Groupoid Models for Diagrams of Groupoid Correspondences

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Higher structures in Noncommutative Geometry and Quantum Algebra

Motivation

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Albandik and Meyer has given an interpretation of higher-rank graphs as dynamical systems

$\rightsquigarrow$ by studying a general kind of dynamical system, to recover the constructions of these groupoids and their C^*-algebras

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- groupoid correspondences are variants of these
- they are spaces with commuting actions of two groupoids
- taking groupoid C^*-algebras induces C^*-correspondences ☺

Groupoid correspondences

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An étale groupoid is *locally compact*: $\mathcal{G}^0$ is locally compact Hausdorff

Groupoid correspondences

Definition (Groupoid actions)

A *right $\mathcal{G}$-space*: $\mathcal{X}$

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Notation

Write $r\colon \mathcal{X} \to \mathcal{G}^0$ for left anchor maps.

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A right $\mathcal{G}$-space is *free* if and only if the map

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It is *proper* if and only if this map is proper.

It is *basic* if and only if this map is a homeomorphism onto its image.

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Proposition

Let $\mathcal{X}$ be a right $\mathcal{G}$-space.

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Proposition

Let $\mathcal{X}$ be a right $\mathcal{G}$-space. TFAE:

(1) *the action of $\mathcal{G}$ on $\mathcal{X}$ is basic and the orbit space $\mathcal{X}/\mathcal{G}$ is Hausdorff;*

(2) *the action of $\mathcal{G}$ on $\mathcal{X}$ is free and proper.*

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such that $s \colon \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism and the right $\mathcal{G}$-action is *basic*.

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When $\mathcal{G}$ and $\mathcal{H}$ are locally compact, and $\mathcal{X}/\mathcal{G}$ is Hausdorff, then it is a *locally compact groupoid correspondence*.

Groupoid correspondences

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A groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ is *proper*:

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It is *tight*: r_* is a homeomorphism.

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A groupoid correspondence $\mathcal{X} \colon \mathcal{H} \leftarrow \mathcal{G}$ is the same as a locally compact Hausdorff $\mathcal{X}$ with a continuous map $r \colon \mathcal{X} \to \mathcal{H}$ and a local homeomorphism $s \colon \mathcal{X} \to \mathcal{G}$.

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If furthermore $\mathcal{H} = \mathcal{G}$, it is a *topological graph* as in Katsura's work.

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A groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ is the same as a set $\mathcal{X}$ with commuting actions, where the right $\mathcal{G}$-action is free.

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Write $A := \mathcal{X}/\mathcal{G}$.

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Left $\mathcal{H}$-action on $A \times \mathcal{G}$:

$$h \cdot (x, g) = (\pi_h(x), \varphi(h, x) \cdot g)$$

for $\pi_h \colon A \to A$, $\varphi \colon \mathcal{H} \times A \to \mathcal{G}$

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$$\varphi(h_1 h_2, x) = \varphi(h_1, h_2 x) \cdot \varphi(h_2, x)$$

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$$\varphi(h_1 h_2, x) = \varphi(h_1, h_2 x) \cdot \varphi(h_2, x)$$

Furthermore, any isomorphism $\mathcal{X} \cong A \times \mathcal{G}$ is unique up to

$$(x, g) \overset{k}{\mapsto} (x, \psi(x) \cdot g)$$

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$$\begin{array}{rl} (x,g) & \overset{k}{\mapsto} (x, \psi(x)\cdot g) \\ h\cdot(x,g) & \mapsto h\cdot(x,\psi(x)\cdot g) = (\pi_h(x), \varphi(h,x)\cdot\psi(x)\cdot g) \\ \| & \\ (\pi_h(x), \varphi(h,x)\cdot g) & \mapsto (\pi_h(x), \psi(\pi_h(x))\cdot \varphi^{\psi}(h,x)\cdot g) \end{array}$$

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$$\|$$

$$(\pi_h(x), \varphi(h, x) \cdot g) \mapsto (\pi_h(x), \psi(\pi_h(x)) \cdot \varphi^{\psi}(h, x) \cdot g)$$

$$\varphi^{\psi}(h, x) := \psi(\pi_h(x))^{-1} \cdot \varphi(h, x) \cdot \psi(x)$$

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The assignment of 'new' 1-cocycle

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is a right action.

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isomorphism classes of $\mathcal{H} \leftarrow \mathcal{G} \leftrightsquigarrow$ equivalence classes of (A,φ)

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$\mathcal{X} \cong \mathcal{G}$ with a base point in $\mathcal{X}$.

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for a group homomorphism $\varphi\colon \mathcal{H} \to \mathcal{G}$.

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If we pick another base point, φ is replaced by

$$\mathrm{Ad}_g^{-1} \circ \varphi$$

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If $\varphi\colon \mathcal{H} \to \mathcal{G}$ is injective, and $\mathcal{G} = \mathcal{H}$, we can get Stammeier's notion.

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If φ is injective, and both $\mathcal{G}$ and $\mathcal{H}$ are Abelian, we recover Cuntz and Vershik's notion.

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$\rightsquigarrow$ a proper $\mathcal{G} \leftarrow \mathcal{G}$ can be viewed as a self-similarity of $\mathcal{G}$

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Set $r(x, y) := r(x)$ and $s(x, y) := s(y)$.

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Proposition (Albandik, Antunes, K., Meyer)

The actions of $\mathcal{H}$ and $\mathcal{K}$ on $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ are well-defined and turn $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ into a groupoid correspondence

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If both $\mathcal{X}$ and $\mathcal{Y}$ are proper or tight, then so is $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$.

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A continuous map $f\colon \mathcal{X} \to \mathcal{Y}$ between two right $\mathcal{G}$-spaces is *$\mathcal{G}$-equivariant*:

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$$\begin{array}{cc} \mathcal{H} \circ_{\mathcal{H}} \mathcal{X} \to \mathcal{X} & \mathcal{X} \circ_{\mathcal{G}} \mathcal{G} \to \mathcal{X} \\ [h, x] \mapsto h \cdot x & [x, g] \mapsto x \cdot g \end{array}$$

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$$\mathcal{X}_1 \circ_{\mathcal{G}_1} (\mathcal{X}_2 \circ_{\mathcal{G}_2} \mathcal{X}_3) \to (\mathcal{X}_1 \circ_{\mathcal{G}_1} \mathcal{X}_2) \circ_{\mathcal{G}_2} \mathcal{X}_3$$

$$[x_1, [x_2, x_3]] \mapsto [[x_1, x_2], x_3]$$

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$$\varphi\colon A \to \mathcal{L}(\mathcal{E})$$

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Definition ($C^*(\mathcal{G})$)

The *groupoid C^*-algebra* $C^*(\mathcal{G})$ of $\mathcal{G}$ is the completion of $\mathfrak{S}(\mathcal{G})$ in the largest C^*-norm.

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Cuntz-Pimsner algebras of C^*-correspondences

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For any C^*-correspondence $\mathcal{E}\colon A \leftarrow A$, we can associate a C^*-algebra, called the *Cuntz-Pimsner algebra* $\mathcal{O}_{\mathcal{E}}$:

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The universal one, $(A, \mathcal{E}) \to \mathcal{O}_{\mathcal{E}}$, gives the Cuntz-Pimsner algebra.

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The norm completion $C^*(\mathcal{X})$ is the C^*-correspondence in Katsura's work used to define topological graph C^*-algebras.

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And the Cuntz-Pimsner algebra $\mathcal{O}_{C^*(\mathcal{X})}$ is Nekrashevych's universal Cuntz-Pimsner algebra $\mathcal{O}_{(\mathcal{G},\mathcal{X})}$ of the self-similar group.

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- composition: completed tensor product

Constructing $\mathfrak{Gr}_{\mathrm{lc}} \to \mathfrak{Corr}$

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Construction

Recall that a 2-morphism in $\mathcal{G}$ is an injective $\mathcal{H}, \mathcal{G}$-equivariant continuous map $\mathcal{X} \xrightarrow{\alpha} \mathcal{Y}$.

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$\rightsquigarrow$ extends uniquely to an isometric $C^*(\mathcal{H}), C^*(\mathcal{G})$-bimodule map

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which extends to an isomorphism of $C^*(\mathcal{K}), C^*(\mathcal{G})$-correspondences

$$\mu_{\mathcal{X},\mathcal{Y}} \colon C^*(\mathcal{X}) \otimes_{C^*(\mathcal{G})} C^*(\mathcal{Y}) \to C^*(\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y})$$

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form a (strictly unital) pseudofunctor $\mathfrak{Gr}_{\mathrm{lc}} \to \mathfrak{Corr}$.

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$$M \to \mathfrak{Gr}_{\mathrm{lc}} \hookrightarrow \mathfrak{Gr} \to \mathfrak{Corr}$$

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$$M \to \mathfrak{Gr}_{\mathrm{lc}} \hookrightarrow \mathfrak{Gr} \to \mathfrak{Corr}$$

is equivalent to a product system over M.

$\rightsquigarrow$ we should study diagrams of groupoid correspondences

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Notation

We can describe a $\mathcal{C}$-shaped diagram by $(\mathcal{G}_x, \mathcal{X}_g, \mu_{g,h})$.

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- if $\gamma \cdot y = \gamma' \cdot y'$, then there is $\eta \in \mathcal{G}$ so that

$$\gamma' = \gamma \cdot \eta \quad \text{and} \quad y = \eta \cdot y'$$

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$$r(\varphi(y)) = r(y) \quad \text{and} \quad \varphi(\gamma \cdot y) = \gamma \cdot \varphi(y)$$

for all $g \in \operatorname{mor} \mathcal{C}, y \in Y_{s(g)}, \gamma \in \mathcal{X}_g$ with $s(\gamma) = r(y)$.

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Definition (Groupoid models)

A *groupoid model* for F-actions is an étale groupoid $\mathcal{U}$ with natural bijections

$$\{\mathcal{U}\text{-actions on } Y\} \leftrightsquigarrow \{F\text{-actions on } Y\}$$

for all spaces Y.

F-actions and groupoid models

In other words, groupoid models encode diagrams of groupoid correspondences:

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$\rightsquigarrow$ for a groupoid C^*-algebra to be defined, the groupoid has to be *locally compact*.

Existence of groupoid models

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The partial homeomorphisms of Y form an *inverse semi-group* $I(Y)$.
Similarly, we can encode a diagram F of groupoid correspondences by an inverse semi-group $I(F)$.

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$$(s_1, y) \sim (s_2, y) \iff \exists \text{ idempotent } e : s_1 e = s_2 e$$

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$$\begin{array}{ccccc} y & \xrightarrow{(s,y)} & \vartheta_s(y) & \xrightarrow{(u,\vartheta_s(y))} & \vartheta_u(\vartheta_s(y)) \\ & & || & & \\ y & & \xrightarrow{(us,y)} & & \vartheta_{us}(y) \end{array}$$

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$\rightsquigarrow$ showing the existence of universal F-actions will imply groupoid models exist

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Theorem (K., Meyer)

Any diagram $F\colon \mathcal{C} \to \mathfrak{Gr}$ of groupoid correspondences has a universal F-action.

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Corollary (K., Meyer)

Any diagram of groupoid correspondences has a groupoid model.

Existence of locally compact groupoid models

Next, we would like to know if the groupoid model is *locally compact*, in order to have applications in C^*-algebras.

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$\rightsquigarrow$ conditions on groupoid correspondences ensuring 'compactness'

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We want a groupoid model to be locally compact

$\Longrightarrow$ the universal F-action should take place on a locally compact Hausdorff space

$\rightsquigarrow$ for each F-action, we find a unique corresponding F-action on a locally compact Hausdorff space

Existence of locally compact groupoid models

Definition (Relative Stone-Čech compactification)

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Remark

The induced $\beta_B X \to B$ is a proper map.

Existence of locally compact groupoid models

Theorem (K., Meyer)

The relative Stone-Čech compactification β_B is left adjoint to the inclusion of proper Hausdorff B-spaces.

$$
\begin{tikzcd}
B\text{-Space} \arrow[r, shift left, "\beta_B"] \arrow[r, phantom, "\perp"] & B\text{-Space-Pro} \arrow[l, hook, shift left]
\end{tikzcd}
$$

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Theorem (K., Meyer)

The relative Stone-Čech compactification β_B is left adjoint to the inclusion of proper Hausdorff B-spaces.

$$\begin{tikzcd} B\text{-Space} \arrow[r, shift left, "\beta_B"] \arrow[r, phantom, "\perp"] & B\text{-Space-Pro} \arrow[l, hook, shift left] \end{tikzcd}$$

Remark

In other words, the category of proper Hausdorff B-spaces is a *reflective subcategory* of the category of B-spaces.

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This construction 'lifts' the previous adjunction to

$$\text{Category of } F\text{-actions} \underset{\hookleftarrow}{\overset{L}{\underset{\perp}{\longrightarrow}}} \begin{array}{l}\text{Category of F-actions on proper}\\ \text{Hausdorff } \mathcal{G}^0\text{-spaces}\end{array}$$

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In particular, terminal objects are a kind of limits.

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Therefore, the groupoid model $I(F) \ltimes \Omega$ of F is locally compact.

Thank you!

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