

파이썬으로 배우는 데이터 구조



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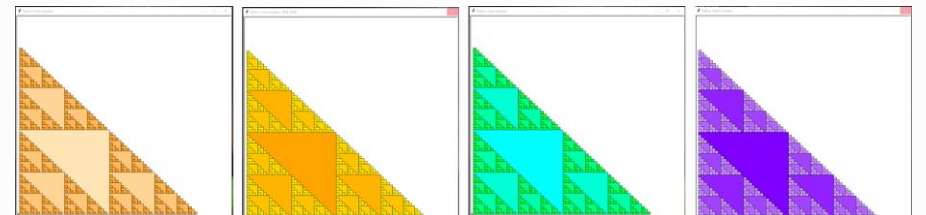
학습 목표

재귀함수를 이용한 핵심적인 예시들의
알고리즘을 이해하고 구현할 수 있다

Data Structures in Python

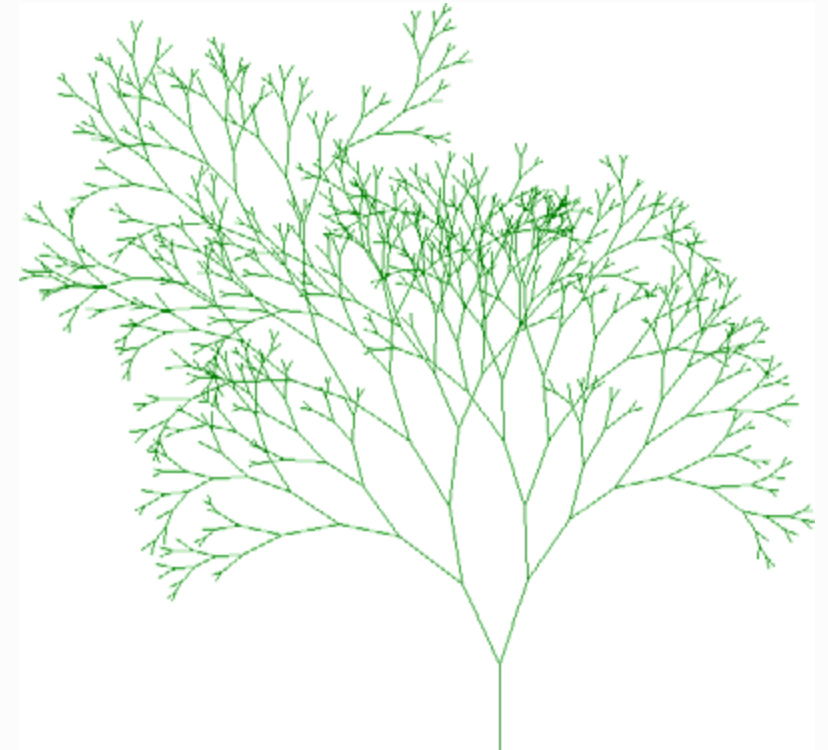
Chapter 4

- Recursion Concepts
- Recursion Stack and Memoization
- **Recursive Algorithms**
- Recursive Graphics
- Exercise - Stacking boxes



Agenda

- Recursion and Stack
- More Examples and Algorithms
 - Radix Conversion
 - The Fibonacci Sequence
 - The Towers of Hanoi



Radix Conversion

- Radix is the base of number representation.
- Examples:
 - Decimal, 10
 - Binary, 2
 - Octal, 8
 - Hexadecimal, 16

Decimal	Binary	Octal	Hexadecimal
20	10100 ₂	24 ₈	14 ₁₆
7	111 ₂	7 ₈	7 ₁₆
32	100000 ₂	40 ₈	20 ₁₆

Radix Conversion

- Radix conversion by division from larger base to a smaller base.
- Example: Convert a decimal number into other bases
 - $\text{radix}(99, 2)$ 1100011
 - $\text{radix}(99, 3)$ 10200
 - $\text{radix}(99, 4)$ 1203
 - $\text{radix}(99, 5)$ 344
 - $\text{radix}(99, 6)$ 243
 - $\text{radix}(99, 7)$ 201
 - $\text{radix}(99, 8)$ 143
 - $\text{radix}(99, 9)$ 120

Radix Conversion

- Radix conversion from other bases to decimal
 - Digits are multiplied by powers of the base or 10, 8, 2, or whatever.

- Decimal numbers multiply digits by powers of 10

$$9507_{10} = 9 \times 10^3 + 5 \times 10^2 + 0 \times 10^1 + 7 \times 10^0$$

- Octal numbers - power of 8

$$\begin{aligned} 1567_8 &= 1 \times 8^3 + 5 \times 8^2 + 6 \times 8^1 + 7 \times 8^0 \\ &= 512 + 320 + 48 + 7 = 887_{10} \end{aligned}$$

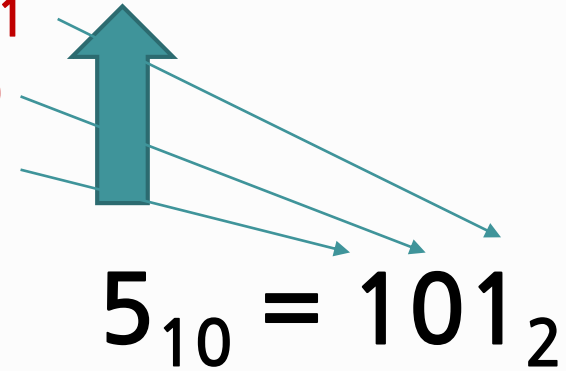
- Binary numbers - power of 2

$$\begin{aligned} 1101_2 &= 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= 8 + 4 + 0 + 1 = 13_{10} \end{aligned}$$

Radix Conversion Example:

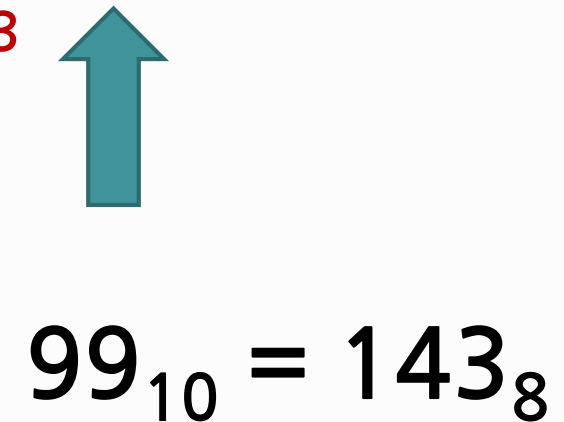
- Convert 5 from base 10 to base 2.

1. Divide 5 by new base 2, then quotient 2 and **remainder 1**
2. Divide quotient 2 by 2, then quotient 1 and **remainder 0**
3. Divide quotient 1 by 2, then quotient 0 and **remainder 1**
Stop when the quotient is 0.


$$5_{10} = 101_2$$

- Convert 99 from base 10 to base 8.

1. Divide 99 by new base 8, then quotient 12 and **remainder 3**
2. Divide quotient 12 by 8, then quotient 1 and **remainder 4**
3. Divide quotient 1 by 8, then quotient 0 and **remainder 1**
Stop when the quotient is 0.


$$99_{10} = 143_8$$

Possible Solutions:

- We could either
 - store remainders in a list by appending.
 - must continue the output until we get the quotient = 0
 - reverse the list
 - return the result as a compact string from the list.
- Iterative Algorithm
 - while the decimal number > 0
 - Divide the decimal number by the new base.
 - Set the decimal number = decimal number divided by the base.
 - Store the remainder to the left of any preceding remainders.

Recursive Algorithm

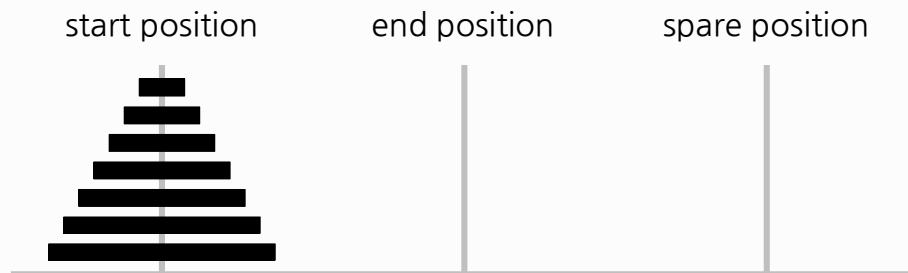
- Base case:
 - if decimal number == 0
 - do nothing (or return "")
- Recursive case
 - if decimal number > 0
 - solve a simpler version of the problem
 - use the quotient as the argument to the next call
 - store the current remainder (number % base) in the correct place

```
def radix(num, base):  
    if num == 0:  
        return ''  
    return radix(num//base, base) + str(num % base)
```

Note: This code does not convert a decimal to a hexadecimal.
It is left as an exercise.

The Towers of Hanoi

- The famous towers of Hanoi consists of n discs and three poles.
 - The discs are of different size and have holes to fit themselves on the poles.
 - Initially all the discs are on one pole, e.g., pole A.
 - The task is to move all n discs to another pole, while obeying the following rules.
 - Move only one disc at a time.
 - Never place a larger disc on a smaller one.
 - One legend says that the world will end when a certain group of monks accomplishes this task in a temple with 64 golden discs on three diamond needles. But how can the monks accomplish the task at all, playing the rules?
 - To solve the problem, **our goal is to issue a sequence of instructions for moving the discs.**

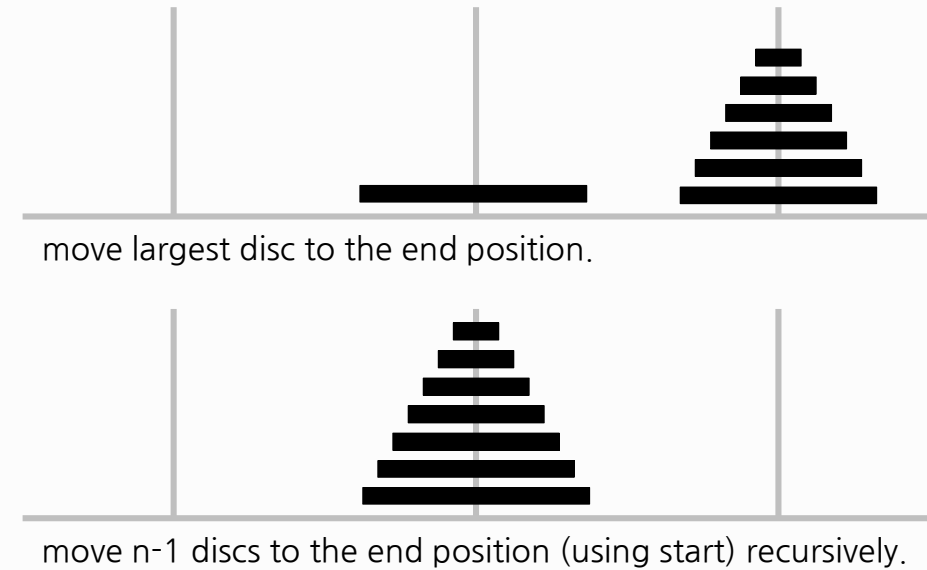
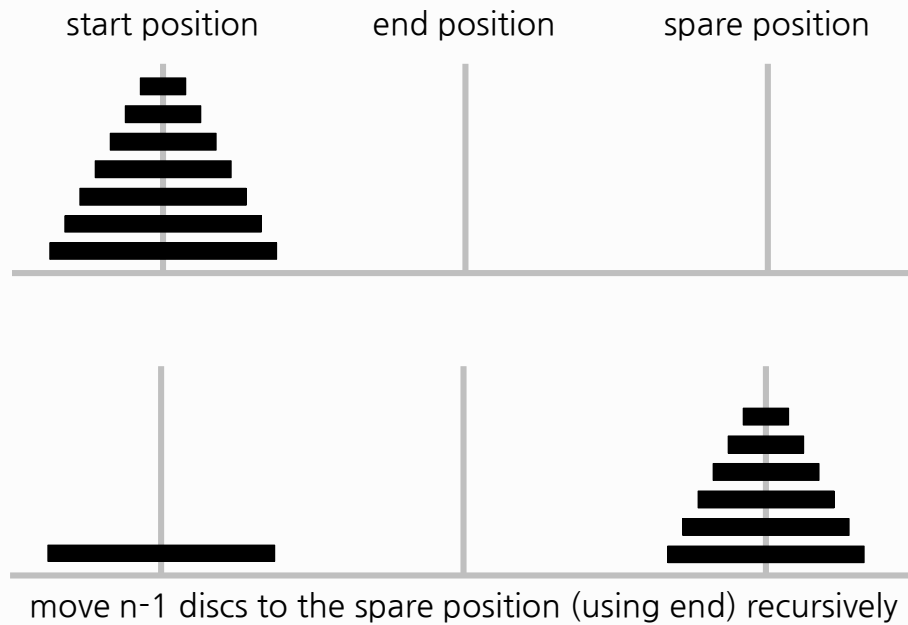


The Towers of Hanoi

- Examples:
 - <https://www.youtube.com/watch?v=q6RicK1FCUs>
 - <https://sikaleo.tistory.com/29> (한국어)

The Towers of Hanoi

- Recursive algorithm:
 1. Move the top **n-1** discs from **start** to **spare** (using **end**), recursively.
 2. Move the **remaining (largest)** disc from **start** to **end**.
 3. Move the **n-1** discs from **spare** to **end** (using **start**), recursively.

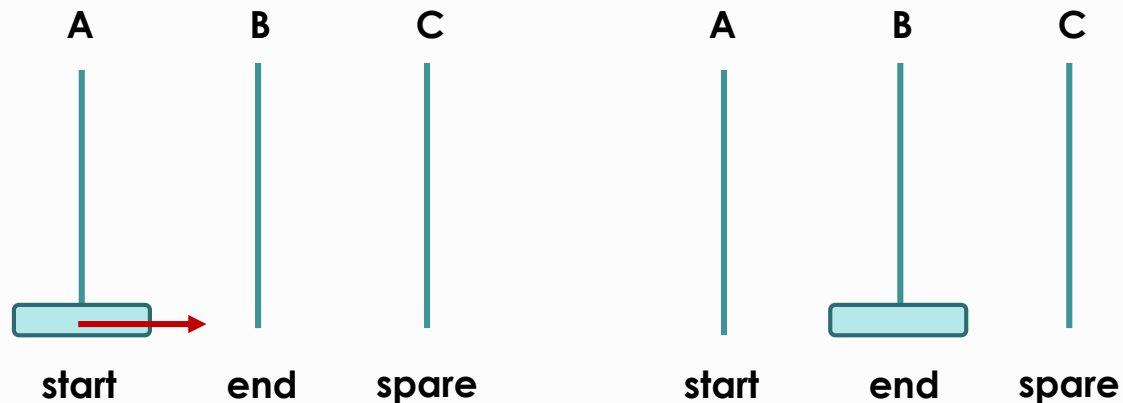


The Towers of Hanoi

- Recursive algorithm:
 1. Move the top **n-1** discs from **start** to **spare** (using **end**), recursively.
 2. Move the **remaining (largest)** disc from **start** to **end**.
 3. Move the **n-1** discs from **spare** to **end** (using **start**), recursively.

One disc case:

(1) move a disc from A to B.

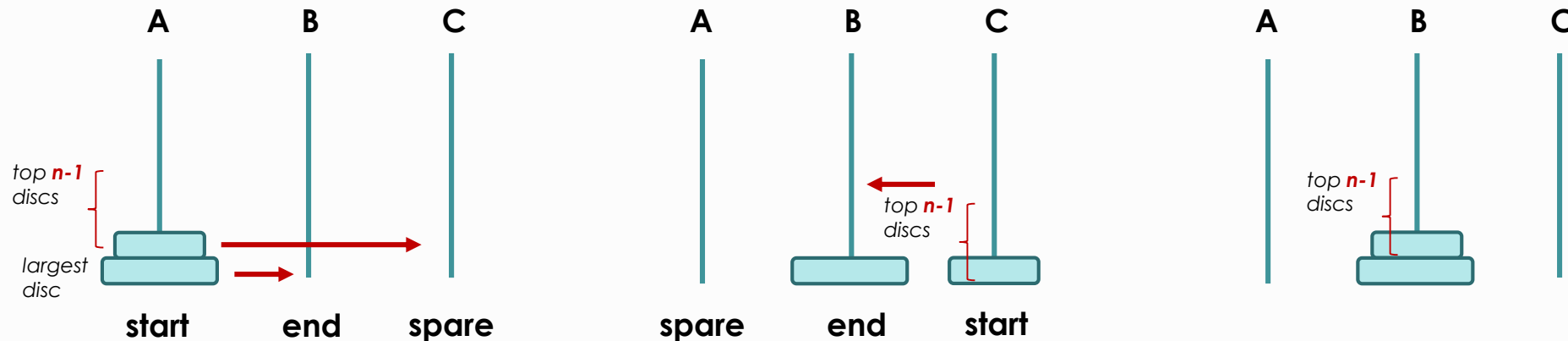


The Towers of Hanoi

- Recursive algorithm:
 1. Move the top $n-1$ discs from **start** to **spare** (using **end**), recursively.
 2. Move the **remaining (largest)** disc from **start** to **end**.
 3. Move the $n-1$ discs from **spare** to **end** (using **start**), recursively.

Two disc case:

- (1) move a disc from A to C **using B**. ← since it is not the end(or destination)
- (2) move a disc from A to B.
- (3) move a disc from C to B **using A**. ←



1 disc case

The Towers of Hanoi

Three discs case:

- (1) move **two discs** from A to C **using B**.
- (2) move a disc from A to B.
- (3) move **two discs** from C to B **using A**

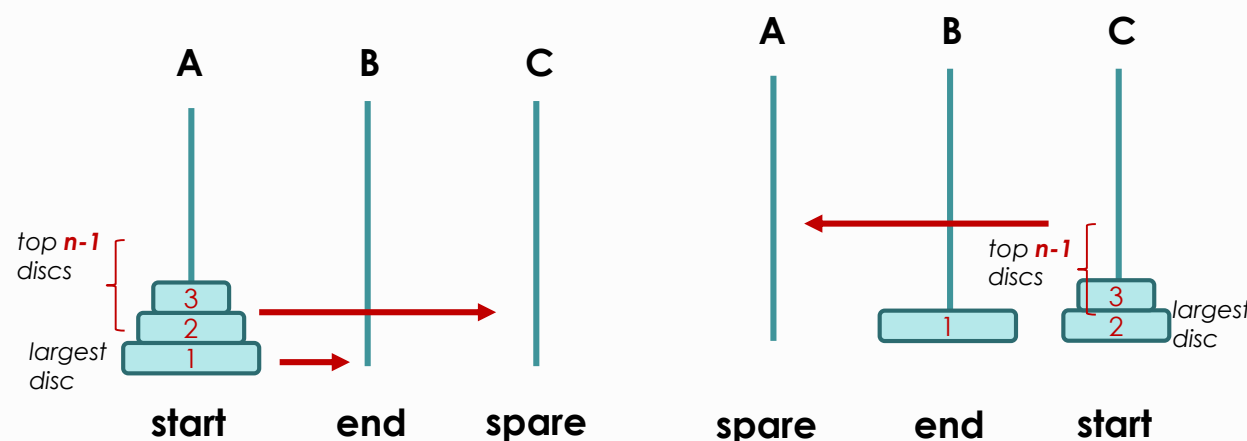
for n discs

for n discs:

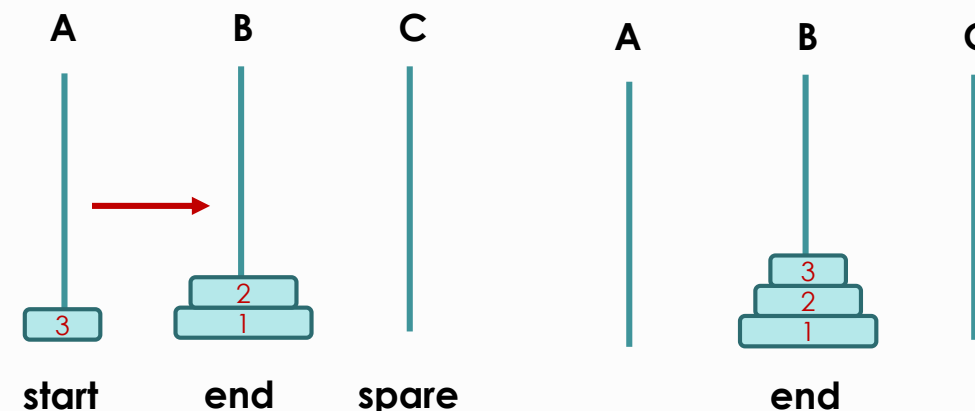
- (1) move **n - 1 discs** from A to C **using B**.
- (2) move a disc from A to B.
- (3) move **n - 1 discs** from C to B **using A**

This is a recursive step.
We already have done this two discs case before.

```
def hanoi(n, start, end, spare):  
    if n >= 1:  
        hanoi(n - 1, start, spare, end)  
        print(f"move disc {n} from {start} to {end}")  
        hanoi(n - 1, spare, end, start)  
if __name__ == '__main__':  
    hanoi(3, 'A', 'B', 'C')
```



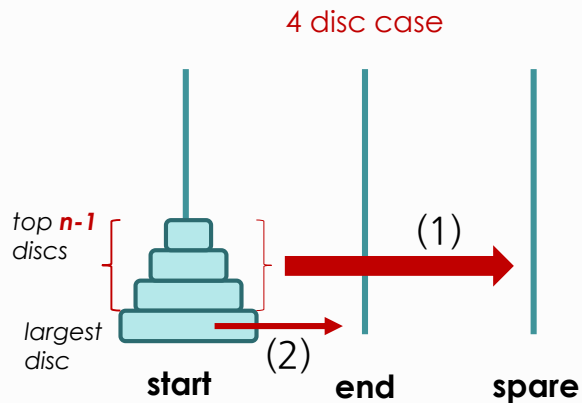
2 disk case



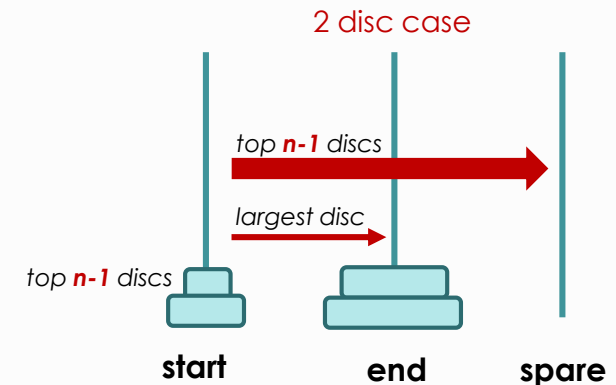
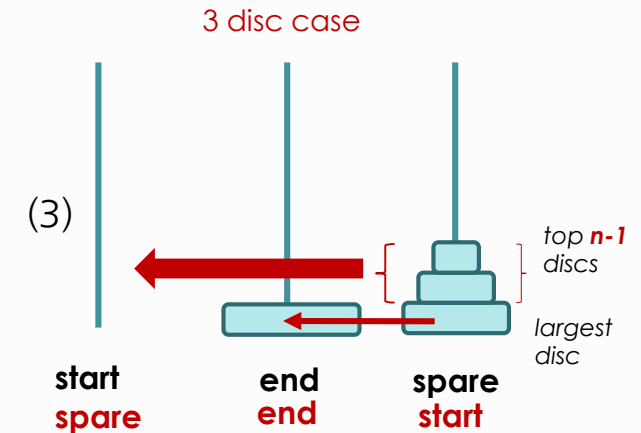
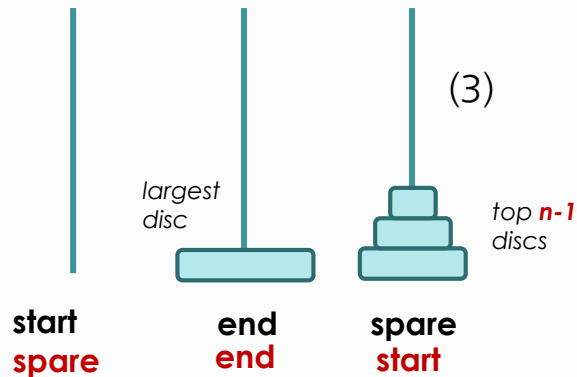
1 disk case

The Towers of Hanoi

- Recursive algorithm:
 1. Move the top $n-1$ discs from **start** to **spare** (using **end**), recursively.
 2. Move the **remaining (largest)** disc from **start** to **end**.
 3. Move the $n-1$ discs from **spare** to **end** (using **start**), recursively.



(3) It becomes a **3 disc case**.
Go back to step 1.
Treat the **spare as start** and the **start as spare**.



The Towers of Hanoi - Algorithm

- Question: How many moves and recursive calls made?

```
def hanoi(n, start, end, spare):  
    if n >= 1:  
        hanoi(n - 1, start, spare, end)  
        print(f"move disc {n} from {start} to {end}")  
        hanoi(n - 1, spare, end, start)  
  
if __name__ == '__main__':  
    hanoi(3, 'A', 'B', 'C')
```

move disc 1 from A to B

move disc 2 from A to C

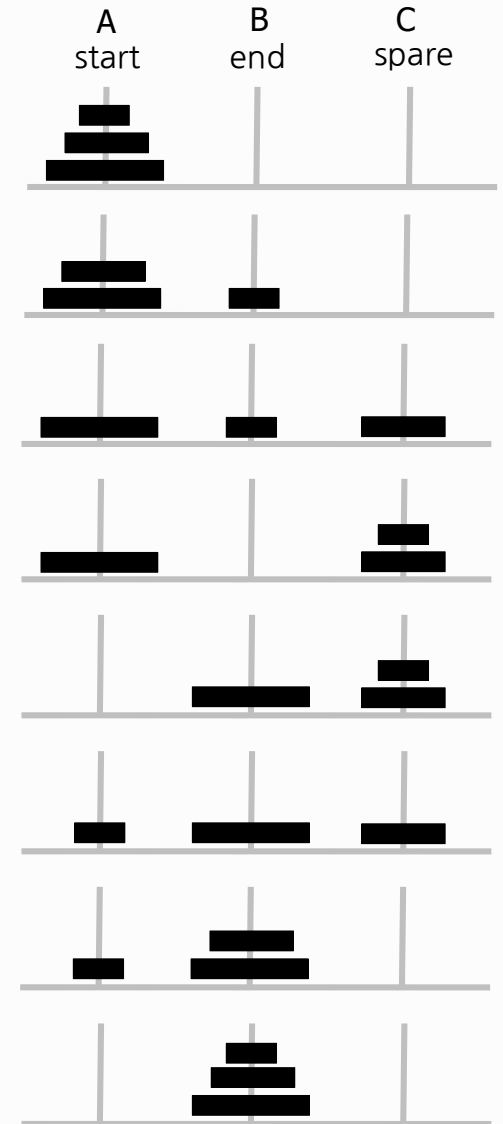
move disc 1 from B to C

move disc 3 from A to B

move disc 1 from C to A

move disc 2 from C to B

move disc 1 from A to B



The Towers of Hanoi - Coding Exercise

- Idea: It is hard to check the correctness of the previous `hanoi()`.
 - Let us use a list to present a disc in a pole and display the result as shown below. The number in a list represents the size of the disc. `tower()` prints the current status of the tower in a list format. Test the cases such as $n = 1, 2, 3, 4, 5, 6$.

```
def hanoi(n, start, end, spare):
```

```
    if n >= 1:
```

```
        None
```

```
def tower(A, B, C):
```

```
    print(None)
```

```
if __name__=='__main__':
```

```
    n = 3
```

```
    A = [* range(1, n+1)]
```

```
    B = []
```

```
    C = []
```

```
    tower(A, B, C)
```

```
    hanoi(n, A, B, C)
```

```
start-[1, 2, 3]
```

```
start-[2, 3]
```

```
start-[3]
```

```
start-[3]
```

```
start-[]
```

```
start-[1]
```

```
start-[1]
```

```
start-[]
```

```
end-[]
```

```
end-[1]
```

```
end-[1]
```

```
end-[]
```

```
end-[3]
```

```
end-[3]
```

```
end-[2, 3]
```

```
end-[1, 2, 3]
```

```
spare-[]
```

```
spare-[]
```

```
spare-[2]
```

```
spare-[1, 2]
```

```
spare-[1, 2]
```

```
spare-[2]
```

```
spare-[]
```

```
spare-[]
```

The Towers of Hanoi - Time complexity

- Recursive algorithm:
 - Move the top $n-1$ discs from start to spare.
 - Move the remaining (largest) disc from start to end.
 - Move the $n-1$ discs from spare to end.

$$\text{hanoi}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 \cdot \text{hanoi}(n-1) + 1 & \text{if } n > 1 \end{cases}$$

Exercise: How many years will take to move 64 discs?

- (1) $\text{hanoi}(1) = 1$
- (2) $\text{hanoi}(2) = 3$
- (3) $\text{hanoi}(3) = 7$
- (4) $\text{hanoi}(4) = 15$
- (5) $\text{hanoi}(5) = 31$
- (6) $\text{hanoi}(32) = 4,294,967,295$
- (7) $\text{hanoi}(64) = 18,446,744,073,709,600,000$

hanoi(n = 4)
$\begin{aligned} \text{hanoi}(4) &= 2 \cdot \text{hanoi}(3) + 1 \\ &= 2 \cdot (2 \cdot \text{hanoi}(2) + 1) + 1 \\ &= 2 \cdot (2 \cdot (2 \cdot \text{hanoi}(1) + 1) + 1) + 1 \\ &= 2 \cdot (2 \cdot (2 \cdot 1 + 1) + 1) + 1 \\ &= 2 \cdot (2 \cdot (3) + 1) + 1 \\ &= 2 \cdot (7) + 1 = 15 \end{aligned}$

The Towers of Hanoi - Time complexity

- Solving the recurrence equation of the Hanoi Tower.

- $T(n) = 2T(n-1) + 1$
 $T(n-1) = 2T(n-2) + 1$
 $T(n-2) = 2T(n-3) + 1$

$T(n)$ can be rewritten some substitutions

$$\begin{aligned} T(n) &= 2(2(2T(n-3) + 1) + 1) + 1 \\ &= 2^3 T(\textcolor{red}{n-3}) + 2^2 + 2^1 + 1 \end{aligned}$$

...

Expand this $T(n)$ until it has $T(\textcolor{red}{n-k})$ term since we know $T(\textcolor{red}{1}) = 1$.

After generalization

- $T(n) = 2^k T(\textcolor{red}{n-k}) + 2^{k-1} + 2^{k-2} + \dots + 2^2 + 2^1 + 1$

Since base condition $T(1) = 1$, and then $\textcolor{red}{n} - \textcolor{red}{k} = 1$, $k = n - 1$

- Replace k with $k = n - 1$.
 - $T(n) = 2^{n-1} T(0) + 2^{n-2} + 2^{n-3} + \dots + 2^2 + 2^1 + 1 = 2^n - 1$
 - The time complexity is $O(2^n)$
 - For 5 discs, $n = 5$, it will take $2^5 - 1 = 31$ moves.

The Towers of Hanoi - Time complexity

- Write a recursive function to compute the number of disc's move first. Then compute the number of years to move 64 discs, while assuming that a group of monks really work diligently to move the disc fast like a computer clock speed or one disc per nano second (10^{-9} sec). Show your code and computation below:

```
def hanoi(n):  
    if n <= 1:  
        return 1  
    return _____  
  
if __name__ == '__main__':  
    for n in [1, 2, 3, 4, 5, 10, 32, 64]:  
        t = hanoi(n)  
        print(n, t)
```

```
1 1  
2 3  
3 7  
4 15  
5 31  
10 1023  
32 4294967295  
64 18446744073709551615  
35096.55 years
```

Summary

- Recursion simplifies program structure at a cost of function calls (using the system stack).
- Understand and learn how to implement the recursive functions for different applications.

학습 정리

- 1) Radix 변환은 변환하고자 하는 수를 밑수(base, radix)로 계속 나누는 방식으로 작동된다
- 2) 하노이 탑(Hanoi Tower) 문제는 재귀 용법으로 접근하면 간단히 풀 수 있는 전형적인 문제이다

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수고했습니다
곧 다음 시간에
다시 뵙겠습니다

