

파이썬으로 배우는 데이터 구조



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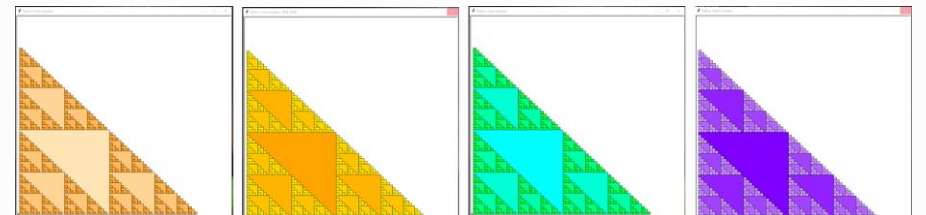
학습 목표

재귀(Recursion)의 개념을 이해하고
간단한 예제들을 통해 깊이 학습한다

Data Structures in Python

Chapter 4

- Recursion Concepts
- Recursion Stack and Memoization
- Recursive Algorithms
- Recursive Graphics
- Exercise - Stacking boxes



Agenda

- Recursion Definition
 - Definitions and Programming
 - Why recursion?
 - Concept Example
 - More Examples

Recursion Definition

- See Recursion

TOP DEFINITION

recursion

See recursion.

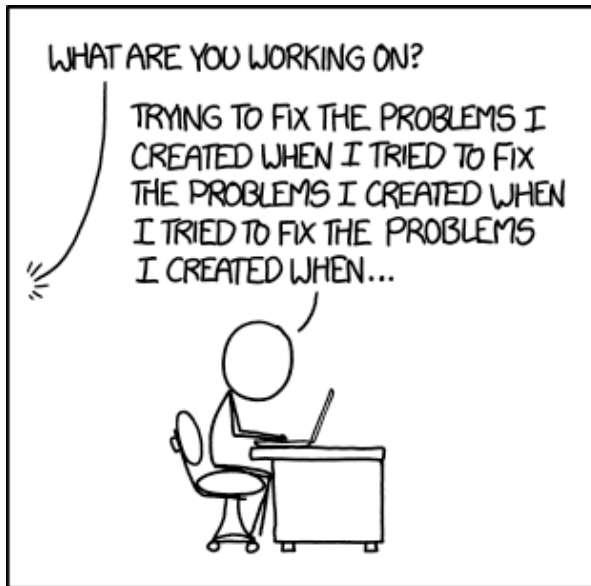
by **Anonymous** December 05, 2002

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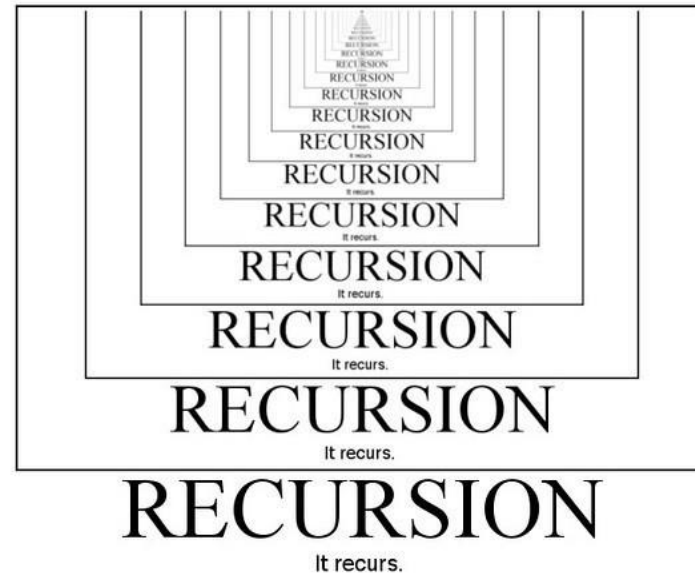
Very descriptive definition

Recursion Definition

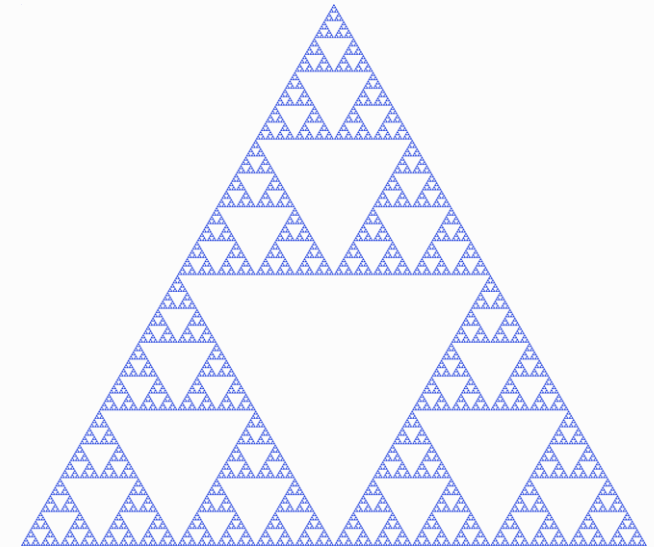
- See Recursion
- Recursion is when a function calls itself
- Recursion simplifies program structure at a cost of function calls



<https://www.xkcd.com/1739/>



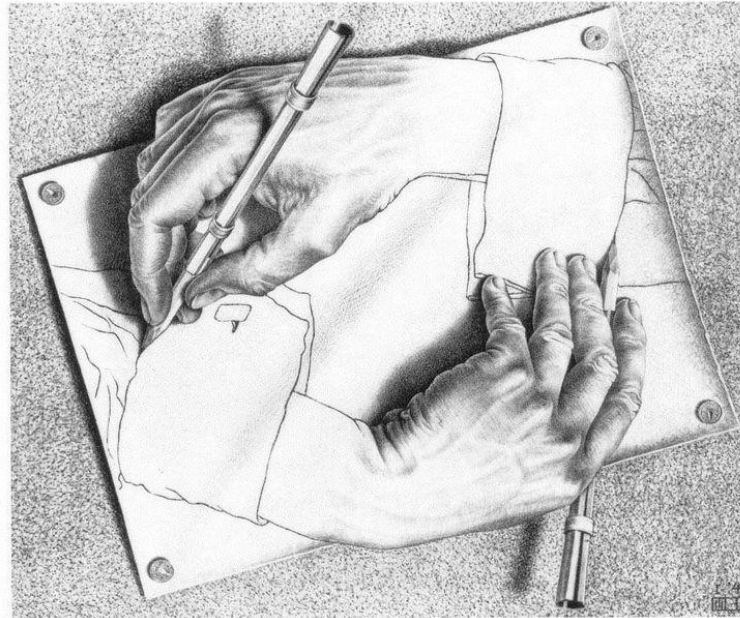
Recursion — Image from AlgoDaily



The **Sierpinski triangle**
a confined recursion of triangles that form a fractal

Recursion Definition

- See Recursion
- Recursion is when a function calls itself
- Recursion simplifies program structure at a cost of function calls
- Recursion vs. Leap of faith



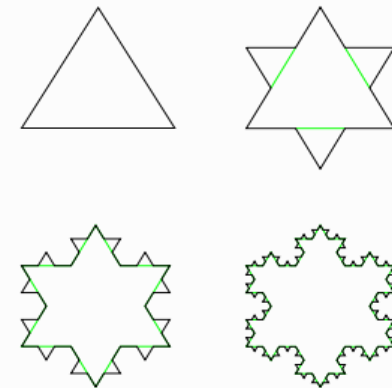
***recursion is
when a function calls itself***

Why recursion?

- A new "cultural experience"
 - A different way of thinking of problems or creative thinking
- It can solve some kinds of problems better than iteration.
- It leads to elegant, simplistic, and short code (when used well).
- Believe it or not, there are some programming languages ("functional" languages such as Scheme, ML, and Haskell) use recursion exclusively (no loops)
- This skill is a key component of the rest of our course.

Recursion

- Recursion is a method where the solution to a problem depends on solutions to smaller instances of the same problem (as opposed to iteration).
- Recursive algorithm is expressed in terms of
 1. **base case(s)** for which the solution can be stated non-recursively,
 2. **recursive case(s)** for which the solution can be expressed in terms of **a smaller version of itself**.



Four stages in the construction of a **Koch snowflake**. The stages are obtained via a recursive definition.

Concept Example

- Pick one of students in the row and ask:
How many students total are next you in your "row"?
 - You have poor vision, so you can see only the people next to you. So, you can't just look the sides and count.
 - But you are allowed to ask questions of two persons next to you.
 - How can we solve this problem, recursively?



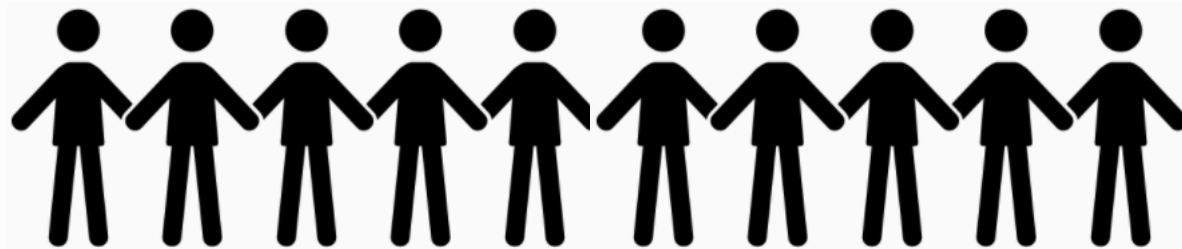
Concept Example: pass the buck

- Number of people on the both sides of me:
 - If there is someone to the left side of me, ask him/her how many people are to the left size of him/her.
 - Do the same to the right side of me.
 - When they respond with a value L from the left and R from the right, then I will answer $L + R + 1$.
- If there is nobody both side of me, I will answer 1.



Recursion and cases

- Every recursive algorithm involves at least 2 cases:
 - **base case:** A simple occurrence that can be answered directly.
 - **recursive case:** A more complex occurrence of the problem that cannot be directly answered but can instead be described in terms of smaller occurrences of the same problem.
- Some recursive algorithms have more than one base or recursive case, but all have at least one of each.
- A crucial part of recursive programming is identifying these cases.



Example 1: Factorial

- Recurrence relation: A mathematical formula that generates the terms in a sequence from previous terms.
 - $\text{factorial}(n) = n * [(n-1) * (n-2) * \dots * 1]$
 - $\text{factorial}(n) = n * \text{factorial}(n-1)$
- Recursive definition of $\text{factorial}(n)$:
 - $$\text{factorial}(n) = \begin{cases} 1, & \text{if } n = 0 \\ n * \text{factorial}(n - 1), & \text{if } n > 0 \end{cases}$$
- Examples:
 - $4! = 4 * 3 * 2 * 1 = 24$
 - $7! = 7 * 6 * 5 * 4 * 3 * 2 * 1 = 5040$

Example 1: Factorial

- Recursive definition of factorial(n)
 - $factorial(n) = \begin{cases} 1, & \text{if } n = 0 \\ n * factorial(n - 1), & \text{if } n > 0 \end{cases}$

factorial(n)

```
function factorial
input: integer  $n$  such that  $n \geq 0$ 
output: [ $n \times (n-1) \times (n-2) \times \dots \times 1$ ]
    1. if  $n$  is 0, return 1
    2. otherwise, return [ $n \times factorial(n-1)$  ]
end factorial
```

factorial($n = 4$)

```
 $f_4 = 4 * f_3$ 
      = 4 * (3 *  $f_2$ )
      = 4 * (3 * (2 *  $f_1$ ))
      = 4 * (3 * (2 * (1 *  $f_0$ )))
      = 4 * (3 * (2 * (1 * 1)))
      = 4 * (3 * (2 * 1))
      = 4 * (3 * 2)
      = 4 * 6
      = 24
```

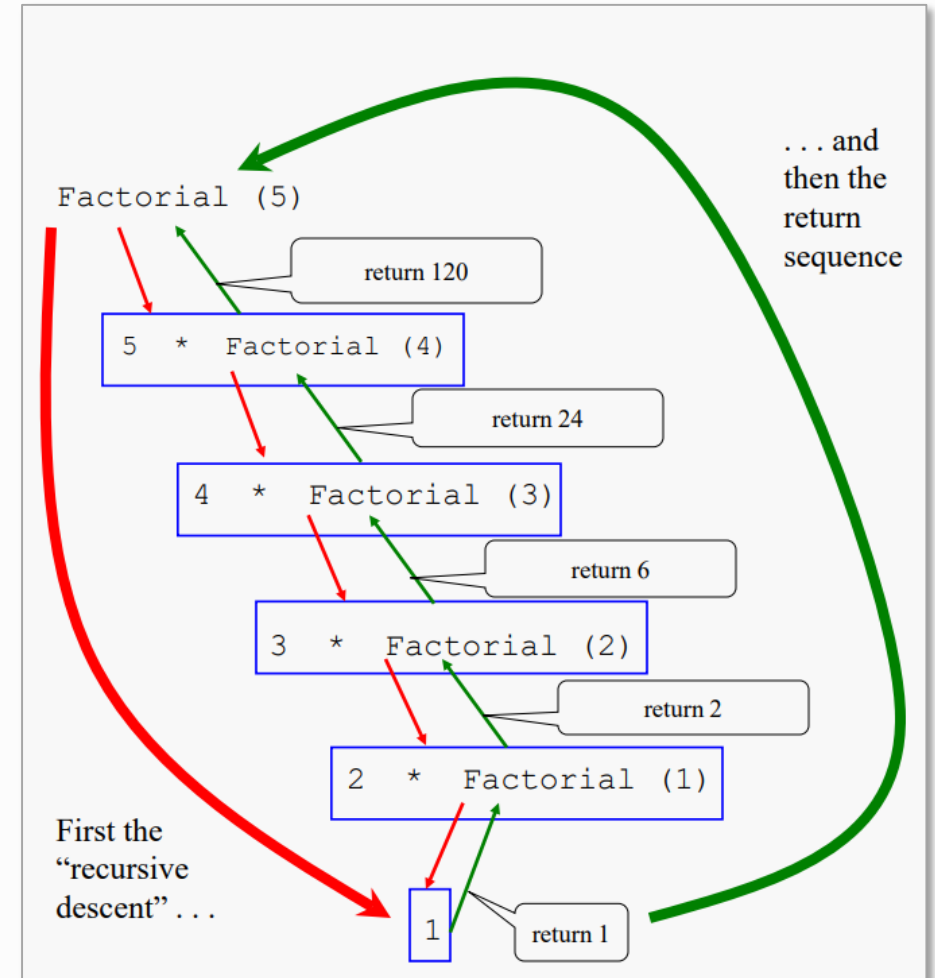
Exercise: With four students, compute 4! using recursion.

Example 1: Factorial

- Recursive definition of factorial(n)
 - $$factorial(n) = \begin{cases} 1, & \text{if } n = 0 \\ n * factorial(n - 1), & \text{if } n > 0 \end{cases}$$

`factorial(n)`

```
function factorial
input: integer  $n$  such that  $n \geq 0$ 
output:  $[n \times (n-1) \times (n-2) \times \dots \times 1]$ 
  1. if  $n$  is 0, return 1
  2. otherwise, return  $[n \times factorial(n-1)]$ 
end factorial
```



Exercise: With four students, compute $4!$ using recursion.

Example 2: print_stars()

- Consider the following function to print a line of * characters:

```
def print_stars(n):  
    """prints a line containing the given number of stars.  
    precondition: n >= 0 """  
    for i in range(n):  
        print('*', end='')  
    print()
```

- Write a recursive version of this method (that calls itself).
 - Solve the problem **without using any loops**.
 - Hint: Your solution should print just one star at a time.

Example 2: print_stars()

- What are the cases to consider?
 - What is a very easy number of stars to print without a loop?

```
def print_stars(n):  
    if n == 1:  
        print('*')  
    else:  
        ...  
    print()
```

Example 2: print_stars()

- Handling additional cases, with no loops (**in a wrong way**):

```
def print_stars(n):  
    if n == 1:  
        print('*')  
    elif n == 2:  
        print '**'  
    elif n == 3:  
        print '***'  
    ...  
    else:  
        ...  
    print()
```

Example 2: print_stars()

- Taking advantage of the repeated pattern (**somewhat better**):

```
def print_stars(n):  
    if n == 1:  
        print('*')  
    elif n == 2:  
        print_stars(2)  
    elif n == 3:  
        print_stars(3)  
    ...  
    else:  
        ...  
    print()
```

Example 2: Using recursion properly

- Condensing the recursive cases into a single case:

```
def print_stars(n):  
    if n == 1:                # base case:  
        print('*')           # print one star  
    else:                     # recursive case:  
        print('*', end='')    # print one and more stars  
        print_stars(n - 1)
```

Example 2: Using recursion properly

- The real, even simpler base case is **an n of 0, not 1**:

```
def print_stars(n):  
    if n == 0:                # base case:  
        print()              # end the output  
    else:                     # recursive case:  
        print('*', end='')    # print one and more stars  
        print_stars(n - 1)
```

Bad Recursion Example 1

- Problem:
 - Compute the sum of all integers from 1 to n

```
def bad_sum(n):  
    return n + bad_sum(n-1)
```

No base case!!!

Bad Recursion Example 2

- Problem:
 - If n is odd, compute the sum of all odd integers from 1 to n ; and if it is even compute sum of all even integers.

```
def bad_sum(n):  
    if n == 0:  
        return 0  
    return n + bad_sum(n-2)
```

Base case cannot be reached!!!

Example 3

- Predict the output of the following code.

```
def foo(n):  
    if n < 1: return  
    print(n, end = ' ')  
    foo(n - 1)  
    print(n, end = ' ')  
    return  
  
if __name__ == '__main__':  
    n = 3  
    foo(n)
```


- When main() calls foo(3), main() and **n=3** are pushed to the system stack. It will finish when foo(3) returned.

Example 3

- Predict the output of the following code.

```
def foo(n):  
    if n < 1: return  
    print(n, end = ' ')  
    foo(n - 1)  
    print(n, end = ' ')  
    return  
  
if __name__ == '__main__':  
    n = 3  
    foo(n)
```

```
def foo(n=3)  
    if n < 1: return  
    print(n, end=' ')  
    foo(2)  
    print(n, end=' ')  
    return
```



- When main() calls foo(3), main() and **n=3** are pushed to the system stack. It will finish when returned from foo(3).
- foo(3) prints '3'** and calls foo(2). Then, **foo(2)** and **n=2** are pushed to the system stack.

Example 3

- Predict the output of the following code.

```
def foo(n):  
    if n < 1: return  
    print(n, end = ' ')  
    foo(n - 1)  
    print(n, end = ' ')  
    return
```

```
if __name__ == '__main__':  
    n = 3  
    foo(n)
```

```
def foo(n=3)  
    if n < 1: return  
    print(n, end=' ')  
    foo(2)  
    print(n, end=' ')  
    return
```

foo(3) calls foo(2)

```
def foo(n=2)  
    if n < 1: return  
    print(n, end=' ')  
    foo(1)  
    print(n, end=' ')  
    return
```

foo(2) calls foo(1)

```
def foo(n=1)  
    if n < 1: return  
    print(n, end=' ')  
    foo(0)  
    print(n, end=' ')  
    return
```

foo(1) calls foo(0)

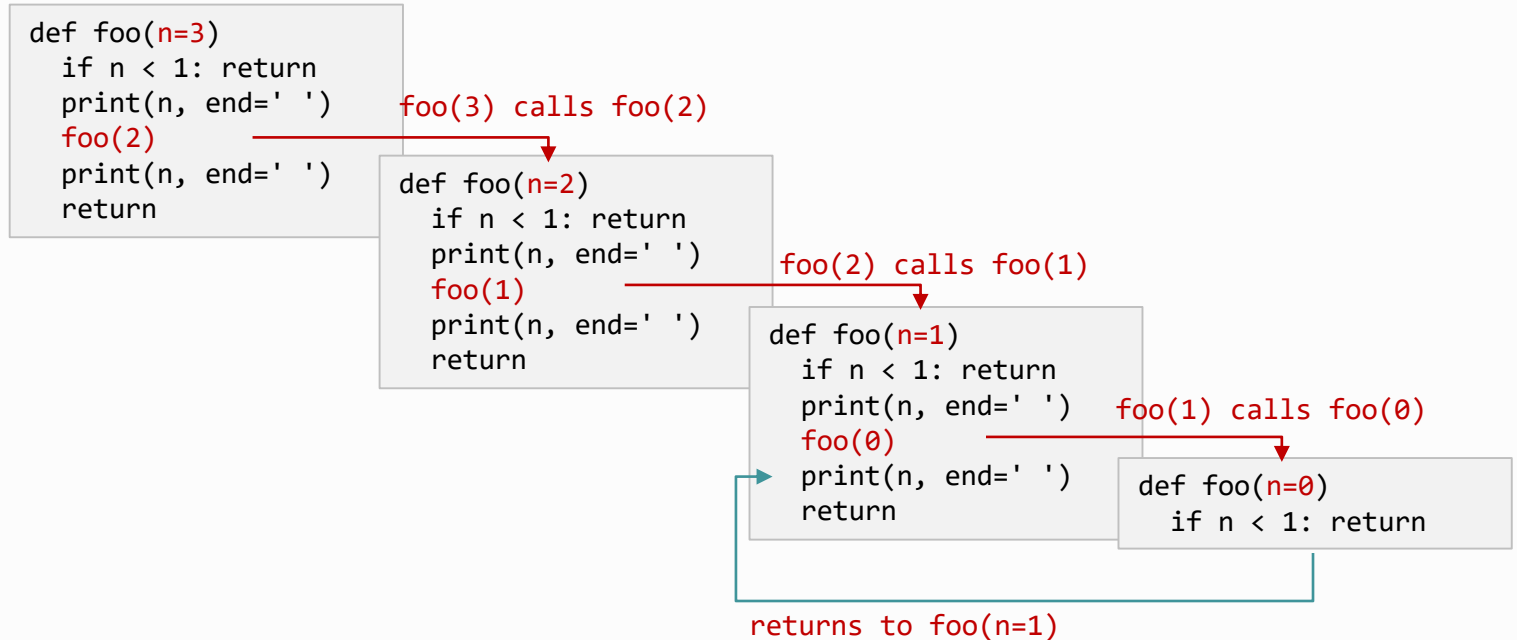
```
def foo(n=0)  
    if n < 1: return
```

- When main() calls foo(3), main() and **n=3** are pushed to the system stack. It will finish when returned from foo(3).
- foo(3) prints '3'** and calls foo(2). Then, **foo(2)** and **n=2** are pushed to the system stack.
- Similarly, **foo(2) prints '2'** and calls foo(1). Then **foo(1) prints '1'** and calls foo(0).

Example 3

- Predict the output of the following code.

```
def foo(n):  
    if n < 1: return  
    print(n, end = ' ')  
    foo(n - 1)  
    print(n, end = ' ')  
    return  
  
if __name__ == '__main__':  
    n = 3  
    foo(n)
```

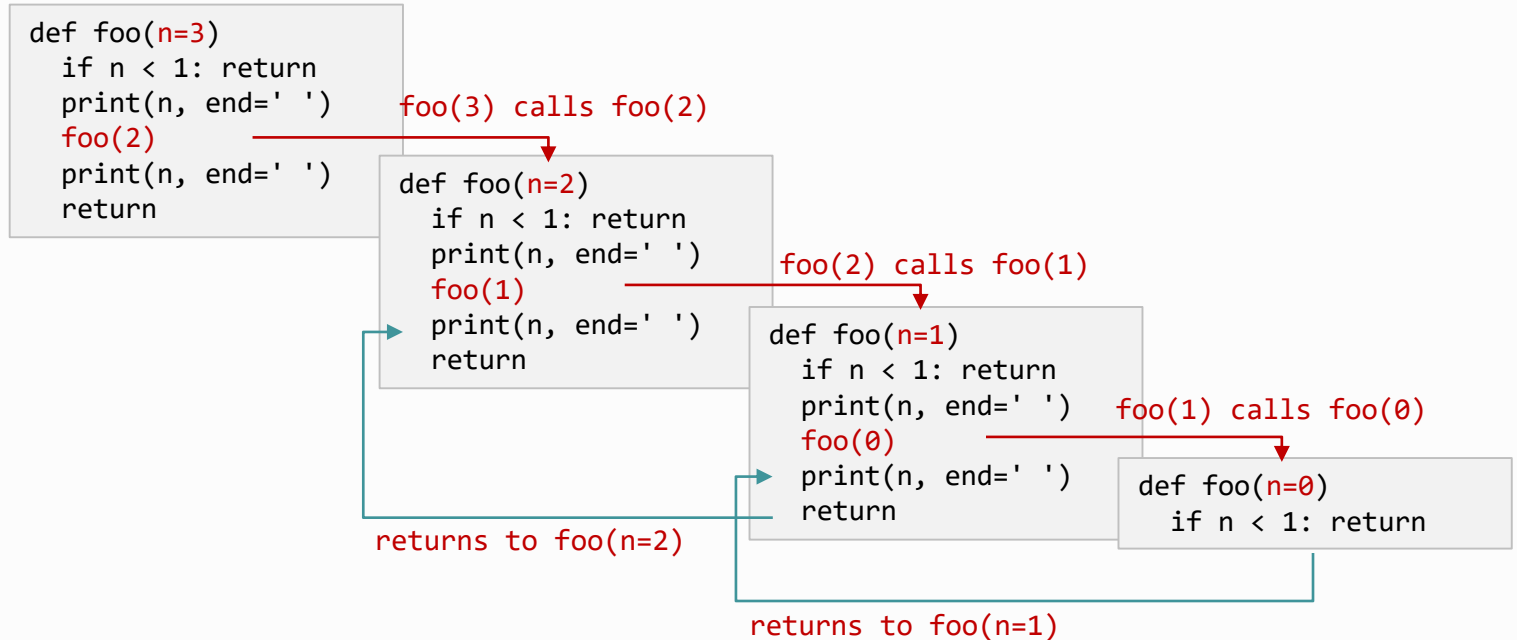


- When `main()` calls `foo(3)`, `main()` and `n=3` are pushed to the system stack. It will finish when returned from `foo(3)`.
- foo(3) prints '3'** and calls `foo(2)`. Then, **foo(2)** and `n=2` are pushed to the system stack.
- Similarly, **foo(2) prints '2'** and calls `foo(1)`. Then **foo(1) prints '1'** and calls `foo(0)`.
- `foo(0)` goes to `if` and returns to `foo(1)`. This is the first return ever. Now **foo(1)** popped from the stack **prints '1'**.

Example 3

- Predict the output of the following code.

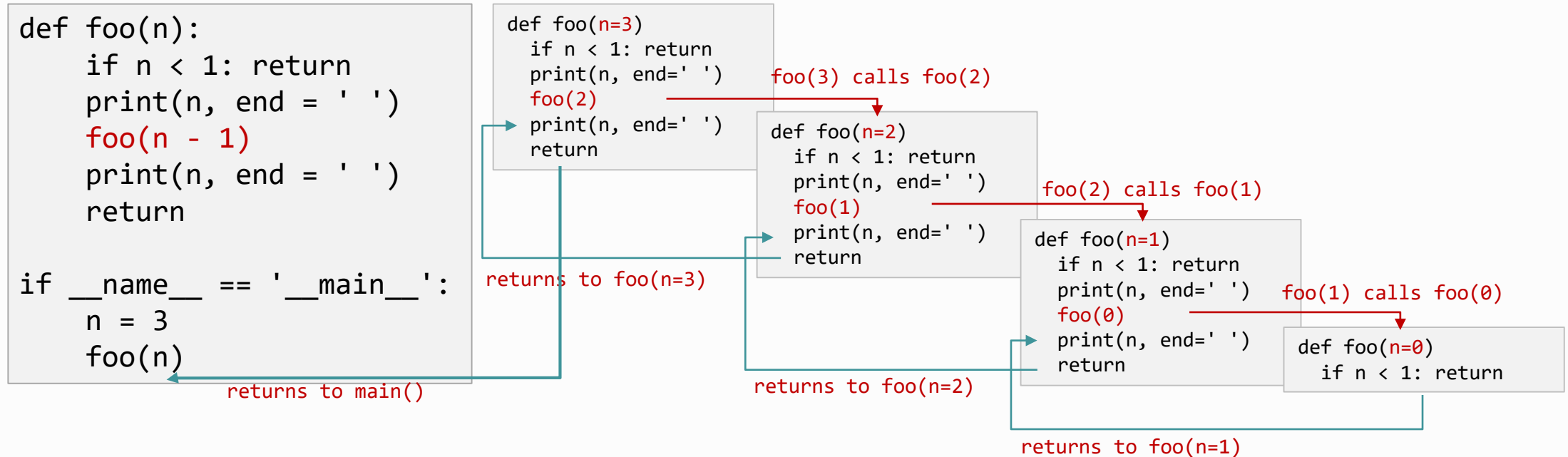
```
def foo(n):  
    if n < 1: return  
    print(n, end = ' ')  
    foo(n - 1)  
    print(n, end = ' ')  
    return  
  
if __name__ == '__main__':  
    n = 3  
    foo(n)
```



- When `main()` calls `foo(3)`, `main()` and `n=3` are pushed to the system stack. It will finish when returned from `foo(3)`.
- foo(3) prints '3'** and calls `foo(2)`. Then, **foo(2)** and `n=2` are pushed to the system stack.
- Similarly, **foo(2) prints '2'** and calls `foo(1)`. Then **foo(1) prints '1'** and calls `foo(0)`.
- `foo(0)` goes to `if` and returns to `foo(1)`. This is the first return ever. Now **foo(1)** popped from the stack **prints '1'**.
- `foo(1)` returns or finishes. Then, `foo(2)` popped from the stack prints '2'. It returns to `foo(3)` popped from the stack.

Example 3

- Predict the output of the following code.



- When `main()` calls `foo(3)`, `main()` and `n=3` are pushed to the system stack. It will finish when returned from `foo(3)`.
- foo(3) prints '3'** and calls `foo(2)`. Then, **foo(2)** and `n=2` are pushed to the system stack.
- Similarly, **foo(2) prints '2'** and calls `foo(1)`. Then **foo(1) prints '1'** and calls `foo(0)`.
- `foo(0)` goes to `if` and returns to `foo(1)`. This is the first return ever. Now **foo(1)** popped from the stack **prints '1'**.
- `foo(1)` returns or finishes. Then, `foo(2)` popped from the stack prints '2'. It returns to `foo(3)` popped from the stack.
- `foo(3)` prints '3' and returns to the `main()`. Then, the `main()` finishes the program.

Reminder

- **Recursion** is a method where the solution to a problem depends on solutions to **smaller instances** of the same problem (as opposed to iteration).

Recursion Exercise 1

- What is the result of the following call `mystery(648)`?
Do it by hands, not running the code, and draw a diagram for the function calls.
- How many kinds of the results will you get if you give many different `n`?

```
def mystery(n):  
    if n < 10:  
        return n  
    else:  
        a = n // 10  
        b = n % 10  
        return mystery(a + b)
```

Recursion Exercise 1

- What is the result of the following call `mystery(648)`? Do it by hands, not running the code.
- How many kinds of the results will you get if you give many different `n`?

```
def mystery(n):  
    if n < 10:  
        return n  
    else:  
        a = n // 10  
        b = n % 10  
        return mystery(a + b)
```

```
mystery(648):  
    a = 648 // 10      # 64  
    b = 648 % 10       # 8  
    return mystery(72) # mystery(72)
```


Recursion Exercise 1

- What is the result of the following call `mystery(648)`? Do it by hands, not running the code.
- How many kinds of the results will you get if you give many different `n`?

```
def mystery(n):  
    if n < 10:  
        return n  
    else:  
        a = n // 10  
        b = n % 10  
        return mystery(a + b)
```

```
mystery(648):  
    a = 648 // 10      # 64  
    b = 648 % 10       # 8  
    return mystery(72) # mystery(72)
```

```
mystery(72):  
    a = 72 // 10      # 7  
    b = 72 % 10       # 2  
    return mystery(9) # mystery(9)
```

Recursion Exercise 1

- What is the result of the following call `mystery(648)`? Do it by hands, not running the code.
- How many kinds of the results will you get if you give many different `n`?

```
def mystery(n):  
    if n < 10:  
        return n  
    else:  
        a = n // 10  
        b = n % 10  
        return mystery(a + b)
```

```
mystery(648):  
    a = 648 // 10      # 64  
    b = 648 % 10       # 8  
    return mystery(72) # mystery(72)
```

```
mystery(72):  
    a = 72 // 10      # 7  
    b = 72 % 10       # 2  
    return mystery(9) # mystery(9)
```

```
mystery(9):  
    return 9
```

Recursion Exercise 2

- What is result of the following call, `mystery(234)` and `mystery(5067)`, respectively? Do it by hands and draw the function call diagrams like the previous example.

```
def mystery(n):  
    if n < 10:  
        return 10 * n + n  
    else:  
        a = mystery(n // 10)  
        b = mystery(n % 10)  
        return a * 100 + b
```

Recursion Exercise 2

- What is result of the following call, `mystery(234)` and `mystery(5067)`, respectively? Do it by hands and draw the function call diagrams like the previous example.

```
def mystery(n):  
    if n < 10:  
        return 10 * n + n  
    else:  
        a = mystery(n // 10)  
        b = mystery(n % 10)  
        return a * 100 + b
```

```
mystery(234):  
    a = ...
```

```
        b = ...
```

```
    return ...
```

Recursion Exercise 3

- Predict the output of the following code.

```
def mystery(n):  
    if n < 10:  
        return 'b-return: ' + str(n)  
    a = n // 10  
    b = n % 10  
    c = mystery(a + b)  
    print(a, b, c)  
    return 'r-return: ' + str(n)  
if __name__ == '__main__':  
    print(mystery(6480))
```

1:

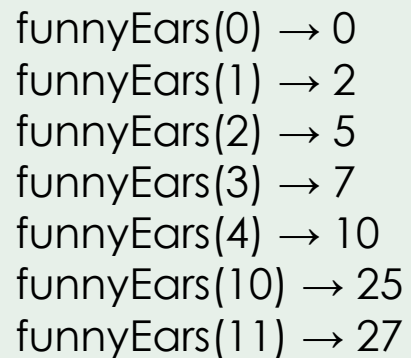
```
1 0 b-return: n=1  
7 3 r-return: n=10  
64 9 r-return: n=73  
648 1 r-return: n=649  
r-return: n=6481
```

2:

```
r-return: n=6481  
648 1 r-return: n=649  
64 9 r-return: n=73  
7 3 r-return: n=10  
1 0 b-return: n=1
```

Recursion Exercise 4


- We have bunnies and funnies standing in a line, numbered 1, 2, ... The odd bunnies (1, 3, ..) have the normal 2 ears. The even funnies (2, 4, ..) we'll say have 3 ears, because they each have a raised foot. Recursively return the number of "ears" in the bunny and funny line 1, 2, ... n (without loops or multiplication).
- Expected Output



```
funnyEars(0) → 0  
funnyEars(1) → 2  
funnyEars(2) → 5  
funnyEars(3) → 7  
funnyEars(4) → 10  
funnyEars(10) → 25  
funnyEars(11) → 27
```

Recursion Exercise 4

- We have bunnies and funnies standing in a line, numbered 1, 2, ... The odd bunnies (1, 3, ..) have the normal 2 ears. The even funnies (2, 4, ..) we'll say have 3 ears, because they each have a raised foot. Recursively return the number of "ears" in the bunny and funny line 1, 2, ... n (without loops or multiplication).
- Expected Output



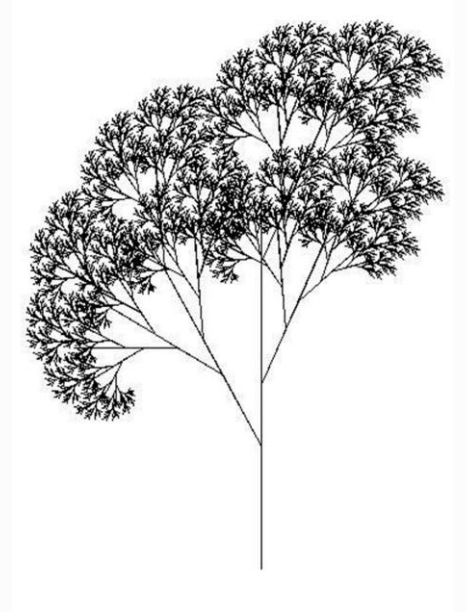
```
funnyEars(0) → 0
funnyEars(1) → 2
funnyEars(2) → 5
funnyEars(3) → 7
funnyEars(4) → 10
funnyEars(10) → 25
funnyEars(11) → 27
```

```
def funnyEars(n):
    if n <= 0 return 0
    if n % 2 == 0:
        return _____
    return _____

if __name__ == '__main__':
    funnies = [0, 1, 2, 3, 4, 10, 11]
    for n in funnies:
        print('n:', funnyEars(n))
```

Summary

- **Recursion:** *see Recursion*
- Recursion is when a function calls itself
 - It can be used to simplify complex solutions to difficult problems.
- A recursive algorithm **passes the buck** repeatedly to the same function.



학습 정리

- 1) Recursion: See Recursion
- 2) 재귀함수(Recursion)를 이용하여 간결한 프로그램을 구현할 수 있다
- 3) 재귀함수는 base case와 recursive case로 구분된다

파이썬으로 배우는 데이터 구조

수고했습니다
곧 다음 시간에
다시 뵙겠습니다

