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BCA II semester
(TBC 204) Discrete Mathematical Structures and Graph Theory

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① Symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix if $A' = A$, ie $A[a_{ij}] = A[a_{ji}]$.

$$A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$$

Now $A + A' = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$ [This is symmetric Matrix]

$$(A+A') = (A+A')'$$

Skew Symmetric Matrix

A square matrix is said to be a skew symmetric matrix if $A' = -A$ ie $A[a_{ij}] = -A[a_{ji}]$

$A - A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ [This is a skew symmetric Matrix]

$$(A-A') = -(A-A')'$$

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② Given system of linear equations is:

$$2x+4y-y=5$$
$$-x+5y-2z=2$$
$$3x-2y+2z=3$$

in matrix form $AX=B$

$$X=A^{-1}B$$

$$A=\begin{bmatrix}2 & 4 & -1\\ -1 & 5 & -2\\ 3 & -2 & 2\end{bmatrix} \quad B=\begin{bmatrix}5\\2\\3\end{bmatrix} \quad x=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$|A| = 2\begin{vmatrix}5 & -2\\ 5 & -2\end{vmatrix} - 4\begin{vmatrix}-1 & -2\\ 3 & 2\end{vmatrix} - 1\begin{vmatrix}-1 & 5\\ 3 & -2\end{vmatrix}$$

$$= 2(10-4) \quad -4 \quad (-2+6) \qquad -1(2-15)$$

$$= 12-16+13=9 \neq 0$$

$-8+5=-3$

$A_{11}=10-4=6$ $\quad A_{21}=8-2=6$ $\quad A_{31}=\cancel{2-15=-13}$

$A_{12}=-2+6=4$ $\quad A_{22}=4+3=7$ $\quad A_{32}=-4-1=-5$

$A_{13}=2-15=-13$ $\quad A_{23}=-4-12=-16$ $\quad A_{33}=10+4=14$

$$adj(A)=C^T=\begin{bmatrix}6 & -6 & -3\\ -4 & 7 & 5\\ -13 & 16 & 14\end{bmatrix}$$

$$A^{-1}=\frac{1}{9}\begin{bmatrix}6 & -6 & -3\\ -4 & 7 & 5\\ -13 & 16 & 14\end{bmatrix}$$

$$X=\frac{1}{9}\begin{bmatrix}6 & -6 & -3\\ -4 & 7 & 5\\ -13 & 16 & 14\end{bmatrix}\begin{bmatrix}5\\2\\3\end{bmatrix}$$

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$$X = \frac{1}{9}\begin{bmatrix} 30 - 12 - 9 \\ -20 + 14 + 15 \\ -65 + 32 + 42 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow x = y = z = 1$$

<u>using rank</u>

augmented matrix $[A:B] = \begin{bmatrix} 2 & 4 & -1 & 5 \\ -1 & 5 & -2 & 2 \\ 3 & -2 & 2 & 3 \end{bmatrix}$

applying $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} -1 & 5 & -2 & 2 \\ 2 & 4 & -1 & 5 \\ 3 & -2 & 2 & 3 \end{bmatrix}$$

applying $R_1 \rightarrow -1 * R_1$

$$\begin{bmatrix} 1 & -5 & 2 & -2 \\ 2 & 4 & -1 & 5 \\ 3 & -2 & 2 & 3 \end{bmatrix}$$

applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & -5 & 2 & -2 \\ 0 & 14 & -5 & 9 \\ 0 & 13 & -4 & 9 \end{bmatrix}$$

applying $R_3 \rightarrow 14R_3 - 13R_2$

$$\begin{bmatrix} 1 & -5 & 2 & -2 \\ 0 & 14 & -5 & 9 \\ 0 & 0 & 9 & 9 \end{bmatrix}$$

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$\because$ Rank([A]) = Rank([A:B]) = 3

$\therefore$ No of solutions = ~~No of~~ unique

The equivalent system of lines is

$$x - 5y + 2z = -2$$
$$14y - 5z = 9$$
$$9z = 9$$

using backward substituting

$z = 1$

$y = 1$

$x = 5 - 2 - 2 = 1$

Hence $x = 1$

$y = 1$

$z = 1$

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③ $A = \begin{vmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix}$

characterotic equation $= |A-\lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)\left[(1-\lambda)^2+4\right] - 3\left[2-2\lambda+2\right] + 1\left[4-1+\lambda\right] = 0$

$\Rightarrow (4-\lambda)\left[\lambda^2-2\lambda+5\right] - 12 + 6\lambda + 3 + \lambda = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 6\lambda + 11 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$ characterotic polynomial equation

__Cayley Hamilton Theorem__

Every square matrix satisfies its characterotic equation

$A^3 - 6A^2 + 6A - 11I = 0$ [To proof]

$$A^2 = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 125 & 84 & -12 \\ 36 & 23 & 0 \\ 48 & 30 & -7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 125 & 84 & -12 \\ 36 & 23 & 0 \\ 48 & 30 & -7 \end{bmatrix} - 6\begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} + 6\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

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$$\begin{bmatrix} 125-138+24-11 & 84-102+18 & -12+6+6 \\ 36-48+12 & 23-18+6-11 & 12-12 \\ 48-54+6 & 30-42+12 & -7+12+6-11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$ Hence verified

Now we know

$$\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$$

using Cayley Hamilton Theorem

$$A^3 - 6A^2 + 6A - 11I = 0$$

Multiplying both sides by A^{-1}

$$\Rightarrow A^2 - 6A + 6I - 11A^{-1} = 0$$

$$\Rightarrow 11A^{-1} = A^2 - 6A + 6I$$

$$\Rightarrow A^{-1} = \frac{1}{11}\left[A^2 - 6A + 6I\right]$$

$$= \frac{1}{11}\begin{bmatrix} 23+6-24 & 17-18 & -1-6 \\ 8-12 & 3+6-6 & -2+12 \\ 9-6 & 7-12 & -2+6-6 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

⑤ (i) $xy'z + xy'z' + xy'z' + xyz'$

$\Rightarrow xy'z + xy'z' + xyz'$

$\Rightarrow xy' + xz'$

		$y'z'$ 00	$y'z$ 01	yz 11	yz' 10
x'	0				
x	1	1	1		1

(ii) $x'y'z' + x'y'z + x'yz + xyz$

$\Rightarrow x'y' + yz$

		$y'z'$ 00	$y'z$ 01	yz 11	yz' 10
x'	0	1	1	1	
x	1			1	

④ Express the function $F = xz + x'y$ as sum of minterms and maxterms.

x	y	z	xz	x'	$x'y$	$x'y+xz$	Minterm	Maxterm
0	0	0	0	1	0	0		$x+y+z$
0	0	1	0	1	0	0		$x+y+z'$
0	1	0	0	1	1	1	$x'yz'$	
0	1	1	0	1	1	1	$x'yz$	
1	0	0	0	0	0	0		$x'+y+z$
1	0	1	1	0	0	1	$xy'z$	
1	1	0	0	0	0	0		$x'+y'+z$
1	1	1	1	0	0	1	xyz	

Sum of Minterms $= (x'yz') + (x'yz) + (xy'z) + (xyz)$

Product of Maxterms $= (x+y+z)(x+y+z')(x'+y+z)(x'+y'+z)$

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⑥ Boolean Algebra

Boolean Algebra is a division of mathematics that deals with operations on logical values and incorporates binary variables.

Laws of Boolean Algebra

Annulment Law - $A \cdot 0 = 0$
$A + 1 = 1$

Identity Law - $A + 0 = A$
$A \cdot 1 = A$

Idempotent Law - $A + A = A$
$A \cdot A = A$

Complement Law - $A \cdot \bar{A} = 0$
$A + \bar{A} = 1$

Commutative Law - $A \cdot B = B \cdot A$
$A + B = B + A$

Double Negation Law - $\bar{\bar{A}} = A$

De Morgan's Laws - $\overline{A+B} = \bar{A} \cdot \bar{B}$
$\overline{A \cdot B} = \bar{A} + \bar{B}$

⑦ Proove that if a graph G (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining the vertices.

Let the graph G is connected, then there exist a path between each pair of vertices. Thus for two vertices for each of both having odd degree, there will be a path between the two vertices

Let the graph G is disconnected, then there let it have exactly two vertics of odd degree. Thus the graph contains some components. We know the number of vertices of odd degree must be even. Since there are exactly two vertices of odd degree then the two vertices must be a part of a component only. As every component is a connected graph, there exist a path between the two vertices.

⑧ Proove that a disconnected simple graph G (without self loops and parallel edges) with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Let there are k components in a graph G containing $n_1, n_2, \ldots, n_k$ vertices each.

Thus $n_1 + n_2 + \ldots + n_k = n$ —①

Let the maximum numbers of edges in the graph $= e$

[maximum number of edges in a complete graph of n vertices $= n(n-1)/2$]

Thus maximum number of edges, $e=$

$$e = \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \frac{n_3}{2}(n_3-1) + \ldots + \frac{n_k}{2}(n_k-1)$$

$$e = \frac{(n_1^2 + n_2^2 + n_3^2 + \ldots + n_k^2) - (n_1 + n_2 + n_3 + \ldots n_k)}{2}$$

$$= \sum_{i=1}^{k} \frac{(n_i^2 - n_i)}{2}$$

from equation ①

$$e = \sum_{i=1}^{k} \frac{n_i^2 - n}{2} \quad \text{---} ②$$

Now we know that

$$\sum_{i=1}^{k} (n_i - 1) = (n_1 - 1) + (n_2 - 1) + (n_3 - 1) + \cdots + (n_k - 1)$$

$$= (n_1 + n_2 + n_3 + \cdots + n_k) - k$$

$$= n - k \quad [\text{from eq}^n ①]$$

squaring both sides

$$\left(\sum_{i=1}^{k} (n_i - 1) \right)^2 = (n - k)^2$$

$$\Rightarrow [(n_1 - 1) + (n_2 - 1) + (n_3 - 1) + \cdots + (n_k - 1)^2]^2 = (n - k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2 \sum_{i=1}^{k} n_i + A \text{ (non negative term)} = (n - k)^2$$

removing the non-negative term

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2 \sum_{i=1}^{k} n_i \le (n - k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2n \le (n - k)^2 \quad [\text{from eq}^n ①]$$

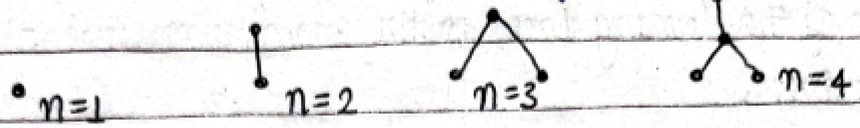
$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le (n - k)^2 + 2n - k \quad \text{---} ③$$

putting value from eq^n ③ in eq^n ②

$$\Rightarrow e \le \frac{1}{2} [(n - k)^2 + 2n - k - n] = \frac{1}{2} [(n - k)^2 + (n - k)]$$

$$\Rightarrow e \le \frac{1}{2} [(n - k)(n - k + 1)]$$

(9) **Tree** - A tree is a ~~complete~~ connected graph without any circuit in which inheritance is followed.

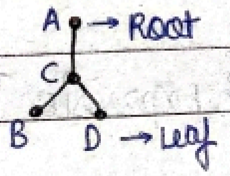


Rooted Tree - A tree with one vertex designated as root in the tree and it is distinguishable from the other's vertices.

Any vertex can be choosen as root.

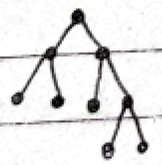
In a rooted tree, the parent of a vertex is the vertex connected to it on the path to the root.

Every vertex except the root have a unique parent.

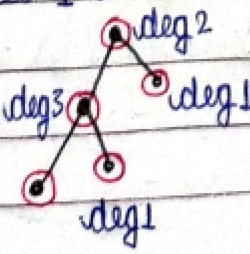


Node without any children is leaf.

Binary Tree - Any tree in which each vertex have either 0, 1 or two children is called a binary tree

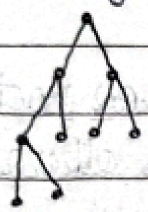


Full Binary Tree - A binary tree is called a full binary tree if there is exactly one vertex of degree 2 and other vertices of degree 1 or 3.



(10) **Height of a Binary Tree** - Distance of a vertex v from the root in a binary tree is called the level of the vertex. The height of the binary tree is the maximum level of any vertex in a binary tree.

Level
0
1
2
3



Minimum height of a full binary tree

Let h be height of a full binary tree with n vertices. The maximum number of vertices upto level h is $2^0 + 2^1 + \dots + 2^h$, since there are n vertices,

$n \le 2^0 + 2^1 + \dots + 2^h$

$\Rightarrow n \le 2^0 \left(\frac{2^{h+1}-1}{2-1}\right)$ [Geometric Expression]

$\Rightarrow n \le 2^{h+1} - 1 \quad \Rightarrow 2^{h+1} \ge n+1$ [taking log on both sides]

$\Rightarrow (h+1)\log 2 \ge \log(n+1)$

$\Rightarrow h+1 \ge \log(n+1) / \log 2$

$\Rightarrow h+1 \ge \log_2(n+1)$

$\Rightarrow h \ge \log_2(n+1) - 1$

Maximum height of a Full Binary Tree

Let h be the maximum height of a full binary tree with n vertices. The maximum hight can be found if each level have 2 vertices only

$n = 1+2+2+\dots+2$ (h times)

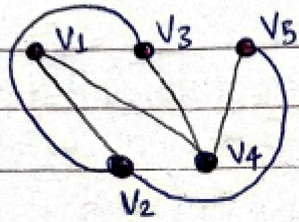
$n = 1+2h$

$2h = n-1$

$h = \frac{n-1}{2}$

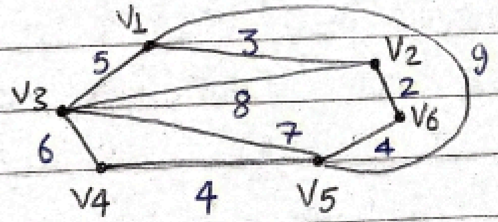
(11) Planar graph

A graph G is said to be a planar graph if there exist some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.



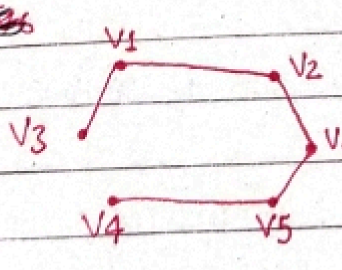
$K_{2,3}$ Hence $K_{2,3}$ is a planar graph.

(12) Find the minimal spanning tree of the graph using Kruskal's Algorithm.



$V = \{V_1, V_2, V_3, V_4, V_5, V_6\}$.

Edge	weight
(V_2, V_6)	2
(V_1, V_2)	3
(V_6, V_5)	4
(V_4, V_5)	4
(V_1, V_3)	5
(V_3, V_4)	6
(V_3, V_5)	7
(V_3, V_2)	8
(V_1, V_5)	9



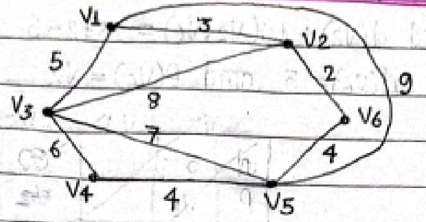
e = edge
d(v) = shortest distance
p(v) = path length

V5, V6}

V5	V6
–	–
∞	∞

d vertices V5, V2, V3 are adjacent

(13)



source vertex = V_1

Initially $T = \emptyset$ and $U = \{V_1, V_2, V_3, V_4, V_5, V_6\}$

	V_1	V_2	V_3	V_4	V_5	V_6
p	–	–	–	–	–	–
d	0	∞	∞	∞	∞	∞

$T = \{V_1\}$ and $U = \{V_2, V_3, V_4, V_5, V_6\}$ and V_5, V_2, V_3 are adjacent

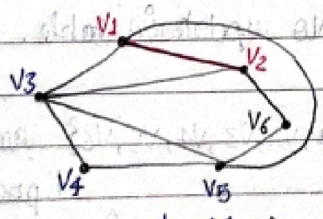
$d(V_2) = \infty$ and $d(V_1) + w(V_1, V_2) = 0 + 3 = 3$
since $3 < \infty$, $d(V_2) = 3$ and $P(V_2) = V_1$

$d(V_3) = \infty$ and $d(V_1) + w(V_1, V_3) = 5$
since $5 < \infty$, $d(V_3) = 5$ and $P(V_3) = V_1$

$d(V_5) = \infty$ and $d(V_1) + w(V_1, V_5) = 9$
since $9 < \infty$, $d(V_5) = 9$ and $P(V_5) = V_1$

	V_2	V_3	V_4	V_5	V_6
d	3	5	∞	9	∞
p	V_1	V_1	–	V_1	–

minimum shortest distance = 3



$T = \{V_1, V_2\}$, $U = \{V_3, V_4, V_5, V_6\}$.
and adjacent vertices are V_3, V_6

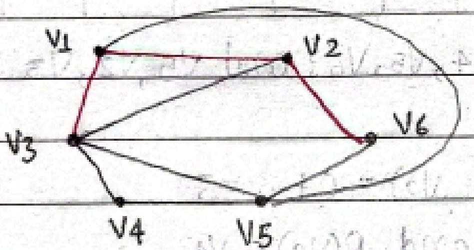
$d(V_3) = 5$, and $d(V_2) + w(V_2, V_3) = 3 + 8 = 11$
since $11 > 5$, no update

$d(V_6) = \infty$, and $d(V_2) + w(V_2, V_6) = 3+2 = 5$
since $5 < \infty$, $d(V_6) = 5$ and $P(V_6) = V_2$

	V_3	V_4	V_5	V_6
d	5		∞	5
p	V_1			V_2

	V_3	V_4	V_5	V_6
d	5	∞	9	5
p	V_1	–	V_1	V_2

Minimum shortest distance = 5

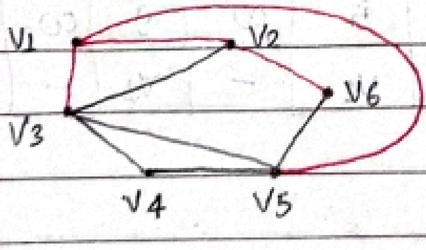


$T = \{V_3, V_1, V_2, V_6\}$, $U = \{V_4, V_5\}$.
adjacent vertices to V_3 are V_4, V_5
adjacent vertices to V_6 is V_5

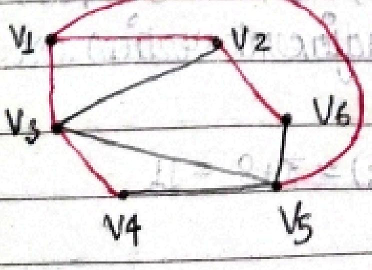
$d(V_3) = 5$, $d(V_3) + w(V_3, V_4) = 5+6 = 11$, $d(V_4) = \infty$
$\because 11 < \infty$, $d(V_4) = 11$, $P(V_4) = V_3$
$\because d(V_3) + w(V_3, V_5) = 5+7 > 9$, so no update in table
$d(V_5) = 9$, $d(V_6) + w(V_6, V_5) = 5+4 = 9$
$\because$ 9 is not less than 9, no update.

	V_4	V_5
d	11	9
P	V_3	V_1

Minimum shortest distance = 9



$T = \{V_1, V_2, V_3, V_5, V_6\}$ $U = \{V_4\}$.
adjacent vertex is V_4
$\because d(V_4) = 11 < d(V_5) + w(V_4, V_5) = 9+4 = 13$
$\therefore$ No update in table.



$T = \{V_1, V_2, V_3, V_4, V_5, V_6\}$ and $U = \phi$
[process terminates]

shortest paths to vertices from V_1

$V_2 = V_1 V_2$ $V_5 = V_1 V_5$
$V_3 = V_1 V_3$ $V_6 = V_1 V_2 V_6$
$V_4 = V_1 V_3 V_4$

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(14) Colouring of a Graph

Graph colouring is a simple way of labelling graph components such as vertices, edges and regions under some constraints. In a graph no two adjacent vertices, adjacent edges, or adjacent regions are coloured with minimum number of colour which is called the chromatic number.

Proper colouring - A vertex colouring or edge colouring of a graph in which ~~no~~ no two adjacent edges or adjacent vertexes have same colours.

Chromatic Number - The chromatic number $X(G)$ of a graph G is the smallest number of colours for $V(G)$ so that the adjacent vertices are coloured differently.

(15) Proove that complete graph of n vertices, the chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$$

The chromatic polynomial P_n of a graph G is a function that takes in a non-negative integer k and returns the number of ways to colour the vertices of G with k colours so that adjacent vertices have different colours.

The Complete Graph K_n

Lets label the vertices $v_0, \ldots, v_{n-1}$ and colour them one by one in given order. when we colour the first vertex v_0 no other vertex has been coloured, and we can use whichever of the k-vertice we like. However when we go to colour v_1, we note that it is adjacent to v_0, and so whatever colour we used for v_0 we can't use it for v_1, and so we have $k-1$ colours to choose for v_1.

Continuing in this way, we see that since all the vertices are adjacent, they all must have different colours and we can't use any of these to colour v_i and so we have $k-i$ choices to colour v_i.

Putting it all together, we see that

$$P_{K_n}(k) = k\cdot(k-1)\cdot(k-2)\cdots(k-n+1)$$

Let's assume $k=\lambda$

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$$