GROTHENDIECK-NEEMAN DUALITY AND THE WIRTHMÜLLER ISOMORPHISM

PAUL BALMER, IVO DELL'AMBROGIO, AND BEREN SANDERS

Abstract. We clarify the relationship between Grothendieck duality à la Neeman and the Wirthmüller isomorphism à la Fausk-Hu-May. We exhibit an interesting pattern of symmetry in the existence of adjoints between rigidly-compactly generated tensor-triangulated categories, which leads to a surprising trichotomy: there exist either exactly three adjoints, exactly five, or infinitely many. A crucial role in this analysis is played by what we call the *relative dualizing object* and we explain how this object gives rise to a duality on a certain canonical subcategory. More generally, we exploit our tensor-triangular setting to develop a duality theory rich enough to capture the main features of Grothendieck duality in algebraic geometry and generalized Pontryagin-Matlis duality à la Dwyer-Greenless-Iyengar in the theory of ring spectra.

Contents

1. Introduction and statement of results

A tale of adjoint functors. Consider a tensor-exact functor $f^* : \mathcal{D} \to \mathcal{C}$ between tensor-triangulated categories. As the notation f^* suggests, one typically obtains such functors by pulling-back representations, sheaves, spectra, etc., along some suitable "underlying" map $f : X \to Y$ of groups, spaces, schemes, etc. (The actual underlying map f is not relevant for our discussion.) We are specifically interested in the existence and properties of adjoints to f^*, of further adjoints to these adjoints, and so on:

$$\begin{array}{c} \mathcal{C} \\ \cdots \quad \Big\uparrow \quad \Big\downarrow \quad \Big\uparrow{\scriptstyle f^*} \quad \Big\downarrow{\scriptstyle f_*} \quad \Big\uparrow \quad \Big\downarrow \quad \cdots \\ \mathcal{D} \end{array} \tag{1.1}$$

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Such questions arise in examples because certain geometric properties of the underlying $f : X \to Y$ can sometimes be translated into the existence, or into properties, of such adjoints. This is illustrated for instance in Neeman's approach [Nee96] to Grothendieck duality. Our main motivation is to provide a systematic treatment of these adjoints in the context of rigidly-compactly generated categories, while simultaneously clarifying the relationship between so-called Wirthmüller isomorphisms and Grothendieck duality. In that respect, our work is a continuation of Fausk-Hu-May [FHM03]. It turns out that the more adjoints exist, the more strongly related they must be to each other. Also remarkable is the existence of a tipping point after which there must exist infinitely many adjoints on both sides. This will happen for instance as soon as we have the six consecutive adjoints pictured in (1.1) above.

Let us be more precise. Here is our basic set-up:

1.2. *Hypothesis.* Throughout the paper, we assume that both tensor-triangulated categories $\mathcal{C}$ and $\mathcal{D}$ are *rigidly-compactly generated*. See Section 2 for details. In short, this means that $\mathcal{C}$ has arbitrary coproducts, its compact objects coincide with the rigid objects (a. k. a. the strongly dualizable objects) and $\mathcal{C}$ is generated by a *set* of those rigid-compacts; and similarly for $\mathcal{D}$. Such categories are the standard "big" tensor-triangulated categories in common use in algebra, geometry and homotopy theory. They are the *unital algebraic stable homotopy categories* of [HPS97] (with "algebraic" understood broadly since it includes, for example, the topological stable homotopy category SH). See Examples 2.9.

Moreover, we assume that $f^* : \mathcal{D} \to \mathcal{C}$ is a tensor-exact functor (*i.e.* strong symmetric monoidal and triangulated) *which preserves arbitrary coproducts.* These hypotheses are quite natural and cover standard examples; see Examples 3.24–3.27 and 4.5–4.7. (Such f^* are called *geometric functors* in [HPS97, Def. 3.4.1].)

By Neeman's Brown Representability Theorem, these basic hypotheses already imply the existence of two layers of adjoints to the right of the given $f^* : \mathcal{D} \to \mathcal{C}$.

1.3. **Theorem** (Cor. 2.10)**.** *Under Hypothesis 1.2, the functor $f^* : \mathcal{D} \to \mathcal{C}$ admits a right adjoint $f_* : \mathcal{C} \to \mathcal{D}$, which itself admits a right adjoint $f^{(1)} : \mathcal{D} \to \mathcal{C}$. Moreover, we have a projection formula $d \otimes f_*(c) \cong f_*(f^*(d) \otimes c)$ and a couple of other relations detailed in Proposition 2.11.*

In other words, we get $f^* \dashv f_* \dashv f^{(1)}$ essentially "for free." This includes the unconditional existence of a special object that we want to single out:

1.4. *Definition.* Writing $\mathbb{1}$ for the $\otimes$-unit, the object $\omega_f := f^{(1)}(\mathbb{1})$ in $\mathcal{C}$ will be called the *relative dualizing object* (for $f^* : \mathcal{D} \to \mathcal{C}$) in reference to the dualizing complexes of derived categories of schemes; see [Lip09] and [Nee96, Nee10]. This object ω_f of $\mathcal{C}$ is uniquely characterized by the existence of a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(f_*(-), \mathbb{1}) \cong \mathrm{Hom}_{\mathcal{C}}(-, \omega_f), \tag{1.5}$$

or equivalently, by the existence of a natural isomorphism

$$\underline{\mathrm{hom}}_{\mathcal{D}}(f_*(-), \mathbb{1}) \cong f_* \, \underline{\mathrm{hom}}_{\mathcal{C}}(-, \omega_f), \tag{1.6}$$

where $\underline{\mathrm{hom}}_{\mathcal{C}}$ and $\underline{\mathrm{hom}}_{\mathcal{D}}$ are the internal hom functors on $\mathcal{C}$ and $\mathcal{D}$ respectively. In other words, ω_f allows us to describe the usual (untwisted) dual $\Delta := \underline{\mathrm{hom}}(-, \mathbb{1})$ of the direct image f_* as the direct image of the ω_f-twisted dual $\Delta_{\omega_f} := \underline{\mathrm{hom}}(-, \omega_f)$.

Armed with this object $\omega_f \in \mathcal{C}$, we return to our three functors $f^* \dashv f_* \dashv f^{(1)}$. We prove that the existence of one more adjoint on either side forces adjoints on both sides $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$ and strong relations between these five functors. This is one of the main clarifications of the paper.

1.7. **Theorem** (Grothendieck-Neeman Duality, Theorem 3.4)**.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ be as in our basic Hypothesis 1.2 and consider the automatic adjoints $f^* \dashv f_* \dashv f^{(1)}$ (Thm. 1.3). Then the following conditions are equivalent:*

(GN1) Grothendieck duality: *There is a natural isomorphism*

$$\omega_f \otimes f^*(-) \cong f^{(1)}(-).$$

(GN2) Neeman's criterion: *The functor f_* preserves compact objects, or equivalently its right adjoint $f^{(1)}$ preserves coproducts, or equivalently by Brown Representability $f^{(1)}$ admits a right adjoint $f_{(-1)}$.*

(GN3) *The original functor $f^* : \mathcal{D} \to \mathcal{C}$ preserves products, or equivalently by Brown Representability f^* admits a left adjoint $f_{(1)}$.*

Moreover, when these conditions hold, the five functors $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$

$$\begin{tikzcd}[row sep=large]
\mathcal{C} \arrow[d, shift right=8, "f_{(1)}"'] \arrow[d, shift right=4, leftarrow, "f^*"] \arrow[d, "f_*"] \arrow[d, shift left=4, leftarrow, "f^{(1)}"] \arrow[d, shift left=8, "f_{(-1)}"] \\
\mathcal{D}
\end{tikzcd}$$

are related by an armada of canonical isomorphisms, detailed in Theorem 3.4 and Remark 3.18. Most notably, we have what we call the ur-Wirthmüller isomorphism

$$f_{(1)}(-) \cong f_*(\omega_f \otimes -) \tag{1.8}$$

and we have a canonical isomorphism $\mathbb{1}_{\mathcal{C}} \cong \underline{\mathrm{hom}}_{\mathcal{C}}(\omega_f, \omega_f)$.

The equivalence between (GN 1) and (GN 2) was established by Neeman [Nee96]. We name the theorem after him since he has been the main architect of compactly generated categories and since several of our techniques have been pioneered by him, if sometimes only in the algebro-geometric context, like in [Nee10]. Our main input is to show that Grothendieck-Neeman duality can be detected on the original functor f^*, namely by the property that f^* preserves products. In other words, the existence of Neeman's right adjoint $f_{(-1)}$ on the far-right is equivalent to the existence of a left adjoint $f_{(1)}$ four steps to the left. Our Lemma 2.6 is the tool which allows us to move from left to right via the duality on the subcategory of compact objects. This lemma is the key to the proof of the new implication (GN 3)$\Rightarrow$(GN 2) above and appears again in the proof of Theorem 1.9 below.

Our ur-Wirthmüller formula (1.8) is also new and connects with similar formulas in [FHM03], as discussed in Remark 1.12 below. In algebraic geometry, an isomorphism as in (1.8) is mentioned in [Nee10, Rem. 4.3]. In that special case, the formula is attributed (without reference) to Lipman and van den Bergh along with the caution "at least for large classes of $f : X \to Y$" indicating that some algebro-geometric technicalities might have obstructed the generality obtained here.

Our Grothendieck-Neeman Duality Theorem 1.7 leaves one question open, made very tempting in view of the isomorphism $\underline{\mathrm{hom}}(\omega_f, \omega_f) \cong \mathbb{1}$: when is the relative dualizing object ω_f $\otimes$-invertible? Amusingly, this is related to another layer of adjoints, on either side of $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$. We reach here the tipping point from which infinitely many adjoints must exist on both sides.

1.9. **Theorem** (Wirthmüller Isomorphism; see Section 4)**.** *Suppose that we have the five adjoints $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$ of Grothendieck-Neeman duality (Thm. 1.7). Then the following conditions are equivalent:*

- (W1) *The left-most functor $f_{(1)}$ admits itself a left adjoint, or equivalently by Brown Representability it preserves arbitrary products.*
- (W2) *The right-most functor $f_{(-1)}$ admits itself a right adjoint, or equivalently by Brown Representability it preserves arbitrary coproducts, or equivalently its left adjoint $f^{(1)}$ preserves compact objects.*
- (W3) *The relative dualizing object ω_f (Def. 1.4) is a compact object of $\mathcal{C}$.*
- (W4) *The relative dualizing object ω_f is $\otimes$-invertible in $\mathcal{C}$.*
- (W5) *There exists a* (strong) Wirthmüller isomorphism *between f_* and $f_{(1)}$; that is, there exists a $\otimes$-invertible object $\omega \in \mathcal{C}$ such that $f_{(1)} \cong f_*(\omega \otimes -)$, or equivalently such that $f_* \cong f_{(1)}(\omega^{-1} \otimes -)$.*
- (W6) *There exists an infinite tower of adjoints on both sides:*

$$\cdots \ \Big\uparrow f^{(-n)} \ \Big\downarrow f_{(n)} \ \cdots \ \Big\uparrow f^{(-1)} \ \Big\downarrow f_{(1)} \ \overset{\mathcal{C}}{\underset{\mathcal{D}}{\Big\uparrow}} f^* \ \ \Big\downarrow f_* \ \ \Big\uparrow f^{(1)} \ \Big\downarrow f_{(-1)} \ \Big\uparrow f^{(2)} \ \cdots \ \Big\uparrow f^{(n)} \ \Big\downarrow f_{(-n)} \ \cdots$$

which necessarily preserve all coproducts, products and compact objects.

Moreover, when these conditions hold, the adjoints of (W6) are necessarily given for all $n \in \mathbb{Z}$ by the formulas

$$f^{(n)} = \omega_f^{\otimes n} \otimes f^* \qquad \text{and} \qquad f_{(n)} = f_*(\omega_f^{\otimes n} \otimes -)\,. \tag{1.10}$$

Finally, (W1)-(W6) hold true as soon as the functor $f_* : \mathcal{C} \to \mathcal{D}$ satisfies, in addition to Grothendieck-Neeman duality, any one of the following three properties (in increasing generality, for (1)⇒(2)⇒(3)):

1. *The functor f_* is faithful (*i.e.* f^* is surjective up to direct summands).*
2. *The functor f_* detects compact objects: any $x \in \mathcal{C}$ is compact if $f_*(x)$ is.*
3. *Any $x \in \mathcal{C}$ is compact if $f_*(x \otimes y)$ is compact for every compact $y \in \mathcal{C}$.*

1.11. *Remark.* We opted for the notation $f^{(n)} \dashv f_{(-n)} \dashv f^{(n+1)}$ after trying everything else. As is well-known, notations of the form $f^!$, $f_!$, $f^\times$, $f_\#$, etc., have flourished in various settings, sometimes with contradictory meanings. Instead of risking collision, we propose a systematic notation which allows for an infinite tower of adjoints, following the tradition that $f^{(n)}$ is numbered with n going up $\cdots f^{(n)}, f^{(n+1)} \cdots$ and $f_{(n)}$ with n going down $\cdots f_{(n)}, f_{(n-1)} \cdots$. Our notation also recalls that $f^{(n)}$ and $f_{(n)}$ are n-fold *twists* of $f^{(0)} = f^*$ and $f_{(0)} = f_*$ by ω_f; see (1.10).

1.12. *Remark.* In the literature, Property (W5) is usually simply called a Wirthmüller isomorphism, referring to the original [Wir74]. Such a strong relation between the left and right adjoints to f^* is very useful. Typically, f_* and $f_{(1)}$ will share all properties which are stable under pre-tensoring with an invertible object (*e.g.*, being full, faithful, etc.). Similarly, most formulas valid for one of them will transpose into a formula for the other one. Here, we temporarily added the adjective "strong" to avoid collision with Fausk-Hu-May's related but different notion of

"Wirthmüller context" [FHM03]; see more in Remark 4.3. Let us simply point out that our Wirthmüller property (W5) is not disjoint from Grothendieck duality but appears as a special case of it. In fact, the Wirthmüller isomorphism itself and the twisting object ω_f are borrowed from the earlier ur-Wirthmüller isomorphism (1.8), since the latter clearly implies (W5) when ω_f is invertible.

1.13. *Remark.* With the very complete statement (W6) above, we may believe to have reached the end of the road. Well, almost. The road actually ends in a loop if we suppose moreover that $\omega_f \cong \mathbb{1}$. This is precisely the case when f^* is a *Frobenius functor* [Mor65], *i.e.* it admits a simultaneous left-and-right adjoint $f^* \dashv f_* \dashv f^*$, a situation which is also called an *ambidextrous adjunction.*

In conclusion, we have the following picture:

$f^* \dashv f_* \dashv f^{(1)}$ always (Thm. 1.3)	$\Leftarrow$	$f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$ Grothendieck-Neeman duality and ur-Wirthmüller (Thm. 1.7)	$\Leftarrow$	$\cdots \dashv f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)} \dashv \cdots$ Grothendieck-Neeman duality with ω_f invertible and Wirthmüller (Thm. 1.9)

1.14. **Corollary** (Trichotomy of adjoints)**.** *If f^* is a coproduct-preserving tensor triangulated functor between rigidly-compactly generated tensor triangulated categories, then* exactly one *of the following three possibilities must hold:*

(1) *There are two adjunctions as follows and no more:* $f^* \dashv f_* \dashv f^{(1)}$.
(2) *There are four adjunctions as follows and no more:* $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$.
(3) *There is an infinite tower of adjunctions in both directions:*

$$\cdots f^{(-1)} \dashv f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)} \cdots f^{(n)} \dashv f_{(-n)} \dashv f^{(n+1)} \cdots$$

* * *

Abstract Grothendieck duality. In the literature, the phrase "Grothendieck duality" can refer to several different things. In its crudest form, it is the isomorphism $\omega_f \otimes f^* \cong f^{(1)}$ of (GN 1) – hence the name *twisted inverse image* for $f^{(1)}$. Grothendieck duality can also refer to the compatibility $\Delta \circ f_* = \underline{\mathrm{hom}}(f_*(-), \mathbb{1}) \cong f_* \, \underline{\mathrm{hom}}(-, \omega_f) = f_* \circ \Delta_{\omega_f}$ given in (1.6) between f_* and the two dualities, although usually this is formulated for certain proper subcategories $\mathcal{C}_0 \subset \mathcal{C}$ and $\mathcal{D}_0 \subset \mathcal{D}$ on which the two duality functors earn their name by inducing equivalences $\mathcal{C}_0^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_0$ and $\mathcal{D}_0^{\mathrm{op}} \xrightarrow{\sim} \mathcal{D}_0$, respectively. Then "Grothendieck duality" refers to the situation where the functor $f_* : \mathcal{C} \to \mathcal{D}$ maps $\mathcal{C}_0$ to $\mathcal{D}_0$ and intertwines the two dualities, as above. A major example occurs with $\mathcal{C}_0 = \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ and $\mathcal{D}_0 = \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y)$ for a suitable morphism of schemes $f : X \to Y$; here X and Y are assumed noetherian and $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ is the bounded derived category of coherent $\mathcal{O}_X$-modules. Since in general $\underline{\mathrm{hom}}_{\mathcal{C}}(-, \mathbb{1})$ might not preserve $\mathcal{C}_0$ (as in the geometric example just mentioned), one might also try to replace the naive duality $\Delta = \underline{\mathrm{hom}}(-, \mathbb{1})$ of (1.6) by a more friendly one, say $\Delta_\kappa := \underline{\mathrm{hom}}(-, \kappa)$ for some object $\kappa \in \mathcal{C}_0$ having the property that $\underline{\mathrm{hom}}_{\mathcal{C}}(-, \kappa) : \mathcal{C}_0^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_0$ is an equivalence. In algebraic geometry, for $\mathcal{C}_0 = \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$, such κ are called *dualizing complexes.*

In Sections 5 and 6, we follow this approach to Grothendieck duality in our abstract setting, with an emphasis on the three levels of adjunction (Cor. 1.14). Let f^* be a functor as in our basic Hypothesis 1.2. As before, we write $\Delta_\kappa := \underline{\mathrm{hom}}(-, \kappa)$ for the κ-twisted duality functor, for any object κ. We prove:

(1) In the general situation, f_* always intertwines dualities: we have
$$\Delta_\kappa \circ f_* \cong f_* \circ \Delta_{\kappa'}$$
where $\kappa \in \mathcal{D}$ is any object and $\kappa' := f^{(1)}(\kappa) \in \mathcal{C}$; see (2.14).

(2) Assume that f^* satisfies Grothendieck-Neeman duality (Theorem 1.7). Let $\mathcal{D}_0 \subset \mathcal{D}$ be a subcategory admitting a dualizing object $\kappa \in \mathcal{D}_0$, in which case κ induces an equivalence $\Delta_\kappa : \mathcal{D}_0^{\mathrm{op}} \xrightarrow{\sim} \mathcal{D}_0$. Provided $\mathcal{D}_0$ is a $\mathcal{D}^c$-submodule (meaning $\mathcal{D}^c \otimes \mathcal{D}_0 \subseteq \mathcal{D}_0$), then the object $\kappa' := f^{(1)}(\kappa) \cong \omega_f \otimes f^*(\kappa)$ is dualizing for the following subcategory of $\mathcal{C}$:
$$\mathcal{C}_0 := \{x \in \mathcal{C} \mid f_*(c \otimes x) \in \mathcal{D}_0 \text{ for all } c \in \mathcal{C}^c\},$$
that is, $\kappa' \in \mathcal{C}_0$ and $\Delta_{\kappa'} : \mathcal{C}_0^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_0$. Thus, by the formula in (1), $f_* : \mathcal{C}_0 \to \mathcal{D}_0$ is a morphism of categories with duality. See Theorem 5.23 for details, and see Theorem 6.4 for a more general relative version.

(3) Assume moreover that we have the Wirthmüller isomorphism of Theorem 1.9. Because of the monoidal adjunction $f^* \dashv f_*$, we may consider $\mathcal{C}$ as an enriched category over $\mathcal{D}$, *i.e.* we may equip $\mathcal{C}$ with Hom objects $\underline{\mathcal{C}}(x, y) := f_* \underline{\mathrm{hom}}_{\mathcal{C}}(x, y)$ in $\mathcal{D}$. Then the equivalence $(-) \otimes \omega_f : \mathcal{C}^c \xrightarrow{\sim} \mathcal{C}^c$ behaves like a *Serre functor relative to* $\mathcal{D}$, meaning that there is a natural equivalence
$$\Delta\underline{\mathcal{C}}(x, y) \cong \underline{\mathcal{C}}(y, x \otimes \omega_f)$$
for all $x, y \in \mathcal{C}^c$, where $\Delta = \underline{\mathrm{hom}}_{\mathcal{D}}(-, \mathbb{1})$ is the plain duality of $\mathcal{D}$. If $\mathcal{D} = \mathrm{D}(\Bbbk)$ is the derived category of a field $\Bbbk$, this reduces to an ordinary Serre functor on the $\Bbbk$-linear category $\mathcal{C}^c$. See Theorem 6.8 and Corollary 6.11.

In algebraic geometry, we prove that if X is a projective scheme over a regular noetherian base then the category $\mathcal{C}_0$ of (2) specializes to $\mathrm{D^b}(\mathrm{coh}\, X)$; see Theorem 5.20. Thus in this case the results in (1) and (2) specialize to the classical algebro-geometric Grothendieck duality. Similarly, (3) specializes to the classical Serre duality for smooth projective varieties (cf. Example 6.13). But of course now these results apply more generally, for instance in representation theory, equivariant stable homotopy, and so on, *ad libitum.*

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Beyond Grothendieck duality. Once more, consider a tensor-exact functor $f^* : \mathcal{D} \to \mathcal{C}$ satisfying Hypothesis 1.2, so that we have the adjoints $f^* \dashv f_* \dashv f^{(1)}$. Instead of starting with a subcategory $\mathcal{D}_0$ of $\mathcal{D}$ equipped with a dualizing object $\kappa \in \mathcal{D}_0$, as we do when treating Grothendieck duality in Sections 5 and 6, we may reverse direction by assuming given a subcategory $\mathcal{C}_0$ of $\mathcal{C}$ with dualizing object κ' and asking under what circumstances we may push the subcategory with duality $(\mathcal{C}_0, \kappa')$ along $f_* : \mathcal{C} \to \mathcal{D}$ to a subcategory with duality in $\mathcal{D}$.

We prove that, if there exists an object $\kappa \in \mathcal{D}$ such that $f^{(1)}(\kappa) \cong \kappa'$, then κ is dualizing for the thick subcategory of $\mathcal{D}$ generated by $f_*(\mathcal{C}_0)$; see Theorem 7.1. This result specializes to classical Matlis duality for commutative noetherian local rings; see Example 7.3. (For this we now allow dualizing objects to be external, *i.e.* we only assume that $\kappa' \in \mathcal{C}$ induces an equivalence $\Delta_{\kappa'} : \mathcal{C}_0 \xrightarrow{\sim} \mathcal{C}_0$ while possibly $\kappa' \notin \mathcal{C}_0$; indeed, even when $\kappa' \in \mathcal{C}_0$, the lift $\kappa \in \mathcal{D}$ need not be in $\mathcal{D}_0$; see Example 7.4.) We conclude that our triangular framework for duality can reach beyond Grothendieck duality. For instance, Dwyer, Greenlees and Iyengar [DGI06] have developed a rich framework which captures several dualities in the style of Pontryagin-Matlis. We show how this connects with our theory at the end of Section 7.

2. Brown representability and the three basic functors

We begin by recollecting some well-known definitions and results.

Perhaps the most basic fact about adjoints of exact functors on triangulated categories is that they are automatically exact; see [Nee01, Lemma 5.3.6].

A triangulated category $\mathcal{T}$ is said to be *compactly generated* if it admits arbitrary coproducts, and if there exists a set of compact objects $\mathcal{G} \subset \mathcal{T}$ such that $\mathcal{T}(\mathcal{G}, t) = 0$ implies $t = 0$ for any $t \in \mathcal{T}$. An object $t \in \mathcal{T}$ is *compact* (a.k.a. *finite*) if the functor $\mathcal{T}(t, -) : \mathcal{T} \to \mathrm{Ab}$ sends coproducts in $\mathcal{T}$ to coproducts of abelian groups. We denote by $\mathcal{T}^c$ the thick subcategory of compact objects of $\mathcal{T}$. A (contravariant) functor $\mathcal{T}^{(\mathrm{op})} \to \mathcal{A}$ to an abelian category is called *(co)homological* if it sends exact triangles to exact sequences. The notion of a compactly generated category is motivated by the following immensely useful result due to Neeman:

2.1. **Theorem** (Brown representability; see [Nee96, Kra02])**.** *Let $\mathcal{T}$ be a compactly generated triangulated category. Then:*

(a) *A cohomological functor $\mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is representable — i.e., is isomorphic to one of the form $\mathcal{T}(-, t)$ for some $t \in \mathcal{T}$ — if and only if it sends coproducts in $\mathcal{T}$ to products of abelian groups.*

(b) *A homological functor $\mathcal{T} \to \mathrm{Ab}$ is corepresentable — i.e., is isomorphic to one of the form $\mathcal{T}(t, -)$ for some $t \in \mathcal{T}$ — if and only if it sends products in $\mathcal{T}$ to products of abelian groups.*

2.2. *Remark.* Theorem 2.1 (a) already implies that $\mathcal{T}$ admits products (apply it to the functor $\prod_i \mathcal{T}(-, t_i)$). In turn, this allows for "dual" statements, such as (b).

Any functor admitting a right (resp. left) adjoint must preserve coproducts (resp. products). Brown representability yields the following converse:

2.3. **Corollary.** *Let $F : \mathcal{T} \to \mathcal{S}$ be an exact functor between triangulated categories, and assume that $\mathcal{T}$ is compactly generated. Then:*

(a) *F admits a right adjoint if and only if it preserves coproducts.*

(b) *F admits a left adjoint if and only if it preserves products.*

Proof. As F is exact, the functors $\mathcal{S}(F(-), s) : \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ and $\mathcal{S}(s, F(-)) : \mathcal{T} \to \mathrm{Ab}$ are (co)homological for each $s \in \mathcal{S}$, so we can feed them to Theorem 2.1. □

2.4. *Remark.* The following is a basic tool for working with a compactly generated category $\mathcal{T}$. In order to show that a natural transformation $\alpha : F \to F'$ between two coproduct-preserving exact functors $F, F' : \mathcal{T} \to \mathcal{S}$ is an isomorphism, it suffices to prove that α_x is an isomorphism for $x \in \mathcal{T}^c$ compact. In some cases, this involves giving an alternative definition of α_x, valid for x compact, and showing by direct computation that the two definitions coincide. Such computations can become rather involved. We have left the easiest of these verifications to the reader but we have sketched the most difficult ones, hopefully to the benefit of the careful reader.

We will also make frequent use of the following two general facts about adjoints on compactly generated categories.

2.5. **Proposition** ([Nee96, Thm. 5.1]). *Let $F : \mathcal{S} \rightleftarrows \mathcal{T} : G$ be an adjoint pair of exact functors between triangulated categories $\mathcal{S}$ and $\mathcal{T}$, and assume $\mathcal{S}$ compactly generated. Then F preserves compact objects iff G preserves coproducts.* □

The second general fact seems new. It will play a crucial role in this paper:

2.6. **Lemma.** *Let $F : \mathcal{S} \rightleftarrows \mathcal{T} : G$ be an adjoint pair of exact functors between triangulated categories, and assume $\mathcal{S}$ compactly generated. Suppose that F preserves compacts and that its restriction $F_0 : \mathcal{S}^c \to \mathcal{T}^c$ admits a right adjoint G_0. Then G preserves compacts, and its restriction to compacts coincides with G_0.*

Proof. For every compact $t \in \mathcal{T}^c$ and every compact $s \in \mathcal{S}^c$, we have a natural bijection $\mathcal{S}^c(s, G_0(t)) \cong \mathcal{T}^c(F_0(s), t) = \mathcal{T}(F(s), t) \cong \mathcal{S}(s, G(t))$. By plugging $s := G_0(t)$, the identity map of $G_0(t)$ corresponds to a certain morphism $\gamma_t : G_0(t) \to G(t)$. Varying $t \in \mathcal{T}^c$, we obtain a natural morphism $\gamma : G_0 \to G|_{\mathcal{T}^c}$ by the naturality in t of the bijection. By its naturality in s, it actually follows that the bijection is obtained by composing maps $f \in \mathcal{S}(s, G_0(t))$ with γ_t. In particular, for any fixed $t \in \mathcal{T}^c$ the induced map $\mathcal{S}(-, \gamma_t) : \mathcal{S}(-, G_0(t)) \to \mathcal{S}(-, G(t))$ is invertible on all $s \in \mathcal{S}^c$ by construction, and since $\mathcal{S}$ is compactly generated, it is therefore invertible on all $s \in \mathcal{S}$ (cf. Remark 2.4). It follows by Yoneda that γ_t is an isomorphism. Hence $G(t) \simeq G_0(t) \in \mathcal{S}^c$ for every $t \in \mathcal{T}^c$, which is the result. □

We now let the tensor $\otimes$ enter the game.

2.7. *Definition.* A tensor-triangulated category $\mathcal{C}$ (*i.e.* a triangulated category with a compatible closed symmetric monoidal structure, see [HPS97, App. A.2]) is called *rigidly-compactly* generated if it is compactly generated and if compact objects and rigid objects coincide; in particular, the tensor unit object $\mathbb{1}$ is compact. An object x is *rigid* if the natural map $\underline{\mathrm{hom}}(x, \mathbb{1}) \otimes y \longrightarrow \underline{\mathrm{hom}}(x, y)$ is an isomorphism for all y. Rigid objects are sometimes called "(strongly) dualizable" but we avoid this terminology to avoid confusion with our "dualizing objects."

2.8. *Remark.* When $\mathcal{C}$ is rigidly-compactly generated, its subcategory of compact objects $\mathcal{C}^c \subset \mathcal{C}$ is a thick subcategory, closed under $\otimes$. It admits the canonical duality $\Delta = \underline{\mathrm{hom}}(-, \mathbb{1}) : (\mathcal{C}^c)^{\mathrm{op}} \to \mathcal{C}^c$ satisfying $\Delta^2 \cong \mathrm{Id}$. See details in [HPS97, App. A] for instance, where our rigidly-compactly generated tensor triangulated categories are called "unital algebraic stable homotopy categories."

2.9. *Examples.* Let us mention at this point a few important examples of rigidly-compactly generated categories $\mathcal{C}$ arising in various fields of mathematics:

(1) Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{C} := \mathrm{D}_{\mathrm{Qcoh}}(X)$ be the derived category of complexes of $\mathcal{O}_X$-modules having quasi-coherent cohomology (see [Lip09]). It is rigidly-compactly generated, and its compact objects are precisely the perfect complexes: $(\mathrm{D}_{\mathrm{Qcoh}}(X))^c = \mathrm{D}^{\mathrm{per}}(X)$ (see [BvdB03]). The latter are easily seen to be rigid for the derived tensor product $\otimes = \otimes^{\mathrm{L}}_{\mathcal{O}_X}$. If moreover X is separated, there is an equivalence $\mathrm{D}_{\mathrm{Qcoh}}(X) \simeq \mathrm{D}(\mathrm{Qcoh}\, X)$ with the derived category of complexes of quasi-coherent $\mathcal{O}_X$-modules (see [BN93]). If $X = \mathrm{Spec}(A)$ is affine, then $\mathrm{D}(\mathrm{Qcoh}\, X) \simeq \mathrm{D}(A\text{-}\mathrm{Mod})$ with compacts $\mathrm{D}(A\text{-}\mathrm{Mod})^c \simeq \mathrm{K}^{\mathrm{b}}(A\text{-}\mathrm{proj})$, the homotopy category of bounded complexes of finitely generated projectives.

(2) Let G be a compact Lie group. Then $\mathcal{C} := \mathrm{SH}(G)$, the homotopy category of "genuine" G-spectra indexed on a complete G-universe (see [HPS97, §9.4]), is

rigidly-compactly generated. The suspension G-spectra $\Sigma^\infty_+ G/H$, with H running through all closed subgroups of G, form a set of rigid-compact generators which includes the tensor unit $\mathbb{1} = \Sigma^\infty_+ G/G$.

(3) Let G be a finite group and let $\Bbbk$ be a field. Then $\mathcal{C} := \mathrm{Stab}(\Bbbk G)$, the stable category of $\Bbbk G$-modules modulo projectives, is rigidly-compactly generated (but not $\mathrm{D}(\Bbbk G)$ in which $\mathbb{1} = k$ is not compact). More generally, G could be a finite group scheme over $\Bbbk$ (see e.g. [HPS97, Theorem 9.6.3]).

(4) Let $\Bbbk$ be a field and let $\mathcal{C} := \mathrm{SH}^{\mathbb{A}^1}(\Bbbk)$ denote the stable $\mathbb{A}^1$-homotopy category. This is rigidly-compactly generated provided $\Bbbk$ has characteristic zero. A set of rigid-compact generators is provided by twists of smooth projective $\Bbbk$-varieties. Although these (compact) objects are always dualizable in $\mathrm{SH}^{\mathbb{A}^1}(\Bbbk)$, without any assumption on the characteristic, the proof that they generate the whole category depends on resolution of singularities (see [Rio05]).

(5) Let A be a "Brave New" commutative ring, that is, a structured commutative ring spectrum. To fix ideas, we can understand A to be a commutative $\mathbb{S}$-algebra in the sense of [EKMM97]. Then its derived category $\mathrm{D}(A)$, *i.e.* the homotopy category of A-modules, is a rigidly-compactly generated category, which is generated by its tensor unit A (see e.g. [HPS97, Example 1.2.3(f)] and [SS03, Example 2.3(ii)]).

2.10. **Corollary.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ be as in our basic Hypothesis 1.2. Then f^* preserves compacts and admits a right adjoint $f_* : \mathcal{C} \to \mathcal{D}$, which itself admits a right adjoint $f^{(1)} : \mathcal{D} \to \mathcal{C}$.*

Proof. Since f^* preserves coproducts by assumption, f_* exists by Brown Representability, Cor. 2.3 (a). Since f^* is symmetric monoidal by assumption, it must send rigid objects of $\mathcal{D}$ to rigid objects of $\mathcal{C}$ (see e.g. [LMSM86, §III.1]). Hence it must preserve compacts (= rigids). By Proposition 2.5, f_* preserves coproducts and we can apply another layer of Brown Representability to f_* in order to get $f^{(1)}$. $\square$

Our three functors $f^* \dashv f_* \dashv f^{(1)}$ automatically satisfy some basic formulas.

2.11. **Proposition.** *Assume we have $f^* \dashv f_* \dashv f^{(1)}$ as in Corollary 2.10. Then there is a canonical natural isomorphism*

$$\pi : x \otimes f_*(y) \xrightarrow{\sim} f_*(f^*(x) \otimes y) \tag{2.12}$$

for all $x \in \mathcal{D}$ and $y \in \mathcal{C}$, obtained from $f^*(x \otimes f_*(y)) \cong f^*(x) \otimes f^* f_*(y) \to f^*(x) \otimes y$ by adjunction. We also have three further canonical isomorphisms as follows:

$$\underline{\mathrm{hom}}_{\mathcal{D}}(x, f_* y) \cong f_* \, \underline{\mathrm{hom}}_{\mathcal{C}}(f^* x, y) \tag{2.13}$$

$$\underline{\mathrm{hom}}_{\mathcal{D}}(f_* x, y) \cong f_* \, \underline{\mathrm{hom}}_{\mathcal{C}}(x, f^{(1)} y) \tag{2.14}$$

$$f^{(1)} \underline{\mathrm{hom}}_{\mathcal{D}}(x, y) \cong \underline{\mathrm{hom}}_{\mathcal{C}}(f^* x, f^{(1)} y)\,. \tag{2.15}$$

2.16. *Terminology.* We call (2.12) the *(right) projection formula*. Equations (2.13) and (2.14) are *internal realizations* of the two adjunctions $f^* \dashv f_* \dashv f^{(1)}$, from which the adjunctions can be recovered by applying $\underline{\mathrm{hom}}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}, -)$. Note that (2.14) specializes to (1.6) by inserting $y = \mathbb{1}_{\mathcal{D}}$.

Proof. The map π is clearly well-defined for all x and y and is automatically invertible whenever x is rigid (cf. [FHM03, Prop. 3.2]). Fixing an arbitrary $y \in \mathcal{C}$, note that both sides of (2.12) are exact and commute with coproducts in the variable x. As $\mathcal{C}$ is generated by its compact (= rigid) objects, π is an isomorphism for all $x \in \mathcal{D}$ (Rem. 2.4). This proves the first isomorphism, *i.e.* the projection formula.

Now we can derive from it two of the other equations by taking adjoints. (Recall that if $F_i \dashv G_i$ for $i = 1, 2$ then $F_1F_2 \dashv G_2G_1$.) First, by fixing x we see two composite adjunctions

$$x \otimes f_* = (x \otimes -) \circ f_* \quad \dashv \quad f^{(1)} \circ \underline{\mathrm{hom}}_{\mathcal{D}}(x, -)$$

and

$$f_*(f^*(x) \otimes -) = f_* \circ (f^*(x) \otimes -) \;\dashv\; \underline{\mathrm{hom}}_{\mathcal{C}}(f^*x, -) \circ f^{(1)} = \underline{\mathrm{hom}}_{\mathcal{C}}(f^*x, f^{(1)}(-))\,.$$

Since π is an isomorphism of the left adjoints, by the uniqueness of right adjoints it induces an isomorphism between the right ones, *i.e.* we get (2.15). (The naturality in x is guaranteed by the fact that the two adjunctions above are actually natural families of adjunctions parametrized by x.) If we fix y instead, we get adjunctions

$$(-) \otimes f_*(y) \quad \dashv \quad \underline{\mathrm{hom}}_{\mathcal{D}}(f_*y, -)$$

and

$$f_*(f^*(-) \otimes y) = f_* \circ (- \otimes y) \circ f^* \;\dashv\; f_* \circ \underline{\mathrm{hom}}_{\mathcal{C}}(y, -) \circ f^{(1)} = f_*\, \underline{\mathrm{hom}}_{\mathcal{C}}(y, f^{(1)}(-))$$

from which we derive the natural isomorphism (2.14). By fixing x in the isomorphism $f^*(x) \otimes f^*(y) \cong f^*(x \otimes y)$ given by the monoidal structure of f^*, we obtain

$$f^*(x) \otimes f^* = (f^*(x) \otimes -) \circ f^* \quad \dashv \quad f_*\, \underline{\mathrm{hom}}_{\mathcal{C}}(f^*x, -)$$

and

$$f^*(x \otimes -) = f^* \circ (x \otimes -) \quad \dashv \quad \underline{\mathrm{hom}}_{\mathcal{D}}(x, -) \circ f_* = \underline{\mathrm{hom}}_{\mathcal{D}}(x, f_*(-))$$

from which we derive the remaining relation (2.13). $\square$

2.17. *Remark.* The reasoning of the previous proof will be used several times, so it is worth spending a little thought on it. Let's say we have some *formula*, by which we mean a natural isomorphism $F_1 \circ \ldots \circ F_n \cong F'_1 \circ \ldots \circ F'_m$ between composite functors, in which every factor is part of an adjunction $F_i \dashv G_i$ and $F'_j \dashv G'_j$. By taking right adjoints on both sides we derive a formula $G_n G_{n-1} \cdots G_1 \cong G'_m G'_{m-1} \cdots G'_1$. Indeed, the two formulas are equivalent, in that we may recover the first one by taking left adjoints in the second one. Following [FHM03], we can say that the two formulas are *conjugate*, or *adjunct*. Note however that if the original formula admits two different factor-decompositions as above, we would obtain a different conjugate formula from each choice of decomposition (though, of course, they are all still equivalent). This is illustrated by the previous proposition, in which (2.14) and (2.15) are obtained from two different decompositions of (2.12). In this case, (2.12) is a formula between functors of two variables x and y, and the two decompositions have been obtained by first fixing either x or y. Note that the tensor formula $f^*(x \otimes y) \cong f^*(x) \otimes f^*(y)$ is symmetric in x and y, hence the two resulting decompositions yield the same conjugate formula (2.13). All our conjugate formulas will come in such couplets or triplets and will be obtained in this way from a starting formula in either one or two variables. The systematic exploitation of this principle will greatly simplify the search for new relations. When repeating this reasoning below we will mostly leave the straightforward details to the reader.

3. Grothendieck-Neeman duality and ur-Wirthmüller

We want to prove Theorem 1.7, for which we need some preparation. Recall the basic set-up as in Hypothesis 1.2 and the three functors $f^* \dashv f_* \dashv f^{(1)}$ (Cor. 2.10). We focus on the new, slightly surprising facts. The following lemma should be compared to the well-known property presented in Proposition 2.5.

3.1. **Lemma.** *If $f^* : \mathcal{D} \to \mathcal{C}$ has a left adjoint $f_{(1)} \dashv f^*$, i.e. if f^* preserves products, then its right adjoint f_* preserves compact objects: $f_*(\mathcal{C}^c) \subseteq \mathcal{D}^c$.*

Proof. Recall that f^* preserves coproducts by our standing hypothesis, hence $f_{(1)}$ preserves compacts (Prop. 2.5). Therefore $f_{(1)} \dashv f^*$ restricts to an adjunction $f_{(1)} : \mathcal{C}^c \rightleftarrows \mathcal{D}^c : f^*$ on compact objects. Since compacts are rigid, duality provides equivalences of (tensor) categories $\Delta := \underline{\hom}_{\mathcal{C}}(-, \mathbb{1}) : (\mathcal{C}^c)^{\mathrm{op}} \to \mathcal{C}^c$ and $\Delta := \underline{\hom}_{\mathcal{D}}(-, \mathbb{1}) : (\mathcal{D}^c)^{\mathrm{op}} \to \mathcal{D}^c$ which are quasi-inverse to themselves (*i.e.* $\Delta^{-1} = \Delta^{\mathrm{op}}$). Moreover, the symmetric monoidal functor f^* preserves rigid objects c and their tensor duals $\Delta(c)$ (cf. [LMSM86, §III.1]), so that we have the following commutative square (up to isomorphism of functors):

$$\begin{tikzcd} (\mathcal{D}^c)^{\mathrm{op}} \arrow[r, "\Delta", "\sim"'] \arrow[d, "(f^*)^{\mathrm{op}}"'] & \mathcal{D}^c \arrow[d, "f^*"] \\ (\mathcal{C}^c)^{\mathrm{op}} \arrow[r, "\Delta", "\sim"'] & \mathcal{C}^c. \end{tikzcd}$$

This self-duality implies that the composite functor $f^c_* := \Delta \circ (f_{(1)})^{\mathrm{op}} \circ \Delta^{-1} = \Delta f_{(1)} \Delta \colon \mathcal{C}^c \to \mathcal{D}^c$ is *right* adjoint to $f^* \colon \mathcal{D}^c \to \mathcal{C}^c$. It follows now from Lemma 2.6 (with $F := f^*$) that the right adjoint f_* to f^* must preserve compact objects. $\square$

We are now going to go in the opposite direction, *i.e.* mostly assume that in our basic trio of functors $f^* \dashv f_* \dashv f^{(1)}$ the first right adjoint f_* preserves compacts. Our next lemma is the abstraction of a result established in algebraic geometry by Neeman [Nee10, Lem. 4.2, p. 325]. It gives us the converse to Lemma 3.1, and provides the other surprising connection between the functors $f^* \dashv f_* \dashv f^{(1)}$.

3.2. **Lemma.** *Suppose that $f_* : \mathcal{C} \to \mathcal{D}$ preserves compacts. Then $f^* : \mathcal{D} \to \mathcal{C}$ preserves products.*

Proof. Let $\{t_i\}_{i \in I}$ be a set of objects in $\mathcal{D}$. Then for every compact c in $\mathcal{C}$, we have

$$\begin{aligned} f_*\big(c \otimes f^*(\prod_{i \in I} t_i)\big) &\cong f_*(c) \otimes \prod_{i \in I} t_i && \text{projection formula Prop. 2.11} \\ &\cong \prod_{i \in I} f_*(c) \otimes t_i && f_*(c) \text{ compact, hence rigid} \\ &\cong \prod_{i \in I} f_*(c \otimes f^*(t_i)) && \text{projection formula Prop. 2.11} \\ &\cong f_*\big(\prod_{i \in I} c \otimes f^*(t_i)\big) && f_* \text{ preserves products (right adjoint)} \\ &\cong f_*\big(c \otimes \prod_{i \in I} f^*(t_i)\big) && c \text{ compact, hence rigid.} \end{aligned}$$

Now apply $\mathcal{D}(\mathbb{1},-)$ on both sides and use the isomorphisms $\mathcal{D}(\mathbb{1}, f_*(c \otimes -)) \cong \mathcal{C}(f^*\mathbb{1}, c \otimes -) \cong \mathcal{C}(\mathbb{1}, c \otimes -) \cong \mathcal{C}(\Delta(c), -)$. Since $\Delta : (\mathcal{C}^c)^{\mathrm{op}} \to \mathcal{C}^c$ is an equivalence, we have shown that $\mathcal{C}(c', f^*(\prod_{i\in I} t_i)) \cong \mathcal{C}(c', \prod_{i\in I} f^*(t_i))$ for every compact $c' \in \mathcal{C}^c$. One verifies that this isomorphism coincides with the image under $\mathcal{C}(c',-)$ of the natural transformation $f^*(\prod t_i) \to \prod f^*(t_i)$. Since $\mathcal{C}$ is compactly generated, we get the result (Remark 2.4 and Yoneda). □

3.3. **Proposition.** *Suppose that $f_* : \mathcal{C} \to \mathcal{D}$ preserves compacts. Then there is a canonical natural isomorphism $f^{(1)}(x) \otimes f^*(y) \xrightarrow{\sim} f^{(1)}(x \otimes y)$ for all $x, y \in \mathcal{D}$.*

Proof. The natural comparison map $f^{(1)}(x) \otimes f^*(y) \to f^{(1)}(x \otimes y)$ can always be constructed out of the counit $\epsilon : f_* f^{(1)} \to \mathrm{Id}_{\mathcal{C}}$ of the adjunction $f_* \dashv f^{(1)}$ as follows:

$$
\begin{array}{rcll}
\epsilon_x \otimes \mathrm{id}_y & \in & \mathcal{D}(f_* f^{(1)}(x) \otimes y, x \otimes y) & \\
 & \cong & \mathcal{D}(f_*(f^{(1)}(x) \otimes f^*(y)), x \otimes y) & \text{projection formula Prop. 2.11} \\
 & \cong & \mathcal{C}(f^{(1)}(x) \otimes f^*(y), f^{(1)}(x \otimes y)) & \text{adjunction } f_* \dashv f^{(1)}\,.
\end{array}
$$

If $y \in \mathcal{D}^c$ is rigid, we have for all $z \in \mathcal{C}$ a natural isomorphism:

$$
\begin{array}{rcll}
\mathcal{C}(z, f^{(1)}(x) \otimes f^*(y)) & \cong & \mathcal{C}(z \otimes \Delta f^*(y), f^{(1)}(x)) & f^*(y) \text{ is rigid} \\
 & \cong & \mathcal{C}(z \otimes f^*\Delta(y), f^{(1)}(x)) & f^*\Delta \cong \Delta f^* \text{ on rigids} \\
 & \cong & \mathcal{D}(f_*(z \otimes f^*\Delta(y)), x) & \text{adjunction } f_* \dashv f^{(1)} \\
 & \cong & \mathcal{D}(f_*(z) \otimes \Delta(y), x) & \text{projection formula Prop. 2.11} \\
 & \cong & \mathcal{D}(f_*(z), x \otimes y) & y \text{ is rigid} \\
 & \cong & \mathcal{C}(z, f^{(1)}(x \otimes y)) & \text{adjunction } f_* \dashv f^{(1)}.
\end{array}
$$

A tedious but straightforward diagram chase verifies that this isomorphism is merely post-composition by the general comparison map $f^{(1)}(x) \otimes f^*(y) \to f^{(1)}(x \otimes y)$ previously defined. Hence, by Yoneda, we conclude that the general comparison map is an isomorphism whenever y is rigid. By Proposition 2.5, the hypothesis on f_* is equivalent to $f^{(1)}$ preserving coproducts. Hence both sides of the comparison map $f^{(1)}(x) \otimes f^*(y) \to f^{(1)}(x \otimes y)$ are coproduct-preserving exact functors in both variables. Hence this comparison map is invertible for all $x, y \in \mathcal{D}$ (Remark 2.4). □

We are now ready to prove our generalized Grothendieck-Neeman duality theorem. Recall from Definition 1.4 that $\omega_f := f^{(1)}(\mathbb{1}) \in \mathcal{C}$ is the *relative dualizing object* associated with the given functor $f^* : \mathcal{D} \to \mathcal{C}$.

3.4. **Theorem.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ be as in our basic Hypothesis 1.2 and consider the adjoints $f^* \dashv f_* \dashv f^{(1)}$ (Cor. 2.10). Then the following conditions are equivalent:*

(a) *The functor f^* admits a left adjoint $f_{(1)}$.*
(b) *The functor f^* preserves products.*
(c) *The functor $f^{(1)}$ admits a right adjoint $f_{(-1)}$.*
(d) *The functor $f^{(1)}$ preserves coproducts.*
(e) *The functor f_* preserves compact objects.*

Furthermore, if (a)-(e) *hold true then* $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$ *satisfy the following additional relations given by canonical natural isomorphisms:*

$$\boxed{f^{(1)} \cong \omega_f \otimes f^*(-)} \qquad \text{(Grothendieck duality)} \tag{3.5}$$

$$f_{(-1)} \cong f_* \, \underline{\mathrm{hom}}_{\mathcal{C}}(\omega_f, -) \tag{3.6}$$

$$f^{(1)}(x \otimes y) \cong f^{(1)}(x) \otimes f^*(y) \tag{3.7}$$

$$\underline{\mathrm{hom}}_{\mathcal{D}}(x, f_{(-1)}y) \cong f_* \, \underline{\mathrm{hom}}_{\mathcal{C}}(f^{(1)}x, y) \tag{3.8}$$

$$\underline{\mathrm{hom}}_{\mathcal{D}}(x, f_{(-1)}y) \cong f_{(-1)} \, \underline{\mathrm{hom}}_{\mathcal{C}}(f^*x, y) \tag{3.9}$$

$$f^*(-) \cong \underline{\mathrm{hom}}_{\mathcal{C}}(\omega_f, f^{(1)}(-)) \tag{3.10}$$

$$\boxed{f_{(1)}(-) \cong f_*(\omega_f \otimes -)} \qquad \text{(ur-Wirthmüller)} \tag{3.11}$$

$$\boxed{x \otimes f_{(1)}(y) \cong f_{(1)}(f^*(x) \otimes y)} \qquad \text{(left projection formula)} \tag{3.12}$$

$$f^* \underline{\mathrm{hom}}_{\mathcal{D}}(x, y) \cong \underline{\mathrm{hom}}_{\mathcal{C}}(f^*x, f^*y) \tag{3.13}$$

$$\underline{\mathrm{hom}}_{\mathcal{D}}(f_{(1)}x, y) \cong f_* \, \underline{\mathrm{hom}}_{\mathcal{C}}(x, f^*y) \tag{3.14}$$

3.15. *Remark.* The existence of *any* natural isomorphism as in Grothendieck duality (3.5) implies that $f^{(1)}$ preserves coproducts (*i.e.* property (d) holds). Hence (3.5) is not only a consequence of, but is equivalent to, Conditions (a)-(e) of the theorem. Similarly, the more general (3.7) is also equivalent to (a)-(e). Finally, if there exists *any* isomorphism as in (3.10), then f^* must preserve products, since so do the left adjoints $\underline{\mathrm{hom}}_{\mathcal{C}}(\omega_f, -)$ and $f^{(1)}$. Hence (3.10) is also an equivalent condition for Theorem 3.4 to hold. We note this for completeness but it is unlikely that such isomorphisms can be established in practice *before* conditions (a)-(e) are known.

3.16. *Remark.* We will see in the proof that each group of equations in (3.5)-(3.14), as displayed above, forms a conjugate set of formulas in the sense of Remark 2.17.

3.17. *Remark.* All of the adjunctions $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$ now have an internal realization in $\mathcal{D}$ by (3.14), (2.13), (2.14), and (3.9), respectively.

3.18. *Remark.* We can further combine the fundamental formulas of Theorem 3.4, for instance by composing Grothendieck duality (3.5) with the ur-Wirthmüller (3.11) isomorphism and then variating by conjugation:

$$\begin{aligned} f_{(1)}(f^*(x) \otimes y) &\cong f_*(f^{(1)}(x) \otimes y) \\ \underline{\mathrm{hom}}_{\mathcal{C}}(f^*x, f^*y) &\cong \underline{\mathrm{hom}}_{\mathcal{C}}(f^{(1)}x, f^{(1)}y) \\ f_* \, \underline{\mathrm{hom}}_{\mathcal{C}}(x, f^*y) &\cong f_{(-1)}\underline{\mathrm{hom}}_{\mathcal{C}}(x, f^{(1)}y) \,. \end{aligned}$$

Or we may plug Grothendieck duality into co-Wirthmüller (3.10) to obtain

$$f^* \cong \underline{\mathrm{hom}}_{\mathcal{C}}(\omega_f, \omega_f \otimes f^*(-))$$

which, when applied to the tensor unit, specializes to the important relation

$$\mathbb{1}_{\mathcal{C}} \cong \underline{\mathrm{hom}}_{\mathcal{C}}(\omega_f, \omega_f) \,. \tag{3.19}$$

We leave further variations to the computer.

Proof of Theorem 3.4. We already know that (a)⇔(b) and (c)⇔(d)⇔(e) by Brown representability (Cor. 2.3) and by Proposition 2.5. We also isolated the non-obvious parts of the equivalences in the above Lemmas 3.1 and 3.2, which give (a)⇒(e) and (e)⇒(b) respectively. So we can assume that (a)-(e) hold true and we now turn to proving Formulas (3.5)-(3.14).

Proposition 3.3 already gives (3.7), which then specializes to (3.5) by setting $x := \mathbb{1}_{\mathcal{D}}$. We now construct the canonical ur-Wirthmüller isomorphism (3.11). Using the unit $\eta : \mathrm{Id} \to f^* f_{(1)}$ of the adjunction $f_{(1)} \dashv f^*$, we get a natural comparison map

$$f_*(\omega_f \otimes x) \to f_{(1)}(x) \tag{3.20}$$

for all $x \in \mathcal{C}$, as follows:

$$\begin{array}{rcll} \eta_x \otimes \mathrm{id}_{\omega_f} & \in & \mathcal{C}(x \otimes \omega_f, f^*(f_{(1)}(x)) \otimes \omega_f) & \\ & \cong & \mathcal{C}(x \otimes \omega_f, f^{(1)} f_{(1)}(x)) & (3.5) \\ & \cong & \mathcal{D}(f_*(x \otimes \omega_f), f_{(1)}(x)) & f_* \dashv f^{(1)}\,. \end{array}$$

(Here of course we do not take *any* isomorphism as in (3.5), but rather the canonical one constructed in Proposition 3.3, specialized at $x = \mathbb{1}$). Explicitly, (3.20) is equal to the composite

$$f_*(x \otimes \omega_f) \xrightarrow{f_*(\eta \otimes 1)} f_*(f^* f_{(1)} x \otimes \omega_f) \cong f_{(1)} x \otimes f_* \omega_f = f_{(1)} x \otimes f_* f^{(1)} \mathbb{1} \xrightarrow{1 \otimes \epsilon} f_{(1)} x \tag{3.21}$$

where the middle map is the right projection formula (2.12). By Remark 2.4, it suffices to show that this map is an isomorphism for $x \in \mathcal{C}^c$ compact, because both sides preserve coproducts (being composed of left adjoints). Now, for every $d \in \mathcal{D}$ and for $x \in \mathcal{C}^c$ rigid we compute:

$$\begin{array}{rcll} \mathcal{D}(d, f_*(x \otimes \omega_f)) & \cong & \mathcal{C}(f^*(d), x \otimes \omega_f) & f^* \dashv f_* \\ & \cong & \mathcal{C}(\Delta(x) \otimes f^*(d), \omega_f) & x \in \mathcal{C}^c \text{ is rigid} \\ & \cong & \mathcal{D}(f_*(\Delta x \otimes f^* d), \mathbb{1}) & \omega_f = f^{(1)}(\mathbb{1}) \text{ and } f_* \dashv f^{(1)} \\ & \cong & \mathcal{D}(f_*(\Delta x) \otimes d, \mathbb{1}) & \text{projection formula (2.12)} \\ & \cong & \mathcal{D}(d, \Delta f_* \Delta(x)) & f_* \Delta(x) \in \mathcal{D}^c \text{ by Lemma 3.1} \\ & \cong & \mathcal{D}(d, f_{(1)}(x)). & \end{array}$$

The last isomorphism holds because, thanks to (e), the adjunction $f_{(1)} \dashv f^* \dashv f_*$ restricts to the categories of compact objects, so that the dual Δ intertwines the two restricted functors: $\Delta f_* \Delta \cong f_{(1)}$ on $\mathcal{C}^c$. By Yoneda, we obtain from the above an isomorphism $f_*(x \otimes \omega_f) \cong f_{(1)}(x)$ for $x \in \mathcal{C}^c$ and we "only" need to show that it coincides with the canonical map (3.21). This is an adventurous diagram chase that we now sketch.

Following through the chain of isomorphisms, we can reduce the problem to checking the commutativity of the following diagram:

$$\begin{tikzcd}
f_*(x\otimes\omega_f) \arrow[r, "\mathrm{coev}\otimes 1"] \arrow[d, "f_*(\eta\otimes 1)"'] & \Delta f_*\Delta x\otimes f_*\Delta x\otimes f_*(x\otimes\omega_f) \arrow[d, "1\otimes\mathrm{lax}"] \\
f_*(f^*f_{(1)}x\otimes\omega_f) \arrow[d, "\pi"'] & \Delta f_*\Delta x\otimes f_*(\Delta x\otimes x\otimes\omega_f) \arrow[d, "1\otimes f_*(\mathrm{ev}\otimes 1)"] \\
f_{(1)}x\otimes f_*\omega_f \arrow[r, "\cong"] & \Delta f_*\Delta x\otimes f_*\omega_f.
\end{tikzcd} \tag{3.22}$$

Using an explicit description of the isomorphism $f_{(1)}x\cong\Delta f_*\Delta x$ in terms of the unit and counit of $f_{(1)}\dashv f^*$ and the duality maps, one can check that the composite along the left and bottom edges is equal to

$$\begin{tikzcd}
f_*(x\otimes\omega_f)\cong f_*(f^*\mathbb{1}\otimes x\otimes\omega_f) \arrow[r, "f_*(f^*\mathrm{coev}\otimes 1\otimes 1)"] & f_*(f^*(\Delta f_*\Delta x\otimes f_*\Delta x)\otimes x\otimes\omega_f) \arrow[d, "f_*(f^*\pi\otimes 1)"] \\
& f_*(f^*f_*(f^*\Delta f_*\Delta x\otimes\Delta x)\otimes x\otimes\omega_f) \arrow[d, "f_*(\epsilon\otimes 1)"] \\
& f_*(f^*\Delta f_*\Delta x\otimes\Delta x\otimes x\otimes\omega_f) \arrow[d, "f_*(1\otimes\mathrm{ev}\otimes 1)"] \\
& f_*(f^*\Delta f_*\Delta x\otimes\omega_f) \arrow[d, "\pi"] \\
& \Delta f_*\Delta x\otimes f_*\omega_f
\end{tikzcd}$$

where π is the projection formula isomorphism (2.12). This composite can then be checked to agree with

$$\begin{tikzcd}
f_*(x\otimes\omega_f)=\mathbb{1}\otimes f_*(x\otimes\omega_f) \arrow[r, "\mathrm{coev}\otimes 1"] & \Delta f_*\Delta x\otimes f_*\Delta x\otimes f_*(x\otimes\omega_f) \arrow[d, "\pi\otimes 1"] \\
& f_*(f^*\Delta f_*\Delta x\otimes\Delta x)\otimes f_*(x\otimes\omega_f) \arrow[d, "\mathrm{lax}"] \\
& f_*(f^*\Delta f_*\Delta x\otimes\Delta x\otimes x\otimes\omega_f) \arrow[d, "f_*(1\otimes\mathrm{ev}\otimes 1)"] \\
& f_*(f^*\Delta f_*\Delta x\otimes\omega_f) \arrow[d, "\pi"] \\
& \Delta f_*\Delta x\otimes f_*\omega_f.
\end{tikzcd}$$

Using this last description, the commutativity of diagram (3.22) can be established. In carrying out these verifications, the commutativity of the following diagrams (3.23)

$$\begin{tikzcd}
f_*a\otimes f_*b \arrow[r, "\pi"] \arrow[dr, "\mathrm{lax}"'] & f_*(f^*f_*a\otimes b) \arrow[d, "f_*(\epsilon\otimes 1)"] \\
& f_*(a\otimes b)
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
a\otimes f_*b\otimes f_*c \arrow[r, "\pi\otimes 1"] \arrow[d, "1\otimes\mathrm{lax}"'] & f_*(f^*a\otimes b)\otimes f_*c \arrow[d, "\mathrm{lax}"] \\
a\otimes f_*(b\otimes c) \arrow[r, "\pi"] & f_*(f^*a\otimes b\otimes c)
\end{tikzcd}$$

will prove to be useful. The details are now left to the careful reader.

We have now established (3.5), (3.7) and (3.11), from which we derive the other ones by the general method of Remark 2.17. Taking right adjoints of the functors in (3.11) yields (3.10). Similarly, taking right adjoints of the functors in (3.7) for each fixed x yields (3.8), and taking right adjoints for each fixed y yields (3.9). (The left-hand-sides of the two latter formulas coincide, because $f^{(1)}(x \otimes y)$ is symmetric in x and y.) On the other hand, taking right adjoints in (3.5) yields (3.6). Also, (3.19) is (3.10) evaluated at $\mathbb{1}_{\mathcal{D}}$. The left projection formula (3.12) follows by conjugating the right projection formula (2.12) by the ur-Wirthmüller isomorphism (3.11):

$$\begin{tikzcd}
x \otimes f_{(1)}(y) \arrow[r, "\cong"] \arrow[d, dotted, "\cong"'] & x \otimes f_*(\omega_f \otimes y) \arrow[d, "\cong"] \\
f_{(1)}(f^*(x) \otimes y) \arrow[r, "\cong"] & f_*(f^*(x) \otimes \omega_f \otimes y).
\end{tikzcd}$$

This new two-variable equation (3.12) has two conjugate formulas (3.13) and (3.14), obtained by taking right adjoints while fixing x or y, respectively.

This finishes the proof of Grothendieck-Neeman duality, Theorem 3.4. $\square$

3.24. *Example* (Algebraic geometry). Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes, as in Example 2.9(1), and consider the (derived) inverse image functor $f^* : \mathcal{D} = \mathrm{D}_{\mathrm{Qcoh}}(Y) \to \mathrm{D}_{\mathrm{Qcoh}}(X) = \mathcal{C}$. It is easy to see that f^* satisfies our basic Hypothesis 1.2; its right adjoint is the derived push-forward $f_* = \mathrm{R}f_*$, whose right adjoint $f^{(1)}$ is the twisted inverse image functor, usually written $f^\times$ or $f^!$ (see [Lip09]). Then the functor f^* satisfies Grothendieck-Neeman duality precisely when the morphism f is *quasi-perfect* [LN07, Def. 1.1]. Indeed, the latter means by definition that $\mathrm{R}f_*$ preserves perfect complexes, *i.e.* compact objects. In this context, our Theorem 3.4 recovers the original results of Neeman that have inspired us; see [LN07, Prop. 2.1] for a geometric statement in the same generality as we obtain here by specializing our abstract methods. Yet, even when specialized to algebraic geometry, our theorem is somewhat stronger, because it includes the extra information about the left adjoint $f_{(1)}$ of f^*, whose existence is equivalent to the quasi-perfection of f and which is necessarily given by the ur-Wirthmüller formula $f_{(1)} \cong \omega_f \otimes f_*$.

The article [LN07] contains a thorough geometric study of quasi-perfection. Among other things, it is shown that f is quasi-perfect iff it is proper and of finite tor-dimension. In particular if $f : X \to Y$ is *finite* then it is quasi-perfect iff $f_*(\mathbb{1}_{\mathcal{C}}) = \mathrm{R}f_*(\mathcal{O}_X)$ is a perfect complex. See Examples 2.2 of *loc. cit.* for more.

3.25. *Example* (Affine case). Let $\phi : B \to A$ be a morphism of commutative rings. We may specialize Example 3.24 to $f := \mathrm{Spec}(\phi) : X := \mathrm{Spec}(A) \to \mathrm{Spec}(B) =: Y$. Since a proper affine morphism is necessarily finite, we see that f is quasi-perfect if and only if A is a finite B-module admitting a finite resolution by finite projective B-modules. Since $\mathrm{R}f_* = f_* \cong {}_BA \otimes_A -$, the right adjoint $f^{(1)}$ has the following reassuring description as a right derived functor: $f^{(1)} = \mathrm{R}\,\mathrm{Hom}_B({}_BA, -)$.

3.26. *Example.* Of course, not all scheme maps $f : X \to Y$ are quasi-perfect. For instance, the affine morphism $f := \mathrm{Spec}(\phi)$, with $\phi : \mathbb{Z}[t]/(t^2) \to \mathbb{Z}$ sending the variable t to zero, is not; see [Nee96, Ex. 6.5]. Note that f is finite, but indeed $\mathrm{R}f_*(\mathcal{O}_X)$ is not compact as $\mathbb{Z}$ has infinite projective dimension over $\mathbb{Z}[t]/(t^2)$.

3.27. *Example.* The affine situation of Example 3.25 can be generalized in the Brave New direction, as in Example 2.9(5). That is, we may consider $\phi : B \to A$ to be a morphism of commutative $\mathbb{S}$-algebras. One easily checks that ϕ induces a functor $f^* := A \otimes_B - \colon \mathrm{D}(B) \to \mathrm{D}(A)$ satisfying our basic Hypotheses. As before, its right adjoint f_* is obtained simply by considering A-modules as B-modules through ϕ and the next right adjoint $f^{(1)}$ is given by the formula $f^{(1)} = \mathrm{Hom}_B(A, -)$. (All functors considered here are derived from appropriate Quillen adjunctions.) Since $\mathrm{D}(A)^c$ is the thick subcategory generated by A, we see by Neeman's criterion that, as before, f^* satisfies Grothendieck-Neeman duality iff $f_*(A)$ is compact.

4. The Wirthmüller isomorphism

When we are in the Grothendieck-Neeman situation, *i.e.* when we have five adjoints $f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$, the relative dualizing object ω_f is remarkably "close" to being $\otimes$-invertible, a fact which perhaps deserves separate statement.

4.1. **Proposition.** *Assume that the hypotheses of Theorem 3.4 hold. Then $f_*(\omega_f)$ is compact in $\mathcal{D}$. Moreover, ω_f is compact in $\mathcal{C}$ if and only if it is $\otimes$-invertible.*

Proof. We have $f_*(\omega_f) \cong f_{(1)}(\mathbb{1})$ by the ur-Wirthmüller isomorphism (3.11). Moreover, $\mathbb{1}$ is assumed to be compact and $f_{(1)}$ preserves compact objects by Proposition 2.5, because its right adjoint f^* preserves coproducts by hypothesis.

Invertible objects are always rigid, hence compact under our assumptions. Conversely, if ω_f is rigid there is an isomorphism $\Delta(\omega_f) \otimes \omega_f \overset{\sim}{\to} \underline{\mathrm{hom}}_{\mathcal{C}}(\omega_f, \omega_f)$ and therefore by (3.19) an isomorphism $\Delta(\omega_f) \otimes \omega_f \overset{\sim}{\to} \mathbb{1}$, hence ω_f is invertible. □

We are now ready to prove Theorem 1.9, which abundantly characterizes the situations when ω_f *does* become invertible.

Proof of the Wirthmüller isomorphism Theorem 1.9. The equivalent formulations of (W1) hold by Corollary 2.3 and similarly for (W2), together with Proposition 2.5. The equivalence (W3)⇔(W4) holds by Proposition 4.1. If $f^{(1)}$ preserves compacts then obviously $\omega_f = f^{(1)}(\mathbb{1})$ is compact, hence we have (W2)⇒(W3). Conversely, we can see from Grothendieck duality $f^{(1)} \cong \omega_f \otimes f^*$ that (W3)⇒(W2), as our f^* always preserves compacts.

Let us show (W1)⇒(W2). Thus we now have the six adjoints $f^{(-1)} \dashv f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)} \dashv f_{(-1)}$ and we want to show that $f^{(1)}$ preserves compacts. Since $f^{(-1)}$, $f_{(1)}$, f^* and f_* have two-fold right adjoints they must preserve compacts by Proposition 2.5 (their right adjoints preserve coproducts). By restricting to compacts, we have four consecutive adjoints $f^{(-1)}|_{\mathcal{D}^c} \dashv f_{(1)}|_{\mathcal{C}^c} \dashv f^*|_{\mathcal{D}^c} \dashv f_*|_{\mathcal{C}^c}$. Now recall from Remark 2.8 that $\Delta = \underline{\mathrm{hom}}(-, \mathbb{1})$ defines a duality on compact objects, hence conjugating with it turns left adjoints into right adjoints. Furthermore, since $f^*\Delta = \Delta f^*$, the original functor $f^*|_{\mathcal{D}^c}$ is fixed by conjugation by Δ. The above four adjoints therefore yield (several isomorphisms, like $\Delta(f_{(1)}|_{\mathcal{C}^c})\Delta \cong f_*|_{\mathcal{C}^c}$, and) the following five consecutive adjoints between $\mathcal{D}^c$ and $\mathcal{C}^c$

$$f^{(-1)}|_{\mathcal{D}^c} \dashv f_{(1)}|_{\mathcal{C}^c} \dashv f^*|_{\mathcal{D}^c} \dashv f_*|_{\mathcal{C}^c} \dashv \Delta(f^{(-1)}|_{\mathcal{D}^c})\Delta\,.$$

The right-most functor is the unpredicted one. We can now apply Lemma 2.6 for $F := f_*$ to show that its right adjoint $f^{(1)}$ preserves compacts, as desired.

Clearly (W4)⇒(W5) because of the ur-Wirthmüller (1.8) isomorphism. Now assume (W5) instead, *i.e.* that there exists a $\otimes$-invertible $\omega \in \mathcal{C}$ such that $f_*(\omega \otimes -)$

is left adjoint to f^*. Then the formulas (1.10) make sense with ω instead of ω_f, yielding well-defined functors $f^{(n)} : \mathcal{D} \to \mathcal{C}$ and $f_{(n)} : \mathcal{C} \to \mathcal{D}$ for all $n \in \mathbb{Z}$:

$$f^{(n)} := \omega^{\otimes n} \otimes f^* \qquad f_{(n)} := f_*(\omega^{\otimes n} \otimes -) \quad \text{with } f^{(0)} := f^* \text{ and } f_{(0)} := f_* \,.$$

Moreover, for all $n \in \mathbb{Z}$ we obtain the required adjunctions $f^{(n)} \dashv f_{(-n)} \dashv f^{(n+1)}$ by variously composing the adjunction $f^* \dashv f_*$ with the appropriate power of $(\omega_f \otimes -) \dashv (\omega_f^{-1} \otimes -)$ or $(\omega_f^{-1} \otimes -) \dashv (\omega_f \otimes -)$. Thus (W5)⇒(W6).

If (W6) holds then we have (W1)-(W2) because then every functor in the tower must preserve products, coproducts and compacts. The uniqueness of adjoints implies that whenever $f^* \dashv f_*$ sprouts a doubly infinite tower of adjoints this is necessarily given by the above formulas, because in that case ω_f is invertible (e.g. by the already proved implication (W6)⇒(W2)⇒(W3)⇒(W4)).

As the alert (or record-keeping) reader must have noticed, we have proved that the conditions (W1)-(W6) are all equivalent, and that they imply (1.10). It remains to verify the claimed sufficient conditions, *i.e.* the "finally" part.

For completeness, let us first recall (see Lemma 4.2 below) that f_* is faithful if and only if f^* is surjective on objects, up to direct summands. Moreover, if this is the case then the counit $\epsilon_x : f^* f_*(x) \to x$ is a split epi for all $x \in \mathcal{C}$. Therefore if $f_*(x)$ is compact then x must be as well, because f^* preserves compacts. This shows the implication (1)⇒(2). Clearly (2) implies (3), as they have the same conclusion but the hypothesis in (2) is weaker. To conclude the proof of the theorem, it remains only to show that (3) implies (W3). As we are in the context of Grothendieck-Neeman duality, we can use the ur-Wirthmüller equation $f_*(\omega_f \otimes x) \cong f_{(1)}(x)$. Since $f_{(1)}$ preserves compacts, it implies that $f_*(\omega_f \otimes x)$ is compact whenever x is a compact object of $\mathcal{C}$, so by (3) we can conclude that ω_f is compact, that is (W3). □

4.2. **Lemma.** *Let $F : \mathcal{S} \rightleftarrows \mathcal{T} : G$ be adjoint exact functors between triangulated categories. Then G is faithful if and only if F is $\oplus$-cofinal, that is, if and only if every object $x \in \mathcal{T}$ is a retract of $F(y)$ for some $y \in \mathcal{S}$. Moreover, this is equivalent to the counit of adjunction admitting an (unnatural) section at each object.*

Proof. (Cf. [Bal11, Prop. 2.10].) For any $x \in \mathcal{T}$, let $FG(x) \xrightarrow{\epsilon_x} x \xrightarrow{\alpha} y \to \Sigma FG(x)$ be an exact triangle containing $\epsilon_x : FG(x) \to x$, the counit of adjunction at x. Then $G(\epsilon_x)$ is a split epi by one of the unit-counit relations. Also, $\alpha\epsilon_x = 0$ hence $G(\alpha)G(\epsilon_x) = 0$. Together these facts imply $G(\alpha) = 0$. Now if G is faithful we have $\alpha = 0$ and therefore, by the exact triangle, ϵ_x is a split epi (cf. [Nee01, Cor. 1.2.7]). In particular, x is a retract of $FG(x)$. Hence G faithful implies that F is $\oplus$-cofinal.

Conversely, assume F is $\oplus$-cofinal: for every $x \in \mathcal{T}$ we can find an $x' \in \mathcal{T}$, a $y \in \mathcal{S}$, and an isomorphism $x \oplus x' \cong F(y)$. By the other unit-counit relation, the morphism $\epsilon_{F(y)} : FGF(y) \to F(y)$ is a split epi. By the naturality of ϵ and the additivity of the functors, the morphisms $\epsilon_{F(y)}$ and $\epsilon_x \oplus \epsilon_{x'}$ are isomorphic, hence ϵ_x must be an epi (indeed, a split epi, as these are the only epis in any triangulated category). As $x \in \mathcal{T}$ was arbitrary, this proves G faithful by [ML98, Thm. IV.3.1]. □

4.3. *Remark.* The article Fausk-Hu-May [FHM03] has a scope similar to ours, although it is not uniquely treating triangulated categories. Their approach is somewhat different, in that they assume given two pairs of adjoints: the original $f^* \dashv f_*$ and another one $f_! \dashv f^!$. This is motivated by the algebro-geometric situation or "Verdier-Grothendieck context". They consider two special cases:

(1) The case $f_! = f_*$, or "Grothendieck context", which reads $\qquad f^* \dashv f_* \dashv f^!$.

(2) The case $f^! = f^*$, or "Wirthmüller context", which reads $f_! \dashv f^* \dashv f_*$.

Note the collision of notation with $f_! \dashv f^!$ if you consider *both cases simultaneously*. Although not insurmountable, this collision unfortunately hides the Trichotomy of Corollary 1.14. In other words, Wirthmüller contexts are not "genuinely different" from Grothendieck contexts. They rather naturally appear as *special cases* thereof, at least for rigidly-compactly generated categories (thus in all examples of interest).

In their Grothendieck context (1), Fausk-Hu-May [FHM03, Thm. 8.4] reprove Grothendieck duality à la Neeman [Nee96], which we also reproved in Theorem 1.7. They do not seem to have observed our new condition (GN 3) in terms of f^*.

In their Wirthmüller context (2), they *assume* the existence of some object C with $f_*(\mathbb{1}) \simeq f_{(1)}(C)$. Then in [FHM03, Thm. 8.1], they establish a Wirthmüller isomorphism $f_* \simeq f_{(1)}(C \otimes -)$ comparable to our (W5). However, in each example, such a "Wirthmüller object" C needs to be constructed *by hand*. For instance, in equivariant stable homotopy, such a construction is done in a separate article [May03], sequel to [FHM03]. The advantage of our approach is that it avoids any *deus ex machina*: we do not need the friendly intervention of a mysterious object C at all. The relative dualizing object ω_f and the ur-Wirthmüller isomorphism (1.8) exist even when f^* has no left adjoint. When it does, *i.e.* under the assumptions of Theorem 1.9, we can simply take C to be the inverse of the relative dualizing object ω_f.

Finally, there is an indeterminacy in the Wirthmüller formalism proposed in [FHM03], in the case where $f_{(1)}$ has a non-zero kernel. Indeed, for any Z such that $f_{(1)}Z = 0$ the object $C' = C \oplus Z$ is as Wirthmüllery as C was. We did not find any example where "the" actual Wirthmüller object C is not rigid, or even $\otimes$-invertible. Hence, the following result essentially subsumes the Wirthmüller context of [FHM03] into ours (again, for rigidly-compactly generated categories).

4.4. **Proposition.** *Suppose that the basic adjunction $f^* : \mathcal{D} \rightleftarrows \mathcal{C} : f_*$ as in Hypothesis 1.2 fits in a "Wirthmüller context" in the sense of [FHM03], i.e. suppose that f^* has a left adjoint $f_{(1)}$ (denoted $f_!$ in [FHM03]) and that there exists an object $C \in \mathcal{C}$ such that $f_{(1)}(C) \simeq f_*(\mathbb{1})$. Then its dual $\underline{\hom}_{\mathcal{C}}(C, \mathbb{1})$ is isomorphic to ω_f.*

If moreover C is rigid (i.e. compact), as is commonly the case in examples, then ω_f and C are invertible and $C \cong \omega_f^{-1}$. In other words, a Wirthüller context with rigid Wirthmüller object C only happens in the case of the infinite tower of adjoints (Theorem 1.9) and then C must be the dual (inverse) of the canonical object ω_f.

Proof. By [FHM03, Thm. 8.1], we have a Wirthmüller isomorphism

$$f_{(1)}(C \otimes -) \simeq f_*.$$

Taking right adjoints (which exist by Theorem 1.3), we get

$$\underline{\hom}_{\mathcal{C}}(C, f^*(-)) \simeq f^{(1)}.$$

Evaluating at $\mathbb{1} \in \mathcal{D}$, we obtain the desired $\underline{\hom}_{\mathcal{C}}(C, \mathbb{1}) \simeq f^{(1)}(\mathbb{1}) \overset{\text{def}}{=} \omega_f$. If moreover C is rigid, then so is its dual ω_f. So, by Theorem 1.9, ω_f must be invertible. $\square$

4.5. *Example* (Equivariant homotopy theory). Let H be a closed subgroup of a compact Lie group G, and let $f^* : \mathrm{SH}(G) \to \mathrm{SH}(H)$ denote the restriction functor from the equivariant stable homotopy category of (genuine) G-spectra to that of H-spectra, as in Example 2.9(2). Then f^* provides an example of Theorem 1.9.

The relative dualizing object ω_f is the H-sphere S^L where L denotes the tangent H-representation of the smooth G-manifold G/H at the identity coset eH (see [LMSM86, Chapter III]). The ur-Wirthmüller isomorphism reads $G_+ \wedge_H X \simeq F_H(G_+, X \wedge S^L)$ and provides the well-known Wirthmüller isomorphism between induction and coinduction, up to a twist by S^L. If H has finite index in G (*e.g.* if G is a finite group) then $L = 0$ and $\omega_f \cong \mathbb{1}$.

4.6. *Example* (Finite group schemes). Let H be a closed subgroup of a finite group scheme G, and consider their stable representation categories, as in Example 2.9(3). As discussed in [Jan87, Chapter 8], the restriction functor $f^* : \mathrm{Stab}(kG) \to \mathrm{Stab}(kH)$ provides another example of Theorem 1.9. If δ_G denotes the unimodular character of the finite group scheme G then the relative dualizing object ω_f is $\delta_G|_H \cdot \delta_H^{-1}$. A finite group scheme is said to be "unimodular" if its unimodular character is trivial, which is equivalent to the group algebra being a symmetric algebra. This is the case for instance for (discrete) finite groups.

4.7. *Example* (Motivic homotopy theory). Let $\Bbbk$ be a field of characteristic zero, and let $\mathrm{SH}^{\mathbb{A}^1}(\Bbbk)$ denote the stable $\mathbb{A}^1$-homotopy category over $\Bbbk$, as in Example 2.9(4). For any finite extension $i : \Bbbk \hookrightarrow \Bbbk'$, the base change functor $i^* : \mathrm{SH}^{\mathbb{A}^1}(\Bbbk) \to \mathrm{SH}^{\mathbb{A}^1}(\Bbbk')$ provides another example of Theorem 1.9. In this example, the relative dualizing object ω_f is the unit object $\mathbb{1}$. See [Hu01] for further details.

Finally, we provide an example of a functor $f^* : \mathcal{D} \to \mathcal{C}$ satisfying Grothendieck-Neeman duality for which ω_f is *not* invertible.

4.8. *Example.* Let $\phi : B \to A$ be a morphism of commutative rings, as in Example 3.25, and assume that the induced pull-back functor $f^* = \mathrm{Spec}(\phi)^* = A \otimes_B^{\mathrm{L}} - : \mathrm{D}(B) \to \mathrm{D}(A)$ satisfies Grothendieck-Neeman duality, *i.e.* that A is (finite and) perfect over B (see Ex. 3.25). For simplicity assume that $B = \Bbbk$ is a field, so that A is a finite-dimensional commutative $\Bbbk$-algebra. In this case ω_f is the A-module $\mathrm{R}\,\mathrm{Hom}_B(A, B) = \mathrm{Hom}_{\Bbbk}(A, \Bbbk) =: A^\star$, the $\Bbbk$-linear dual of A, seen as an object of $\mathrm{D}(A)$. As A has zero Krull dimension, the Picard group of A is trivial, so ω_f is invertible (*i.e.* perfect!) only if $A^\star \cong A$ as A-modules. For an explicit example where this is *not* the case, we can take the three-dimensional $\Bbbk$-algebra $A := \Bbbk[t, s]/(s^2, t^2, st)$, which has the basis $\{1, s, t\}$; then $A^\star$ has the dual basis $\{1^\star, s^\star, t^\star\}$ and the A-action determined by $s{\cdot}s^\star = 1^\star$, $t{\cdot}t^\star = 1^\star$, $s{\cdot}t^\star = s{\cdot}1^\star = t{\cdot}s^\star = t{\cdot}1^\star = 0$. Since $(s \cdot A^\star) \cap (t \cdot A^\star) \ni 1^\star \neq 0$, this intersection is non-zero. On the other hand, $(s \cdot A) \cap (t \cdot A) = 0$, hence $A^\star \not\cong A$ as an A-module. (From a traditional point of view: if A is a finite-dimensional algebra, $A^\star$ has finite A-injective dimension and in fact any of its finite injective resolutions is a dualizing complex for A, in the classical sense; see [Bou07, Prop. X.9.1(b) and Ex. X.9.10(b)].)

5. Grothendieck duality on subcategories

In this section we consider subcategories $\mathcal{C}_0 \subset \mathcal{C}$ admitting a dualizing object κ, and how such structures behave with respect to our functors f^*.

5.1. *Definition.* Let $\mathcal{C}_0 \subset \mathcal{C}$ be a $\mathcal{C}^c$-*submodule*, *i.e.* a thick triangulated subcategory of our big category $\mathcal{C}$ such that $c \otimes x \in \mathcal{C}_0$ for all $x \in \mathcal{C}_0$ and all compact $c \in \mathcal{C}^c$.

An object $\kappa \in \mathcal{C}_0$ is called a *dualizing object for* $\mathcal{C}_0$ if the κ*-twisted duality* $\Delta_\kappa := \underline{\mathrm{hom}}_{\mathcal{C}}(-,\kappa)$ defines an anti-equivalence on $\mathcal{C}_0$:

$$\Delta_\kappa = \underline{\mathrm{hom}}_{\mathcal{C}}(-,\kappa) : \mathcal{C}_0^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_0. \tag{5.2}$$

In Section 7, we will consider the more general situation of an "external" dualizing object $\kappa \in \mathcal{C}$ by dropping the assumption that κ belongs to the subcategory $\mathcal{C}_0$ itself. (Note that if $\mathbb{1}$ belongs to $\mathcal{C}_0$ then necessarily $\kappa \cong \Delta_\kappa(\mathbb{1}) \in \mathcal{C}_0$.)

5.3. *Remark.* Because $\mathcal{C}(x, \Delta_\kappa(y)) \cong \mathcal{C}(x \otimes y, \kappa) \cong \mathcal{C}(y \otimes x, \kappa) \cong \mathcal{C}(y, \Delta_\kappa(x)) \cong \mathcal{C}^{\mathrm{op}}(\Delta_\kappa(x), y)$, we see that Δ_κ is adjoint to itself and we have a canonical natural morphism

$$\varpi_\kappa : x \longrightarrow \Delta_\kappa \Delta_\kappa(x) \tag{5.4}$$

for all $x \in \mathcal{C}$, which is both the unit and the counit of this self-adjunction. It satisfies $\Delta_\kappa(\varpi) \circ \varpi_{\Delta_\kappa} = \mathrm{id}_{\Delta_\kappa} : \Delta_\kappa \to \Delta_\kappa\Delta_\kappa\Delta_\kappa \to \Delta_\kappa$ by the unit-counit relation. We say that $x \in \mathcal{C}$ is κ*-reflexive* if this morphism ϖ_κ is an isomorphism.

5.5. **Lemma.** *For a $\mathcal{C}^c$-submodule $\mathcal{C}_0 \subset \mathcal{C}$, an object $\kappa \in \mathcal{C}_0$ is a dualizing object if and only if the following two conditions are satisfied:*

(i) *For every $x \in \mathcal{C}_0$, the κ-twisted dual $\Delta_\kappa(x) = \underline{\mathrm{hom}}_{\mathcal{C}}(x,\kappa)$ belongs to $\mathcal{C}_0$.*
(ii) *Every $x \in \mathcal{C}_0$ is κ-reflexive, i.e. $\varpi_\kappa : x \xrightarrow{\sim} \Delta_\kappa\Delta_\kappa(x)$.*

Proof. Conditions (i)-(ii) clearly imply the equivalence $\Delta_\kappa : \mathcal{C}_0^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_0$. Conversely, suppose that $\Delta_\kappa(\mathcal{C}_0) \subset \mathcal{C}_0$ as in (i). Then $\Delta_\kappa : \mathcal{C}_0^{\mathrm{op}} \rightleftarrows \mathcal{C}_0 : \Delta_\kappa$ is an adjunction whose unit and counit are isomorphisms by (ii). Hence Δ_κ is an equivalence. □

5.6. *Example.* For the subcategory $\mathcal{C}_0 = \mathcal{C}^c$ of all rigids, an object $\kappa \in \mathcal{C}_0$ is dualizing if and only if it is invertible (cf. [FHM03, Cor. 5.9]). In particular $\mathcal{C}_0 = \mathcal{C}^c$ always admits $\kappa = \mathbb{1}$ as dualizing object.

5.7. **Lemma.** *There is a canonical natural isomorphism $\Delta_\kappa(x) \otimes \Delta(c) \cong \Delta_\kappa(x \otimes c)$ for all $x, \kappa \in \mathcal{C}$ and all rigid $c \in \mathcal{C}^c$, where $\Delta = \Delta_{\mathbb{1}}$ is the usual dual.*

Proof. This is standard; see [HPS97, Thm. A.2.5.(d)]. The map $\underline{\mathrm{hom}}(x,\kappa) \otimes \underline{\mathrm{hom}}(c,\mathbb{1}) \to \underline{\mathrm{hom}}(x \otimes c, \kappa)$ is adjoint to the map $\underline{\mathrm{hom}}(x,\kappa) \otimes \underline{\mathrm{hom}}(c,\mathbb{1}) \otimes x \otimes c \cong \underline{\mathrm{hom}}(x,\kappa) \otimes x \otimes \underline{\mathrm{hom}}(c,\mathbb{1}) \otimes c \xrightarrow{\mathrm{ev} \otimes \mathrm{ev}} \kappa \otimes \mathbb{1} \cong \kappa$. □

5.8. *Remark.* It follows that if κ is a dualizing object for $\mathcal{C}_0$ then so is $\kappa \otimes u$ for every $\otimes$-invertible u. In algebraic geometry, dualizing complexes are unique up to tensoring by an invertible object; see [Nee10, Lem. 3.9]. For a general $\mathcal{C}_0$ this seems to be over-optimistic, although we can prove the following variant, replacing an equivalence of the form $u \otimes -$ by one of the form $\underline{\mathrm{hom}}_{\mathcal{C}}(u,-)$.

5.9. **Proposition.** *Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a $\mathcal{C}^c$-submodule containing $\mathcal{C}^c$ (that is, $\mathbb{1} \in \mathcal{C}_0$). Let κ and κ' be two dualizing objects for $\mathcal{C}_0$. Let $u := \Delta_{\kappa'}(\kappa) = \underline{\mathrm{hom}}_{\mathcal{C}}(\kappa,\kappa')$ and $v := \Delta_\kappa(\kappa') = \underline{\mathrm{hom}}_{\mathcal{C}}(\kappa',\kappa)$, both in $\mathcal{C}_0$. Then $v \cong \Delta u$ and $u \cong \Delta v$ and the restrictions to $\mathcal{C}_0$ of the functors $F_u := \underline{\mathrm{hom}}_{\mathcal{C}}(u,-)$ and $F_v := \underline{\mathrm{hom}}_{\mathcal{C}}(v,-)$ yield mutually inverse equivalences $F_u : \mathcal{C}_0 \overset{\sim}{\longleftrightarrow} \mathcal{C}_0 : F_v$ such that $\kappa \cong F_u(\kappa')$. Moreover, we have $\Delta_{\kappa'} \cong F_v \circ \Delta_\kappa \cong \Delta_\kappa \circ F_u$.*

Proof. Let us write $[-,-]$ for $\underline{\mathrm{hom}}_{\mathcal{C}}(-,-)$. From the equivalence $[-,\kappa] : \mathcal{C}_0^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_0$, one deduces a natural isomorphism

$$\beta : [x,y] \xrightarrow{\sim} \big[[y,\kappa],[x,\kappa]\big] \tag{5.10}$$

for $x, y \in \mathcal{C}_0$. Indeed, the morphism β is (double) adjoint to the (double) evaluation $[x, y] \otimes [y, \kappa] \otimes x \longrightarrow \kappa$. When tested on $\mathrm{Hom}_{\mathcal{C}}(c, -)$ for $c \in \mathcal{C}^c$, we obtain

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(c,[x,y]) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(c,\beta)"] & \cong & \mathrm{Hom}_{\mathcal{C}}(c\otimes x, y) \arrow[r, dotted] & \mathrm{Hom}_{\mathcal{C}}([y,\kappa],[c\otimes x,\kappa]) \\
\mathrm{Hom}_{\mathcal{C}}(c,\big[[y,\kappa],[x,\kappa]\big]) \;\cong\; \mathrm{Hom}_{\mathcal{C}}(c\otimes[y,\kappa],[x,\kappa]) & & & \;\cong\; \mathrm{Hom}_{\mathcal{C}}([y,\kappa], \Delta c\otimes[x,\kappa]). \arrow[u, "\cong"']
\end{tikzcd}$$

The dotted arrow thus obtained is nothing but the map on morphisms sets induced by the contravariant functor $[-, \kappa] : \mathcal{C}_0^{\mathrm{op}} \to \mathcal{C}_0$. Since the latter is an equivalence, this dotted map is a bijection for $c \in \mathcal{C}^c$ arbitrary, hence the morphism β is an isomorphism since $\mathcal{C}$ is compactly generated. As the same is true for $[-, \kappa']$ we have a natural isomorphism

$$\big[[y,\kappa],[x,\kappa]\big] \cong [x,y] \cong \big[[y,\kappa'],[x,\kappa']\big] \tag{5.11}$$

for all $x, y \in \mathcal{C}_0$. Since $\mathbb{1} \in \mathcal{C}_0$, we have $[\kappa, \kappa] \cong \Delta_\kappa^2(\mathbb{1}) \cong \mathbb{1}$ and similarly $[\kappa', \kappa'] \cong \mathbb{1}$. Thus, plugging $x = \kappa$ and $y = \kappa'$ (resp. $x = \kappa'$ and $y = \kappa$) in (5.11) we get the announced isomorphisms $u \cong \Delta v$ and $v \cong \Delta u$. Plugging instead $x = \mathbb{1}$ and $y = \kappa$ (resp. $y = \kappa'$) in (5.11), we obtain

$$\kappa \cong [u, \kappa'] \qquad \text{and} \qquad \kappa' \cong [v, \kappa]. \tag{5.12}$$

Now compute the composite equivalence $\mathcal{C}_0 \xrightarrow[\simeq]{[-,\kappa]} \mathcal{C}_0^{\mathrm{op}} \xrightarrow[\simeq]{[-,\kappa']} \mathcal{C}_0$. For all $x \in \mathcal{C}_0$,

$$\big[[x,\kappa],\kappa'\big] \overset{(5.12)}{\cong} \big[[x,\kappa],[v,\kappa]\big] \overset{(5.10)}{\cong} [v, x].$$

This proves that $F_v := [v, -] \cong \Delta_{\kappa'}\Delta_\kappa$ and $F_u := [u, -] \cong \Delta_\kappa\Delta_{\kappa'}$ define equivalences on $\mathcal{C}_0$, which satisfy the desired relations since $\Delta_\kappa^2 \cong \mathrm{Id} \cong \Delta_{\kappa'}^2$. □

5.13. *Remark.* One can deduce from Proposition 5.9 that $u = \underline{\mathrm{hom}}_{\mathcal{C}}(\kappa, \kappa')$ is $\otimes$-invertible, and that $\kappa \otimes u \cong \kappa'$, if $\mathcal{C}_0$ satisfies any of the following properties:

(i) $u \otimes \mathcal{C}_0 \subset \mathcal{C}_0$ (for instance if u belongs to $\mathcal{C}^c$);
(ii) $\mathcal{C}_0$ cogenerates $\mathcal{C}$, *i.e.* for $t \in \mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(t, x) = 0$ for all $x \in \mathcal{C}_0$ then $t = 0$;
(iii) if an object $a \in \mathcal{C}$, not necessarily in $\mathcal{C}_0$, admits a natural isomorphism $\underline{\mathrm{hom}}_{\mathcal{C}}(a, x) \cong x$ for all $x \in \mathcal{C}_0$, then $a \cong \mathbb{1}$.

This is left as an easy exercise for the interested reader. Note in particular that condition (i) holds if $\mathcal{C}_0$ satisfies $\mathcal{C}_0 \otimes \mathcal{C}_0 \subseteq \mathcal{C}_0$. This assumption appears, for example, in [BD13]; however, $\mathcal{C}_0 \otimes \mathcal{C}_0 \subset \mathcal{C}_0$ is not true in algebraic geometry for $\mathcal{C}_0 = \mathrm{D^b}(\mathrm{coh}\, X)$. Nevertheless, this important example can also be derived from Proposition 5.9:

5.14. **Corollary** ([Nee10, Lem. 3.9])**.** *If X is a noetherian scheme admitting two dualizing complexes κ and κ', then there exists a $\otimes$-invertible $\ell \in \mathrm{D^{perf}}(X)$ (a shift of a line bundle on each connected component of X) such that $\kappa' \cong \kappa \otimes \ell$.*

Proof. According to the definition used in [Nee10], which is slightly more general than the classical one, a dualizing complex for X is an (internal) dualizing object for $\mathcal{C}_0 = \mathrm{D^b}(\mathrm{coh}\, X)$. It suffices to prove that $\ell := u = \underline{\mathrm{hom}}(\kappa, \kappa')$ is $\otimes$-invertible, since Proposition 5.9 then implies that $\kappa \cong \underline{\mathrm{hom}}(u, \kappa') \cong \Delta u \otimes \kappa' = u^{-1} \otimes \kappa'$. To this end, we appeal to the following criterion: over a noetherian scheme X, a complex $x \in \mathrm{D^b}(\mathrm{coh}\, X)$ is $\otimes$-invertible iff both

(a) x is ($\mathbb{1}$-)reflexive, $x \xrightarrow{\sim} \Delta^2(x)$, and
(b) $\mathbb{1}$ is x-reflexive, $\mathbb{1} \xrightarrow{\sim} \Delta_x^2(\mathbb{1}) = \underline{\mathrm{hom}}(x,x)$.

For the affine case, see [AIL10, Cor. 5.7]. The criterion globalizes because the canonical morphism ϖ commutes with localization to an open subscheme [AIL11, §1.3 and Rem. 1.5.5]; cf. [CH09, Thm. 4.1.2].

By Proposition 5.9, we indeed have isomorphisms $u \cong \Delta(v) \cong \Delta^2(u)$ and $\mathbb{1} \cong \underline{\mathrm{hom}}(u,u)$. In order to conclude that u is $\otimes$-invertible by the above criterion, we must prove that the latter isomorphisms are instances of the canonical maps ϖ. This verification is easy for the second map, but for the first one it appears to be rather involved. Fortunately, we can avoid it altogether: by the local nature of ϖ, we may reduce to the affine case where, by [AIL10, Prop. 2.3], the existence of *any* isomorphism $x \cong \Delta_y^2(x)$ implies that x is y-reflexive (for $x, y \in \mathrm{D^b}(\mathrm{coh}\, X)$). □

* * *

Let us now "move" the above subcategories with duality under a general tensor-exact functor $f^* : \mathcal{D} \to \mathcal{C}$ as in Hypothesis 1.2. Recall that by Corollary 2.10 we have the two adjunctions $f^* \dashv f_* \dashv f^{(1)}$ as well as their internal realizations (2.13) and (2.14). Note that the latter can be written as a natural isomorphism

$$\Delta_\kappa(f_* x) \cong f_* \Delta_{f^{(1)}(\kappa)}(x) \tag{5.15}$$

for all $x \in \mathcal{C}$ and $\kappa \in \mathcal{D}$.

5.16. *Remark.* It can be proved that the isomorphism (5.15) is compatible with the canonical maps ϖ of Δ_κ and $\Delta_{\kappa'}$, thus turning f_* into a duality-preserving functor in the technical sense of [CH09]. We will make this fact precise when it will be needed, in Section 7.

5.17. *Definition.* If $\mathcal{D}_0$ is a $\mathcal{D}^c$-submodule of $\mathcal{D}$, we define its *compact pull-back along* f_* as the following full subcategory of $\mathcal{C}$:

$$f^\#(\mathcal{D}_0) := \{x \in \mathcal{C} \mid f_*(c \otimes x) \in \mathcal{D}_0 \text{ for all } c \in \mathcal{C}^c\}.$$

One sees immediately that $f^\#(\mathcal{D}_0)$ is a $\mathcal{C}^c$-submodule of $\mathcal{C}$, because $\mathcal{C}^c$ is one.

We can use this compact pull-back $f^\#$ to rephrase Grothendieck-Neeman duality:

5.18. **Proposition.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ be as in Hypothesis 1.2.*

(a) *The functor f^* satisfies Grothendieck-Neeman duality (Theorem 3.4) if and only if the following inclusion holds: $\mathcal{C}^c \subset f^\#(\mathcal{D}^c)$.*
(b) *If $\mathcal{C}^c = f^\#(\mathcal{D}^c)$ then we have the Wirthmüller isomorphism (Theorem 1.9).*

Proof. Hypothesis (GN 2) of Theorem 1.7 reads $f_*(c) \in \mathcal{D}^c$ for all $c \in \mathcal{C}^c$. Since $\mathbb{1} \in \mathcal{C}^c$ and since $\mathcal{C}^c$ is stable under the tensor product, this condition is equivalent to $f_*(c \otimes x) \in \mathcal{D}^c$ for all $x, c \in \mathcal{C}^c$. The latter exactly means $\mathcal{C}^c \subset f^\#(\mathcal{D}^c)$ by Definition 5.17. Hence (a). For (b), it suffices to note that $\mathcal{C}^c \supset f^\#(\mathcal{D}^c)$ is precisely the sufficient condition (3) in Theorem 1.9. □

Furthermore, compact pullback is compatible with composition of functors:

5.19. **Proposition.** *Consider two composable functors $\mathcal{E} \xrightarrow{g^*} \mathcal{D} \xrightarrow{f^*} \mathcal{C}$, both satisfying Hypothesis 1.2 and with composite $f^* g^* =: (gf)^*$, and let $\mathcal{E}_0$ be any $\mathcal{E}^c$-submodule of $\mathcal{E}$. Then $(gf)^\#(\mathcal{E}_0) = f^\#(g^\#(\mathcal{E}_0))$.*

Proof. Notice that the composite $(gf)^*$ also satisfies Hypothesis 1.2, and that its right adjoint must be $(gf)_* \cong g_*f_*$ by the uniqueness of adjoints. Thus $x \in \mathcal{C}$ belongs to $(gf)^{\#}(\mathcal{E}_0)$ iff $g_*f_*(c \otimes x) \in \mathcal{E}_0$ for all $c \in \mathcal{C}^c$.

On the other hand: $x \in f^{\#}(g^{\#}(\mathcal{E}_0))$ iff $f_*(c \otimes x) \in g^{\#}(\mathcal{E}_0)$ for all $c \in \mathcal{C}^c$, iff $g_*(d \otimes f_*(c \otimes x)) \in \mathcal{E}_0$ for all $c \in \mathcal{C}^c$ and $d \in \mathcal{D}^c$. By the projection formula (2.12), we see that $g_*(d \otimes f_*(c \otimes x)) \cong g_*f_*(f^*(d) \otimes c \otimes x)$.

Since each $f^*(d) \otimes c$ as above is compact in $\mathcal{C}$, and since every compact of $\mathcal{C}$ has this form (choose $d = \mathbb{1}_{\mathcal{D}}$), the two conditions on x are equivalent. □

Let us give an example of this compact pullback $f^{\#}$ in algebraic geometry.

5.20. **Theorem.** *Let $f : X \to Y$ be a morphism of noetherian schemes and let $f^* : \mathcal{D} := \mathrm{D}_{\mathrm{Qcoh}}(Y) \longrightarrow \mathrm{D}_{\mathrm{Qcoh}}(X) =: \mathcal{C}$ be the induced functor, which satisfies Hypothesis 1.2 by Example 3.24.*

(a) *Suppose that $f : X \to Y$ is proper. Then $f_* : \mathcal{C} \to \mathcal{D}$ maps $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ into $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y)$. Moreover, for every object $x \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ and every perfect $c \in \mathrm{D}_{\mathrm{Qcoh}}(X)^c$ we have $f_*(c \otimes x) \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y)$.*

(b) *Suppose that $f : X \to Y$ is projective. Then the following converse to (a) holds: If $x \in \mathrm{D}_{\mathrm{Qcoh}}(X)$ is such that $f_*(c \otimes x) \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y)$ for every perfect $c \in \mathrm{D}_{\mathrm{Qcoh}}(X)^c$ then $x \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$.*

In the notation of Definition 5.17, we have for $f : X \to Y$ projective that

$$f^{\#}\big(\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y)\big) = \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X).$$

Proof. For (a), the question being local in the base Y, we can assume that $Y = \mathrm{Spec}(A)$ is affine. For any coherent sheaf $\mathcal{F} \in \mathrm{coh}(X)$, the A-modules $\mathrm{R}^i f_*\mathcal{F}$ are finitely generated and vanish for $i \gg 0$, by [Gro63, III.3.2.3]. It follows that $f_*\mathcal{F} = \mathrm{R}f_*\mathcal{F}$ is bounded coherent. Hence so is $f_*(x)$ for any $x \in \mathrm{D}_{\mathrm{Qcoh}}(X)$ contained in the thick subcategory generated by coherent sheaves, which is precisely $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$. The "moreover part" follows immediately since $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ is a $\mathrm{D}^{\mathrm{perf}}(X)$-submodule of $\mathrm{D}_{\mathrm{Qcoh}}(X)$, where $\mathrm{D}^{\mathrm{perf}}(X) = \mathrm{D}_{\mathrm{Qcoh}}(X)^c$.

For (b), in view of Proposition 5.19, and since we can decompose f into a closed immersion followed by the structure morphism $\mathbb{P}^n_Y \to Y$, we treat the two cases separately. For $f : X \to Y$ a closed immersion, the result is straightforward since f_* itself detects boundedness and coherence. (One can reduce to Y affine – in any case, f_* commutes with taking homology and detects coherence, so $f_*(x) \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y)$ forces x to be bounded with coherent homology, *i.e.* to be in $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$.) For the projection $f : \mathbb{P}^n_Y \to Y$, we can use the resolution of the diagonal $\mathcal{O}_{\Delta}$ in $\mathbb{P}^n \times_Y \mathbb{P}^n$ by objects of the form $p_1^*d_i \otimes p_2^*c_i$ where $p_i : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ are the two projections and $d_i = \Omega^i(i)$ and $c_i = \mathcal{O}(-i)$ are vector bundles over $\mathbb{P}^n$ and in particular compact objects; see [Beĭ84]. Hence, since every $x \in \mathrm{D}_{\mathrm{Qcoh}}(X)$ is isomorphic to $(p_2)_*(\mathcal{O}_{\Delta} \otimes p_1^*(x))$, we see that x belongs to the thick subcategory generated by the $(p_2)_*\big(p_1^*d_i \otimes p_2^*c_i \otimes p_1^*(x)\big)$. Computing the latter using the projection formula and flat base change (in the cartesian square for $\mathbb{P}^n \times_Y \mathbb{P}^n$ over Y), we get

$$(p_2)_*\big(p_1^*d_i \otimes p_2^*c_i \otimes p_1^*(x)\big) \;\cong\; (p_2)_*(p_1^*(x \otimes d_i)) \otimes c_i \;\cong\; f^*(f_*(x \otimes d_i)) \otimes c_i\,.$$

In particular, as soon as $f_*(x \otimes d_i) \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y)$ then all the objects above belong to $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathbb{P}^n_Y)$, hence so does x itself. □

5.21. **Corollary.** *Let $f : X \to S$ be a projective morphism of noetherian schemes, with S regular (for instance $S = \mathrm{Spec}(\Bbbk)$ for $\Bbbk$ a field). Then $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ is equal to $\big\{\, x \in \mathrm{D}_{\mathrm{Qcoh}}(X) \,\big|\, f_*(c \otimes x) \in \mathrm{D}^{\mathrm{perf}}(S) \text{ for all } c \in \mathrm{D}^{\mathrm{perf}}(X) \,\big\}$.*

Proof. In this case, $\mathrm{D}^{\mathrm{perf}}(S) = \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, S)$ and we can apply Theorem 5.20. □

* * *

We now turn to the interaction between the two notions discussed above, namely that of dualizing object and that of compact pullback.

5.22. **Theorem.** *Let $f^*\colon \mathcal{D} \to \mathcal{C}$ be as in Hypothesis 1.2. Let $\mathcal{D}_0 \subset \mathcal{D}$ be a $\mathcal{D}^c$-submodule equipped with a dualizing object $\kappa \in \mathcal{D}_0$ (Def. 5.1) and consider the following two possible properties of an object $x \in \mathcal{C}$:*

(i) *$x \in f^{\#}(\mathcal{D}_0)$.*
(ii) *x is $f^{(1)}(\kappa)$-reflexive: $x \xrightarrow{\sim} \Delta_{f^{(1)}(\kappa)}\Delta_{f^{(1)}(\kappa)}(x)$.*

Then we have :

(a) *If $\mathbb{1} \in \mathcal{D}_0$ then (i) implies (ii).*
(b) *If $\mathcal{D}_0$ consists precisely of the κ-reflexive objects of $\mathcal{D}$, then (ii) implies (i).*

Proof. Let us prove (i)⇒(ii) when $\mathbb{1} \in \mathcal{D}_0$, and write $\kappa' := f^{(1)}(\kappa)$ for short. Let $x, y, c \in \mathcal{C}$ with c compact, y arbitrary, and $x \in f^{\#}(\mathcal{D}_0)$, which implies in particular that $f_*(x \otimes \Delta c) \in \mathcal{D}^c$. We obtain the following isomorphism:

$$\begin{array}{rcll}
\mathcal{C}(c, x) & \cong & \mathcal{C}(\mathbb{1}, x \otimes \Delta c) & c \in \mathcal{C}^c \text{ is rigid} \\
& \cong & \mathcal{D}\big(\mathbb{1}, f_*(x \otimes \Delta c)\big) & \mathbb{1} \cong f^*\mathbb{1} \text{ and } f^* \dashv f_* \\
& \cong & \mathcal{D}\big(\Delta_\kappa f_*(x \otimes \Delta c), \kappa\big) & \mathbb{1} \text{ and } f_*(x \otimes \Delta c) \in \mathcal{D}_0 \text{ and } \Delta_\kappa : \mathcal{D}_0^{\mathrm{op}} \xrightarrow{\sim} \mathcal{D}_0 \\
& \cong & \mathcal{D}\big(f_*\Delta_{\kappa'}(x \otimes \Delta c), \kappa\big) & \text{by (5.15), special case of (2.14)} \\
& \cong & \mathcal{C}\big(\Delta_{\kappa'}(x \otimes \Delta c), \kappa'\big) & f_* \dashv f^{(1)} \text{ and } f^{(1)}(\kappa) = \kappa' \\
& \cong & \mathcal{C}\big(\Delta_{\kappa'}(x) \otimes c, \kappa'\big) & c \in \mathcal{C}^c \text{ is rigid and Lemma 5.7} \\
& \cong & \mathcal{C}(c, \Delta_{\kappa'}\Delta_{\kappa'}(x)) & \otimes \dashv \underline{\mathrm{hom}}_{\mathcal{C}}\,.
\end{array}$$

By following through this composite isomorphism, one can check that it is induced by the canonical map (5.4). Indeed, choosing an arbitrary morphism $\varphi : c \to x$ and writing $[-,-]$ for $\underline{\mathrm{hom}}(-,-)$ to save space on the page, the relevant diagram can be checked using the naturality of the isomorphism $f_*[a, f^{(1)}b] \cong [f_*a, b]$ in (2.14) with respect to the morphism $\mathbb{1} \xrightarrow{\eta} c \otimes \Delta c \xrightarrow{\varphi \otimes 1} x \otimes \Delta c$, together with the following three diagrams:

$$\begin{tikzcd}
{[c,\kappa'] \otimes c} \arrow[r, "\sim", "5.7"'] \arrow[d, "\mathrm{ev}"] & {[c \otimes \Delta c, \kappa']} \arrow[d, "{[\eta,1]}"] \\
\kappa' \arrow[r, "\sim"] & {[\mathbb{1}, \kappa']}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\mathbb{1},\kappa']} \arrow[r, "f_* \dashv f^{(1)}"'] \arrow[d, "\sim"] & f^{(1)} f_*[\mathbb{1},\kappa'] \arrow[r, "\sim", "(2.14)"'] & f^{(1)}[f_*\mathbb{1}, \kappa] \arrow[d, "f^* \dashv f_*"'] \\
\kappa' \arrow[r, equal] & f^{(1)}\kappa \arrow[r, "\sim"] & f^{(1)}[\mathbb{1}, \kappa]
\end{tikzcd}$$

$$\begin{tikzcd}
c \arrow[r, "\varphi"] \arrow[d, "\mathrm{coev}"] & x \arrow[r, "\mathrm{coev}"] & {[[x,\kappa'],[x,\kappa'] \otimes x]} \arrow[d, "{[1,\mathrm{ev}]}"] \\
{[[x,\kappa'],[x,\kappa'] \otimes c]} \arrow[r, "{[1,[\varphi,1]\otimes 1]}"] & {[[x,\kappa'],[c,\kappa'] \otimes c]} \arrow[r, "{[1,\mathrm{ev}]}"] & {[[x,\kappa'],\kappa']].}
\end{tikzcd}$$

The first diagram can be checked using the definition of $[c,\kappa']\otimes c\cong[c\otimes\Delta c,\kappa']$ (cf. Lemma 5.7) and dinaturality of coevaluation with respect to $\eta:\mathbb{1}\to c\otimes\Delta c$. Similarly, the second diagram can be checked using the definition of $f_*[\mathbb{1},\kappa']\cong[f_*\mathbb{1},\kappa]$ together with dinaturality of coevaluation with respect to $\mathbb{1}\to f_*\mathbb{1}$. Finally, the last diagram follows from dinaturality of (co)evaluation and naturality of evaluation with respect to $\varphi:c\to x$. As $\mathcal{C}$ is compactly generated, the canonical map (5.4) is invertible for any x, as claimed.

In order to prove the conditional implication (ii)⇒(i), assume that $x\in\mathcal{C}$ is κ'-reflexive and let $c\in\mathcal{C}^c$ be a compact object. We must show that $f_*(x\otimes c)$ belongs to $\mathcal{D}_0$. By the extra hypothesis, it suffices to show that $f_*(x\otimes c)\in\mathcal{D}$ is κ-reflexive. We have the following composite isomorphism starting with κ'-reflexivity of x and $\mathbb{1}$-reflexivity of c:

$$\begin{array}{rcll} f_*(x\otimes c) & \cong & f_*\big(\Delta_{\kappa'}\Delta_{\kappa'}(x)\otimes\Delta\Delta(c)\big) & \\ & \cong & f_*\Delta_{\kappa'}\Delta_{\kappa'}(x\otimes c) & \text{Lemma 5.7 twice} \\ & \cong & \Delta_\kappa\Delta_\kappa f_*(x\otimes c) & \text{(5.15) twice.} \end{array}$$

To check that this isomorphism $f_*(x\otimes c)\xrightarrow{\sim}\Delta_\kappa\Delta_\kappa f_*(x\otimes c)$ coincides with the canonical map (5.4), it suffices to check that it is adjoint to the evaluation map $[f_*(x\otimes c),\kappa]\otimes f_*(x\otimes c)\xrightarrow{\text{ev}}\kappa$. This can be accomplished from the definitions by using dinaturality of evaluation with respect to $f_*\Delta_{\kappa'}(x\otimes c)\cong\Delta_\kappa(f_*(x\otimes c))$ together with the following two diagrams:

$$\begin{tikzcd} f_*[a,f^{(1)}(b)]\otimes f_*(a) \arrow[r, "\text{lax}"] \arrow[d, "\cong"] & f_*([a,f^{(1)}(b)]\otimes a] \arrow[r, "f_*\,\text{ev}"] & f_*f^{(1)}(b) \arrow[d, "\epsilon"] \\ {[f_*(a),b]}\otimes f_*(a) \arrow[rr, "\text{ev}"] & & b \end{tikzcd}$$

and

$$\begin{tikzcd} \Delta_\kappa(x\otimes c)\otimes x\otimes c \arrow[r] \arrow[ddd, "\cong"'] & \Delta_\kappa(x\otimes c)\otimes\Delta_\kappa^2x\otimes\Delta^2c \arrow[d, "\text{switch}"] \\ & \Delta_\kappa^2x\otimes\Delta^2c\otimes\Delta_\kappa(x\otimes c) \arrow[d, "\cong"'] \\ & \\ \kappa & \Delta_\kappa^2(x\otimes c)\otimes\Delta_\kappa(x\otimes c). \arrow[l, "\text{ev}"'] \end{tikzcd}$$

The first diagram can be checked using the definition of the maps $x\to\Delta_\kappa^2x$, $c\to\Delta^2c$ together with the fact that

$$\begin{tikzcd} {[a,b]}\otimes[a',b']\otimes a\otimes a' \arrow[r, "\text{switch}"] \arrow[d, "\cong"'] & {[a,b]}\otimes a\otimes[a',b']\otimes a' \arrow[d, "\text{ev}\otimes\text{ev}"] \\ {[a\otimes a',b\otimes b']}\otimes a\otimes a' \arrow[r, "\text{ev}"] & b\otimes b' \end{tikzcd}$$

commutes (which can be checked from the definition). The second diagram can be checked in a straightforward manner by using the definition of $f_*[a,f^!b]\cong[f_*a,b]$ in (2.14).

We conclude that indeed $f_*(x\otimes c)$ is κ-reflexive and therefore belongs to $\mathcal{D}_0$. $\square$

The next theorem is the main result of this section.

5.23. **Theorem** (Grothendieck duality). *Let $f^*\colon \mathcal{D} \to \mathcal{C}$ be as in our basic Hypothesis 1.2 and let $\kappa \in \mathcal{D}$. Recall $f^* \dashv f_* \dashv f^{(1)}$ from Corollary 2.10 and set $\kappa' := f^{(1)}(\kappa)$. Then we have a natural isomorphism*

$$\Delta_\kappa(f_* x) \cong f_* \Delta_{\kappa'}(x)$$

for all $x \in \mathcal{C}$. Suppose moreover that f^* satisfies Grothendieck-Neeman duality (Theorem 1.7) and that $\mathcal{D}_0 \subset \mathcal{D}$ is a $\mathcal{D}^c$-submodule which admits $\kappa \in \mathcal{D}_0$ as dualizing object (Definition 5.1). Then

$$\kappa' \cong \omega_f \otimes f^*(\kappa)$$

is a dualizing object for the $\mathcal{C}^c$-submodule $f^\#(\mathcal{D}_0) \subset \mathcal{C}$ (Definition 5.17). In particular, $f_* : f^\#(\mathcal{D}_0) \longrightarrow \mathcal{D}_0$ is a duality-preserving exact functor between categories with duality, where $f^\#(\mathcal{D}_0)$ is equipped with the duality $\Delta_{\kappa'}$ and $\mathcal{D}_0$ with Δ_κ.

Proof. The first formula is (5.15). Assuming Grothendieck-Neeman duality, we have $f^{(1)} \cong \omega_f \otimes f^*$ by formula (3.5). So $\kappa' \cong \omega_f \otimes f^*(\kappa)$ in that case. Moreover, by Theorem 5.22, every object of $f^\#(\mathcal{D}_0)$ is κ'-reflexive, hence by Lemma 5.5 it remains to prove that $\kappa' = \omega_f \otimes f^*\kappa$ belongs to $f^\#(\mathcal{D}_0)$ and that $\Delta_{\kappa'}$ preserves $f^\#(\mathcal{D}_0)$. Indeed for every compact $c \in \mathcal{C}^c$, we can use the projection formula (2.12) and the ur-Wirthmüller formula (1.8) to compute

$$f_*(\omega_f \otimes f^*\kappa \otimes c) \;\;\cong\;\; f_*(\omega_f \otimes c) \otimes \kappa \;\;\cong\;\; f_{(1)}(c) \otimes \kappa\,;$$

this object belongs to $\mathcal{D}_0$ because κ does, since $f_{(1)}$ preserves compacts and $\mathcal{D}_0$ is a $\mathcal{C}^c$-submodule. Hence $\kappa' \in f^\#(\mathcal{D}_0)$. Finally, let us show that $\Delta_{\kappa'}$ preserves $f^\#(\mathcal{D}_0)$. For $x \in f^\#(\mathcal{D}_0)$ and $c \in \mathcal{C}^c$ we compute, using first Lemma 5.7 and the rigidity of $c \in \mathcal{C}^c$ and then (5.15):

$$f_*(c \otimes \Delta_{\kappa'}(x)) \;\;\cong\;\; f_*(\Delta_{\kappa'}(x \otimes \Delta c)) \;\;\cong\;\; \Delta_\kappa(f_*(x \otimes \Delta c))\,.$$

The latter belongs to $\mathcal{D}_0$ since $f_*(x \otimes \Delta c)$ does by definition of $x \in f^\#(\mathcal{D}_0)$ and since Δ_κ preserves $\mathcal{D}_0$ by hypothesis. □

6. Categories over a base and relative compactness

We now want to analyse a relative setting.

6.1. *Definition.* Let $\mathcal{B}$ be a rigidly-compactly generated tensor triangulated category that we call the "base." We say that $\mathcal{C}$ is a $\mathcal{B}$-*category* if it comes equipped with a *structure* functor $p^* : \mathcal{B} \to \mathcal{C}$ satisfying Hypothesis 1.2.

6.2. *Definition.* A *morphism of $\mathcal{B}$-categories* $f^* : (\mathcal{D}, q^*) \to (\mathcal{C}, p^*)$ is a functor $f^* : \mathcal{D} \to \mathcal{C}$ satisfying Hypothesis 1.2 together with an isomorphism $f^*q^* \cong p^*$. By the uniqueness property of adjoint functors, this canonically spawns isomorphisms $q_* f_* \cong p_*$ and $f^{(1)} q^{(1)} \cong p^{(1)}$. In particular, we have an isomorphism in $\mathcal{C}$:

$$f^{(1)}(\omega_q) = f^{(1)} q^{(1)}(\mathbb{1}_\mathcal{B}) \cong p^{(1)}(\mathbb{1}_\mathcal{B}) = \omega_p\,. \tag{6.3}$$

We can then prove the following generalization of Theorem 5.23, in which we do not assume that f^* satisfies Grothendieck-Neeman duality, but only that its source and target do, with respect to their base.

6.4. **Theorem.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ be a morphism of $\mathcal{B}$-categories (Def. 6.2). Assume that the structure morphisms $p^* : \mathcal{B} \to \mathcal{C}$ and $q^* : \mathcal{B} \to \mathcal{D}$ satisfy Grothendieck-Neeman duality (Thm. 1.7). Let $\mathcal{B}_0 \subset \mathcal{B}$ be a $\mathcal{B}^c$-subcategory with dualizing object $\kappa \in \mathcal{B}_0$ (Def. 5.1). Let $\mathcal{C}_0 = p^{\#}\mathcal{B}_0$ and $\mathcal{D}_0 = q^{\#}\mathcal{B}_0$ be its compact pullbacks in $\mathcal{C}$ and $\mathcal{D}$ respectively (Def. 5.17), which admit the dualizing objects*

$$\gamma := \omega_p \otimes p^*(\kappa) \in \mathcal{C}_0 \qquad \text{and} \qquad \delta := \omega_q \otimes q^*(\kappa) \in \mathcal{D}_0$$

respectively, by Theorem 5.23. Then $f^{\#}(\mathcal{D}_0) = \mathcal{C}_0$ and f_* restricts to a well-defined exact functor $f_* : \mathcal{C}_0 \to \mathcal{D}_0$ which is duality-preserving with respect to Δ_γ and Δ_δ.

Proof. Note that $\gamma = \omega_p \otimes p^*(\kappa) \cong p^{(1)}(\kappa) \cong f^{(1)}q^{(1)}(\kappa) \cong f^{(1)}(\delta)$. So we already know from (5.15) that f_* will be compatible with the dualities Δ_γ and Δ_δ. By Proposition 5.19, we know that $\mathcal{C}_0 = p^{\#}\mathcal{B}_0 = f^{\#}q^{\#}\mathcal{B}_0 = f^{\#}\mathcal{D}_0$. It follows from this and the definition of $f^{\#}\mathcal{D}_0$ that $f_*(\mathcal{C}_0) \subseteq \mathcal{D}_0$ yielding the desired $f_* : \mathcal{C}_0 \to \mathcal{D}_0$. □

* * *

An example of the above relative discussion over a base category $\mathcal{B}$ is the situation where $\mathcal{B}_0 = \mathcal{B}^c$ and $\kappa = \mathbb{1}$. In other words, we can assume that the base is sufficiently simple that the duality question over $\mathcal{B}$ is solved in the "trivial" way, as in Example 5.6. This is interesting in algebraic geometry when $\mathcal{B} = \mathrm{D}_{\mathrm{Qcoh}}(S)$ for S regular, as we saw in Corollary 5.21, for instance when $S = \mathrm{Spec}(\Bbbk)$ for $\Bbbk$ a field. In that case, it is not a restriction to consider the trivial duality on $\mathcal{B}$, with $\mathcal{B}_0 = \mathcal{B}^c$ and $\kappa = \mathbb{1}$. We can then pull it back to obtain a more interesting subcategory with duality $\mathcal{C}_0 = p^{\#}(\mathcal{B}_0)$ in $\mathcal{C}$.

6.5. *Definition.* Let $\mathcal{C}$ be a $\mathcal{B}$-*category* as in Definition 6.1. We define the full subcategory of $\mathcal{C}$ of $\mathcal{B}$-*relatively compact* objects to be

$$\mathcal{C}^{c/p} := p^{\#}(\mathcal{B}^c) = \left\{ x \in \mathcal{C} \,\middle|\, p_*(c \otimes x) \in \mathcal{B}^c \text{ for all compact } c \in \mathcal{C}^c \right\}.$$

We can then rephrase the above results in this setting.

6.6. **Corollary.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ be a morphism of $\mathcal{B}$-categories (Def. 6.2) with structure morphisms $p^* : \mathcal{B} \to \mathcal{C}$ and $q^* : \mathcal{B} \to \mathcal{D}$.*

(a) *Compact pullback (Def. 5.17) preserves the subcategories of $\mathcal{B}$-relatively compact objects: $f^{\#}(\mathcal{D}^{c/q}) = \mathcal{C}^{c/p}$.*
(b) *Suppose that $p^* : \mathcal{B} \to \mathcal{C}$ satisfies Grothendieck-Neeman duality. Then $\omega_p = p^{(1)}(\mathbb{1})$ is a dualizing object for the subcategory of relatively compact objects $\mathcal{C}^{c/p}$ (Def. 6.5).*
(c) *Suppose that p^* and q^* satisfy Grothendieck-Neeman duality. Then the functor f_* restricts to an exact functor $f_* : \mathcal{C}^{c/p} \to \mathcal{D}^{c/q}$ which is duality-preserving with respect to the dualities Δ_{ω_p} and Δ_{ω_q}.*

Proof. Proposition 5.19 gives (a). Theorem 5.23 applied to p^* gives (b). Theorem 6.4 gives (c). □

* * *

6.7. *Remark.* If $f^* : \mathcal{D} \rightleftarrows \mathcal{C} : f_*$ is an adjunction between closed tensor categories with f^* a tensor functor, then $\mathcal{C}$ inherits an enrichment over $\mathcal{D}$ (see [Kel05]): the Hom-objects are given by $\underline{\mathcal{C}}(x,y) := f_*\,\underline{\mathrm{hom}}_{\mathcal{C}}(x,y) \in \mathcal{D}$, and the unit and composition morphisms $\mathbb{1}_{\mathcal{D}} \to \underline{\mathcal{C}}(x,y)$ and $\underline{\mathcal{C}}(y,z) \otimes_{\mathcal{D}} \underline{\mathcal{C}}(x,y) \to \underline{\mathcal{C}}(x,z)$ in $\mathcal{D}$ are obtained by adjunction in the evident way from the $\mathcal{C}$-internal unit and composition maps.

6.8. **Theorem** (Relative Serre duality)**.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ be a functor as in Hypothesis 1.2 and let $\underline{\mathcal{C}}$ denote the resulting $\mathcal{D}$-enriched category as in Remark 6.7. Then there is a canonical natural isomorphism in $\mathcal{D}$*

$$\sigma_{x,y} : \Delta\underline{\mathcal{C}}(x,y) \xrightarrow{\sim} \underline{\mathcal{C}}(y, x \otimes \omega_f) \tag{6.9}$$

for all $x \in \mathcal{C}^c$ and $y \in \mathcal{C}$, where we recall $\Delta := \underline{\mathrm{hom}}_{\mathcal{D}}(-, \mathbb{1})$. In particular, if we have the Wirthmüller isomorphism (Theorem 1.9), the pair $(\mathbb{S} := (-) \otimes \omega_f, \sigma)$ defines a Serre functor *on $\mathcal{C}^c$ relative to $\mathcal{D}^c$, by which we mean that $\mathbb{S}$ is an equivalence $\mathbb{S} : \mathcal{C}^c \xrightarrow{\sim} \mathcal{C}^c$ and that σ is a natural isomorphism $\Delta\underline{\mathcal{C}}(x,y) \cong \underline{\mathcal{C}}(y, \mathbb{S}x)$ in the tensor-category $\mathcal{D}^c$ for all $x, y \in \mathcal{C}^c$.*

Proof. Under our basic hypothesis, we have the adjunction $f_* \dashv f^{(1)}$ and its internal version. If x is compact, and hence rigid, we obtain an isomorphism

$$\begin{aligned}
\Delta\,\underline{\mathcal{C}}(x,y) &= \underline{\mathrm{hom}}_{\mathcal{D}}\big(f_*\,\underline{\mathrm{hom}}_{\mathcal{C}}(x,y), \mathbb{1}\big) && \text{by definition}\\
&\cong f_*\,\underline{\mathrm{hom}}_{\mathcal{D}}\big(\underline{\mathrm{hom}}_{\mathcal{C}}(x,y), \omega_f\big) && (2.14)\\
&\cong f_*\,\underline{\mathrm{hom}}_{\mathcal{D}}\big(\Delta(x) \otimes y, \omega_f\big) && x \in \mathcal{C}^c\\
&\cong f_*\,\underline{\mathrm{hom}}_{\mathcal{D}}\big(y, x \otimes \omega_f\big) && x \in \mathcal{C}^c\\
&= \underline{\mathcal{C}}(y, x \otimes \omega_f)
\end{aligned}$$

(the second isomorphism uses [HPS97, Thm. A.2.5] again). This is the claimed natural isomorphism σ. When the object ω_f is invertible (Thm. 1.9), the functor $\mathbb{S} = (-) \otimes \omega_f$ restricts to a self-equivalence on compacts. □

6.10. *Remark.* Usually, what one means by a "Serre functor" is a self-equivalence $\mathbb{S}$ on a $\mathbb{k}$-linear (triangulated) category $\mathcal{C}$ together with an isomorphism as in (6.9), where $\mathbb{k}$ is a field and Δ should be replaced by the $\mathbb{k}$-linear dual. We can easily deduce such a structure from our result when the target category is $\mathcal{D} = \mathrm{D}(\mathbb{k})$.

6.11. **Corollary** (Serre duality)**.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ satisfy the Wirthmüller isomorphism (Theorem 1.9), and assume moreover that $\mathcal{D} = \mathrm{D}(\mathbb{k})$ is the derived category of a field $\mathbb{k}$. Then $\mathcal{C}^c$ is $\mathbb{k}$-linear and endowed with a Serre functor*

$$\mathbb{S} = (-) \otimes \omega_f : \mathcal{C}^c \xrightarrow{\sim} \mathcal{C}^c \qquad \sigma : \mathcal{C}(x,y)^\star \xrightarrow{\sim} \mathcal{C}(y, \mathbb{S}x)$$

in the sense of [BK89], where $(-)^\star = \mathrm{Hom}_{\mathbb{k}}(-, \mathbb{k})$ denotes the $\mathbb{k}$-linear dual.

Proof. Apply $H^0 = \mathrm{D}(\mathbb{k})(\mathbb{1}, -)$ to (6.9) and note that $H^0 \circ \Delta \cong (-)^\star \circ H^0$. □

6.12. *Remark.* If it exists, a Serre functor $(\mathbb{S}, \sigma)$ on $\mathcal{C}^c$ relative to $\mathcal{D}^c$ as in Theorem 6.8 is unique. More precisely, if $(\mathbb{S}, \sigma)$ and $(\mathbb{S}', \sigma')$ are two of them then by Yoneda there is a unique isomorphism $\mathbb{S} \xrightarrow{\sim} \mathbb{S}'$ of functors $\mathcal{C}^c \to \mathcal{C}^c$ inducing $(\sigma'\sigma^{-1})_*$ on the Hom sets. A similar remark holds for the usual Serre functors.

In order to illustrate our results within a single coherent picture, we end this section by specializing them to a classical example.

6.13. *Example* (Projective varieties). Let $f : X \to Y = \mathrm{Spec}(\mathbb{k})$ be a projective variety over a field $\mathbb{k}$, and let $f^* : \mathcal{C} = \mathrm{D}(\mathrm{Qcoh}\,X) \to \mathrm{D}(\mathbb{k}) = \mathcal{D}$ be the pull-back functor. Then $\mathcal{C}^c = \mathrm{D}^{\mathrm{perf}}(X)$, and moreover $\mathcal{C}^{c/f} = \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X)$ by Theorem 5.20. Thus we have the inclusion $\mathcal{C}^c = \mathrm{D}^{\mathrm{perf}}(X) \subset \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X) = \mathcal{C}^{c/f} \overset{\text{def.}}{=} f^\#(\mathcal{B}^c)$ and therefore f^* must satisfy Grothendieck-Neeman duality by Proposition 5.18. We conclude that f^* is quasi-perfect. Moreover, by Theorem 5.22 we know that the subcategory $\mathrm{D^b}(\mathrm{coh}\,X)$ consists of ω_f-reflexive objects in $\mathrm{D}(\mathrm{Qcoh}\,X)$. Hence by our Grothendieck duality Theorem 5.23, the object ω_f is dualizing for $\mathrm{D^b}(\mathrm{coh}\,X)$, *i.e.* in more classical language, it is a *dualizing complex for* X (as defined in [Nee10]). But what is ω_f?

If X is Gorenstein (e.g. regular, or a complete intersection), then by [Har66, p. 299] the structure sheaf $\mathcal{O}_X$ is also a dualizing complex for X. But then, by the uniqueness of dualizing complexes (see Corollary 5.14), there exists a tensor invertible $\ell \in \mathrm{D}(\mathrm{Qcoh}\,X)$ and an isomorphism $\omega_f \cong \mathcal{O}_X \otimes \ell = \ell$, so in this case ω_f is invertible and therefore f^* satisfies the Wirthmüller isomorphism (Thm. 1.9). Indeed, it can be shown in general that Gorenstein varieties are characterized by having an invertible dualizing complex (see [AIL10, §8.3]). Still, this does not yet determine ω_f up to isomorphism.

To this end, assume now that X is *regular*, so that we have the equality $\mathcal{C}^c = \mathrm{D^{perf}}(X) = \mathrm{D^b}(\mathrm{coh}\,X) = \mathcal{C}^{c/f}$. In this case, by condition (3) of Theorem 1.9, ω_f must be invertible. Moreover, Theorem 6.11 applies so that $-\otimes\omega_f$ yields a Serre functor on $\mathcal{C}^c$. But it is a basic classical result that $\mathcal{C}^c$ also admits a Serre functor $-\otimes\Sigma^n\omega_X$, where $\omega_X = \Lambda^n\Omega_{X/\Bbbk}$ is the canonical sheaf on X (see e.g. [Rou10, Lemma 4.18]; here we assume X is of pure dimension n, for simplicity). Therefore $\omega_f \cong \Sigma^n\omega_X$ by Remark 6.12. In conclusion, one can identify ω_f up to isomorphism from very general principles.

7. Matlis duality

We conclude our article by showing that Matlis duality over a commutative noetherian local ring also falls under the scope of our theory.

7.1. **Theorem.** *Let $f^* : \mathcal{D} \to \mathcal{C}$ be a functor satisfying our basic Hypothesis 1.2. Let $\mathcal{C}_0$ be a subcategory of $\mathcal{C}$ admitting a (possibly external) dualizing object $\kappa' \in \mathcal{C}$ (see Def. 5.1). Assume moreover that κ' admits a* Matlis lift *κ, that is, assume that there exists some object $\kappa \in \mathcal{D}$ such that $f^{(1)}(\kappa) \cong \kappa'$. Then κ is a (possibly external) dualizing object for the subcategory $\mathcal{D}_0 := \mathrm{thick}(f_*(\mathcal{C}_0))$, the thick subcategory generated by the image of $\mathcal{C}_0$ under push-forward.*

Proof. Since $\mathrm{Id}_\mathcal{D}$ and $\Delta^2_{\kappa'}$ are triangulated functors, it suffices to show that the natural transformation $\varpi_{f_*(x)} \colon f_*(x) \to \Delta^2_{\kappa'}(f_*x)$ of (5.4) is invertible whenever x belongs to $\mathcal{C}_0$. For this, it suffices to show that the following square commutes

$$\begin{tikzcd} f_*(x) \arrow[r, "f_*(\varpi_x)"] \arrow[d, "\varpi_{f_*(x)}"'] & f_*\Delta_{\kappa'}\Delta_{\kappa'}(x) \arrow[d, "\zeta_{\Delta_{\kappa'}(x)}", "\cong"'] \\ \Delta_\kappa\Delta_\kappa f_*(x) \arrow[r, "\cong", "\Delta_\kappa(\zeta)"'] & \Delta_\kappa f_*\Delta_{\kappa'}(x) \end{tikzcd} \tag{7.2}$$

where ζ denotes the natural isomorphism (5.15). Since the top horizontal map is invertible for $x \in \mathcal{C}_0$, the commutativity of (7.2) will imply that $\varpi_{f_*(x)}$ is also invertible, as desired.

It is far from obvious to verify that (7.2) commutes, but fortunately this has already been proved, in even greater generality, in [CH09, Thm. 4.2.9]. Let us explain. The commutativity of (7.2) states precisely that the functor $f_* : \mathcal{C} \to \mathcal{D}$, or rather the pair (f_*, ζ), is a "duality-preserving functor" in the sense of [CH09, Def. 2.2.1] between the "categories with (weak) duality" $(\mathcal{C}, \Delta_{\kappa'})$ and $(\mathcal{D}, \Delta_\kappa)$. The hypotheses $(\mathbf{A}_f)$, $(\mathbf{B}_f)$ and $(\mathbf{C}_f)$ of the cited theorem are all satisfied in our situation by virtue of Corollary 2.10 (note that our $f^{(1)}$ is denoted $f^!$ in *loc. cit.*). To see that the conclusion of the cited theorem applies here, we must still verify that the natural maps we denote by π and ζ coincide with the homonymous maps of [CH09]. For π, it suffices to inspect the definitions in Prop. 2.11 and [CH09, Prop. 4.2.5] and note that they agree. For ζ, we must compare our definition of (2.14) as a conjugate of π (which then specializes to (5.15)) with the definition of ζ given in [CH09, Thm. 4.2.9]. In more detail, we must show that our map (2.14) coincides with the composite

$$f_*\underline{\mathrm{hom}}(x, f^{(1)}y) \to \underline{\mathrm{hom}}(f_*x, f_*f^{(1)}y) \xrightarrow{\underline{\mathrm{hom}}(1,\epsilon)} \underline{\mathrm{hom}}(f_*x, y)$$

where the first map is the canonical map $f_*\underline{\mathrm{hom}}(a, b) \to \underline{\mathrm{hom}}(f_*a, f_*b)$ induced by the (lax) monoidal structure on f_*. This is readily checked from the definition of (2.14) in terms of π, together with the first diagram in (3.23). □

7.3. *Example* (Matlis duality). Let R be a commutative noetherian local ring, let $R \to k$ be the quotient map to the residue field at the maximal ideal, and let $f^* : \mathrm{D}(R) \to \mathrm{D}(k)$ be the induced functor as in Example 3.25. Then $I := E(k)$, the injective hull of the R-module k, is a Matlis lift of k: $f^{(1)}(E(k)) = \mathrm{RHom}_R(k, E(k)) \cong k$ in $\mathrm{D}(k)$. By Theorem 7.1, the functor $\Delta_{E(k)} = \mathrm{RHom}_R(-, E(k))$ induces a duality on the thick subcategory of $\mathrm{D}(R)$ generated by $f_*(k)$, *i.e.* on complexes whose cohomology is bounded and consists of finite length modules. As $E(k)$ is injective, we may restrict this duality to the category of finite length modules.

7.4. *Example* (Pontryagin duality). The dualizing object $I = E(k)$ of Example 7.3 is typically external, *i.e.* it often lies outside the subcategory it dualizes: $E(k) \notin \mathrm{thick}(f_*(k))$. This already happens in the archetypical example of (discrete p-local) Pontryagin duality, where $R \to k$ is the quotient map $\mathbb{Z}_{(p)} \to \mathbb{Z}/p$ and $E(k)$ is the Prüfer group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, which has infinite length.

7.5. *Example* (Generalized Matlis duality). Let $R \to k$ be a morphism of commutative $\mathbb{S}$-algebras and consider the three induced functors

$$k \otimes_R (-) \;=\; f^* \uparrow \quad \begin{array}{c} \mathcal{C} := \mathrm{D}(k) \\ \downarrow f_* \quad \uparrow f^{(1)} \\ \mathcal{D} := \mathrm{D}(R) \end{array} \;=\; \mathrm{Hom}_R(k, -) \tag{7.6}$$

as in Example 3.27. We write $R \to k$ rather then $B \to A$ in order to be consistent with the notation of Dwyer-Greenlees-Iyengar [DGI06]. In *loc. cit.*, a *Matlis lift of* k is defined to be a (structured) R-module I such that $\mathrm{D}(k)(x, k) \cong \mathrm{D}(R)(f_*x, I)$ naturally in $x \in \mathrm{D}(k)$, *i.e.* by Yoneda, such that $f^{(1)}(I) \cong k$ (see [DGI06, Def. 6.2 and Rem. 6.3]). Moreover, I is required to be "effectively constructible from k", a property somewhat stronger then I belonging to the localizing subcategory generated by $f_*(k)$. In particular, a Matlis lift of k in the sense of Dwyer-Greenlees-Iyengar is also a Matlis lift, in the more modest sense of Theorem 7.1, of the dualizing object $\kappa' := k$ for the subcategory $\mathcal{C}_0 := \mathrm{D}(k)^c$ of $\mathrm{D}(k)$. Hence by Theorem 7.1 we immediately obtain the following generalization of Matlis duality.

7.7. **Corollary.** *Let $R \to k$ be a morphism of commutative $\mathbb{S}$-algebras, and assume that the R-module I is a Matlis lift of k in the sense of* [DGI06]*. Then I is a (possibly external) dualizing object for the thick subcategory of $\mathrm{D}(k)$ generated by $f_*(k)$.* $\square$

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