios where deterministic rankings fall short.

The rest of the paper is organized as follows: Section 2 provides a review of related work, Section 3 describes our proposed fuzzy ranking algorithm, Section 4 discusses optimization strategies, Section 5 presents our experimental evaluation, and Section 6 concludes with future research directions.

II. Related work

Fuzzy ranking algorithms have emerged as a significant area of research, particularly in addressing the limitations of traditional deterministic ranking systems. These methods aim to incorporate uncertainty, imprecision, and variability into the ranking process, making them more suitable for complex, real-world applications.

A. Fuzzy Logic-Based Approaches

Early work in fuzzy ranking systems leveraged fuzzy logic to handle imprecise and vague data. Zadeh's foundational theory of fuzzy sets [1] provided the groundwork for representing uncertainty through membership functions, enabling the development of fuzzy ranking models. These approaches use fuzzy numbers to represent scores, and defuzzification techniques to extract a final ranking. For instance, Chen and Hwang (1992) [2] proposed methods to rank fuzzy numbers by comparing the expected values and spread of fuzzy sets, allowing for more flexible decision-making.

B. Multi-Criteria Decision-Making (MCDM)

Fuzzy ranking has also been extensively explored within the context of Multi-Criteria Decision-Making (MCDM) [2]. Methods like the fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) and fuzzy AHP (Analytic Hierarchy Process) [4] extend classical MCDM frameworks by incorporating fuzzy set theory to handle subjective judgments and uncertain information. These models assign fuzzy weights to criteria and alternatives, facilitating more robust decision-making in areas such as supplier evaluation, project selection, and performance appraisal.

C. Probabilistic Fuzzy Ranking

More recently, probabilistic fuzzy ranking methods have gained attention for their ability to combine fuzzy logic with probability theory [5]. Unlike traditional fuzzy models that rely solely on membership functions, these algorithms incorporate probabilistic distributions to capture the likelihood of various ranking outcomes. Research by Torra and Narukawa (2004) [3] introduced models that integrate fuzzy sets with probabilistic frameworks, enhancing the ability to model complex systems with inherent randomness.

D. Optimization Techniques in Fuzzy Ranking

Optimization algorithms have been applied to enhance the performance of fuzzy ranking systems, particularly in handling large datasets and improving computational efficiency. Genetic algorithms, particle swarm optimization, and other metaheuristic techniques have been utilized to fine-tune fuzzy ranking parameters [6]. These optimization strategies help in dynamically adjusting weights, reducing computational complexity, and improving the accuracy of the fuzzy ranking outcomes.

E. Applications and Challenges

Fuzzy ranking algorithms have found applications in various fields, including sports ranking, financial risk assessment, and recommendation systems [7]. However, challenges remain in standardizing these methods, ensuring interpretability, and managing computational overhead. Future research continues to explore hybrid models that combine machine learning with fuzzy ranking to address these issues and expand their applicability.

F. Summary

In summary, fuzzy ranking algorithms provide a powerful tool for handling uncertainty in ranking tasks. By integrating fuzzy logic, probabilistic models, and optimization techniques, these approaches offer a more nuanced and flexible alternative to deterministic methods. This paper builds on these foundations, proposing a novel probabilistic fuzzy ranking algorithm that addresses existing limitations and demonstrates its effectiveness through real-world application.

III. The Proposed Fuzzy Ranking Algorithm

A. Distribution Rankings

Consider a set of n entities, each represented by a random variable X_i (where $i = 1, 2, \ldots, n$). Each random variable X_i follows a Gaussian (normal) distribution, characterized by its mean μ_i and standard deviation σ_i:

$$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2) \tag{1}$$

Here, μ_i represents the expected score of the i-th entity, while σ_i indicates the variability or uncertainty associated with that score. This probabilistic representation allows us to model the inherent uncertainty in the ranking process.

If all of the μ_i and σ_i values are known, then the ranking will look like the plot below,

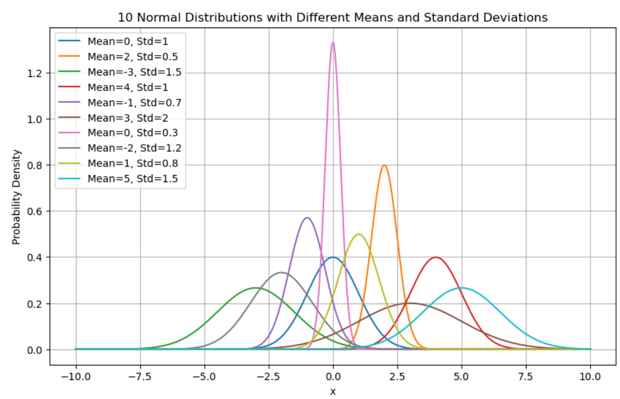

Fig. 1. Fuzzy Ranking of 10 Random Variables

This ranking result presents the ordering of the means of the 10 random variables along with their standard deviations. In a deterministic ranking system, only the mean values are considered: a variable with a higher mean is ranked higher than one with a lower mean. However, in a fuzzy ranking system, the result incorporates the full probability distribution of each variable. This means that, in some cases, a variable with a higher mean may not always be preferable to one with a lower mean. For example, a variable with a lower standard deviation might be more desirable due to its more stable performance. Conversely, a variable with a larger standard deviation might occasionally outperform others because it has a certain probability of achieving exceptionally high results.

B. Sample Rankings

A sample ranking is a system that ranks entities based on a single random sample from each. Examples include most competitions, student exams, sports games, and various evaluation methods.

In practice, many ranking systems rely on multiple sample rankings, with each sample ranking producing its own individual ranking outcome.

Consider a set of m sample rankings. The k-th ranking outcome is represented by an integer ranking function $R(x_i^{(k)})$ that returns the rank index of the k-th sample set $x_i^{(k)}$. All of the ranking outcomes are represented by

$$\{R(x_i^{(k)})\} \quad \text{for} \quad i = 1, 2, \ldots, n \quad \text{and} \quad k = 1, 2, \ldots, m \tag{2}$$

where $x_i^{(k)}$ represents a sample from the variable X_i in the k-th sample set. $R_k(x_i^{(k)})$ outputs the rank index number for $x_i^{(k)}$ in the n samples of the k-th sample set. To keep it simple, let smaller index number represent smaller x_i value

$$R(x_i^{(k)}) < R(x_j^{(k)}) \Leftrightarrow x_i^{(k)} < x_j^{(k)} \tag{3}$$

For example, one sample set from 10 random variables are,

$$(0, 2, -3, -4, -1, 3, 0.3, -2, 1, 5)$$

Their ranking outcome is the index of the sorted positions by the sample value set,

$$R(0) = 5, R(2) = 8, R(-3) = 2, R(-4) = 1, \ldots$$

So the ranking outcome is

$$(5, 8, 2, 1, 4, 9, 6, 3, 7, 10)$$

Apparently, multiple sample rankings may generate multiple inconsistent rankings.

C. Probability of Sample Rankings

In the case of lack of data, one can not accurately estimate the μ_i and σ_i values, then it is hard to generate the distribution rankings. However, in many cases, the sample rankings are available.

The objective of the proposed algorithm is to generate the distribution ranking of these n entities by finding the implied μ_i and σ_i values from the sample rankings. This approach aims to provide a more robust and flexible ranking system that accounts for variability and uncertainty in the data and handles the inconsistent sample rankings properly.

Define the k-th set of samples from each entity

$$x^{(k)} = \{x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\}$$

and their corresponding random variables X_i, each with unknown parameters (μ_i, σ_i) to be found.

The ranking outcome at the k-th sample set is,

$$R(x^{(k)}) = \{R(x_1^{(k)}), R(x_2^{(k)}), \ldots, R(x_n^{(k)})\}$$

For each pair of samples $(x_i^{(k)}, x_j^{(k)})$, the ranking outcome $(R(x_i^{(k)}), R(x_j^{(k)}))$ implies the probability distribution of $X_i - X_j$.

Since $X_i - X_j$ is also normally distributed with mean $\mu_i - \mu_j$ and variance $\sigma_i^2 + \sigma_j^2$:

$$X_i - X_j \sim \mathcal{N}(\mu_i - \mu_j, \sigma_i^2 + \sigma_j^2)$$

The probability $P(X_i > X_j)$ when $R(x_i^{(k)}) > R(x_j^{(k)})$ can be computed as

$$\begin{aligned} P(X_i > X_j) &= P(X_i - X_j > 0) \\ &= 1 - \Phi\left(\frac{\mu_j - \mu_i}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right) \\ &= \Phi\left(-\frac{\mu_j - \mu_i}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right) \end{aligned} \tag{4}$$

If $R(x_i^{(k)}) < R(x_j^{(k)})$, then

$$\begin{aligned} P(X_i < X_j) &= P(X_i - X_j < 0) \\ &= \Phi\left(\frac{\mu_j - \mu_i}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right) \end{aligned} \tag{5}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution.

If $R(x_i^{(k)}) = R(x_j^{(k)})$ in some ranking systems, then

$$\begin{aligned} P(X_i = X_j) &= P(X_i - X_j = 0) \\ &= \Phi\left(0 \times \frac{\mu_j - \mu_i}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right) \\ &= \frac{1}{2} \end{aligned} \tag{6}$$

It indicates that if X_i and X_j have the same rank in this sample ranking, then they are not distinguishable. In other words, they are symmetric, so the probability is $1/2$.

The constant result indicates that this probability doesn't rely on their μ and σ values, so we safely ignore the case of $R(x_i^{(k)}) = R(x_j^{(k)})$ in the optimization.

Combine (4) and (5) into a more general format

$$\begin{aligned} P_k(X_i, X_j) &= P_k(\text{sign}(R(x_j^{(k)}) - R(x_i^{(k)}))(X_i - X_j) < 0) \\ &= \Phi\left(\text{sign}(R(x_j^{(k)}) - R(x_i^{(k)}))\frac{\mu_j - \mu_i}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right) \end{aligned} \tag{7}$$

D. From Sample Rankings to Distribution Ranking by Maximum Likelihood Estimation [8]

Combine all the probabilities together, we get the total probability

$$\begin{aligned} &P(\mu_1, \mu_2, \ldots, \mu_n, \sigma_1, \sigma_2, \ldots, \sigma_n) \\ &= \prod_{k=1}^{m} \prod_{i=1}^{n} \prod_{j=i+1}^{n} P_k(X_i, X_j) \\ &= \prod_{k=1}^{m} \prod_{i=1}^{n} \prod_{j=i+1}^{n} \Phi\left(\text{sign}(R(x_j^{(k)}) - R(x_i^{(k)}))\frac{\mu_j - \mu_i}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right) \end{aligned} \tag{8}$$

In order to find the μ_i and σ_i values from the sample rankings, one will need to maximize the total probability (8)

$$\underset{\substack{\mu_1,\mu_2,\ldots,\mu_n\\ \sigma_1,\sigma_2,\ldots,\sigma_n}}{\arg\max}\; P(\mu_1,\mu_2,\ldots,\mu_n,\sigma_1,\sigma_2,\ldots,\sigma_n) \tag{9}$$

which is equivalent to minimize

$$\begin{aligned}&-\ln P(\mu_1,\mu_2,\ldots,\mu_n,\sigma_1,\sigma_2,\ldots,\sigma_n)\\ &= -\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\Phi\left(\operatorname{sign}(R(x_j^{(k)})-R(x_i^{(k)}))\frac{\mu_j-\mu_i}{\sqrt{\sigma_i^2+\sigma_j^2}}\right)\end{aligned} \tag{10}$$

IV. Optimization Strategies

A. Construct Objective Function

- Search Space: We want to find the distribution ranking by combining sample rankings but we don't know the range of μ_i and σ_i. We can simply set the range $[0, \mu_{\max}]$ for μ_i and the range $[0, \sigma_{\max}]$ for σ_i because we are only interested in the final distribution ranking.

$$0 \le \mu_i \le \mu_{\max}, \quad 0 \le \sigma_i \le \sigma_{\max}$$

- $L2$ Regularization: Whenever possible, the solution favors the smaller value of μ_i and σ_i. This approach is based on the assumption that an entity's average performance μ_i and standard deviation σ_i represent the minimum values satisfying (9), as entities tend to showcase their best performance in competitive settings. Under this framework, the ranks reflect the highest potential level an entity can achieve, while a minimum standard deviation indicates the most stable performance.

$$L2 = \beta_1\sum_{i=1}^{n}\mu_i^2 + \beta_2\sum_{i=1}^{n}\sigma_i^2$$

where β_1 and β_2 are positive hyper-parameters. Larger β_1 and β_2 values enourage the model to favor smaller μ_i and σ_i values. The process of setting β_1 and β_2 is often iterative. Start with an initial guess, evaluate the results, and adjust the values to favor the most stable results.
- The Objective Function

$$\begin{aligned}&L(\mu_1,\mu_2,\ldots,\mu_n,\sigma_1,\sigma_2,\ldots,\sigma_n)\\ &= -\ln P(\mu_1,\mu_2,\ldots,\mu_n,\sigma_1,\sigma_2,\ldots,\sigma_n) + L2\\ &= -\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\\ &\quad \ln\Phi\left(\operatorname{sign}(R(x_j^{(k)})-R(x_i^{(k)}))\frac{\mu_j-\mu_i}{\sqrt{\sigma_i^2+\sigma_j^2}}\right)\\ &\quad +\beta_1\sum_{i=1}^{n}\mu_i^2+\beta_2\sum_{i=1}^{n}\sigma_i^2\end{aligned} \tag{11}$$

B. Initialization

In order to efficiently find the solution, we need to estimate a good initial values for $(\mu_1,\mu_2,\ldots,\mu_n,\sigma_1,\sigma_2,\ldots,\sigma_n)$.

A reasonable guess of $(\mu_1,\mu_2,\ldots,\mu_n)$ would be the average sample ranks and then allocate in the range $[0, \mu_{\max}]$ by the average sample ranks. The average sample ranks are,

$$\bar{R}(X) = \{\bar{R}(X_1), \bar{R}(X_2), \ldots, \bar{R}(X_n)\}$$

$$\bar{R}(X_i) = \frac{1}{m}\sum_{k=1}^{m} R(x_i^{(k)})$$

where $x_i^{(k)}$ is the k-th sample of X_i.

Then the initial values of μ_i are

$$\mu_i^{(0)} = \frac{\bar{R}(X_i)}{n}\mu_{\max} \tag{12}$$

Similarly we can estimate the initial values for σ_i. The standard deviation of the sample ranks are,

$$\tilde{R}(X) = \{\tilde{R}(X_1), \tilde{R}(X_2), \ldots, \tilde{R}(X_n)\}$$

$$\tilde{R}(X_i) = \sqrt{\frac{1}{m}\sum_{k=1}^{m}\left(R(x_i^{(k)}) - \bar{R}(X_i)\right)^2}$$

Then the initial values of σ_i are

$$\sigma_i^{(0)} = \frac{\tilde{R}(X_i)}{n}\mu_{\max} \tag{13}$$

Note that $\mu_{\max}$ is in this formula because (13) is scaled by the same value as in (12).

Another simple way to initialize σ_i is to assign random numbers in uniform distribution in the range $[0, \sigma_{\max}]$

$$\sigma_i^{(0)} \sim \mathcal{U}(0, \sigma_{\max}) \tag{14}$$

C. Continuous Optimization

The gradient of $L(\mu_1,\mu_2,\ldots,\mu_n,\sigma_1,\sigma_2,\ldots,\sigma_n)$ is

$$\nabla_{\mu,\sigma}L = \left[\frac{\partial L}{\partial\mu_1},\frac{\partial L}{\partial\mu_2},\ldots,\frac{\partial L}{\partial\mu_n},\frac{\partial L}{\partial\sigma_1},\frac{\partial L}{\partial\sigma_2},\ldots,\frac{\partial L}{\partial\sigma_n}\right] \tag{15}$$

Then update μ_i and σ_i with the above gradient

$$\begin{aligned}&(\mu_1^{(t+1)},\mu_2^{(t+1)},\ldots,\mu_n^{(t+1)},\sigma_1^{(t+1)},\sigma_2^{(t+1)},\ldots,\sigma_n^{(t+1)})\\ &= (\mu_1^{(t)},\mu_2^{(t)},\ldots,\mu_n^{(t)},\sigma_1^{(t)},\sigma_2^{(t)},\ldots,\sigma_n^{(t)}) - \alpha\times\nabla_{\mu,\sigma}L\end{aligned} \tag{16}$$

where α is the learning rate.

This process repeats until it gets the optimal solution.

V. Experiments: Student Rankings

Typically, a student's performance is evaluated using their GPA, which is calculated as a grade point average based on a standard formula [9]. However, a higher GPA doesn't always indicate that one student is better than another with a lower GPA [10]. For instance, a student with a high GPA might have taken easier courses, while a student with a lower GPA may have tackled more challenging subjects. So, how can we fairly rank students based on their course final grades?

A. The Sample Data Set

Let's start with simplified final grades report as below. There are 5 students $\{X_1, X_2, X_3, X_4, X_5\}$ and their final grades in the 5 courses $\{C_1, C_2, C_3, C_4, C_5\}$, assuming each course have one credit, to be simple.

Note that some students didn't take certain courses, so they don't have grades, which is very normal. The GPAs are computed in the last column.

	C_1	C_2	C_3	C_4	C_5	GPA
X_1	A	A	A	A	B	3.80
X_2	A	A		A	C	3.50
X_3	B	A	B	C	F	2.40
X_4		B	A	B		3.00
X_5	A	A	A			4.00

TABLE I
5 Students' Grades in 5 Courses

From this report, one can tell that the best student is X_5 with GPA 4.0. However, X_5 only took three courses $\{C_1, C_2, C_3\}$. By the grade distributions in the three courses, all students got good grades (A or B), so these courses are either easy courses, or the professors are very generous in grading.

In contrast, the second best student X_1 with GPA 3.8 appears to be the actual best student for three reasons.

- This student took more courses, so this student's rank is more accurate.
- This student got As in all courses except the course C_5, which appears to be a very difficult course because the students in this course got low grades $\{B, C, F\}$.
- The course C_4 is not an easy course too. This student got A in this course.

B. The Sample Rankings

The ranking in each course can be treated as a sample ranking. By the ranking function (2), the sample rankings in each course are,

	C_1	C_2	C_3	C_4	C_5
X_1	2	2	2	3	3
X_2	2	2		3	2
X_3	1	2	1	1	1
X_4		1	2	2	
X_5	2	2	2		

TABLE II
Sample Rankings in 5 Courses

C. Construct the Objective Function

Assume that each student's performance X_i follows a Gaussian (normal) distribution, defined by its mean μ_i and standard deviation σ_i, as expressed in (1).

In the course C_1, the expression (10) becomes

$$-\ln \Phi\left(\frac{\mu_1 - \mu_3}{\sqrt{\sigma_1^2 + \sigma_3^2}}\right) - \ln \Phi\left(\frac{\mu_2 - \mu_3}{\sqrt{\sigma_2^2 + \sigma_3^2}}\right) - \ln \Phi\left(\frac{\mu_5 - \mu_3}{\sqrt{\sigma_5^2 + \sigma_3^2}}\right) \tag{17}$$

Similarly, in the course C_2, we have

$$\begin{aligned} &-\ln \Phi\left(\frac{\mu_1 - \mu_4}{\sqrt{\sigma_1^2 + \sigma_4^2}}\right) - \ln \Phi\left(\frac{\mu_2 - \mu_4}{\sqrt{\sigma_2^2 + \sigma_4^2}}\right) \\ &-\ln \Phi\left(\frac{\mu_3 - \mu_4}{\sqrt{\sigma_3^2 + \sigma_4^2}}\right) - \ln \Phi\left(\frac{\mu_5 - \mu_4}{\sqrt{\sigma_5^2 + \sigma_4^2}}\right) \end{aligned} \tag{18}$$

By following the same approach for each course, one can add all the items together to get the complete form of (10).

$L2$ regularization is,

$$L2 = \beta_1 \sum_{i=1}^{5} \mu_i^2 + \beta_2 \sum_{i=1}^{5} \sigma_i^2$$

If you want to enforce stronger regularization on the means (μ_i) compared to the standard deviations (σ_i), set $\beta_1 > \beta_2$. Conversely, if you want to prioritize controlling the variability (σ_i), set $\beta_2 > \beta_1$. Since the goal is to make informed decisions on inconsistent rankings, one can adjust the values to favor the most stable and explainable results.

D. The solution

The solution doesn't have to be the truly optimal solution since the decision-making from the fuzzy ranking isn't sensitive to the minor differences among the solutions.

$$(\mu_i, \sigma_i) = \{(8.1, 0.3), (7.4, 0.7), (3.1, 0.8), (2.8, 1.5), (6.0, 1.2)\}$$

The plot of the fuzzy ranking outcome is

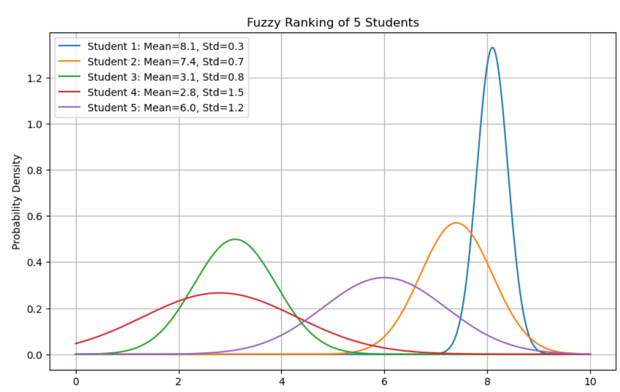

Fig. 2. Fuzzy Ranking of 5 Students

This result indicates that student X_1 has the highest mean with a small standard deviation. This conclusion is different from GPA ranking but the new conclusion is very reasonable:

student X_1 has been the top student in all the courses, regardless of the difficulty level.

In contrast, student X_5 ranks third in terms of mean values, because of the $L2$ regularization. However, since X_5 has only taken three courses, there is greater uncertainty about the performance, reflected by a much larger standard deviation. This high standard deviation implies that while X_5 has a chance of achieving the best performance, the current data doesn't provide enough certainty to confirm this.

This example shows that this method can produce rankings that differ significantly from GPA-based rankings but offer a more reasonable assessment by considering course difficulty and performance consistency. Optimization techniques, such as L2 regularization, enhance the algorithm's accuracy and stability, particularly when dealing with small or incomplete datasets.

VI. Discussion and Conclusions

The proposed fuzzy ranking algorithm introduces a novel approach to handling uncertainty and variability in ranking systems, addressing the limitations of traditional deterministic methods. By representing entities as random variables with Gaussian distributions, the algorithm provides a more nuanced ranking that considers both performance expectations (means) and variability (standard deviations).

A. The Key Advantages

This approach offers several key advantages:

- Robustness to Variability: Unlike deterministic systems that rely solely on mean values, our method incorporates the standard deviation, making it more resilient to fluctuations in performance. This is particularly useful in scenarios where the reliability of outcomes matters as much as the magnitude of the scores.
- Handling Inconsistent Data: The algorithm effectively handles inconsistent rankings by leveraging probabilistic comparisons. This is particularly beneficial in real-world situations, such as student or team evaluations, where performance data might be incomplete or noisy.
- Enhanced Decision-Making Transparency: By visualizing fuzzy ranks through probability distributions, stakeholders gain a clearer understanding of the inherent uncertainties. This transparency helps in making informed decisions, reducing controversies often associated with deterministic rankings.

B. Computational Complexity Analysis

Traditional deterministic ranking methods, such as those based on simple score aggregation (e.g., GPA calculation), involves a simple weighted average of scores, which can be done in $O(n)$ time for n entities. This makes traditional methods significantly faster and more scalable for large datasets. In contrast, the proposed fuzzy ranking algorithm, while offering more nuanced and probabilistic rankings, incurs a higher computational cost. The computational complexity of the algorithm can be broken down as follows:

- Objective Function Evaluation: The evaluation of the objective function (10) involves computing the cumulative distribution function (CDF) of the standard normal distribution (Φ) for each pair of entities in each sample ranking over m, n, and n, leading to a time complexity of $O(m \times n^2)$. This operation is computationally expensive, especially for large n and m, making the algorithm less suitable for applications involving thousands or millions of entities, such as large-scale recommendation systems.
- Gradient Computation: The gradient of the objective function (15) requires partial derivatives with respect to each μ_i and σ_i. This involves additional computations of the CDF and its derivative, further increasing the time complexity. The gradient computation also scales as $O(m \times n^2)$.
- Optimization Iterations: The optimization process typically requires multiple iterations to converge, especially for large datasets. Each iteration involves both the evaluation of the objective function and the computation of its gradient. Therefore, the overall time complexity of the optimization process is $O(T \times m \times n^2)$, where T is the number of iterations required for convergence.

In conclusion, probabilistic fuzzy ranking represents a significant advancement over traditional deterministic methods, offering a more realistic and transparent approach to handling uncertainty. Future research can expand this method to Non-Gaussian Distributions and parallelization since the algorithm's structure allows for parallelization, particularly in the evaluation of the objective function and gradient computation. Leveraging parallel computing resources, such as GPUs or distributed computing frameworks, could significantly reduce the runtime for large datasets. On the other hand, combining the proposed fuzzy ranking algorithm with traditional deterministic methods could offer a balance between accuracy and computational efficiency.

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