KK-duality for Temperley–Lieb subproduct systems

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A better title would have been *KK-duality from quantum group symmetries*

1. KK-theory and KK-duality

2. Subproduct systems

3. KK-duality for Temperley–Lieb subproduct systems

Kasparov '80s: For two C^*-algebras, a graded abelian group $KK_*(A,B)$, with an associative, bilinear product

$$KK_i(A,B) \times KK_j(B,C) \to KK_{i+j}(A,C).$$

The Kasparov KK-theory groups recover K-theory and K-homology:

$$KK_*(\mathbb{C},A) \simeq K_*(A) \qquad KK_*(A,\mathbb{C}) \simeq K^*(A).$$

The Atiyah–Singer index pairing is a special case of the KK-product:

$$KK_0(\mathbb{C},A) \times KK_0(A,\mathbb{C}) \to KK_0(\mathbb{C},\mathbb{C}) = \mathbb{Z}.$$

Semi-split extensions of C*-algebras

$$0 \to I \to E \to A \to 0$$

give rise to elements in $KK_1(A,I)$.
Alternative description through the Busby invariant $\tau : A \to \mathcal{Q}(I) = M(I)/I$.

The notion of KK-duality is a noncommutative analogue of the Spanier–Whitehead duality which relates the homology of a finite complex with the cohomology of some dual finite complex.

Definition

Let A and B be separable C*-algebras. We say that A and B are *KK-dual* if there is a K-homology class $\Delta \in KK_i(A \otimes B, \mathbb{C})$ (the *fundamental class*) and a K-theory class $\delta \in KK_i(\mathbb{C}, A \otimes B)$ (the *dual class*) such that

$$\delta \otimes_A \Delta = (-1)^i 1_{KK(B,B)}, \qquad \delta \otimes_B \Delta = 1_{KK(A,A)}.$$

One obtains isomorphisms

$$- \otimes_A \Delta : K_j(A) \to K^{j+i}(B), \qquad \delta \otimes_B - : K^j(B) \to K_{j+i}(A).$$

Theorem (Kasparov 1988)

For any compact Riemannian manifold X, the C^*-algebra $C(X)$ is KK-dual to the (graded) algebra of sections of the associated Clifford bundle.

Theorem (Kaminker–Putnam 1997)

Let $A \in \mathrm{Mat}_n\{0,1\}$ with no row or column consisting entirely of zeros. The Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A^t}$ are KK-dual (with degree shift 1).

In this talk: a new class of examples of KK-dual algebras and can be viewed as a quantum analogue of the result of Kaminker and Putnam.

Joint work with D.M. Gerontogiannis (Leiden) and S. Neshveyev (Oslo), with thanks to E. Habbestad. arXiv:2401.01725.

Defined by Shalit and Solel for correspondences, and independently by Bhat and Mukherjee for Hilbert spaces. Connections to dilation theory, free probability, function theory.

Definition

Let $\{E_m\}_{m\in\mathbb{N}_0}$ a sequence of Hilbert spaces with isometries $\iota_{k,m} : E_{k+m} \to E_k \otimes E_m$ for every $k, m \in \mathbb{N}_0$. We say that (E, ι) is a *standard subproduct system* over $\mathbb{C}$ whenever for all $k, l, m \in \mathbb{N}_0$:

1. $E_0 = \mathbb{C}$;
2. The maps $\iota_{0,m} : E_m \to E_0 \otimes E_m$ and $\iota_{m,0} : E_m \to E_m \otimes E_0$ are the canonical identifications; and
3. We have *associativity*:

$$(1_k \otimes \iota_{l,m}) \circ \iota_{k,l+m} = (\iota_{k,l} \otimes 1_m) \circ \iota_{k+l,m}$$

on $E_{k+l+m} \to E_k \otimes E_l \otimes E_m$, where 1_k and 1_m denote the identity operators on E_k and E_m, respectively.

Let E be a standard subproduct system of Hilbert spaces over $\mathbb{N}_0$.
The direct sum Hilbert space $F_E := \bigoplus_{m \geq 0} E_m$ is called the *Fock space* of the subproduct system.
For each $\xi \in E_k$, we define the Toeplitz operator on F_E by compression by the shift.

$$T_\xi : F_E \to F_E \quad T_\xi(\zeta) := \iota^*_{k,m}(\xi \otimes \zeta), \quad \zeta \in E_m \subseteq F_E.$$

Definition (Shalit–Solel)

The *Toeplitz algebra* of the subproduct system E, denoted $\mathcal{T}_E$ is the smallest C^*-subalgebra of $\mathcal{L}(F_E)$ that contains all the shift operators, i.e.

$$T_\xi \in \mathcal{T}_E \quad \text{for all } \xi \in E_k \ , \ \ k \in \mathbb{N}_0.$$

If E_1 is finite dimensional with basis $\{\xi_i\}_{i=1}^n$ then $\mathcal{T}_E$ is generated by the T_{ξ_i}'s.

If E is a subproduct system of finite dimensional Hilbert spaces, then $\mathcal{K}(F_E) \subseteq \mathcal{T}_E$ (cf. Viselter 2012).

Definition (Viselter)

Let E be a subproduct system of f.d. Hilbert spaces. The Cuntz–Pimsner algebra of E is the quotient

$$0 \longrightarrow \mathcal{K}(F_E) \longrightarrow \mathcal{T}_E \xrightarrow{q} \mathbb{O}_E \longrightarrow 0. \tag{1}$$

Example

Let $d \in \mathbb{N}_0$, and consider, for every m, the symmetric tensors

$$E_m := \mathrm{Sym}^m(\mathbb{C}^d) \subseteq (\mathbb{C}^d)^{\otimes m}.$$

The resulting extension is the Toeplitz extension for odd spheres due to Arveson.

$$0 \longrightarrow \mathcal{K}(\mathcal{F}_{\mathrm{sym}}(\mathbb{C}^d)) \longrightarrow \mathcal{T}_d \longrightarrow C(S^{2d-1}) \longrightarrow 0\ . \tag{2}$$

Let $X_n := \{x_1, \dots, x_n\}$ be a finite set of variables. Let $\mathbb{C}\langle X\rangle := \mathbb{C}\langle x_1, \dots, x_n\rangle$ denote the complex free associative algebra with unit generated by X_n. An element of $\mathbb{C}\langle X\rangle$ is called a *noncommutative polynomial*; $f \in \mathbb{C}\langle X\rangle$ is *homogeneous of degree m* if $f \in \mathbb{C}X^m$.

Theorem (Shalit–Solel 2009)

Let E be an n-dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^n$. Then there is a bijective inclusion-reversing correspondence between proper homogeneous ideals $J \triangleleft \mathbb{C}\langle x_1, \dots, x_n\rangle$ and standard subproduct systems $E = \{E_m\}_{m\in\mathbb{N}_0}$ with $E_1 \subseteq E$.

Notation: for an ideal $J \triangleleft \mathbb{C}\langle X\rangle$ we write E^J for the corresponding subproduct system.
For a polynomial $p = \sum c_\alpha x^\alpha$, we write $p(e) = \sum c_\alpha e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n}$.

$$J \triangleleft \mathbb{C}\langle X\rangle \quad \longrightarrow \quad E^J_m := E^{\otimes m} \ominus \{p(e) | p \in J^{(m)}\}$$

$$E^J = \{E_m\}_{m\in\mathbb{N}_0} \quad \longrightarrow \quad J^E = \mathrm{span}\{p \in \mathbb{C}\langle X\rangle \mid \exists m > 0 : p(e) \in E^{\otimes m} \ominus E_m\}$$

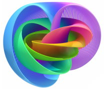
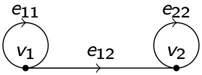
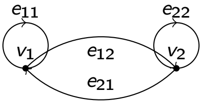
Goal: study the C*-algebra generated by a row contraction ($\sum_{i=1}^n T_i T_i^* \leq 1_H$) satisfying polynomial equations.
Let $T = (T_i)_{i=1}^n$ be an n-tuple of operators acting on a Hilbert space. If $\alpha = (\alpha_1, \ldots, \alpha_m) \in X_m$ is a length m word, then

$$T^\alpha := T_1^{\alpha_1} \ldots T_m^{\alpha_m},$$

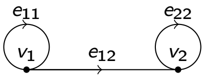
with the convention that $T^1 = 1_H$.

If $p(x) = \sum c_\alpha x^\alpha \in \mathbb{C}\langle X \rangle$ is a complex polynomial, by $p(T)$ we mean the linear combination $p(T) := \sum c_\alpha T^\alpha$.

Remark In the commutative case, one can interpret the Toeplitz algebra $\mathcal{T}_J$ as the universal C*-algebra generated by a row-contraction subject to polynomial relations.
Connections to the **Douglas–Arveson conjecture**.

Ideal	CP-algebra $\mathbb{O}_E$	Noncommutative space
$\langle x_1x_2 - x_2x_1, \quad i \neq j\rangle \subseteq \mathbb{C}\langle x_1, x_2\rangle$	$C(S^3)$	
$\langle [x_i, x_j] := x_ix_j - x_jx_i, \quad i \neq j\rangle \subseteq \mathbb{C}\langle x_1, \ldots, x_d\rangle$	$C(S^{2d-1})$	
$\langle q^{-1/2}x_1x_2 - q^{1/2}x_2x_1\rangle \subseteq \mathbb{C}\langle x_1, x_2\rangle$	$C(S^3_q) = C(SU_q(2))$	 subshift of finite type/ TMC
$\langle 0\rangle \subseteq \mathbb{C}\langle x_1, x_2\rangle$	$\mathcal{O}_2$	 full 2-shift

The algebra $SU_q(2)$ is sometimes called a quantum three sphere. Structure of quantum group. Example of C*-algebra associated to a topological Markov chain.



Directed graph with incidence matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Let Ω_A denote the space of one-sided infinite A-admissible words (words in $0, 1$ not containing 10 as subword).
Then Ω_A is the boundary of the tree $\mathcal{V}_A$ of finite admissible words:
Let $\sigma : \Omega \to \Omega$ denote the shift map. Then

$$SU_q(2) \simeq C(\Omega) \rtimes_\sigma \mathbb{N} \simeq C(\partial\mathcal{V}_A) \rtimes_\sigma \mathbb{N}.$$

Definition (Habbestad–Neshveyev)

Let H be a finite-dimensional Hilbert space of dimension $m \geq 2$. A nonzero vector $P \in H \otimes H$ is called *Temperley–Lieb* if there is $\lambda > 0$ such that the orthogonal projection $e\colon H \otimes H \to \mathbb{C}P$ satisfies

$$(e \otimes 1)(1 \otimes e)(e \otimes 1) = \frac{1}{\lambda}(e \otimes 1) \quad \text{in} \quad B(H \otimes H \otimes H).$$

The standard subproduct system $\mathcal{H}_P = \{H_n\}_{n \in \mathbb{Z}_+}$ defined by the ideal $\langle P \rangle \subset T(H)$ generated by P is called a Temperley–Lieb subproduct system.

We write $\mathcal{F}_P = \mathcal{F}_{\mathcal{H}_P}$, $\mathcal{T}_P = \mathcal{T}_{\mathcal{H}_P}$ and $\mathbb{O}_P = \mathbb{O}_{\mathcal{H}_P}$.

Fix an orthonormal basis in H and identify $H^{\otimes n}$ with the space of homogeneous noncommutative polynomials of degree n in variables $X_1, \dots, X_m$. A vector $P \in H \otimes H$ as a noncommutative polynomial $P = \sum_{i,j=1}^{m} a_{ij} X_i X_j$. Consider the matrix $A = (a_{ij})_{i,j}$.

Lemma (Habbestad–Neshveyev)

P is Temperley–Lieb if and only if the matrix $A\bar{A}$ is unitary up to a (nonzero) scalar factor.

Since the ideal generated by P does not change if we multiply P by a nonzero factor, we may always assume that $A\bar{A}$ is unitary.

For every Temperley–Lieb polynomial $P = \sum_{i,j} a_{ij} X_i X_j$ we consider Mrozinski's compact quantum group $\tilde{O}_P^+$.

Definition

We define the algebra $\mathbb{C}[\tilde{O}_P^+]$ of regular functions on $\tilde{O}_P^+$ as the universal unital $*$-algebra generated by a unitary element d and elements v_{ij}, $1 \leq i, j \leq m$, such that

$$V = (v_{ij})_{i,j} \text{ is unitary and } VAV^t = dA.$$

This is a Hopf $*$-algebra with comultiplication defined on generators as

$$\Delta(d) = d \otimes d, \quad \Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}.$$

When $A\bar{A} = \pm 1$ it decomposes as $O_P^+ \times \mathbb{T}$, whereO_P^+ is the universal orthogonal free quantum group.
The subproduct system H_P and the algebras $\mathcal{T}_P$ and $\mathbb{O}_P$ are $\tilde{O}_P^+$-**equivariant**.

Let $P = \sum_{i=1}^{n} a_i X_i X_{n-i+1}$, $a_i \bar{a}_{n-i+1} = -\tau \in \{-1, 1\}$ and $q \in (0, 1]$ satisfying $\sum_{i=1}^{n} |a_i|^2 = Tr(A) = q + q^{-1}$. The following facts hold:

- The boundary algebra $\mathbb{O}_P$ algebra is isomorphic to the linking algebra (cf. Bichon–de Rijdt–Vaes)

$$B(SU_{\tau q}(2), O_P^+).$$

- $\mathbb{O}_P$ can be identified with the boundary $C(\partial \mathbb{F} O_P^+)$ where $\mathbb{F} O_P^+$ is the dual discrete quantum group to O_P^+.

For *general* Temperley Lieb P, the boundary algebra can be interpreted as a crossed product of that boundary by a hyperbolic transformation:

$$\mathbb{O}_P \simeq C(\partial \mathbb{F} O_P^+) \rtimes_\phi \mathbb{N}.$$

Theorem (Habbestad–Neshveyev)

For every Temperley–Lieb polynomial P, the embedding $\mathbb{C} \to \mathcal{T}_P$ is an O_P^+-equivariant KK-equivalence.

Relies on Baum–Connes for $SU_q(2)$.

$$K_0(\mathbb{O}_P) \cong \mathbb{Z}/(n-1)\mathbb{Z} \qquad K_1(\mathbb{O}_P) \cong \begin{cases} \mathbb{Z} & n = 1, \\ \{0\} & \text{otherwise.} \end{cases} \tag{3}$$

Theorem (A.–Gerontogiannis–Neshveyev)

Let $A = (a_{ij})_{i,j} \in \mathrm{GL}_m(\mathbb{C})$, with $m \geq 2$, be such that $A\bar{A}$ is unitary. Consider the noncommutative quadratic polynomials defined respectively by A and A^t,

$$P = \sum_{i,j=1}^{m} a_{ij} X_i X_j, \qquad P^\dagger = \sum_{i,j=1}^{m} a_{ji} X_i X_j.$$

Then the Cuntz–Pimsner algebras $\mathbb{O}_P$ and $\mathbb{O}_{P^\dagger}$, associated with the subproduct systems defined by P and $P^\dagger$, are KK-dual with dimension shift 1.

As a consequence of nuclearity the Toeplitz extension

$$0 \longrightarrow \mathcal{K}(\mathcal{F}_P) \xrightarrow{j} \mathcal{T}_P \xrightarrow{q} \mathbb{O}_P \longrightarrow 0. \tag{4}$$

is semi-split for any $P = \sum_{i=1}^{n+1} a_i X_i X_{n-i+1}$ Temperley–Lieb.
As a consequence, using the KK-equivalences, we have a dual exact sequence in K-homology:

$$0 \longrightarrow K^0(\mathbb{O}_P) \xrightarrow{i^*} K^0(\mathbb{C}) \longrightarrow K^0(\mathbb{C}) \longrightarrow K^1(\mathbb{O}_P) \longrightarrow 0$$

so that

$$K^1(\mathbb{O}_P) \cong \mathbb{Z}/(n-1)\mathbb{Z} \qquad K^0(\mathbb{O}_P) \cong \begin{cases} \mathbb{Z} & n = 1, \\ \{0\} & \text{otherwise.} \end{cases} \tag{5}$$

Fundamental class: built from the $*$-homomorphisms $\tau_P\colon \mathbb{O}_P \to \mathcal{Q}(\mathcal{F}), \tau_{P^\dagger}\colon \mathbb{O}_{P^\dagger} \to \mathcal{Q}(\mathcal{F})$ into the Calkin algebra $\mathcal{Q}(\mathcal{F})$ given by

$$\tau_P(s_i) := L_i + \mathcal{K}(\mathcal{F}), \qquad \tau_{P^\dagger}(t_i) := R_i + \mathcal{K}(\mathcal{F}). \tag{6}$$

The images of these maps commute. Since $\mathbb{O}_P$ and $\mathbb{O}_{P^\dagger}$ are nuclear, multiplication yields a $*$-homomorphism

$$\tau\colon \mathbb{O}_P \otimes \mathbb{O}_{P^\dagger} \to \mathcal{Q}(\mathcal{F}) \tag{7}$$

which yields an extension class $[\tau] \in \mathrm{Ext}(\mathbb{O}_P \otimes \mathbb{O}_{P^\dagger}, \mathbb{C}) \cong KK_1(\mathbb{O}_P \otimes \mathbb{O}_{P^\dagger}, \mathbb{C})$.

Definition

The *fundamental class* Δ is the image of $[\tau]$ in $KK_1(\mathbb{O}_P \otimes \mathbb{O}_{P^\dagger}, \mathbb{C})$.

Dual class: the operator

$$w := \sum_{i=1}^{m} \begin{pmatrix} t_i^* \otimes s_i & q^{1/2} a_i t_i \otimes s_{m-i+1} \\ q^{1/2} \overline{a}_i t_i^* \otimes s_{m-i+1}^* & q a_i \overline{a}_{m-i+1} t_i \otimes s_i^* \end{pmatrix} \in M_2(\mathbb{O}_{P^\dagger} \otimes \mathbb{O}_P). \tag{8}$$

is unitary.

Let $\beta \in KK_1(\mathbb{C}, C_0(\mathbb{R}))$ is the Bott class and $[\overline{w}] \in KK(C_0(\mathbb{R}), \mathbb{O}_{P^\dagger} \otimes \mathbb{O}_P)$ is given by the $*$-homomorphism $\overline{w}\colon C_0(\mathbb{R}) \to M_2(\mathbb{O}_{P^\dagger} \otimes \mathbb{O}_P)$,

$$\overline{w}(z - 1_{C(\mathbb{T})}) = w^* - 1,$$

and where $z - 1_{C(\mathbb{T})}$ is viewed as a function in $C(\mathbb{T})$ that generates a copy of $C_0(\mathbb{R})$.

Definition

The *dual class* $\delta \in KK_1(\mathbb{C}, \mathbb{O}_{P^\dagger} \otimes \mathbb{O}_P)$ is defined as $\beta \otimes_{C_0(\mathbb{R})} [\overline{w}]$, .

- The Kasparov products of those classes lift to $\tilde{O}_P^+$-equivariant KK-theory.
- By a result of Mrozinski , $\tilde{O}_P^+$ is monoidally equivalent to $U_q(2)$ for a uniquely defined $0 < q \leq 1$:

$$\mathrm{Rep}(\tilde{O}_P^+) \simeq \mathrm{Rep}(U_q(2))$$

  - This allows us to reduce the proof of duality to the polynomials $q^{-1/2}X_1X_2 - q^{1/2}X_2X_1$.
- Moreover, by considering equivariant KK-classes we get more tools for manipulating with them, such as Frobenius reciprocity.

We obtain $U_q(2)$ equivariant KK-duality for $C(S^3)$.

Thank you for your attention!