Module categories over tensor categories: some structures and some applications

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based on work with many people

October 8, 2024

Overview

Chapter 1

Finite module categories

Finite tensor categories

General setting: k-linear abelian categories with finiteness properties. k a field, say $k = \mathbb{C}$.

Definition (Finite category)

A k-linear abelian category $\mathcal{C}$ is finite, if

1. $\mathcal{C}$ has finite-dimensional spaces of morphisms.
2. Every object of $\mathcal{C}$ has finite length.
3. $\mathcal{C}$ has enough projectives.
4. There are finitely many isomorphism classes of simple objects.

Remark

Linear abelian category $\mathcal{A}$ finite $\Leftrightarrow$ $\mathcal{A} \cong A\text{-mod}_{fd}$ with A finite-dimensional k-algebra.

Definition (Finite tensor category)

A finite tensor category is a finite rigid monoidal linear category.
A fusion category is a semisimple finite tensor category.

Rigidity

Definition

1. Let $\mathcal{C}$ be a monoidal category. A right dual to an object $a \in \mathcal{C}$ consists of an object $a^\vee \in \mathcal{C}$ and morphisms $1 \to a \otimes a^\vee$ and $a^\vee \otimes a \to 1$ that fulfill the appropriate snake relations; left duals are defined similarly.
   Having a dual is a property.
2. A monoidal category is right rigid, if every object has a right dual. A monoidal category is rigid, if every object has a left and a right dual.

Remarks

1. Example: category of finite-dimensional modules over a finite-dimensional k-Hopf algebra.
2. Consequence of rigidity: the internal hom
   $\mathrm{Hom}_\mathcal{C}(V \otimes W, Z) \cong \mathrm{Hom}_\mathcal{C}(V, \underline{\mathrm{Hom}}(W, Z))$
   and coHom
   $\mathrm{Hom}_\mathcal{C}(Z, V \otimes W) \cong \mathrm{Hom}_\mathcal{C}(\underline{\mathrm{coHom}}(W, Z), V)$
   exist.
3. In particular, the tensor product of a rigid category is exact in both arguments.

Module categories

Definition (Module categories, Pareigis, Bernstein, ...)

Let $\mathcal{A}$ and $\mathcal{B}$ be linear monoidal categories.

1. A left $\mathcal{A}$-module category is a linear category $\mathcal{M}$ with a bilinear functor $\otimes : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms

$$\alpha_{a_1,a_2,m} : \quad a_1.(a_2.m) \xrightarrow{\sim} (a_1 \otimes a_2).m \qquad \lambda_m : \quad 1.m \xrightarrow{\sim} m$$

   satisfying obvious pentagon and triangle axioms. We write $a.m := a \otimes m$.
2. Right module categories are defined analogously.
3. An $\mathcal{A}$-$\mathcal{B}$ bimodule category is a linear category $\mathcal{D}$, with the structure of a left $\mathcal{A}$ and right $\mathcal{B}$-module category and a natural associator isomorphism $(a.d).b \cong c.(d.b)$.
4. Module functors, module natural transformations defined in obvious way.

Definition (Finite module categories)

Let $\mathcal{A}$ be a finite tensor category over k. A left $\mathcal{A}$-module category is finite, if the underlying category is a finite abelian category over k and the action is k-linear in each variable and right exact in the first variable.

Examples

Examples

1. Let H be a finite-dimensional k-Hopf algebra. The category vect_{fd} is a finite module category over H-mod_{fd}.
2. More generally, for A a comodule algebra over a Hopf algebra, A-mod is an H-module category.
3. Let $(A, \mu : A \otimes A \to A)$ be an associative monoid in $\mathcal{A}$. Then the category $\mathrm{mod}_{\mathcal{A}} - A$ of right A-modules is a left $\mathcal{A}$-module category by considering $a \otimes m$ for $a \in \mathcal{A}$ and $m \in \mathrm{mod}_{\mathcal{A}} - A$ as a right A module.
4. Let G be a finite group and vect_G be the fusion category of finite-dimensional G-graded $\mathbb{C}$-vector spaces.
   For a subgroup $H \subset G$ the object $A = \oplus_{g \in H} \mathbb{C}_g$ is endowed by any choice of a 2-cocycle $\psi \in Z^2(H, \mathbb{C}^*)$ with an associative multiplication. All module categories are of this form (Ostrik).
   This example motivates the term "quantum subgroup".
5. Let $\mathcal{C}$ be the fusion category associated to $\mathrm{sl}(2)$ at level $k \in \mathbb{N}$. Semisimple module categories are classified by an ADE pattern (Kirillov-Ostrik)

Exact module categories

The example of the fusion category vect shows that, in general, classifying all module categories is hopeless.

Definition (Etingof-Ostrik)

An exact module category $\mathcal{M}$ over a finite tensor category $\mathcal{A}$ with enough projectives is a module category such that $p.m$ is projective for all projective $p \in \mathcal{A}$ and all $m \in \mathcal{M}$.

Remarks

- $\mathcal{A}$ as a left module category over itself is an exact $\mathcal{A}$-module category.
- Module functors out of an exact module category are exact.
- It is expected that $\mathrm{mod}_{\mathcal{A}}\text{-}A$ is exact, iff A does not have non-trivial two-sided ideals. (Stroiński, Etingof-Ostrik)
- An exact module category has internal Homs and coHoms, e.g.

  $$\mathrm{Hom}_{\mathcal{M}}(a.m_1, m_2) \cong \mathrm{Hom}_{\mathcal{A}}(a, \underline{\mathrm{Hom}}_{\mathcal{M}}(m_1, m_2))$$

  The internal Hom of an exact module category is an exact functor and a strong module functor.

Morita contexts from exact module categories

Exact module categories are closely related to (categorical) Morita contexts:

Definition

1. A categorical Morita context is a bicategory with two objects

$$\begin{tikzcd} \bullet \arrow[loop left, "\mathcal{A}"] \arrow[r, bend left, "\mathcal{M}"] & \bullet \arrow[l, bend left, "\mathcal{N}"] \arrow[loop right, "\mathcal{B}"] \end{tikzcd}$$

   where $\mathcal{A}, \mathcal{B}$ a finite tensor categories and ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$ are finite bimodule categories.

2. A Morita context is called strong, iff

$$\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \cong \mathcal{A} \quad \text{and } \mathcal{N} \boxtimes_{\mathcal{A}} \mathcal{M} \cong \mathcal{B}$$

   Here, $\boxtimes_{\mathcal{A}}$ is the relative Deligne product.

Exact module categories are the source of Morita contexts:

Theorem (Fuchs, Galindo, Jaklitsch, CS)

1. *Let $\mathcal{A}$ be a finite tensor category and ${}_{\mathcal{A}}\mathcal{M}$ be an exact module category. Then*

$$\begin{tikzcd} \bullet \arrow[loop left, "\mathcal{A}"] \arrow[r, bend left, "\mathcal{M}"] & \bullet \arrow[l, bend left, "\overline{\mathcal{M}}"] \arrow[loop right, "Fun_{\mathcal{A}}(\mathcal{M},\mathcal{M})"] \end{tikzcd}$$

with $\overline{\mathcal{M}} := Fun_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ is an strong Morita context.

2. *Any strong Morita context is of this form.*

Morita equivalent algebras have isomorphic centers.

Definition (Half-braiding, Drinfeld center)

Let $\mathcal{A}$ be a monoidal category.
A half-braiding for $V \in \mathcal{A}$ is a natural isomorphism

$$\sigma_V : V \otimes - \to - \otimes V$$

such that $\sigma_V(X \otimes Y) = (\mathrm{id}_X \otimes \sigma_V(Y)) \circ (\sigma_V(X) \otimes \mathrm{id}_Y)$ for all $X, Y \in \mathbb{C}$.
The Drinfeld center $\mathcal{Z}(\mathcal{A})$ has pairs (V, σ_V) as objects.

$\mathcal{Z}(\mathcal{A})$ is a braided monoidal category. There is a forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ which is monoidal and exact.x

Proposition (Schauenburg, Shimizu)

Two finite tensor categories are Morita equivalent, iff their Drinfeld centers are braided equivalent.

Definition

Let $\mathbb{M}$ be a bicategory. A right dual (or right adjoint) to a 1-morphism $a \in \mathbb{M}(x, y)$ consists of a 1-morphism $a^\vee \in \mathbb{M}(y, x)$ and 2-morphisms $1_y \to a \circ a^\vee$ and $a^\vee \circ a \to 1_x$ that fulfill the appropriate snake relations; left duals are defined similarly. (Having a dual is a property.)

Proposition

In the two-object bicategory given by an exact module category, we have duals:

$$m^\vee = \underline{\mathrm{Hom}}^{\mathcal{A}}_{\mathcal{M}}(m, -) \in \overline{\mathcal{M}} \quad \textit{and} \quad {}^\vee m = \underline{\mathrm{coHom}}^{\mathcal{A}}_{\mathcal{M}}(m, -) \in \overline{\mathcal{M}}$$

Natural question: compute the biduals and investigate when can the bidual be trivialized.

Chapter 2

Pivotal structures and relative Serre functors

Pivotal structures

Definition

A pivotal structure on a rigid monoidal category is a monoidal isomorphism $\mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} (-)^{\vee\vee}$.

Remarks

1. The category of finite-dimensional vector spaces is pivotal.
2. A pivotal structure allows to identify the left and the right dual, ${}^{\vee}a \cong a^{\vee}$.
3. Invariant tensors $\mathrm{Hom}(a_1 \otimes a_2 \otimes \ldots \otimes a_n, 1)$ are cyclic invariant.

Dichotomy

framed TFT	tensor categories, algebras
oriented TFT	pivotal tensor categories, Frobenius algebras

We need a tool: relative Serre functors.

Relative Serre functors

Definition (Fuchs, Schaumann, CS)

Let $\mathcal{M}$ be a $\mathcal{A}$-module. A right/left relative Serre functor is an endofunctor $\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ / $\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}$ of $\mathcal{M}$ together with a family

$$\begin{aligned} \underline{\mathrm{Hom}}(m,n)^{\vee} = \underline{\mathrm{coHom}}(m,n) &\xrightarrow{\cong} \underline{\mathrm{Hom}}(n, \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)) \\ {}^{\vee}\underline{\mathrm{Hom}}(m,n) &\xrightarrow{\cong} \underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(n), m) \end{aligned}$$

of isomorphisms natural in $m, n \in \mathcal{M}$.

- Relative Serre functors exist, iff $\mathcal{M}$ is an exact module category.
- Relative Serre functors are equivalences of categories.
- Relative Serre functors are twisted module functors:

$$\phi_{a,m} : \ \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(a.m) \longrightarrow a^{\vee\vee}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \quad \text{and} \quad \tilde{\phi}_{a,m} : \ \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(a.m) \longrightarrow {}^{\vee\vee}a.\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(m)$$

Pivotal module categories

Serre functors are twisted module functors:

$$\phi_{a,m}: \quad \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(a.m) \longrightarrow a^{\vee\vee}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \quad \text{and} \quad \tilde{\phi}_{a,m}: \quad \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(a.m) \longrightarrow {}^{\vee\vee}a.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)\,.$$

Definition (Schaumann 2015, Shimizu 2019)

A pivotal structure on an exact module category $\mathcal{M}$ over a pivotal finite tensor category $(\mathcal{A}, \pi)$ is an isomorphism of functors $\tilde{\pi}: \mathrm{id}_{\mathcal{M}} \to \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ such that the following diagram commutes for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$:

$$\begin{tikzcd} a.m \arrow[rr, "\pi_a.\tilde{\pi}_m"] \arrow[dr, "\tilde{\pi}_{a.m}"'] & & a^{\vee\vee}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \\ & \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(a.m) \arrow[ur, "\phi_{a,m}"'] & \end{tikzcd}$$

- For an indecomposable exact module category, the pivotal structure is unique up to scalar, if it exists.
- The algebra $\underline{\mathrm{Hom}}(m, m) \in \mathcal{A}$ for m in a pivotal module category has the structure of symmetric Frobenius algebras with Frobenius form

$$\underline{\mathrm{Hom}}(m, m) \cong \underline{\mathrm{Hom}}(m, \mathrm{S}^{\mathrm{r}}_{\mathcal{M}} m) \cong \underline{\mathrm{Hom}}(m, m)^{\vee} \xrightarrow{\mathrm{coev}^{\vee}_m} 1$$

Biduals and the relative Serre functor

Back to Morita contexts:

Definition

Let $\mathbb{M}$ be a bicategory with duals. It comes with a pseudofunctor $\mathbb{M} \to \mathbb{M}^{op,op}$ sending $x \mapsto x$ for objects and on 1-morphisms

$$(x \xrightarrow{a} y) \mapsto (y \xrightarrow{a^\vee} x)$$

A pivotal structure on $\mathbb{M}$ is a pseudo-natural isomorphism

$$P : \quad \mathrm{id}_{\mathbb{M}} \to (-)^{\vee\vee}$$

such that $P_x = \mathrm{id}_x$ for all objects $x \in \mathbb{M}$.
A Morita context is called pivotal, iff the associated bicategory is pivotal.

Proposition

1. *Let $\mathcal{M}$ be an exact module category. Then*

$$m^{\vee\vee} = \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \ \ \textit{and} \ \ {}^{\vee\vee}m = \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(m) \ .$$

2. *The Morita context given by a pivotal module category is pivotal.*

Chapter 3

Nakayama functors and modified traces

Non-degenerate traces are an important ingredient in TFT constructions.

Eilenberg-Watts calculus

In many situations, Serre functors are related to Nakayama functors. We need a categorical formulation of a classical result about finite categories:

Eilenberg-Watts calculus

Proposition (Eilenberg-Watts)

Let A-mod and B-mod finite categories. Let

$$G : A\text{-mod} \to B\text{-mod}$$

be a right exact functor. Then $G \cong G({}_AA_A) \otimes_A -$.
The B-A-bimodule $G({}_AA_A)$ is a right A-module via the image of right multiplication $r_A : A \to A$ under $\mathrm{End}_A(A) \xrightarrow{G} \mathrm{End}_B(G(A))$.
A similar statement allows to express left exact functors in terms of bimodules.

Morita-invariant formulation: triangle of explicit adjoint equivalences:

$$\begin{tikzcd}
 & \mathcal{A}^{opp} \boxtimes \mathcal{B} \arrow[dl, shift left, "\Phi^{l}"] \arrow[dr, shift left, "\Phi^{r}"] & \\
\mathcal{L}ex(\mathcal{A},\mathcal{B}) \arrow[ur, shift left, "\Psi^{l}"] \arrow[rr, shift left, "\Gamma_{lr}"] & & \mathcal{R}ex(\mathcal{A},\mathcal{B}) \arrow[ul, shift left, "\Psi^{r}"] \arrow[ll, shift left, "\Gamma_{rl}"]
\end{tikzcd}$$

In particular, $\mathrm{id}_{\mathcal{A}} \in \mathcal{L}ex(\mathcal{A},\mathcal{A})$ is mapped to the right exact endofunctor

$$N^r_{\mathcal{A}} := \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(-,a)^* \otimes a\,.$$

Nakayama functors

$$N^r_{\mathcal{A}} := \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(-,a)^* \otimes a \quad \text{and} \quad N^l_{\mathcal{A}} := \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,-) \otimes a$$

Lemma

For $\mathcal{A} = A\text{-mod}$:
$N^r_{\mathcal{A}} = A^* \otimes_A - \cong \mathrm{Hom}_A(-,A)^*$ *and* $N^l_{\mathcal{A}} = \mathrm{Hom}_A(A^*,-)$.

Proof:
Suppose $\mathcal{A} \cong A\text{-mod}$.

- Since $N^r_{\mathcal{A}}$ is right exact, the Eilenberg-Watts theorem implies

$$N^r_{\mathcal{A}} \cong N^r({}_AA_A) \otimes_A -$$

- Thus compute the bimodule $N^r({}_AA_A)$:

$$N^r_{\mathcal{A}}({}_AA_A) = \int^{y\in\mathcal{A}} \mathrm{Hom}_A(A,y)^* \otimes y \cong \int^{y\in\mathcal{A}} y^* \otimes y \cong ({}_AA_A)^*$$

where in the last step, we used a Peter-Weyl theorem.

Nakayama functors

$$N^r_{\mathcal{A}} := \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(-,a)^* \otimes a \quad \text{and} \quad N^l_{\mathcal{A}} := \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,-) \otimes a$$

For this reason, we call $N^r_{\mathcal{A}}$ and $N^l_{\mathcal{A}}$ Nakayama functors.

Proposition

1. *The Nakayama functors are adjoints,* $N^l_{\mathcal{A}} \dashv N^r_{\mathcal{A}}$.
2. $N^l_{\mathcal{A}}$ *equivalence* $\Leftrightarrow$ $N^r_{\mathcal{A}}$ *equivalence.* $\Leftrightarrow$ $\mathcal{A}$ *is selfinjective.*

Theorem

Let $\mathcal{M}$ *be an exact* $\mathcal{A}$*-module and* $D_{\mathcal{A}} := N^r_{\mathcal{A}}(1)$ *the canonical invertible object of* $\mathcal{A}$*. Then*

$$N^l_{\mathcal{M}} \cong D_{\mathcal{A}}.\mathrm{S}^{\mathrm{l}}_{\mathcal{M}} \quad \textit{and} \quad N^r_{\mathcal{M}} \cong D^{-1}_{\mathcal{A}}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$$

Nakayama functors

$$N^r_{\mathcal{A}} := \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(-,a)^* \otimes a \quad \text{and} \quad N^l_{\mathcal{A}} := \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,-) \otimes a$$

Theorem

Let $\mathcal{M}$ be an exact $\mathcal{A}$-module and $D_{\mathcal{A}} := N^r_{\mathcal{A}}(1)$ the canonical invertible object of $\mathcal{A}$. Then

$$N^l_{\mathcal{M}} \cong D_{\mathcal{A}}.\mathrm{S}^{\mathrm{l}}_{\mathcal{M}} \quad \textit{and} \quad N^r_{\mathcal{M}} \cong D_{\mathcal{A}}^{-1}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$$

Proposition

The canoncial invertible object of $\mathcal{A}^*_{\mathcal{M}} := Fun_{\mathcal{A}}(\mathcal{M},\mathcal{M})$ is the module endofunctor $D_{\mathcal{A}^*_{\mathcal{M}}} = N^l_{\mathcal{M}} \circ \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}$. There is a natural isomorphism

$$r_{\mathcal{M}} : \quad D_{\mathcal{A}}.-.D_{\mathcal{A}^*_{\mathcal{M}}} \ \to\ \mathrm{S}^{\mathrm{r}}_{\mathcal{M}} \circ \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$$

of twisted bimodule functors. Summarize for the bicategory $\mathbb{M}$ in terms of a pseudo-natural equivalence $r_{\mathbb{M}} : \mathrm{id}_{\mathbb{M}} \xrightarrow{\sim} S^2$,

Radford's S^4-theorem

For linear functors, we have

Theorem (Fuchs, Schaumann, CS)

Let $\mathcal{A}, \mathcal{B}$ be finite categories. Let $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{B})$ such that F^{la} is left exact so that F^{lla} exists. Assume that F^{lla} is left exact as well. Then there is a natural isomorphism

$$\varphi^l_F : \quad N^l_{\mathcal{B}} \circ F \cong F^{lla} \circ N^l_{\mathcal{A}}$$

that is coherent with respect to composition of functors.

Theorem (Fuchs, Schaumann, CS)

Let $\mathcal{A}, \mathcal{B}$ be finite tensor categories and $\mathcal{M}$ an $\mathcal{A}$-$\mathcal{B}$ bimodule. Then the Nakayama functor has the structure of a twisted bimodule functor:

$$N^l_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^l_{\mathcal{M}}(m).{}^{\vee\vee}b$$

Non-degenerate traces

Let $\mathcal{C}$ be a finite category, $p \in \mathcal{A}$ projective.

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(p, N^r(x)) &= \int^{y \in \mathcal{A}} \mathrm{Hom}(p, \mathrm{Hom}(x,y)^* y) \\ &= \int^{y \in \mathcal{A}} \mathrm{Hom}(p, y) \otimes \mathrm{Hom}(x,y)^* = \mathrm{Hom}(x,p)^* \end{aligned}$$

Definition (Shibata-Shimizu; CS, Woike)

Image of $\mathrm{id}_p \in \mathrm{Hom}(p,p)$ is $t_p : \mathrm{Hom}(p, N^r(p)) \to k$ is called the modified trace.

Defined only for p projective and on Nakayama twisted morphisms, but without the need of a monoidal structure.

Proposition (CS, Woike)

1. *Non-gedenerate and cyclic:*

$$t_q\left(q \xrightarrow{g} p \xrightarrow{f} N^r(q)\right) = t_p\left(p \xrightarrow{f} N^r(q) \xrightarrow{N^r(g)} N^r(p)\right)$$

2. *If $\mathcal{A}$ is even a bimodule category, partial trace property with respect to traces on the monoidal categories.*

Traces for endomorphisms: sphericality

Suppose that $\mathcal{A}$ is unimodular, $1 \cong D_{\mathcal{A}}$, and pivotal. Consider endomorphisms in the following diagram:

$$\begin{tikzcd}
 & & X \\
 & & {}^{\vee\vee}X \otimes D_{\mathcal{A}} \arrow[u, "\pi_{{}^{\vee\vee}X}"'] \arrow[dd, "R_X"] \\
X \arrow[uurr] \arrow[r, "g"] \arrow[ddrr] & N^r_{\mathcal{A}}(X) \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & \\
 & & D_{\mathcal{A}} \otimes X^{\vee\vee} \arrow[d, "\pi_X^{-1}"] \\
 & & X
\end{tikzcd}$$

Call a pivotal category spherical, if "the pivotal structure squares to the Radford".

Spherical module categories

Definition

A pivotal module $\mathcal{M}$ over a spherical category $\mathcal{A}$ is called spherical iff the diagram

$$\begin{tikzcd} \mathrm{id}_{\mathcal{M}} \arrow[rr, Rightarrow, "r_{\mathcal{M}}"] \arrow[dr, Rightarrow, "\tilde{p}"'] & & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \\ & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[ur, Rightarrow, "\mathrm{id} \circ \tilde{p}"'] & \end{tikzcd}$$

commutes, where $\widetilde{p}$ is the pivotal structure of $\mathcal{M}$.

Proposition (Fuchs, Galindo, Jaklisch, CS)

Let $\mathcal{M}$ be a spherical $\mathcal{A}$-module category. Then the following hold:

1. *The natural pivotal structure on the dual tensor category $\mathcal{A}^*_{\mathcal{M}} = Fun_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$ is also spherical.*
2. *$\mathcal{M}$ is a spherical $\mathcal{A}^*_{\mathcal{M}}$-module category.*
3. *$\overline{\mathcal{M}}$ is a spherical $\mathcal{A}^*_{\mathcal{M}}$-bimodule category.*

Spherical module categories

Definition

A pivotal module $\mathcal{M}$ over a spherical category $\mathcal{A}$ is called spherical iff the diagram

$$\begin{tikzcd} \mathrm{id}_{\mathcal{M}} \arrow[rr, Rightarrow, "r_{\mathcal{M}}"] \arrow[dr, Rightarrow, "\tilde{p}"'] & & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \circ \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \\ & \mathbb{S}^{\mathcal{A}}_{\mathcal{M}} \arrow[ur, Rightarrow, "\mathrm{id} \circ \tilde{p}"'] & \end{tikzcd}$$

commutes, where $\widetilde{p}$ is the pivotal structure of $\mathcal{M}$.

Remarks

1. For a spherical fusion category, there is a graphical calculus on spheres.
2. Let $\mathcal{A}$ and $\mathcal{B}$ be two pivotal tensor categories that are pivotal Morita equivalent. $\mathcal{A}$ is (unimodular) spherical if and only if $\mathcal{B}$ is (unimodular) spherical.

Chapter 4

Duality beyond rigidity: Grothendieck-Verdier categories

Consequences of rigidity

In nature, rigidity is not generic:

Examples

- A a finite-dimensional k-algebra, $A\text{-}\mathrm{bimod}$ the category of finite-dimensional A-bimodules. Then $B \otimes_A -$ is not necessarily exact.
- Vertex algebras to which HLZ tensor product theory applies: Example $\mathcal{W}_{2,3}$ with $c = 0$: tensor product is not exact.
- The category of finite-dimensional modules over a finite-dimensional Hopf algebroid is, in general, not rigid.

Bimodules revisited

k a field, A a finite-dimensional k-algebra.
(All vector spaces will be finite-dimensional.)

Facts

1. If B is an A_1-A_2-bimodule, its linear dual $B^* = \mathrm{Hom}_k(B, k)$ with action
$$(a_2.\beta.a_1)(b) := \beta(a_1.b.a_2)$$
is an A_2-A_1-bimodule. In particular, A^* is an A-bimodule. A^* is, in general, not isomorphic to A.
2. The tensor product $B \otimes_A \tilde{B}$ of two bimodules B and $\tilde{B}$ is a coequalizer
$$B \otimes A \otimes \tilde{B} \rightrightarrows B \otimes \tilde{B} \to B \otimes_A \tilde{B} \to 0$$
and thus right exact. Monoidal unit is A.

A second tensor product for bimodules

Facts

- A^* is a coalgebra, any $M \in A\text{-bimod}$ is a bicomodule. Equalizer

$$0 \to B \otimes^A \tilde{B} \to B \otimes \tilde{B} \rightrightarrows B \otimes A^* \otimes \tilde{B}$$

  gives left exact tensor product with monoidal unit A^*.

- One has

$$\mathrm{Hom}_{A\text{-bimod}}(M_1 \otimes_A M_2, A^*) \cong \mathrm{Hom}_{A\text{-bimod}}(M_1, (M_2)^*)$$

  A^* is a dualizing object for $A\text{-bimod}$

The second tensor product is very useful:

Facts

- Eilenberg-Watts:
  $F : A_1\text{-mod} \to A_2\text{-mod}$ right exact, then $F(-) \cong F(A_1) \otimes_{A_1} -$
  $H : A_1\text{-mod} \to A_2\text{-mod}$ left exact, then $H(-) \cong H(A_1^*) \otimes^{A_1} -$

Contragredient duals for modules over vertex algebras

Setting:

- V vertex algebra to which the HLZ theory of logarithmic tensor products applies.
- (Generalized weak) Module $M = \bigoplus_{h \in \mathbb{C}, b \in B} M_h^{(b)}$ is strongly B-graded.

Contragredient module $M' := \bigoplus_{h \in \mathbb{C}, b \in B} \left(M_h^{(b)}\right)^*$ with action

$$\langle Y_{M'}(v,z)\phi, m\rangle = \langle \phi, Y_M^{opp}(v,z)m\rangle$$

with

$$Y_M^{opp}(v,z) := Y_M(e^{zL_1}(-z^{-2})^{L_0} v, z^{-1})$$

(depends on conformal structure on V)

Facts

1. The contragredient module V' is not necessarily isomorphic to V.
2. Example: $\mathcal{W}_{2,3}$-model: V is not simple $0 \to \mathcal{W}(2) \to V \to \mathcal{W}(0) \to 0$ with simples $\mathcal{W}(2)$ and $\mathcal{W}(0)$. The dual is

$$0 \to \mathcal{W}(0) \to \mathcal{W}' \to \mathcal{W}(2) \to 0$$

Same character, but not isomorphic.

GV categories

Definition

1. Let $(\mathcal{C}, \otimes, 1, \alpha, l, r)$ be a monoidal category.
   An object $K \in \mathcal{C}$ is called a dualizing object if, for every $Y \in \mathcal{C}$, the functor
   $$X \mapsto \mathrm{Hom}(X \otimes Y, K)$$
   is representable by some object $GY \in \mathcal{C}$ and the contravariant functor $G : \mathcal{C} \to \mathcal{C}$ is an anti-equivalence.
   $$\mathrm{Hom}(X \otimes Y, K) \cong \mathrm{Hom}(X, GY)\,.$$
   G is called the duality functor with respect to K.
2. A Grothendieck-Verdier category or GV-category, is a monoidal category $(\mathcal{C}, \otimes, 1)$ together with a choice of a dualizing object $K \in \mathcal{C}$.

Examples of GV categories

Remarks

1. Symmetric GV categories are known since the seventies as $*$-autonomous categories and correspond to linearly distributive categories with negation.
2. The choice of dualizing object is structure (in contrast to rigid duality)
3. Let V be a conformal vertex algebra and $\mathcal{C}$ be a category of V-modules to which the HLZ tensor product theory applies. Then $\mathcal{C}$ has a natural structure of a ribbon GV category [Allen, Lentner, CS, Wood].
4. Let H be a Hopf algebroid with finite-dimensional base algebra A and an invertible antipode S. Then the category of finite-dimensional H-modules is a GV category.
   A dualizing object is given by the vector space dual of the base algebra A. [Allen]

Properties of GV categories

Definition

Internal Hom: $\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z))$.
Internal coHom: $\mathrm{Hom}(X, Y \otimes Z) \cong \mathrm{Hom}(\underline{\mathrm{coHom}}(Z, X), Y)$.

Remarks

1. In a GV category, internal Homs

$$\underline{\mathrm{Hom}}(X, Z) \cong G(X \otimes G^{-1}Z)$$

   exist. Thus $\otimes$ is cocontinuous.
2. Second tensor product $X \bullet Y := G^{-1}(GY \otimes GX)$ with the dualizing object K as a monoidal unit has internal coHoms and is thus continuous.
3. Distributors

$$\delta^r : \quad c_1 \otimes (c_2 \bullet c_3) \to (c_1 \otimes c_2) \bullet c_3 \quad \text{and} \quad \delta^l : (c_1 \bullet c_2) \otimes c_3 \to c_1 \bullet (c_2 \otimes c_3)$$

   are not necessarily isomorphisms, but obey pentagon diagrams.
4. Double dual G^2 is monoidal equivalence $\rightsquigarrow$ Notion of pivotal GV category.
5. Notion of braided GV category, ribbon GV category exist.

GV categories and quantum topology

Theorem (Brochier, Müller, Woike, 2020-2024)

1. *Cyclic associative algebras in Lex^f $\leftrightarrow$ pivotal GV categories*
2. *Cyclic framed E_2-algebras in Lex^f $\leftrightarrow$ ribbon GV categories.*
3. *Cyclic framed E_2-algebra $\leftrightarrow$ ansular functor.*
   (in the case in which Lex^f is the target category, in ribbon GV categories).
4. *Under precise conditions, the ansular functor can be extended to a modular functor. This conditions holds for modular categories, but also the Feigin-Fuchs boson.*

Chapter 6

Module categories over Grothendieck-Verdier categories

Motivations:

- Full CFT needs module categories.
- Frobenius algebras obtained from module categories are candidates for field objects.
- Mathematical goal: understand (and generalize) distributors

$$\delta^r : \quad c_1 \otimes (c_2 \bullet c_3) \to (c_1 \otimes c_2) \bullet c_3 \quad \text{and} \quad \delta^l : (c_1 \bullet c_2) \otimes c_3 \to c_1 \bullet (c_2 \otimes c_3)$$

GV module categories

Definition

A left GV-module category over a GV-category $(\mathcal{C}, \otimes, K)$ is a left module category $(\mathcal{M}, \triangleright)$ over $(\mathcal{C}, \otimes)$ such that all functors

$$- \triangleright m : {}_{\mathcal{C}}\mathcal{C} \to \mathcal{M} \text{ with } c \mapsto c \triangleright m \quad \text{and} \quad c \triangleright - : \mathcal{M} \to \mathcal{M} \text{ with } m \mapsto c \triangleright m$$

admit a right adjoint.

Call $R_c : \mathcal{M} \to \mathcal{M}$ the right adjoint to $c \triangleright -$.

Proposition (Fuchs, Schaumann, S, Wood)

Let $(\mathcal{M}, \triangleright)$ be a left GV-module category over a GV-category $(\mathcal{C}, \otimes)$. Then the bifunctor

$$\blacktriangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M} \quad \text{with} \quad c \blacktriangleright m := R_{Gc}(m)$$

is left exact in each variable and defines a left module category structure over $(\mathcal{C}, \bullet)$.

$\Rightarrow$ Existence of inner Homs for the action $\triangleright$ and inner coHoms for the action $\blacktriangleright$.

Lax module functors

Fact

$\mathcal{C}$ k-linear monoidal category, $\mathcal{M}, \mathcal{N}$ left $\mathcal{C}$-modules

1. *Linear functor $F\colon \mathcal{M} \to \mathcal{N}$ with $G\colon \mathcal{N} \to \mathcal{M}$ right adjoint.*
   Canonical bijection
   Oplax $\mathcal{C}$-module functor structures on F $\leftrightarrow$ Lax $\mathcal{C}$-module structures on G
   such that the adjunction φ with components $\varphi_{m,n} : \mathrm{Hom}(F(m), n) \to \mathrm{Hom}(m, G(n))$ is an isomorphism of $\mathcal{C}$-module profunctors.
2. *F strong module functor $\Rightarrow$ G unique structure of a lax $\mathcal{C}$-module functor*
   such that the unit and counit of the adjunction φ are module natural transformations.
3. *Apply to the module functor $c \mapsto c \triangleright m$*
   $\Rightarrow$ for all $m \in \mathcal{M}$ the functor $\underline{\mathrm{Hom}}(m, -)$ is a lax module functor
   $\Rightarrow$ coherent morphisms
   $$\delta^r : c \otimes \underline{\mathrm{Hom}}(m, n) \to \underline{\mathrm{Hom}}(m, c \triangleright n) \qquad \text{for } c \in \mathcal{C} \text{ and } m, n \in \mathcal{M}\ .$$
4. *Special case: for $\mathcal{M} = \mathcal{C}$ obtain distributors*
   $$\delta^r : c \otimes \underline{\mathrm{Hom}}(x, y) = c \otimes (y \bullet G(x)) \to (c \otimes y) \bullet G(x) = \underline{\mathrm{Hom}}(x, c \otimes y)$$
   obeying pentagon diagrams.

Strong module functors

Proposition (Fuchs, Schaumann, S, Wood)

Let $(\mathcal{C}, \otimes)$ be a right closed monoidal category and $x \in \mathcal{C}$.
The lax module functor $\underline{\mathrm{Hom}}(x, -)\colon \mathcal{C} \to \mathcal{C}$ is a strong module functor if and only if x has a right dual object $x^\vee$.
Then $x^\vee \cong \underline{\mathrm{Hom}}(x, 1)$ as objects, and $\underline{\mathrm{Hom}}(x, -) \cong - \otimes x^\vee$ as module functors.

Lemma

Let A be a finite-dimensional k-algebra, and let ${}_AM_A \in A\text{-}\mathrm{bimod}$ be a finite-dimensional A-bimodule.

The following statements are equivalent:

Strong module functors

Proposition (Fuchs, Schaumann, S, Wood)

Let $(\mathcal{C}, \otimes)$ be a right closed monoidal category and $x \in \mathcal{C}$.
The lax module functor $\underline{\mathrm{Hom}}(x, -) \colon \mathcal{C} \to \mathcal{C}$ is a strong module functor if and only if x has a right dual object $x^\vee$.
Then $x^\vee \cong \underline{\mathrm{Hom}}(x, 1)$ as objects, and $\underline{\mathrm{Hom}}(x, -) \cong - \otimes x^\vee$ as module functors.

Lemma

Let A be a finite-dimensional k-algebra, and let ${}_AM_A \in A\text{-bimod}$ be a finite-dimensional A-bimodule.

The following statements are equivalent:

(i) $\underline{\mathrm{Hom}}(M, -)$ *is a strong module functor.*

(ii) M *has an $\otimes_A$-right dual.*

(iii) M_A *is projective as a right A-module.*

(iv) *For all $X, Y \in A\text{-bimod}$ the distributor*
$\delta^r \colon X \otimes_A (Y \otimes^A M^*) \to (X \otimes_A Y) \otimes^A M^*$ *is an isomorphism.*

Important subcategories

Definition

1. An object $m \in \mathcal{M}$ is called $\otimes$-admissible if
   1. The functor $\underline{\mathrm{Hom}}(m, -)$ is a strong $\triangleright$- module functor.
   2. The functor $\underline{\mathrm{Hom}}(m, -)$ has a right adjoint.
2. $\bullet$-admissible objects are defined using $\underline{\mathrm{coHom}}$.
3. $\widehat{\mathcal{M}}^{\otimes} / \widehat{\mathcal{M}}^{\bullet}$: full subcategories of $\mathcal{M}$ of $\otimes / \bullet$-admissible objects.
4. The subcategories $\widehat{\mathcal{C}}^{\otimes} / \widehat{\mathcal{C}}^{\bullet}$ of $\mathcal{C}$ are obtained by considering $\mathcal{C}$ as a left GV-module category over itself.

Proposition (Fuchs, Schaumann, CS, Wood)

1. *Let $\mathcal{C}$ be a GV-category. The subcategories $\widehat{\mathcal{C}}^{\otimes}$ and $\widehat{\mathcal{C}}^{\bullet}$ of $\mathcal{C}$ are unital monoidal subcategories.*
2. *Let $\mathcal{M}$ be a left GV-module category over $\mathcal{C}$. By restriction, the category $\widehat{\mathcal{M}}^{\otimes}$ is a left $\widehat{\mathcal{C}}^{\otimes}$-module category and $\widehat{\mathcal{M}}^{\bullet}$ is a left $\widehat{\mathcal{C}}^{\bullet}$-module category.*

Algebras for module categories

$\mathcal{C}$ finite abelian GV-category, $\mathcal{M}$ a finite abelian GV-module over $\mathcal{C}$.
For any $m \in \mathcal{M}$,

- Internal End $A_m := \underline{\mathrm{Hom}}(m, m)$ is associative unital algebra in $(\mathcal{C}, \otimes)$,
- Internal coEnd $C_m := \underline{\mathrm{coHom}}(m, m)$ is a coass. counital coalgebra in $(\mathcal{C}, \bullet)$.
- Natural functor

$$\begin{array}{rccc} \underline{\mathrm{Hom}}(m_0, -): & \mathcal{M} & \to & \mathrm{mod}\text{-}A_{m_0}\,, \\ & m & \mapsto & \underline{\mathrm{Hom}}(m_0, m)\,. \end{array}$$

Similarly there is a functor $\underline{\mathrm{coHom}}(m_0, -)$ from $\mathcal{M}$ to C_{m_0}-comodules.

Proposition (Fuchs, Schaumann, S, Wood)

1. *Let $m_0 \in \widehat{\mathcal{M}}^{\otimes}$. If for every $m \in \mathcal{M}$ there exists an object $c \in \mathcal{C}$ with an epimorphism $c \triangleright m_0 \longrightarrow m$, then the functor $\underline{\mathrm{Hom}}(m_0, -)$ is an equivalence of $(\mathcal{C}, \otimes)$-module categories.*
2. *A similar statement can be formulated for comodules.*

Partially defined relative Serre functors

Proposition

There exists an equivalence $S\colon \widehat{\mathcal{M}}^{\otimes} \to \widehat{\mathcal{M}}^{\bullet}$ called the relative Serre functor such that

$$\underline{\mathrm{Hom}}(m,-) \cong \underline{\mathrm{coHom}}(Sm,-)$$

as an equivalence of $\otimes$- and $\bullet$-module functors.

Proposition

Let $\mathcal{C}$ be a GV-category and $\mathcal{M}$ a left GV-module category over $\mathcal{C}$.

1. *The relative Serre functor of $\mathcal{C}$ is canonically a monoidal equivalence*

$$S_{\mathcal{C}} : \widehat{\mathcal{C}}^{\otimes} \to \widehat{\mathcal{C}}^{\bullet}$$

2. *The relative Serre functor $S_{\mathcal{M}}$ of $\mathcal{M}$ is a twisted module functor,*

$$S_{\mathcal{M}}(c \triangleright m) \cong S_{\mathcal{C}}(c) \blacktriangleright S_{\mathcal{M}}(m)$$

for $c \in \widehat{\mathcal{C}}^{\otimes}$ and $m \in \widehat{\mathcal{M}}^{\otimes}$.

Frobenius algebras

- For field objects in full CFTs, we need Frobenius algebras.
- Internal Homs in pivotal module categories $\rightsquigarrow$ Frobenius algebras.
- There is a natural notion of a GV Frobenius algebra.

Theorem (Fuchs, Schaumann, S, Wood)

Let $m \in \widehat{\mathcal{M}}^{\otimes}$. For every choice of isomorphism $p\colon m \to Sm$ in $\mathcal{M}$, $\underline{\mathrm{Hom}}(m, m)$ is a GV-Frobenius algebra in $\mathcal{C}$ with Frobenius form

$$\lambda : \underline{\mathrm{Hom}}(m, m) \xrightarrow{\underline{\mathrm{Hom}}(m,p)} \underline{\mathrm{Hom}}(m, Sm) \xrightarrow{\mathrm{tr}_m} K \,.$$

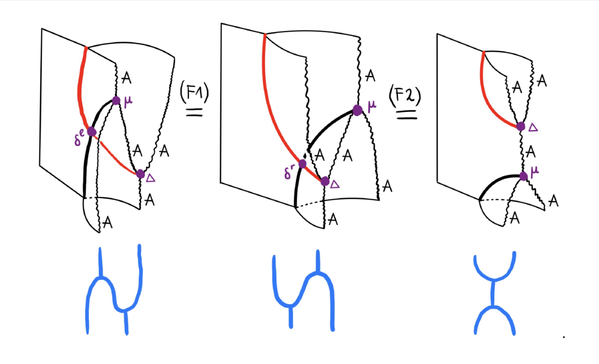
where

$$\mathrm{tr}_m := \big(\underline{\mathrm{Hom}}(m, S(m)) \longrightarrow G(\underline{\mathrm{Hom}}(m, m)) \xrightarrow{G(u_m)} G(1) = K \big) \,.$$

More results

Theorem (M. Demirdilek)

The equality of the lhs and rhs implies all other equalities:



More results

Theorem (M. Demirdilek)

Let $U : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor between closed monoidal categories. Let $K \in \mathcal{C}$ be an object such that $U(K) \in \mathcal{D}$ is dualizing. If the functor U is isomorphism-reflecting and closed, then K is dualizing.

Proposition (M. Demirdilek)

Let T be a Hopf monad on a GV-category $(\mathcal{C}, \otimes, 1, K)$. Any T-module structure $\rho : T(K) \to K$ on the dualizing object K yields a dualizing object (K, ρ) in the monoidal category of T-modules.

Summary and outlook

Summary

- Nakayama and relative Serre functors are highly useful tools for representation theory and TFT
  - pivotal/spherical structures (defects, Morita-completed theories)
  - "modified" traces (non-semisimple)
- Grothendieck-Verdier duality is a natural algebraic structure.
  - Interesting subcategories with partially defined Serre functors
  - Good theory of Frobenius algebras, Frobenius-Schur indicators, lifting results

Outlook

- Develop more methods for quantum topology.
- Understand CFT correlators