Tensor C^*-categories and their representations

① Operator algebras

② $2-W^*-$categories

③ Examples

④ Further results + Prospects

① Operator algebras.

Algebra A (over Field k) $\longleftrightarrow$ $*$-algebra A (over $\mathbb{C}$)

$$\begin{cases} (a+\lambda b)^* = a^* + \bar{\lambda} b \\ (ab)^* = b^* a^* \\ a^{**} = a \end{cases}$$

E.g. $k[\Gamma]$, group Γ

E.g. $\mathbb{C}[\Gamma]$, group Γ

$g^* = g^{-1}$

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Module M $\rightsquigarrow$ $*$-representation

$A \to \mathrm{End}_k M$ $A \to \mathcal{B}(\mathcal{H})$

$\hookrightarrow$ Hilbert space

$A = k[\Gamma] \Rightarrow \Gamma$-modules $A = \mathbb{C}[\Gamma] \Rightarrow$ unitary Γ-rep's

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Within $*$-algebra context, we have

unitaries	self-adjoints	positive elements
$u^*u = 1 = uu^*$	$x^* = x$	$x = y^*y$

isometries	projections
$v^*v = 1$	$p^2 = p = p^*$

Finite-dimensional
~~Semisimple~~
algebra

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"Semisimple"
$*$-algebra

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<u>Example</u>: quantum $ax+b$-groups

<u>Algebraic</u> (Taft '71)

$q \in \mathbb{C}$, $q^n = 1$, $q \neq 1$, $q = e^{2\pi i/n}$

$$A = \mathbb{C}\left\langle a, b \;\middle|\; \begin{array}{l} ab = q^2 ba \\ a^n = 1 \\ b^n = 0 \end{array} \right\rangle$$

$\Delta(a) = a \otimes a$

$\Delta(b) = a \otimes b + b \otimes 1$

1. $S^2 \neq id$
2. Left integral $\varphi \neq$ Right integral ψ

   $\psi(a) = \varphi(a\delta)$, $\delta = a^{1-n}$
3. Left integral φ not tracial

   $\varphi(ab) = \varphi(b\sigma(a))$, $\sigma \neq id$.

Finite-dimensional ~~Semisimple~~ algebra

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Algebraic (Taft '71)	Analytic (Woronowicz-Zakrzewski '02, Ip '13)
$q \in \mathbb{C}$, $q^n = 1$, $q \neq 1$, $q = e^{2\pi i/n}$	$q \in \mathbb{C}$, $\lvert q\rvert = 1$, $q \neq 1 \rightsquigarrow q = e^{2\pi i \hbar}$ $(\hbar > 0)$
$A = \mathbb{C}\left\langle a, b \,\middle\vert\, \begin{array}{l} ab = q^2 ba \\ a^n = 1 \\ b^n = 0 \end{array} \right\rangle$	$A = B(L^2(\mathbb{R}))$, $a = e^x$, $b = e^{2\pi i \hbar \, d/dx}$
$\Delta(a) = a \otimes a$	$\Delta(a) = a \otimes a$
$\Delta(b) = a \otimes b + b \otimes 1$	$\Delta(b) = a \otimes b + b \otimes 1$
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(3) Left integral φ not tracial; $\varphi(ab) = \varphi(b\sigma(a))$, $\sigma \neq id$.	(3) Left integral φ not tracial; "$\varphi(ab) = \varphi(b\sigma(a))$", $\sigma \neq id$.

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$\rightsquigarrow$

"Small quantum groups"

Analytic (Woronowicz-Zakrzewski '02, Ip '13)

$q \in \mathbb{C}$, $|q| = 1$, $q \neq 1 \rightsquigarrow q = e^{2\pi i \hbar}$ $(\hbar > 0)$

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$a = e^x$, $b = e^{2\pi i \hbar\, d/dx}$

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"Positive representations"

(Fock, Goncharov, Shen, Shapiro, Schrader, Ip, Kim, Frenkel, Bytsko, Teschner, Kashaev, ...)

Operator algebras:

(1) C^*- algebra : $A \subseteq \mathcal{B}(\mathcal{H})$ normclosed $*$- algebra

$\Leftrightarrow \|a^* a\| = \|a\|^2$ $\qquad a \mapsto \|a\|$

E.g. 1) $A = C_0(X)$, X l.c. Hd. space

2) $A = C^*_r(G)$, G l.c. Hd. group

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② W^*- algebra : $A \subseteq B(H)$ WOT closed $*$- algebra

= von Neumann algebra $\qquad a \mapsto \langle \xi, a\eta \rangle$.

E.g. 1) $A = L^\infty(X,\mu)$, X l.c. Hd. space
μ Radon measure

2) $A = L(G)$, G l.c. Hd. group

Fact : $A \subseteq B(H)$ is W^*- algebra

$\Leftrightarrow$ $A'' = A$, where $A' = Z_{B(H)}(A)$

$\Leftrightarrow$ A is dual Banach space
and $A \subseteq B(H)$ <u>normal</u> (respects weaker topologies)

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Slogan: If A is W^*-algebra, then

$\hookrightarrow$ normal $*$-rep

A-modules ${}_A\mathcal{H}$ are boring ... $\rightsquigarrow$ If $A \subseteq B(H)$, take

$\mathcal{H} = \oplus p_i H$, $p_i \in \mathrm{Proj}(A')$.

A-bimodules ${}_A\mathcal{H}_A$ are interesting!

$\hookrightarrow$ separately normal $*$-reps

(2) 2-W^*-categories.

C^*-category : Normed $\mathbb{C}$-linear category w/

"dagger structure" $*$: Hom $(X,Y) \to$ Hom (Y,X)

Is involutive, anti-linear, anti-multiplicative

s.t. 1) Each Hom (X,Y) normcomplete

2) $\| x^* x \| = \|x\|^2$

3) Each $x^* x$ positive

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W^*-category : C^*-category s.t. $\mathrm{Hom}(X,Y)$ are dual Banach spaces

(weak) 2-W^*-category : (Yamagami '07)

$$\begin{tikzcd} \alpha \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=D}] & \beta \arrow[from=U, to=D, Rightarrow, "T"'] \end{tikzcd}$$

s.t. 1) Each $\mathrm{Hom}(\alpha,\beta)$ W^*-category

2) Each $X\otimes(Y\otimes Z) \cong (X\otimes Y)\otimes Z$ unitary

3) Each $X\otimes -$ and $-\otimes X$ normal $*$-functor

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Bi-involutive 2-W^*-category : (Hayashi-Yamagami '99, Penneys-Henriques '17)

$$\begin{tikzcd} \alpha \arrow[r, bend left=50, "X"] \arrow[r, bend right=50, "Y"'] \arrow[r, phantom, "T \Downarrow"] & \beta \end{tikzcd} \Longrightarrow \begin{tikzcd} \alpha & \beta \arrow[l, bend right=50, "\overline{X}"'] \arrow[l, bend left=50, "\overline{Y}"] \arrow[l, phantom, "\overline{T} \Downarrow"] \end{tikzcd}$$

so $\overline{} : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(\overline{X},\overline{Y})$, $*$-Functor

anti-linear

with given $\overline{\overline{X}} \cong X$, $\overline{X \otimes Y} \cong \overline{Y} \otimes \overline{X}$ + coherence

"Large" examples of bi-involutive 2-W^*-categories

(1) Corr:

$$\begin{tikzcd} A \arrow[r, bend left=50, "{}_A\mathcal{H}_B"] \arrow[r, bend right=50, "{}_A\mathcal{G}_B"'] \arrow[r, phantom, "\Downarrow T"] & B \end{tikzcd}$$

Correspondence = sep. normal bimodule

bounded intertwiner

W^*- algebra

Composition:

Connes Fusion product

$${}_A\mathcal{H}_B \underset{B}{\boxtimes} {}_B\mathcal{G}_C$$

Involution:

$\overline{\mathcal{H}}$ complex conjugate,

$$b\,\overline{\xi}\,a = \overline{a^*\xi b^*}$$

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② Rep(G) for G l.c. (q.) group.

$$\begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=60, "\mathcal{H}"] \arrow[r, bend right=60, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow T"] & \bullet \end{tikzcd}$$

cont. G-rep

bounded intertwiner

Composition:

$\mathcal{H} \otimes \mathcal{G} \cong$ diagonal rep.

Involution

$$\overline{\mathcal{H}} \cong g\cdot\bar{\xi} = \overline{g\xi}$$

"Large" examples of bi-involutive 2-W*-categories

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③ Corr^G

(DC-De Ro '24)

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Monoidal W^*-category: clear!
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Unitary tensor W^*-category:

Monoidal W^*-category with duals and simple unit.

Facts: ① All $\mathrm{Hom}(X,Y)$ f.d.

$$\Longrightarrow \mathrm{End}(X) \cong \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$$

② After adding $\oplus$ and subobjects:

$$X \cong \oplus X_i \quad \text{w/} \quad X_i \text{ irreducible}$$

③ Canonical bi-involutive structure with

$$\overline{X} \cong X^\vee \cong {}^\vee X .$$

E.g.: $\mathrm{Rep}(G)$ for G cpt (q.) group.

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Unitary Fusion W^*-category.

Unitary tensor W^*-category

w/ Finitely many simples (after adding $\oplus$ and subobjects)

E.g. $\mathrm{Rep}(G)$ for G finite (q.) group.

Representations of $\otimes$-W^*-categories

$\mathcal{D}$ W^*-category

$\Rightarrow$ $\mathrm{End}_{norm}(\mathcal{D})$ is monoidal W^*-category:

1) $F : \mathcal{D} \to \mathcal{D}$ normal $*$-Functor

2) $F \xrightarrow{n} G$ uniformly bounded nat. transformations.

$\sup_X \| n_X : F(X) \to G(X) \| < \infty$

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If $\mathcal{C}$ monoidal W^*-category:

Module W^*-category $\mathcal{C} \curvearrowright \mathcal{D}$ via $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$

$\Updownarrow$

$\mathcal{C} \xrightarrow{\otimes} \mathrm{End}_{\mathrm{norm}}(\mathcal{D})$

$\hookrightarrow$ strongly monoidal normal $*$-Functor

$\sim$ unitary coherence conditions

Intertwiners

$\mathcal{C}$ monoidal W*- category , $\mathcal{D}$ is $\mathcal{C}$- module W*- category

$$\Downarrow$$

Monoidal W*- category $\mathrm{End}_{\mathcal{C}}(\mathcal{D})$:

$$F \in \mathcal{E} := \mathrm{End}(\mathcal{D}) \ \overline{w}\ \ F(X \otimes Y) \underset{\substack{\text{unitary} \\ v_{X,Y} + \text{coherence}}}{\cong} X \otimes F(Y) \quad \begin{matrix} X \in \mathcal{C} \\ Y \in \mathcal{D} \end{matrix}$$

$$\Updownarrow$$

$$F \in \mathcal{E} \ \ \overline{w}\ \ F \otimes X \underset{\substack{\text{unitary} \\ \text{half braiding}}}{\cong} X \otimes F$$

$$\rightsquigarrow \ {}_{\mathcal{C}}\mathrm{End}(\mathcal{D}) = \mathcal{C}' \cap \mathcal{E}.$$

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Joined together: 2-W^*-category

$${}_{\mathcal{C}}\mathrm{Hom}(\mathcal{D},\mathcal{D}) \overset{!!}{=} \mathcal{C}' \cap \mathcal{E}$$

$$\begin{tikzcd} \bullet \arrow[d, bend left=60, "\mathcal{D}^\vee := {}_{\mathcal{C}}\mathrm{Hom}(\mathcal{D},\mathcal{C})"] \\ \bullet \arrow[u, bend left=60, "{}_{\mathcal{C}}\mathrm{Hom}(\mathcal{C},\mathcal{D}) \cong \mathcal{D}"] \end{tikzcd}$$

$$(\mathcal{C}, \otimes^{op}) \cong {}_{\mathcal{C}}\mathrm{Hom}(\mathcal{C},\mathcal{C})$$

③ Examples

Ⓐ If G compact (q.) group,

A vN algebra and $G \curvearrowright A$,

then $\mathcal{C} = \mathrm{Rep}(G)$, $\mathcal{D} = \mathrm{Rep}^G_{\mathrm{norm}}(A)$:

$$\mathrm{End}_{\mathcal{C}}(\mathcal{D}) = {}_A\mathrm{Corr}^G_A \qquad \text{(De Ro, '24)}$$

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<u>Note</u>: In case $\mathcal{C}$ is (Ind-completion of) unitary tensor W^*-cat:

$$\left(\mathcal{C} \curvearrowright \mathcal{D} + \text{object } M \in \mathcal{D}\right) \longleftrightarrow \text{Internal } W^*\text{-algebra } A_M \in \mathcal{C}$$

(Corey + Penneys '17, Hataishi + Yamashita '22)

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Ⓑ If G locally compact quantum group

$$\mathrm{End}_{\mathrm{Rep}(G)}(\mathrm{Hilb}) \cong \mathrm{Rep}(\widehat{G}) \qquad \text{(cf. Müger '03)}$$

(C) If $0 < q < 1$:

$$\mathcal{Z}(\mathrm{Rep}(G_q)) \cong \mathrm{Rep}(G_q^{\mathbb{C}}) \quad , \quad G \text{ cpt Lie}$$

(0+1 connected)

E.g. $\mathcal{Z}(\mathrm{Rep}(SU_q(N))) \cong \mathrm{Rep}(GL_q(N,\mathbb{C}))$

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More generally :

$$\mathrm{End}_{\mathrm{Rep}(G_q)}(K_q) \cong \mathrm{Rep}(L_q)$$

(DC - Dzokou Talla '24
DC '24)

where

$$\begin{tikzcd}[column sep=small, row sep=small] & G^{\mathbb{C}} & \\ L \arrow[ur, phantom, "\subseteq" sloped] & & G \arrow[ul, phantom, "\supseteq" sloped] \\ & K \arrow[ul, phantom, "\supseteq" sloped] \arrow[ur, phantom, "\subseteq" sloped] & \end{tikzcd}$$

Cartan square

symmetric pair nc. type

symmetric pair cpt. type

e.g. $\mathrm{End}_{\mathrm{Rep}(SU_q(N))}(\mathrm{Rep}(SO_q(N))) = \mathrm{Rep}(SL_q(N,\mathbb{R}))$

$\rightsquigarrow$

$$\begin{tikzcd} & \widehat{G_q^{\mathbb{C}}} \arrow[dl] \arrow[dr] & \\ \widehat{L_q} \arrow[dr] & & \widehat{G_q} \arrow[dl] \\ & \widehat{K_q} & \end{tikzcd}$$

D $\Gamma = (V,E)$ connected, locally finite graph

$$\Rightarrow \mathrm{Rep}\left(\widehat{QAut(\Gamma)}\right) = \left\{ \left(\mathcal{H}, u = (u_{vw})_{vw}\right) \;\middle|\; \begin{array}{l} u \in \mathcal{U}(\ell^2(V) \otimes \mathcal{H}) \\ u_{v,w} = u_{v,w}^* = u_{v,w}^2 \\ u A_\Gamma u^* = A_\Gamma \end{array} \right\}$$

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Define $\operatorname{Rep}(QAut(\Gamma)) := \operatorname{Rep}\left(\widehat{QAut(\Gamma)}\right)' \cap \mathrm{Hilb}$

Then $\operatorname{Rep}(QAut(\Gamma)) \curvearrowright \ell^\infty(V)$ Corr !

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Then $\mathrm{Rep}(QAut(\Gamma)) \curvearrowright {}_{\ell^\infty(V)}\mathrm{Corr}$!

One has $QAut(\Gamma)$ locally compact quantum group: quantum automorphism group of V (Rollier - Vaes '24)

We then have $\mathcal{C} := \mathrm{Rep}(QAut(\Gamma))' \cap {}_{\ell^\infty(V)}\mathrm{Corr}_{\ell^\infty(V)}$

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(D) $\Gamma = (V,E)$ connected, locally finite graph

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$\hookrightarrow$ (1) has "dense" rigid 2-W^*-category
$\rightsquigarrow$ 0-cells are quantum orbits.

(2) Particular objects $\ell^2(\ )^{\otimes n}$:

Hom-spaces via planar graph morphisms
(cf. Mančinska-Roberson)

(E) W^*-algebra is II_1-factor if trivial center + pos. trace.

E.g. (the) hyperfinite II_1-factor

$$R \cong L(S_\infty) \cong \tau\text{-}\underrightarrow{\lim} \bigotimes_1^n M_2(\mathbb{C})$$

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1 Every s.c. locally compact group

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Slightly better: cocycle-twisted action

(discrete) $G \overset{\text{"}\curvearrowright\text{"}}{\underset{\alpha}{}} A$ $+ \{u_{g,h} \in \mathcal{U}(A)\}$: $\alpha_{gh}(a) = U_{g,h}\, \alpha_g \alpha_h(a)\, U_{g,h}^*$

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$$\Rightarrow \mathcal{C} = \{G\text{- graded Hb. spaces}\} \to {}_A\mathrm{Corr}_A$$

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(Popa '94, Hayashi-Yamagami '99)

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Jones for TL
$\rightsquigarrow$ Origin Jones pd.!

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(4) Every unitary $\otimes$ - W^*- category (Yamagami '03; Brothier, Hartglass, Penneys '12; Giorgetti-Yuan '23)

$$\mathcal{C} \subseteq {}_A\mathrm{Corr}_A \quad (\text{some } A)$$

F $\quad \mathcal{C} = TL_t = \mathrm{Rep}(SU_q(2)) \quad , \quad t = -[2]_q \quad , \; 0<|q|\leq 1$

 $^{*} = -\mathrm{sgn}(q)$ 

Classification irreducible

 $= -(|q|+|q|^{-1})$.

$TL_t \curvearrowright \mathcal{D} \quad \longleftrightarrow \quad$ Ergodic $SU_q(2) \curvearrowright A \big/_{\text{Morita}}$

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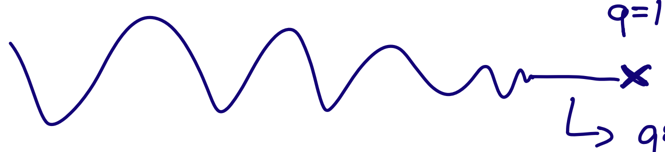
⇝ weighted symm. graphs
↳ depends on q

w_e w_e^{-1} $\forall v: \sum_{\text{edges}} w_1, w_2, w_3, w_4 = q+q^{-1}$



$q=1$: extended ADE

↳ $q\approx 1$: extended ADE
+ extra 1-par. d.o.f.
(coideal case)

wild behaviour

Higher rank? Also exotic representations occur!

E.g. $\mathrm{Rep}(SO_q(5)) \cong \mathrm{Rep}(Q\mathrm{Aut}(\text{Higman-Sims}))$
non-trivial Fiber Functor!

But no systematic results (e.g. $\mathrm{Rep}\, SU_q(3)$?)

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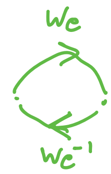
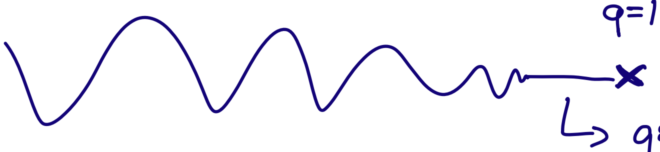


$\forall v: \sum_{\text{edges } v}$ 

$= q+q^{-1}$

$q=1$: extended ADE



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At root of unity: higher rank results by

Ocneanu, Evans, Gannon, Edie-Michell, ...

4 Further results + Prospects

Further results:

- Dynamical properties of unitary W^*-categories + boundary theory: Neshveyer, Yamashita, Hataishi, Habbestad, ...

- Actions unitary W^*-$\otimes$-cat's on C^*-algebras:
  ⇝ Evington's talk!

- 2-C^*-Cat and balanced Deligne-Kelley product:
  Antoun + Voigt

- Links w/ e.g. conformal field theory:
  Longo, Kawahigashi, Giorgetti, ...

- Links w/ (classification of) subfactors:
  Jones, Popa, Ocneanu, Morrison, Snyder, Penneys, ...

Prospects

(A) What are "good" monoidal W^*-categories / 2-W^*-categories?

→ Existence of bi-involutive structure ✓

→ Existence of monoidal sub-W^*-category

$$\mathcal{C}_{\text{coarse}} \subseteq \mathcal{C} \qquad (?)$$

→ Existence of Fell topology (?)

(B) Classification results $\mathrm{IrrRep}(\mathrm{Rep}(G_q))$?

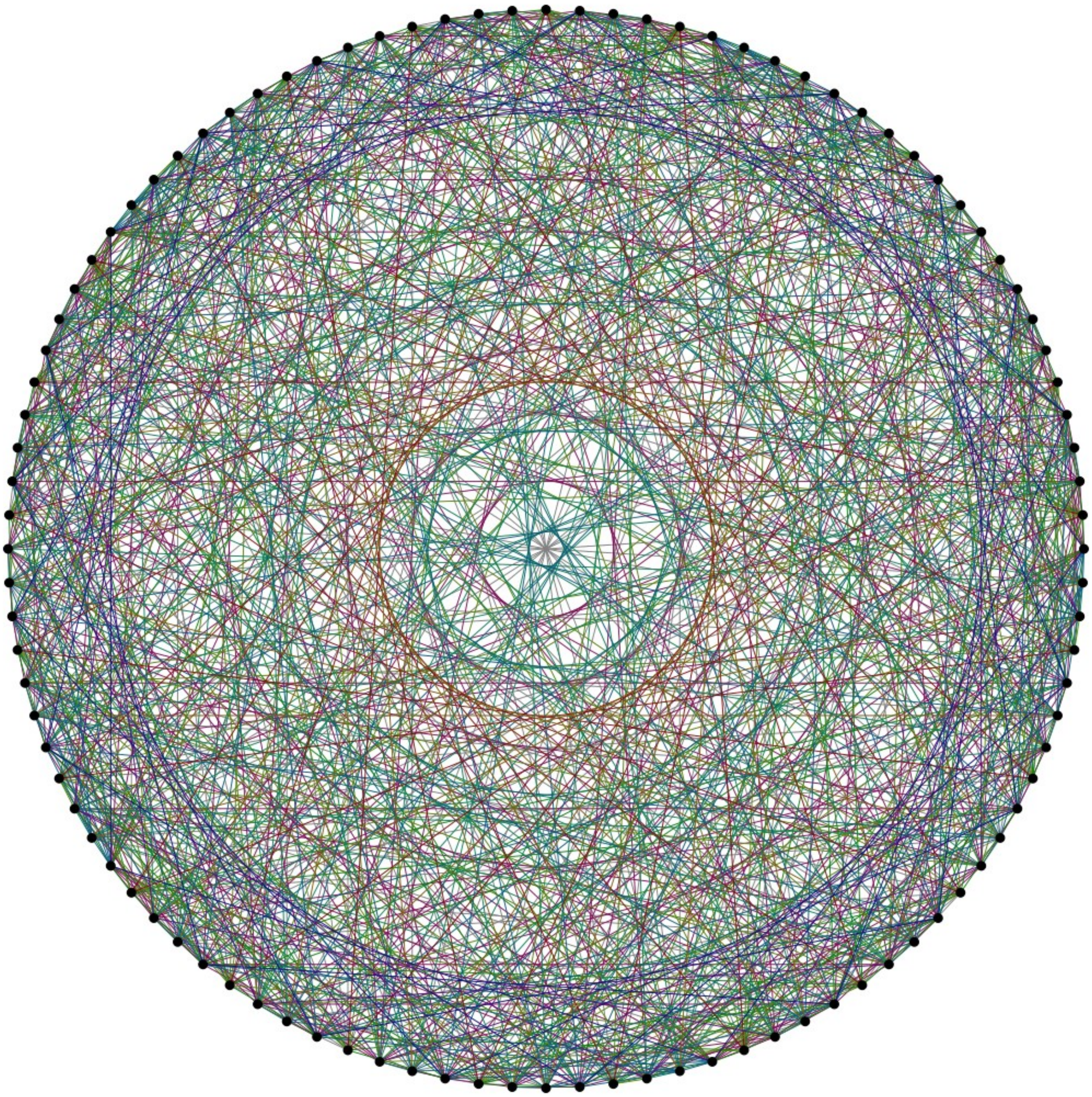
"Exotic" cases ?

(C) Approximation theories for non-discrete setting ?

Thanks

for your

attention :)



In mathematical graph theory, the **Higman–Sims graph** is a 22-regular undirected graph with 100 vertices and 1100 edges. It is the unique strongly regular graph srg(100,22,0,6), where no neighboring pair of vertices share a common neighbor and each non-neighboring pair of vertices share six common neighbors.[2] It was first constructed by Mesner (1956)[3] and rediscovered in 1968 by Donald G. Higman and Charles C. Sims as a way to define the Higman–Sims group, a subgroup of index two in the group of automorphisms of the Hoffman–Singleton graph.[4]