# Equivariant homological algebra in tensor-triangulated categories

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# Relative homological algebra

Christensen [1998], Beligiannis [2000], Meyer–Nest [2010], ...

Fix a (big) triangulated category $\mathcal{T}$.

To do relative homological algebra in $\mathcal{T}$, we can axiomatize either:

- a class $\mathcal{P}$ of relative projective objects, or
- an ideal $\mathcal{J}$ of relative phantom maps, or
- a class $\mathcal{E}$ of relative exact distinguished triangles.

Under mild hypotheses, these data are all equivalent as follows:

$$\begin{tikzcd} & & P \in \mathcal{P} \arrow[d, "\forall"'] \arrow[dr, "0"] & \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r, "\in \mathcal{J}"'] & \Sigma X \end{tikzcd} \qquad \in \mathcal{E}$$

# Relative homological algebra

A good situation: when $\mathcal{T}$ is *compactly generated*, and the relative projectives

$$\mathcal{P} := \mathsf{Add}(\mathcal{G})$$

is additively generated by a $\Sigma$-stable set of compact objects $\mathcal{G} \overset{\text{full}}{\subseteq} \mathcal{T}^c$.

Then:

- The relative phantoms form the kernel of the **restricted Yoneda functor**

$$\begin{aligned} h_{\mathcal{G}} \quad : \quad \mathcal{T} &\longrightarrow \mathsf{Mod}\,\mathcal{G} := \mathsf{Fun}_{\mathsf{add}}(\mathcal{G}^{\mathsf{op}}, \mathsf{Ab}) \\ X &\longmapsto \mathcal{T}(-, X)|_{\mathcal{G}} \end{aligned}$$

- The latter is the universal homological functor which kills the phantoms.
- One can compute in the abelian category $\mathsf{Mod}\,\mathcal{G}$ in order to approximate $\mathcal{T}$, or at least $\mathsf{Loc}(\mathcal{G})$ ...

# Examples: two extreme cases

- $\mathcal{G} = \mathcal{T}^c$: take all the compacts!

  For $\mathcal{T} = Ho(Sp)$ the homotopy category of spectra: original phantom maps!

  For $\mathcal{T} = D(R)$ the derived category of a ring, linked to purity.

- If $\mathcal{T}$ is a tensor-triangulated category with unit object $\mathbf{1}$, we can set:

$$\mathcal{G} = \{\Sigma^n \mathbf{1} \mid n \in \mathbb{Z}\}$$

  Then $h_{\mathcal{G}}$ simply computes the homotopy/cohomology groups as a module over the graded endomorphism ring $\mathrm{End}^*_{\mathcal{T}}(\mathbf{1})$.

  Here the relative phantoms are sometimes called *ghosts*.

- In this talk I am interested in "equivariant ghosts"...

# Equivariant settings

Let $G$ be a finite group.

Suppose now $\mathcal{T} = \mathcal{T}(G)$ is a tensor-triangulated category associated to $G$.

Examples:

- $\mathcal{T}(G) = D(kG)$ : the derived category of $k$-linear $G$-representations.
- $\mathcal{T}(G) = StMod(kG)$ : the stable module category.
- $\mathcal{T}(G) = Ho(Sp^G)$ : the homotopy category of genuine $G$-spectra.
- More generally:
  $\mathcal{T}(G) = Ho(Mod_A)$, where $A$ is a (highly structured) commutative algebra in $G$-spectra $Sp^G$.

In all these cases, homological algebra involves

Mackey functors (a "$G$-equivariant version" of abelian groups), and

Green functors (the corresponding "$G$-equivariant version" of rings).

**GOAL:** explain this phenomenon, by axiomatizing a "$G$-equivariant version" of tensor-triangulated categories!

# Highly structured vs algebraic explanations

There *are* explanations, involving $\infty$-categories and parametrized topology (Barwick and collaborators; Cnossen–Lenz–Linskens [2023-25]).

However these use very heavy topological machinery, and it is hard to recognize examples this way.

Today's approach is purely algebraic (up to 2-categorical) and the axioms are relatively easy to check!

**Common idea:** parametrize everything over $G$-sets, hence replace the point by the collection of all orbits $G/H$ for $H \leq G$.

$\rightsquigarrow$ in a "$G$-equivariant tt-category $\mathcal{T}$", we should have a set of "orbits" $\mathbb{1}(G/H)$ generalizing the tensor unit $\mathbb{1} = \mathbb{1}(G/G)$. For "$G$-equivariant ghosts", we should set

$$\mathcal{G} = \{\, \Sigma^n \mathbb{1}(G/H) \mid H \leq G, n \in \mathbb{Z} \,\}$$

and $\operatorname{Mod} \mathcal{G}$ should be some category of Mackey functors.

# Mackey functors (for a fixed finite group $G$)

Green [1971], Dress [1973], Lindner [1976].

A **Mackey functor** $M$ is ...

- *Original definition*:
  - a family of abelian groups $M(H)$ for all subgroups $H \leq G$
  - with restriction, induction and conjugation additive maps between them
  - satisfying many relations (functoriality, commutativity, Mackey formula ...)

- *"Motivic" definition*:
  - just an additive functor

$$M \colon \mathrm{Span}(G\text{-set}) \to \mathrm{Ab}$$

  - on the additive category of spans of finite $G$-sets (with $U \oplus V = U \sqcup V$):

$$\color{red}{\mathrm{Span}(G\text{-set})} := \begin{cases} \text{objects} = \text{finite } G\text{-sets} \\ \mathrm{Hom}(U,V) = \mathbb{Z} \otimes_{\mathbb{N}} \left\{ U \leftarrow W \rightarrow V \text{ in } G\text{-set} \right\} / \mathrm{iso} \\ \text{composition via pullbacks.} \end{cases}$$

The correspondence: via $M(H) \leftrightsquigarrow M(G/H)$, and a presentation of $\mathrm{Span}(G\text{-set})$.

# Green functors

A **Green functor** $R$ is ...

- *Original definition*:
  - a Mackey functor $R$ such that each $R(H)$ is a ring
  - the restriction and conjugation maps are ring morphisms
  - satisfying the Frobenius formulas (or projection formulas):

$$ind_K^H(x \cdot res_K^H(y)) = ind_K^H(x) \cdot y \qquad \text{for all } K \leq H \leq G, \text{ etc.}$$

- *Motivic definition*:
  - just a monoid in the abelian tensor category Mack of Mackey functors!
  - Indeed, the Cartesian product $U \times V$ of $G$-sets induces a tensor product in Span($G$-set) which by Day convolution extends on the category of Mackey functors.

**Modules over a Green functor** $R$ can be defined easily in both pictures, giving rise to an abelian category:

$R$-Mack

# Back to our "$G$-equivariant triangulated categories"

Let $\mathcal{T}(G)$ be one of our triangulated categories "of $G$-objects".

We have the analog category $\mathcal{T}(H)$ for each subgroup $H \leq G$, and extra structure:

- The **restriction**, **induction** and **conjugation** functors ($K \leq H \leq G \ni g$):

$$\begin{tikzcd} \mathcal{T}(H) \arrow[d, "Res"] & \\ \mathcal{T}(K) \arrow[u, bend left, "Ind"] \arrow[r, "Conj_g", "\sim"'] & \mathcal{T}({}^gK) \end{tikzcd}$$

- The two **adjunctions** $Ind \dashv Res \dashv Ind$.
- **Conjugation natural isos** between composites functors, e.g.

$$conj_g : \; Conj_g \circ Res^G_H \cong Res^G_{{}^gH}$$

- The **Mackey formula** (for $K, L \leq G$):

$$Res^G_L \circ Ind^G_K \cong \bigoplus_{[g] \in L \backslash G / K} Ind^L_{L \cap {}^gK} \circ Conj_g \circ Res^K_{L^g \cap K} \,.$$

# Axiomatization: Mackey 2-functors

Each family $\mathcal{T} = \{\mathcal{T}(H), \mathit{Ind}, \mathit{Res}, \ldots\}_{H \leq G, \ldots}$ satisfies the following:

## Definition [Balmer-D. 2020]

A **Mackey 2-functor** is a 2-functor or pseudo-functor

$$\mathcal{T} \colon (G\text{-set})^{\mathrm{op}} \longrightarrow \mathit{ADD}$$

to the 2-category of additive categories, satisfying the following axioms:

1. Additivity: $\mathcal{T}(U \sqcup V) \xrightarrow{\sim} \mathcal{T}(U) \times \mathcal{T}(V)$.
2. Every "restriction functor" $f^* := \mathcal{T}(f) \colon \mathcal{T}(V) \to \mathcal{T}(U)$ along a map $f \colon U \to V$ of $G$-sets admits a left-and-right adjoint $f_*$.
   (Then e.g. $\mathrm{Res}^G_H = f^*$ and $\mathrm{Ind}^G_H = f_*$ for $f \colon G/H \to G/G$.)
3. Left and right adjunctions satisfy Beck–Chevalley for pullback squares.
   (This provides the Mackey formula.)

We say the Mackey 2-functor $\mathcal{T}$ is **triangulated** when it takes values in the 2-category of triangulated categories and exact functors.

# Axiomatization: Green 2-functors

Each of our families $\mathcal{T}$ actually consists of *tensor* triangulated categories. Their tensor and Mackey 2-functor structures are compatible as follows:

## Definition [D. 2022]

A **(symmetric) Green 2-functor** is a Mackey 2-functor $\mathcal{T}$ equipped with a lifting

$$\begin{tikzcd} & & PsMon(ADD) \arrow[d, "forget"] \\ (G\text{-set})^{\mathrm{op}} \arrow[rru, dashed] \arrow[rr, "\mathcal{T}"] & & ADD \end{tikzcd}$$

to (symmetric) monoidal additive categories, satisfying the **projection formulas:**

$$f_*(\, f^*(X) \otimes Y \,) \;\cong\; X \otimes f_*(Y) \qquad (\, X \in \mathcal{T}(U), Y \in \mathcal{T}(V)\,)$$

for every map $f\colon U \to V$ of $G$-sets, via the evident mates for the left and for the right adjunctions $f_* \dashv f^* \dashv f_*$.

Again, the Green 2-functor $\mathcal{T}$ is **triangulated** if *ADD* is replaced by triangulated categories.

# We made it!

So, our precise axiomatization of "$G$-equivariant tensor-triangulated category" $\mathcal{T}$, or ***G*-tt-category** for short, is as a *triangulated symmetric Green 2-functor.*

Remarks:

- These axioms are purely algebraic at the 1- and 2-categorical level.
- They are relatively easy to check!

But we already get:

### Theorem [D. 2025]

If $\mathcal{T}$ is a (compactly generated) $G$-tt-category and if we set

$$\mathcal{G} := \{\, \Sigma^n \operatorname{Ind}_H^G(\mathbb{1}) \mid H \leq G, n \in \mathbb{Z} \,\},$$

then restricted Yoneda lands in the category of Mackey modules

$$\operatorname{Mod} \mathcal{G} \simeq R^*_{\mathcal{T}}\text{-Mack}$$

over a graded commutative Green functor $R^*_{\mathcal{T}}$ such that $R^*_{\mathcal{T}}(H) = \operatorname{End}^*_{\mathcal{T}(H)}(\mathbb{1})$ for all $H \leq G$.

# Explanation

This follows from:

## Theorem [D. 2025]

The following data are equivalent:

1. A $G$-tt-category $\mathcal{T}$, as before.
2. A symmetric monoidal additive functor $\mathbb{A}\colon \mathrm{Span}(G\text{-set}) \to \mathcal{D}$ into some tensor-triangulated category $\mathcal{D}$.

The correspondence:

- Given $\mathcal{T}$, set $\mathcal{D} := \mathcal{T}(G)$ and define $\mathbb{A}$ on objects by $\mathbb{A}(U) := f_*(\mathbf{1})$ where $f\colon U \to G/G$ is the unique map of $G$-sets (or by: $\mathbb{A}(G/H) := \mathrm{Ind}_H^G(\mathbf{1})$). On maps, use the units and counits of the adjunctions $f_* \dashv f^* \dashv f_*$.
- Given $\mathbb{A}$ and for each $H \leq G$, set

$$\mathcal{T}(H) := \mathbb{A}(G/H)\text{-}\mathrm{Mod}_{\mathcal{D}}$$

using that $G/H$ is canonically a (special Frobenius) commutative algebra in $\mathrm{Span}(G\text{-sets})$, hence $\mathbb{A}(G/H)$ is one in $\mathcal{D}$.

# Idea of proof

- To recover $\mathcal{T}$ from the associated functor $\mathbb{A}$, use monadicity of restriction functors (Balmer–D.–Sanders [2015]):

$$\begin{tikzcd} \mathcal{T}(G) \arrow[d, "\mathrm{Res}^G_H"'] \arrow[dr, "\text{free module}"] & \\ \mathcal{T}(H) \arrow[r, dashed, "\simeq"] & \mathbb{A}(G/H)\text{-}\mathrm{Mod}_{\mathcal{T}(G)} \end{tikzcd}$$

- Difficult part: to use the limited coherence requirements on the $G$-tt-category $\mathcal{T}$ to show that the formulas for $\mathbb{A}$ actually define a functor!

Then, for the previous theorem: Once we have a tensor functor $\mathbb{A}$ on Span($G$-set), it is easy to produce Mackey (and Green) functors by post-composing $\mathbb{A}$ with additive (and lax monoidal additive) functors defined on $\mathcal{T}(G)$.

*Thank you for your attention!*

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