CQF Exam One Solution

January 2025 Cohort

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Optimal Portfolio Allocation

1)

We formulate the minimum variance portfolio as:

$\arg\min_w \frac{1}{2}$ w' Σw

`Subject to:`

$w'1 = 1, \quad \mu_\pi = w'\mu = m$

The Lagrangian multiplier of this minimum variance portfolio is:

$$L(w, \lambda, \gamma) = \frac{1}{2} w'\Sigma w + \lambda(1 - w'1) + \gamma(m - w'\mu)$$

The partial derivatives are:

$$\frac{\partial L(w, \lambda, \gamma)}{\partial w} = \Sigma w - \lambda 1 - \gamma \mu \quad (1)$$

$$\frac{\partial L(w, \lambda, \gamma)}{\partial \lambda} = 1 - w'1 \quad (2)$$

$$\frac{\partial L(w, \lambda, \gamma)}{\partial \gamma} = m - w'\mu \quad (3)$$

Optimal Weight Allocation   From equation (1),

- The optimal weight allocation is:

$$w^* = \Sigma^{-1}(\lambda 1 + \gamma \mu)$$

$$Substituting \quad the \quad value \quad of \quad w^* \quad in \quad the \quad constraints:$$

μ' w = m   and $1'$ w = 1

$$\mu'\Sigma^{-1}(\lambda 1 + \gamma \mu) = \lambda \mu'\Sigma^{-1} 1 + \gamma \mu'\Sigma^{-1}\mu = m$$

$$1'\Sigma^{-1}(\lambda 1 + \gamma\mu) = \lambda 1'\Sigma^{-1}1 + \gamma 1'\Sigma^{-1}\mu = 1$$

Solving for λ and γ :

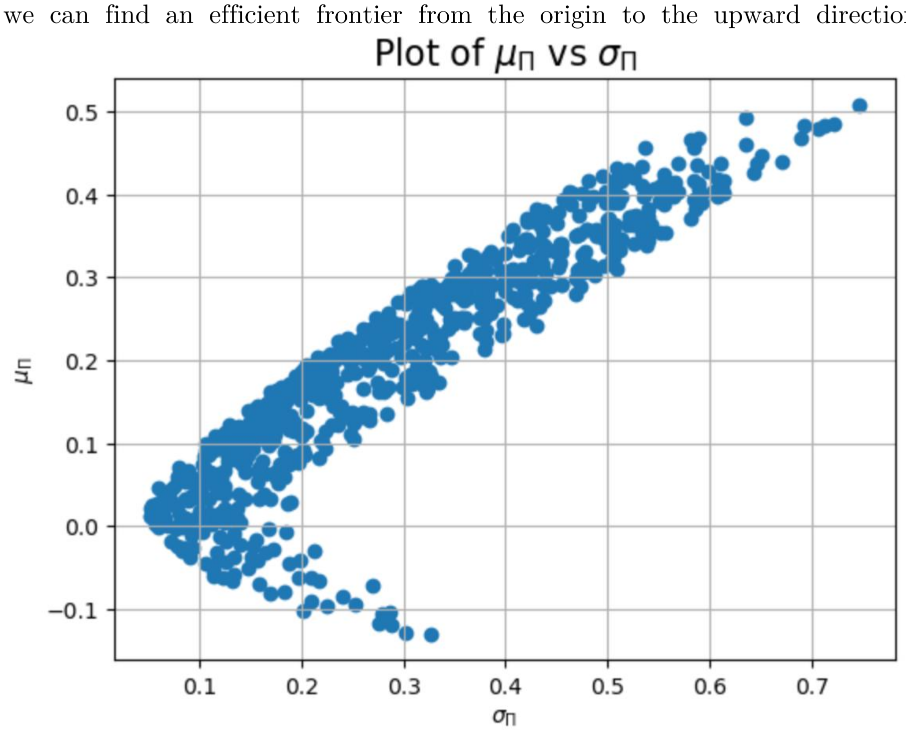
$$\lambda = \frac{(\mu'\Sigma^{-1}\mu) - (1'\Sigma^{-1}\mu)\cdot m}{(1'\Sigma^{-1}1)(\mu'\Sigma^{-1}\mu) - (1'\Sigma^{-1}\mu)^2}$$

$$\gamma = \frac{(1'\Sigma^{-1}1)\cdot m - (1'\Sigma^{-1}\mu)}{(1'\Sigma^{-1}1)(\mu'\Sigma^{-1}\mu) - (1'\Sigma^{-1}\mu)^2}$$

$$w^* = \begin{bmatrix} -7.92630392 \\ -0.98068924 \\ 0.18958165 \\ 0.47620601 \end{bmatrix}$$

2)

The shape of the plot μ vs σ is found to be Elliptical. In the plot, we can find an efficient frontier from the origin to the upward direction.



Products and Market Risk

3)

The VaR with regard to each asset is calculated as:

$$\frac{\partial VaR(w)}{\partial w_i} = \mu_i + \text{Factor} \times \frac{(\Sigma w)_i}{\sqrt{w^T \Sigma w}}$$

where the factor is determined by the standard normal distribution,

$$\frac{\partial VaR(w)}{\partial w_i} = \mu_i + \Phi(1 - 0.99) \times \frac{(\Sigma w)_i}{\sqrt{w^T \Sigma w}}$$

Similarly, the Expected Shortfall (ES) with regard to each asset is calculated as:

$$\frac{\partial ES(w)}{\partial w_i} = \mu_i - \frac{\phi(\text{Factor})}{1 - c} \times \frac{(\Sigma w_i)}{\sqrt{w^T \Sigma w}}$$

$$\frac{\partial ES(w)}{\partial w_i} = \mu_i - \frac{\phi\left(\Phi(1 - 0.99)\right)}{1 - 0.99} \times \frac{(\Sigma w)_i}{\sqrt{w^T \Sigma w}}$$

VaR and ES Sensitivity Table

Asset	VaR Sensitivity	ES Sensitivity
1	-0.6838647463691414	-0.7834795763585853
2	-0.3867988051998241	-0.4431416674758421
3	-0.22070940469615322	-0.25285893417929894

4)

The Expected Shortfall (ES) is calculated as:

$$ES_c(X) = \mu - \sigma \frac{\phi(\Phi^{-1}(1 - c))}{1 - c}$$

where (c) is the range of percentiles:

$$[99.95, 99.75, 99.5, 99.25, 99, 98.5, 98, 97.5]$$

Expected Shortfall Table

Percentile	Expected Shortfall
99.95	-3.55438
99.75	-3.10436
99.5	-2.89195
99.25	-2.76124
99	-2.66521
98.5	-2.5247
98	-2.42091
97.5	-2.3378

5)

Given Condition:

$$\alpha_{i+1} = \lambda\alpha_i \quad \text{where } \lambda \in (0,1)$$

$$\alpha_2 = \lambda\alpha_1$$

$$\alpha_3 = \lambda\alpha_2 = \lambda^2\alpha_1$$

$$\alpha_4 = \lambda\alpha_3 = \lambda^3\alpha_1$$

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Since α is the weight of assets,

$$\sum_{i=1}^{n->\infty} \alpha_i = 1$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \cdots = 1$$

$$\alpha_1 + \lambda\alpha_1 + \lambda^2\alpha_1 + \lambda^3\alpha_1 + \cdots = 1$$

Since the above equation is the sum of a geometric progression,

$$\frac{\alpha_1}{1-\lambda} = 1$$

$$\alpha_1 = 1 - \lambda \qquad (1)$$

Deriving the EWMA Model Equation:

$$\sigma_t^2 = \alpha_1 u_{n-1}^2 + \alpha_2 u_{n-2}^2 + \alpha_3 u_{n-3}^2 + \alpha_4 u_{n-4}^2 + ...$$

$$= \alpha_1 u_{n-1}^2 + \lambda\alpha_1 u_{n-2}^2 + \lambda^2\alpha_1 u_{n-3}^2 + ... \quad (2)$$

Shifting the Equation (2) by (t-1)

$$\sigma_{t-2}^2 = \alpha_2 u_{n-2}^2 + \alpha_2 u_{n-3}^2 + \alpha_3 u_{n-4}^2 + ...$$

$$\lambda\sigma_{t-2}^2 = \lambda\alpha_1 u_{n-2}^2 + \lambda\alpha_2 u_{n-3}^2 + \lambda\alpha_3 u_{n-4}^2 + ...$$

$$= \lambda\alpha_1 u_{n-2}^2 + \lambda^2\alpha_1 u_{n-3}^2 + \lambda^3\alpha_1 u_{n-4}^2 + ...$$

(3)

Substituting Equation (3) into (2)

$$\sigma_t^2 = \alpha_1 u_{t-1}^2 + \lambda\sigma_{t-1}^2$$

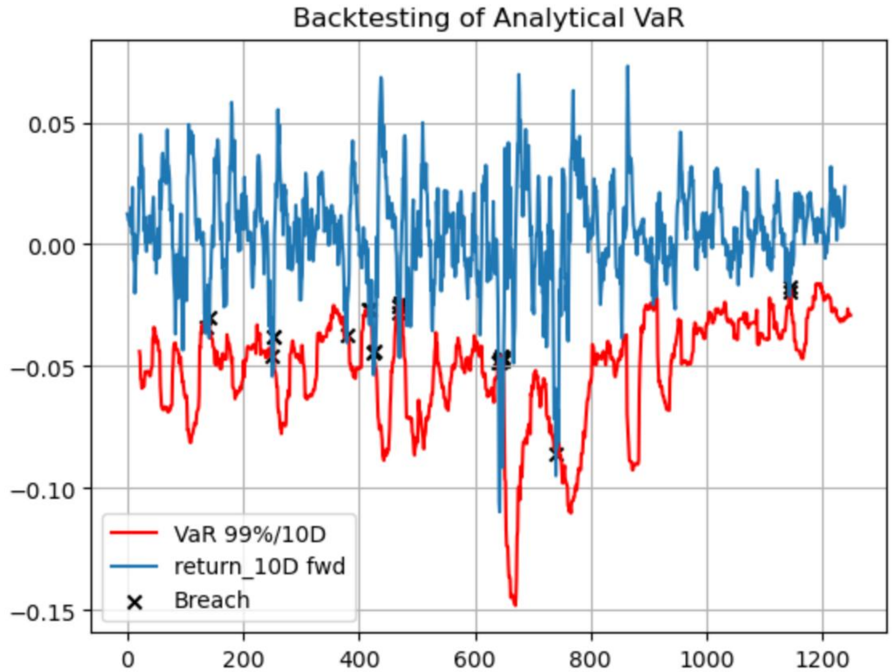
$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)u_{t-1}^2$$

[From Equation (1)]

VaR Backtesting

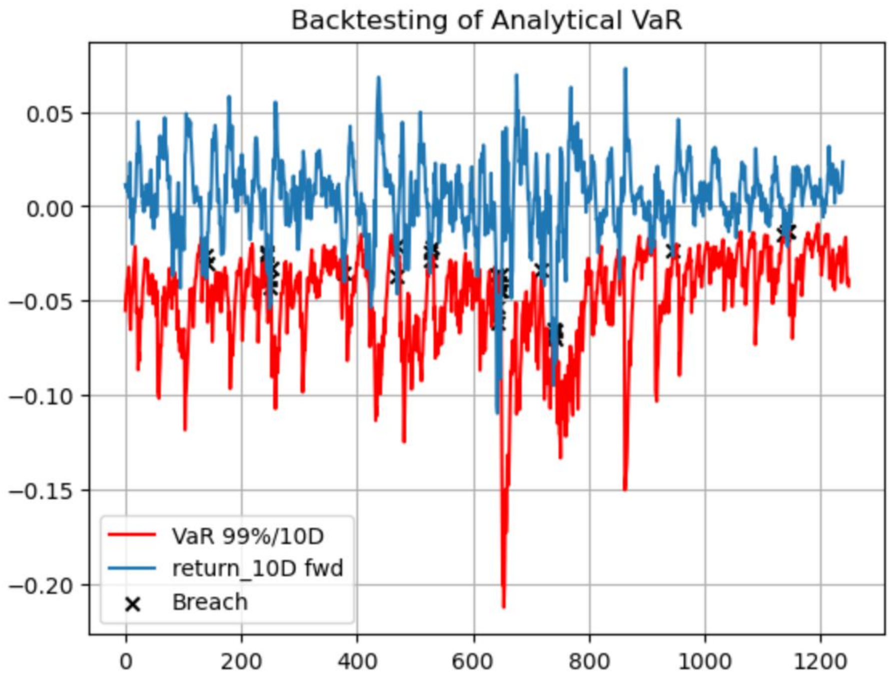
6)

The total number of VaR breaches is 25 and the percentage of VaR breaches is 2.0508%. And, the total number of Consecutive VaR breaches is 14 and the perecentage of Consecutive VaR breaches is 1.1484%.



7)

The total number of VaR breaches is 32 and the percentage of VaR breaches is 2.5620%. And, the total number of Consecutive VaR breaches is 17 and the perecentage of consecutive VaR breaches is 1.3610%.



Backtesting of Analytical VaR
0.05
0.00
−0.05
−0.10
−0.15
−0.20
0
200
400
600
800
1000
1200
VaR 99%/10D
return_10D fwd
Breach