A braided monoidal 2-category from Soergel bimodules

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Background:

Categorification:
tangle invariants and TQFTs

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$U_q(\mathfrak{g}) \rightsquigarrow$ interesting braided monoidal categories

Applications: topological invariants
- knot / tangle invariants
- TQFTs

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Modern way to formulate TQFT: monoidal functor Z

Bordism hypothesis: Z determined by $Z(pt)$

$U_q(\mathfrak{g}) \rightsquigarrow$ interesting braided monoidal categories

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Modern way to formulate TQFT: monoidal functor Z

Bordism hypothesis: Z determined by $Z(pt)$

RTW - theory	$Z(pt)$ some braided monoidal category
"categorified RTW - theory"	$Z(pt)$ some "braided monoidal bicategory"

Revisit quantum groups

$U_q(sl_2) \circlearrowright V$ natural representation

$\rightsquigarrow$ braided monoidal category generated by V

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$\rightsquigarrow$ braided monoidal category generated by V

objects: $V^{\otimes a}$

morphisms: $\mathrm{Hom}(V^{\otimes a}, V^{\otimes b}) = \begin{cases} \{0\} & \text{if } a+b \text{ odd} \\ \mathrm{Hom}(V^{\otimes \frac{a+b}{2}}, V^{\otimes \frac{a+b}{2}}) & \text{otherw.} \end{cases}$

$$\Uparrow$$

$H_q(S_n)$ Hecke algebra

$$\Uparrow \quad \beta_i^2 = 1 + (q^{-1} - q)\beta_i$$

$\mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[q,q^{-1}](Br_n)$ group algebra of braid group

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$\mathbb{Z}[q, q^{-1}](Br_n)$ group algebra of braid group

Interchange law

$$\begin{matrix} g \\ \circ \\ f \end{matrix} \otimes \begin{matrix} g' \\ \circ \\ f' \end{matrix} = \begin{matrix} g \otimes g' \\ \circ \\ f \otimes f' \end{matrix}$$

NATURAL isos

$c_{X,Y} : X \otimes Y \to Y \otimes X$

satisfying hexagons

Universal braided monoidal category $\mathcal{H}$ behind ALL $U_q(\mathfrak{sl}_k)$

object: $\mathbb{N}_0$

morphisms: $\mathrm{Hom}(m,n) = \begin{cases} \{0\} & m \neq n \\ H_q(S_n) & \text{if } m=n \end{cases}$

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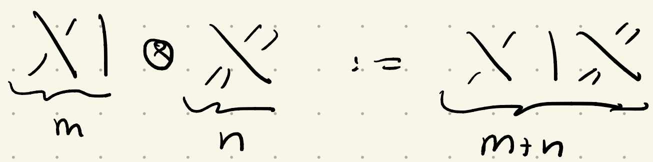
monoidal: $m \otimes n := m+n$



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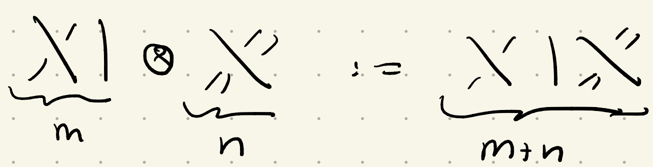
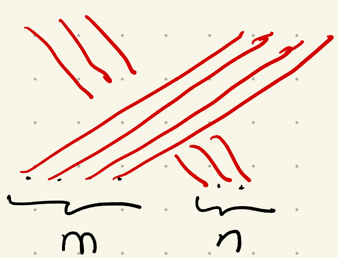


braided

$c_{m,n}$:



$n \otimes m$

$\uparrow$

$m \otimes n$

positive (m,n) - shuffle braid

Categorify?

$\mathbb{C} \otimes_{\mathbb{Z}} K_0$

vector spaces	categories	algebras
linear maps	(exact) functors	bimodules
	natural transformations	bimodule maps
Categories	2-categories	bicategories

Categorify?

$\mathcal{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0$

$V^{\otimes n}$	vector spaces	categories	algebras	graded algebras R_n
$H_q(n)$	linear maps	(exact) functors	bimodules	certain graded bimodules
		natural transformations	bimodule maps	bimodule maps
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Categorify!

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Soergel bimodules!

∞-categorical model:

A BRAIDED MONOIDAL $(\infty, 2)$-CATEGORY OF SOERGEL BIMODULES

YU LEON LIU, AARON MAZEL-GEE, DAVID REUTTER, CATHARINA STROPPEL, AND PAUL WEDRICH

Abstract. The Hecke algebras for all symmetric groups taken together form a braided monoidal category that controls all quantum link invariants of type A and, by extension, the standard canon of topological quantum field theories in dimension 3 and 4. Here we provide the first categorification of this Hecke braided monoidal category, which takes the form of an $\mathbb{E}_2$-monoidal $(\infty, 2)$-category whose hom-$(\infty, 1)$-categories are k-linear, stable, idempotent-complete, and equipped with $\mathbb{Z}$-actions. This categorification is designed to control homotopy-coherent link homology theories and to-be-constructed topological quantum field theories in dimension 4 and 5.

Our construction is based on chain complexes of Soergel bimodules, with monoidal structure given by parabolic induction and braiding implemented by Rouquier complexes, all modelled homotopy-coherently. This is part of a framework which allows to transfer the toolkit of the categorification literature into the realm of ∞-categories and higher algebra. Along the way, we develop families of factorization systems for (∞, n)-categories, enriched ∞-categories, and ∞-operads, which may be of independent interest.

As a service aimed at readers less familiar with homotopy-coherent mathematics, we include a brief introduction to the necessary ∞-categorical technology in the form of an appendix.

dg-categorical model:

(in the progress of writing up)

NATURALITY FOR THE BRAIDING ON TYPE A SOERGEL BIMODULES

CATHARINA STROPPEL AND PAUL WEDRICH

Abstract. We consider categories of Soergel bimodules for the symmetric groups S_n in their $\mathfrak{gl}_n$-realizations for all n in $\mathbb{N}_0$ and assemble them into a locally linear monoidal bicategory. Chain complexes of Soergel bimodules likewise form a locally dg-monoidal bicategory which can be equipped with the structure of a braiding, whose data includes the Rouquier complexes of shuffle braids. The braiding, together with a uniqueness result, was established in an infinity-categorical setting in recent work with Yu Leon Liu, Aaron Mazel-Gee and David Reutter.

In the present article we construct this braiding explicitly and describe its requisite coherent naturality structure in a concrete dg-model for the morphism categories.

First step: not quantised $\quad H_{q=1}(S_n) = \mathbb{Z}[S_n]$

Prop:

$$K_0\left(\begin{array}{l}\text{additive } \otimes\text{-category} \\ \text{generated by} \\ \text{permutation } R_n\text{-bimodules}\end{array}\right) \cong \mathbb{Z}[S_n]$$

$$[M \oplus N] = [M] + [N] \qquad [M \otimes_{R_n} N] = [M][N]$$

First step: not quantised $H_{q=1}(S_n) = \mathbb{Z}[S_n]$

$R_n = \mathbb{C}[x_1, \dots, x_n]$ clear: R_n is R_n-bimodule

$\omega \in S_n$ $\quad R_n^{(\omega)}$ right action twisted by ω

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$\omega \in S_n$ $\quad R_n^{(\omega)}$ right action twisted by ω

Lemma: $R_n^{(x)} \otimes_{R_n} R_n^{(y)} \cong R_n^{(xy)}$ $\quad x, y \in S_n$

$$\boxed{\text{Prop: } K_0 \left(\begin{array}{l} \text{additive } \otimes\text{-category} \\ \text{generated by} \\ \text{permutation } R_n\text{-bimodules} \end{array} \right) \cong \mathbb{Z}[S_n]}$$

$[M \oplus N] = [M] + [N] \qquad [M \otimes N] = [M][N]$

How to quantise? Interesting braiding?

Step I: Grading on R_n $\deg(x_i) = 2$

$\mathbb{Z}[S_n]$

$\Big\}$ Step I

$\mathbb{Z}[q, q^{-1}][S_n]$

How to quantise? Interesting braiding?

$$\mathbb{Z}[S_n] \xrightarrow{\text{Step I}} \mathbb{Z}[q,q^{-1}][S_n]$$

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Observation: Have short exact sequence of graded bimodules

$$R_n^{\langle 1\rangle} \hookrightarrow R_n \otimes_{R_n^{s_i}} R_n^{\langle -1\rangle} \xrightarrow{\text{mult}} R_n^{\langle -1\rangle}$$

Analog: 

How to quantise? Interesting braiding?

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Observation: Have short exact sequence of graded bimodules

$$R_n^{s_i}\langle 1\rangle \hookrightarrow \underbrace{R_n \otimes_{R_n^{s_i}} R_n\langle -1\rangle}_{\substack{\shortparallel \\ B_i}} \xrightarrow{\text{mult}} R_n\langle -1\rangle$$

Analog:

Theorem (Soergel, Rouquier)

$$K_0\left(\begin{array}{l} \overset{\mathrm{BSbim}_n}{\overset{\shortparallel}{\text{additive } \otimes\text{-category}}} \\ \text{generated by graded} \\ \text{Soergel } R_n\text{-bimodules } B_i \\ \text{and their grading shifts} \end{array}\right) \cong H_q[S_n]$$

Theorem (S.-Wedrich)

There exists a semistrict monoidal 2-category

Objects: $\mathbb{N}_0$

homs: $\mathrm{Hom}(m,n) = \begin{cases} \{0\} & m \neq n \\ \mathrm{BSbim}_n & m = n \end{cases}$

monoidal: on objects $m \otimes n := m + n$

on morphisms $\boxtimes := \otimes_{\mathbb{C}}$ (extended to complexes)

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Interchange law:

$$(M \otimes_{R_n} M') \boxtimes (N \otimes_{R_n} N') \cong (M \boxtimes N) \otimes_{R_{m+n}} (M' \boxtimes N')$$

Theorem (Soergel, Rouquier)

$$K_0\left(\begin{array}{c} \overset{\text{BSbim}_n}{\text{additive } \otimes\text{-category}} \\ \text{generated by graded} \\ \text{Soergel } R_n\text{-bimodules } B_i \\ \text{and their grading shifts} \end{array}\right) \cong H_q[S_n]$$

$$\| $$

$$K_0\left(K^b(\text{BSbim})\right)$$

$$B_i^{-1} \neq B_i = \Big(\underbrace{R_n \otimes_{R_n^{s_i}} R_n}_{=: B_i}\langle -1\rangle \xrightarrow{\text{mult}} R_n\langle -1\rangle \Big) \in K^b(\text{BSbim}_n)$$

Rouquier complex

Theorem (S. - Wedrich)

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braiding: braiding isomorphism $c_{m,n}$ given by the Rouquier complex associated to positive (m,n)-shuffle braid

(ALL coherence and naturality constraints are satisfied).

Tetracategories

Notes by Todd Trimble

The terminology for tricategories is taken from

R. Gordon, A. J. Power, and Ross Street, Coherence for Tricategories. Memoirs of the AMS, vol. 117, no. 558, September 1995.

A <u>tetracategory</u> is given by the following data:

(1) A collection of <u>0-cells</u> $a, b, c, \ldots$

(2) For each pair of 0-cells (a, b), a tricategory $C(a, b)$. Objects of these tricategories, called <u>1-cells</u> of the tetracategory C, are denoted by letters x, y, z, u, v, w. 1-morphisms, 2-morphisms, and 3-morphisms of the local tricategories $C(a,b)$ are called 2-cells, 3-cells, and 4-cells of the tetracategory C.

(3) For each triple of 0-cells (a, b, c), a trifunctor

$$\otimes_{a,b,c} : C(a,b) \times C(b,c) \longrightarrow C(a,c)$$

called <u>composition</u>, and for each 0-cell a, a 1-cell $I_a \in C(a,a)$, called a <u>unit</u>.

In what follows, the notation $\otimes$ is usually suppressed to save space. Thus, instead of writing $\otimes_{a,b,c}(x, y)$ where $x \in C(a,b)$, $y \in C(b,c)$ are 1-cells, we write xy. Similarly, we typically suppress the index of unit objects I_a, and write simply I.

(4) For each 4-tuple of 0-cells (a, b, c, d), a triequivalence

$$\alpha_{a,b,c,d} : \otimes_{a,c,d}(\otimes_{a,b,c} \times 1) \longrightarrow \otimes_{a,b,d}(1 \times \otimes_{b,c,d})$$

called the <u>associativity</u>. This is comprised of 2-cells of the form $(xy)z \to x(yz)$ and attendant structural data ("product cells") for a tritransformation, all of which are equivalences at the appropriate level.

Also, for each pair of 0-cells (a,b), there are triequivalences

$$\lambda_{a,b} : \otimes_{a,a,b}(I_a \times 1) \longrightarrow 1$$

$$\rho_{a,b} : \otimes_{a,b,b}(1 \times I_b) \longrightarrow 1$$

comprised of 2-cells of the form $\lambda : Ix \to x$, $\rho : xI \to I$, called <u>left</u> and <u>right</u> <u>unit</u> <u>actions</u>.

(5) For each 5-tuple of 0-cells, a trimodification π called the <u>pentagonator</u>, which is a local equivalence comprised of 3-cells

$$\begin{tikzcd}
 & (xy)(zw) \arrow[r, "\alpha"] & x(y(zw)) \\
((xy)z)w \arrow[ur, "\alpha"] \arrow[d, "\alpha w"'] \arrow[r, phantom, "\Downarrow\pi"] & & \\
(x(yz))w \arrow[r, "\alpha"'] & x((yz)w) \arrow[uur, "x\alpha"'] &
\end{tikzcd}$$

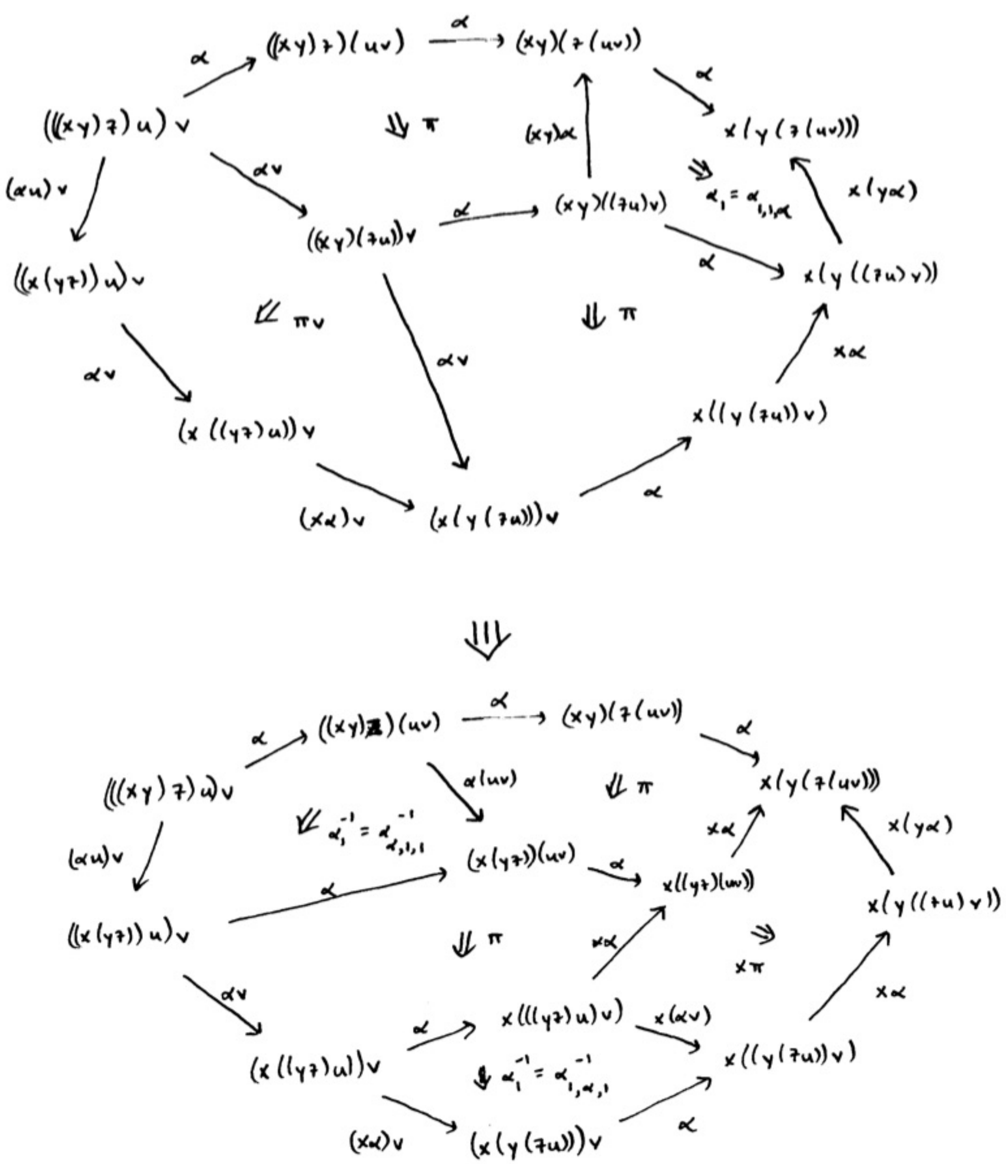
and attendant structural data ("product cells") for a trimodification that are invertible.

Also, for each triple of 0-cells, trimodifications l, m, r called <u>unit</u> <u>mediators</u>, that are local equivalences comprised of 3-cells

$$\begin{tikzcd}
(Ix)y \arrow[rr, "\alpha"] \arrow[dr, "\lambda y"'] & \arrow[d, phantom, "l \Rrightarrow"] & I(xy) \arrow[dl, "\lambda"] \\
 & xy &
\end{tikzcd}
\qquad
\begin{tikzcd}
(xI)y \arrow[rr, "\alpha"] \arrow[dr, "\rho y"'] & \arrow[d, phantom, "m \Rrightarrow"] & x(Iy) \arrow[dl, "x\lambda"] \\
 & xy &
\end{tikzcd}
\qquad
\begin{tikzcd}
(xy)I \arrow[rr, "\alpha"] \arrow[dr, "\rho"'] & \arrow[d, phantom, "r \Rrightarrow"] & x(yI) \arrow[dl, "x\rho"] \\
 & xy &
\end{tikzcd}$$

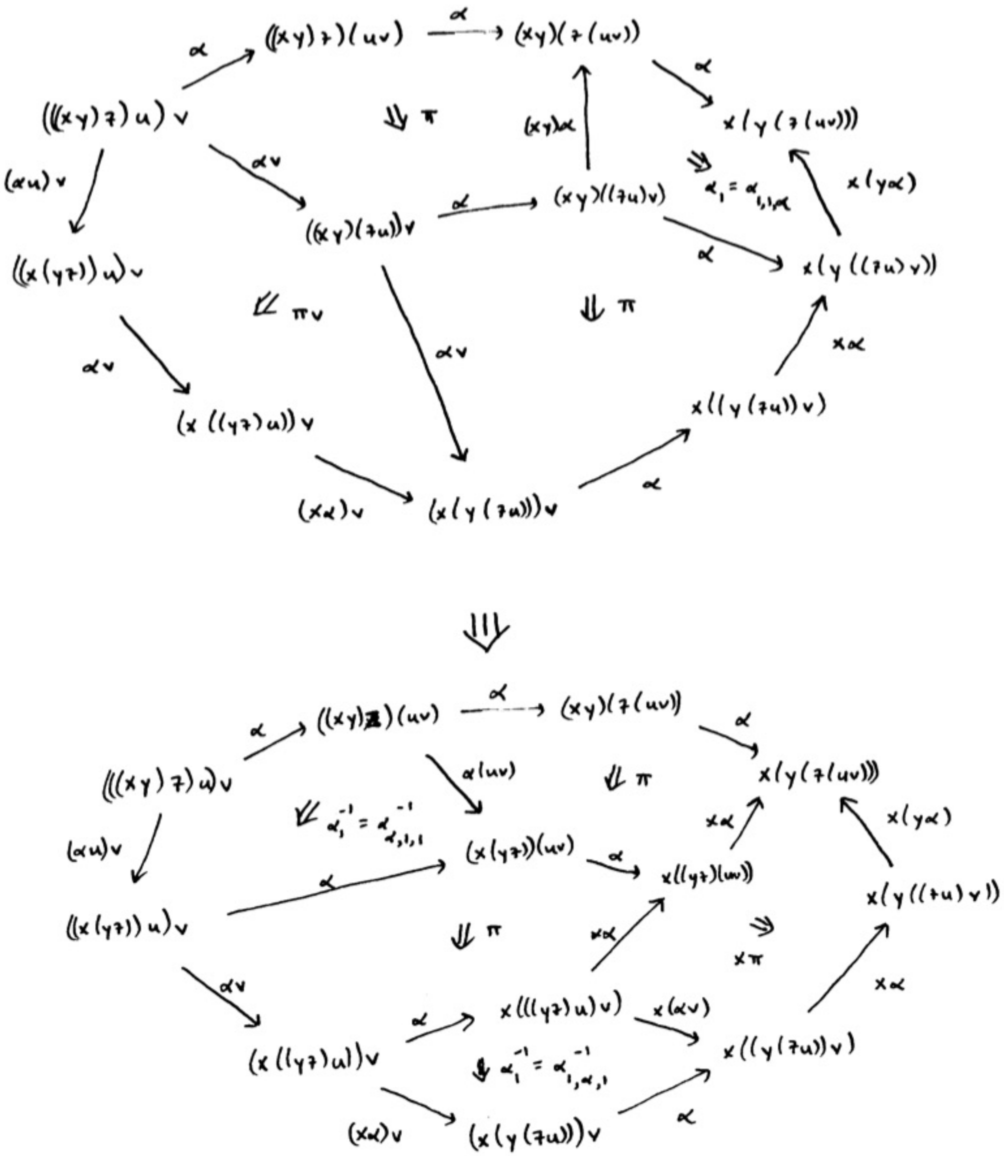
and attendant structural data for a trimodification that are invertible.

(6) For each 6-tuple of 0-cells, a pertubation K_5 called a (non-abelian) <u>4-cocycle</u>, comprised of invertible 4-cells of the form



Still data ...

Also, for each 4-tuple of 0-cells, four perturbations $U_{4,1}$, $U_{4,2}$, $U_{4,3}$, $U_{4,4}$ called <u>unit cocycle</u> perturbations, comprised of invertible 4-cells of the form



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have all data!

These data are subject to four equations:

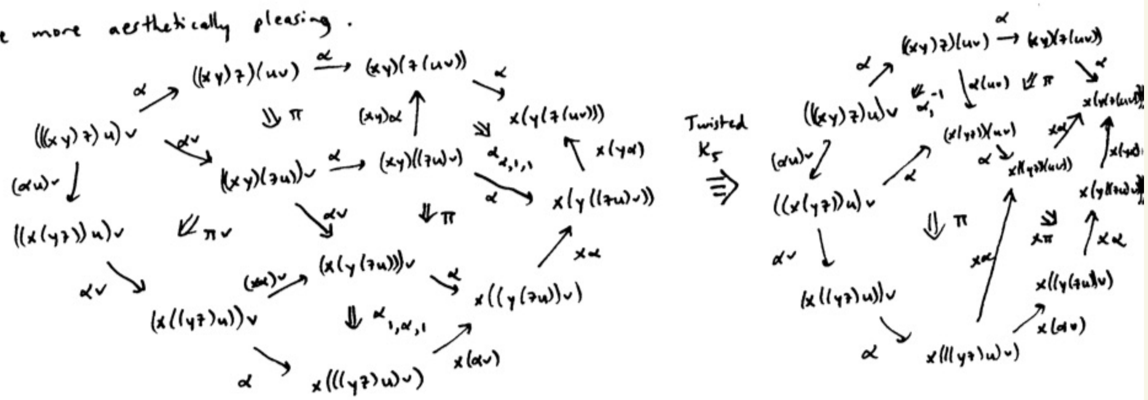
- K_6 Associativity Condition
- $U_{5,2}$ Unit Condition
- $U_{5,3}$ Unit Condition
- $U_{5,4}$ Unit Condition

which are described by the 4-cell pasting diagrams which follow. The interpretations of the pasting composites as 4-cells in local hom-tricategories is (essentially) unambiguous due to the coherence theorem for tricategories stated by Gordon-Power-Street.

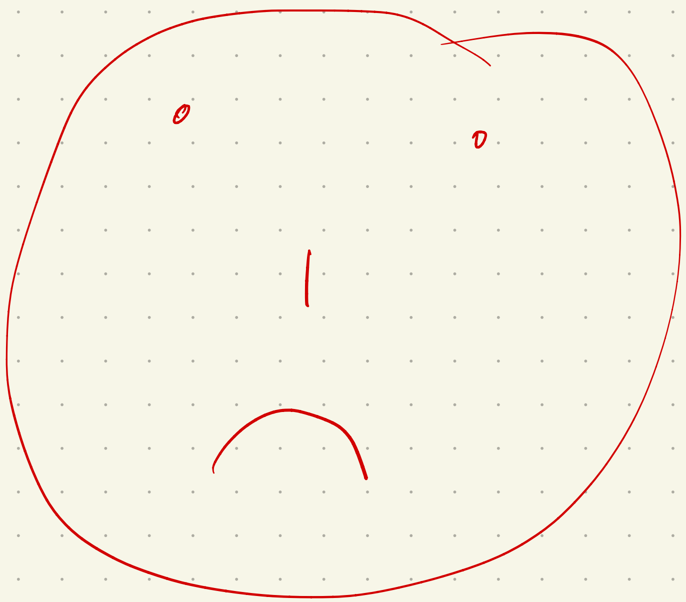
anounces coherence conditions ...

Note The original diagrams, drawn up in 1995 at the request of Ross Street, involved the K_5 perturbation as presented on page 3. Only the K_6 associativity survived the span of years 1995 - 2006; the unit diagrams were redrawn in 2006, but using a slightly "twisted" version of K_5 — in the opinion of the author, this leads to unit diagrams which are more aesthetically pleasing.

end of long history ?



Various twists occur frequently throughout the pasting diagrams, and seem to be an unavoidable fact of life, due to an apparent inexistence of a lift of the Stasheff operad to the level of parity complexes. (Possibly related to similar difficulties in early attacks on the Deligne conjecture.)



__Theorem__ (S.-Wedrich)

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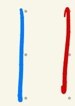
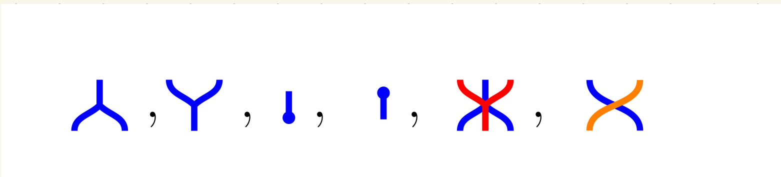
(ALL coherence and naturality constraints are satisfied)

Turn $BSbim_n$ into diagrams?

(Elias–Khovanov, Elias–Williamson)

Draw B_i as $B_i \otimes_{R_n} B_j$ as 

generating morphisms:



modulo some list of relations

Theorem

$$BSbim_n \cong BSbim'_n$$

diagrammatical category

Theorem (S.-Wedrich)

The $BSbim_n$ assemble into a semistrict monoidal (via $\boxtimes$) 2-category

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Theorem (S.-Wedrich) $\mathcal{C}$ strict monoidal k-linear dg-category

$\Rightarrow$ $Ch^b(\mathcal{C})'$, $K^b(\mathcal{C})'$ strict monoidal

Idea:

objects: $\bigoplus_{i\in I} \Sigma^{a_i} X^i := \{ \Sigma^{a_i} X^i \}$ $a_i \in \mathbb{N}_0$

finite totally ordered set

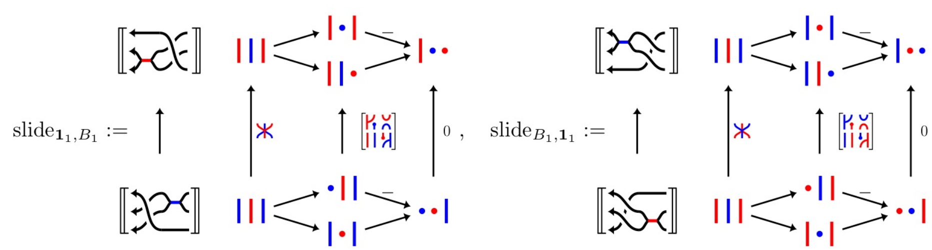
$\mathrm{Hom}^{\mathcal{C}}\left(\bigoplus_{i\in I} \Sigma^{a_i} X^i, \bigoplus_{j\in J} \Sigma^{b_j} Y^j \right)$ $J \times I$ matrices of morphisms

$$\left(\bigoplus_{i\in I} \Sigma^{a_i} X^i \right) \otimes \bigoplus_{j\in J} \left(\Sigma^{b_j} Y^j \right) := \bigoplus_{(i,j)\in I\times J} \Sigma^{a_i + b_j} X^i \otimes Y^j$$

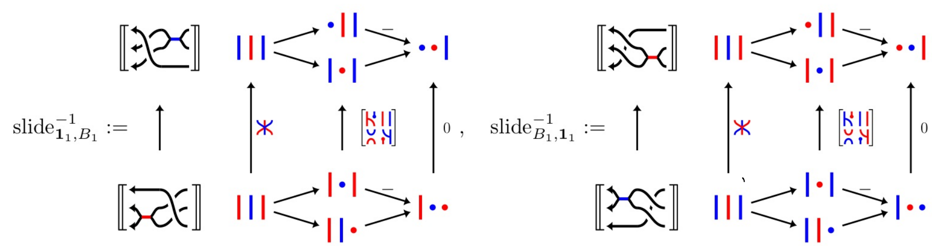
lexicographic ordering!

Braiding is natural!

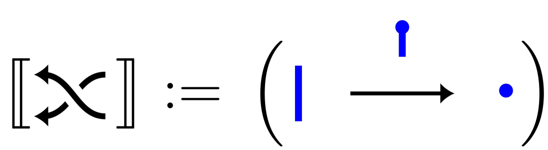
Lemma 2.5.5. *There are slide chain maps*



which are invertible up to homotopy. The inverses are given by the chain maps



1-Morphisms and
2-Morphisms slide
through!



Rouquier complex

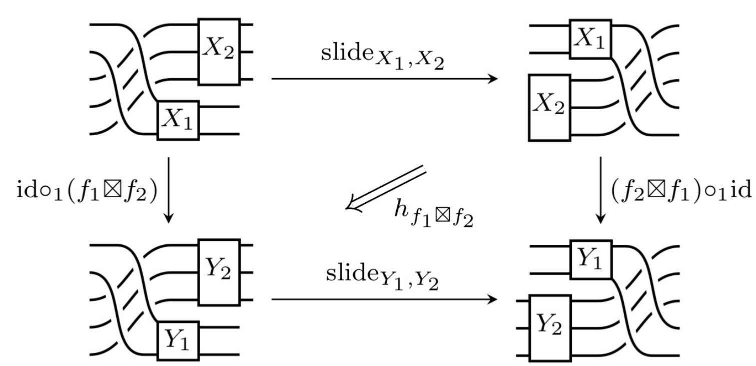
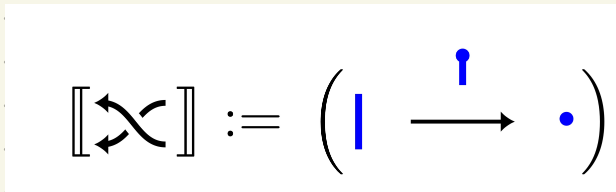


FIGURE 1. Slide chain maps and slide homotopies.

multiplication map



B_i

regular bimodule