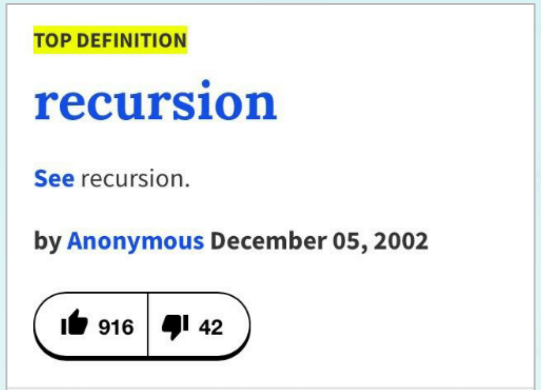
Data Structures
Chapter 1

1. **Recursion**
   - **Recursion**
   - Mergesort
2. Performance Analysis
3. Asymptotic Analysis

Recursion

- See Recursion


TOP DEFINITION

recursion

See recursion.

by **Anonymous** December 05, 2002

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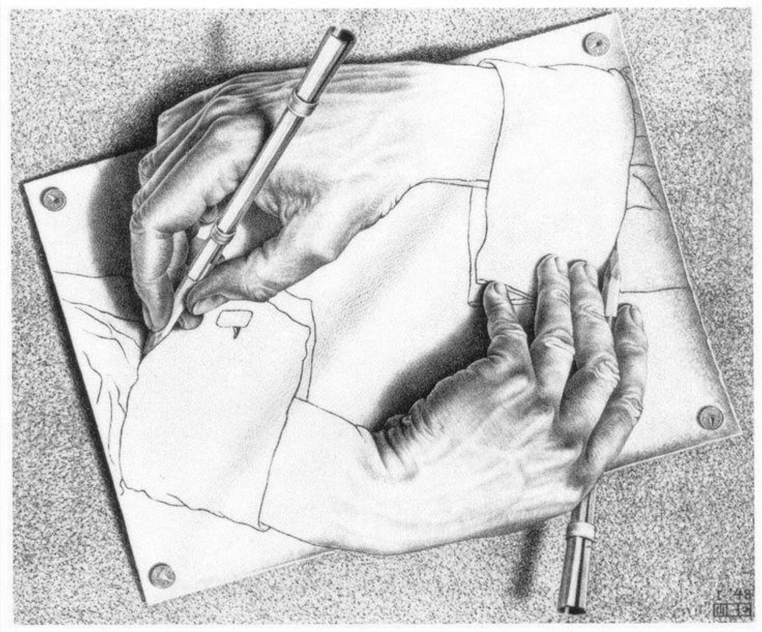
Very descriptive definition

Recursion

- See Recursion
- Recursion is when a function calls itself
- Recursion simplifies program structure at a cost of function calls
- Recursion vs. Leap of faith



recursion is
when a function *calls itself*

Example 1: BunnyEars

- What is the output of the function `bunnyEars()`?
- What is the output of the function `main()`?
- How many times does two returns execute, respectively?

```
int bunnyEars(int n) {
    cout << n << endl;
    if (n > 0)
        return bunnyEars(n - 1) + 2;
    return 0;
}
```

```
int main() {
    cout << bunnyEars(4) << endl;
}
```

1	bunnyEars(4)	
2	cout << 4	
3	bunnyEars(3)	
4	cout << 3	
5	bunnyEars(2)	
6	cout << 2	
7	bunnyEars(1)	
8	cout << 1	
9	bunnyEars(0)	
10	cout << 0	return 0

system stack

Example 1: BunnyEars

- What is the output of the function `bunnyEars()`?
- What is the output of the function `main()`?
- How many times does two returns execute, respectively?

```
int bunnyEars(int n) {
    cout << n << endl;
    if (n > 0)
        return bunnyEars(n - 1) + 2;
    return 0;
}
```

```
int main() {
    cout << bunnyEars(4) << endl;
}
```

1	bunnyEars(4)	
2	cout << 4	
3	bunnyEars(3)	
4	cout << 3	
5	bunnyEars(2)	
6	cout << 2	
7	bunnyEars(1)	
8	cout << 1	
9	bunnyEars(0)	
10	cout << 0	return 0

```
f(n=1)
f(n=2)
f(n=3)
f(n=4)
```

system
stack

Example 1: BunnyEars

- What is the output of the function `bunnyEars()`?
- What is the output of the function `main()`?
- How many times does two returns execute, respectively?

```
int bunnyEars(int n) {
    cout << n << endl;
    if (n > 0)
        return bunnyEars(n - 1) + 2;
    return 0;
}
```

```
int main() {
    cout << bunnyEars(4) << endl;
}
```

1	bunnyEars(4)	
2	cout << 4	return f(3) + 2
3	bunnyEars(3)	
4	cout << 3	return f(2) + 2
5	bunnyEars(2)	
6	cout << 2	return f(1) + 2
7	bunnyEars(1)	
8	cout << 1	return f(0) + 2
9	bunnyEars(0)	
10	cout << 0	return **0**

system stack

Example 2: FunnyEars

We have bunnies and funnies standing in a line, numbered 1, 2, ... The odd bunnies (1, 3, ..) have the normal 2 ears. The even funnies (2, 4, ..) we'll say have 3 ears, because they each have a raised foot. Recursively return the number of "ears" in the bunny and funny line 1, 2, ... n (without loops or multiplication).

Expected results:

funnyEars(0) → 0
funnyEars(1) → 2
funnyEars(2) → 5
funnyEars(3) → 7
funnyEars(4) → 10
funnyEars(10) → 25
funnyEars(11) → 27

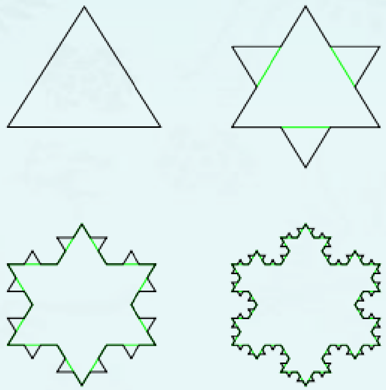
```
int funnyEars(int n) {
    if (n <= 0) return 0;
    if (n % 2 == 0)
        return ___________________________;

    return ____________________;
}
```

```
int main() {
    int funnies[] = {0, 1, 2, 3, 4, 10, 11};
    for (auto n: funnies)
        cout << funnyEars(n) << endl;
}
```

Recursion

- **Recursion** is a method where the solution to a problem depends on solutions to smaller instances of the same problem (as opposed to iteration).
- **Recursive algorithm** is expressed in terms of
  1. **base case(s)** for which the solution can be stated **non-recursively**,
  2. **recursive case(s)** for which the solution can be expressed in terms of a smaller version of itself.



Four stages in the construction of a **Koch snowflake**. The stages are obtained via a recursive definition.

Example 3: Factorial

$$\text{fact}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{fact}(n-1) & \text{if } n > 0 \end{cases}$$

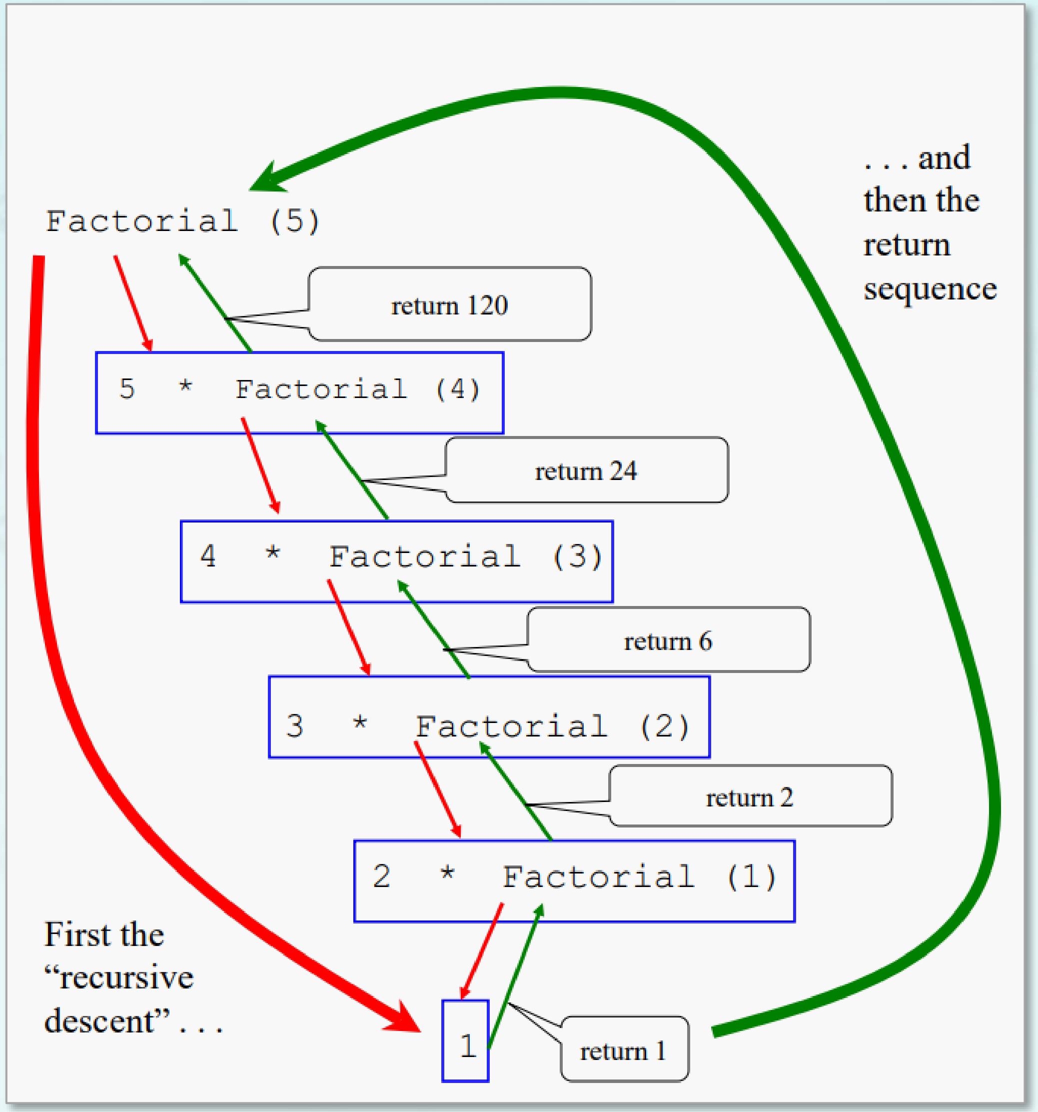
factorial(n)
function factorial **input**: integer *n* such that *n* >= 0 **output**: [*n* × (*n*-1) × (*n*-2) × … × 1] 1. if *n* is 0, **return** 1 2. otherwise, **return** [*n* × factorial(*n*-1)] **end** factorial

factorial (n = 4)
$f_4 = 4 * f_3$ $= 4 * (3 * f_2)$ $= 4 * (3 * (2 * f_1))$ $= 4 * (3 * (2 * (1 * \mathbf{f_0})))$ $= 4 * (3 * (2 * (1 * \mathbf{1})))$ $= 4 * (3 * (2 * 1))$ $= 4 * (3 * 2)$ $= 4 * 6$ $= 24$

Example 3: Factorial

$$\text{fact}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{fact}(n-1) & \text{if } n > 0 \end{cases}$$

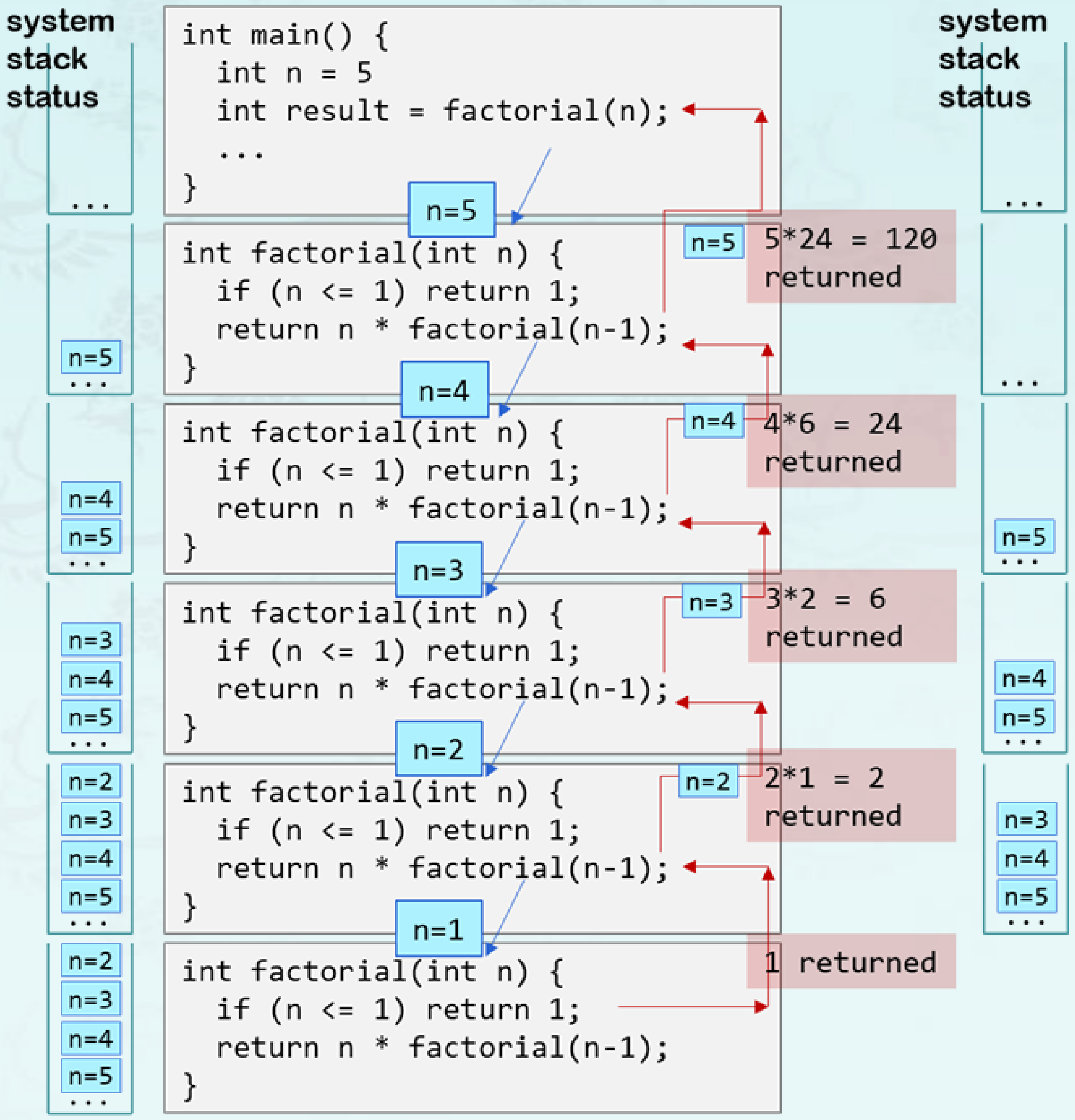
```
int factorial(int n) {
    if (n <= 1) return 1;
    return n * factorial(n - 1);
}
```



Example 3: Factorial

$$\text{fact}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{fact}(n-1) & \text{if } n > 0 \end{cases}$$

```
int factorial(int n) {
    if (n <= 1) return 1;
    return n * factorial(n – 1);
}
```



Example 4: foo

- Predict the output of the following code.

```
#include <iostream>
using namespace std;
void foo(int n) {
    if (n < 1) return;
    cout << n << " ";
    foo(n - 1);
    cout << n << " ";
    return;
}
int main() {
  int x = 3
  foo(x);
}
```

- When main() calls foo(3), main() and **x=3** are pushed to the system stack. It will finish when returned from foo(3).

Example 4: foo

- Predict the output of the following code.

```
#include <iostream>
using namespace std;
void foo(int n) {
    if (n < 1) return;
    cout << n << " ";
    foo(n - 1);
    cout << n << " ";
    return;
}
int main() {
  int x = 3
  foo(x);
}
```

```
void foo(n=3)
  if (n < 1) return;
  cout << n << " ";
  foo(2);
  cout << n << " ";
  return;
```

`foo(3) calls foo(2)`

- When main() calls foo(3), main() and **x=3** are pushed to the system stack. It will finish when returned from foo(3).
- **foo(3) prints '3'** and calls foo(2). Then, **foo(2)** and **n=2** are pushed to the system stack.

Example 4: foo

- Predict the output of the following code.

```
#include <iostream>
using namespace std;
void foo(int n) {
    if (n < 1) return;
    cout << n << " ";
    foo(n - 1);
    cout << n << " ";
    return;
}
int main() {
  int x = 3
  foo(x);
}
```

```
void foo(n=3)
  if (n < 1) return;
  cout << n << " ";
  foo(2);
  cout << n << " ";
  return;
```

foo(3) calls foo(2)

```
void foo(n=2)
  if (n < 1) return;
  cout << n << " ";
  foo(1);
  cout << n << " ";
  return;
```

foo(2) calls foo(1)

```
void foo(n=1)
  if (n < 1) return;
  cout << n << " ";
  foo(0);
  cout << n << " ";
  return;
```

foo(1) calls foo(0)

```
void foo(n=0)
  if (n < 1) return;
```

- When main() calls foo(3), main() and **n=3** are pushed to the system stack. It will finish when retured from foo(3).
- **foo(3) prints '3'** and calls foo(2). Then, **foo(2)** and **n=2** are pushed to the system stack.
- Similarly, **foo(2) prints '2'** and calls foo(1). Then **foo(1) prints '1'** and calls foo(0).

Example 4: foo

- Predict the output of the following code.

```
#include <iostream>
using namespace std;
void foo(int n) {
    if (n < 1) return;
    cout << n << " ";
    foo(n - 1);
    cout << n << " ";
    return;
}
int main() {
  int x = 3
  foo(x);
}
```

```
void foo(n=3)
  if (n < 1) return;
  cout << n << " ";
  foo(2);
  cout << n << " ";
  return;
```

foo(3) calls foo(2)

```
void foo(n=2)
  if (n < 1) return;
  cout << n << " ";
  foo(1);
  cout << n << " ";
  return;
```

foo(2) calls foo(1)

```
void foo(n=1)
  if (n < 1) return;
  cout << n << " ";
  foo(0);
  cout << n << " ";
  return;
```

foo(1) calls foo(0)

```
void foo(n=0)
  if (n < 1) return;
```

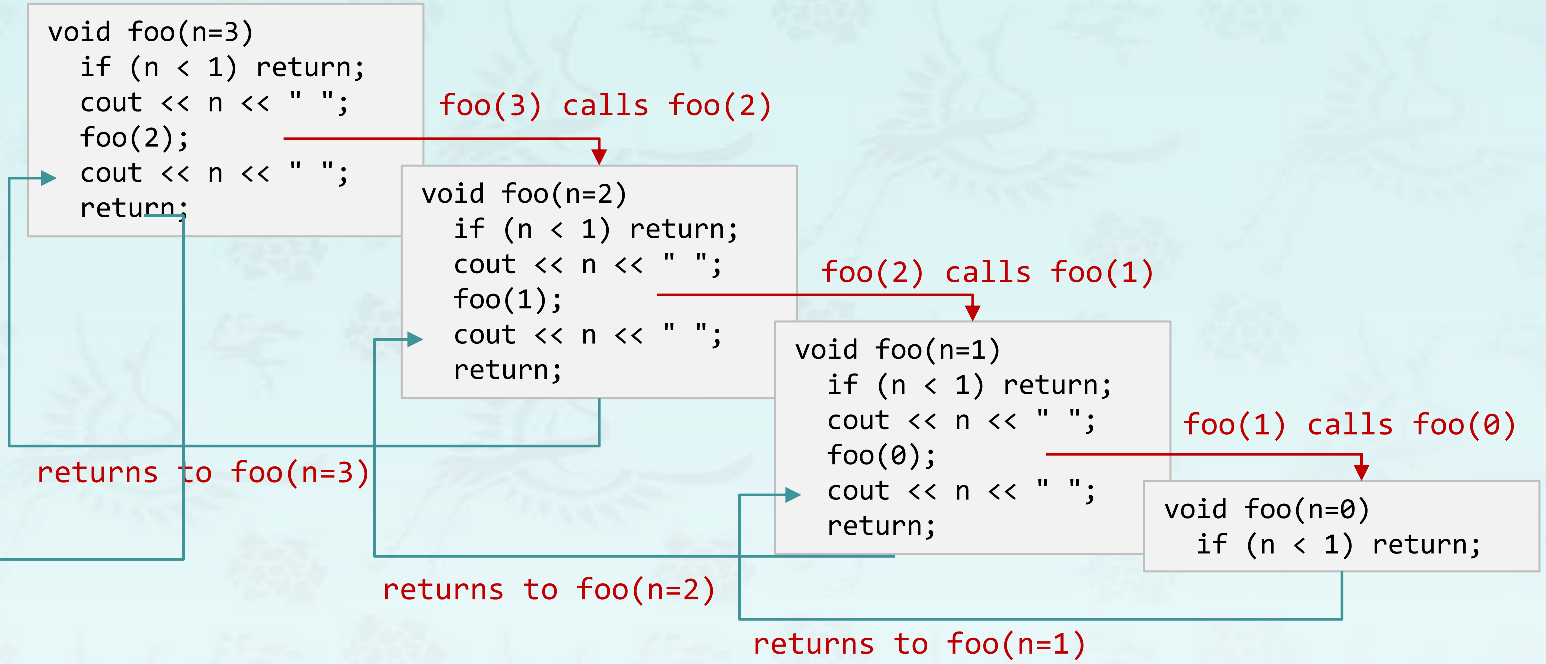
returns to foo(n=1)

- When main() calls foo(3), main() and **n=3** are pushed to the system stack. It will finish when retured from foo(3).
- **foo(3) prints '3'** and calls foo(2). Then, **foo(2)** and **n=2** are pushed to the system stack.
- Similarly, **foo(2) prints '2'** and calls foo(1). Then **foo(1) prints '1'** and calls foo(0).
- foo(0) goes to if and returns to foo(1). This is the first return ever. Now **foo(1)** popped from the stack **prints '1'.**
- foo(1) returns or finishes. Then, foo(2) popped from the stack prints '2'. It returns to foo(3) popped from the stack.

Example 4: foo

- Predict the output of the following code.

```
#include <iostream>
using namespace std;
void foo(int n) {
    if (n < 1) return;
    cout << n << " ";
    foo(n - 1);
    cout << n << " ";
    return;
}
int main() {
  int x = 3
  foo(x);
}
```



- When main() calls foo(3), main() and **n=3** are pushed to the system stack. It will finish when retured from foo(3).
- **foo(3) prints '3'** and calls foo(2). Then, **foo(2)** and **n=2** are pushed to the system stack.
- Similarly, **foo(2) prints '2'** and calls foo(1). Then **foo(1) prints '1'** and calls foo(0).
- foo(0) goes to if and returns to foo(1). This is the first return ever. Now **foo(1)** popped from the stack **prints '1'.**
- foo(1) returns or finishes. Then, foo(2) popped from the stack prints '2'. It returns to foo(3) popped from the stack.
- foo(3) prints '3' and returns to the main(). Then, the main() finishes the program.

Reminder

- **Recursion** is a method where the solution to a problem depends on solutions to smaller instances of the same problem (as opposed to iteration).

Example 5: Recursive binary search

- Binary search is an efficient algorithm for finding an item from **a sorted list** of items.
  - It works by repeatedly **dividing in half the portion of the list** that could contain the item,
  - until you've **narrowed down the possible locations to just one.**

	0	1	2	3	4	5	6	7	8	9
key=23	2	5	8	9	16	23	31	56	62	71

Example 5: Recursive binary search

- For instance, we want to search "23" from the array. If we find it, we return its array index; otherwise, -1 or something else.

	0	1	2	3	4	5	6	7	8	9
key=23	2	5	8	9	16	23	31	56	62	71

	lo=0	1	2	3	mi=4	5	6	7	8	hi=9
key>mi 23>16	2	5	8	9	16	23	31	56	62	71

	0	1	2	3	4	lo=5	6	mi=7	8	hi=9
key<mi 23<56	2	5	8	9	16	23	31	56	62	71

	0	1	2	3	4	mi=5 lo=5	hi=6	7	8	9
key=mi 23=23	2	5	8	9	16	23	31	56	62	71

```
int binarySearch(int list[], int key,
                             int lo, int hi) {
  if (lo > hi) return -1;

  mi = (lo + hi)/2;
  if (key == list[mi]) return mi;
  if (key <  list[mi])
    return binarySearch(list,                 );
  else
    return binarySearch(list,                 );
}
```

Example 5: Recursive binary search

- For instance, we want to search "23" from the array. If we find it, we return its array index; otherwise, -1 or something else.

	0	1	2	3	4	5	6	7	8	9
key=23	2	5	8	9	16	23	31	56	62	71

	lo=0	1	2	3	mi=4	5	6	7	8	hi=9
key>mi 23>16	2	5	8	9	16	23	31	56	62	71

	0	1	2	3	4	lo=5	6	mi=7	8	hi=9
key<mi 23<56	2	5	8	9	16	23	31	56	62	71

	0	1	2	3	4	mi=5 lo=5	hi=6	7	8	9
key=mi 23=23	2	5	8	9	16	23	31	56	62	71

```
int binarySearch(int list[], int key,
                              int lo, int hi) {
  if (lo > hi) return -1;

  mi = (lo + hi)/2;
  if (key == list[mi]) return mi;
  if (key <  list[mi])
    return binarySearch(list, key, lo, mi - 1);
  else
    return binarySearch(list, key, mi + 1, hi);
}
```

Example 5: Recursive binary search

- How many times is the `binarySearch()` called in terms of n?
- In one call to `binarySearch()`, we eliminate at least half the elements from consideration. Hence, it takes $\log_2 n$ (the base 2 logarithm of n) `binarySearch()` calls to compare down the possibilities to one. Therefore binarySearch takes time proportional to $\log_2 n$.

```
int binarySearch(int list[], int key, int lo, int hi) {
  if (lo > hi) return -1;

  mi = (lo + hi)/2;
  if (key == list[mi]) return mi;                    // base case
  if (key <  list[mi])                               // recursive case
    return binarySearch(list, key, lo, mi - 1);
  else
    return binarySearch(list, key, mi + 1, hi);
}
```

Example 6: Recursive binary search 100

- Given the numbers 1 to 100, what is the minimum number of guesses needed to find a specific number if you are given the hint 'higher' or 'lower' for each guess you make?
  - Since the numbers are sequential (or sorted), we can use **binary search**.
  - Look at the middle element: if it's after than the number we're looking for, search the first half. If it's before the number we're looking for, look at the second half.
  - Each check cuts the size of the list numbers in half; how many times can we do this?
  - If we think backwards, in terms of doubling the list, we'll need n doublings to generate a list of length 2^n = 100. What is the value of n?
  - Since $2^6 = 64$ and $2^7 = 128$ (or $log_2 64 = 6$, $log_2 128 = 7$), n = 6.x
    Therefore n = 7 guesses will be enough.
- Then, it requires $ceil(\log_2 100) = 7$ guesses.

Example 6: Recursive binary search 1000

- Given the numbers 1 to 1000, what is the minimum number of guesses needed to find a specific number if you are given the hint 'higher' or 'lower' for each guess you make?
  - For an array whose length is 1000, the closest lower power of 2 is 512, which is 2^9.
  - We can thus estimate that $log_2 1000$ is a number greater than 9 and less than 10, or use a calculator to see that its about 9.97. Adding one to that yields about 10.97.
  - In the case of a decimal number, we round down to find the actual number of guesses.
  - Therefore, for a 1000-element array, binary search would require at most **10 guesses.**
- Then, it requires $ceil(\log_2 1000) = 10$ guesses.

Reference: https://www.geeksforgeeks.org/complexity-analysis-of-binary-search/

Example 5: Recursive binary search

0	1	2	3	4	5	6	7	8	9
2	5	8	9	16	23	31	56	62	71

lo=0	1	2	3	mi=4	5	6	7	8	hi=9
2	5	8	9	16	23	31	56	62	71

0	1	2	3	4	lo=5	6	mi=7	8	hi=9
2	5	8	9	16	23	31	56	62	71

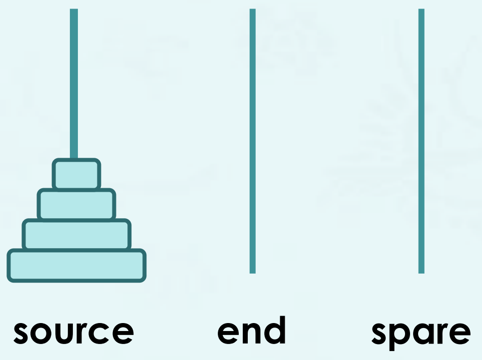
0	1	2	3	4	mi=5 lo=5	hi=6	7	8	9
2	5	8	9	16	23	31	56	62	71

	Stack	Stack	Heap
search()	lo=5 hi=6 mi=5	key=23 list[.]	
search()	lo=5 hi=9 mi=7	key=23 list[.]	
search()	lo=0 hi=9 mi=4	key=23 list[.]	
search()	key=23	list[.]	[2 5 8 9 16 23 31 56 62 71]
main()		args[.]	args[]

Most operating systems give a program enough stack space for a few thousand stack frames. If you use a recursive procedure to walk through a million-node list, the program will try to create a million stack frames, and **the stack will run out of space**. The result is a run-time error.

Example 6: Tower of Hanoi

- Given three pegs, one with a set of N disks of increasing size, determine the minimum (optimal) number of steps it takes to move all the disks from their initial position to a single stack on another peg *without placing a larger disk on top of a smaller one. Only one disk can be moved at any time.*



Example 6: Tower of Hanoi

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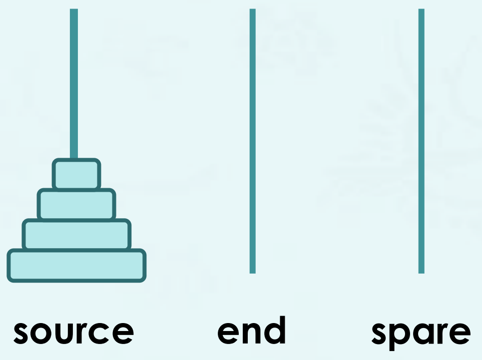
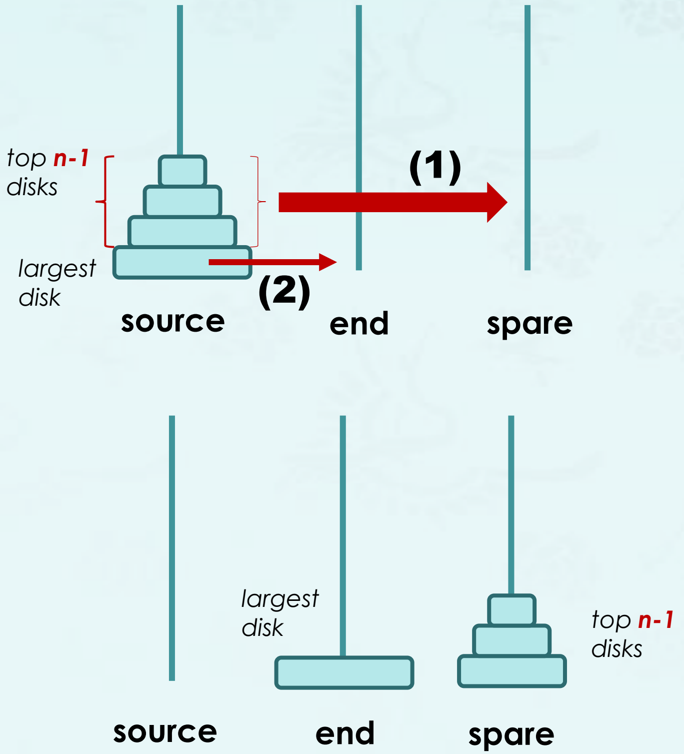
- **Recursive algorithm:**
  (1) *Move the top **n-1** disks from **source** to **spare**.*
  (2) *Move the remaining (**largest**) disk from **source** to **end.***
  (3) *Move the **n-1** disks from **spare** to **end.***



Example 6: Tower of Hanoi

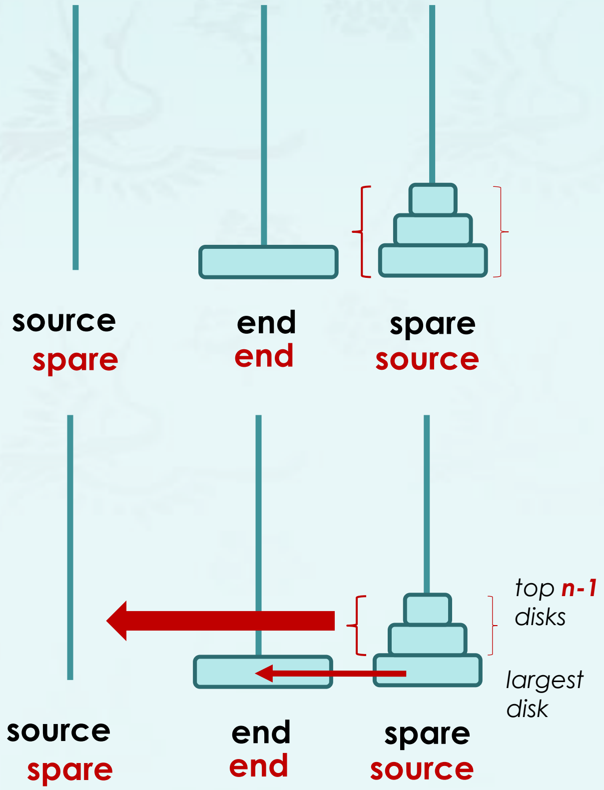
- **Recursive algorithm:**
  - *(1) Move the top **n-1** disks from **source** to **spare**.*
  - *(2) Move the remaining (**largest**) disk from **source** to **end.***
  - *(3) Move the **n-1** disks from **spare** to **end.***

4 disk problem



(3) It becomes a **3 disk problem.**
Go back to step 1.
Treat the **spare as source** and
the **source as spare**.

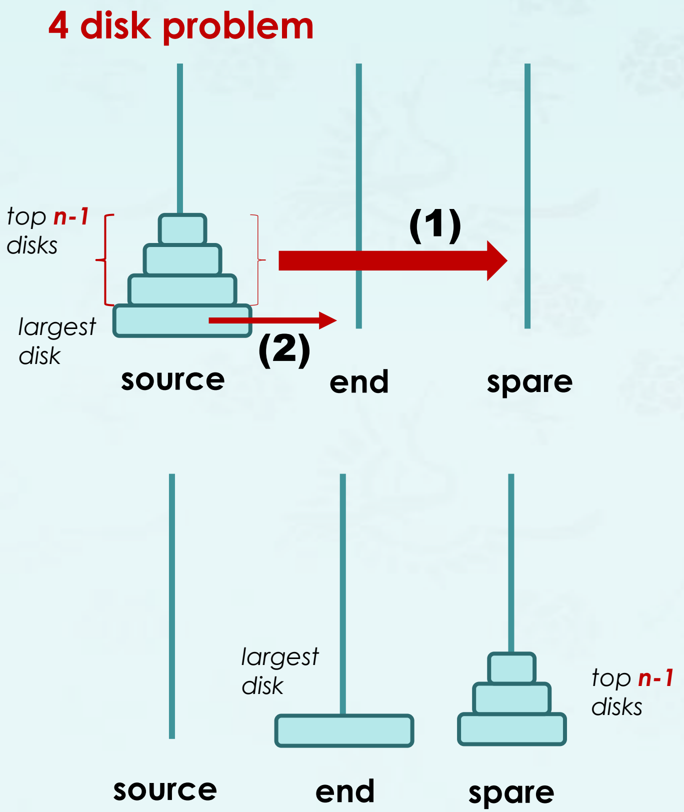
3 disk problem



Example 6: Tower of Hanoi

```
void hanoi(int n, char source, char end, char spare) {
  if (n == 1)
    printf("Disk 1 from %c to %c\n", source, end);
  else {
    hanoi(n - 1, _____, _____, _____);
    printf("Disk %d from %c to %c\n", n, source, end);
    hanoi(n - 1, _____, _____, _____);
  }
}
```

- **Recursive algorithm:**
  - *(1) Move the top **n-1** disks from **source** to **spare**.*
  - *(2) Move the remaining (**largest**) disk from **source** to **end**.*
  - *(3) Move the **n-1** disks from **spare** to **end**.*



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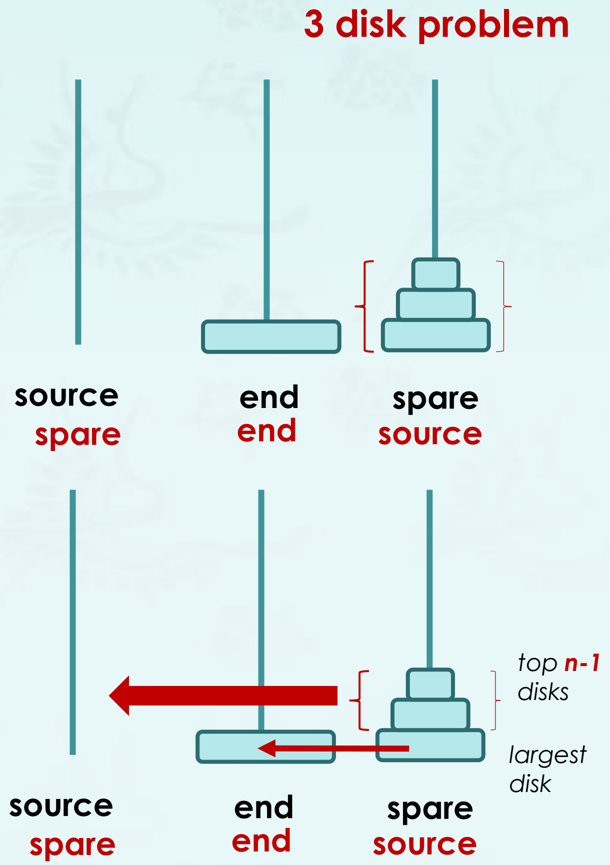
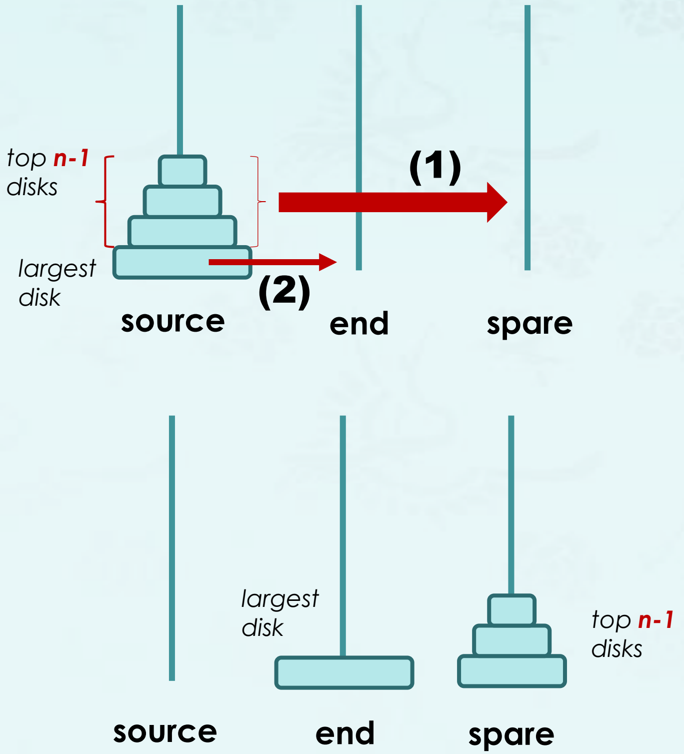


Example 6: Tower of Hanoi

```
void hanoi(int n, char source, char end, char spare) {
  if (n == 1)
    printf("Disk 1 from %c to %c\n", source, end);
  else {
    hanoi(n - 1, _____, _____, _____);
    printf("Disk %d from %c to %c\n", n, source, end);
    hanoi(n - 1, _____, _____, _____);
  }
}
```

- **Recursive algorithm:**
  - (1) *Move the top **n-1** disks from **source** to **spare**.*
  - (2) *Move the remaining (**largest**) disk from **source** to **end.***
  - (3) *Move the **n-1** disks from **spare** to **end.***

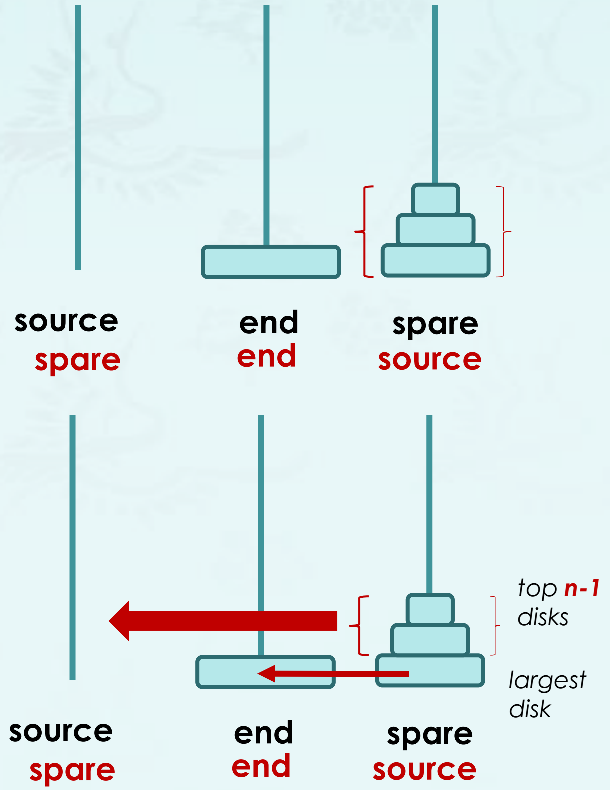
4 disk problem



3 disk case
Expected Output

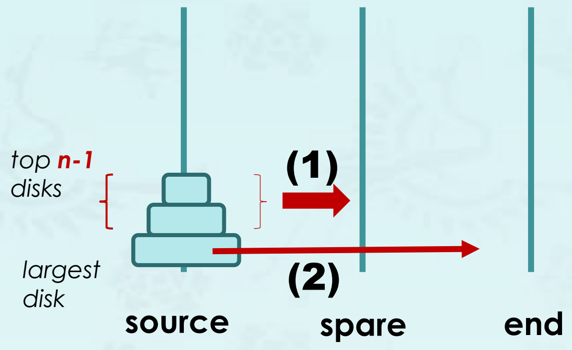
1. Disk 1 from A to B
2. Disk 2 from A to C
3. Disk 1 from B to C
4. Disk 3 from A to B
5. Disk 1 from C to A
6. Disk 2 from C to B
7. Disk 1 from A to B.

3 disk problem



Example 6: Tower of Hanoi

- **Recursive algorithm:**
  - *(1) Move the top **n-1** disks from **source** to **spare**.*
  - *(2) Move the remaining (**largest**) disk from **source** to **end.***
  - *(3) Move the **n-1** disks from **spare** to **end.***

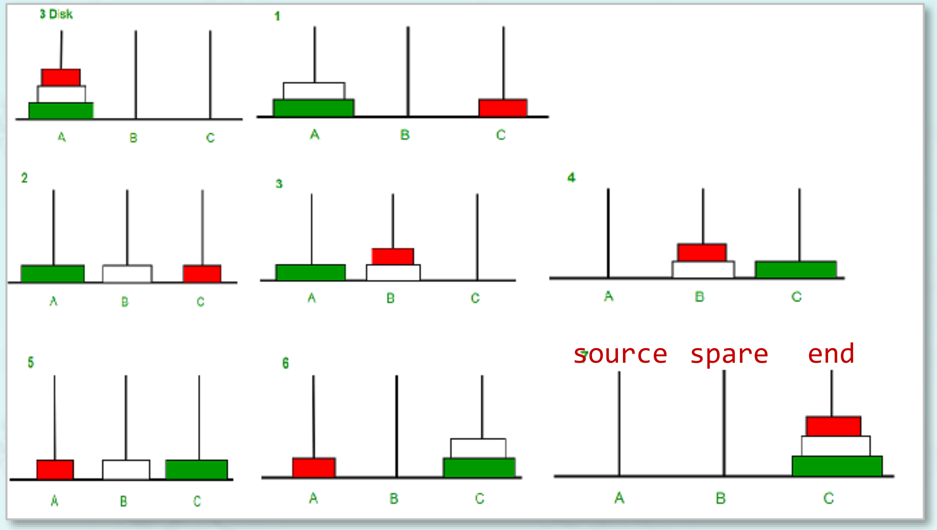


- **How do you program this to have the output as shown below?**
  - *(1) Disk 1 from A to C*
  - *(2) Disk 2 from A to B*
  - *(3) Disk 1 from C to B*
  - *(4) Disk 3 from A to C*
  - *(5) Disk 1 from B to A*
  - *(6) Disk 2 from B to C*
  - *(7) Disk 1 from A to C*

```
void hanoi(int n, char source, char spare, char end) {
  if (n == 1)
    printf("Disk 1 from %c to %c\n", source, end);
  else {
    hanoi(n - 1, source, end, spare);
    printf("Disk %d from %c to %c\n", n, source, end);
    hanoi(n - 1, spare, source, end);
  }
}
```

Example 6: Tower of Hanoi

- **Recursive algorithm:**
  - *(1) Move the top **n-1** disks from **source** to **spare**.*
  - *(2) Move the remaining (**largest**) disk from **source** to **end.***
  - *(3) Move the **n-1** disks from **spare** to **end.***



- **How do you program this to have the output as shown below?**
  - *(1) Disk 1 from A to C*
  - *(2) Disk 2 from A to B*
  - *(3) Disk 1 from C to B*
  - *(4) Disk 3 from A to C*
  - *(5) Disk 1 from B to A*
  - *(6) Disk 2 from B to C*
  - *(7) Disk 1 from A to C*

```
void hanoi(int n, char source, char spare, char end) {
  if (n == 1)
    printf("Disk 1 from %c to %c\n", source, end);
  else {
    hanoi(n – 1, source, end, spare);
    printf("Disk %d from %c to %c\n", n, source, end);
    hanoi(n – 1, spare, source, end);
  }
}
```

Example 6: Time complexity of Tower of Hanoi - $O(2^n)$

$$\mathrm{hanoi}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 \cdot \mathrm{hanoi}(n-1) + 1 & \text{if } n > 1 \end{cases}$$

Exercise: How many years will it take to move 64 disks while assuming one move per pico second at the clock speed of a super fast machine?

(1) hanoi(1) = 1
(2) hanoi(2) = 3
(3) hanoi(3) = 7
(4) hanoi(4) = 15
(5) hanoi(5) = 31
(6) hanoi(32) = 4,294,967,295
(7) hanoi(64) = 18,446,744,073,709,600,000

hanoi(n = 4)
hanoi(4) = 2*hanoi(3) + 1 = 2*(2*hanoi(2) + 1) + 1 = 2*(2*(2*hanoi(1) + 1) + 1) + 1 = 2*(2*(2*1 + 1) + 1) + 1 = 2*(2*(3) + 1) + 1 = 2*(7) + 1 = 15

Recursion

Q: Is the recursive version usually **faster?**

A: No -- it's usually slower (due to the overhead of maintaining the stack)

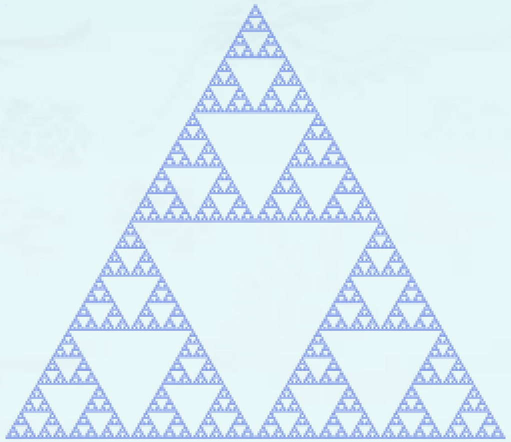
Q: Does the recursive version usually use **less memory?**

A: No -- it usually uses ***more*** memory (for the stack).

Q: *Then* ***why*** use recursion?

A: Sometimes it is much simpler to write the recursive version.

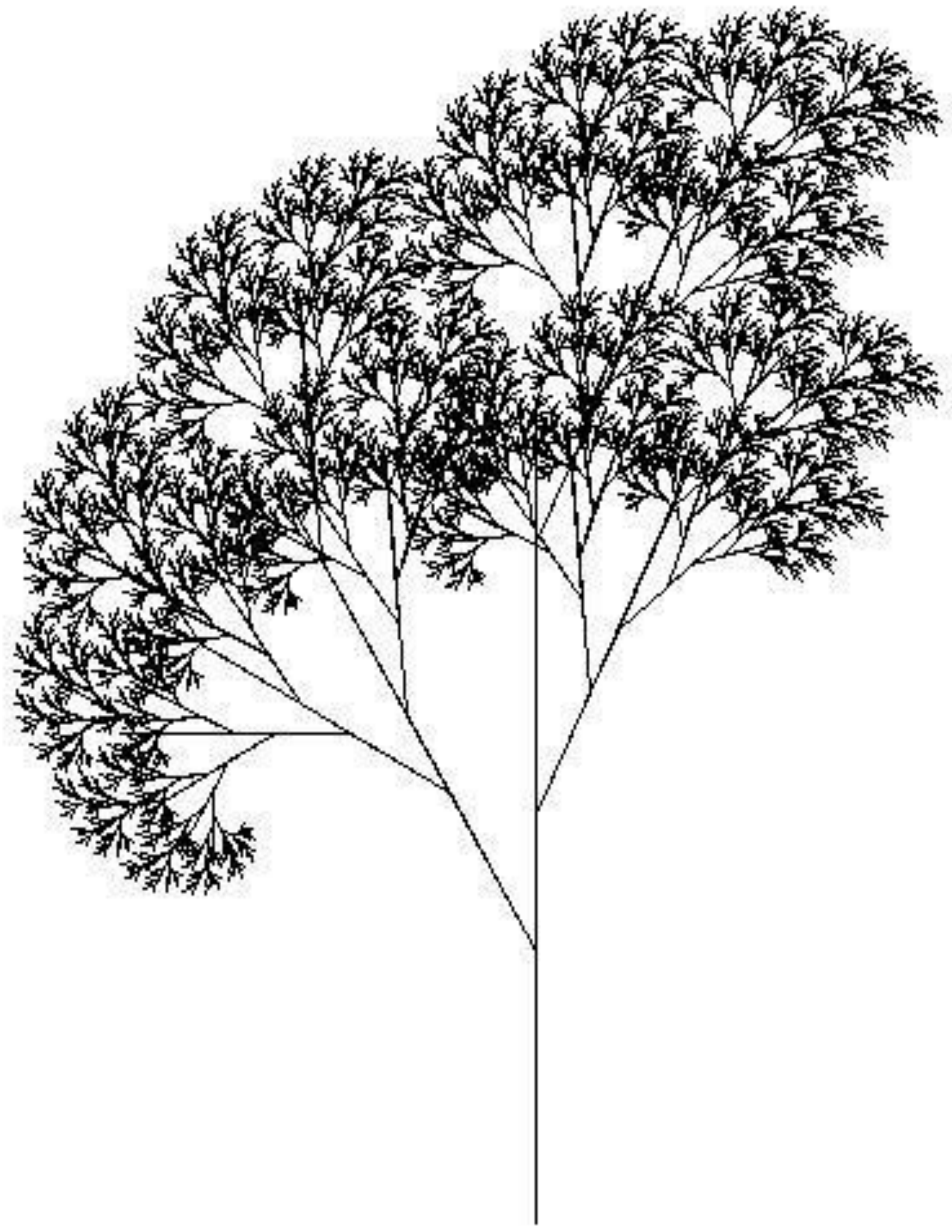
*Because the recursive version causes an **activation record** to be pushed onto the system stack for every call, it is also more limited than the iterative version (it will fail, with a "stack overflow" error), for large values of N.*



Sierpinski Triangle:
a confined recursion of triangles to form a geometric lattice

Recursion

Recursion *see Recursion*



Data Structures
Chapter 1

1. **Recursion**
   - **Recursion**
   - Mergesort
2. Performance Analysis
3. Asymptotic Analysis