On Krull-Schmidt bicategories

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Plan: (1) Classical KS-theory
(2) Direct sums in a bicategory
(3) KS bicategories
(4) Examples & applications

for more, see:

I.D., "On KS bicat's", Theory Appl. Categ. 2022

(1) Classical Krull-Schmidt theory

$\mathcal{C}$: an additive category

Def: $\mathcal{C}$ is Krull-Schmidt if $\forall X \in Ob(\mathcal{C})$ $\exists$ iso $X \cong X_1 \oplus \cdots \oplus X_n$ where each object X_i is strongly indecomposable, i.e.: $End(X_i)$ is a local ring.

Rmk: X strongly indec. $\Rightarrow$ X indec. (for $\oplus$)

Main results:

The Krull-Schmidt thm: $\mathcal{C}$ KS implies:

1) $\forall X \in \mathcal{C}$, X indec. $\Leftrightarrow$ X strongly indec.
2) $\forall X \in \mathcal{C}$, the decomposition into finitely many indecomposable objects
$$X \cong X_1 \oplus \cdots \oplus X_n$$
is unique, up to permuting the X_i's and replacing them with isomorphic objects.

Very useful! But how to recognize KS-cat's?

Characterization

$\mathcal{C}$ is KS iff:

a) $\mathcal{C}$ is idempotent-complete

and

b) $\forall X \in \mathcal{C}$, the ring $\mathrm{End}(X)$ is semi-perfect.

Recall:

- $\mathcal{C}$ is idempotent complete if:
  $\forall\, f = f^2 \circlearrowright X$ in $\mathcal{C}$, have $X \cong \mathrm{Im}(f) \oplus \mathrm{Ker}(f)$.

- A ring R is semi-perfect if:
  $R/\mathrm{rad}\,R$ semi-simple & can lift idempotents.
  $\iff$ every simple R-module has a proj. cover.
  $\iff$ $\mathrm{fgproj}(R)$ is a KS-category!
  $\vdots$

Rmk: we have the strict implications:

$$\begin{array}{ccccccc} & & \text{division rings} & \Rightarrow & \text{local} & & \\ & & & & & \searrow & \\ & & & & & & \boxed{\text{semi-perfect}} \Rightarrow \text{semi-local} \\ & & & & & \nearrow & \\ \text{f. dim. algebras} & \Rightarrow & \text{Artinian rings} & & & & \end{array}$$

Cor. The following categories are all KS:
- semi-simple + abelian
- Hom-finite/field + abelian (or just id.c.)

Examples of KS-categories:

- $\mathcal{C} = \mathrm{fdmod}(A)$ for A an algebra / field k
  e.g. $A = kG$, G a group.

- also $\mathcal{C} = D^b(\mathrm{fdmod}\, A)$

- [Atiyah '56, SGA6 '71]
  $\mathcal{C} = \mathrm{Coh}\, \mathcal{O}_X$ for X a scheme f.t. / field
  also $D^b(\mathrm{Coh}\, \mathcal{O}_X)$ if X regular & proper.

⚠ The KS property is subtle:

- [Facchini, Herbera, Levy, Vámos 1995]
  "KS fails for Artinian modules"

  in fact, already over certain R
  semi-local, commutative & noetherian!

② Direct sums in a bicategory (= 2-cat. today ☺)

Question: for an additive category $\mathcal{A}$, when do we have uniqueness of a decomposition

$$\mathcal{A} \simeq \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \quad ?$$

Idea: fin. products are direct sums in the 2-category

$$\mathrm{ADD} = \begin{cases} \text{additive categories } \mathcal{A}, \mathcal{A}', \ldots \\ \text{additive functors } F : \mathcal{A} \to \mathcal{A}' \\ \text{natural transf. } \alpha : F_1 \Rightarrow F_2 \end{cases}$$

$\rightsquigarrow$ KS theory for such bicategories ?!

Def.: a bicat. $\mathcal{B}$ is **additive** if:

1) Hom-categories are additive,
2) composition functors $\mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \xrightarrow{\circ} \mathcal{B}(X,Z)$ are additive in both variables,
3) $\exists$ finite **direct sums** of objects:

$$X_1 \underset{I_1}{\overset{P_1}{\rightleftarrows}} X_1 \oplus X_2 \underset{I_2}{\overset{P_2}{\rightleftarrows}} X_2 \qquad \leftsquigarrow \left(\begin{matrix}\text{the case} \\ n = 2\end{matrix}\right)$$

$$I_1 P_1 \oplus I_2 P_2 \cong \mathrm{Id}_{X_1 \oplus X_2}, \qquad P_i I_j \cong \begin{cases} 0 & (i \neq j) \\ \mathrm{Id}_{X_i} & (i = j) \end{cases}.$$

Rmk : direct sums are both products & coproducts (in a weak, up to equiv. sense).

In particular: $0 = \bigoplus_{\emptyset}$ is initial & final.

Examples :

- ADD is additive:

$$\begin{tikzcd} & \mathcal{A}_1 \oplus \mathcal{A}_2 \arrow[d, phantom, "\wr"] & \\ \mathcal{A}_1 \arrow[r, dashed, shift right, "I_1"'] & \mathcal{A}_1 \times \mathcal{A}_2 \arrow[l, shift right, "P_1"'] \arrow[r, shift left, "P_2"] & \mathcal{A}_2 \arrow[l, dashed, shift left, "I_2"] \end{tikzcd}$$

$$x \longmapsto (x, 0)$$

- Any 2-full sub-2-category of ADD closed under dir. sums of of obj & functors

③ KS theory for bicategories

$\mathcal{B}$: an additive bicat.

Recall : $\forall X \in \mathcal{B}$, $\mathrm{End}(\mathrm{Id}_X)$ is a comm. ring!
(Eckmann-Hilton argument)

Ex : $\mathcal{B} = \mathrm{ADD} \ni \mathcal{A} \rightsquigarrow \mathrm{End}(\mathrm{Id}_{\mathcal{A}}) = Z(\mathcal{A})$ "center"

e.g. if $\overset{\shortparallel}{\mathrm{Mod}(R)} \rightsquigarrow Z(\mathcal{A}) \cong Z(R)$.

Def: $X \in \mathcal{B}$ strongly indecomposable if $\mathrm{End}(\mathrm{Id}_X)$ is a connected (comm.) ring.

This is reasonable:

$$\boxed{\begin{array}{l}
\textbf{Proposition:}\\
\text{For } 0 \neq X \in \mathcal{B}\text{, we have:}\\[1ex]
\boxed{1}\ X \text{ strongly indec.}\\
\Downarrow\\
\boxed{2}\ \mathrm{Id}_X \text{ indec. object of } \mathrm{End}_{\mathcal{B}}(X).\\
\Downarrow\\
\boxed{3}\ \mathrm{End}_{\mathcal{B}}(X) \text{ is ``local'':}\\
\qquad F_1, F_2 : X \to X \text{ not equivalences}\\
\qquad \Longrightarrow F_1 \oplus F_2 : X \to X \text{ neither.}\\
\Downarrow\\
\boxed{4}\ \mathrm{End}_{\mathcal{B}}(X) \text{ is indecomposable}\\
\qquad \text{as an additive } \otimes\text{-category.}\\[1ex]
\text{These are all equivalent if } \mathrm{End}_{\mathcal{B}}(X)\\
\text{is idempotent complete!}
\end{array}}$$

Def. $\mathcal{B}$ is Krull-Schmidt if $\forall X \in \mathcal{B}$, $\exists$ equivalence $X \simeq X_1 \oplus \cdots \oplus X_n$ for finitely many str. indec. X_i's.

KS-theorem $\mathcal{B}$ KS implies:

1) str. indec $\Longleftrightarrow$ indec. for $\oplus$

2) The decomp. of each X as a sum of indec's is unique up to permutation & equivalence of the factors.

Characterization An additive bicategory $\mathcal{B}$ is KS iff:

a) $\mathcal{B}$ is weakly block-complete: $\forall$ decomp. $1_{\mathrm{Id}_X} = e_1 + e_2 : \mathrm{Id}_X \Rightarrow \mathrm{Id}_X$ in orthog. idempots. of $\mathrm{End}(\mathrm{Id}_X)$, $\exists$ $X \simeq X_1 \oplus X_2$ s.t. $1_{\mathrm{Id}_{X_i}} \leftrightarrow e_i$.

b) $\forall X$, $\mathrm{End}(\mathrm{Id}_X)$ is semi-connected, i.e: a finite product of (comm.) connected rings.

Crucial idea of the proofs ($\nexists$ for 1-cat's!):

An equivalence $X \simeq X_1 \oplus \dots \oplus X_n$ in $\mathcal{B}$

$$\Rightarrow \quad \mathrm{End}_{\mathcal{B}}(X) \simeq \bigoplus_{i,j} \mathcal{B}(X_j, X_i) \quad \text{of add. cat's}$$

$$\Rightarrow \quad \mathrm{End}(\mathrm{Id}_X) \cong \mathrm{End}(\mathrm{Id}_{X_1}) \times \dots \times \mathrm{End}(\mathrm{Id}_{X_n})$$

iso of commutative rings!

$$e_i = e_i^2 \longleftrightarrow 1_{\mathrm{Id}_{X_i}} \text{ (and } 0 \text{ for } j \neq i)$$

- Main technical lemma:

  this idempotent determines the "summand" X_i (with P_i, I_i, adjunctions ...) uniquely up to equiv.

- Then to get uniqueness of the decomp. $X \simeq X_1 \oplus \dots \oplus X_n$, just recall that in any comm. ring

$$1 = e_1 + \dots + e_n$$

in orthog. primitive idempotents in at most one way!

④ Examples & applications

- ADD is <u>not</u> KS! (both (a), (b) fail)

no!

full $\cup$

- $\mathrm{ADD}_{\text{id-c.}} = \{ \mathcal{A} \text{ idempotent complete cat's} \}$

no!

(a) ✓ but not (b), e.g.:

$\mathcal{A} = \mathrm{Mod}(R)$ with $Z(R) = Z(\mathcal{A})$ a non semi-conn. ring! $(R = \prod_{\mathbb{N}} \mathbb{Z})$

full $\cup$

KS yes!

- $\{ \mathcal{A} \text{ id-complete with } Z(\mathcal{A}) \text{ semi-conn.} \}$

$\cup$

- $\{ \mathcal{A} \mid \mathcal{A} \simeq \mathrm{Mod}(R), \text{ some } R \text{ with } Z(R) \text{ semi-conn.} \}$

2-full $\cup$

- $\mathrm{ADD}^{fin}_{k} := \left\{ \begin{array}{l} \mathcal{A} \simeq \mathrm{fdmod}(A), \exists A \text{ f.d alg} / k \\ + \ k\text{-linear functors} \end{array} \right\}$

the finite k-linear cat's [Etingof-Ostrik]

full $\cup$

- $2\mathrm{FVect}_{k} := \{ \text{finite semi-simple abelian } k\text{-cat's} \}$

the (finite) 2-vector spaces [Kapranov-Voevodsky-Neuchl]

Using our characterization:

Thm.
$\boxed{\begin{array}{l} k \text{ a field} \\ \mathcal{B}, \mathcal{D} \text{ two } k\text{-linear bicategories with:} \\ \mathcal{D} \text{ is KS}, \ \mathcal{B} \text{ has fin. many objects/equiv.} \\ \Longrightarrow \ \mathrm{PsFun}_k(\mathcal{B}, \mathcal{D}) \text{ is KS!} \end{array}}$

<u>Ex.1</u> fin. dim'l 2-representations of a 2-groupoid $\mathcal{G}$ with fin. many conn. components:

$$\mathrm{PsFun}(\mathcal{G}, 2\mathrm{FVect}) \simeq \mathrm{PsFun}_k(\underbrace{k\mathcal{G}}_{\text{free } k\text{-lineariz.}}, 2\mathrm{FVect}) \quad \checkmark$$

<u>Ex.2</u> finite category modules over any k-linear tensor category $\mathcal{V}$:

$$\mathrm{fin.mod}(\mathcal{V}) \simeq \mathrm{PsFun}_k(\underbrace{B\mathcal{V}}_{\text{delooping}}, \mathrm{ADD}_k^{fin}) \quad \checkmark$$

Thank you for your attention!