

THE SPECTRUM OF EQUIVARIANT KASPAROV THEORY FOR CYCLIC GROUPS OF PRIME ORDER

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ABSTRACT. We compute the Balmer spectrum of the equivariant bootstrap category of separable G - C^* -algebras when G is a group of prime order.

1. INTRODUCTION AND RESULTS

Let G be a second countable locally compact group. Kasparov's G -equivariant KK-theory of complex separable G - C^* -algebras [Kas88] defines a tensor triangulated category KK^G . This fact was first recognized by Meyer and Nest [MN06], who used it to reformulate the Baum–Connes assembly map elegantly and study its functorial properties. The spectrum of an (essentially small) tensor triangulated category \mathcal{T} is a certain topological space $\mathrm{Spc}(\mathcal{T})$. This important invariant was introduced by Balmer [Bal05]. In [Del10], it was observed that the Baum–Connes conjecture would follow from the (also conjectural) surjectivity of the canonical map $\sqcup_H \mathrm{Spc}(\mathrm{KK}^H) \rightarrow \mathrm{Spc}(\mathrm{KK}^G)$, where H runs through the compact subgroups of G .

Unfortunately, a geometric result such as the above surjectivity still appears well out of reach; this is despite recent major advances of tensor triangular geometry, the theory and computational techniques concerned with the spectrum of tensor triangulated categories (see the survey [Bal10b]). While numerous spectra have been computed in fields ranging from topology to modular representation theory, algebraic geometry and motivic theory (see [Bal19] for a large catalogue), in non-commutative geometry we are still striving to understand the most basic examples. The most general fact known to date is the existence, when G is a compact Lie group, of a canonical continuous surjective map $\mathrm{Spc} \mathrm{KK}^G \rightarrow \mathrm{Spec} R(G)$ onto the Zariski spectrum of the complex character ring $R(G)$; and this holds for rather formal reasons (see [Bal10a, Cor. 8.8]). In order to obtain sharper results, one must severely restrict the kind of groups and algebras under study.

The present article contributes the first complete computation of a truly equivariant example in this context, as well as techniques which may prove useful in other cases. Because of its good generation properties, we consider as in [Del14] the subcategory $\mathrm{Cell}(G) \subset \mathrm{KK}^G$ of G -cell algebras, that is, the localizing subcategory of KK^G generated by the function algebras $C(G/H)$ for $H \leq G$; and we aim at computing the spectrum of the tensor triangulated subcategory $\mathrm{Cell}(G)^c$ of compact objects (see Remark 2.3). In topology, this is analogous to considering the

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spectrum of the equivariant stable homotopy category $\mathrm{SH}(G)^c$ of finite G -spectra, as done in [BS17]. Moreover, we restrict attention to finite groups G . In this case there is also a continuous surjection

$$\rho_G: \mathrm{Spc} \, \mathrm{Cell}(G)^c \rightarrow \mathrm{Spec} \, \mathrm{R}(G),$$

which is known to admit a canonical continuous splitting (see [Del10, Thm. 1.4]). This map was conjectured by the first author to be invertible.

The main result of the present paper settles some cases of the conjecture:

1.1. Theorem (See Section 6). *Let $G = \mathbb{Z}/p\mathbb{Z}$ for a prime number p . Then the canonical comparison map is a homeomorphism $\rho_G: \mathrm{Spc} \, \mathrm{Cell}(G)^c \xrightarrow{\sim} \mathrm{Spec} \, \mathrm{R}(G)$.*

The representation ring $\mathrm{R}(G)$ of \mathbb{Z}/p is isomorphic to the group ring:

$$\mathrm{R}(G) \cong \mathbb{Z}[G] \cong \mathbb{Z}[x]/(x^p - 1).$$

A *thick tensor-ideal subcategory* of $\mathrm{Cell}(G)^c$ is a full subcategory \mathcal{C} that is closed under taking isomorphism – that is, equivariantly KK-equivalent – objects, mapping cones, suspensions, retracts and tensor products with arbitrary objects. A subset of a topological space is called *specialization closed* if it is a union $S = \bigcup_i Z_i$ of closed subsets Z_i .

Since $\mathrm{Cell}(G)^c$ is rigid and idempotent complete by Remark 2.3, Theorem 1.1 may be combined with general abstract results (namely, [Bal10b, Thm. 14, Rmks. 12 and 23]) to classify the thick tensor-ideal subcategories of $\mathrm{Cell}(G)^c$:

1.2. Corollary. *If $G \cong \mathbb{Z}/p\mathbb{Z}$, then there is a canonical bijection between*

- (1) *thick tensor-ideal subcategories \mathcal{C} of $\mathrm{Cell}(G)^c$ and*
- (2) *specialization closed subsets S of the Zariski spectrum $\mathrm{Spec}(\mathbb{Z}[x]/(x^p - 1))$.*

The classification assigns to a thick tensor-ideal \mathcal{C} the union of the *supports* $\mathrm{supp}(A)$ of all algebras A in \mathcal{C} , and to a specialization closed subset S the subcategory of all A with $\mathrm{supp}(A) \subseteq S$. Morally, the support of an object is the subset of $\mathrm{Spec}(\mathbb{Z}[x]/(x^p - 1))$ on which it ‘lives’ (see Section 5 for the abstract definition and [Del10, §6] for a more concrete description). As a consequence of the corollary, a compact G -cell algebra A can be built from another one B using the tensor-triangular operations if and only if $\mathrm{supp}(A) \subseteq \mathrm{supp}(B)$.

If $G = 1$ is the trivial group, then $\mathrm{Cell}(1) \subset \mathrm{KK}$ is just the usual bootstrap category and $\mathrm{Cell}(1)^c$ consists of the C^* -algebras in the bootstrap category with finitely generated K-theory groups. In this case, our results are already known (see [Del10, Thm. 1.2] and also [Del11]).

Theorem 1.1 is significantly harder to prove than the case of the trivial group and requires new ingredients. Our proof strategy is roughly inspired by that of [BS17], which computes the spectrum of the above-mentioned equivariant stable homotopy category $\mathrm{SH}(G)^c$. Specifically, we divide the problem in two halves (see the beginning of Section 6) and settle the first half thanks to the separable monadicity of restriction functors proved in [BDS15]. For the second half, however, our proofs diverge because of fundamental structural differences between equivariant stable homotopy and KK-theory (see Remarks 6.8 and 6.9). The point is to compute the spectrum of a certain localization $\mathcal{Q}(G)$ of $\mathrm{Cell}(G)$ (see Section 4). In order to do this, we show that the tensor triangulated category $\mathcal{Q}(G)$ is ‘weakly regular’, so that we may apply the results of [DS16]. (Incidentally, this method *cannot* be applied to the equivariant stable homotopy category by Remark 6.9.) The weak

regularity of $\mathcal{Q}(G)$ is shown using our last crucial ingredient, Köhler's universal coefficient theorem [Köh10].

This strategy may well provide a proof of the bijectivity of ρ_G for more general finite groups. Köhler's result, however, only holds for groups of prime order and hence would need to be replaced by a more general argument.

2. THE EQUIVARIANT BOOTSTRAP CATEGORY

We collect here some structural results on equivariant KK-theory and cell algebras. Our sources are [MN06] and [Del14].

Let G be a finite group (for simplicity, as many of the statements hold much more generally). First of all, recall from [MN06] that the *Kasparov category* of separable G - C^* -algebras, KK^G , is a *tensor triangulated category*, that is, it is triangulated in the sense of Verdier and carries a symmetric monoidal structure. The exact tensor functor is the minimal tensor product of C^* -algebras, equipped with the diagonal group action. The tensor unit object $\mathbb{1}$ is the algebra \mathbb{C} of complex numbers with the trivial G -action. The category KK^G is essentially small and has all countable coproducts, provided by C_0 -direct sums. The suspension functor $\Sigma = C_0(\mathbb{R}) \otimes -$ of the triangulated structure of KK^G is 2-periodic, that is, there is a natural isomorphism $\Sigma^2 \cong \mathrm{id}_{\mathrm{KK}^G}$, thanks to the *Bott isomorphism* $\beta: \mathbb{C} \xrightarrow{\sim} C_0(\mathbb{R}^2)$. Let $R(G)$ be the representation ring of G . The graded endomorphism ring of the unit is

$$\mathrm{End}_{\mathrm{KK}^G}(\mathbb{1})_* \cong R(G)[\beta^\pm].$$

2.1. *Definition* ([Del14]). Let

$$\mathrm{Cell}(G) := \mathrm{Loc}(\{C(G/H) \mid H \leq G\})$$

be the *localizing subcategory* of KK^G generated by the G - C^* -algebras $C(G/H)$ of complex functions for all subgroups $H \leq G$, with G -action induced by that of G on its cosets G/H . That is, $\mathrm{Cell}(G)$ is the smallest triangulated subcategory of KK^G containing all $C(G/H)$ and closed under forming arbitrary (countable) coproducts. The algebras in $\mathrm{Cell}(G)$ are called *G -cell algebras*.

2.2. *Remark* ([DEM14, §3.1]). The *G -equivariant bootstrap category* \mathfrak{B}^G is defined similarly as the localizing subcategory of KK^G generated by certain objects. It is shown in [DEM14, Sec. 3.1] that \mathfrak{B}^G consists precisely of those separable G - C^* -algebras that are equivariantly KK-equivalent to a G -action on a C^* -algebra of Type I. If G is a finite cyclic group, then $\mathrm{Cell}(G) = \mathfrak{B}^G$. For more general finite groups, there is only an inclusion $\mathrm{Cell}(G) \subseteq \mathfrak{B}^G$.

2.3. *Remark*. By [Del14, Prop. 2.9], $\mathrm{Cell}(G)$ is a *rigidly-compactly generated* tensor triangulated category, in a countable sense of the term (as here there are only countable coproducts to work with). This entails, in particular, that its *compact* objects (those G -cell algebras A such that $\mathrm{KK}^G(A, -)$ commutes with arbitrary countable direct sums) coincide with its *rigid* objects (those A admitting an inverse A^\vee with respect to the tensor product). It follows that $\mathrm{Cell}(G)^c$, the full subcategory of rigid-compact objects, is itself a tensor triangulated category. In addition, $\mathrm{Cell}(G)^c$ is essentially small, *rigid* (it admits the self-duality $A \mapsto A^\vee$), *idempotent complete* (every idempotent endomorphism splits), and it is generated as a thick subcategory by the same set of objects:

$$\mathrm{Cell}(G)^c = \mathrm{Thick}(\{C(G/H) \mid H \leq G\}).$$

Recall that a subcategory is *thick* if it is triangulated and closed under retracts.

2.4. *Remark.* For finite groups, the usual functors between the equivariant Kasparov categories, such as restriction Res_H^G , induction Ind_H^G , and inflation Inf_1^G , preserve cell algebras and also compact cell algebras. See [Dell14, §2].

3. KÖHLER'S UNIVERSAL COEFFICIENT THEOREM

From now on, we restrict attention to $G = \mathbb{Z}/p\mathbb{Z}$ for a prime number p . For these groups we may use the universal coefficient theorem (“UCT”) due to Köhler [Köh10] in order to compute G -equivariant KK-theory groups by algebraic means, at least in principle. We recall how this works and refer to [Köh10] as well as [Mey19, §4–5] for all proofs.

Let $\Sigma C(G) \rightarrow D \rightarrow \mathbb{1} \rightarrow C(G)$ be the mapping cone sequence (distinguished triangle in $\text{Cell}(G)$) for the unit map $\mathbb{1} \rightarrow C(G)$, that is, the embedding of \mathbb{C} as constant functions on G . Let $B := \mathbb{C} \oplus C(G) \oplus D$ be the sum of the three vertices of the triangle. Define the $\mathbb{Z}/2$ -graded ring

$$\mathfrak{K} := (\text{End}_{\text{Cell}(G)}(B)_*)^{\text{op}}.$$

Given any G - C^* -algebra $A \in \text{KK}^G$, its *Köhler invariant* $F_*(A)$ is defined as

$$(3.1) \quad F_*(A) := \text{KK}_*^G(B, A) = \underbrace{\text{KK}_*^G(\mathbb{1}, A)}_{F_*(A)_0} \oplus \underbrace{\text{KK}_*^G(C(G), A)}_{F_*(A)_1} \oplus \underbrace{\text{KK}_*^G(D, A)}_{F_*(A)_2},$$

considered with its evident structure of $\mathbb{Z}/2$ -graded countable left \mathfrak{K} -module. The assignment $A \mapsto F_*(A)$ extends to a functor $F_* = \text{KK}_*^G(B, -)$ from KK^G to $\mathbb{Z}/2$ -graded countable left \mathfrak{K} -modules.

Köhler's UCT says that there is a natural short exact sequence

$$(3.2) \quad 0 \rightarrow \text{Ext}_{\mathfrak{K}}^1(F_{*+1}A, F_*A') \rightarrow \text{KK}_*^G(A, A') \xrightarrow{F_*} \text{Hom}_{\mathfrak{K}}(F_*A, F_*A') \rightarrow 0$$

of $\mathbb{Z}/2$ -graded abelian groups for all $A \in \text{Cell}(G)$ and $A' \in \text{KK}^G$. Here $\text{Hom}_{\mathfrak{K}}(-, -)$ denotes the graded Hom, which in degree zero is the group of degree-preserving \mathfrak{K} -linear maps and in degree one the group of degree-reversing maps; and similarly for the extension group $\text{Ext}_{\mathfrak{K}}^1$. The map denoted F_* is part of the functor F_* .

3.3. *Remark.* A standard consequence of the UCT is that every isomorphism between the Köhler invariants $F_*(A)$ and $F_*(B)$ of two G -cell algebras A and B lifts to an isomorphism $A \cong B$ in $\text{Cell}(G)$. Köhler also characterizes the essential image of the functor F_* among the \mathfrak{K} -modules. Namely, it consists precisely of those countable $\mathbb{Z}/2$ -graded \mathfrak{K} -modules which are *exact* in the sense that two sequences built out of M and the internal structure of \mathfrak{K} are exact. For M of the form $F_*(A)$, the two sequences are those arising by applying $\text{KK}^G(-, A)$ to the triangle linking $\mathbb{1}, C(G), D$ and its dual triangle under Baaĵ–Skandalis duality. It follows that $A \mapsto F_*(A)$ induces a bijection between isomorphism classes of G -cell algebras and isomorphism classes of exact countable $\mathbb{Z}/2$ -graded left \mathfrak{K} -modules.

3.4. *Remark.* We have displayed in (3.1) the canonical decomposition of $F_*(A)$ into three parts. Note that any abstract \mathfrak{K} -module similarly decomposes as

$$M = M_0 \oplus M_1 \oplus M_2,$$

where the three direct summands are the images of the idempotent elements of \mathfrak{K} corresponding to the units of the endomorphism rings $\text{End}(\mathbb{1})$, $\text{End}(\text{C}(G))$ and $\text{End}(D)$, respectively. The three parts are precisely the terms appearing in each of the two sequences defining the exactness of M as in Remark 3.3.

3.5. *Remark.* Both endomorphism rings $\text{End}(\mathbb{1}) = \text{R}(G)$ and $\text{End}(\text{C}(G))$ are canonically isomorphic to $\mathbb{Z}[x]/(x^p - 1)$. So M_0 and M_1 are $\mathbb{Z}[x]/(x^p - 1)$ -modules for any \mathfrak{K} -module M . In view of later sections, let us detail what happens if we invert p . The polynomial $x^p - 1$ has two irreducible factors, $x - 1$ and $\Phi_p(x) := 1 + x + \dots + x^{p-1}$. After inverting p , they give rise to the product decomposition

$$\mathbb{Z}[x, p^{-1}]/(x^p - 1) \cong \underbrace{\mathbb{Z}[x, p^{-1}]/(x - 1)}_{\cong \mathbb{Z}[p^{-1}]} \times \underbrace{\mathbb{Z}[x, p^{-1}]/(\Phi_p)}_{\cong \mathbb{Z}[\vartheta, p^{-1}]}$$

(see Remark 6.4 below for more on this). Here we write ϑ for the ‘abstract’ primitive p -th root of unity, that is, the image of x in the right-hand side quotient. Now suppose that a \mathfrak{K} -module M is *uniquely p -divisible*, that is, it is acted on invertibly by p . In other words, suppose it is a $\mathfrak{K}[p^{-1}]$ -module. Then each of the two parts M_0 and M_1 decomposes into a sum of a $\mathbb{Z}[p^{-1}]$ -module and a $\mathbb{Z}[\vartheta, p^{-1}]$ -module, via the above actions and decomposition.

4. A SIMPLIFICATION OF THE BOOTSTRAP CATEGORY

Let $G = \mathbb{Z}/p\mathbb{Z}$. We will use the UCT for G -cell algebras recalled in Section 3, together with closely related results from [Mey19], in order to study the following localization of the G -equivariant bootstrap category.

4.1. *Notation.* Let $\text{Loc}(\text{C}(G)) \subset \text{Cell}(G)$ be the localizing subcategory generated by the G - C^* -algebra $\text{C}(G)$. Let

$$Q_G : \text{Cell}(G) \longrightarrow \text{Cell}(G) / \text{Loc}(\text{C}(G)) =: \mathcal{Q}(G)$$

be the Verdier quotient of the bootstrap category $\text{Cell}(G)$ by the localizing subcategory generated by the G - C^* -algebra $\text{C}(G)$.

The quotient $\mathcal{Q}(G)$ is related to $\text{Cell}(G)$ by a nice localization sequence of tensor triangulated categories:

$$\begin{array}{ccccc} \text{Loc}(\text{C}(G)) & \xrightarrow{\text{incl}} & \text{Cell}(G) & \xrightarrow{Q_G} & \mathcal{Q}(G) \\ & \searrow \perp & & \searrow \perp & \\ & S & & R & \end{array}$$

The following omnibus lemma makes this more precise.

4.2. **Lemma.** *The quotient functor Q_G admits a coproduct-preserving fully faithful right adjoint R . The inclusion of its full kernel $\text{Ker}(Q_G) = \text{Loc}(\text{C}(G))$ also admits a coproduct-preserving right adjoint S .*

(1) *The full essential image of R , which we denote by $\mathcal{N} := \text{Im}(R) = \text{Im}(R \circ Q_G)$, is equal to the right orthogonal of $\text{Loc}(\text{C}(G))$:*

$$\mathcal{N} = \text{Loc}(\text{C}(G))^\perp = \text{C}(G)^\perp := \{A \in \text{Cell}(G) \mid \text{KK}_*^G(\text{C}(G), A) = 0\}.$$

(2) *The kernel $\text{Ker}(Q_G) = \text{Loc}(\text{C}(G))$ equals the left orthogonal of \mathcal{N} :*

$$\text{Loc}(\text{C}(G)) = {}^\perp \mathcal{N} = \{A \in \text{Cell}(G) \mid \text{KK}_*^G(A, B) = 0 \text{ for all } B \in \mathcal{N}\}.$$

(3) *There is an essentially unique distinguished triangle in $\text{Cell}(G)$ of the form*

$$(4.3) \quad \Sigma N \longrightarrow P \longrightarrow \mathbb{1} \longrightarrow N,$$

where $P := \text{incl} \circ S(\mathbb{1}) \in \text{Loc}(\text{C}(G))$ and $N := R \circ Q_G(\mathbb{1}) \in \text{Loc}(\text{C}(G))^\perp$.

(4) *There are isomorphisms of endofunctors of $\text{Cell}(G)$*

$$\text{incl} \circ S \cong P \otimes - \quad \text{and} \quad R \circ Q_G \cong N \otimes -.$$

Proof. This all follows from standard results on rigidly-compactly generated categories, modulo the fact that we are in the *countably generated* setting. All claims can be deduced from [Del10, §2] for the case $\alpha = \aleph_1$.

More precisely, we begin by noticing that $\text{Thick}(\text{C}(G))$ is a tensor ideal in $\text{Cell}(G)^c$ and $\text{Loc}(\text{C}(G))$ is a tensor ideal in $\text{Cell}(G)$. Both follow from the isomorphisms $\text{C}(G) \otimes \text{C}(G/H) \cong \bigoplus_{G/H} \text{C}(G)$ for all $H \leq G$ and [Del14, Lem. 2.5].

Then we apply [Del10, Thm. 2.28] to the rigidly-compactly generated category $\mathcal{T} := \text{Cell}(G)$ (see Remark 2.3) and its tensor ideal of compact objects $\mathcal{J} := \text{Thick}(\text{C}(G))$ to conclude that $\mathcal{L} := \text{Loc}(\text{C}(G))$ and $\mathcal{L}^\perp = \mathcal{N}$ form a pair of localizing tensor ideals of $\text{Cell}(G)$ which are *complementary* as in [Del10, Def. 2.7]. All the remaining claims then follow from [Del10, Prop. 2.26]. \square

By construction, $\mathcal{Q}(G)$ enjoys the same structural properties as $\text{Cell}(G)$ but with an extra simplification: it is generated by its tensor unit.

4.4. Corollary. *The category $\mathcal{Q}(G)$ is a tensor triangulated category that is rigidly-compactly generated (in the countable sense). It is generated by its tensor unit, that is, $\mathcal{Q}(G) = \text{Loc}(\mathbb{1})$, and also $\mathcal{Q}(G)^c = \text{Thick}(\mathbb{1})$. The quotient functor Q_G is an exact tensor functor and preserves coproducts. Moreover, $Q_G: \text{Cell}(G) \rightarrow \mathcal{Q}(G)$ preserves compact objects; the image $Q_G(\text{Cell}(G)^c) \subseteq \mathcal{Q}(G)^c$ identifies with the Verdier quotient $\text{Cell}(G)^c / \text{Thick}(\text{C}(G))$ and embeds fully faithfully in $\mathcal{Q}(G)^c$. Its image is dense, that is, any object of $\mathcal{Q}(G)^c$ is a retract of one in $Q_G(\text{Cell}(G)^c)$.*

Proof. Again, these are standard consequences. Clearly, being the quotient of $\text{Cell}(G)$ by a localizing tensor ideal, the category $\mathcal{Q}(G)$ inherits from $\text{Cell}(G)$ a tensor triangulated structure and countable coproducts, and these are preserved by the quotient functor Q_G .

The functor $\mathcal{Q}(G)$ has a coproduct-preserving right adjoint. A short computation using this shows that it preserves compact objects. One verifies similarly that $\mathcal{Q}(G)$ is generated by the image under Q_G of the compact generators of $\text{Cell}(G)$. Since $Q_G(\text{C}(G)) \cong 0$ by construction, the compact-rigid object $\mathbb{1}_{\mathcal{Q}(G)} = Q_G(\mathbb{1}_{\text{Cell}(G)})$ suffices. The remaining claims, which are harder, are all part of Neeman's localization theorem, in its countable form (see [Del10, Thm. 2.10]). \square

Let us explicitly record another, immediate consequence of Lemma 4.2:

4.5. Corollary. *The right adjoint R of Q_G restricts to a canonical equivalence $\mathcal{Q}(G) \xrightarrow{\sim} \text{Im}(R) = \mathcal{N}$ of triangulated categories, which further restricts to an isomorphism $\text{End}_{\mathcal{Q}(G)}(\mathbb{1})_* \xrightarrow{\sim} \text{End}_{\text{Cell}(G)}(N)_*$ of graded endomorphism rings.* \square

In the remainder of this section, we move beyond abstract generalities. The next two propositions classify the objects of $\mathcal{Q}(G)$ algebraically and compute the graded endomorphism ring of its unit. We will actually work within \mathcal{N} (thanks to Corollary 4.5) and use Köhler's UCT as well as a refinement due to the second

author (see [Mey19]). Köhler's invariant $F_*: \mathrm{KK}^G \rightarrow \mathfrak{K}\text{-Mod}_\infty^{\mathbb{Z}/2}$ is recalled in Section 3. Its target is the category of $\mathbb{Z}/2$ -graded countable left \mathfrak{K} -modules. Recall that any \mathfrak{K} -module decomposes canonically as $M = M_0 \oplus M_1 \oplus M_2$.

4.6. Lemma. *Suppose that $M \in \mathfrak{K}\text{-Mod}_\infty^{\mathbb{Z}/2}$ is exact as in Remark 3.3 and satisfies $M_1 = 0$. Then M is uniquely p -divisible and is entirely determined by M_0 with its natural structure of graded $\mathbb{Z}[\vartheta, p^{-1}]$ -module. Similarly, every \mathfrak{K} -linear map of such modules is uniquely determined by its restriction to their M_0 -parts. This gives an equivalence*

$$\{M \text{ exact and } M_1 = 0\} \xrightarrow{\sim} \mathbb{Z}[\vartheta, p^{-1}]\text{-Mod}_\infty^{\mathbb{Z}/2}, \quad M \mapsto M_0,$$

between the full subcategory of exact \mathfrak{K} -modules M with $M_1 = 0$ and the category of $\mathbb{Z}/2$ -graded countable $\mathbb{Z}[\vartheta, p^{-1}]$ -modules.

Proof. Let M be an exact \mathfrak{K} -module whose M_1 -part vanishes. In particular, and trivially, the abelian group M_1 is p -divisible. Then [Mey19, Thm. 7.2] applies to M and says that the whole group M is uniquely p -divisible and that its \mathfrak{K} -module structure is of the form described in [Mey19, Ex. 7.1]. In particular,

$$M_0 = X \oplus Y, \quad M_1 = X \oplus Z, \quad M_2 = Y \oplus \Sigma Z,$$

where X is some $\mathbb{Z}/2$ -graded $\mathbb{Z}[p^{-1}]$ -module and Y, Z are some $\mathbb{Z}/2$ -graded $\mathbb{Z}[\vartheta, p^{-1}]$ -modules. The decompositions of M_0 and M_1 arise from the action of $\mathfrak{K}[p^{-1}]$ as in Remark 3.5. Since $M_1 = 0$ in our case, it follows that $X = Z = 0$. So

$$(4.7) \quad M_0 = Y, \quad M_1 = 0, \quad M_2 = Y.$$

The construction in [Mey19, Ex. 7.1] shows that the whole $\mathfrak{K}[p^{-1}]$ -action on such a \mathfrak{K} -module M is determined by the $\mathbb{Z}[\vartheta, p^{-1}]$ -action on $Y = M_0$. Similarly, every $\mathbb{Z}[x]/(x^p - 1)$ -linear map $Y \rightarrow Y'$ admits a unique \mathfrak{K} -linear extension $M \rightarrow M'$ to the corresponding \mathfrak{K} -modules. This proves the lemma. \square

4.8. Proposition. *The functor F_* restricts between \mathcal{N} and the full subcategory of $\mathfrak{K}\text{-Mod}_\infty^{\mathbb{Z}/2}$ of those exact M such that $M_1 = 0$ as in Lemma 4.6. In particular, the G -equivariant K -theory functor*

$$A \mapsto F_*(A)_0 := \mathrm{KK}_*^G(\mathbb{1}, A)$$

induces a bijection between the isomorphism classes of objects in \mathcal{N} and those in $\mathbb{Z}[\vartheta, p^{-1}]\text{-Mod}_\infty^{\mathbb{Z}/2}$.

Proof. We know from Köhler's classification (Remark 3.3) that F_* induces a bijection between the isomorphism classes of G -cell algebras and those of exact countable graded modules. If $A \in \mathcal{N} = \mathrm{Loc}(\mathrm{C}(G))^\perp$, then

$$(4.9) \quad F_*(A)_1 := \mathrm{KK}_*^G(\mathrm{C}(G), A) = 0.$$

So $F_*(A)$ is a \mathfrak{K} -module as in part (1). Hence the functor F_* restricts as claimed. Moreover, any (graded countable) $\mathbb{Z}[\vartheta, p^{-1}]$ -module gives rise to a unique exact (countable $\mathbb{Z}/2$ -graded) \mathfrak{K} -module of the form (4.7). Therefore, Köhler's classification combines with Lemma 4.6 to yield the claimed classification for \mathcal{N} . \square

We conclude the section with this central computation:

4.10. Proposition. *As above, let $G = \mathbb{Z}/p\mathbb{Z}$ for a prime number p . The graded endomorphism ring of the tensor unit $\mathbb{1} \in \mathcal{Q}(G)$ is given by*

$$\mathrm{End}_{\mathcal{Q}(G)}(\mathbb{1})_* \cong \mathbb{Z}[\vartheta, p^{-1}, \beta^{\pm 1}],$$

where ϑ is a primitive p -th root of unity (in the sense of Remark 3.5 and set in degree zero) and where β is the invertible Bott element (in degree two). More precisely, the restriction $\mathrm{End}_{\mathrm{Cell}(G)}(\mathbb{1})_* \rightarrow \mathrm{End}_{\mathcal{Q}(G)}(\mathbb{1})_*$ of the localization functor Q_G identifies with the canonical grading-preserving ring map which inverts p and kills the ideal generated by $\Phi_p(x) = 1 + x + \dots + x^{p-1}$:

$$\mathbb{Z}[x]/(x^p - 1)[\beta^{\pm 1}] \longrightarrow \mathbb{Z}[x, p^{-1}]/(\Phi_p) [\beta^{\pm 1}] = \mathbb{Z}[\vartheta, p^{-1}, \beta^{\pm 1}].$$

Proof. Recall the algebra $P \in \mathrm{Loc}(\mathrm{C}(G))$ of Lemma 4.2 (3). For any A belonging to $\mathcal{N} = \mathrm{Loc}(\mathrm{C}(G))^{\perp}$ we must have

$$(4.11) \quad \mathrm{KK}_*^G(P, A) = 0 \quad \text{and} \quad F_*(A)_1 := \mathrm{KK}_*^G(\mathrm{C}(G), A) = 0.$$

The first vanishing group implies that the map $\mathbb{1} \rightarrow N$ of the distinguished triangle (4.3) induces a natural isomorphism

$$(4.12) \quad \mathrm{KK}_*^G(N, A) \xrightarrow{\sim} \mathrm{KK}_*^G(\mathbb{1}, A) = F_*(A)_0$$

for all such $A \in \mathcal{N}$. Specializing this to the case $A = N$, we see that

$$\mathrm{KK}_*^G(N, N) \cong F_*(N)_0.$$

We claim that $F_*(N)_0$, and therefore $\mathrm{KK}_*^G(N, N)$, is a free $\mathbb{Z}[\vartheta, p^{-1}]$ -module of rank one and concentrated in degree zero. By Corollary 4.5 and after unwinding $\mathbb{Z}/2$ -gradings by Bott periodicity, this would prove the proposition.

By Proposition 4.8, there is an object $R \in \mathcal{N}$ such that $F_*(R)_0$ (and thus $F_*(R)_2$) is a free $\mathbb{Z}[\vartheta, p^{-1}]$ -module of rank one concentrated in degree zero. To prove the claim, it now suffices to prove that N and R are isomorphic in \mathcal{N} .

Consider the UCT exact sequence (3.2) for an arbitrary object $A \in \mathcal{N}$:

$$(4.13) \quad 0 \longrightarrow \mathrm{Ext}_{\mathfrak{K}}^1(F_{*+1}R, F_*A) \longrightarrow \mathrm{KK}_*^G(R, A) \xrightarrow{F_*} \mathrm{Hom}_{\mathfrak{K}}(F_*R, F_*A) \longrightarrow 0.$$

By construction, $F_*(R)_0$ is a free $\mathbb{Z}[\vartheta, p^{-1}]$ -module of rank one in degree zero. Hence by Lemma 4.6, the Hom-term in (4.13) reduces to

$$\mathrm{Hom}_{\mathbb{Z}[\vartheta, p^{-1}]}(F_*(R)_0, F_*(A)_0) \cong F_*(A)_0.$$

Now, suppose that the Ext-term in (4.13) vanishes. We would obtain an isomorphism

$$F_* : \mathrm{KK}_*^G(R, A) \xrightarrow{\sim} F_*(A)_0$$

natural in $A \in \mathcal{N}$. By combining it with the isomorphism (4.12) and applying the Yoneda Lemma for the category \mathcal{N} , this would show that $R \cong N$ as wished. Thus it only remains to show that the Ext-term of (4.13) vanishes.

Consider an arbitrary extension

$$(4.14) \quad 0 \longrightarrow F_*(A) \longrightarrow M \longrightarrow F_{*+1}(R) \longrightarrow 0$$

of graded \mathfrak{K} -modules. By applying to (4.14) the idempotent element of \mathfrak{K} corresponding to the identity map of $\mathrm{C}(G)$ we obtain an exact sequence

$$0 \longrightarrow F_*(A)_1 \longrightarrow M_1 \longrightarrow F_{*+1}(R)_1 \longrightarrow 0$$

where both outer terms vanish by hypothesis. Then $M_1 = 0$. In particular, M_1 is uniquely p -divisible. Since M is exact as an extension of two exact \mathfrak{K} -modules, we may apply [Mey19, Thm. 7.2] to it. Thus M also has the special form of Lemma 4.6 and is uniquely determined by its M_0 -part, viewed as a $\mathbb{Z}[\vartheta, p^{-1}]$ -module. Now consider the extension of graded $\mathbb{Z}[\vartheta, p^{-1}]$ -modules

$$0 \longrightarrow F_*(A)_0 \longrightarrow M_0 \longrightarrow F_{*+1}(R)_0 \longrightarrow 0$$

obtained by hitting (4.14) with the idempotent of \mathfrak{K} corresponding to the identity of $\mathbb{1}$. This extension must split because $F_{*+1}(R)_0$ is a free module. Moreover, by Lemma 4.6 once again, any $\mathbb{Z}[\vartheta, p^{-1}]$ -linear section of $M_0 \rightarrow F_{*+1}(R)_0$ extends to a \mathfrak{K} -linear section of $M \rightarrow F_{*+1}(R)$. Thus the original extension (4.14) of \mathfrak{K} -modules splits as well. As the latter extension was arbitrary, this implies that $\text{Ext}_{\mathfrak{K}}^1(F_{*+1}R, F_*A) = 0$ as required. \square

5. THE BALMER SPECTRUM

We very briefly recall some basic notions of tensor triangular geometry, referring to [Bal10b] and the original references therein for more details.

Let \mathcal{K} be an essentially small tensor triangulated category, with tensor \otimes and unit object $\mathbb{1}$. Its *spectrum* $\text{Spc } \mathcal{K}$ is the set of its *prime* \otimes -ideals \mathcal{P} , that is, those proper, full and thick subcategories $\mathcal{P} \subsetneq \mathcal{K}$ which are prime tensor ideals for the tensor product: $A \otimes B \in \mathcal{P} \Leftrightarrow A \in \mathcal{P} \text{ or } B \in \mathcal{P}$, for any objects $A, B \in \mathcal{K}$. The spectrum is endowed with the ‘Zariski’ topology, which has the family of subsets

$$(5.1) \quad \text{supp}(A) := \{\mathcal{P} \in \text{Spc } \mathcal{K} \mid A \notin \mathcal{P}\} \quad (A \in \mathcal{K})$$

as a basis of closed subsets.

The ring $\text{End}_{\mathcal{K}}(\mathbb{1})$ is commutative. So we can consider the usual Zariski spectrum of its prime ideals, $\text{Spec } \text{End}_{\mathcal{K}}(\mathbb{1})$. The assignment

$$\mathcal{P} \mapsto \rho_{\mathcal{K}}(\mathcal{P}) := \{f \in \text{End}_{\mathcal{K}}(\mathbb{1}) \mid \text{cone}(f) \notin \mathcal{P}\}$$

defines a continuous map between the two spectra:

$$(5.2) \quad \rho_{\mathcal{K}}: \text{Spc } \mathcal{K} \longrightarrow \text{Spec } \text{End}_{\mathcal{K}}(\mathbb{1})$$

(see [Bal10a, Thm. 5.3]). In general, this comparison map is neither injective nor surjective. Surjectivity is more common and often much easier to prove.

5.3. *Remark.* The spectrum is functorial: Any exact tensor functor $F: \mathcal{K} \rightarrow \mathcal{L}$ defines a continuous map $\text{Spc } F: \text{Spc } \mathcal{K} \rightarrow \text{Spc } \mathcal{L}$ sending a prime \mathcal{P} of \mathcal{K} to $(\text{Spc } F)(\mathcal{P}) := F^{-1}\mathcal{P}$. Moreover, $\text{Spc}(F_2 \circ F_1) = \text{Spc } F_1 \circ \text{Spc } F_2$ and $\text{Spc } \text{Id} = \text{Id}$.

5.4. *Remark.* The comparison map (5.2) is natural (see [Bal10a, Thm. 5.3 (c)]); that is, if $F: \mathcal{K} \rightarrow \mathcal{L}$ is an exact tensor functor, then the square

$$\begin{array}{ccc} \text{Spc } \mathcal{L} & \xrightarrow{\text{Spc } F} & \text{Spc } \mathcal{K} \\ \rho_{\mathcal{L}} \downarrow & & \downarrow \rho_{\mathcal{K}} \\ \text{Spec } \text{End}_{\mathcal{L}}(\mathbb{1}) & \xrightarrow{\text{Spec } F} & \text{Spec } \text{End}_{\mathcal{K}}(\mathbb{1}) \end{array}$$

commutes; here the bottom arrow is the continuous map $\mathfrak{p} \mapsto F^{-1}\mathfrak{p}$ induced between Zariski spectra by the ring homomorphism $F: \text{End}_{\mathcal{K}}(\mathbb{1}) \rightarrow \text{End}_{\mathcal{L}}(\mathbb{1})$.

6. COMPUTATION OF THE SPECTRUM

Finally, we are ready to prove Theorem 1.1.

Let $G = \mathbb{Z}/p\mathbb{Z}$ as before. Let $\text{Cell}(G) \subset \text{KK}^G$ be the tensor triangulated category of G -cell algebras (Definition 2.1). Let $\text{Cell}(G)^c \subset \text{Cell}(G)$ be the tensor triangulated subcategory of its compact objects (Remark 2.3). Write

$$\rho_G := \rho_{\text{Cell}(G)^c} : \text{Spc Cell}(G)^c \rightarrow \text{Spec R}(G)$$

for the canonical map (5.2) comparing the triangular spectrum of $\text{Cell}(G)^c$ to the Zariski spectrum of the representation ring $\text{R}(G) = \text{End}_{\text{Cell}(G)^c}(\mathbb{1}) = \mathbb{Z}[x]/(x^p - 1)$.

Our goal is to show that ρ_G is a homeomorphism. To this end, we will adapt the ‘divide and conquer’ strategy of [BS17] to the two exact tensor functors

$$(6.1) \quad \text{Cell}(1) \xleftarrow{\text{Res}_1^G} \text{Cell}(G) \xrightarrow{Q_G} \mathcal{Q}(G)$$

given by restriction to the trivial group, Res_1^G , and the Verdier quotient functor Q_G studied in Section 4.

6.2. Lemma. *The graded commutative ring $\text{End}_{\mathcal{Q}(G)}(\mathbb{1})_* \cong \mathbb{Z}[\vartheta, p^{-1}, \beta^{\pm 1}]$ computed in Proposition 4.10 is Noetherian and regular.*

Proof. The degree zero subring $\mathbb{Z}[\vartheta, p^{-1}]$ is Noetherian because it is finitely generated and commutative. It is regular because it is a localization of the classical Dedekind domain $\mathbb{Z}[\vartheta] \cong \mathbb{Z}[\zeta_p] \subset \mathbb{C}$, where ζ_p is a complex primitive p -th root of unit. It follows that our graded ring is also Noetherian and regular. \square

6.3. Remark. If R_* is a \mathbb{Z} -graded commutative ring, we can consider the Zariski spectrum $\text{Spec}^h R_*$ of its *homogeneous* prime ideals. The inclusion of the zero degree subring $R_0 \subset R_*$ induces a continuous restriction map $\text{Spec}^h R_* \rightarrow \text{Spec} R_0$. If the graded ring R_* is 2-periodic and concentrated in even degrees – that is, $R_* = R_0[\beta^{\pm 1}]$ with $|\beta| = 2$ – the latter map is easily seen to be a homeomorphism $\text{Spec}^h R_* \xrightarrow{\sim} \text{Spec} R_0$. Of course, this applies to our endomorphism rings.

6.4. Remark. Consider the Zariski spectrum of $\text{R}(G) = \mathbb{Z}[x]/(x^p - 1)$. It has two irreducible components, namely, the images of the two embeddings

$$\text{Spec } \mathbb{Z} \xrightarrow{\psi} \text{Spec } \mathbb{Z}[x]/(x^p - 1) \xleftarrow{\varphi} \text{Spec } \mathbb{Z}[x]/(\Phi_p)$$

induced by the two ring quotients

$$\mathbb{Z} \longleftarrow \mathbb{Z}[x]/(x^p - 1) \longrightarrow \mathbb{Z}[x]/(\Phi_p)$$

given by killing $x - 1$ and $\Phi_p = 1 + x + \dots + x^{p-1}$, respectively; these are the two irreducible factors of $x^p - 1$ in $\mathbb{Z}[x]$. Thus the maps ψ and φ are jointly surjective. The intersection of their images can be shown to consist exactly of one point lying above p , namely, the maximal ideal $\psi((p)) = (p, x - 1) = (p, \Phi_p) = \varphi((p))$. By inverting p in $\mathbb{Z}[x]/(\Phi_p)$ we get rid of precisely the preimage under φ of this common point, thus eliminating the redundancy. In particular, we obtain a decomposition

$$\text{Spec R}(G) \cong \text{Spec } \mathbb{Z} \sqcup \text{Spec } \mathbb{Z}[x, p^{-1}]/(\Phi_p)$$

as sets. As before, we write $\mathbb{Z}[\vartheta, p^{-1}] := \mathbb{Z}[x, p^{-1}]/(\Phi_p)$. (This is all well-known; see, for instance, [BD08] and the references therein for explanations and context.)

Now we know everything we need about the geometry of Q_G . As for the restriction functor $\text{Res}_1^G: \text{Cell}(G)^c \rightarrow \text{Cell}(1)^c$, we need the following:

6.5. Lemma. *The equality $\text{supp } C(G) = \text{Im}(\text{Spc } \text{Res}_1^G)$, between the support of the object $C(G)$ and the image of the map $\text{Spc}(\text{Res}_1^G)$, holds in $\text{Spc } \text{Cell}(G)^c$.*

Proof. By [BDS15, Thm. 1.2] (see also [BD20, §2.4]), the restriction functor Res_1^G is a *finite separable extension*. More precisely, this is proved in [BDS15] for the restriction functor $\text{KK}^G \rightarrow \text{KK}(1)$ between the *whole* Kasparov categories. But the result also holds for (compact) cell algebras. Let us briefly recall this. The two-sided adjunction between Ind_1^G and Res_1^G restricts to a two-sided adjunction between $\text{Cell}(G)$ and $\text{Cell}(1)$ (Remark 2.4), and provides us with a separable commutative monoid A_1^G in $\text{Cell}(G)$ whose underlying object is $\text{Ind}_1^G(1) = C(G)$. Exactly the same proofs as in [BDS15] yield a canonical equivalence of tensor triangulated categories

$$\text{Cell}(1) \simeq A_1^G\text{-Mod}_{\text{Cell}(G)}$$

between the bootstrap category $\text{Cell}(1)$ and the Eilenberg–Moore category of modules in $\text{Cell}(G)$ over the monoid A_1^G . This equivalence identifies Res_1^G with the ‘free module’ functor $F := A_1^G \otimes -: \text{Cell}(G) \rightarrow A_1^G\text{-Mod}_{\text{Cell}(G)}$. It may also be restricted to compact objects: $\text{Cell}(1)^c \simeq (A_1^G\text{-Mod}_{\text{Cell}(G)})^c = A_1^G\text{-Mod}_{\text{Cell}(G)^c}$.

As with any separable monoid, [Bal16, Thm. 1.5] shows the equality $\text{Im}(\text{Spc } F) = \text{supp } A_1^G$. Hence $\text{Im}(\text{Spc } \text{Res}_1^G) = \text{supp } C(G)$ by the above identifications. \square

Proof of Theorem 1.1. We already know from [Del10, Thm. 1.4] that the map ρ_G admits a continuous section – even for any finite group G . To show that it is a homeomorphism, it will therefore suffice to prove its injectivity.

We claim that the two functors (6.1) induce the following commutative diagram:

$$(6.6) \quad \begin{array}{ccccc} \text{Spc } \text{Cell}(1)^c & \xrightarrow{\text{Spc } \text{Res}_1^G} & \text{Spc } \text{Cell}(G)^c & \xleftarrow{\text{Spc } Q_G} & \text{Spc } Q(G)^c \\ \rho_1 \downarrow & & \rho_G \downarrow & & \rho_{Q(G)^c} =: \rho_Q \downarrow \\ \text{Spec } \mathbb{Z} & \xrightarrow{\psi} & \text{Spec } \mathbb{Z}[x]/(x^p - 1) & \xleftarrow{\varphi} & \text{Spec } \mathbb{Z}[\vartheta, p^{-1}] \end{array}$$

Indeed, the top row is obtained by restricting the two functors to rigid-compact objects and applying the functoriality of $\text{Spc}(-)$ (Remark 5.3). The three vertical maps are all instances of the canonical comparison (5.2). The two squares commute because the latter is natural (see Remark 5.4). The bottom row is as in Remark 6.4: Indeed, the right arrow is given by inverting p and killing Φ_p (by Proposition 4.10) and the left arrow by mapping $x \mapsto 1$ (because it corresponds to the rank homomorphism $R(G) \rightarrow R(1) = \mathbb{Z}$).

By Remark 6.4, the bottom row of (6.6) is a disjoint-union decomposition of the set $\text{Spec } \mathbb{Z}[x]/(x^p - 1)$.

We claim that the top row of (6.6) provides a similar decomposition, that is, it consists of two injective and jointly surjective maps with disjoint images, so that

$$(6.7) \quad \text{Spc } \text{Cell}(G)^c = \text{Im}(\text{Spc } \text{Res}_1^G) \sqcup \text{Im}(\text{Spc } Q_G).$$

Indeed, we obviously have

$$\text{Spc } \text{Cell}(G)^c = \{\mathcal{P} \mid C(G) \notin \mathcal{P}\} \sqcup \{\mathcal{P} \mid C(G) \in \mathcal{P}\}.$$

Moreover, by Corollary 4.4 the restriction of Q_G on compact objects factors as a Verdier quotient with kernel $\text{Thick}(\mathcal{C}(G))$ followed by a full dense embedding:

$$\text{Cell}(G)^c \longrightarrow \text{Cell}(G)^c / \text{Thick}(\mathcal{C}(G)) \hookrightarrow \mathcal{Q}(G)^c.$$

By basic tensor-triangular results [Bal05, Prop. 3.11 and Prop. 3.13], the induced map $\text{Spc } Q_G$ is therefore injective with image

$$\text{Im}(\text{Spc } Q_G) = \{\mathcal{P} \mid \text{Thick}(\mathcal{C}(G)) \subseteq \mathcal{P}\} = \{\mathcal{P} \mid \mathcal{C}(G) \in \mathcal{P}\}.$$

Let us consider the other half of the decomposition. Recall the inflation functor $\text{Inf}_1^G: \text{Cell}(1)^c \rightarrow \text{Cell}(G)^c$ (Remark 2.4), which endows each $A \in \text{Cell}(1)^c$ with the trivial G -action. It is an exact tensor functor such that $\text{Res}_1^G \circ \text{Inf}_1^G = \text{Id}$. Since Spc is a contravariant functor (Remark 5.3), we deduce that

$$\text{Spc}(\text{Inf}_1^G) \circ \text{Spc}(\text{Res}_1^G) = \text{Spc}(\text{Res}_1^G \circ \text{Inf}_1^G) = \text{Spc}(\text{Id}_{\text{Cell}(1)^c}) = \text{Id}.$$

Thus $\text{Spc}(\text{Res}_1^G)$ is injective. By Lemma 6.5 and the definition of support (5.1), its image is

$$\text{Im}(\text{Spc } \text{Res}_1^G) = \text{supp } \mathcal{C}(G) = \{\mathcal{P} \mid \mathcal{C}(G) \notin \mathcal{P}\}.$$

This concludes the proof of the decomposition (6.7).

The above decompositions of the triangular and Zariski spectra and the commutative diagram (6.6) imply that the middle vertical map ρ_G is bijective if and only if both ρ_1 and ρ_Q are. We know from [Del10, Thm. 1.2] that ρ_1 is bijective. As for $\rho_{Q(G)}$, we will appeal to [DS16].

The comparison map has a graded version

$$\rho_Q^*: \text{Spc } \mathcal{Q}(G)^c \longrightarrow \text{Spec}^h \text{End}(\mathbb{1})_*,$$

whose target is the *homogeneous* spectrum of the *graded* endomorphism ring of $\mathcal{Q}(G)$ (see Remark 6.3); this follows from [Bal10a, Thm. 5.3] for the choice $u = \Sigma(\mathbb{1})$ of grading object. By Corollary 4.4, $\mathcal{Q}(G)^c$ is generated by its tensor unit as a thick subcategory. By Lemma 6.2, the graded ring $\text{End}_{\mathcal{Q}(G)^c}(\mathbb{1})_*$ is Noetherian and regular. Therefore, [DS16, Thm. 1.1] shows that the graded comparison map ρ_Q^* is bijective. The map ρ_Q is bijective as well because the isomorphism $\text{Spec}^h \text{End}(\mathbb{1})_* \cong \text{Spec } \text{End}(\mathbb{1})$ of Remark 6.3 identifies ρ_G^* with ρ_Q (see [Bal10a, Cor. 5.6.(b)]). This completes the proof. \square

6.8. *Remark.* As already mentioned, the present proof of Theorem 1.1 is loosely inspired by the analogous determination (as a set) of the spectrum of $\text{SH}(G)^c$, the stable homotopy category of compact G -spectra; more precisely, by the proof of [BS17, Thm. 4.9]. The latter argument works for any finite group G by induction on its order, and this induction *could* be adapted to yield a homeomorphism $\rho: \text{Spc } \text{Cell}(G)^c \xrightarrow{\sim} \text{Spec } R(G)$ for general G , *provided* we knew that a certain ring is regular for all G (in order to invoke [DS16]). The ring in question is the graded endomorphism ring of the tensor unit in the tensor triangulated category

$$\text{Cell}(G) / \text{Loc}(\{\mathcal{C}(G/H) \mid H \leq G\})$$

which we currently do not know how to compute. In other words, we would need to find a general replacement for our use of Köhler's UCT in Section 4.

6.9. *Remark.* Besides the proofs' analogies, the result in [BS17] is actually quite different from ours. Most strikingly, the comparison map $\rho_{\text{SH}(G)^c}$ is very far from being injective. In particular, [DS16] cannot be applied to the case of G -spectra;

in this case, the role of [DS16] in the proof's structure is played instead by the fact that the composite of inflation followed by localization

$$\mathrm{SH} \xrightarrow{\mathrm{Inf}_1^G} \mathrm{SH}(G) \longrightarrow \mathrm{SH}(G)/\mathrm{Loc}(\{\Sigma_+^\infty G/H \mid H \not\leq G\})$$

is an equivalence of tensor triangulated categories (see [BS17, §2 (H)]). The analogous result for KK-theory is false. By Proposition 4.10, it fails for $G \cong \mathbb{Z}/p\mathbb{Z}$.

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