파이썬으로 배우는 데이터 구조

한동대학교
전산전자공학부
김영섭 교수

학습 목표

힙(Heap)과 BST와의 차이점을 이해하고

힙(Heap)의 시간복잡도를 알 수 있다

Data Structures in Python
Chapter 8

- **Heap and Priority Queue**
- Heap Coding
- Min/MaxHeap and Heap sort

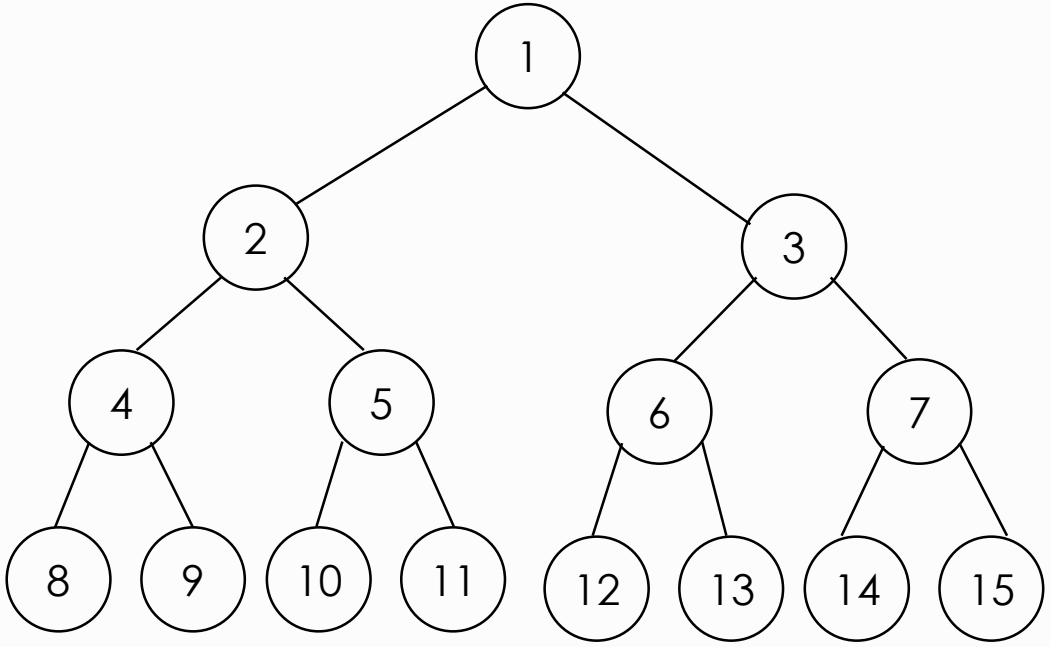
Agenda & Readings

- Heap and Priority Queue
  - Complete Binary Tee (Review)
  - Heap and Priority Queue
  - Heap ADT
  - Time Complexity

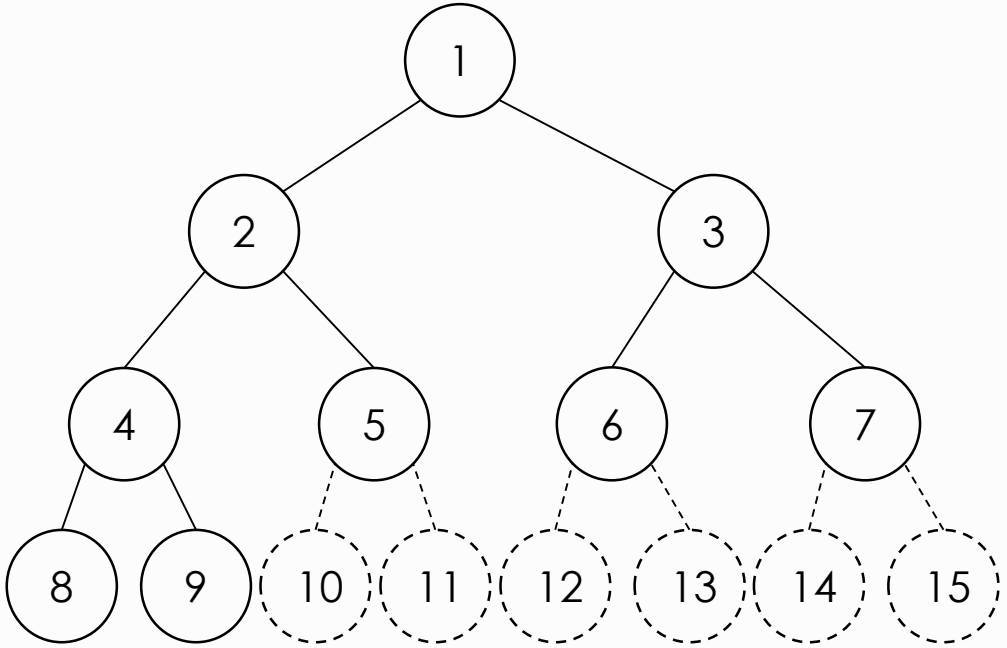
- Reference:
  - Problem Solving with Algorithms and Data Structures

Binary trees - Properties

- **Definition:** *A* **full** binary tree of level k is a binary tree having **2^k - 1 nodes**, $k \geq 0$.
- **Definition**: A binary tree with n nodes and level k is **complete** if and only if its nodes correspond to the nodes numbered from 1 to n in the full binary tree of level k.



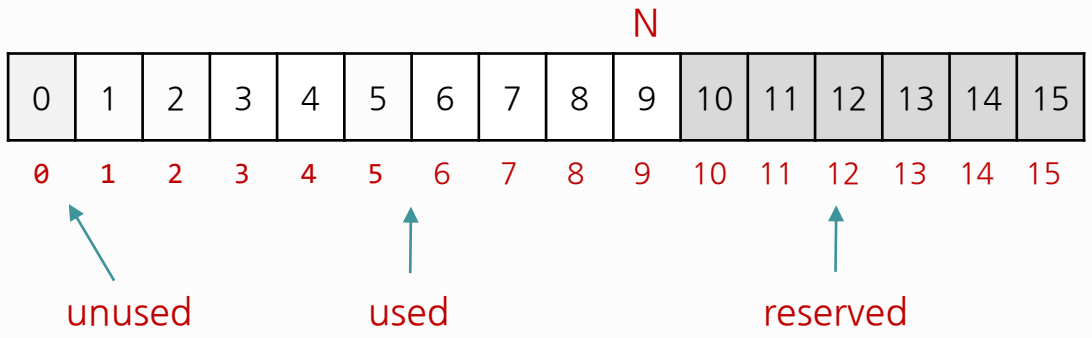
*A **full** binary tree*

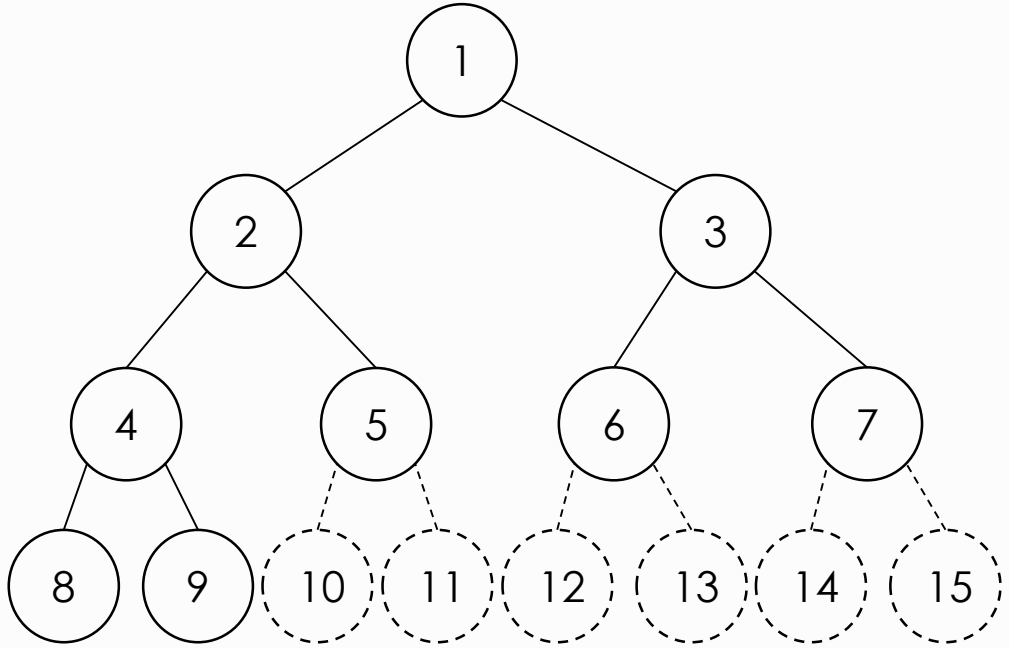


*A **complete** binary tree*

Binary trees - Array representation

- A **complete** binary tree with n nodes, any node index i, $1 \leq i \leq n$, we have
  - `parent(i) is at` $\lfloor i/2 \rfloor$ `If i = 1, i is at the root and has no parent`
  - `leftChild(i) is at 2i if 2i <= n. If 2i > n, then i has no left child.`
  - `rightChild(i) is at 2i+1 if 2i+1 <= n. If 2i+1 > n, then i has no right child.`





*A **complete** binary tree*

Heaps & Priority Queues

- **Heaps** are frequently used to implement **priority queues.**
  - Because it provides an efficient implementation for **priority queues.**

Heaps & Priority Queues

- **Heaps** are frequently used to implement **priority queues.**
  - Because it provides an efficient implementation for **priority queues.**

- **Priority queues.**
  - Queues with priorities associated to.
  - **Example:** A line waiting to be served at a bank and served FIFO except if a senior or a disabled person arrives in the line. They are served first. Seniors and disabled persons have higher priority than others.

Heaps & Priority Queues

- **Heaps** are frequently used to implement **priority queues.**
  - Because it provides an efficient implementation for **priority queues.**

- **Priority queues.**
  - Queues with priorities associated to.
  - **Example:** A line waiting to be served at a bank and served FIFO except if a senior or a disabled person arrives in the line. They are served first. Seniors and disabled persons have higher priority than others.

- **A typical ADT for Priority Queue**
  - Get the top priority element (min or max)
  - Insert an element
  - Delete the top priority element
  - Decrease the priority of an element

- O(1)
- O(log n)
- O(log n)
- O(log n)

Heaps & Priority Queues

- Challenge: Find the largest M items in a stream of N items.
- Constraints: Not enough memory to store N items.

N huge, M large

Order of insert of finding the largest M **in a stream of N items**

implementation	insert	delete	min/max
unordered array	1	N	N
ordered array	N	1	1
goal	**log N**	**log N**	1 or log N

N huge

min/max

priority queue

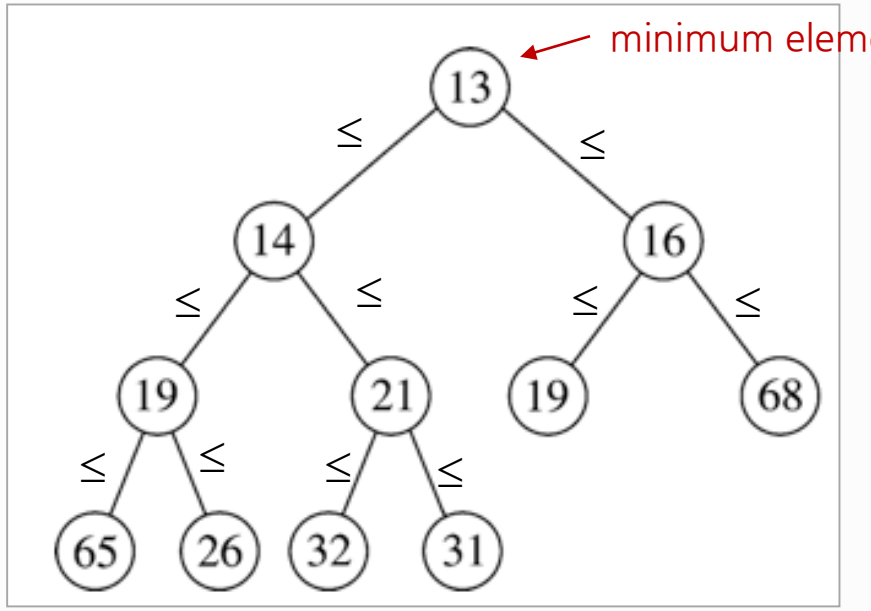
Mission Impossible?

Binary heap

- **Binary heap**: array representation of a **heap-ordered** complete binary tree
- **Properties:**
  - **Heap-ordered:**
    Parent's key **no smaller** than children's keys for max-heap. (no greater for min-heap)
  - **Heap-structure**:
    A complete binary tree

Binary heap

- **Binary heap**: array representation of a **heap-ordered** complete binary tree
- **Properties:**
  - **Heap-ordered:**
    Parent's key **no smaller** than children's keys for max-heap. (no greater for min-heap)
  - **Heap-structure**:
    A complete binary tree

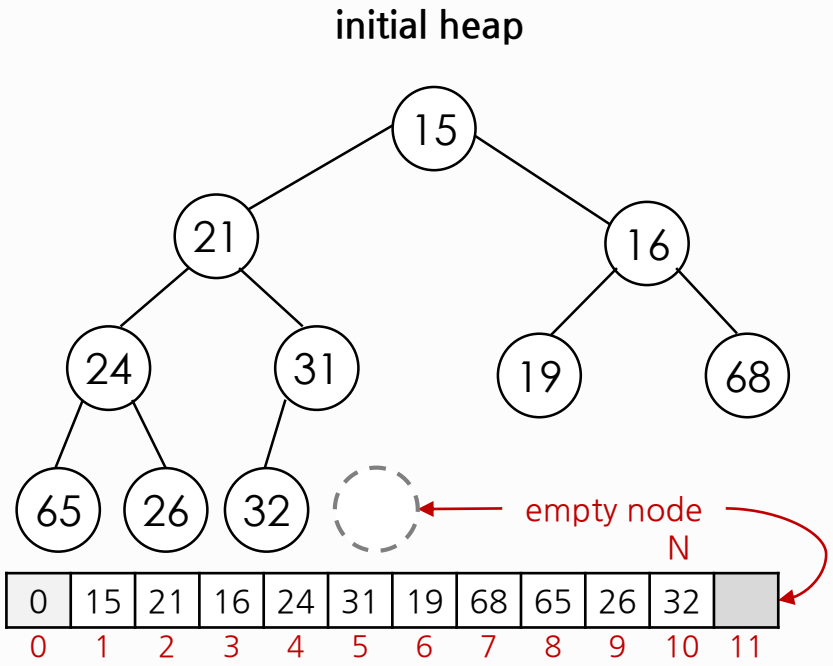


- Duplicates are allowed
- No order implied for elements which do not share ancestor-descendant relationship

min-heap: insert(heap, 14)

Algorithm:

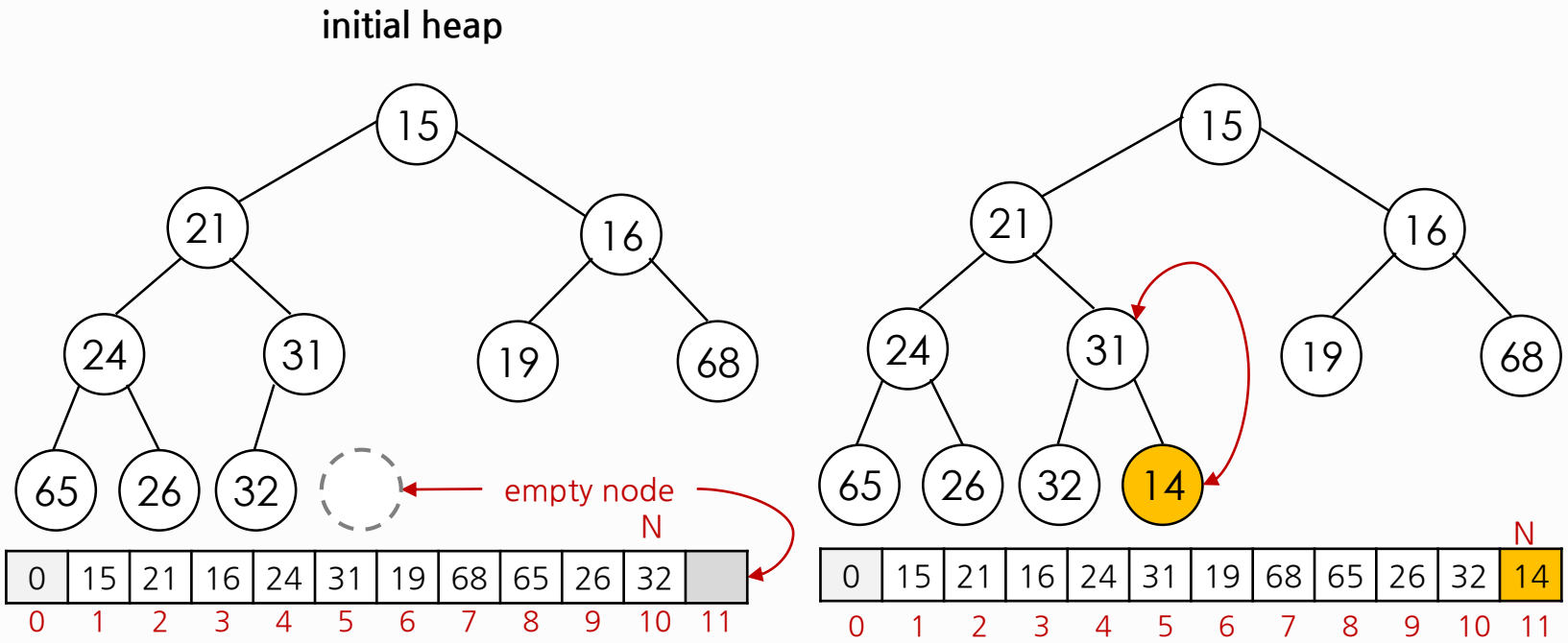
- Insert a new element **while maintaining a heap-structure**
- Move the element up the heap **while not satisfying heap-ordered**



min-heap: insert(heap, 14)

Algorithm:

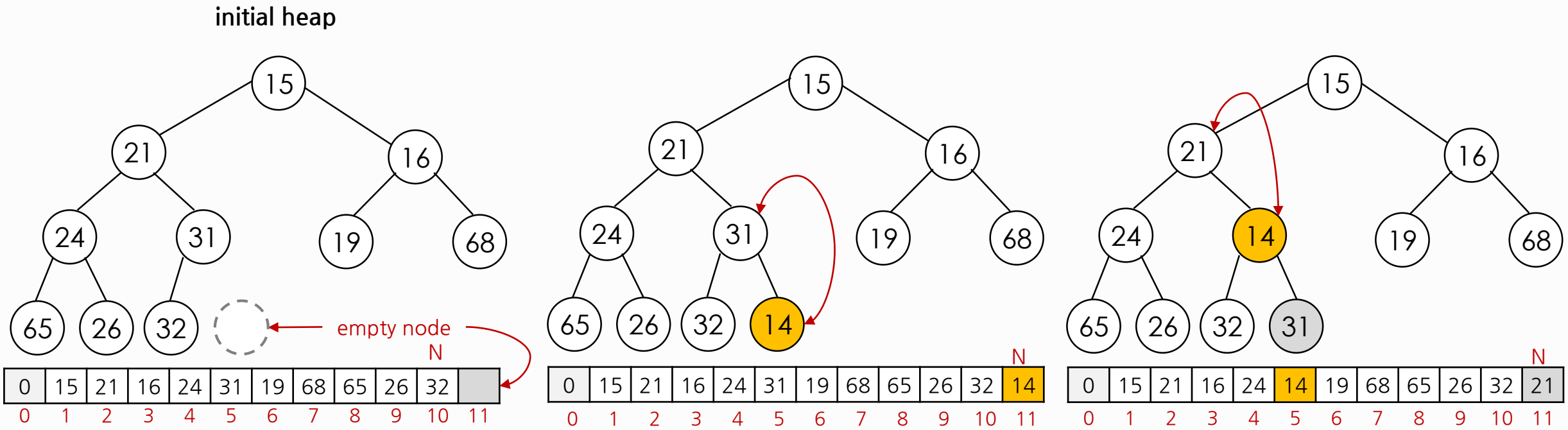
- Insert a new element **while maintaining a heap-structure**
- Move the element up the heap **while not satisfying heap-ordered**



min-heap: insert(heap, 14)

Algorithm:

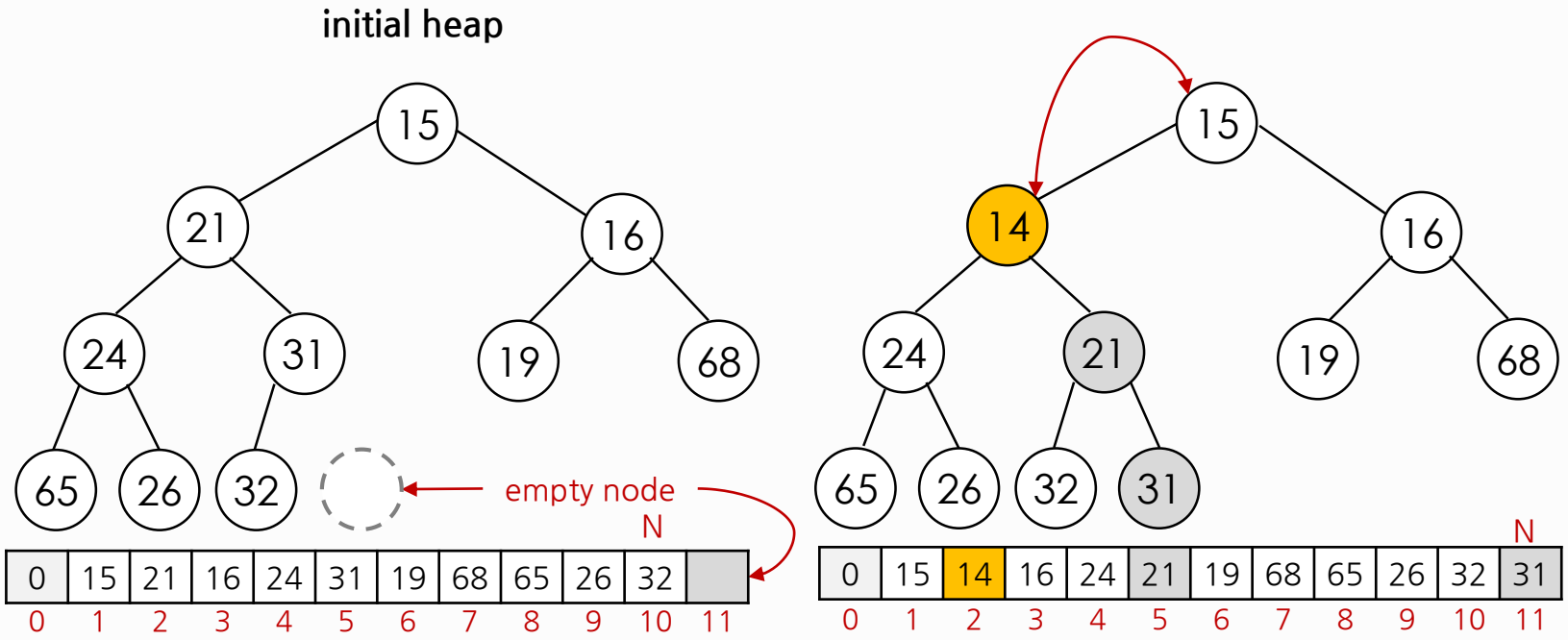
- Insert a new element **while maintaining a heap-structure**
- Move the element up the heap **while not satisfying heap-ordered**



min-heap: insert(heap, 14)

Algorithm:

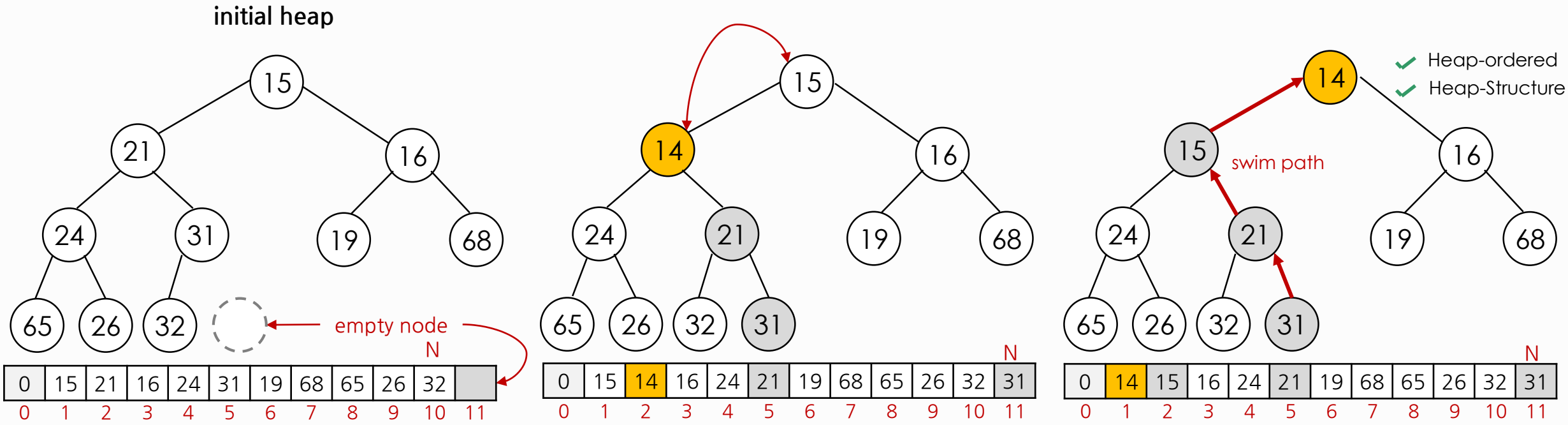
- Insert a new element **while maintaining a heap-structure**
- Move the element up the heap **while not satisfying heap-ordered**



min-heap: insert(heap, 14)

Algorithm:

- Insert a new element **while maintaining a heap-structure**
- Move the element up the heap **while not satisfying heap-ordered**

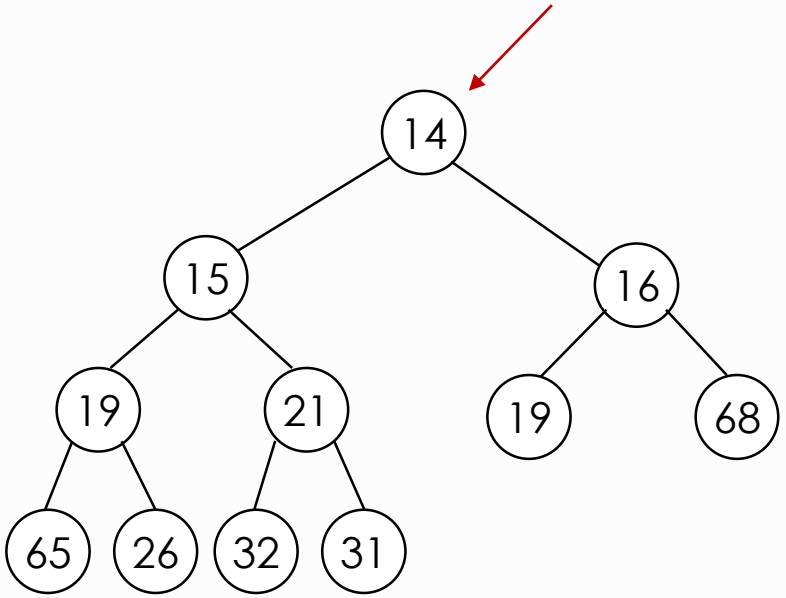


min-heap: delete() or dequeue()

- Swap the root and the last element.
- Heap decreases by one in size.
- **Move down (sink) the root** while not satisfying heap-ordered.
  - Minimum element is always at the root (by min-heap definition).

min-heap: delete() or dequeue()

- Which position of the node will be empty?

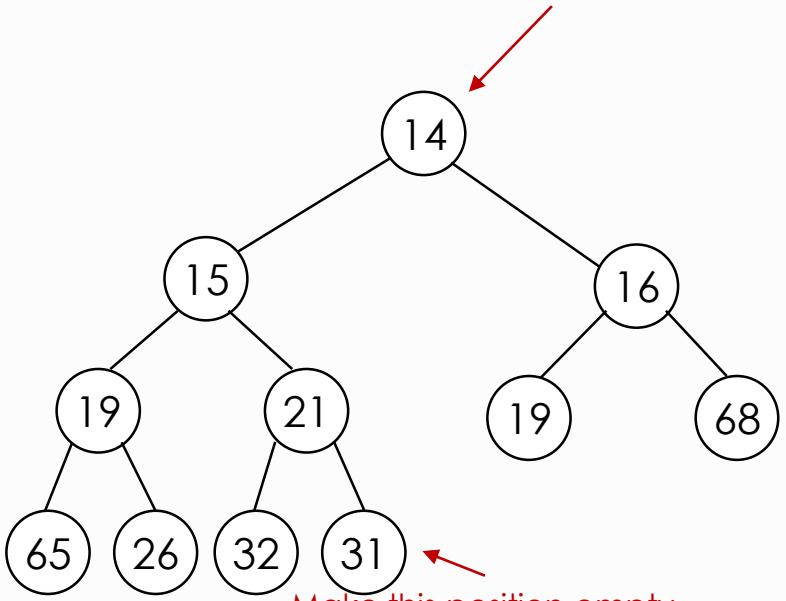


N

0	14	15	16	24	21	19	68	65	26	32	31
0	1	2	3	4	5	6	7	8	9	10	11

min-heap: delete() or dequeue()

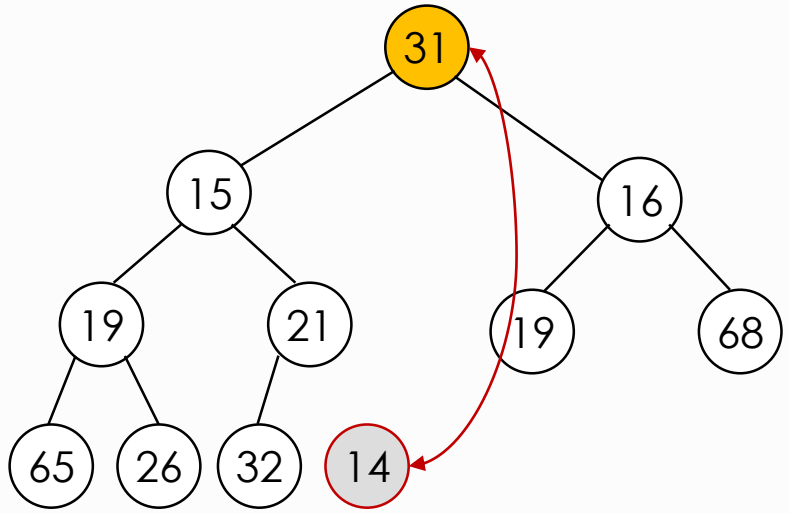
- Which position of the node will be empty?



	0	14	15	16	24	21	19	68	65	26	32	31
	0	1	2	3	4	5	6	7	8	9	10	11

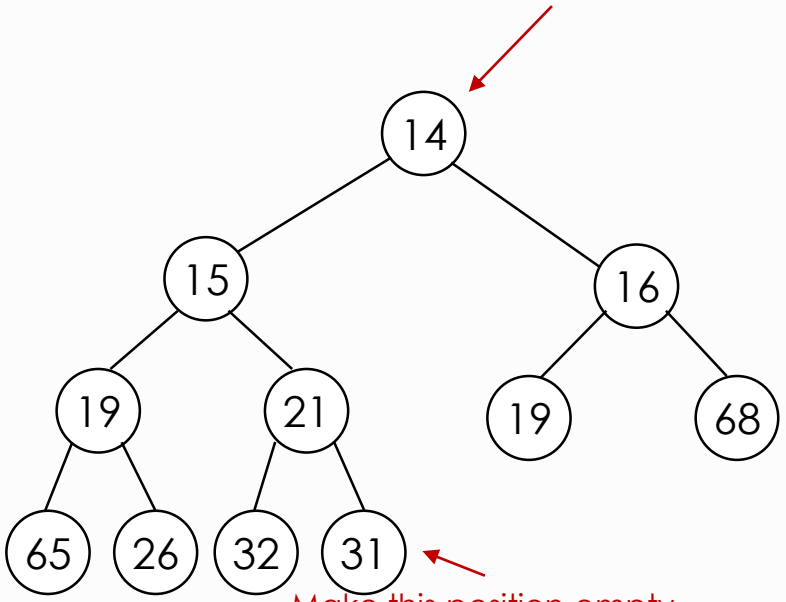
N

- swap the root & last
- **N--**



min-heap: delete() or dequeue()

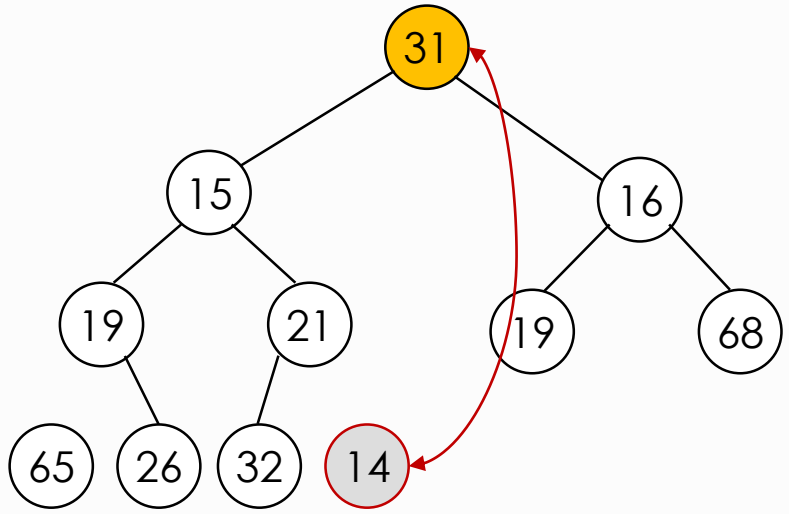
- Which position of the node will be empty?



	0	1	2	3	4	5	6	7	8	9	10	11
	0	14	15	16	24	21	19	68	65	26	32	31

N

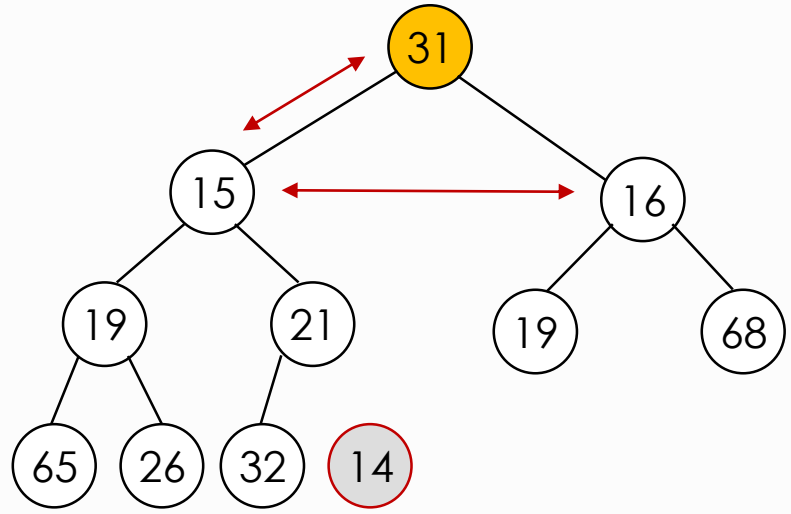
- swap the root & last
- **N--**



	0	1	2	3	4	5	6	7	8	9	10	11
	0	31	15	16	24	21	19	68	65	26	32	14

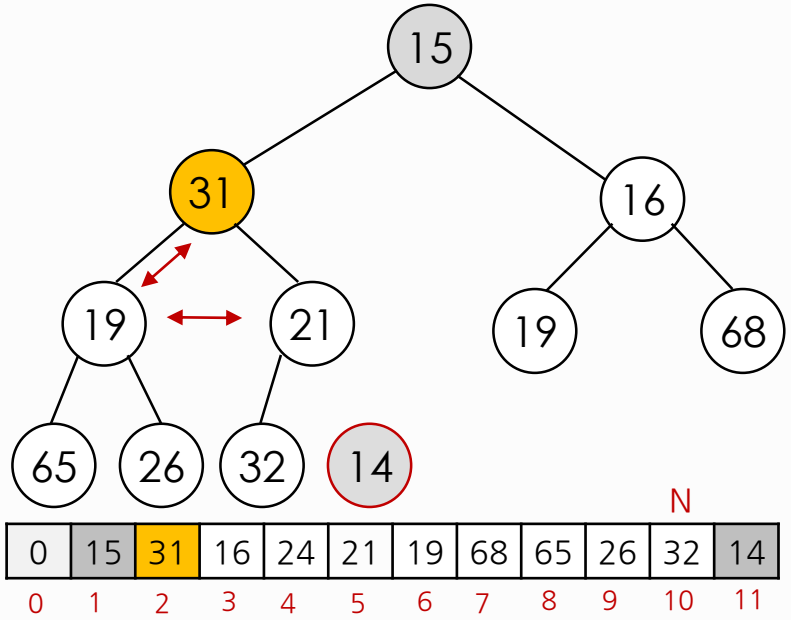
N

- heap-ordered?
- sink(): select one of two children and compare



min-heap: delete() or dequeue()

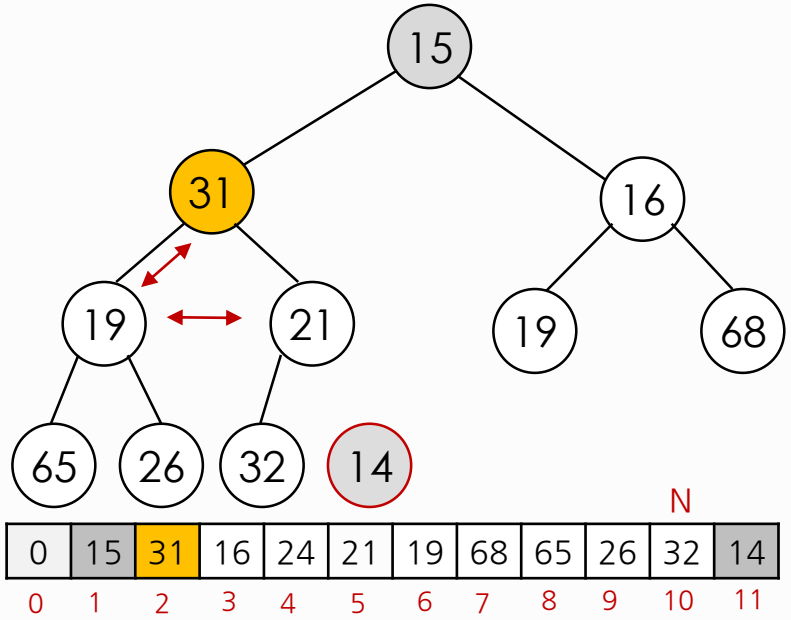
- heap-ordered?
- sink(): select one of two children and compare



- Is 31 > min(14,16)?
- Yes - swap 31 with min(14,16)

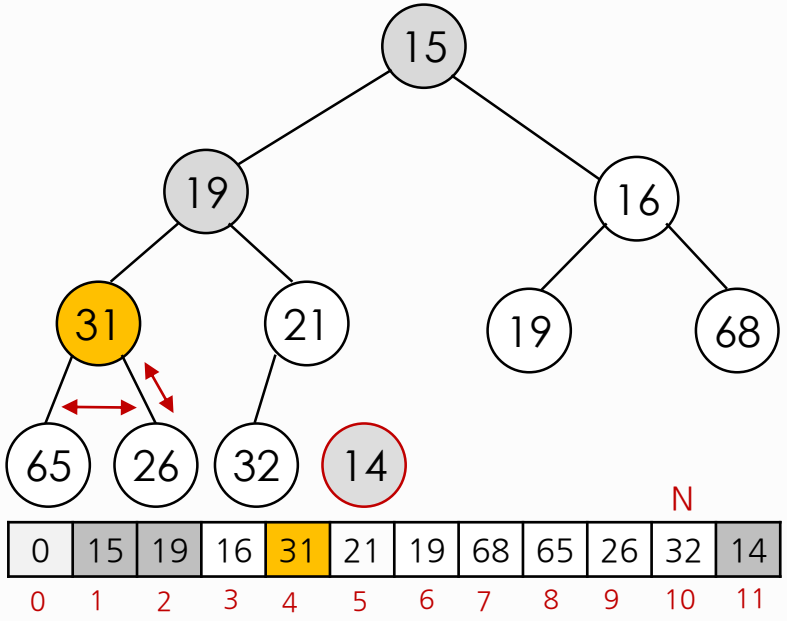
min-heap: delete() or dequeue()

- heap-ordered?
- sink(): select one of two children and compare



- Is 31 > min(14,16)?
- Yes - swap 31 with min(14,16)

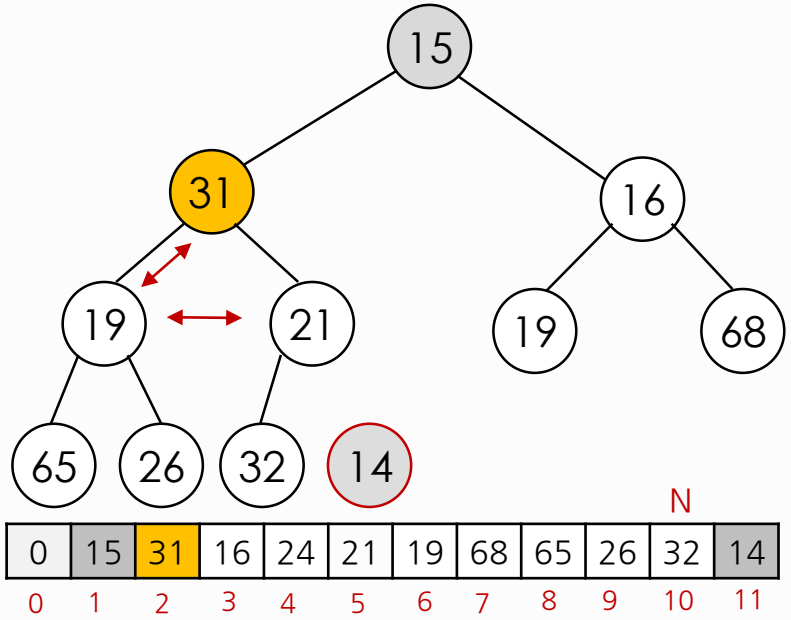
- heap-ordered?
- sink(): select one of two children and compare



- Is 31 > min(19,21)?
- Yes - swap 31 with min(19,21)

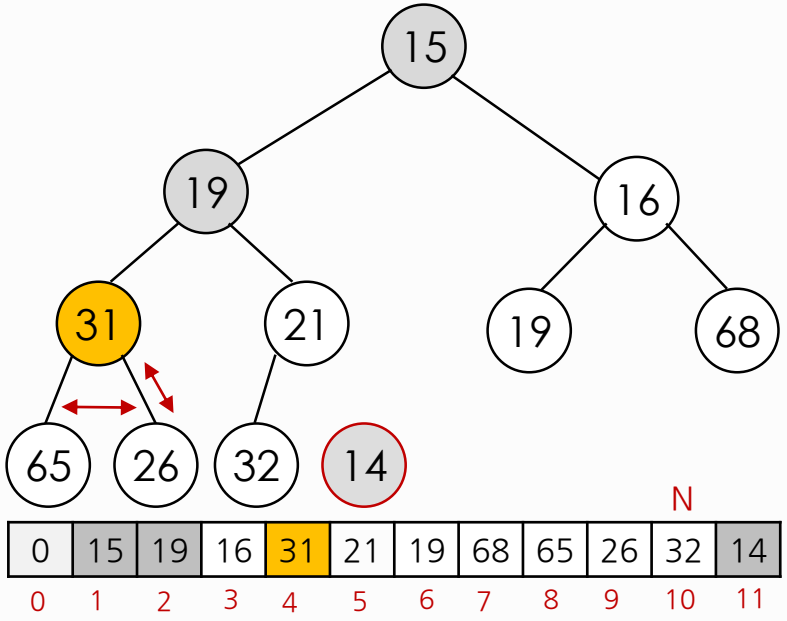
min-heap: delete() or dequeue()

- heap-ordered?
- sink(): select one of two children and compare



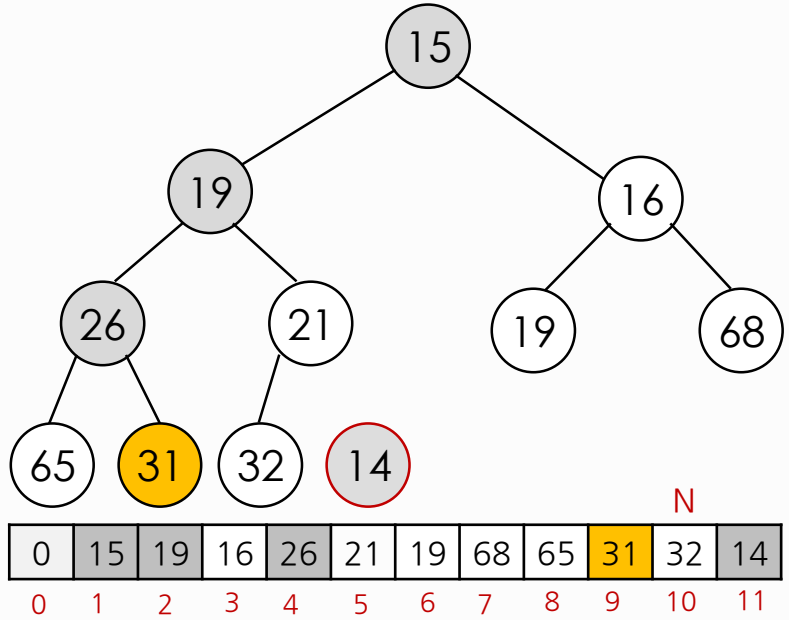
- Is 31 > min(14,16)?
- Yes - swap 31 with min(14,16)

- heap-ordered?
- sink(): select one of two children and compare



- Is 31 > min(19,21)?
- Yes - swap 31 with min(19,21)

- heap-ordered?
- sink(): select one of two children and compare



- Is 31 > min(65,26)?
- Yes - swap 31 with min(65,26)

✓ Heap-ordered

✓ Heap-Structure

Binary heap: Time complexity:

- Level of heap is $\lfloor \log_2 N \rfloor$
- insert: O(log N) for each insert
  - In practice, expect less
- delete: O(log N) // deleting root node or any node
- increase/decrease key: O(log N)

Implementation	Insert	Delete	min/max
Unordered array	1	N	N
Ordered array	N	1	1
Binary heap	**log N**	**log N**	1 or log N

Mission Completed

min/max

priority queue

학습 정리

1) 힙(Heap)은 우선순위 큐(Priority queue)에 주로 사용된다

2) 힙(Heap)에서의 삽입과 삭제 작업은 최대 $O(\log n)$ 시간복잡도로 수행할 수 있다

파이썬으로 배우는 데이터 구조

수고했습니다
곧 다음 시간에
다시 뵙겠습니다

Data Structures in Python

- Heap and Priority Queue
- **Heap Coding**
- Min/MaxHeap and Heap sort

Proof:

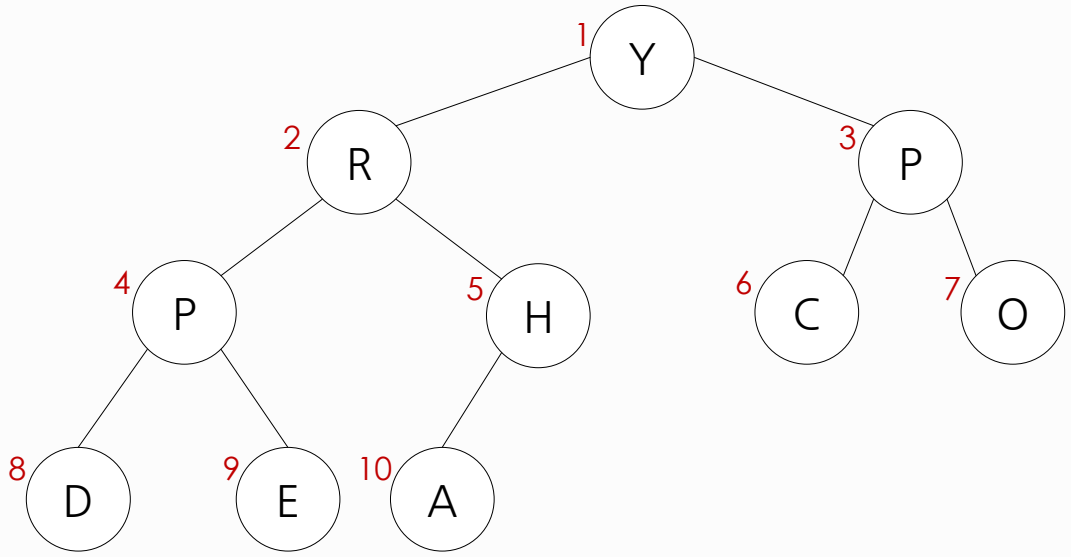
https://stackoverflow.com/questions/9755721/how-can-building-a-heap-be-on-time-complexity
https://www.insertingwiththeweb.com/data-structures/binary-heap/build-heap-proof/
https://www.quora.com/How-is-the-time-complexity-of-building-a-heap-is-o-n

References in Korean:

https://ratsgo.github.io/data%20structure&algorithm/2017/09/27/heapsort/
https://zeddios.tistory.com/56

max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**



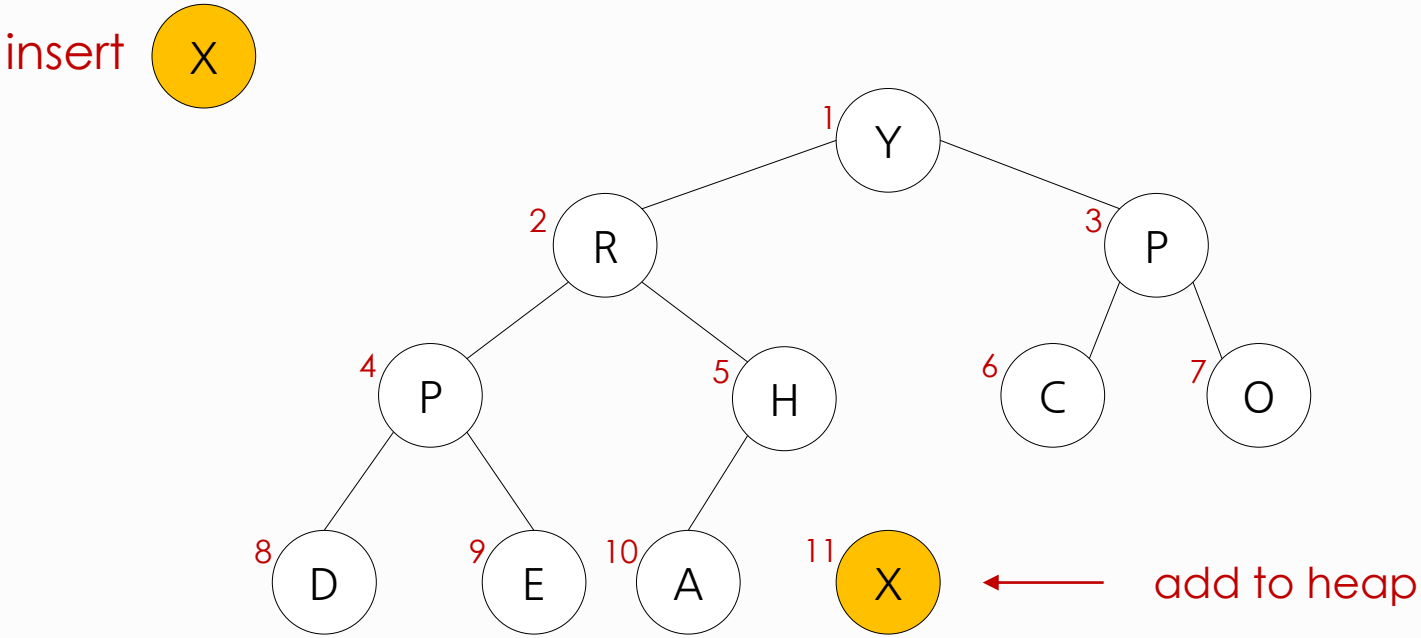
heap-ordered

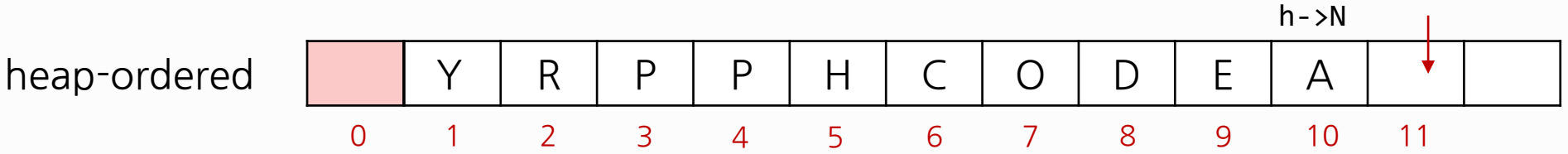
	Y	R	P	P	H	C	O	D	E	A		
0	1	2	3	4	5	6	7	8	9	10	11	

N

max-heap example

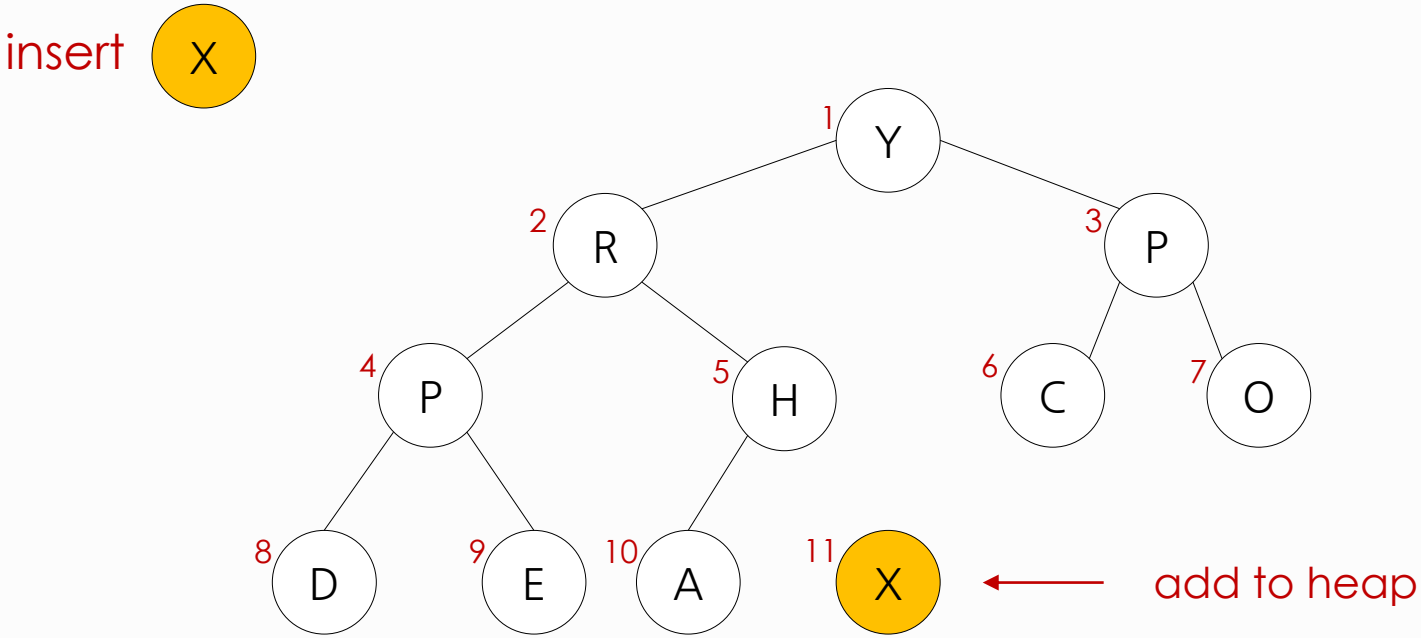
- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.

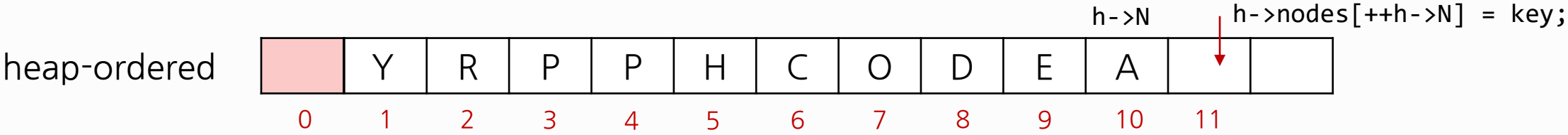




max-heap example

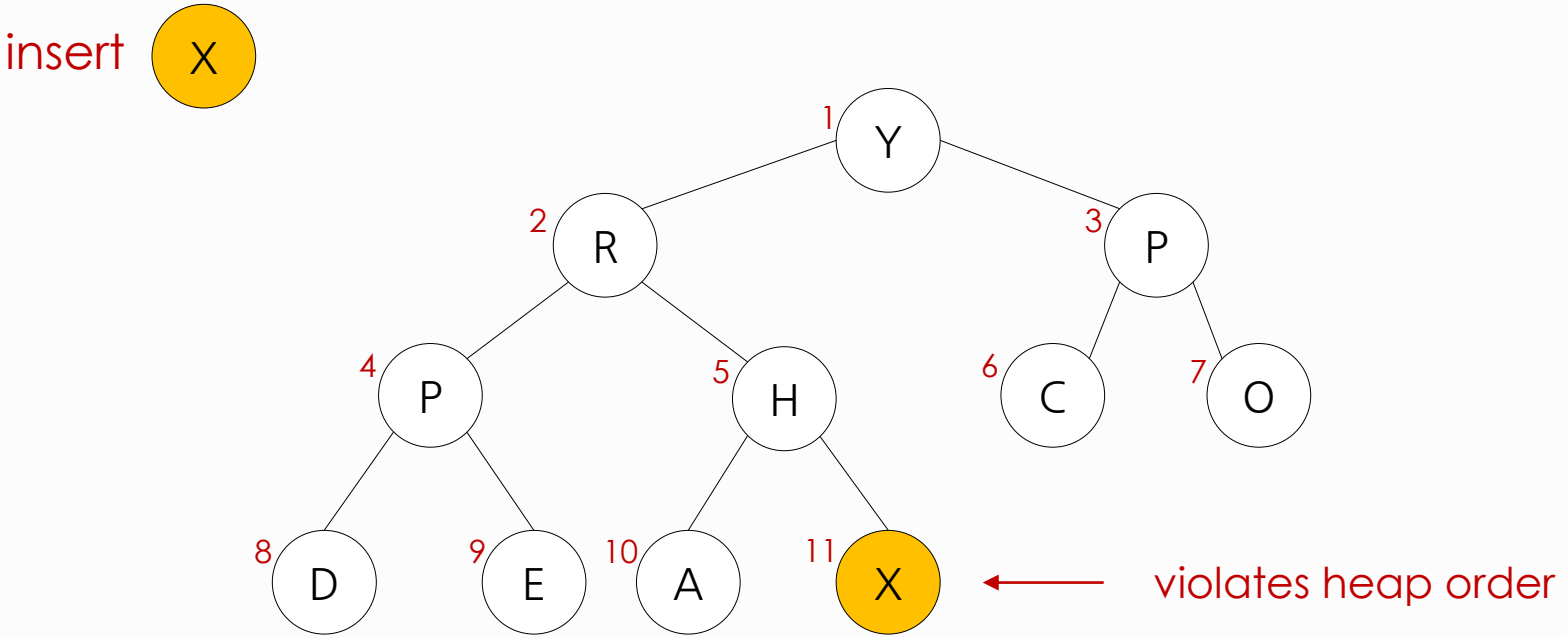
- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.





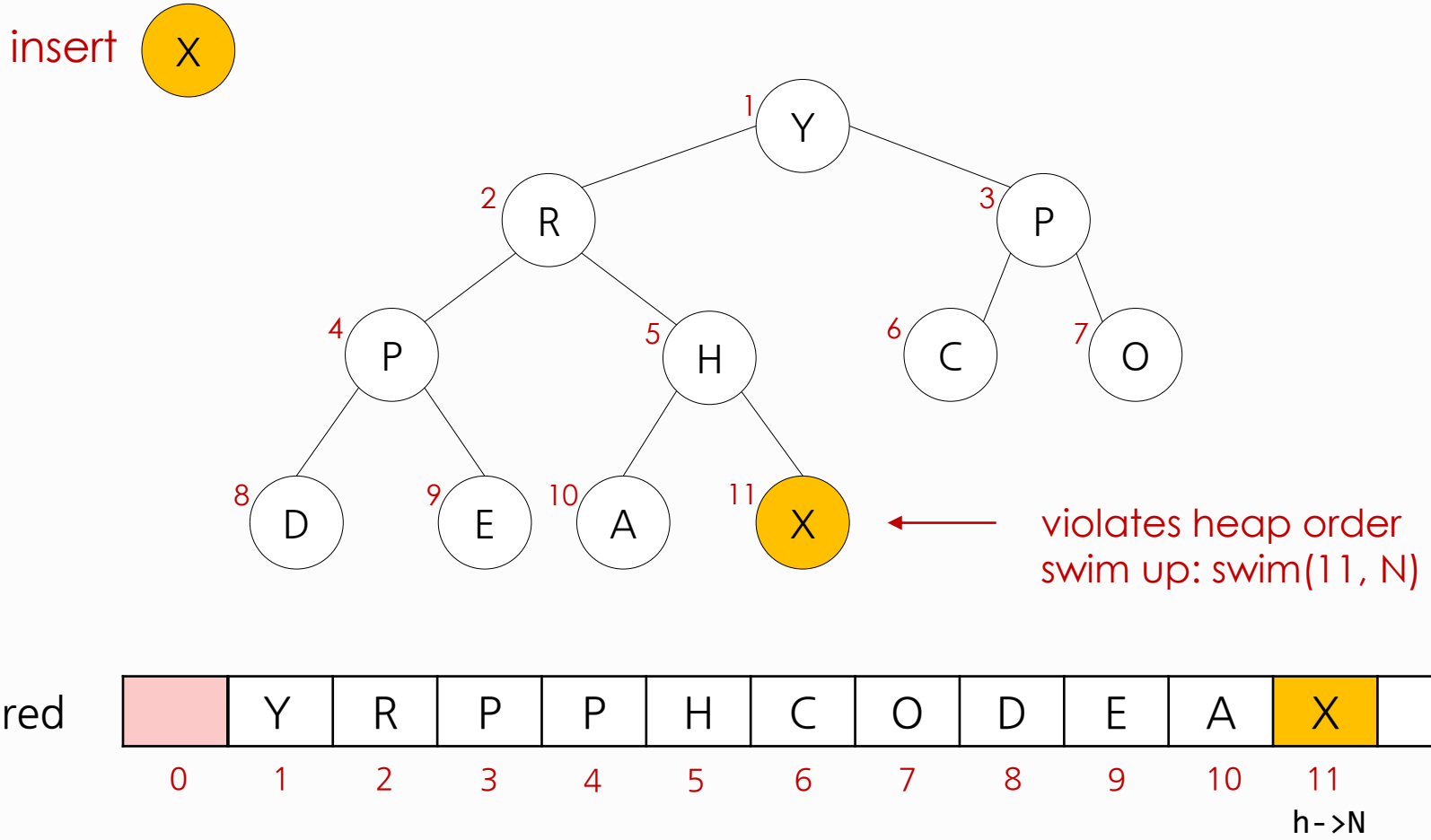
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**



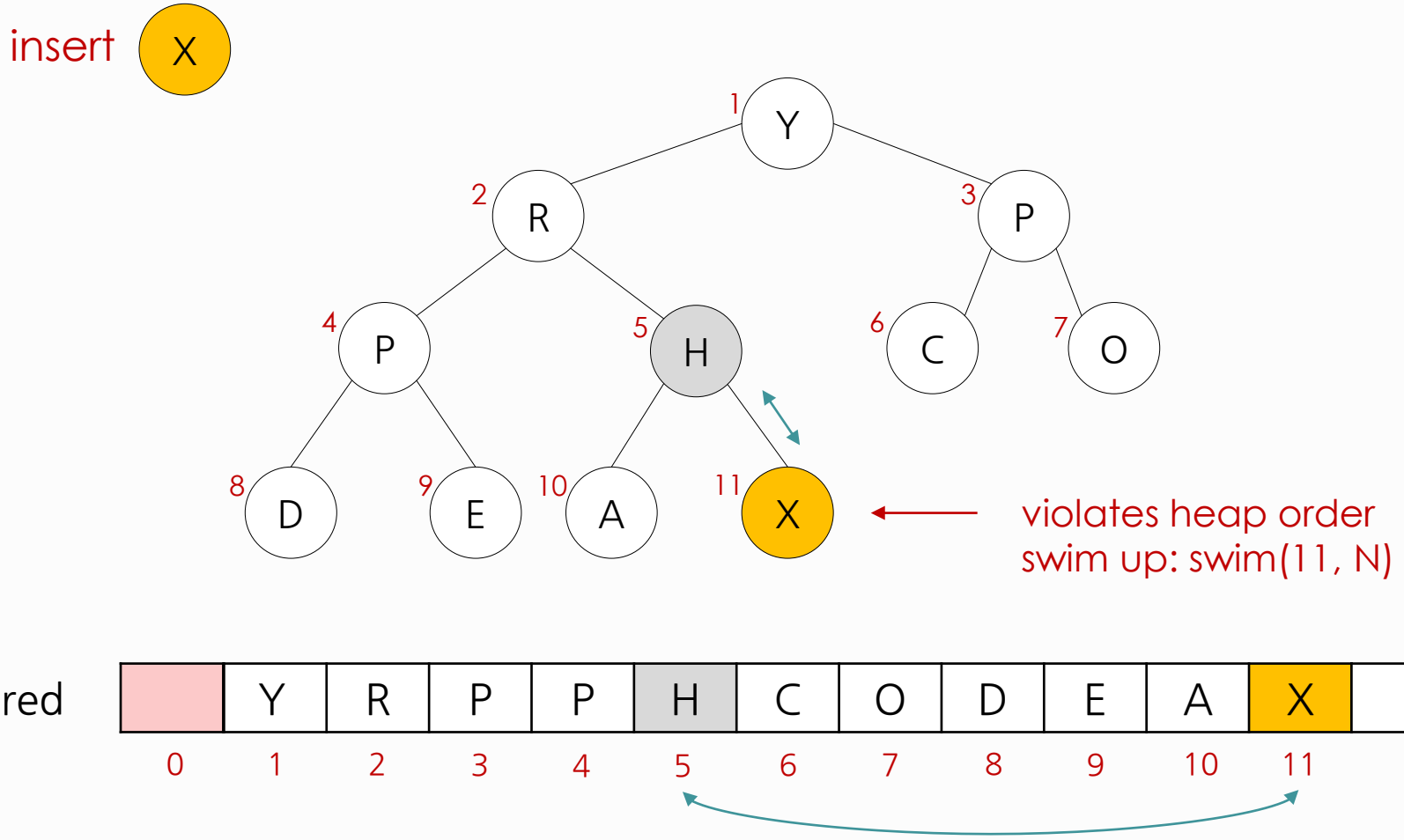
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.



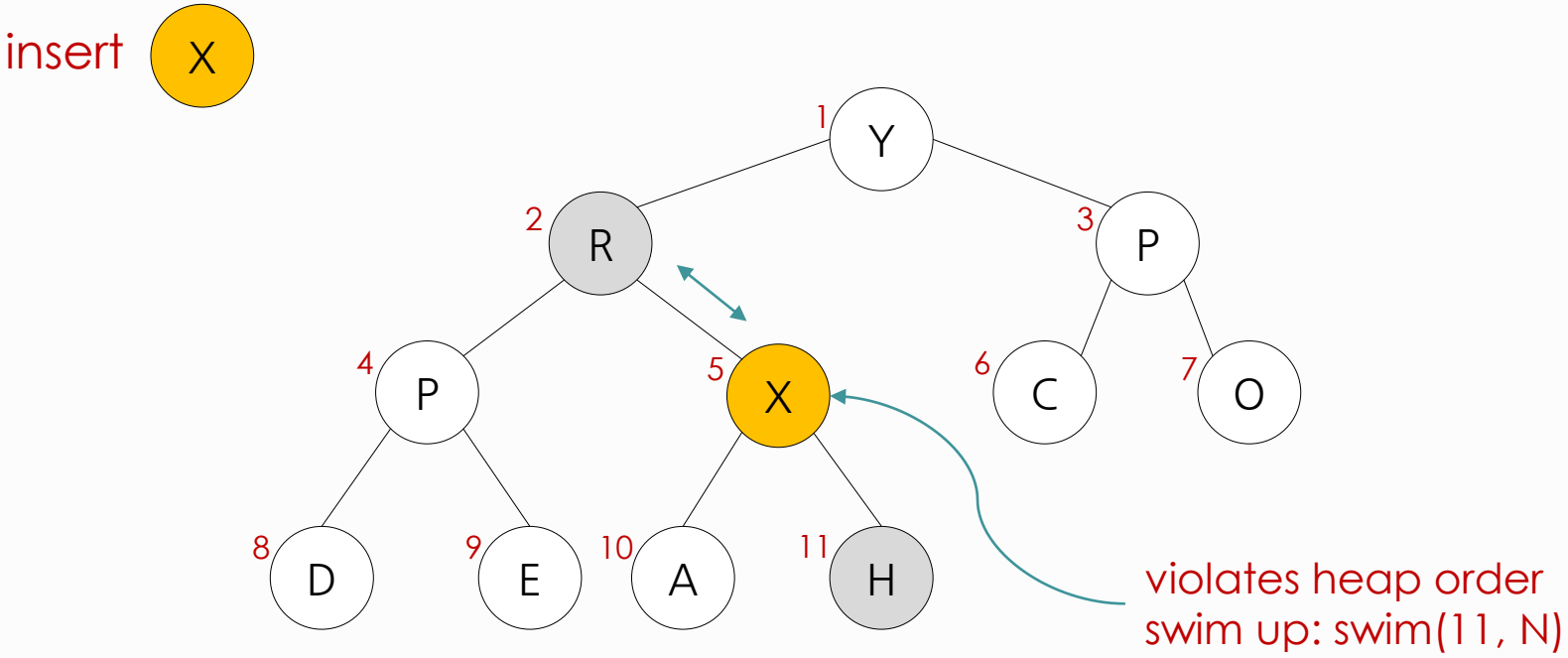
max-heap example

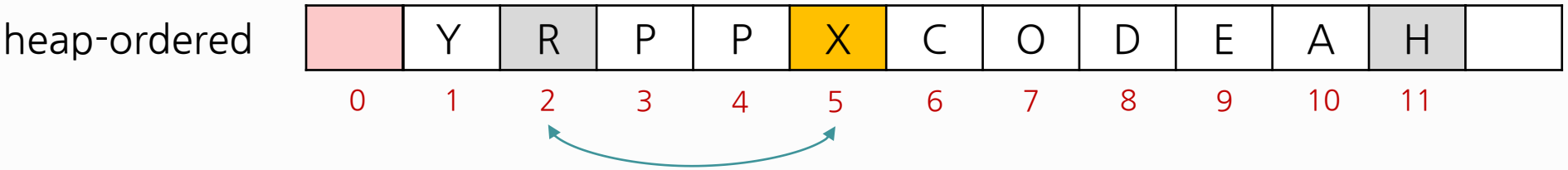
- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.



max-heap example

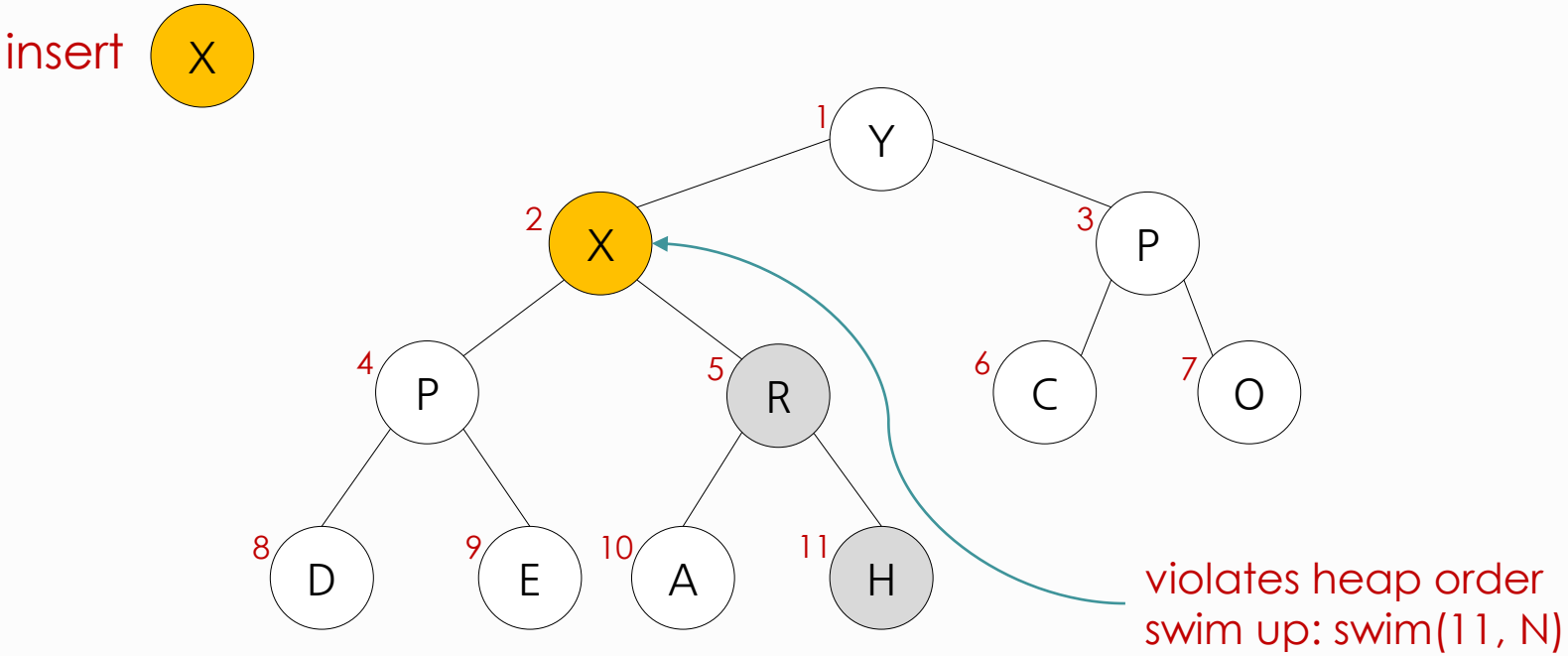
- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.





max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.

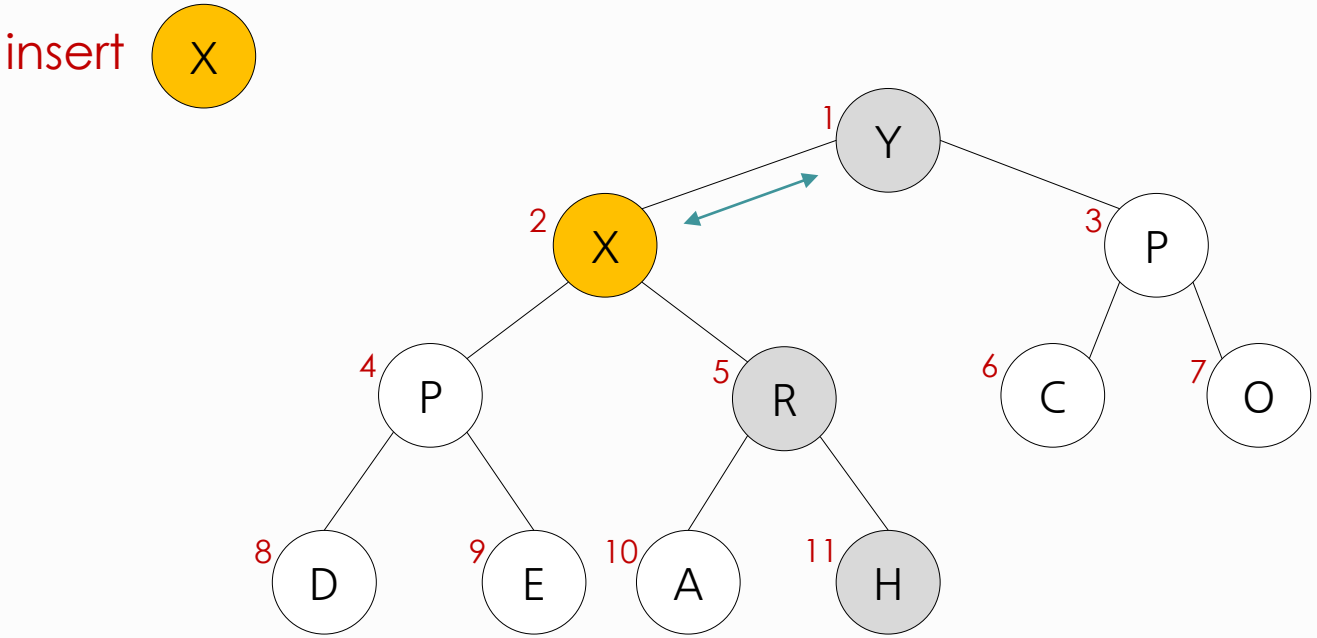


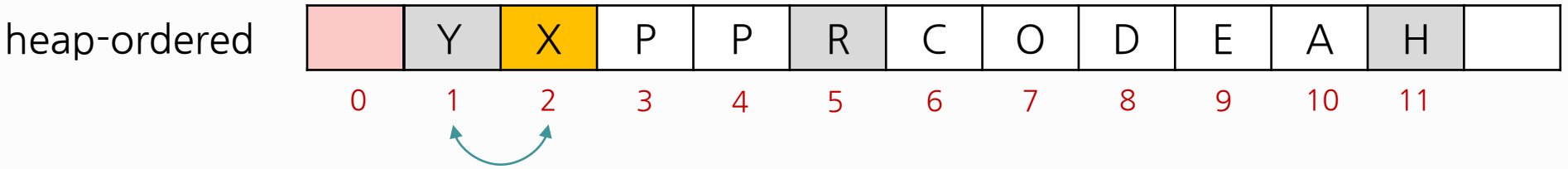
heap-ordered

	Y	X	P	P	R	C	O	D	E	A	H	
0	1	2	3	4	5	6	7	8	9	10	11	

max-heap example

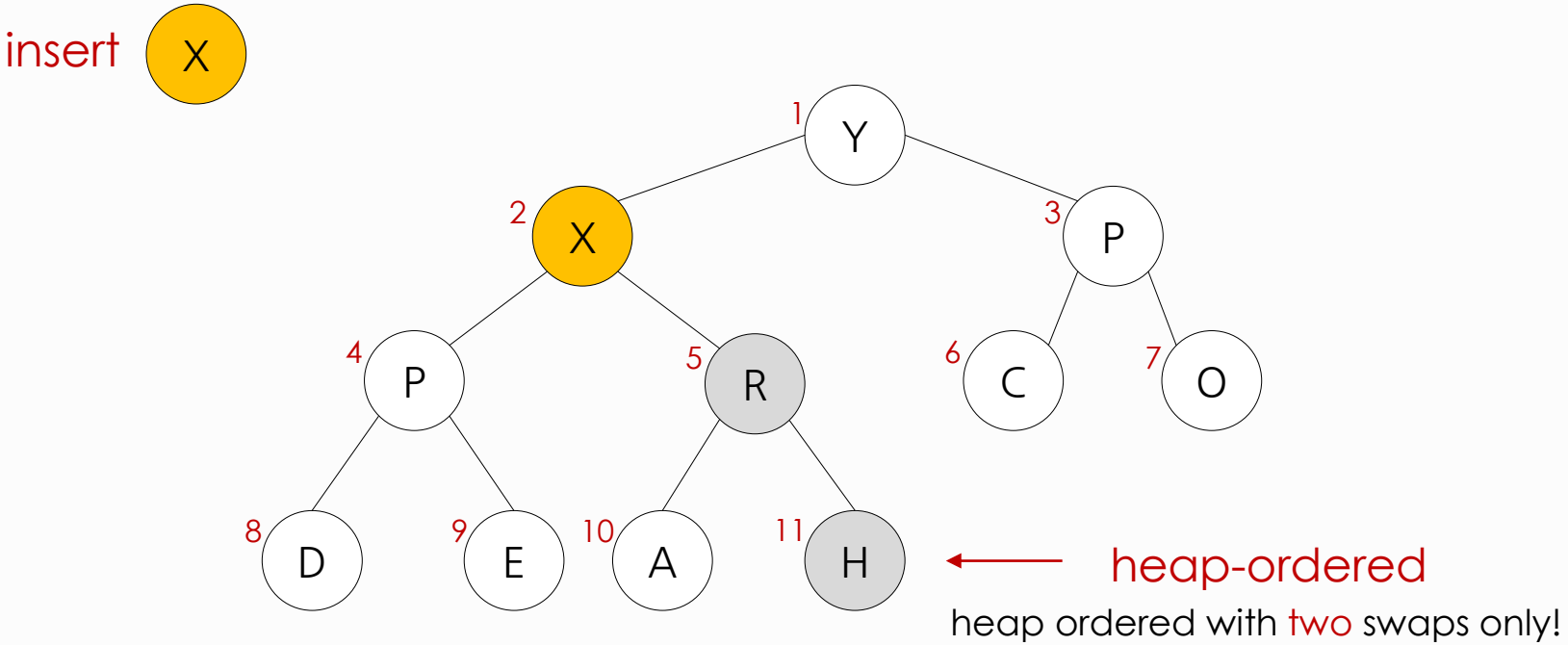
- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**





max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.



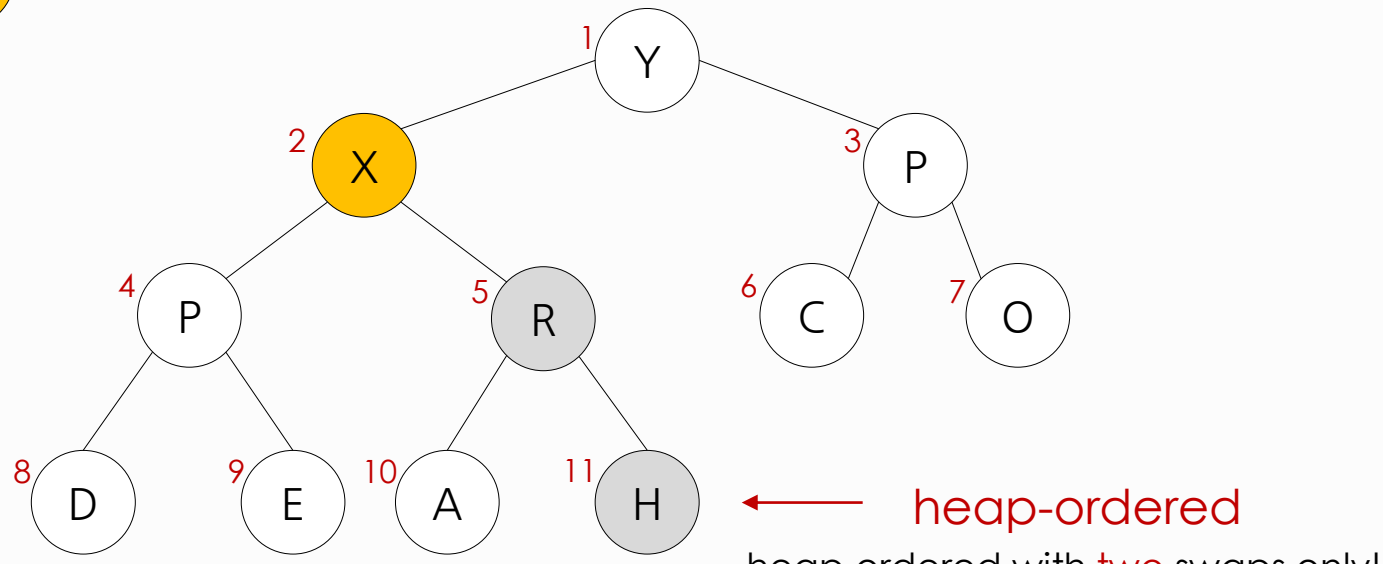
heap-ordered

	Y	X	P	P	R	C	O	D	E	A	H	
0	1	2	3	4	5	6	7	8	9	10	11	

max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

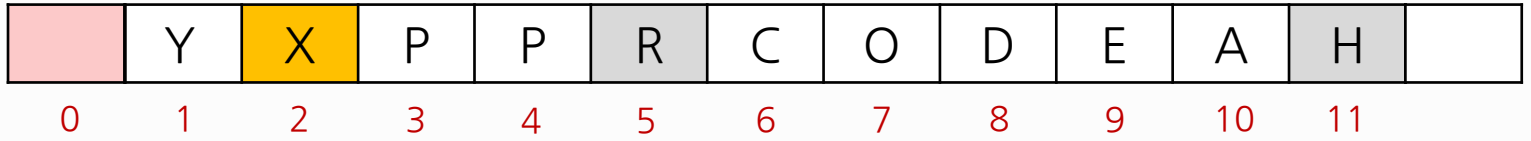
How to code insert 



heap-ordered

heap ordered with two swaps only!

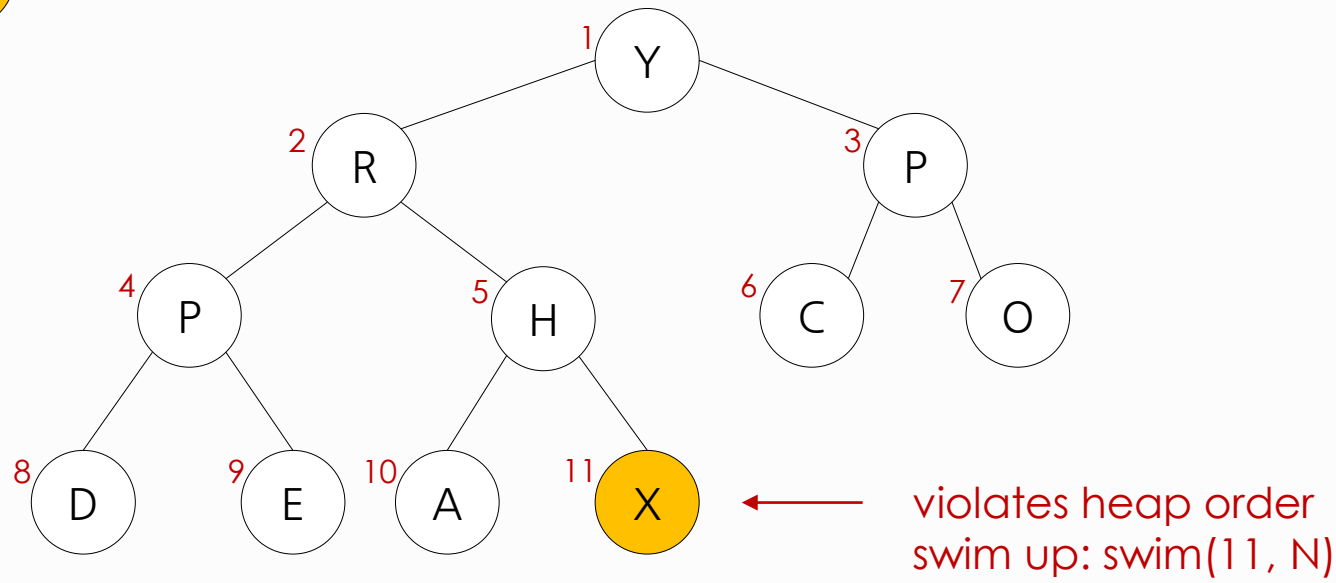
heap-ordered



max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

How to code insert 



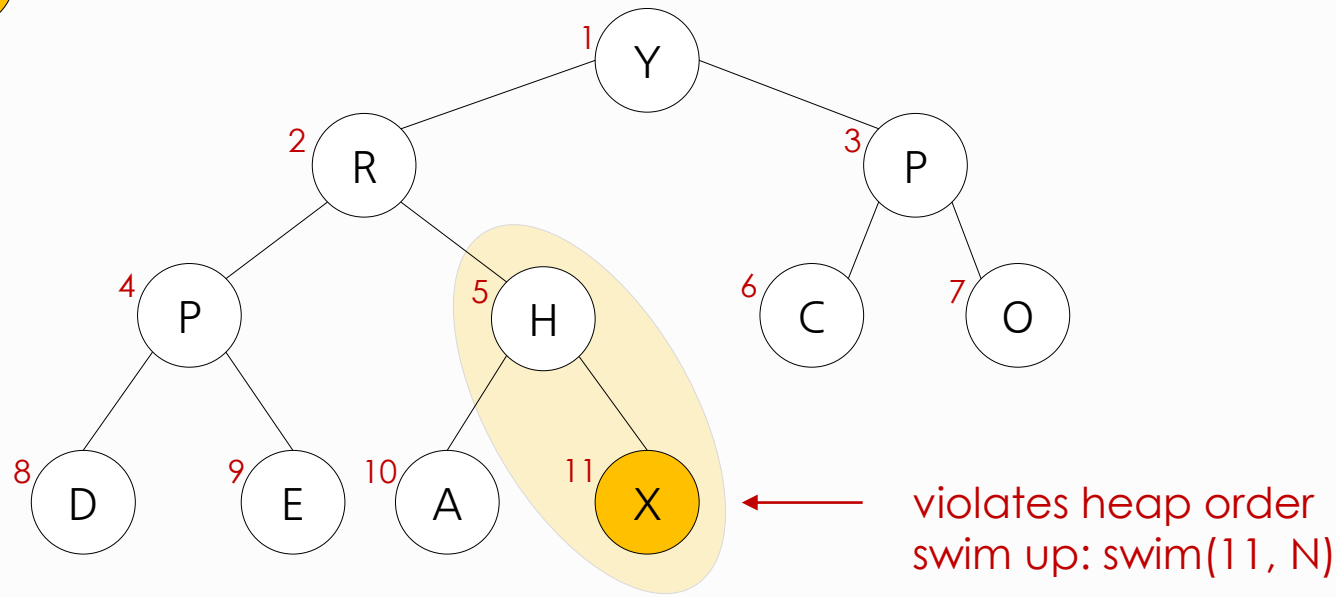
heap-ordered

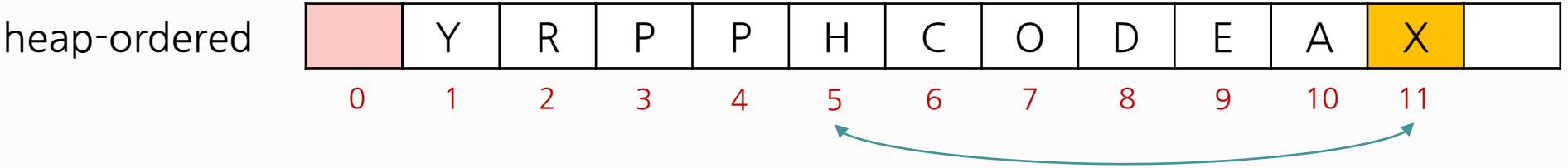
	Y	R	P	P	H	C	O	D	E	A	X	
0	1	2	3	4	5	6	7	8	9	10	11	

max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

How to code insert 

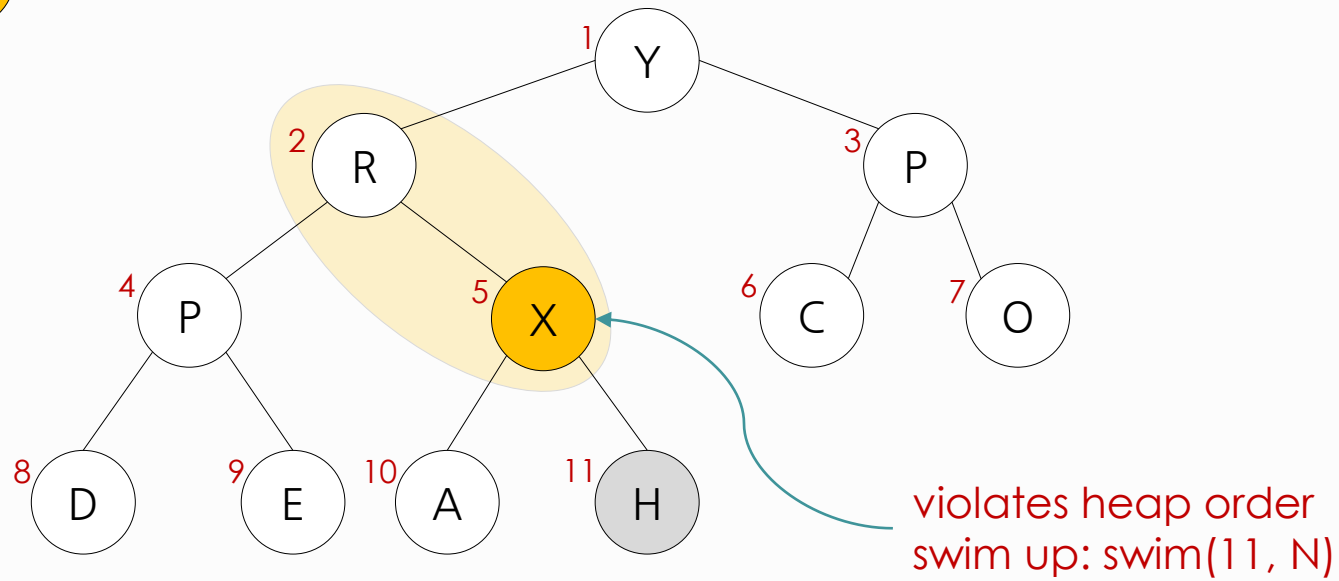




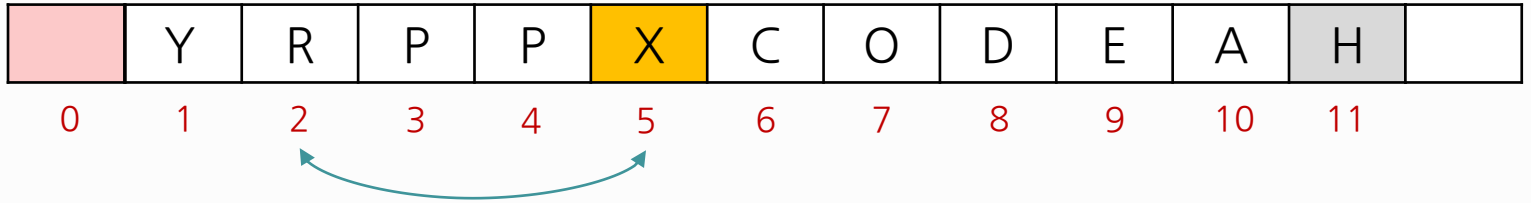
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.

How to code insert 



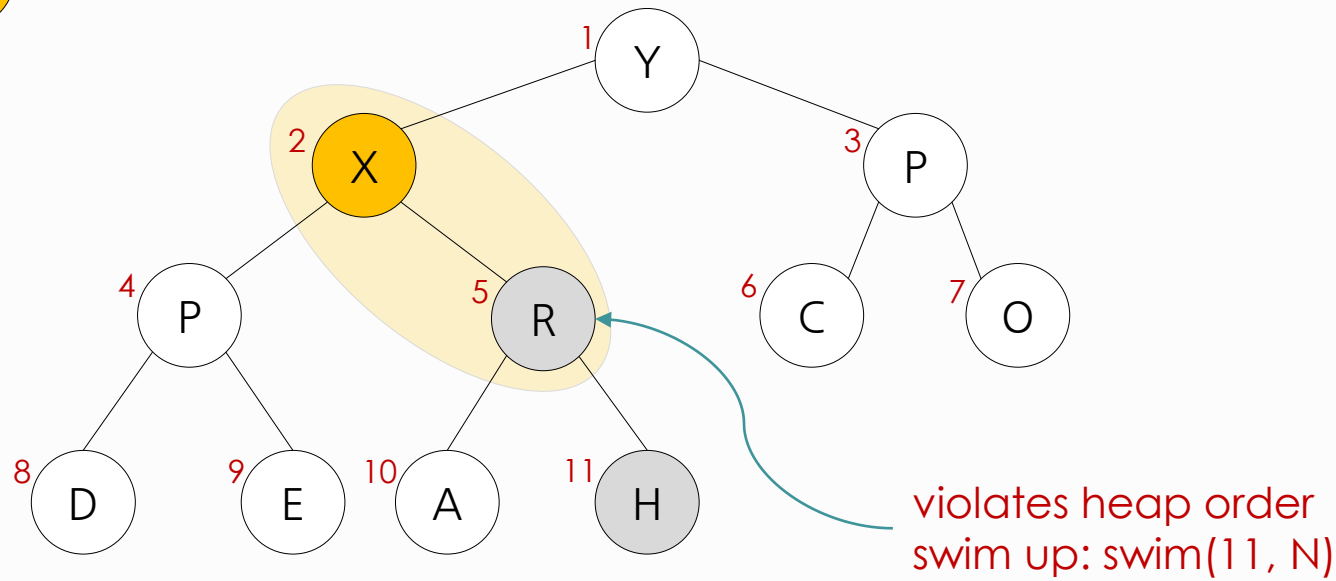
heap-ordered



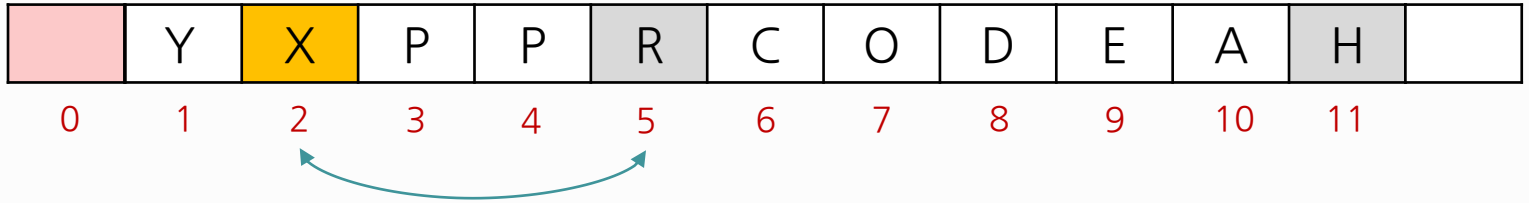
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.

How to code insert 



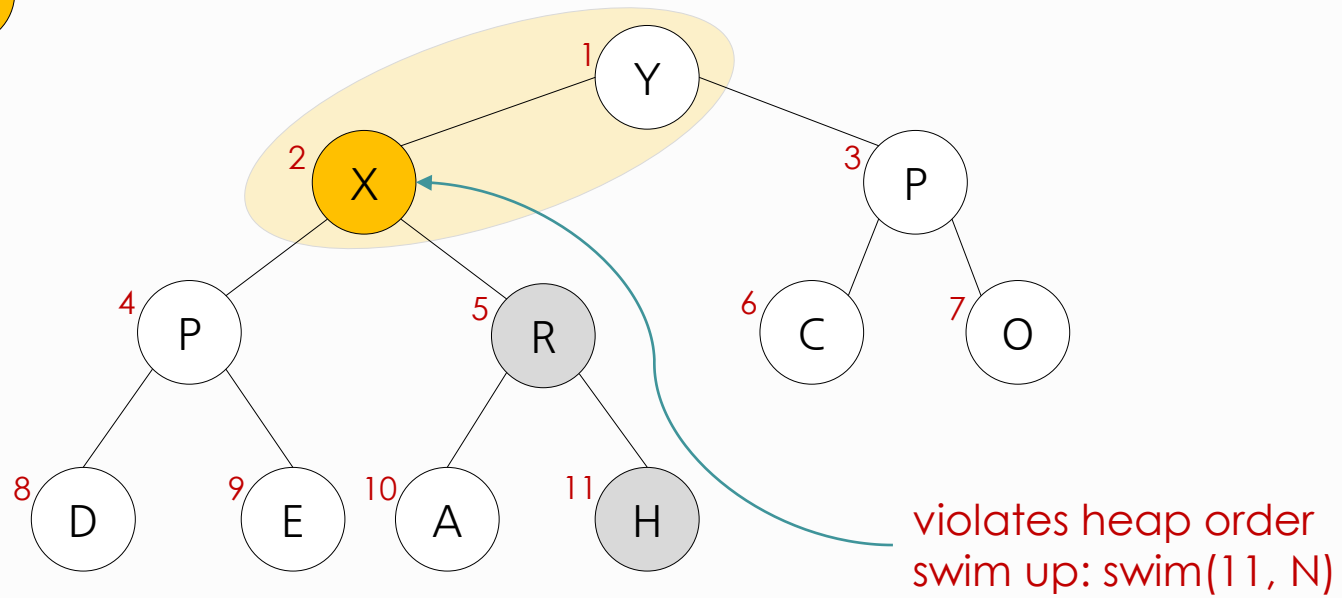
heap-ordered



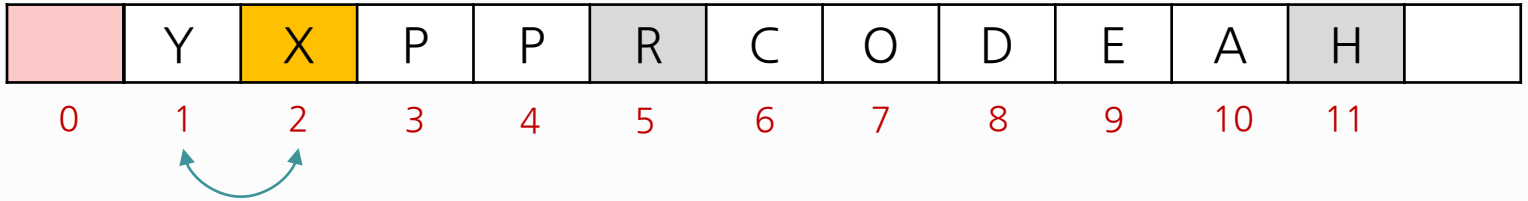
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.

How to code insert 



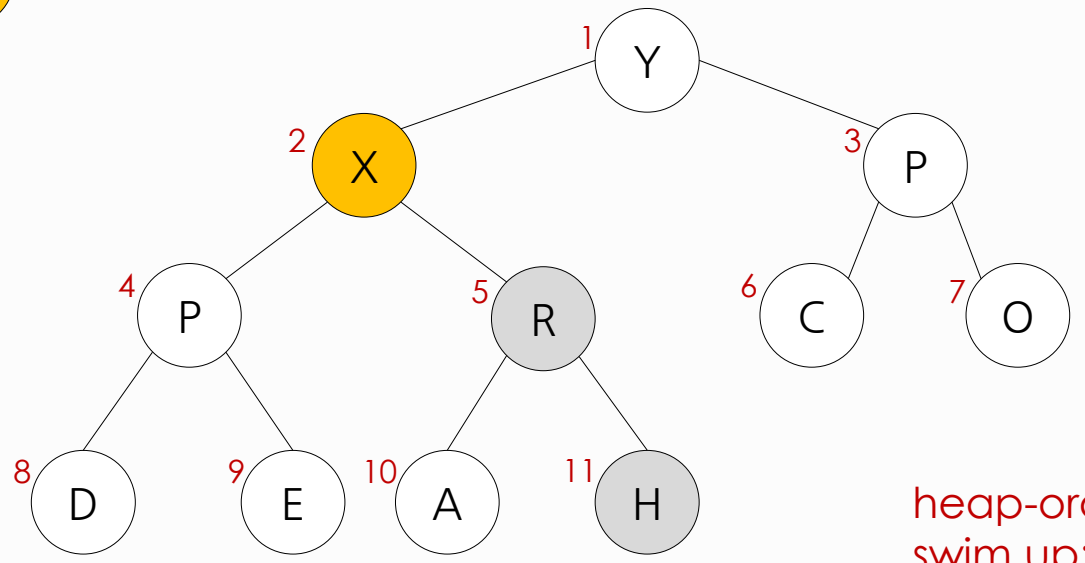
heap-ordered



max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

How to code insert 



heap-ordered
swim up: swim(11, N)

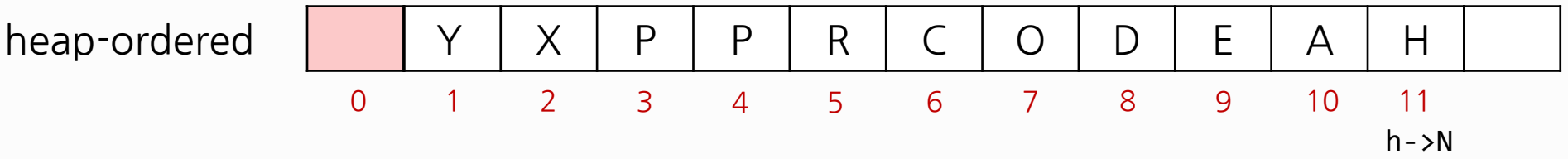
heap-ordered

	Y	X	P	P	R	C	O	D	E	A	H	
0	1	2	3	4	5	6	7	8	9	10	11	

max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

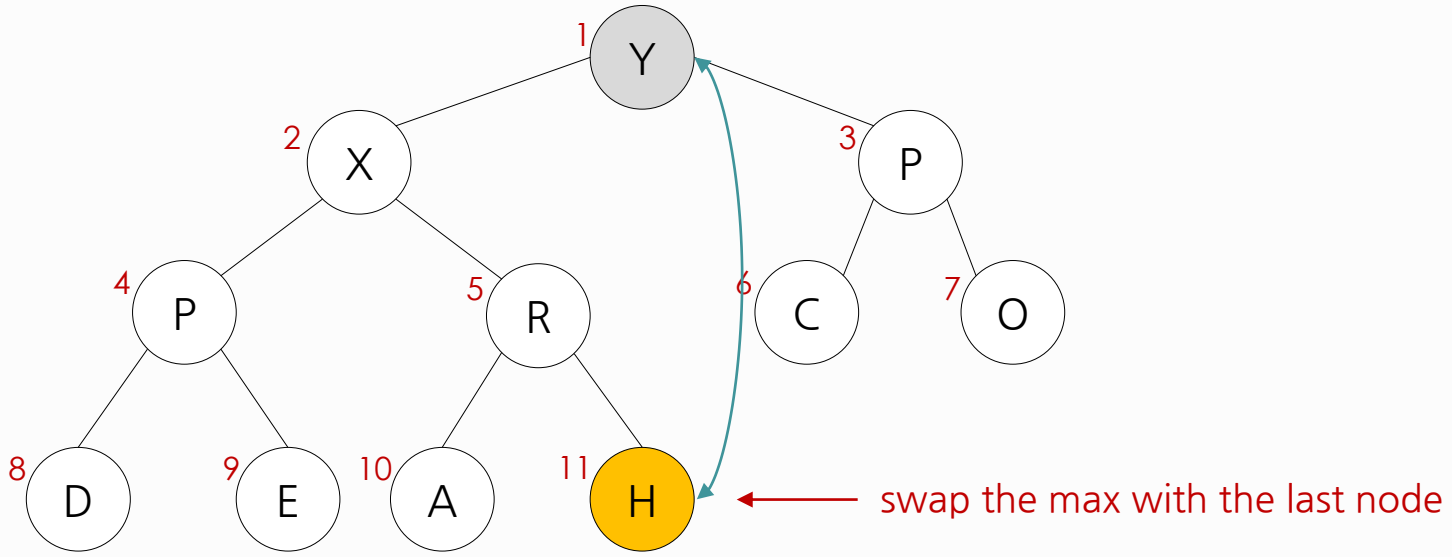
remove the max (root) – pop in priority queue



max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



heap-ordered

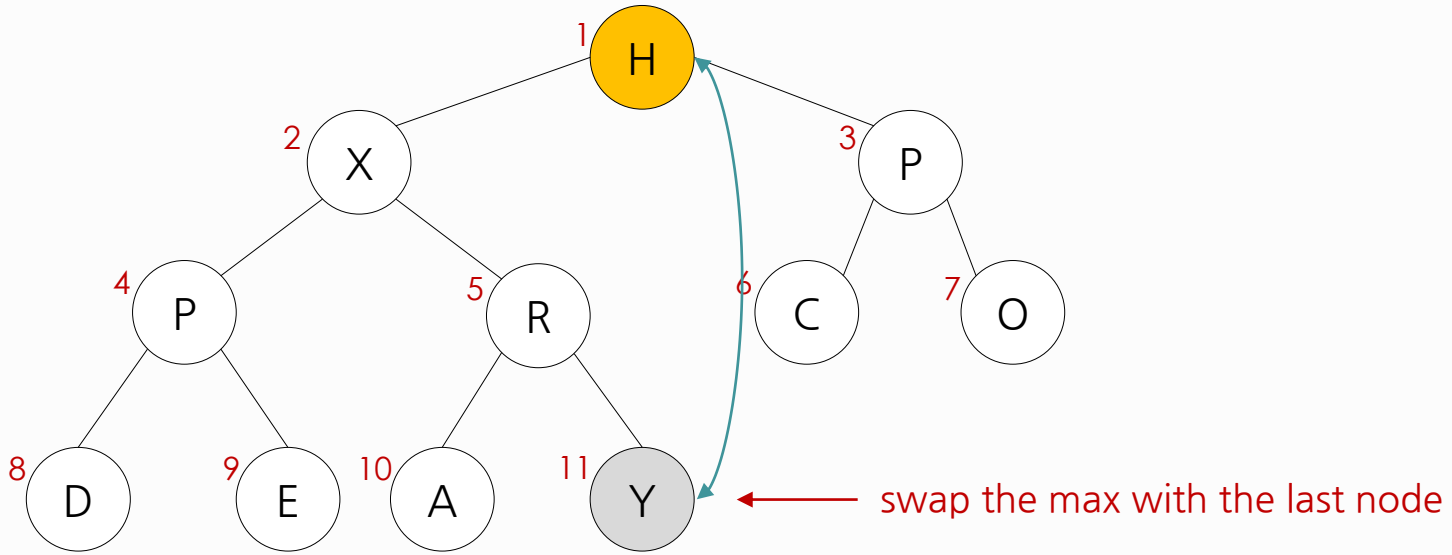
	Y	X	P	P	R	C	O	D	E	A	H	
0	1	2	3	4	5	6	7	8	9	10	11	

h->N

max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



heap-ordered

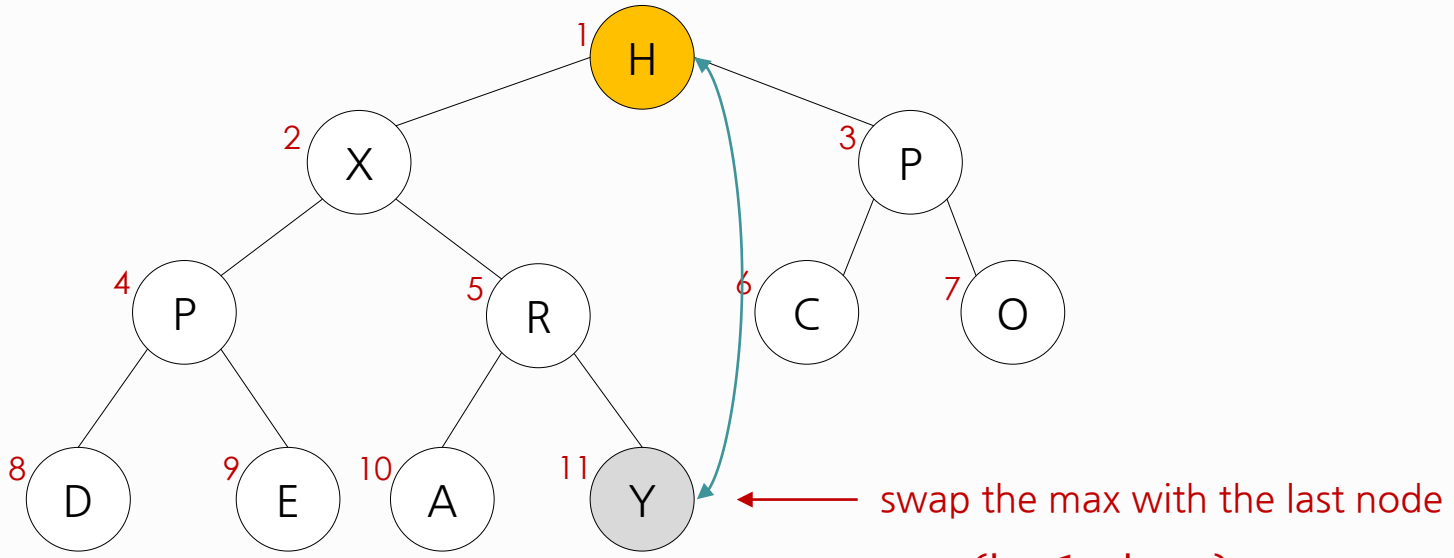
	H	X	P	P	R	C	O	D	E	A	Y	
0	1	2	3	4	5	6	7	8	9	10	11	

h->N

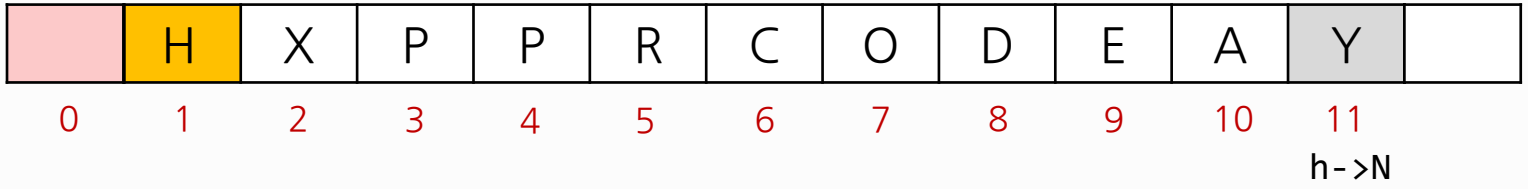
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



heap-ordered

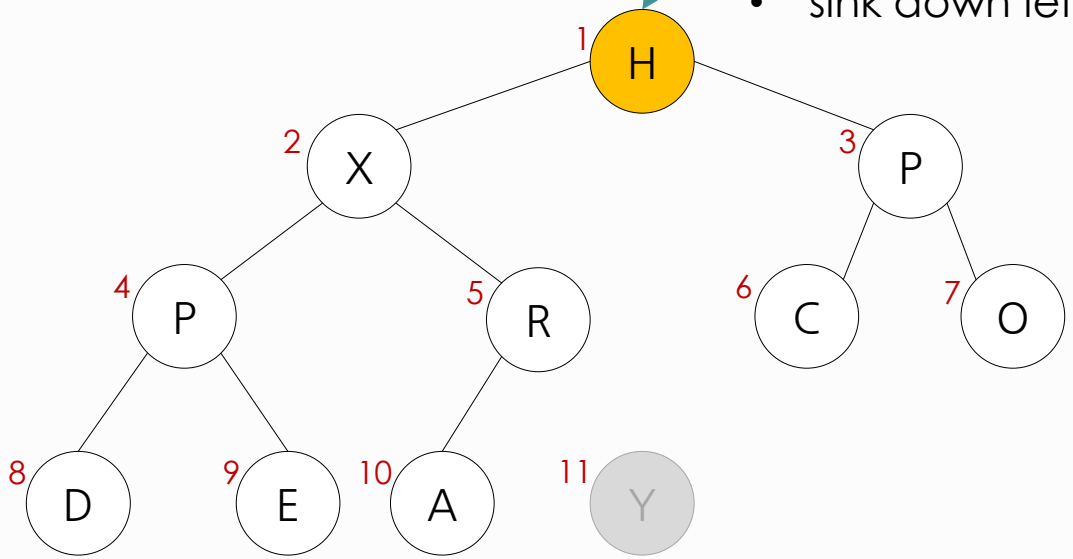


max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)

- violates heap order: sink(1, N)
- sink down left or right?



swap(h, 1, h->N--)

heap-ordered

	H	X	P	P	R	C	O	D	E	A	Y	
0	1	2	3	4	5	6	7	8	9	10	11	

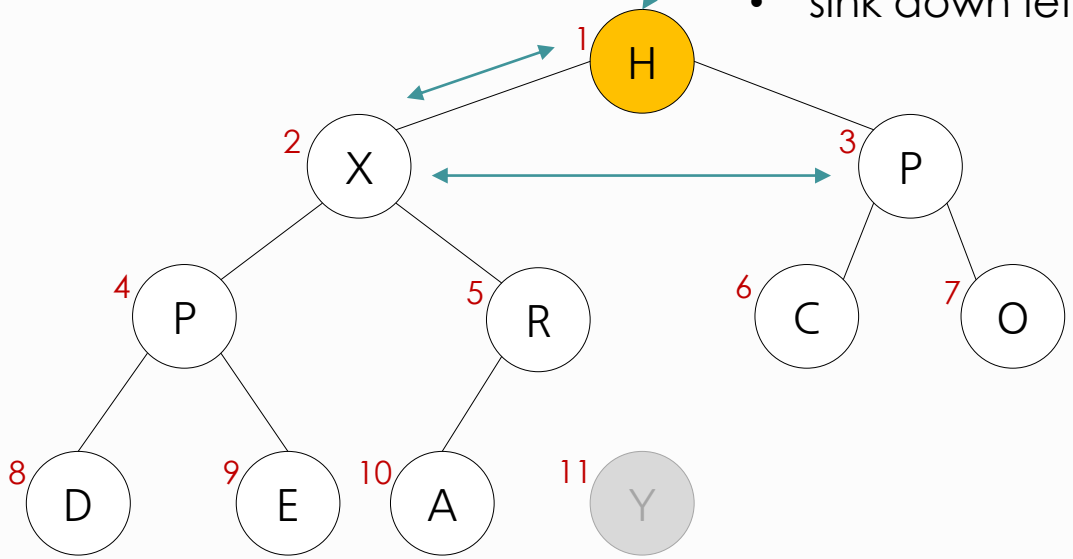
h->N heap size decreased by one

max-heap example

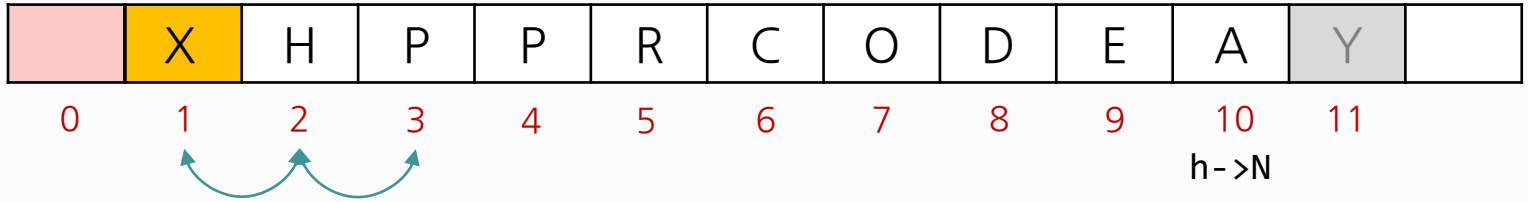
- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)

- violates heap order: sink(1, N)
- sink down left or right?



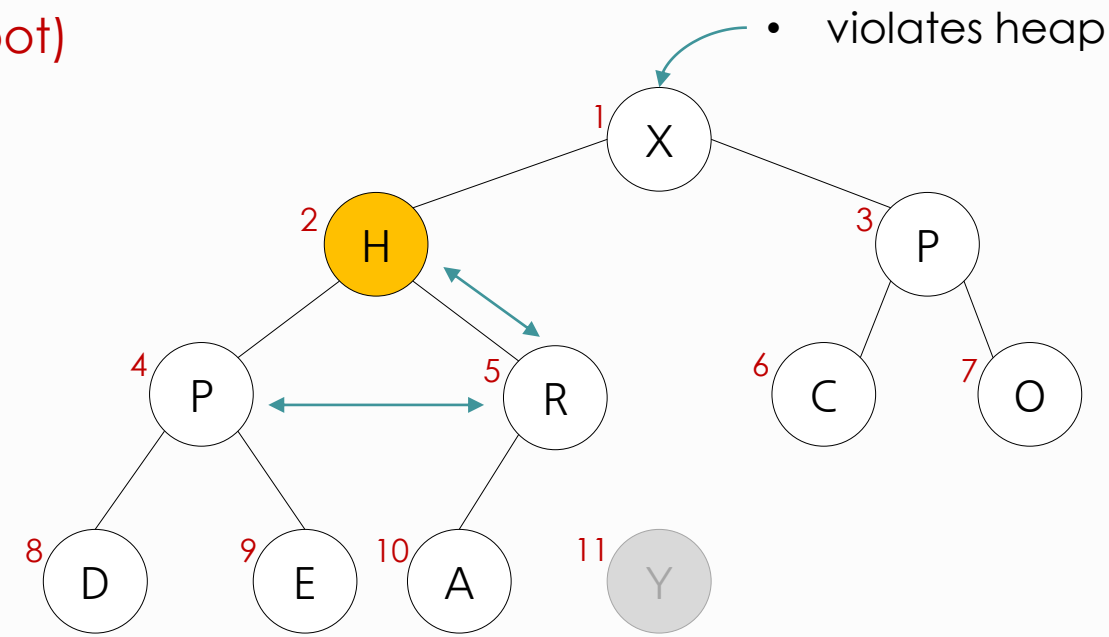
heap-ordered



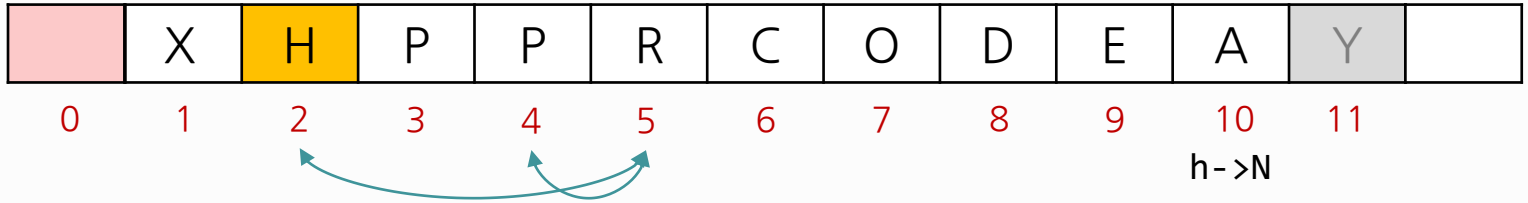
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



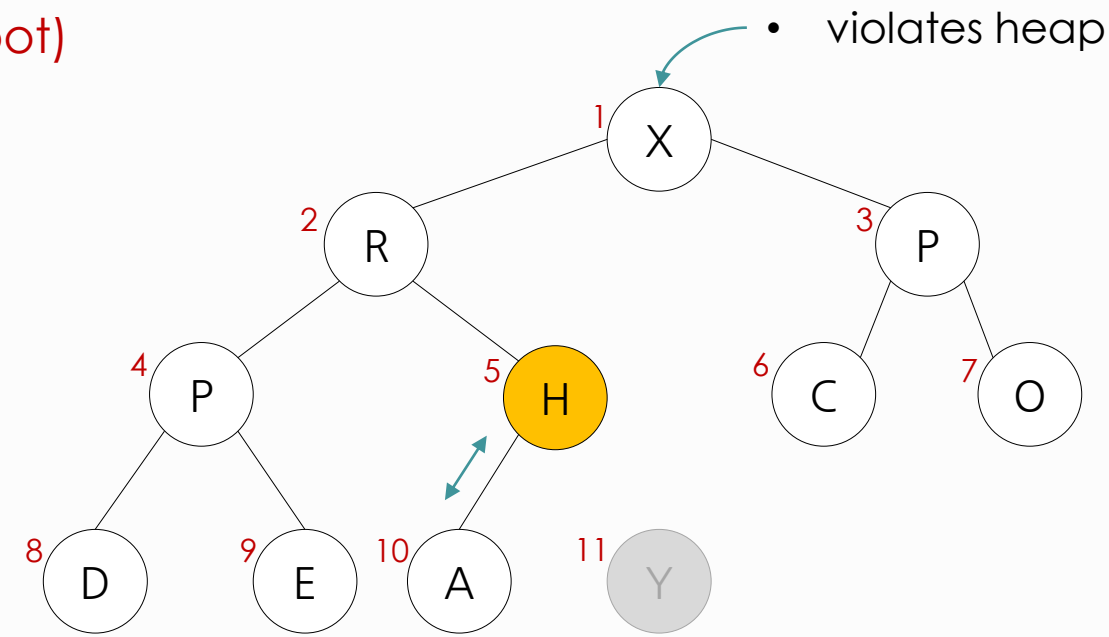
heap-ordered



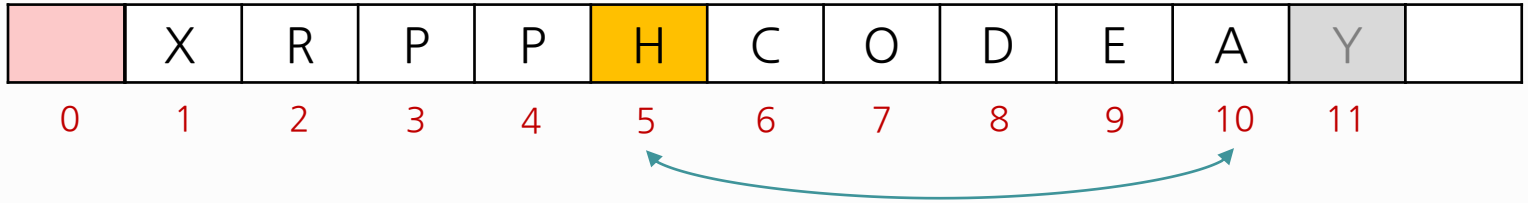
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



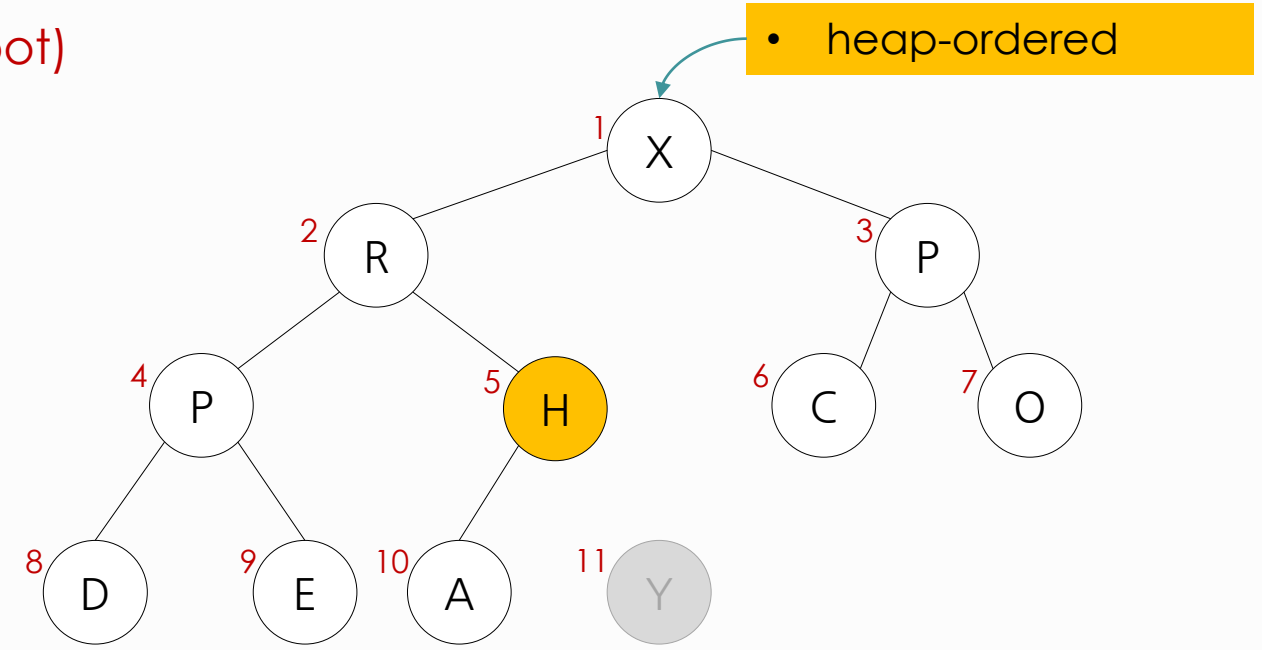
heap-ordered



max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



heap-ordered

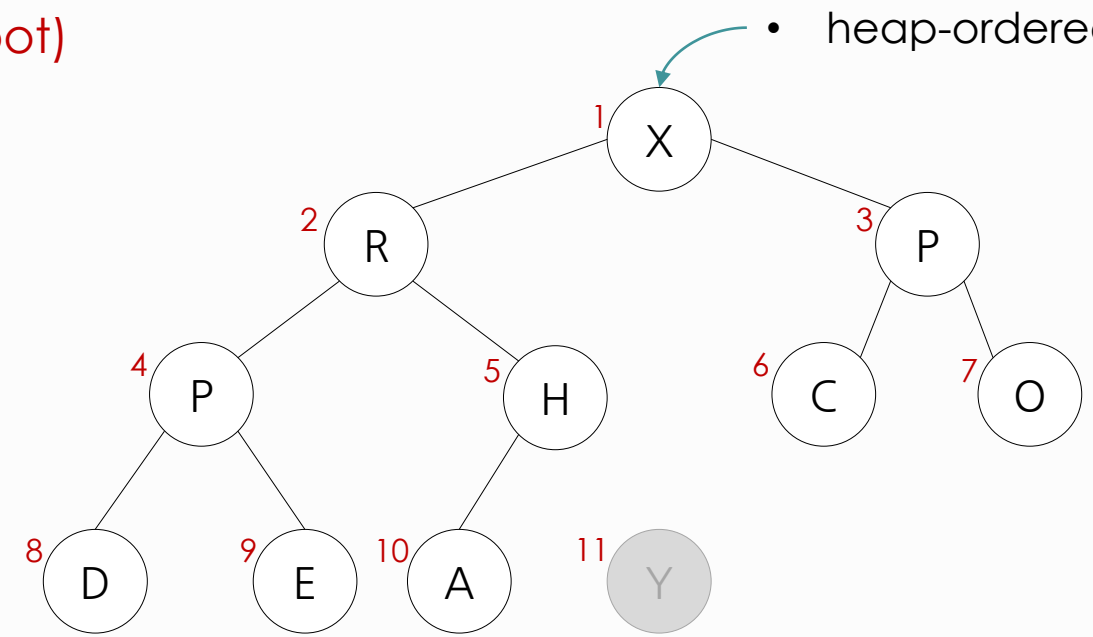
	X	R	P	P	H	C	O	D	E	A	Y	
0	1	2	3	4	5	6	7	8	9	10	11	

h->N

max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



heap-ordered

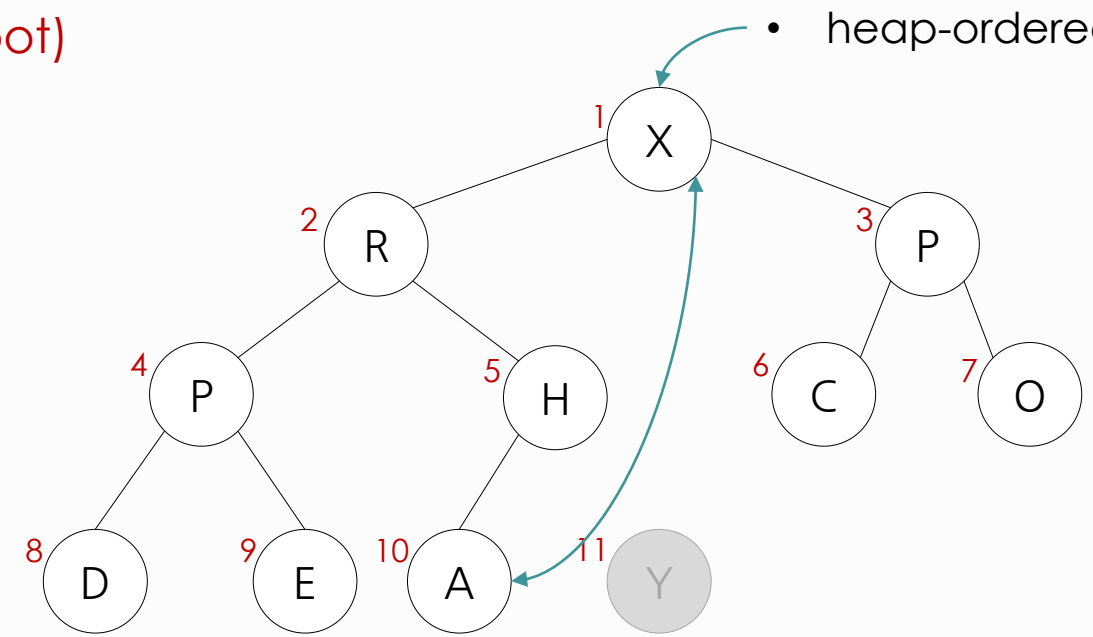
	X	R	P	P	H	C	O	D	E	A	Y	
0	1	2	3	4	5	6	7	8	9	10	11	

h->N

max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



swap(h, 1, h->N--)

heap-ordered

	X	R	P	P	H	C	O	D	E	A	Y	
0	1	2	3	4	5	6	7	8	9	10	11	

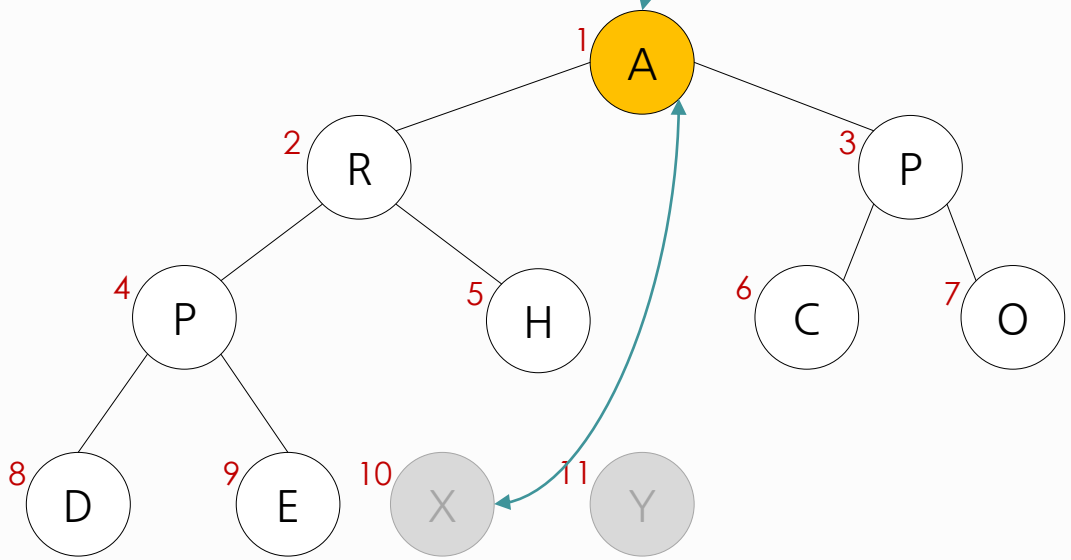
h->N heap size decreased by one

max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)

- violates heap order: sink(1, N)



swap(h, 1, h->N--)

heap-ordered

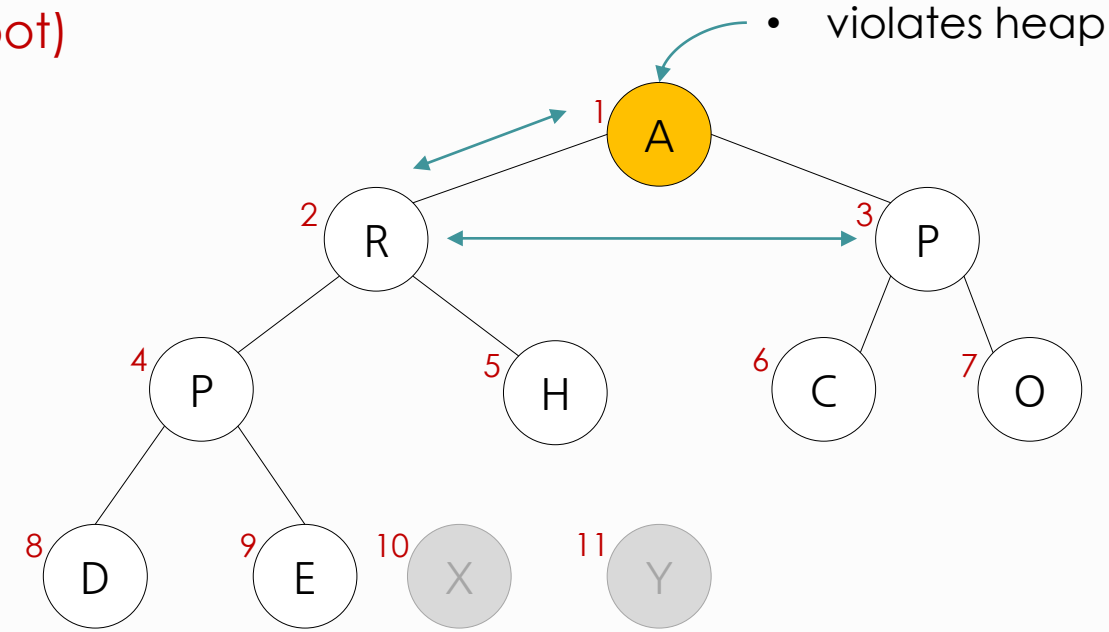
	A	R	P	P	H	C	O	D	E	X	Y	
0	1	2	3	4	5	6	7	8	9	10	11	

h->N heap size decreased by one

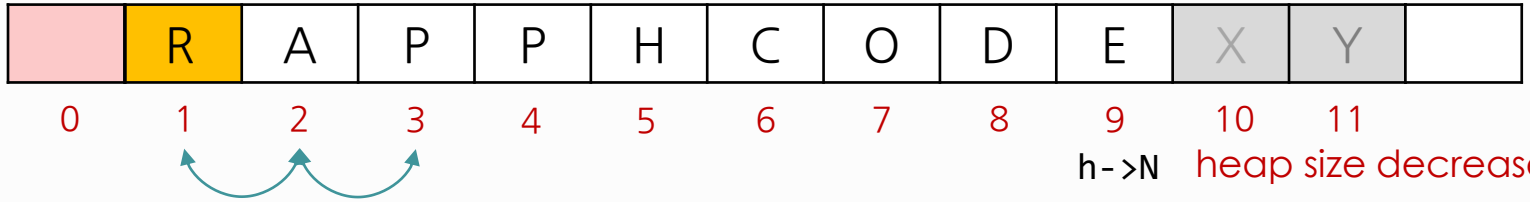
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



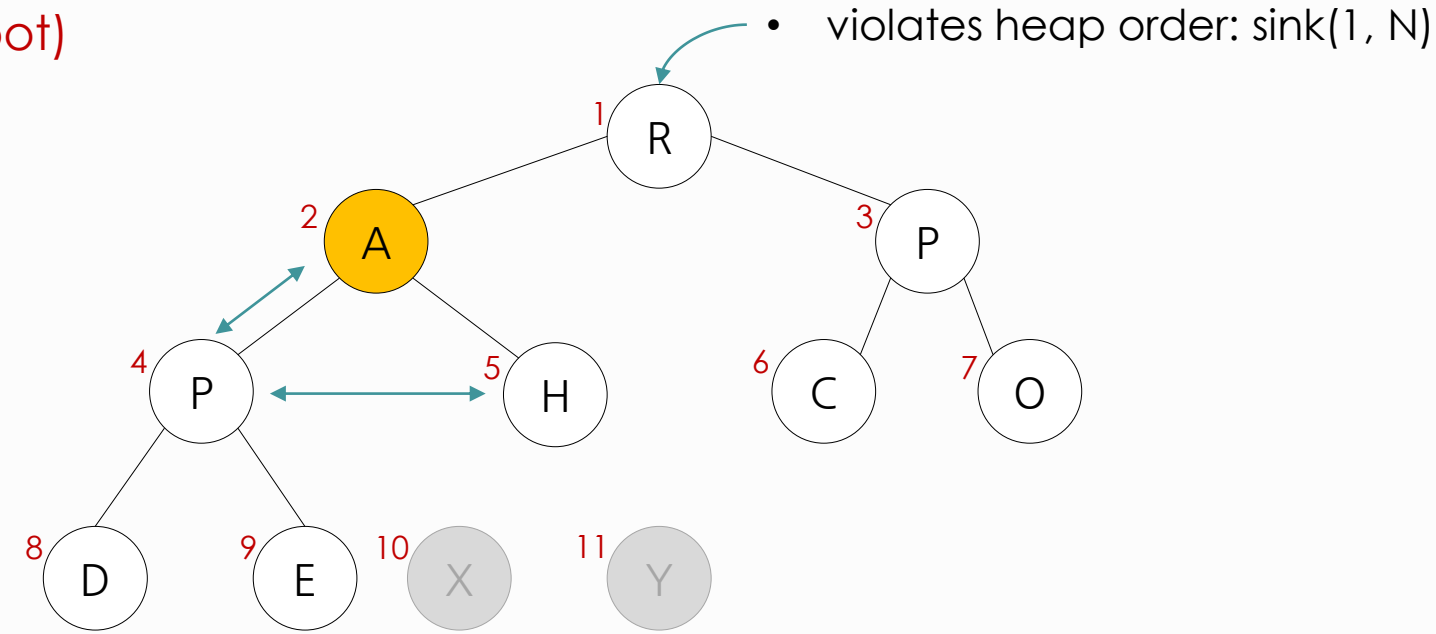
heap-ordered



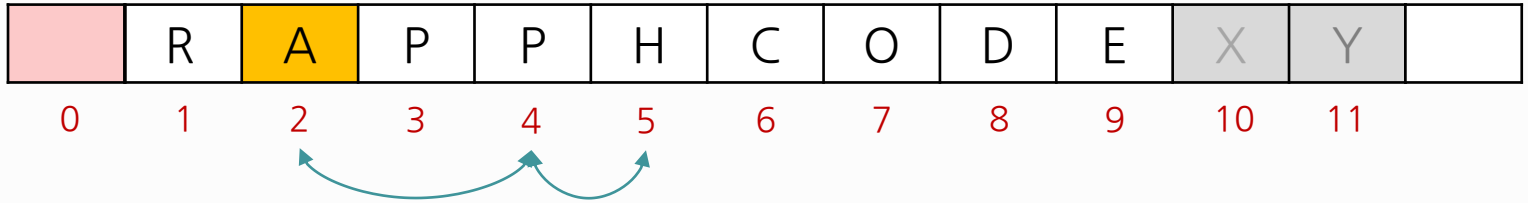
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



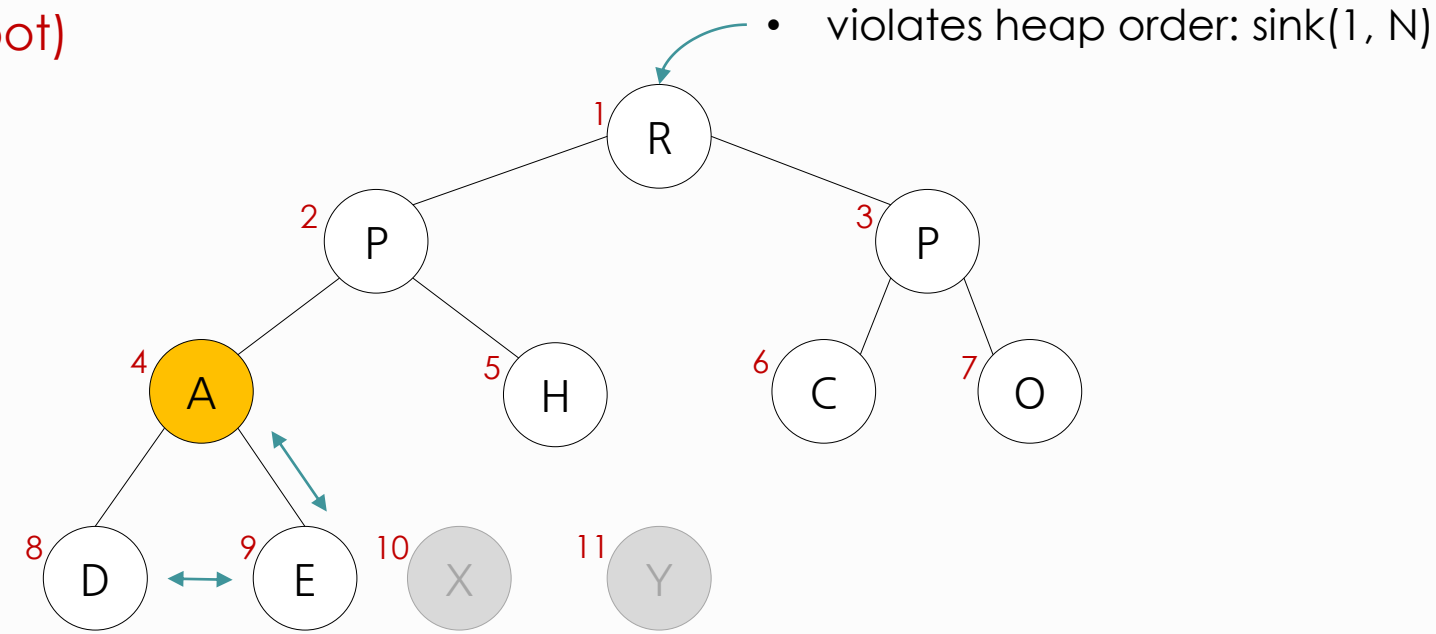
heap-ordered

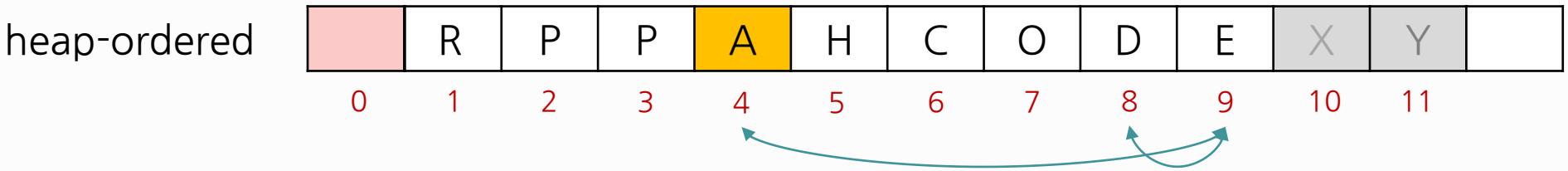


max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)

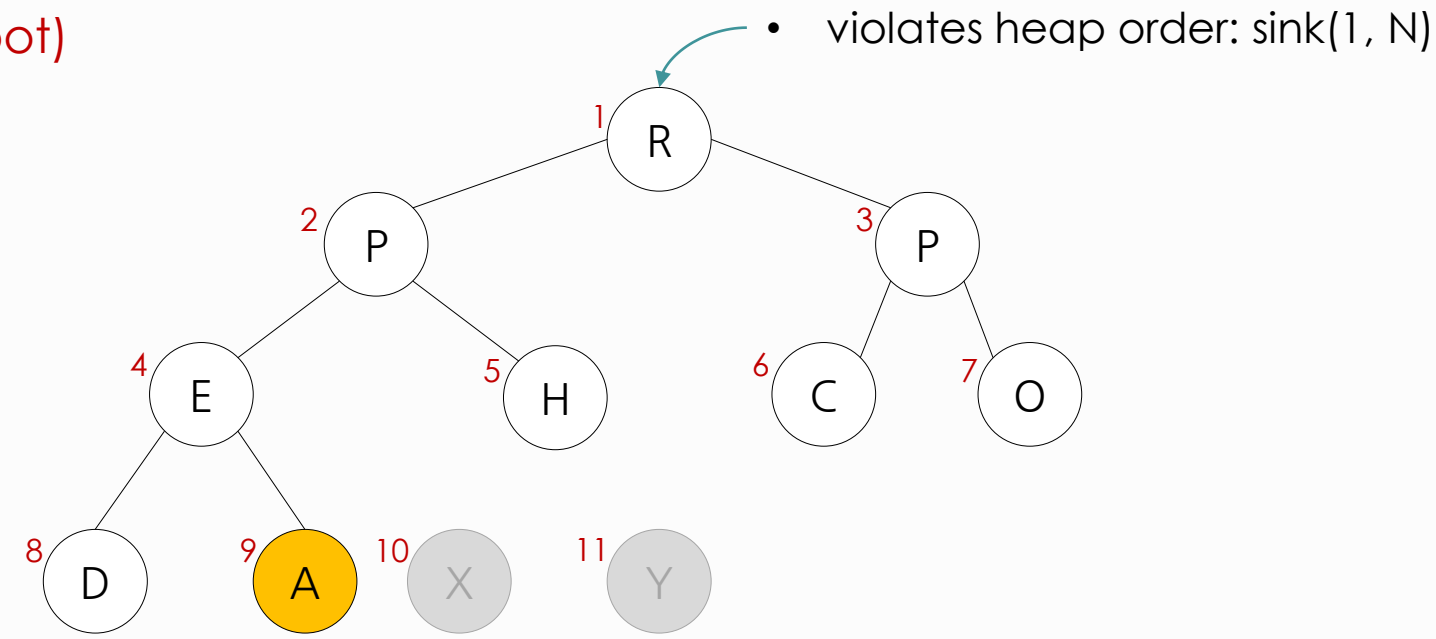




max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)



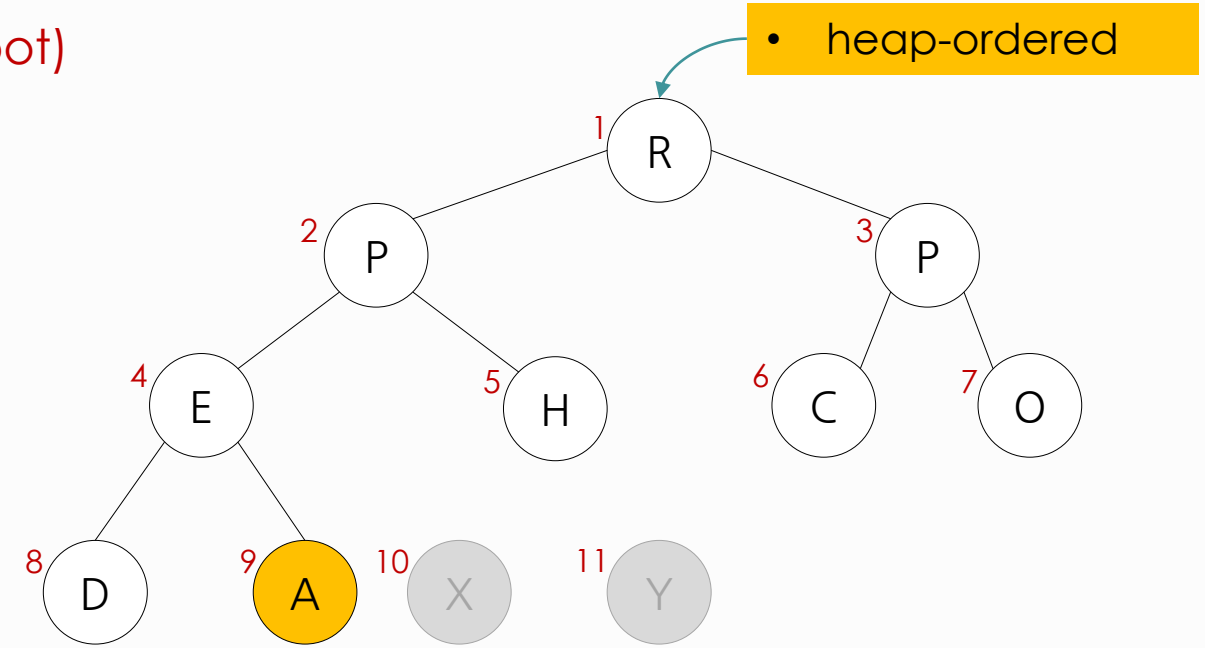
heap-ordered

	R	P	P	E	H	C	O	D	A	X	Y
0	1	2	3	4	5	6	7	8	9	10	11

max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**

remove the max (root)

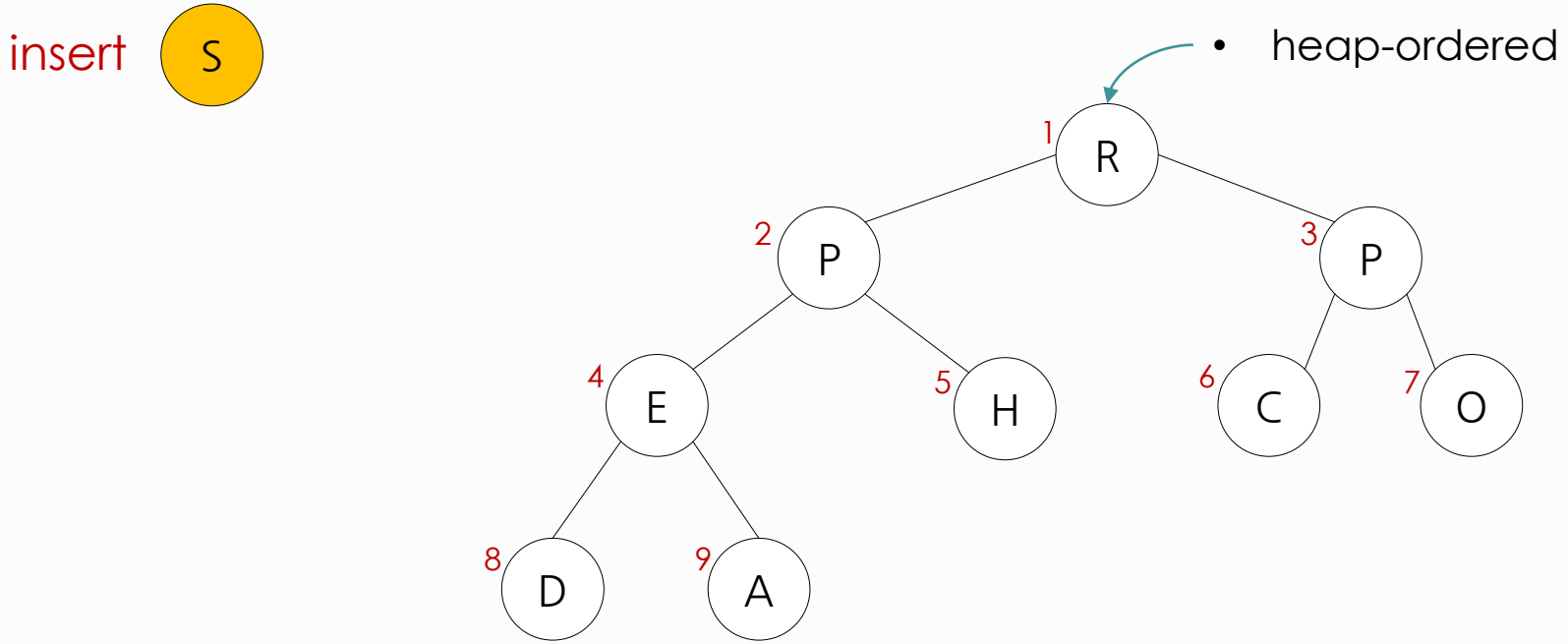


heap-ordered

	R	P	P	E	H	C	O	D	A	X	Y	
0	1	2	3	4	5	6	7	8	9	10	11	

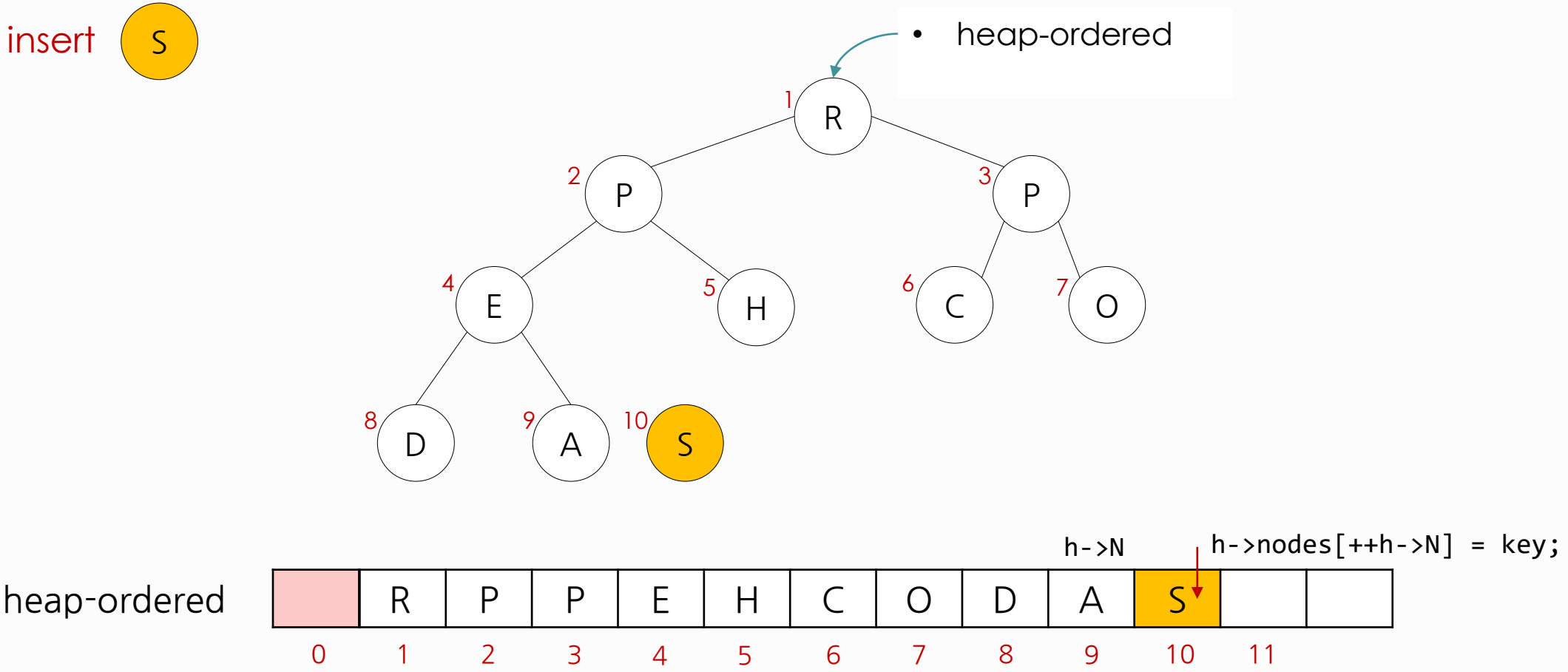
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**



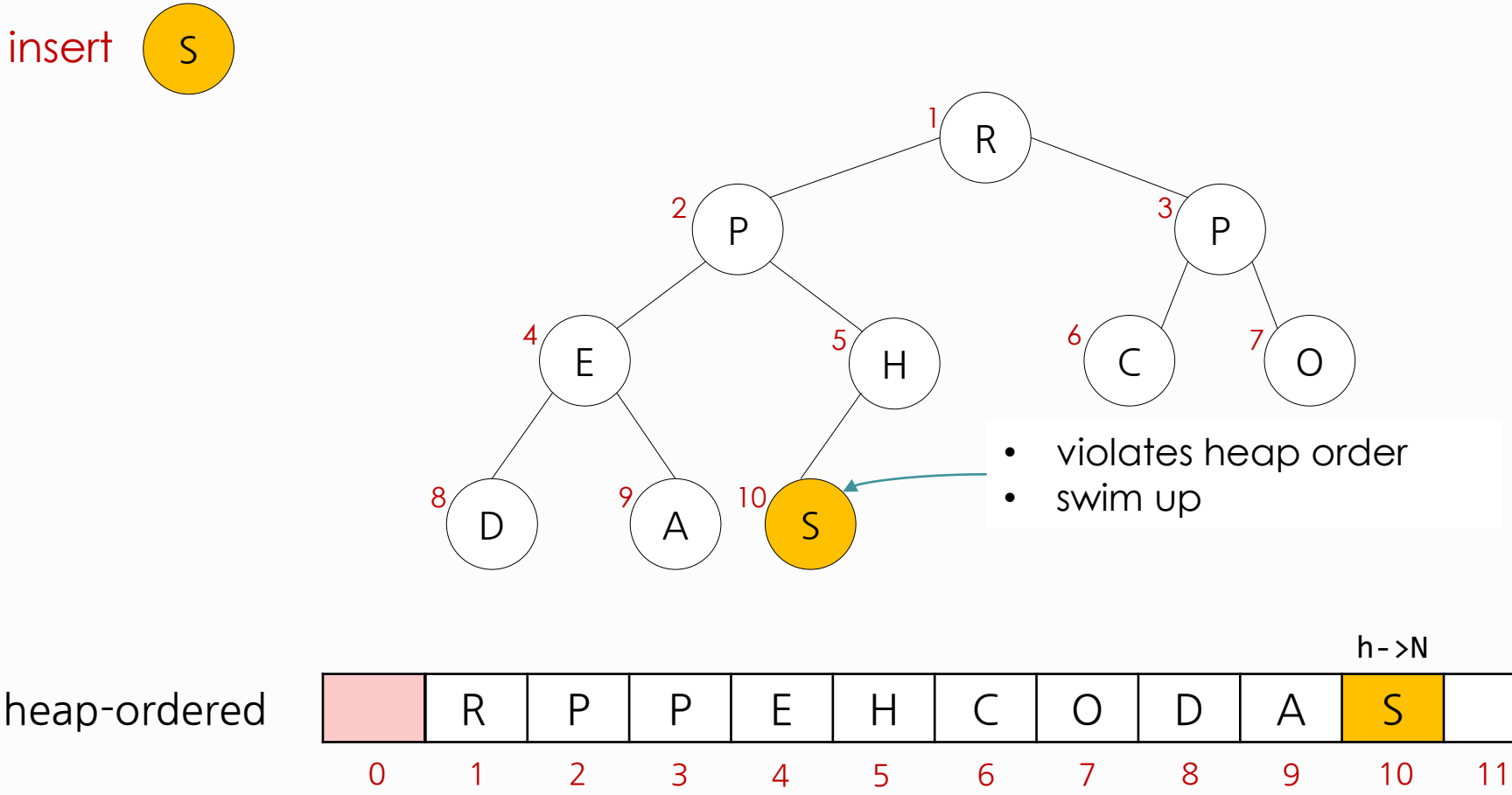
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**



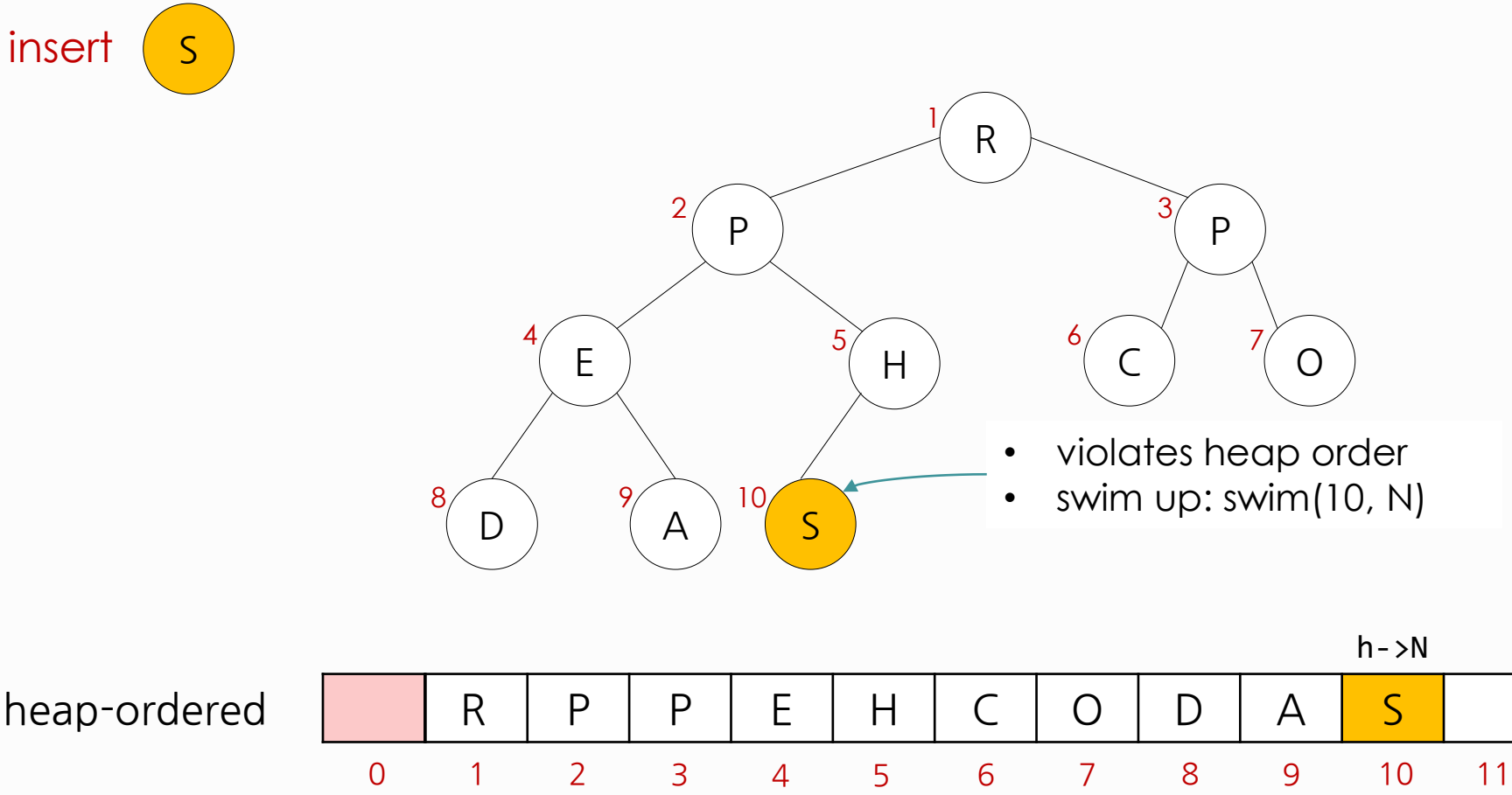
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**



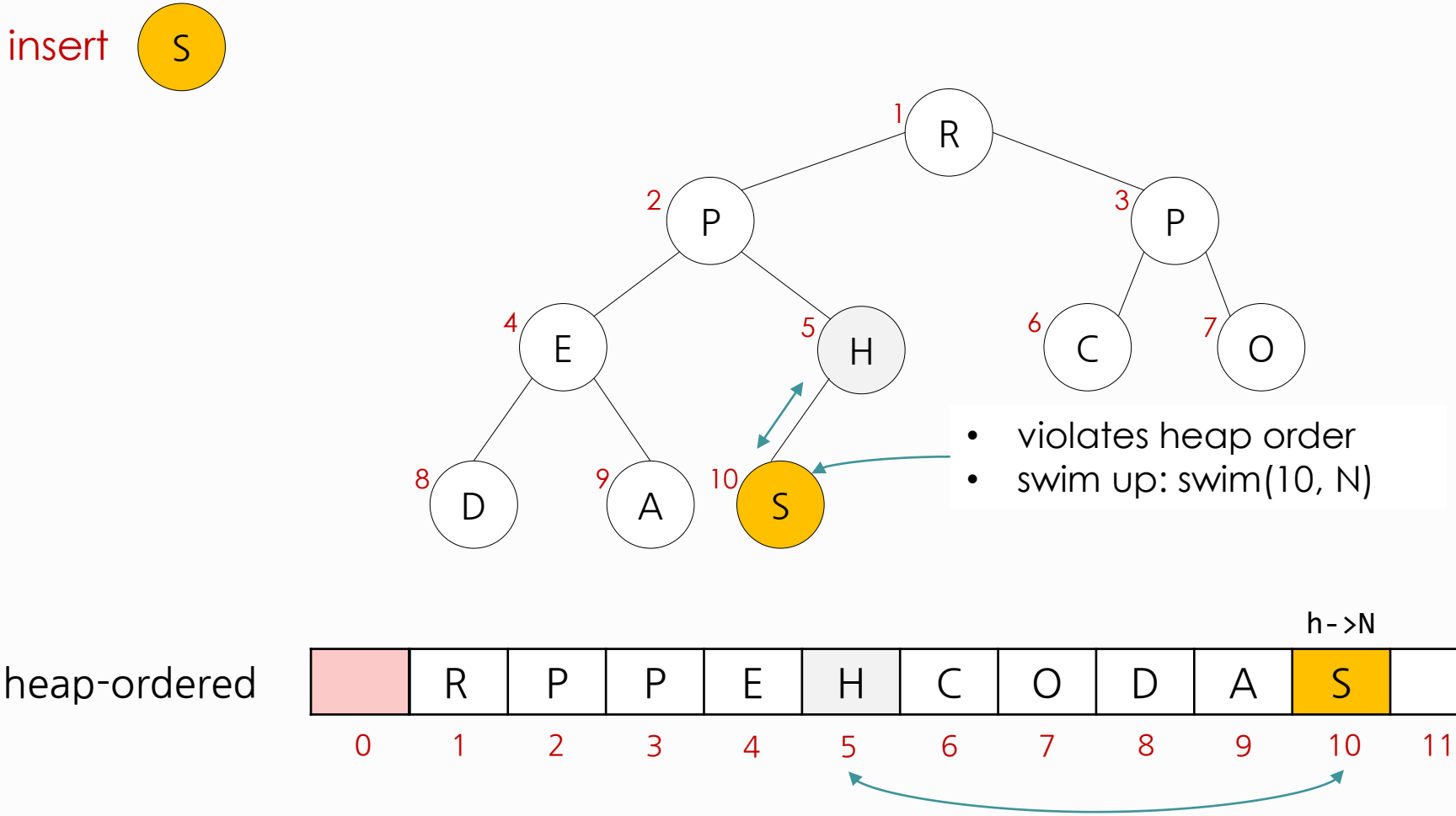
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**



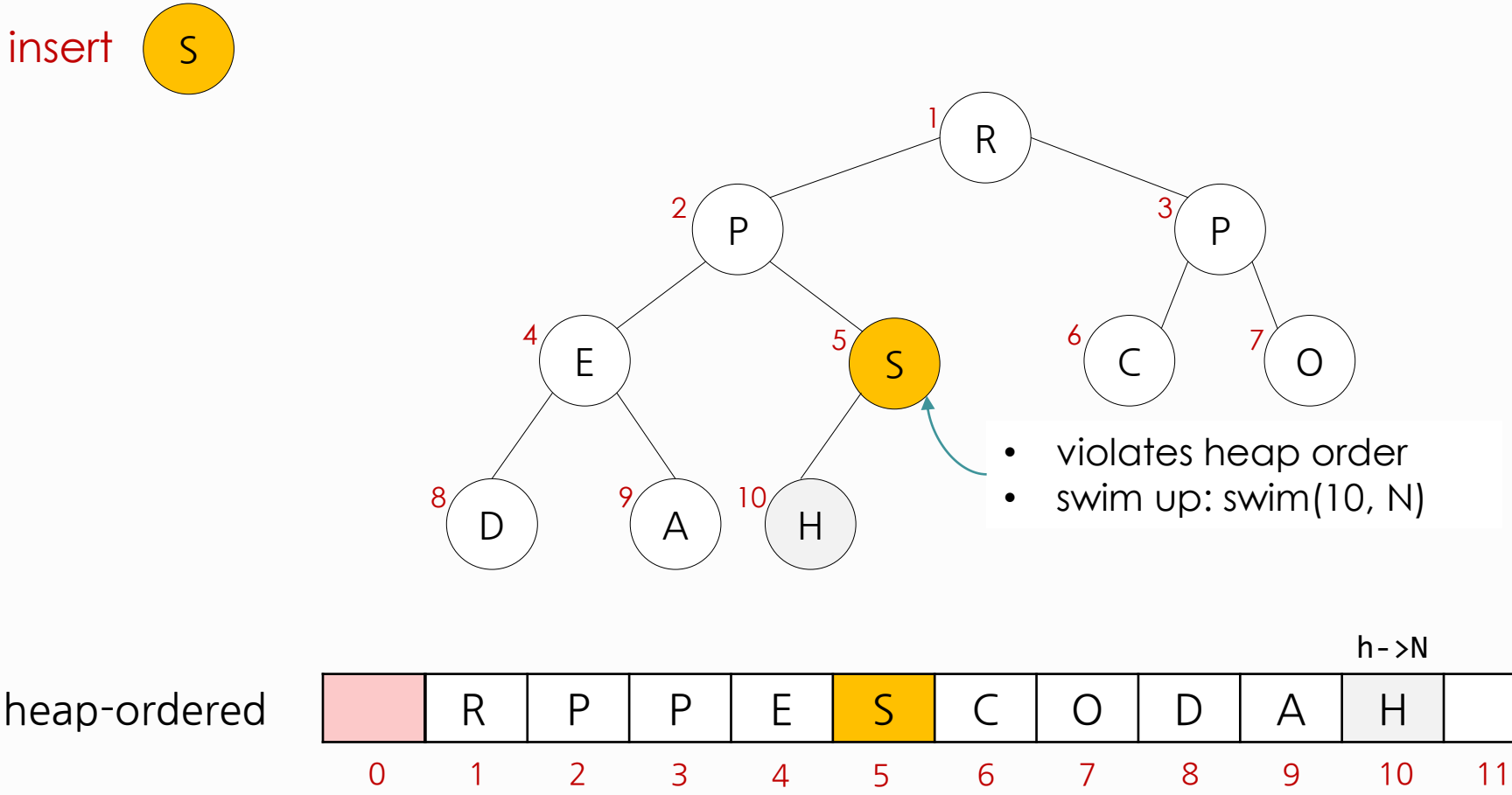
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.



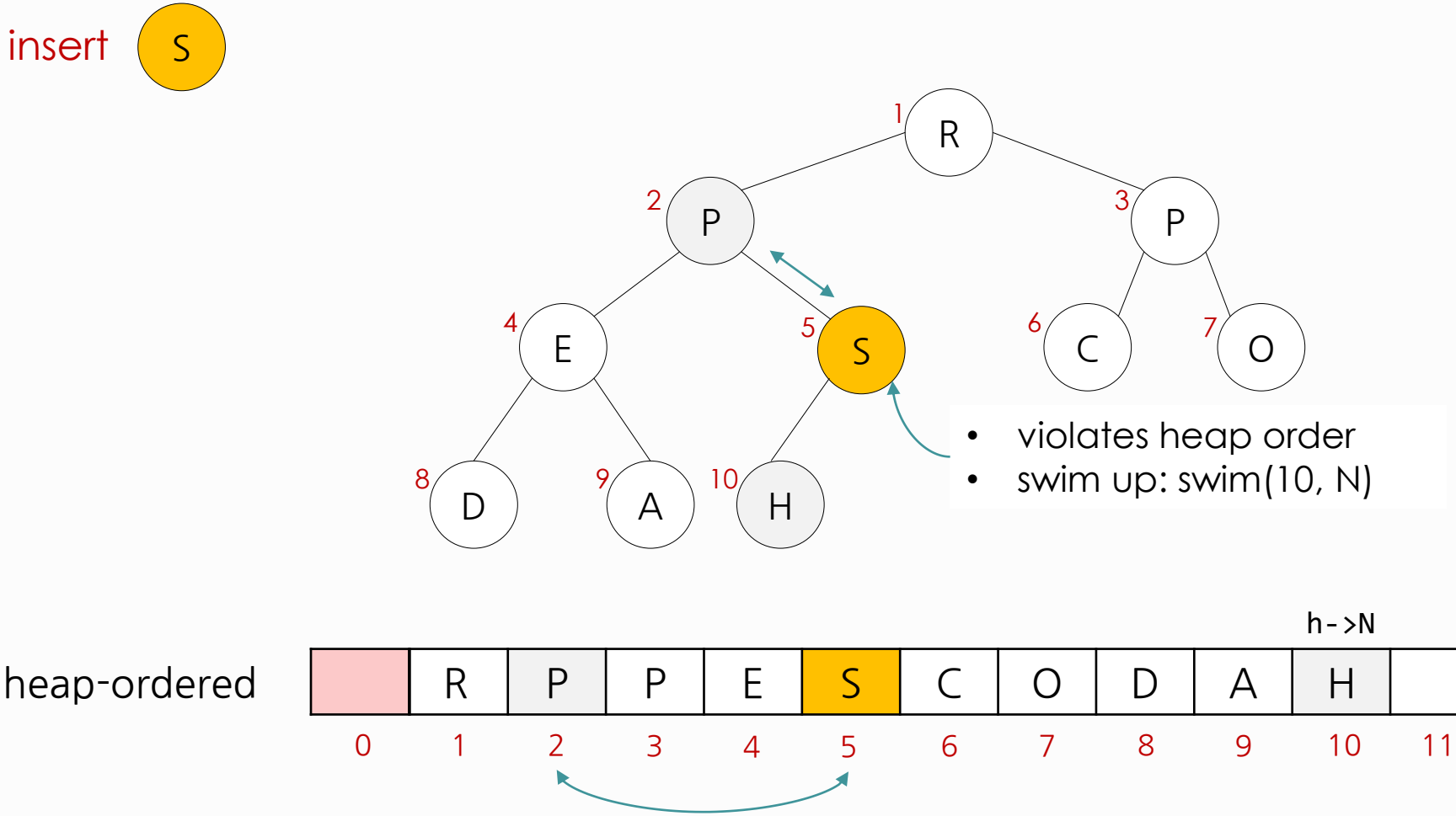
max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**



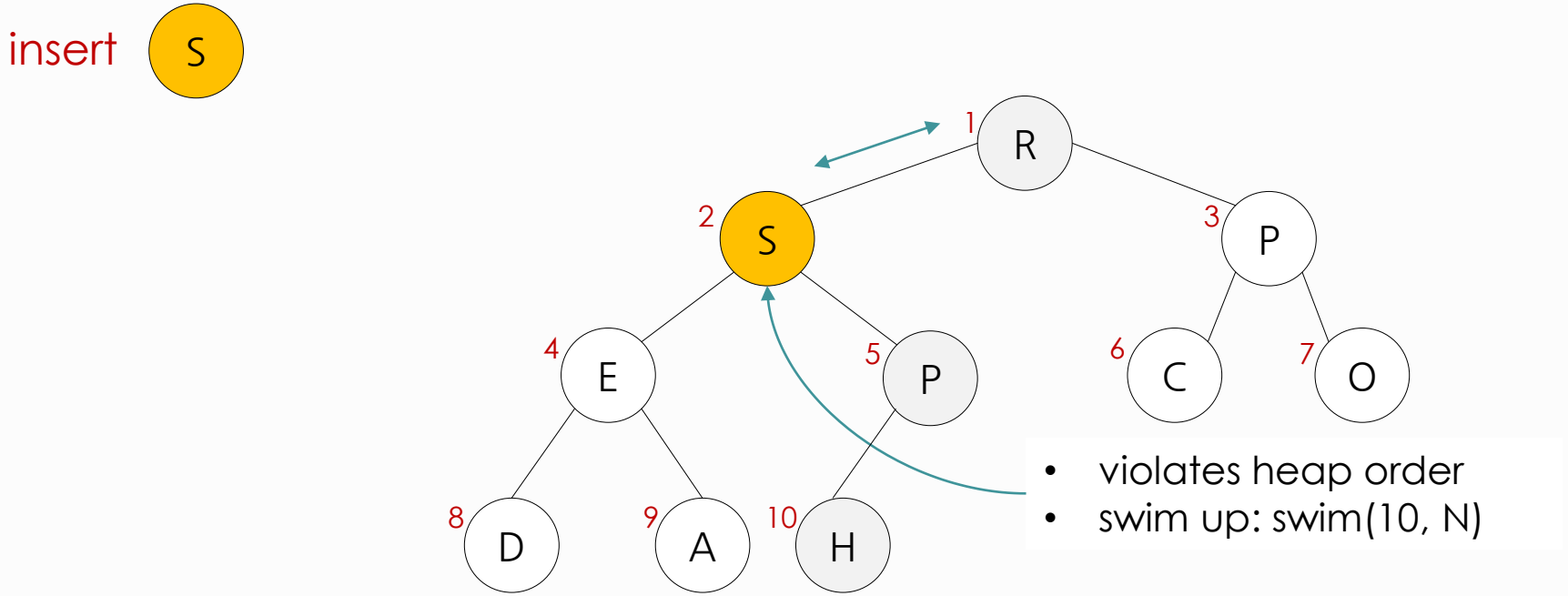
max-heap example

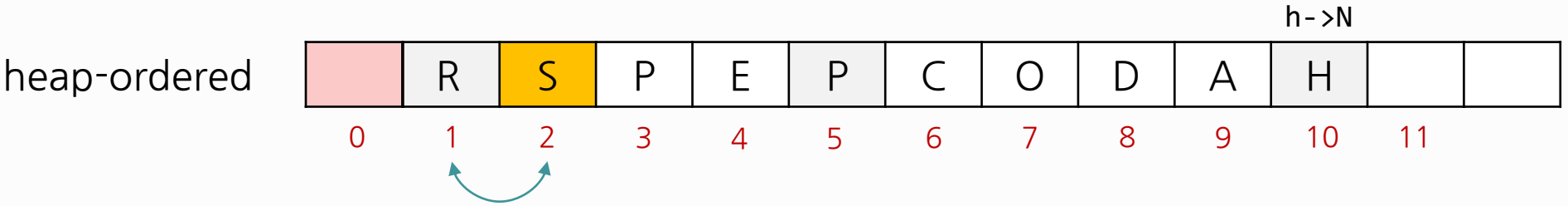
- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**



max-heap example

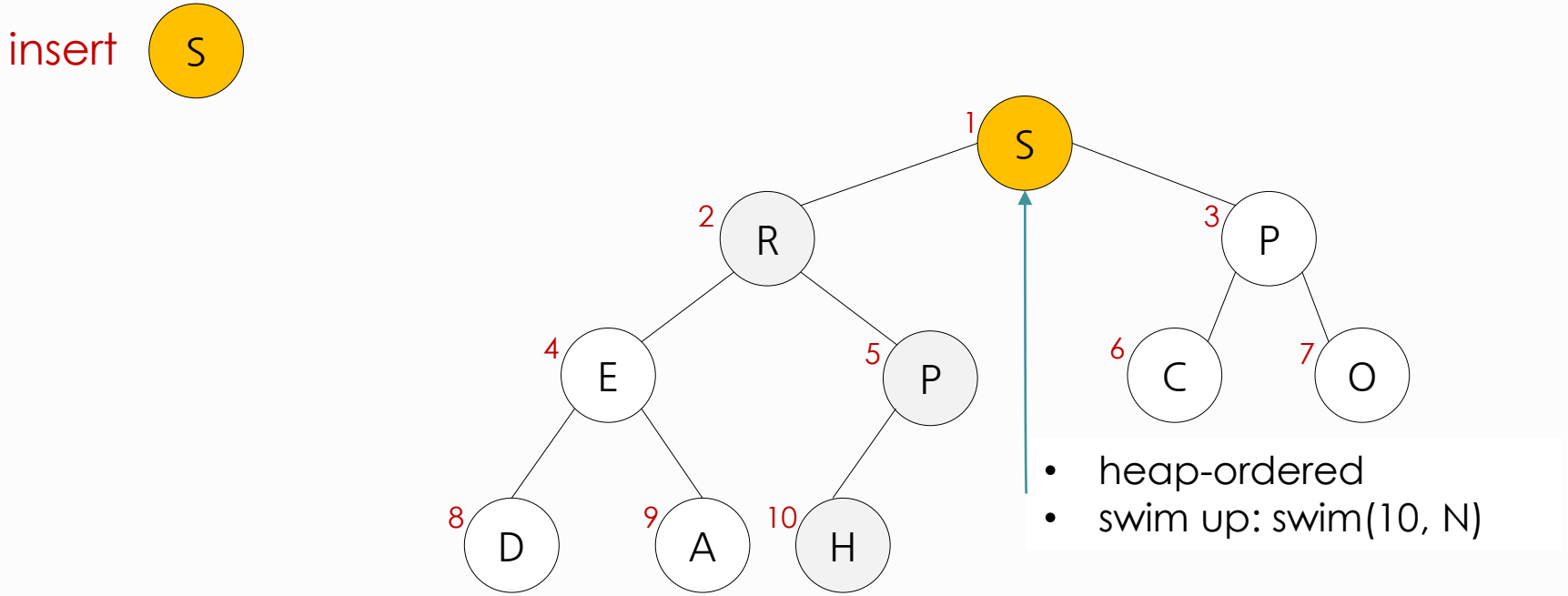
- **Insert**: Add node at end, then swim it up.
- **Remove**: Swap root with last node, then sink down.

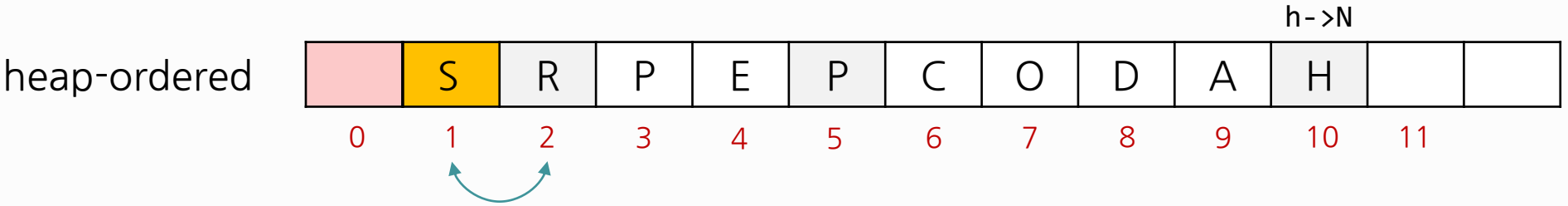




max-heap example

- **Insert**: Add node at end, then swim it up.
- **Remove: Swap root with last node, then sink down.**





Binary heap operations time complexity with N items:

- Level of heap is $\lfloor \log_2 N \rfloor$
- insert: O(log N) for each insert
  - In practice, expect less
- delete: O(log N) // deleting root node or any node
- increase/decrease key: O(log N)
- **Heapify():** O(N)
- **Heapsort():** O(N log N)
- Because O(N) heapify + O(log N) delete = O(N log N)

- **Proof:**
  - https://stackoverflow.com/questions/9755721/how-can-building-a-heap-be-on-time-complexity
  - https://www.insertingwiththeweb.com/data-structures/binary-heap/build-heap-proof/
  - https://www.quora.com/How-is-the-time-complexity-of-building-a-heap-is-o-n
- **References in Korean:**
  - https://ratsgo.github.io/data%20structure&algorithm/2017/09/27/heapsort/
  - https://zeddios.tistory.com/56