Functorial precomposition with applications to Mackey functors and biset functors

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Introduction

Summary

The main goal of this thesis was to give a new proof of, and enhance, a theorem by Nakaoka, in [20], comparing *global Mackey functors* and *biset functors*. (See respectively [27] and [6].) This brought us to prove a new abstract monadicity theorem for functor categories, and to prove the existence, and study a number of properties, of a pseudofunctor $\mathcal{R}$ from the bicategory of spans in finite groupoids to the bicategory of finite bimodules between finite groupoids.

Let us fix a base commutative ring $\Bbbk$, which typically is $\mathbb{Z}$ or a field. By *tensor category* we will always mean a symmetric monoidal category with unit object 1. By *tensor functor* we mean a strong symmetric monoidal functor.

Main Theorem 1 (Tensor monadicity for functor categories; see 1.3.13)**.** *Let $F\colon \mathbb{C} \to \mathbb{D}$ be a $\Bbbk$-linear tensor functor between two small $\Bbbk$-linear tensor categories. By standard constructions, we obtain the following diagram*

$$\begin{tikzcd}
& \mathbb{V}^{\mathbb{C}} \arrow[dl, shift right, "\mathrm{Lan}_F"'] \arrow[dr, shift left, "\mathrm{Free}"'] & \\
\mathbb{V}^{\mathbb{D}} \arrow[ur, shift right, "F^*"'] \arrow[rr, "E"'] & & A - \mathrm{Mod}_{\mathbb{V}^{\mathbb{C}}} \arrow[ul, shift left, "U"']
\end{tikzcd}$$

where $\mathbb{V} := \Bbbk - \mathrm{Mod}$ *is the category of* $\Bbbk$-*modules and* $\mathbb{V}^{\mathbb{C}}$ *and* $\mathbb{V}^{\mathbb{D}}$ *denote the categories of* $\Bbbk$*-linear functors equipped with the Day convolution tensor product; where A is the commutative monoid $F^*(1)$ in $\mathbb{V}^{\mathbb{C}}$ and $A - \mathrm{Mod}_{\mathbb{V}^{\mathbb{C}}}$ its category of modules; and where E is the Eilenberg-Moore comparison functor between the precomposition–left Kan extension adjunction* (Lan_F, F^*) *and the free module–forgetful adjunction* (Free, U)*. If F is full and essentially surjective and $\mathbb{C}$ (and therefore also $\mathbb{D}$) is rigid, then E is an equivalence of tensor categories.*

The monoid structure on $A = F^*(1)$ comes from the lax structure of the right adjoint F^*, and – since F is full and essentially surjective – the theorem identifies its module category with $\mathbb{V}^{\mathbb{D}}$ as full subcategories of $\mathbb{V}^{\mathbb{C}}$.

The primordial example of this is the case where F is a surjective homomorphism $f\colon R \to S$ of commutative $\Bbbk$-algebras, in which case the theorem identifies the tensor category of S-modules (in the usual sense) with the tensor category of ${}_R S$-modules *inside the category* $R-\mathrm{Mod}$, that is: those R-modules whose action factors through S.

Another example is the following one, which had previously been studied by Yoshida [28] and Panchadcharam-Street [22]. (This one is far less obvious: Webb [27] calls it "perhaps the most striking result about cohomological Mackey functors"). We can directly obtain their results from our abstract theorem above by taking F to be the full tensor functor ${}_G X \mapsto \Bbbk[X]$ comparing $\Bbbk$-linear spans of G-sets with permutation $\Bbbk G$-modules:

Corollary 1 (Cohomological vs ordinary Mackey functors; see 2.2.17)**.** *For every finite group G, there is an equivalence of tensor categories between:*

- *The category of modules over the fixed-point Green functor $FP_{\Bbbk}$;*
- *Representations of the category of permutation $\Bbbk G$-modules.*

Moreover, both categories identify canonically with the full subcategory of ordinary Mackey functors for G satisfying the 'cohomological axiom' ($L \leq H \leq G$):

$$\mathrm{ind}_L^H \circ \mathrm{res}_L^H = [H : L] \cdot \mathrm{Id}.$$

For our second main result, consider the 2-category $\mathcal{G}$ of finite groupoids, functors and natural transformations. There are two well-known ways to 'symmetrize' $\mathcal{G}$, using spans (a.k.a. correspondences) or bimodules (a.k.a. bisets or profunctors). They give rise, respectively, to the bicategories $\mathcal{S}$ and $\mathcal{B}$. Our result is the following precise comparison of these two bicategories (a similar construction was considered by Hoffnung, see Claim 13 in [13], but without giving any proofs).

Main Theorem 2 (Comparison of spans and bimodules; see § 4.2 and 4.3)**.** *There exists a pseudo-functor $\mathcal{R}\colon \mathcal{S} \to \mathcal{B}$ which is the identity on objects (that is, finite groupoids), and realizes a span $Y \overset{b}{\leftarrow} S \overset{a}{\rightarrow} X$ as the coend*

$$b_* \otimes a^* := \int^{s \in S} Y(bs, -) \times X(-, as).$$

It enjoys several properties. In particular, after 1-truncation τ_1 (identify isomorphic 1-cells) and $\Bbbk$-linearization, it induces a functor

$$F := \Bbbk\tau_1\mathcal{R} \quad : \quad \mathbb{C} = \Bbbk\tau_1\mathcal{S} \longrightarrow \Bbbk\tau_1\mathcal{B} = \mathbb{D}$$

satisfying the hypotheses of the Main Theorem 1.

As our second application, we achieve the goal that was initially set as a corollary: indeed, we can now directly apply the Main Theorem 1 to the functor F of the Main Theorem 2 to obtain the following result, most of which (safe the monoidal part) has already appeared in work of Nakaoka [20] [21], with very different proofs:

Corollary 2 (Biset functors vs global Mackey functors; see 4.4.5)**.** *There is an equivalence of tensor categories between:*

- *The category of modules over a certain global Green functor (see [21]);*
- *The category of biset functors $\mathcal{F}$ of Bouc [6], that is, representations of the category of bisets.*

Moreover, both categories identify canonically with the reflexive full subcategory of global Mackey functors satisfying the 'deflation axiom' ($N \trianglelefteq G$):

$$\mathrm{def}^G_{G/N} \circ \mathrm{inf}^G_{G/N} = \mathrm{Id}.$$

Other than our first Main Theorem, this second application uses the identification of Bouc's category $\mathcal{F}$ of biset functors with the category of representations of bimodules between groupoids; this is easily seen because $\tau_1\mathcal{B}$ is the additive completion of the category of bisets between finite groups. Indeed, a biset is the same thing as a bimodule between groups, and every groupoid is isomorphic in $\tau_1\mathcal{B}$ to a direct sum of groups. (See section 4.4 for details.)

Conventions

In order to be as light as possible when it comes to diagrams, polygon diagrams will be commutative by default. For the same purpose, not every arrow will have a label: arrows with no label are either the obvious arrow, i.e. the only arrow with such source and target that has been defined, or the arrow defined by the commutativity of the diagram. Always in a wish for readability, we will often omit subscripts when ambiguity is improbable due to context.

Chapter 1

Functorial precomposition

In this chapter we present all the constructions necessary to understand the first Main Theorem of the Introduction, then we prove it.

1.1 Tensor categories

In this section, we recall the basic notions of monoidal category theory that will be useful in this work.

Definition 1.1.1. A *symmetric monoidal category with unit*, or simply *tensor category*, is a category $\mathbb{C}$ equipped with:

1. a functor $- \otimes_{\mathbb{C}} - : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, called the *tensor*,
2. an object $1_{\mathbb{C}} \in \mathbb{C}$, called *tensor unit*,
3. a natural isomorphism called *associator* $a_{\mathbb{C}}$ (for $x, y, z \in \mathbb{C}$):
$$a_{\mathbb{C},(x,y,z)} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$$
4. a natural isomorphism $\mathrm{lun}_{\mathbb{C}}$, called *left unitor* ($x \in \mathbb{C}$):
$$\mathrm{lun}_{\mathbb{C},x} : 1_{\mathbb{C}} \otimes x \to x$$
5. a natural isomorphism $\mathrm{run}_{\mathbb{C}}$, called *right unitor* ($x \in \mathbb{C}$):
$$\mathrm{run}_{\mathbb{C},x} : x \otimes 1_{\mathbb{C}} \to x$$
6. a natural transformation $\sigma_{\mathbb{C}}$, called *symmetry*,
$$\sigma_{\mathbb{C},(x,y)} : x \otimes_{\mathbb{C}} y \to y \otimes_{\mathbb{C}} x$$
such that
$$\sigma_{\mathbb{C},(x,y)} \circ \sigma_{\mathbb{C},(y,x)} = \mathrm{Id}_{x \otimes_{\mathbb{C}} y}$$
for any two objects in $\mathbb{C}$.

(When there will be no risk of misunderstanding, the subscript will be omitted and we will simply write $\otimes$, a, 1, etc.)

This data is subject to the following coherence conditions, called *triangle and pentagon identities*: for any objects x, y in $\mathbb{C}$,

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "a_{x,1,y}"] \arrow[dr, "\mathrm{run}_x \otimes \mathrm{Id}_y"'] & & x \otimes (1 \otimes y) \arrow[dl, "\mathrm{Id}_x \otimes \mathrm{lun}_y"] \\
& x \otimes y &
\end{tikzcd}$$

and for any objects s, t, u, v in $\mathbb{C}$,

$$\begin{tikzcd}
& (s \otimes t) \otimes (u \otimes v) \arrow[dr, "a_{(s,t,u\otimes v)}"] & \\
((s \otimes t) \otimes u) \otimes v \arrow[ur, "a_{(s\otimes t,u,v)}"] \arrow[d, "a_{(s,t,u)} \otimes \mathrm{Id}"] & & s \otimes (t \otimes (u \otimes v)) \\
(s \otimes (t \otimes u)) \otimes v \arrow[rr, "a_{(s,t\otimes u,v)}"] & & s \otimes ((t \otimes u) \otimes v) \arrow[u, "\mathrm{Id} \otimes a_{(t,u,v)}"']
\end{tikzcd}$$

The following compatibility between the symmetry and the associator and unitors is also required:

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a_{(x,y,z)}"] \arrow[d, "\sigma_{(x,y)} \otimes \mathrm{Id}"'] & x \otimes (y \otimes z) \arrow[r, "\sigma_{(x,y\otimes z)}"] & (y \otimes z) \otimes x \arrow[d, "a_{(y,z,x)}"] \\
(y \otimes x) \otimes z \arrow[r, "a_{(y,x,z)}"] & y \otimes (x \otimes z) \arrow[r, "\mathrm{Id} \otimes \sigma_{(x,z)}"] & y \otimes (z \otimes x)
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes 1 \arrow[rr, "\sigma_{x,1}"] \arrow[dr, "\mathrm{run}_x"'] & & 1 \otimes x \arrow[dl, "\mathrm{lun}_x"] \\
& x &
\end{tikzcd}$$

(it can actually be shown that the second axiom follows from all the others, see [15] Proposition 2.1).

The tensor category is said to be *strict* if the associator, left and right unitors (but not necessarily the symmetry!) are identities.

Throughout this section and this chapter, $\mathbb{C}$ will be a tensor category.

Definition 1.1.2. Let $(\mathbb{C}, \otimes_{\mathbb{C}}, 1_{\mathbb{C}})$ and $(\mathbb{D}, \otimes_{\mathbb{D}}, 1_{\mathbb{D}})$ be two tensor categories. A *lax monoidal functor* between them is given by:

1. a functor $F : \mathbb{C} \to \mathbb{D}$,
2. a morphism $\mathrm{lax}_1^F : 1_{\mathbb{D}} \to F(1_{\mathbb{C}})$,

3. a natural transformation $\mathrm{lax}^F_{x,y} : F(x) \otimes_{\mathbb{D}} F(y) \to F(x \otimes_{\mathbb{C}} y)$ (for $x, y \in \mathbb{C}$).
   This data must satisfy the following associativity and unitality axioms:

$$\begin{tikzcd}
(F(x) \otimes F(y)) \otimes F(z) \arrow[r, "a_{\mathbb{D}}"] \arrow[d, "\mathrm{lax}^F_{x,y} \otimes \mathrm{Id}"'] & F(x) \otimes (F(y) \otimes F(z)) \arrow[d, "\mathrm{Id} \otimes \mathrm{lax}^F_{y,z}"] \\
F(x \otimes y) \otimes F(z) \arrow[d, "\mathrm{lax}^F_{x \otimes y, z}"'] & F(x) \otimes F(y \otimes z) \arrow[d, "\mathrm{lax}^F_{x, y \otimes z}"] \\
F((x \otimes y) \otimes z) \arrow[r, "F(a_{\mathbb{C}})"] & F(x \otimes (y \otimes z))
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes F(x) \arrow[r, "\mathrm{lax}^F_1 \otimes \mathrm{Id}"] \arrow[d, "\mathrm{lun}_{\mathbb{D}, F(x)}"'] & F(1) \otimes F(x) \arrow[d, "\mathrm{lax}^F_{1,x}"] \\
F(x) & F(1 \otimes x) \arrow[l, "F(\mathrm{lun}_{\mathbb{C},x})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(x) \otimes 1 \arrow[r, "\mathrm{Id} \otimes \mathrm{lax}^F_1"] \arrow[d, "\mathrm{run}_{\mathbb{D}, F(x)}"'] & F(x) \otimes F(1) \arrow[d, "\mathrm{lax}^F_{x,1}"] \\
F(x) & F(x \otimes 1) \arrow[l, "F(\mathrm{run}_{\mathbb{C},x})"']
\end{tikzcd}$$

It is a *symmetric* lax monoidal functor if the following holds:

$$\begin{tikzcd}
F(x) \otimes F(y) \arrow[r, "\sigma_{Fx,Fy}"] \arrow[d, "\mathrm{lax}_{x,y}"'] & F(y) \otimes F(x) \arrow[d, "\mathrm{lax}_{y,x}"] \\
F(x \otimes y) \arrow[r, "F(\sigma_{x,y})"] & F(y \otimes x)
\end{tikzcd}$$

It is a *strong* monoidal functor if both the morphisms lax^F_1 and $\mathrm{lax}^F_{x,y}$ are isomorphisms (in which case we will also write 'strg' instead of 'lax').

By *tensor functor* we will mean a strong symmetric monoidal functor. By a *lax tensor functor* we mean a symmetric lax monoidal functor.

A *tensor equivalence* is a strong tensor functor whose underlying functor F is an equivalence of categories. In this case, one can check that any pseudo-inverse F^{-1} is also a strong tensor functor.

Theorem 1.1.3. *Every tensor category is tensor equivalent to a strict tensor category.*

Proof. A proof and an explanation can be found in [19] VII. First one shows, forgetting the symmetry, that the given monoidal category is monoidal equivalent to a strict one (where associators and unitors are identities). Then one shows that the symmetry can be transferred along this equivalence. □

Remark 1.1.4. A consequence of the coherence theorem for monoidal categories is that the abusive bracket-free notations, such as $x \otimes y \otimes z$, in a monoidal category that is not associative "on the nose" for instance, can be used without causing real damages: any two ways of re-introducing associators and unitors to make things precise will yield the same morphisms, and so there is no real ambiguity when using such short-hand notations. We will commit such notational abuses in this work.

Proposition 1.1.5. *The right adjoint to a strong tensor functor is a lax tensor functor.*

Proof. Let $F : \mathbb{C} \to \mathbb{D}$ be a strong tensor functor, and let G be its right adjoint. We denote η and ε the unit and counit of this adjunction. We define the following morphisms:

$$\mathrm{lax}_1^G := G(\mathrm{strg}_1^F)^{-1} \circ \eta_1 : 1_{\mathbb{C}} \to G(1_{\mathbb{D}}),$$

$$\mathrm{lax}_{y_1,y_2}^G := G(\varepsilon_{y_1} \otimes \varepsilon_{y_2}) \circ G(\mathrm{strg}_{G(y_1),G(y_2)}^{F-1}) \circ \eta_{G(y_1)\otimes G(y_2)},$$

$$G(y_1) \otimes G(y_2) \to G \circ F(G(y_1) \otimes G(y_2)) \to G(FG(y_1) \otimes FG(y_2)) \to G(y_1 \otimes y_2).$$

It is lenghty but straightforward [1] to check that these morphisms give a tensor structure to G. ☐

Definition 1.1.6. A tensor category $\mathbb{C}$ is *rigid* if every object $x \in \mathbb{C}$ has a *dual* object $x^\vee$ together with morphisms $\eta_x : 1 \to x \otimes x^\vee$ and $\varepsilon_x : x^\vee \otimes x \to 1$, such that

$$\mathrm{run}_x \circ (\mathrm{Id}_x \otimes \varepsilon_x) \circ a_{x,x^\vee,x} \circ (\eta_x \otimes \mathrm{Id}_x) \circ \mathrm{lun}_x^{-1} = \mathrm{Id}_x,$$

and

$$\mathrm{lun}_{x^\vee} \circ (\varepsilon_x \otimes \mathrm{Id}_{x^\vee}) \circ a_{x^\vee,x,x^\vee}^{-1} \circ (\mathrm{Id}_{x^\vee} \otimes \eta_x) \circ \mathrm{run}_{x^\vee}^{-1} = \mathrm{Id}_{x^\vee}.$$

Definition 1.1.7. A tensor category $\mathbb{C}$ is *closed* if for any object x of $\mathbb{C}$, the functor $- \otimes x : \mathbb{C} \to \mathbb{C}$ has a right adjoint, usually denoted $[x, -]$ and called *internal hom*.

Remark 1.1.8. A rigid category $\mathbb{C}$ is closed, with internal hom given by $[x, -] := x^\vee \otimes -$. In particular, we have the adjunction isomorphism

$$\mathbb{C}(y \otimes x, z) \simeq \mathbb{C}(y, x^\vee \otimes z),$$

which sends $y \otimes x \xrightarrow{f} z$ to $y \simeq y \otimes 1 \xrightarrow{\mathrm{Id}\otimes\eta_x} y \otimes x \otimes x^\vee \xrightarrow{f\otimes\mathrm{Id}} z \otimes x^\vee \xrightarrow{\sigma} x^\vee \otimes z$.

Definition 1.1.9. A *monoid in* $\mathbb{C}$ is an object R together with morphisms $m_R : R\otimes R \to R$ (*multiplication*) and $u_R : 1 \to R$ (*unit*) such that the associative law, the left unit law and the right unit law all hold:

$$\begin{tikzcd}
R \otimes (R \otimes R) \arrow[r, "\mathrm{Id}\otimes m"] \arrow[d, "a"'] & R \otimes R \arrow[dr, "m"] & \\
(R \otimes R) \otimes R \arrow[r, "m\otimes\mathrm{Id}"'] & R \otimes R \arrow[r, "m"'] & R
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes R \arrow[rr, "u_R\otimes\mathrm{Id}"] \arrow[dr, "\mathrm{lun}_R"'] & & R \otimes R \arrow[dl, "m_R"] \\
& R &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \otimes 1 \arrow[rr, "\mathrm{Id}\otimes u_R"] \arrow[dr, "\mathrm{run}_R"'] & & R \otimes R \arrow[dl, "m_R"] \\
& R &
\end{tikzcd}$$

Furthermore, a monoid R in $\mathbb{C}$ is said to be *commutative* if $m_R \circ \sigma_{R,R} = m_R$.

[1] Details can be found at https://ncatlab.org/nlab/show/monoidal+adjunction.

Proposition 1.1.10. *If $F : \mathbb{C} \to \mathbb{D}$ is a (strong) tensor functor that is essentially surjective, and if $\mathbb{C}$ is rigid, then $\mathbb{D}$ is also rigid.*

Proof. We denote by η and ε the rigidity unit and counit in $\mathbb{C}$.

Let $F(x)^\vee := F(x^\vee)$. We can now construct a unit and a counit for this object, using the structure morphims of F:

$$\eta_{\mathbb{D},F(x)} := \left(1_{\mathbb{D}} \simeq F(1) \xrightarrow{F(\eta)} F(x \otimes x^\vee) \simeq F(x) \otimes F(x)^\vee\right)$$

and

$$\varepsilon_{\mathbb{D},F(x)} := \left(F(x)^\vee \otimes F(x) \simeq F(x^\vee \otimes x) \xrightarrow{F(\varepsilon)} F(1) \simeq 1\right).$$

We need to check the commutativity of

$$\begin{tikzcd}
F(x) \arrow[rrr, equal] \arrow[d, "\mathrm{lun}^{-1}"'] & & & F(x) \\
1_{\mathbb{D}} \otimes F(x) \arrow[r, "\eta_{\mathbb{D}} \otimes \mathrm{Id}"] & (F(x) \otimes F(x)^\vee) \otimes F(x) \arrow[rr, "a_{\mathbb{D}}"] & & F(x) \otimes (F(x)^\vee \otimes F(x)), \arrow[u, "\mathrm{run} \circ (\mathrm{Id} \otimes \varepsilon_{\mathbb{D}})"']
\end{tikzcd}$$

i.e. that of the outer rectangle of the following diagram:

$$\begin{tikzcd}
F(x) \arrow[rr, equal] \arrow[d, "\mathrm{lun}_{\mathbb{D}}^{-1}"'] \arrow[dr, "F(\mathrm{lun}^{-1})"] & & F(x) & F(x) \otimes 1_{\mathbb{D}} \arrow[l, "\mathrm{run}"'] \\
1_{\mathbb{D}} \otimes F(x) \arrow[d, "\mathrm{strg}_1^F \otimes \mathrm{Id}"'] & F(1 \otimes x) \arrow[r, dashed, "\varphi"'] \arrow[dl, "\mathrm{strg}^{-1}"] & F(x \otimes 1) \arrow[u, "F(\mathrm{run})"'] & F(x) \otimes F(1) \arrow[l, "\mathrm{strg}"] \arrow[u, "\mathrm{Id} \otimes \mathrm{strg}_1^{-1}"'] \\
F(1) \otimes F(x) \arrow[d, "F(\eta) \otimes \mathrm{Id}"'] & & & F(x) \otimes F(x^\vee \otimes x) \arrow[u, "\mathrm{Id} \otimes F(\varepsilon)"'] \\
F(x \otimes x^\vee) \otimes F(x) \arrow[r, "\mathrm{strg}^{-1} \otimes \mathrm{Id}"'] & (F(x) \otimes F(x^\vee)) \otimes F(x) \arrow[rr, "a_{\mathbb{D}}"'] & & F(x) \otimes (F(x^\vee) \otimes F(x)). \arrow[u, "(\mathrm{Id} \otimes \mathrm{strg})"']
\end{tikzcd}$$

To show the commutativity of the outer rectangle, it suffices to show that of the four inner polygons, where φ is defined in the next diagram. The right and left polygons commute thanks to the unitary axioms of lax monoidal functors (1.1.2) applied to the strong monoidal case. If we apply F to the first axiom of a dual object (1.1.6), we see that the top polygon also commutes. The remaining octagon is equal to the perimeter of the next diagram:

$$\begin{tikzcd}
F(1 \otimes x) \arrow[r, "\mathrm{strg}^{-1}"] \arrow[d, "F(\eta \otimes \mathrm{Id})"] \arrow[ddd, bend right=60, "\varphi :="'] & F(1) \otimes F(x) \arrow[r, "F(\eta)\otimes \mathrm{Id}"] & F(x \otimes x^{\vee}) \otimes F(x) \arrow[d, "\mathrm{strg}^{-1}\otimes \mathrm{Id}"] \\
F((x \otimes x^{\vee}) \otimes x)) \arrow[urr, dashed, "\mathrm{strg}^{-1}_{x\otimes x^{\vee},x}"'] \arrow[d, "F(a_{\mathbb{C}})"] & & (F(x) \otimes F(x^{\vee})) \otimes F(x) \arrow[d, "a_{\mathbb{D}}"] \\
F(x \otimes (x^{\vee} \otimes x)) \arrow[rr, dashed, "(\mathrm{Id}\otimes \mathrm{strg}^{-1}_{x^{\vee},x}) \circ \mathrm{strg}^{-1}_{x,x^{\vee}\otimes x}"] \arrow[d, "F(\mathrm{Id}\otimes\varepsilon)"] & & F(x) \otimes (F(x^{\vee}) \otimes F(x)) \arrow[d, "\mathrm{Id}\otimes \mathrm{strg}"] \\
F(x \otimes 1) & F(x \otimes 1) \arrow[l, "\mathrm{strg}"] & F(x) \otimes F(x^{\vee} \otimes x) \arrow[l, "\mathrm{Id}\otimes F(\varepsilon)"]
\end{tikzcd}$$

By the hexagonal axiom in 1.1.2, the trapezoid in the middle is commutative. The triangle at the top and the rectangle at the bottom are commutative by naturality of strg.

The proof that the second rigidity axiom is similar and is left to the reader.

Since F is essentially surjective, every object $y \in \mathbb{D}$ is isomorphic to one of the form $F(x)$, and the dual constructed above for $F(x)$ yields a dual for y via any chosen isomorphism. Hence, $\mathbb{D}$ is rigid. □

Definition 1.1.11. If $R = (R, m_R, u_R)$ is a monoid in $\mathbb{C}$, *a (left) R-module $M = (M, \lambda_M)$ in $\mathbb{C}$* is an object M equipped with an action morphism $\lambda_M : R \otimes M \to M$ such that the expected associativity and unitality axioms hold:

$$\begin{tikzcd}
R \otimes (R \otimes M) \arrow[rr, "\mathrm{Id}\otimes\lambda_M"] & & R \otimes M \arrow[d, "\lambda_M"] \\
(R \otimes R) \otimes M \arrow[u, "a_{\mathbb{C}}"] \arrow[r, "m\otimes \mathrm{Id}"'] & R \otimes M \arrow[r, "\lambda_M"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes M \arrow[r, "u_R\otimes \mathrm{Id}"] \arrow[d, "\mathrm{lun}_M"'] & R \otimes M \arrow[d, "\lambda_M"] \\
M \arrow[r, equal] & M.
\end{tikzcd}$$

Proposition 1.1.12. *A lax monoidal functor sends monoid objects to monoid objects. If the functor is symmetric, it sends commutative monoids to commutative monoids. Furthermore, the image of a module over a monoid is a module over the image of the monoid.*

Proof. Let $F : \mathbb{C} \to \mathbb{D}$ be a lax monoidal functor, R be a monoid in $\mathbb{C}$ and M be a R-module in $\mathbb{C}$. Let's define three morphisms:

$$m_{F(R)} := F(m_R) \circ \mathrm{lax}^F_{R,R},$$

$$u_{F(R)} := F(u_R) \circ \mathrm{lax}^F_1,$$

$$\lambda_{F(M)} := F(\lambda_M) \circ \mathrm{lax}^F_{R,M}.$$

It follows easily from the axioms of 1.1.2 that $(F(R), m_{F(R)}, u_{F(R)})$ is a monoid and $(F(M), \lambda_{F(M)})$ is an $F(R)$-module in $\mathbb{D}$. If R is commutative and F is symmetric, it follows immediatly from the definition that $F(R)$ is commutative as well.

See again [19] VII for more details. ☐

Proposition 1.1.13. *$1_{\mathbb{C}}$ is a commutative monoid in $\mathbb{C}$, the multiplication morphism being given by $\mathrm{lun}_{1_{\mathbb{C}}} = \mathrm{run}_{1_{\mathbb{C}}}$ and the unit morphism by the identity of $1_{\mathbb{C}}$. Any object x of $\mathbb{C}$ is a $1_{\mathbb{C}}$-module with action also given by the left unitor lun_x. Furthermore, these monoid and module structure are unique up to a canonical isomorphism.*

Proof. The fact that $(1_{\mathbb{C}}, \mathrm{lun}_1, \mathrm{Id}_1)$ is a commutative monoid is easy to check thanks to the coherence theorem. The same is true of the $1_{\mathbb{C}}$-module structure of (x, lun_x) for that monoid structure on $1_{\mathbb{C}}$.

The unicity of those structures is quite the exercise. It uses the coherence theorem, as well as the fact that the monoid for composition $\mathrm{End}_{\mathbb{C}}(1_{\mathbb{C}})$ is commutative. ☐

Proposition 1.1.14. *If $\mathbb{C}$ and $\mathbb{D}$ are tensor categories and $F : \mathbb{C} \to \mathbb{D}$ is a symmetric lax monoidal functor, then $F(1)$ is a commutative monoid object inside $\mathbb{D}$, and the images of objects of $\mathbb{C}$ by F are $F(1)$-modules.*

Proof. This is immediate from 1.1.13 and 1.1.12. ☐

Definition 1.1.15. Let R be a monoid object in $\mathbb{C}$. We denote by $R{-}\mathrm{Mod}_{\mathbb{C}}$ the category of R-modules in $\mathbb{C}$, and morphisms in $\mathbb{C}$ between them that commute with actions.

Proposition 1.1.16. *If $\mathbb{C}$ has enough colimits (coequalizers suffice), and if the monoid R is commutative, the category $R - \mathrm{Mod}_{\mathbb{C}}$ is a symmetric monoidal category.*

Proof. This are all standard constructions, which we now partly recall.

The monoidal product in $R - Mod_{\mathbb{C}}$, denoted by $- \otimes_R -$, is defined by the following coequalizer in $\mathbb{C}$

$$\begin{tikzcd} M \otimes R \otimes M' \arrow[r, shift left, "\mathrm{Id}\otimes\lambda'"] \arrow[r, shift right, "(\lambda\otimes\mathrm{Id})\circ(\sigma\otimes\mathrm{Id})"'] & M \otimes M \arrow[r, "p_{M,M'}"] & M \otimes_R M' \end{tikzcd} \tag{1.1}$$

where $\lambda : R \otimes M \to M$ and $\lambda' : R \otimes M' \to M'$ are the given actions.

As a monoid object, R is an R-module, and from its module structure emerges the unitary structure of the category.

The symmetry σ_R is the collection of arrows induced by the graph below via universal property

$$\begin{tikzcd} M \otimes R \otimes M' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\sigma\otimes\mathrm{Id})\circ\sigma"'] & M \otimes M' \arrow[r] \arrow[d, "\sigma"'] & M \otimes_R M' \arrow[d, "!"] \\ M' \otimes R \otimes M \arrow[r, shift left] \arrow[r, shift right] & M' \otimes M \arrow[r] & M' \otimes_R M. \end{tikzcd}$$

Using the universal property of coequalizers, one can construct associativity and unit morphisms in a straightforward way, and check that the axioms hold because they hold for the symmetric monoidal category $\mathbb{C}$. $\square$

The following lemma will be usefull in section 1.4:

Lemma 1.1.17. *Let (M, λ) be a left module over a monoid (A, m_A, u_A) in a monoidal category $\mathbb{C}$. Then the following diagram is a coequalizer in $\mathbb{C}$:*

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, shift left, "\mathrm{Id}\otimes\lambda"] \arrow[r, shift right, "m\otimes\mathrm{Id}"'] & A \otimes M \arrow[r, "\lambda"'] & M \end{tikzcd}$$

Proof. One can check that the Universal Property of coequalizers holds: let $\alpha : A \otimes M \to T$ be such that $\alpha \circ (\mathrm{Id}_A \otimes \lambda) = \alpha \circ (m_A \otimes \mathrm{Id}_M)$.

We must show that there is a unique $\beta : M \to T$ such that $\beta \circ \lambda = \alpha$.

The uniqueness follows from the fact that λ is a split epimorphism, as implied by the unit axiom $\lambda \circ (u_A \otimes \mathrm{Id}_M) \circ \mathrm{lun}_M^{-1} = \mathrm{Id}_M$. The existence is obtained as follows: let $\beta := \alpha \circ (u_A \otimes \mathrm{Id}_M) \circ \mathrm{lun}_M^{-1}$, then the following diagram shows that $\beta \circ \lambda = \alpha$:

$$\begin{tikzcd}
A \otimes M \arrow[ddd, "\alpha"'] & & & A \otimes M \arrow[lll, "\mathrm{Id}"'] \arrow[ddd, "\lambda"] \\
& A \otimes A \otimes M \arrow[ul, "m_A\otimes\mathrm{Id}"'] \arrow[d, "\mathrm{Id}\otimes\lambda"'] & 1 \otimes A \otimes M \arrow[l, "u_A\otimes\mathrm{Id}"'] \arrow[ur, "\mathrm{lun}"] \arrow[d, "\mathrm{Id}\otimes\lambda"] \arrow[dl, "u\otimes\lambda"'] & \\
& A \otimes M \arrow[dl, "\alpha"] & 1 \otimes M \arrow[l, "u_A\otimes\mathrm{Id}"'] \arrow[dr, "\mathrm{lun}"'] & \\
T & & & M. \arrow[lll, "\beta"]
\end{tikzcd}$$

Commutativity of the trapezoids: the left one commutes by the property of α, the right one commutes by naturality of the left unitor, the bottom one commutes by definition of β and the top one is the unit axiom for the monoid A. The middle square commutes by functoriality of $- \otimes -$. $\square$

Proposition 1.1.18. *Let R be a monoid object in $\mathbb{C}$. There is a faithful functor $U : R - \mathrm{Mod}_{\mathbb{C}} \to \mathbb{C}$ that sends (M, ρ_M) to M and any morphism to "itself". We call this functor the forgetful functor.*

Proof. See [19] VII 4. $\square$

Proposition 1.1.19. *Let R be a monoid object in $\mathbb{C}$. There is a functor* Free : $\mathbb{C} \longrightarrow R - \mathrm{Mod}_{\mathbb{C}}$ *that sends M to $(R \otimes M, (m_R \otimes \mathrm{Id}_M) \circ a_{R,R,M}^{-1})$ and any morphism $\alpha : M \to N$ to $\mathrm{Id}_R \otimes \alpha$. It is left adjoint to the forgetful functor U.*

Proof. See [19] VII 4. $\square$

1.2 Day convolution

In this section, and in the next one, we cite G.M. Kelly's [16]. However, Kelly's work is much more general than what we need here, namely results of the theory of categories enriched over $\Bbbk$-modules. We try to give explicit definitions and simple, specific proofs of our statement, but sometimes we will refer to that greater generality.

Definition 1.2.1. Let $\Bbbk$ be a commutative ring. We set $\mathbb{V} := \Bbbk - \mathrm{Mod}$, the category of $\Bbbk$-modules and $\Bbbk$-linear maps.

We recall the following well known fact:

Proposition 1.2.2. *When equipped with the tensor product $\otimes_\Bbbk$ and the internal Hom $\mathrm{Hom}_\Bbbk(-,-)$, $\mathbb{V}$ becomes a symmetric closed monoidal category which is both complete and cocomplete.* □

Definition 1.2.3. A category $\mathbb{C}$ is $\Bbbk$-*linear*, or $\mathbb{V}$-*enriched*, if for any two objects x, y, the hom-set $\mathbb{C}(x,y)$ is an object in $\mathbb{V}$ (a $\Bbbk$-module) and if there are composition maps $c : \mathbb{C}(y,z) \otimes_\Bbbk \mathbb{C}(x,y) \to \mathbb{C}(x,z)$ and identity maps $\mathrm{Id}_x : \Bbbk \to \mathbb{C}(x,x)$ in $\mathbb{V}$ ($\Bbbk$-homomorphisms) such that the associativity condition holds

$$\begin{tikzcd}
(\mathbb{C}(z,t) \otimes_\Bbbk \mathbb{C}(y,z)) \otimes_\Bbbk \mathbb{C}(x,y) \arrow[rr, "a_{\mathbb{V}}", "\cong"'] \arrow[d, "c \otimes_\Bbbk \mathrm{Id}"'] & & \mathbb{C}(z,t) \otimes_\Bbbk (\mathbb{C}(y,z) \otimes_\Bbbk \mathbb{C}(x,y)) \arrow[d, "\mathrm{Id} \otimes_\Bbbk c"] \\
\mathbb{C}(y,t) \otimes_\Bbbk \mathbb{C}(x,y) \arrow[dr, "c"'] & & \mathbb{C}(z,t) \otimes_\Bbbk \mathbb{C}(x,z) \arrow[dl, "c"] \\
& \mathbb{C}(x,t) &
\end{tikzcd}$$

and identities are left/right neutral elements for composition.

A tensor (that is, symmetric monoidal) structure on a $\Bbbk$-category is $\Bbbk$-*linear* if its tensor functor $- \otimes -$ is a $\Bbbk$-linear functor of each variable (see definition 1.2.7 below).

Definition 1.2.4. A category $\mathbb{C}$ is said to be *semi-additive* if there is

- a zero object $0 \in \mathrm{Ob}(\mathbb{C})$, that is an object which is both initial and final in $\mathbb{C}$,
- a commutative monoid structure $+ : \mathbb{C}(X,Y) \times \mathbb{C}(X,Y) \to \mathbb{C}(X,Y)$ on every hom-set, with neutral element $0_{XY} = X \xrightarrow{!} 0 \xrightarrow{!} Y$, such that composition is bilinear with respect to that structure,
- binary biproducts that is, for every two objects X and Y, there is an object

$X \sqcup Y$ and four morphisms that make the following diagram commutative

$$\begin{tikzcd}
X \arrow[dd, equal] \arrow[dr, "i_X"] & & Y \arrow[dd, equal] \arrow[dl, "i_Y"'] \\
& X \sqcup Y \arrow[dl, "p_X"] \arrow[dr, "p_Y"'] & \\
X & & Y,
\end{tikzcd}$$

and such that $p_Y \circ i_X = 0_{XY}$, $p_X \circ i_Y = 0_{YX}$, and $i_X \circ p_X + i_Y \circ p_Y = \mathrm{Id}_{X \sqcup Y}$.

Proposition 1.2.5. *Given a semi-additive category $\mathbb{C}$, one obtains a $\mathbb{V}$-category $\Bbbk\mathbb{C}$ by considering the category with the same objects* $\mathrm{Ob}(\mathbb{C})$ *and where*

$$\Bbbk\mathbb{C}(X,Y) := \Bbbk \otimes_{\mathbb{Z}} \mathrm{Gr}\big(\mathbb{C}(X,Y)\big)$$

is the extension of scalars from $\mathbb{Z}$ to $\Bbbk$ of Grothendieck's group completion $\mathrm{Gr}\big(\mathbb{C}(X,Y)$ *of the commutative monoid $(\mathbb{C}(X,Y),+)$, and composition is extended $\Bbbk$-linearily.*

Proof. This is well-known and straightforward. □

Proposition 1.2.6. *Given two semi-additive categories $\mathbb{C}$ and $\mathbb{D}$, and given a functor $F : \mathbb{C} \to \mathbb{D}$ that respects sums of morphisms (and therefore also biproducts of objects), there is a unique way to extend F to $\Bbbk\mathbb{C} \to \Bbbk\mathbb{D}$ with respect to the $\Bbbk$-linearization process described above.*

Proof. This is well-known and straightforward. □

Definition 1.2.7. A functor $F : \mathbb{C} \to \mathbb{D}$ between two $\mathbb{V}$-enriched categories is $\Bbbk$-*linear*, or a $\mathbb{V}$-*enriched functor*, or $\mathbb{V}$-*functor*, if it is compatible with the linear structure, meaning that each map $F : \mathbb{C}(x,y) \to \mathbb{D}(Fx, Fy)$ is $\Bbbk$-linear.

Remark 1.2.8. In enriched category theory over a general base category $\mathbb{V}$, there is a notion of $\mathbb{V}$-*natural transformation*, but in our context, with $\mathbb{V} = \Bbbk - \mathrm{Mod}$, it is easy to check that ordinary natural transformations between $\mathbb{V}$-functors are always compatible with the extra structure in the required sense.

Proposition 1.2.9. *For every (essentially small) $\mathbb{V}$-category $\mathbb{C}$, the collection of all $\mathbb{V}$-functors $\mathbb{C} \to \mathbb{V}$ and all natural transformations between them forms a complete and co-complete $\mathbb{V}$-category, which we denote by $\mathbb{V}^{\mathbb{C}}$.*

Proof. It is a straightforward verification that $\mathbb{V}^{\mathbb{C}}$ forms a $\mathbb{V}$-category. It inherits its properties from the completeness and co-completeness of $\mathbb{V}$, since one can always construct limits and colimits pointwise. □

In the following, we will use the convenient notation of ($\Bbbk$-linear) coends:

Definition 1.2.10. Let $H\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{V}$ be a functor which is $\Bbbk$-linear in both variables, with $\mathbb{C}$ any (essentially) small $\Bbbk$-category. Then its *($\mathbb{V}$-enriched) coend* is an object $\int^{c\in\mathbb{C}} H \in \mathbb{V}$ defined by the following formula

$$\int^{u\in\mathbb{C}} H := \left(\bigoplus_{u\in\mathrm{Ob}(\mathbb{C})} H(u,u)\right)/T,$$

where T is the $\Bbbk$-submodule of the coproduct generated by the elements $x - x'$, where $x \in H(u,u)$ and $x' \in H(u',u')$ such that there exists an $f \in \mathbb{C}(u,u')$ and $y \in H(u',u)$ such that $H(f,\mathrm{Id}_u)(y) = x$ and $H(\mathrm{Id}_{u'},f)(y) = x'$. By construction, the coend comes equipped with a canonical map $H(v,v) \to \int^{u\in\mathbb{C}} H$ for every $v \in \mathbb{C}$.

Below, we give two lemmas that will be useful for coend calculations, which appear a lot in this work. The next 'Fubini lemma' is one good reason to use the integral notation:

Lemma 1.2.11. *Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be two $\mathbb{V}$-categories. If*

$$H : (\mathbb{C}_1 \times \mathbb{C}_2)^{op} \times (\mathbb{C}_1 \times \mathbb{C}_2) \to \mathbb{V}$$

is a $\mathbb{V}$-functor, then its (enriched) coend can be computed one variable at a time, that is there is a canonical isomorphism

$$\int^{c_1,c_2\in\mathbb{C}_1\times\mathbb{C}_2} H \simeq \int^{c_1\in\mathbb{C}_1}\int^{c_2\in\mathbb{C}_2} H,$$

given by the obvious map $[x] \mapsto [x]$ ($x \in H(c_1,c_2,c_1,c_2)$) which is the identity on representatives.

Proof. The reader can find more details and a proof in chapter 8 of [19], for the non-enriched version of the result. The same proof works in the $\Bbbk$-linear (or more general enriched) context, see also [24]. ☐

The next lemma is known as 'co-Yoneda':

Lemma 1.2.12. *For every functor $M : \mathbb{C} \to \mathbb{V}$, there is a canonical isomorphism of functors:*

$$\int^{x\in\mathbb{C}} \mathbb{C}(x,-) \otimes_{\mathbb{V}} M(x) \simeq M$$

$$[f : x \to c, \zeta \in M(x)] \mapsto M(f)(\zeta) \in M(c)$$

Here, the left-hand side notation denotes the functor sending $c \in \mathbb{C}$ to the coend $\int^{x\in\mathbb{C}} \mathbb{C}(x,c) \otimes_{\mathbb{V}} M(x)$ as in 1.2.10, with the obvious induced functoriality.

Proof. See example 1.2.4 in [24]. ☐

Coends can also be used to compute left Kan extensions. Recall that ($\Bbbk$-linear) left Kan extensions are defined as follows.

Definition 1.2.13. Given two $\mathbb{V}$-functors $T : \mathbb{A} \to \mathbb{E}$ and $K : \mathbb{A} \to \mathbb{B}$, the *left Kan extension of T along K* is a $\mathbb{V}$-functor $\mathrm{Lan}_K T : \mathbb{B} \to \mathbb{E}$, together with a natural transformation η as in the following diagram

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "T"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathbb{E} \\ & \mathbb{B} \arrow[ur, dashed, "\mathrm{Lan}_K T"'] & \end{tikzcd}$$

having the following universal property: for every $\Bbbk$-linear $L : \mathbb{B} \to \mathbb{E}$, the map

$$\mathbb{E}^{\mathbb{B}}(\mathrm{Lan}_K T, L) \xrightarrow{\sim} \mathbb{E}^{\mathbb{A}}(T, L \circ K) \tag{1.2}$$

sending β to $(\beta * K) \circ \eta$ is an isomorphism.

Proposition 1.2.14. *When $\mathbb{E}$ is co-complete and both $\mathbb{A}$ and $\mathbb{B}$ are essentially small, the left Kan extension defined above is given by the following coend:*

$$(\mathrm{Lan}_K T)(b) = \int^{a \in \mathbb{A}} \mathbb{B}(Ka, b) \otimes_{\Bbbk} T(a) \tag{1.3}$$

for $b \in \mathbb{B}$.

Proof. See [19, X.4]. See also [24, §1.2 and §7.6] for the easy adaptations to the $\Bbbk$-linear case. ☐

Let us return to the main topic of this section: coends also allow us to define a tensor product on the functor category $\mathbb{V}^{\mathbb{C}}$, as follows.

Definition 1.2.15 (Day convolution). For any two functors $M, N \in \mathbb{V}^{\mathbb{C}}$, we define their *Day convolution product* (see [9]) as the following functor $\mathbb{C} \to \mathbb{V}$:

$$M \boxtimes_{\mathbb{C}} N := \int^{u,v \in \mathbb{C}} \mathbb{C}(u \otimes v, -) \otimes_{\Bbbk} M(u) \otimes_{\Bbbk} N(v)$$

Unpacking definition 1.2.10, for each $c \in \mathbb{C}$ this functor is computed as the following quotient of $\Bbbk$-modules

$$\int^{u,v \in \mathbb{C}} \mathbb{C}(u \otimes v, c) \otimes_{\Bbbk} M(u) \otimes_{\Bbbk} N(v) = \left(\bigoplus_{u,v \in \mathbb{C}} \mathbb{C}(u \otimes v, c) \otimes_{\Bbbk} M(u) \otimes_{\Bbbk} N(v) \right) / T$$

where T is the sub-$\Bbbk$-module generated by the elements

$$f \otimes x \otimes y - f' \otimes x' \otimes y'$$

(with $f \in \mathbb{C}(u \otimes v, c)$, $x \in M(u)$, $y \in M(v)$, $f' \in \mathbb{C}(u' \otimes v', c)$, $x' \in M(u')$ and $y' \in M(v')$) whenever there exist $a \in \mathbb{C}(u, u')$ and $b \in \mathbb{C}(v, v')$ such that:

$$\begin{tikzcd} u \otimes v \arrow[d, "a \otimes b"'] \arrow[dr, "f"] & \\ u' \otimes v' \arrow[r, "f'"'] & c \end{tikzcd} \quad ; \quad M(a)(x) = x' \quad ; \quad N(b)(y) = y' \tag{1.4}$$

We will write $[f,x,y]$ for the equivalence class of $f\otimes x\otimes y$ in the quotient, and we will call *basic relation* an equality $[f,x,y]=[f',x',y']$ witnessed by a and b as in (1.4).

The functoriality of $M\boxtimes_{\mathbb{C}}N$ is induced in the evident way: given $g\in\mathbb{C}(c,c')$, the map $(M\boxtimes_{\mathbb{C}}N)(c)\to(M\boxtimes_{\mathbb{C}}N)(c')$ is given by $[f,x,y]\mapsto[g\circ f,x,y]$.

Proposition 1.2.16. *Day convolution equips $\mathbb{V}^{\mathbb{C}}$ with a $\Bbbk$-linear closed tensor structure, given by:*

1. *The functor $-\boxtimes_{\mathbb{C}}-:\mathbb{V}^{\mathbb{C}}\times\mathbb{V}^{\mathbb{C}}\to\mathbb{V}^{\mathbb{C}}$ given on objects by*
$$(M,N)\mapsto M\boxtimes_{\mathbb{C}}N:=\int^{u,v\in\mathbb{C}}\mathbb{C}(u\otimes v,-)\otimes_{\Bbbk}M(u)\otimes_{\Bbbk}N(v)$$
and on morphisms (that is, pairs of natural transformations $\alpha:M\to M'$ and $\beta:N\to N'$) by
$$(\alpha,\beta)\mapsto\left\{\ [f\in\mathbb{C}(u\otimes v,c),x,y]\mapsto[f,\alpha_u(x),\beta_v(y)]\ \right\}_{c\in\mathbb{C}}.$$

2. *The tensor unit $1_{Day}:=\mathbb{C}(1_{\mathbb{C}},-)$.*

3. *The associator $\mathfrak{a}_{\mathbb{C}}$, obtained by using the Fubini lemma 1.2.11:*
$$\mathfrak{a}_{\mathbb{C},(L,M,N)}:(L\boxtimes_{\mathbb{C}}M)\boxtimes_{\mathbb{C}}N\to L\boxtimes_{\mathbb{C}}(M\boxtimes_{\mathbb{C}}N)$$
with, at $c\in\mathbb{C}$,
$$((L\boxtimes_{\mathbb{C}}M)\boxtimes_{\mathbb{C}}N)(c)=\int^{u,r,s,t}\mathbb{C}(u\otimes t,c)\otimes_{\Bbbk}\mathbb{C}(r\otimes s,u)\otimes_{\Bbbk}L(r)\otimes_{\Bbbk}M(s)\otimes_{\Bbbk}N(t)$$
$$(L\boxtimes_{\mathbb{C}}(M\boxtimes_{\mathbb{C}}N))(c)=\int^{o,v,p,q}\mathbb{C}(o\otimes v,c)\otimes_{\Bbbk}\mathbb{C}(p\otimes q,v)\otimes_{\Bbbk}L(o)\otimes_{\Bbbk}M(p)\otimes_{\Bbbk}N(q)$$
$$\mathfrak{a}_{\mathbb{C},c}:[f,g,x,y,z]\to[f\circ(g\otimes\mathrm{Id}_t),\mathrm{Id}_v,x,y,z]$$
at $o=r$, $p=s$, $q=t$ and $v=s\otimes t$.

4. *The left unitor $\mathfrak{lun}_{\mathbb{C}}$, given at M and c by*
$$\begin{aligned}(\mathbb{C}(1_{\mathbb{C}},-)\boxtimes_{\mathbb{C}}M)(c)&=\int^{u,v}\mathbb{C}(u\otimes v,c)\otimes_{\Bbbk}\mathbb{C}(1,u)\otimes_{\Bbbk}M(v)\\&\longrightarrow\int^{v}\mathbb{C}(v,c)\otimes_{\Bbbk}M(v)\overset{1.2.12}{\simeq}M(c)\end{aligned}$$
$$\left[u\otimes v\xrightarrow{f}c,1\xrightarrow{g}u,x\right]\mapsto\left[\underbrace{v\xrightarrow{\sim}1\otimes v\xrightarrow{g\otimes\mathrm{Id}}u\otimes v\xrightarrow{f}c}_{=:h},x\right]\mapsto M(h)(x).$$

5. *The right unitor $\mathfrak{run}_{\mathbb{C}}:M\boxtimes_{\mathbb{C}}\mathbb{C}(1_{\mathbb{C}},-)\to M$ (at M) is given by the analogous formula, using the right unitor of $\mathbb{C}$ instead of the left one.*

6. *The symmetry isomorphism* $\mathfrak{s}_{M,N} \colon M \boxtimes_{\mathbb{C}} N \simeq N \boxtimes_{\mathbb{C}} M$ *is the map*
$$[f, x, y] \mapsto [f \circ \sigma, y, x]$$
induced by the symmetries of $\mathbb{C}$ *and* $\mathbb{V}$.

7. *The internal Hom is given by the following end:*
$$[M, N]_{Day}(c) := \int_{c_1, c_2 \in \mathbb{C}} \mathbb{V}\big(\mathbb{C}(c \otimes c_1, c_2),\ \mathbb{V}(M(c_1), N(c_2))\big)$$

Proof. See the original work of Day [9] for these results (in a somewhat old-fashioned terminology), or also the explanations of the nCat lab:
https://ncatlab.org/nlab/show/Day+convolution.
One can also verify this directly from the above formulas. □

Proposition 1.2.17. *The Yoneda embedding* $y_{\mathbb{C}} : \mathbb{C}^{\mathrm{op}} \to \mathbb{V}^{\mathbb{C}},\ c \mapsto \mathbb{C}(c, -)$ *is a* $\Bbbk$*-linear strong tensor functor.*

Proof. This can also be found on the nCat lab, as proposition 3.3 of:
https://ncatlab.org/nlab/show/Day+convolution.
Concretely, the map $\mathfrak{strg}^y_1$ is simply the identity, and for $c_1, c_2 \in \mathbb{C}$ the map $\mathfrak{strg}^y_{c_1,c_2}$ is given by
$$\int^{u,v \in \mathbb{C}} \mathbb{C}(u \otimes v, c) \otimes_{\Bbbk} \mathbb{C}(c_1, u) \otimes_{\Bbbk} \mathbb{C}(c_2, v) \longrightarrow \mathbb{C}(c_1 \otimes c_2, c)$$
$$[f, g, h] \mapsto f \circ (g \otimes h)$$
at each $c \in \mathbb{C}$. □

The following lemma is of crucial importance for the computations in later sections:

Lemma 1.2.18. *If* $\mathbb{C}$ *is moreover rigid, then we have the formulas:*
$$\begin{aligned} M \boxtimes N \ &\simeq \ \int^{v \in \mathbb{C}} M(v^\vee \otimes -) \otimes_{\Bbbk} N(v) \\ [f, x, y] \ &\mapsto \ [M(\tilde{f})(x), y] \end{aligned}$$
and
$$\begin{aligned} M \boxtimes N \ &\simeq \ \int^{u \in \mathbb{C}} M(u) \otimes_{\Bbbk} N(u^\vee \otimes -) \\ [f, x, y] \ &\mapsto \ [x, N(\bar{f})(y)] \end{aligned}$$
for all $M, N \in \mathbb{V}^{\mathbb{C}}$*, where* $\tilde{f}$ *and* $\bar{f}$ *are the evident composites, as follows:*
$$\tilde{f} := \sigma_{c,v^\vee} \circ (f \otimes \mathrm{Id}_{v^\vee}) \circ (\mathrm{Id}_u \otimes \eta_v) \circ \mathrm{run}_u^{-1}$$
$$\bar{f} := (\mathrm{Id}_{u^\vee} \otimes f) \circ (\sigma \otimes \mathrm{Id}_v) \circ (\eta_u \otimes \mathrm{Id}_v) \circ \mathrm{lun}_v^{-1}$$

Proof. The two formulas having similar proofs, and since only the first one will be of use in this work, we only prove the first one. For any object c in $\mathbb{C}$, we have:

$$\begin{aligned}
(M \boxtimes N)(c) &= \int^{u,v\in\mathbb{C}} \mathbb{C}(u\otimes v, c)\otimes_{\Bbbk} M(u)\otimes_{\Bbbk} N(v) & \\
&\simeq \int^{u,v\in\mathbb{C}} \mathbb{C}(u, v^{\vee}\otimes c)\otimes_{\Bbbk} M(u)\otimes_{\Bbbk} N(v) & \text{by 1.1.8} \\
&\simeq \int^{v\in\mathbb{C}}\int^{u\in\mathbb{C}} \mathbb{C}(u, v^{\vee}\otimes c)\otimes_{\Bbbk} M(u)\otimes_{\Bbbk} N(v) & \text{by 1.2.11} \\
&\simeq \int^{v\in\mathbb{C}} N(v)\otimes_{\Bbbk}\int^{u\in\mathbb{C}} \mathbb{C}(u, v^{\vee}\otimes c)\otimes_{\Bbbk} M(u) & \text{see below} \\
&\simeq \int^{v\in\mathbb{C}} M(v^{\vee}\otimes c)\otimes_{\Bbbk} N(v) & \text{by 1.2.12}
\end{aligned}$$

In the fourth line we use that for every $V \in \mathbb{V}$ the functor $V \otimes_{\Bbbk} -$, being a left adjoint, commutes with colimits, and that the coend on the right (like any coend) is a colimit in $\mathbb{V}$.

We follow these isomorphisms and see that they are given by

$$\begin{aligned}
[f, x, y] &\mapsto [\tilde{f}, x, y] \\
&\mapsto [y, \tilde{f}, x] \\
&\mapsto [M(\tilde{f})(x), y]
\end{aligned}$$

which is the straightforward shuffling of the given information. □

1.3 Precomposition

In this section, as before, we denote $\mathbb{V} = \Bbbk - \mathrm{Mod}$ for a commutative ring $\Bbbk$. We will denote by $\mathbb{C}$ and $\mathbb{D}$ two essentially small $\mathbb{V}$-enriched tensor categories, that is symmetric monoidal and with unit. We denote by $F : \mathbb{C} \to \mathbb{D}$ a $\mathbb{V}$-functor. This section will culminate with theorem 1.3.13, one of our main results.

Proposition 1.3.1. *F induces a $\mathbb{V}$-functor $F^* : \mathbb{V}^{\mathbb{D}} \to \mathbb{V}^{\mathbb{C}}$ by precomposition.*

Proof. F^* maps any functor M to the functor MF, and any natural transformation $\alpha : M \Rightarrow N$ to the whiskered natural transformation $\alpha * \mathrm{Id}_F$. Since the composite of two $\Bbbk$-linear functors is $\Bbbk$-linear, this is well defined.

Functoriality is a direct consequence of the Interchange Law for natural transformations. It is immediate to see that F^* is $\Bbbk$-linear. □

Proposition 1.3.2. *Let F be essentially surjective on objects and full. Then F^* is fully faithful.*

Proof. Let $M, N \in \mathbb{V}^{\mathbb{D}}$, and let

$$\beta = \{\beta_c : MF(c) \to NF(c) | c \in \mathrm{Ob}(\mathbb{C})\} \in \mathbb{V}^{\mathbb{C}}(MF, NF).$$

Let d be an object of $\mathbb{D}$. By essential surjectivity of F, there is an object c_d of $\mathbb{C}$ such that there is an isomorphism $f_d \in \mathbb{D}(F(c_d), d)$. We choose a couple (c_d, f_d) for every object of $\mathbb{D}$, with $f_d = \mathrm{Id}_d$ when d is in the image of F. Now, we can set

$$\alpha_d := N(f_d) \circ \beta_{c_d} \circ M(f_d^{-1}) : M(d) \to N(d)$$
$$\alpha = \{\alpha_d | d \in \mathrm{Ob}(\mathbb{D})\}$$

Now, let $g \in \mathbb{D}(d, d')$. By fullness of F, there is $a \in \mathbb{C}(c_d, c_{d'})$ such that

$$F(a) = f_{d'}^{-1} \circ g \circ f_d : F(c_d) \to F(c_{d'})$$

Hence, by naturality of β, we obtain the naturality of α:

$$\begin{aligned}
\alpha_{d'} \circ M(g) &= N(f_{d'}) \circ \beta_{c_{d'}} \circ M(f_{d'}^{-1}) \circ M(g) \\
&= N(f_{d'}) \circ \beta_{c_{d'}} \circ M(F(a)) \circ M(f_d^{-1}) \\
&= N(f_{d'}) \circ NF(a) \circ \beta_{c_d} \circ M(f_d^{-1}) \\
&= N(g) \circ N(f_d) \circ \beta_{c_d} \circ M(f_d^{-1}) \\
&= N(g) \circ \alpha_d
\end{aligned}$$

Now, it is an easy check that $F^*(\alpha) = \beta$, and we've proven F^* to be full.

Now, let $\alpha = \{\alpha_d\}_{d\in\mathbb{D}}, \alpha' = \{\alpha'_d\}_{d\in\mathbb{D}} \in \mathbb{V}^{\mathbb{D}}(M, N)$, and suppose that $F^*(\alpha) = F^*(\alpha')$. Then, for any $c \in \mathrm{Ob}(\mathbb{C})$, $\alpha_{F(c)} = \alpha'_{F(c)}$ Let d be an object of $\mathbb{D}$ and (c_d, f_d) be a couple chosen as before. By naturality of α, we have

$$\alpha_d \circ M(f_d) = N(f_d) \circ \alpha_{F(c)},$$

and by naturality of α', we have

$$\alpha'_d \circ M(f_d) = N(f_d) \circ \alpha'_{F(c)}.$$

By hypothesis, the left-hand sides are equals. And since f_d is an isomorphism, we have $\alpha_d = \alpha'_d$ for any d, and F^* is faithful. □

Proposition 1.3.3. *The precomposition* $\mathbb{V}$*-functor* $F^* \colon \mathbb{V}^{\mathbb{D}} \to \mathbb{V}^{\mathbb{C}}$ *has both left and right adjoint* $\mathbb{V}$*-functors: they are given by sending* M *to the left Kan extension* $\mathrm{Lan}_F M$*, respectively the right Kan extension* $\mathrm{Ran}_F M$*. In particular,* F^* *preserves all limits and colimits.*

Proof. Recall left Kan extensions from 1.2.13; right Kan extensions being the dual notion, we only consider here the left adjoint.

For every $M \in \mathbb{V}^{\mathbb{C}}$, define a $\mathbb{V}$-functor $\tilde{F}(M) : \mathbb{D} \to \mathbb{V}$ by the following coend formula (see 1.2.10):

$$\tilde{F}(M) := \int^{x\in\mathbb{C}} M(x) \otimes_{\Bbbk} \mathbb{D}(Fx, -) \tag{1.5}$$

By (1.3), this is the left Kan extension of M along F, that is: $\tilde{F}(M) = \mathrm{Lan}_F M$. By varying M, this induces a $\mathbb{V}$-functor $\mathbb{V}^{\mathbb{C}} \to \mathbb{V}^{\mathbb{D}}$. By (1.2) we have a natural isomorphism

$$\mathbb{V}^{\mathbb{D}}(\mathrm{Lan}_F M, N) \simeq \mathbb{V}^{\mathbb{C}}(M, NF)$$

which tells us that $\tilde{F} = \mathrm{Lan}_F(-)$ is left adjoint to F^*. We can also check that the unit η and counit ε of the adjunction $\mathrm{Lan}_F \dashv F^*$ are given by:

1. For $K \in \mathbb{V}^{\mathbb{C}}$, we use (1.5) and define the trivial enough map

$$\eta_{K,c} : K(c) \longrightarrow ((F^* \circ \mathrm{Lan}_F)(K))(c) = \int^{u \in \mathbb{C}} \mathbb{D}(Fu, Fc) \otimes K(u)$$

$$x \mapsto [\mathrm{Id}_{Fc}, x]$$

2. For $J \in \mathbb{V}^{\mathbb{D}}$, we define an evaluation map

$$\varepsilon_{J,d} : ((\mathrm{Lan}_F \circ F^*)(J))(d) = \int^{u \in \mathbb{C}} \mathbb{D}(Fu, d) \otimes J(Fu) \longrightarrow J(d)$$

$$[f : Fu \to d, x] \mapsto J(f)(x)$$

Indeed, naturality is straightforward, and we check that both triangle identities hold:

$$\begin{tikzcd}
\mathrm{Lan}_F(K)(d) \arrow[dr, "\mathrm{Lan}_F(\eta_K)"] \arrow[dd, equal] & \\
& \mathrm{Lan}_F\big(F^* \circ \mathrm{Lan}_F\big)(K)(d) \arrow[dl, "\varepsilon_{\mathrm{Lan}_F(K)}"] \\
\mathrm{Lan}_F(K)(d) &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[x \in K(c), f : Fc \to d]} \arrow[dr, mapsto] \arrow[dd, equal] & \\
& {[\mathrm{Id}_{Fc}, x, f]} \arrow[dl, mapsto] \\
{[\mathrm{Lan}_F(K)(\mathrm{Id}_{Fc})(x) = x, f]} &
\end{tikzcd}$$

$$\begin{tikzcd}
F^*(J)(c) \arrow[dr, "\eta_{F^*(J)}"] \arrow[dd, equal] & \\
& (F^* \circ \mathrm{Lan}_F)(F^*(J))(c) \arrow[dl, "F^*(\varepsilon_J)"] \\
F^*(J)(c) \ , &
\end{tikzcd}
\qquad
\begin{tikzcd}
x \in J(Fc) \arrow[dr, mapsto] & \\
& {[\mathrm{Id}_{Fc}, x]} \arrow[dl, mapsto] \\
x &
\end{tikzcd}$$

and we've established that those are the unit and counit □

Proposition 1.3.4. *There is a canonical isomorphism $y_{\mathbb{D}} \circ F \simeq \mathrm{Lan}_F \circ y_{\mathbb{C}}$, given concretely by*

$$\mathbb{D}(Fc, d) \longrightarrow \int^{x \in \mathbb{C}} \mathbb{D}(Fx, d) \otimes_{\mathbb{k}} \mathbb{C}(c, x)$$

$$f \mapsto [f, \mathrm{Id}_c]$$

at each $c \in \mathbb{C}$.

Proof. Let M be any functor in $\mathbb{V}^{\mathbb{C}}$. By Yoneda's lemma applied to (1.5), we have

$$\mathrm{Lan}_F(M) = \int^x \mathbb{V}^{\mathbb{C}}(y_{\mathbb{C}}(x), M) \otimes_{\Bbbk} (y_{\mathbb{D}} \circ F)(x)$$

and thus, by abstracting away M and by (1.3) we obtain:

$$\mathrm{Lan}_F = \int^x \mathbb{V}^{\mathbb{C}}(y_{\mathbb{C}}(x), -) \otimes_{\Bbbk} (y_{\mathbb{D}} \circ F)(x) = \mathrm{Lan}_{y_{\mathbb{C}}}(y_{\mathbb{D}} \circ F)$$

Yoneda's embedding $y_{\mathbb{C}}$ is a fully faithful functor, hence according to (the dual of) [19, Corollary X.3], the structural natural transformation of the left Kan extension along $y_{\mathbb{C}}$

$$y_{\mathbb{D}} \circ F \Rightarrow \mathrm{Lan}_F \circ y_{\mathbb{C}}$$

is an isomorphism. One can check that the structural map is indeed given as in the statement. (Alternatively, one can check directly that the latter is invertible.) □

Proposition 1.3.5. *Let $F : \mathbb{C} \to \mathbb{D}$ be as above, and moreover suppose that it is a strong tensor functor. Then Lan_F is a strong tensor functor with respect to the Day convolutions. The structure maps are as in* (1.6) *and* (1.7) *below.*

Proof. Let us describe the structure maps

$$\mathrm{strg}_1^{\mathrm{Lan}_F} : 1_{\mathbb{V}^{\mathbb{D}}} \xrightarrow{\simeq} \mathrm{Lan}_F(1_{\mathbb{V}^{\mathbb{C}}})$$

and

$$\mathrm{strg}_{M,N}^{\mathrm{Lan}_F} : \mathrm{Lan}_F(M) \boxtimes \mathrm{Lan}_F(N) \xrightarrow{\simeq} \mathrm{Lan}_F(M \boxtimes N)$$

for any $M, N \in \mathbb{V}^{\mathbb{C}}$. We will give their component at $d \in \mathbb{D}$.

For the first one, we take the composite isomorphism

$$\begin{array}{ccc} \mathbb{D}(1,d) \simeq \mathbb{D}(F1,d) & \xrightarrow{\simeq} & \displaystyle\int^{x\in\mathbb{C}} \mathbb{D}(Fx,d) \otimes_{\Bbbk} \mathbb{C}(1,x) \\ f & \mapsto & [f \circ (\mathrm{strg}_1^F)^{-1}, \mathrm{Id}_1] \end{array} \tag{1.6}$$

given by the isomorphism $\mathrm{strg}_1^F : 1 \simeq F(1)$ of the tensor functor F and the isomorphism of proposition 1.3.4, that is, the fact that Lan_F preserves representables.

As for the second one, let us compute its domain and codomain explicitly. According to the coend formula 1.2.15 for Day convolution and the coend formula (1.5) for the left Kan extension Lan_F, combined with Fubini 1.2.11, we obtain for each $d \in \mathrm{Ob}(\mathbb{D})$:

$$\big(\mathrm{Lan}_F(M) \boxtimes \mathrm{Lan}_F(N)\big)(d) =$$

$$\int^{s,t\in\mathbb{D}\ u,v\in\mathbb{C}} \mathbb{D}(s \otimes t, d) \otimes_{\Bbbk} \mathbb{D}(Fu, s) \otimes_{\Bbbk} \mathbb{D}(Fv, t) \otimes_{\Bbbk} M(u) \otimes_{\Bbbk} N(v)$$

and

$$\mathrm{Lan}_F(M \boxtimes N)(d) = \int^{o,p,q\in\mathbb{C}} \mathbb{D}(Fq,d) \otimes_{\Bbbk} \mathbb{C}(o\otimes p, q) \otimes_{\Bbbk} M(o) \otimes_{\Bbbk} N(p)$$

In view of this, we consider the rather obvious map

$$\mu := \mathrm{strg}^{\mathrm{Lan}_F}_{M,N,d} : [f,g,h,x,y] \mapsto [f\circ(g\otimes h)\circ(\mathrm{strg}^F_{u,v})^{-1}, \mathrm{Id}_{u\otimes v}, x, y] \tag{1.7}$$

with image sitting at $o := u$, $p := v$ and $q := u \otimes v$.

We check that μ is well-defined. Let $[f,g,h,x,y]$ and $[f',g',h',x',y']$ be equal via a basic relation: there are $\ell \in \mathbb{D}(r,r')$, $k \in \mathbb{D}(s,s')$, $p \in \mathbb{C}(u,u')$ and $q \in \mathbb{C}(v,v')$ such that

$$\begin{tikzcd} r\otimes s \arrow[d, "\ell\otimes k"'] \arrow[dr, "f"] & \\ r'\otimes s' \arrow[r, "f'"'] & d \end{tikzcd} \quad \begin{tikzcd} Fu \arrow[r, "g"] \arrow[d, "Fp"'] & r \arrow[d, "\ell"] \\ Fu' \arrow[r, "g'"'] & r' \end{tikzcd} \quad \begin{tikzcd} Fv \arrow[r, "h"] \arrow[d, "Fq"'] & s \arrow[d, "k"] \\ Fv' \arrow[r, "h'"'] & s' \end{tikzcd} \quad \begin{array}{c} M(p)(x) = x' \\ \\ N(q)(y) = y' \ . \end{array}$$

Then, we obtain from this a basic relation between the two images:

$$\begin{tikzcd} F(u\otimes v) \arrow[r, "\cong"] \arrow[d, "F(p\otimes q)"] & Fu\otimes Fv \arrow[r, "g\otimes h"] \arrow[d, "Fp\otimes Fq"] & r\otimes s \arrow[r, "f"] \arrow[d, "\ell\otimes k"] & d \arrow[d, equal] \\ F(u'\otimes v') \arrow[r, "\cong"'] & Fu'\otimes Fv' \arrow[r, "g'\otimes h'"'] & r'\otimes s' \arrow[r, "f'"'] & d \end{tikzcd},$$

hence $\mu\big([f,g,h,x,y]\big) = \mu\big([f',g',h',x',y']\big)$, so that μ is well-defined.

Now we claim that the inverse map μ^{-1} is as follows:

$$\begin{aligned} &\Big[F(u\otimes v) \xrightarrow{a} d, u\otimes v \xrightarrow{b} w, x, y\Big] \\ \mapsto\quad &\Big[Fu\otimes Fv \simeq F(u\otimes v) \xrightarrow{Fb} Fw \xrightarrow{a} d, \mathrm{Id}_{Fu}, \mathrm{Id}_{Fv}, x, y\Big] \ . \end{aligned} \tag{1.8}$$

To see that it is well-defined, assume that we have $[a,b,x,y] = [a',b',x',y']$ via a basic relation in its source coend:

$$\exists \quad \begin{tikzcd} u \arrow[d, "k"'] \\ u' \end{tikzcd} \quad \begin{tikzcd} v \arrow[d, "\ell"'] \\ v' \end{tikzcd} \quad \begin{tikzcd} w \arrow[d, "m"'] \\ w' \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} Fw \arrow[r, "a"] \arrow[d, "Fm"] & d \arrow[d, equal] \\ Fw \arrow[r, "a'"'] & d \end{tikzcd} \quad \begin{tikzcd} u\otimes v \arrow[r, "b"] \arrow[d, "k\otimes\ell"] & w \arrow[d, "m"] \\ u'\otimes v' \arrow[r, "b'"'] & w \end{tikzcd} \quad \begin{tikzcd} x \arrow[d, mapsto, "M(k)"] \\ x' \end{tikzcd} \quad \begin{tikzcd} y \arrow[d, mapsto, "N(\ell)"] \\ y' \end{tikzcd}$$

Then their images under μ^{-1} are equivalent in its target coend by the basic relation witnessed by $Fk, F\ell, k, \ell$ and the following diagrams:

$$\begin{tikzcd} Fu\otimes Fv \arrow[r, "\simeq"] \arrow[d, "Fk\otimes F\ell"'] & F(u\otimes v) \arrow[r, "Fb"] \arrow[d, "F(k\otimes\ell)"'] & Fw \arrow[r, "a"] \arrow[d, "Fm"'] & d \arrow[d, equal] \\ Fu'\otimes Fv' \arrow[r, "\simeq"'] & F(u'\otimes v') \arrow[r, "Fb'"'] & Fw' \arrow[r, "a'"'] & d \end{tikzcd} \quad \begin{tikzcd} Fu \arrow[r, equal] \arrow[d, "Fk"] & Fu \arrow[d, "Fk"] \\ Fu' \arrow[r, equal] & Fu' \end{tikzcd} \quad \begin{tikzcd} Fv \arrow[r, equal] \arrow[d, "F\ell"] & Fv \arrow[d, "F\ell"] \\ Fv' \arrow[r, equal] & Fv' \end{tikzcd} \quad \begin{tikzcd} x \arrow[d, mapsto, "M(k)"] \\ x' \end{tikzcd} \quad \begin{tikzcd} y \arrow[d, mapsto, "N(\ell)"] \\ y' \end{tikzcd}$$

Thus μ^{-1} is a well-defined map.

Let us check that $\mu^{-1}\mu$ is the identity. For every $[f, g, x, h, y]$ in its source, the maps $g, h, \mathrm{Id}_u, \mathrm{Id}_v$ and the commutative diagrams

$$\begin{tikzcd}
Fu \otimes Fv \arrow[r, "\simeq"] \arrow[d, "g\otimes h"'] \arrow[rr, bend right, "\mathrm{Id}"'] & F(u\otimes v) \arrow[r, "\simeq"] & Fu \otimes Fv \arrow[r, "g\otimes h"] & r \otimes s \arrow[r, "f"] & d \arrow[d, equal] \\
r \otimes s \arrow[rrrr, "f"] & & & & d
\end{tikzcd}
\qquad
\begin{tikzcd}
Fu \arrow[r, "\mathrm{Id}"] \arrow[d, equal] & Fu \arrow[d, "g"] \\
Fu \arrow[r, "g"] & r
\end{tikzcd}
\qquad
\begin{tikzcd}
Fv \arrow[r, "\mathrm{Id}"] \arrow[d, equal] & Fv \arrow[d, "h"] \\
Fv \arrow[r, "h"] & s
\end{tikzcd}$$

show that indeed $\mu^{-1}\mu([f, g, x, h, y]) = [f, g, x, h, y]$, as wished.

For the other composite $\mu\mu^{-1}$, let $[a, b, x, y]$ be any element of the source coend of μ^{-1}, as above. Then the maps $b, \mathrm{Id}_u, \mathrm{Id}_v$ and the commutative squares

$$\begin{tikzcd}
F(u\otimes v) \arrow[d, "Fb"'] \arrow[rrr, bend right, "\mathrm{Id}"'] & Fu \otimes Fv \arrow[l, "\simeq"'] \arrow[r, "\mathrm{Id}_{Fu}\otimes \mathrm{Id}_{Fv}"] & Fu \otimes Fv \arrow[r, "\simeq"] & F(u\otimes v) \arrow[r, "Fb"] & w \arrow[r, "a"] & d \arrow[d, equal] \\
F(u\otimes v) \arrow[rrrrr, "a"'] & & & & & d
\end{tikzcd}$$

and

$$\begin{tikzcd}
u \otimes v \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}_u\otimes \mathrm{Id}_v"'] & u \otimes v \arrow[d, "b"] \\
u \otimes v \arrow[r, "b"] & w
\end{tikzcd}$$

show that $\mu\mu^{-1}([a, b, x, y]) = [a, b, x, y]$ in the other coend, as required.

Finally, the verifications that $(\mathrm{Lan}_F, \mathrm{strg}^{\mathrm{Lan}_F}_{M,N}, \mathrm{strg}^{\mathrm{Lan}_F}_1)$ satisfies the axioms of a tensor functor are similarly straightforward and left to the reader. □

Corollary 1.3.6. *The precomposition functor F^* is a lax tensor functor, with structure maps* $\mathrm{lax}^{F^*}_{M,N} : F^*(M) \boxtimes_{\mathbb{C}} F^*(N) \to F^*(M \boxtimes_{\mathbb{D}} N)$ *given by*

$$[f, x, y] \mapsto [F(f) \circ \mathrm{strg}^F_{u,v}, x, y]$$

and $\mathrm{lax}^{F^*}_1 : 1_{\mathbb{V}^{\mathbb{C}}} = \mathbb{C}(1, -) \to \mathbb{D}(1, F-) = F^*(1_{\mathbb{V}^{\mathbb{D}}})$ *given by*

$$f \mapsto F(f) \circ \mathrm{strg}^F_1 .$$

Proof. Indeed, it is the right adjoint to a strong monoidal functor, so it is lax according to 1.1.5. Considering the construction by adjunction of the lax structure, which uses the strong monoidal structure of Lan_F as well as the unit η and the counit ε of the adjunction $\mathrm{Lan}_F \dashv F^*$ (see the proof of 1.3.3), we get:

$$\mathrm{lax}^{F^*}_{M,N} := F^*(\varepsilon \boxtimes \varepsilon) \circ F^*(\mu^{-1}) \circ \eta,$$

where $\mu := \mathrm{strg}^{\mathrm{Lan}_F}_{M,N}$. We recall the various components:

1. For $K \in \mathbb{V}^{\mathbb{C}}$,

$$\eta_{K,c} : K(c) \longrightarrow ((F^* \circ \mathrm{Lan}_F)(K))(c) = \int^{u\in\mathbb{C}} \mathbb{D}(Fu, Fc) \otimes K(u)$$
$$x \mapsto [\mathrm{Id}_{Fc}, x]$$

2. For $J \in \mathbb{V}^{\mathbb{D}}$,

$$\varepsilon_{J,d} : ((\mathrm{Lan}_F \circ F^*)(J))(d) = \int^{u\in\mathbb{C}} \mathbb{D}(Fu, d) \otimes J(Fu) \longrightarrow J(d)$$
$$[f : Fu \to d, x] \mapsto J(f)(x)$$

3. For $M, N \in \mathbb{V}^{\mathbb{D}}$,

$$\mu : (\mathrm{Lan}_F(M) \boxtimes \mathrm{Lan}_F(N))(d) \to \mathrm{Lan}_F(M \boxtimes N)(d)$$
$$[f, g, x, h, y] \mapsto [f \circ (g \otimes h) \circ (\mathrm{strg}^F_{u,v})^{-1}, \mathrm{Id}_{u,v}, x, y]$$

Now, one can put the pieces together, and check that $\mathrm{lax}^{F^*}_{M,N,d}$ is given by

$$\int^{u,v\in\mathbb{C}} \mathbb{C}(u\otimes v, c) \otimes_{\Bbbk} M(Fu) \otimes_{\Bbbk} N(Fv) \longrightarrow \int^{r,s\in\mathbb{D}} \mathbb{D}(r\otimes s, Fc) \otimes_{\Bbbk} M(r) \otimes_{\Bbbk} N(s)$$
$$[f, x, y] \mapsto [F(f) \circ \mathrm{strg}^F_{u,v}, x, y]$$

as announced. The formula for $\mathrm{lax}^{F^*}_1$ is given by 1.1.5 as well. □

Corollary 1.3.7. *$F^*(1_{\mathbb{V}^{\mathbb{D}}})$ is a commutative monoid in $\mathbb{V}^{\mathbb{C}}$.*

Proof. Indeed, $1_{\mathbb{V}^{\mathbb{D}}}$ is a commutative monoid in $\mathbb{V}^{\mathbb{D}}$ by 1.1.13, we just saw that F^* is lax symmetric monoidal, and we apply 1.1.12. □

Notation 1.3.8. From now on, when it is seen as a monoid, we will denote

$$A := F^*(1_{\mathbb{V}^{\mathbb{D}}}).$$

Lemma 1.3.9. *If F is full, then each component $\eta_M : M \to F^* \circ \mathrm{Lan}_F M$ of the unit of the adjunction $\mathrm{Lan}_F \dashv F^*$ is an epimorphism in $\mathbb{V}^{\mathbb{C}}$.*

Proof. We recall from above that η_M is given at $c \in \mathrm{Ob}(\mathbb{C})$ by the following map:

$$x \in M(c) \mapsto [\mathrm{Id}_{Fc}, x]$$

Now, let $[f, x]$ be any element of the target, with $f \in \mathbb{D}(Fu, Fc)$, and $x \in M(u)$. Since F is full, there exists $g \in \mathbb{C}(u, c)$ such that $Fg = f$. Hence, if one sets $y := M(g)(x)$, one has $[f, x] = [\mathrm{Id}_{Fc}, y]$ via a basic relation:

$$\begin{tikzcd} Fu \arrow[d, "Fg"'] \arrow[dr, "f"] & \\ Fc \arrow[r, equal] & Fc \end{tikzcd} \quad ; \quad y = M(g)(x)$$

This shows that every $\eta_{M,c}$ is surjective, hence η_M is an epimorphism. □

Corollary 1.3.10. *Let $F : \mathbb{C} \to \mathbb{D}$ be a strong tensor functor, and consider the associated monoidal adjunction, as above. If F is full, the unit map*

$$u_A : 1_{\mathbb{V}^{\mathbb{C}}} \to F^*(1_{\mathbb{V}^{\mathbb{D}}}) = A$$

of the monoid A is an epimorphism in $\mathbb{V}^{\mathbb{C}}$.

Proof. It follows from the lemma 1.3.9 above, since the unit map is an instance of the unit of the adjunction: by definition, we have an isomorphism $u_A \simeq \eta_1$ thanks to the fact that Lan_F is strong monoidal. □

Corollary 1.3.11. *On any functor $M \in \mathbb{V}^{\mathbb{C}}$, there is at most one possible structure of A-module.*

Proof. Let $\lambda_1, \lambda_2 : A \boxtimes M \to M$ be two such structures. By the unit axiom of modules, we have that

$$\lambda_i \circ (u_A \boxtimes \mathrm{Id}_M) : 1 \boxtimes M \to M$$

is equal to the canonical isomorphism $\mathfrak{lun}_M$, for $i \in \{1, 2\}$. But since u_A is an epimorphism, by 1.3.10, and since $- \boxtimes \mathrm{Id}_M$ is right exact (indeed, it has a right adjoint by part 7 of proposition 1.2.16), then $u_A \boxtimes \mathrm{Id}_M$ is also an epimorphism, hence the equality implies $\lambda_1 = \lambda_2$. □

Corollary 1.3.12. *If (M, λ) and (M', λ') are A-modules in $\mathbb{V}^{\mathbb{C}}$, any morphism $\psi : M \to M'$ preserves the actions. Thus, the forgetful functor $(M, \lambda) \mapsto M$ is not only faithful but also full.*

Proof. Let's consider the following diagram.

$$\begin{tikzcd}
1 \boxtimes M \arrow[rr, "\mathfrak{lun}_M"'] \arrow[dr, "u \boxtimes \mathrm{Id}_M"'] \arrow[ddd, "\mathrm{Id}_1 \boxtimes \psi"'] & & M \arrow[ddd, "\psi"] \\
& A \boxtimes M \arrow[ur, "\lambda"'] \arrow[d, "\mathrm{Id}_A \boxtimes \psi"'] & \\
& A \boxtimes M' \arrow[dr, "\lambda'"] & \\
1 \boxtimes M' \arrow[ur, "u \boxtimes \mathrm{Id}_{M'}"] \arrow[rr, "\mathfrak{lun}_{M'}"'] & & M'
\end{tikzcd}$$

The unit axiom ensures the commutativity of both triangles. The left trapezoid commutes by functoriality of the tensor product. The outward square commutes by naturality of the left unitor. Only the right trapezoid remains.

By 1.3.10, $u_A \boxtimes \mathrm{Id}_M$ is an epimorphism, hence the following calculation yields the commutativity:

$$\begin{aligned}
\psi \circ \lambda \circ (u_A \boxtimes \mathrm{Id}_M) &= \psi \circ \mathfrak{lun}_M && (1.9)\\
&= \mathfrak{lun}_{M'} \circ (\mathrm{Id}_1 \boxtimes \psi) && (1.10)\\
&= \lambda' \circ (u_A \boxtimes \mathrm{Id}_{M'}) \circ (\mathrm{Id}_1 \boxtimes \psi) && (1.11)\\
&= \lambda' \circ (\mathrm{Id}_A \boxtimes \psi) \circ (u_A \boxtimes \mathrm{Id}_M)
\end{aligned}$$

The equality (1.9) holds by the unit axiom for the action λ ; (1.10) holds by naturality of the left unitor ; (1.11) holds by the unit axiom for the action λ'. $\square$

We can now state the main theorem of this chapter:

Theorem 1.3.13. *Let $F\colon \mathbb{C} \to \mathbb{D}$ be a strong tensor $\mathbb{V}$-functor. By standard constructions that we have recalled above, we obtain a (solid-arrow) diagram as follows:*

$$\begin{tikzcd} & \mathbb{V}^{\mathbb{C}} \arrow[dl, shift right, "\mathrm{Lan}_F"'] \arrow[dr, shift left, "\mathrm{Free}"'] & \\ \mathbb{V}^{\mathbb{D}} \arrow[ur, shift right, "F^*"'] \arrow[rr, dashed, "E"'] & & A-\mathrm{Mod}_{\mathbb{V}^{\mathbb{C}}} \arrow[ul, shift left, "U"'] \end{tikzcd}$$

Then there exists a unique functor E such that $U \circ E = F^*$. If moreover F is full and essentially surjective and $\mathbb{C}$ is rigid, then E is an equivalence of tensor categories identifying the adjunction (Lan_F, F^*) between the left Kan extension (see 1.3.3) and precomposition with the adjunction (Free, U) between the free module functor (see 1.1.19) and the forgetful functor.

The proof is given in the next section along with the construction of E (see 1.12).

We note that all the constructions in the theorem are rather standard, including the comparison functor E. There are also many variants of such results, saying that E is an equivalence in certain nice situations.

Remark 1.3.14. We stress that our theorem also says that E is a *tensor* equivalence. If one only wants E to be a plain equivalence of categories, then it is not necessary to assume the fullness of F; the result can be obtained from a general monadicity theorem for abelian categories (see [8]) together with an explicit computation showing that the natural transformation $A \otimes (-) \Rightarrow F^*\mathrm{Lan}_F$ obtained from the lax structure of F^* (which is automatically a morphisms of monads on $\mathbb{V}^{\mathbb{C}}$ by Lemma 2.8 of [2]) is in fact invertible. This only uses the essential surjectivity of F and the rigidity of $\mathbb{C}$ and $\mathbb{D}$.

1.4 The comparison functor E

Let us now prove theorem 1.3.13. We will note from which point on we will need the various hypotheses.

Notation 1.4.1. When there will be no risk of confusion, we will denote by 1 the tensor unit for Day convolution, which is $\mathbb{D}(1_{\mathbb{D}}, -)$ or $\mathbb{C}(1_{\mathbb{C}}, -)$, depending on the situation.

Both adjunctions in theorem 1.3.13 have been studied already, so the first step of the proof of the theorem is the construction of E, which is as follows.

Recall that for every $M \in \mathbb{V}^{\mathbb{D}}$, there is a (unique) structure of $\mathbb{D}(1_{\mathbb{D}}, -)$-module on M (see 1.1.13), given by the left unitor $\mathfrak{lun}_M : 1 \boxtimes_{\mathbb{D}} M \xrightarrow{\sim} M$. By proposition 1.3.5, F^* is lax monoidal, hence by 1.1.12, it sends 1 to a monoid $A = F^*(1)$, and also the left 1-module M to a left $F^*(1)$-module $F^*(M)$ in $\mathbb{V}^{\mathbb{C}}$ with action

$$\lambda_M := F^*(\mathfrak{lun}_M) \circ \mathrm{lax}^{F^*}_{1,M} : F^*(1) \boxtimes_{\mathbb{C}} F^*(M) \longrightarrow F^*(M)\,.$$

For any $\alpha : M \Rightarrow M'$, one can check that $F^*(\alpha)$ is an $F^*(1)$-module morphism, that is, that the outer square commutes:

$$\begin{tikzcd}
F^*(1) \boxtimes_{\mathbb{C}} F^*(M) \arrow[rr, "\lambda_M"'] \arrow[dr, "\mathrm{lax}^{F^*}"'] \arrow[ddd, "\mathrm{Id}\boxtimes_{\mathbb{C}} F^*(\alpha)"'] & & F^*(M) \arrow[ddd, "F^*(\alpha)"] \\
& F^*(1 \boxtimes M) \arrow[ur, "F^*(\mathfrak{lun}_M)"'] \arrow[d, "F^*(1\boxtimes\alpha)"'] & \\
& F^*(1 \boxtimes_{\mathbb{D}} M') \arrow[dr, "F^*(\mathfrak{lun}_{M'})"] & \\
F^*(1) \boxtimes_{\mathbb{C}} F^*(M') \arrow[rr, "\lambda_{M'}"'] \arrow[ur, "\mathrm{lax}^{F^*}"'] & & F^*(M')
\end{tikzcd}$$

Indeed, the triangles commute by definition, the left trapezoid commutes by naturality of lax^{F^*}, and the right trapezoid commutes by naturality of $\mathfrak{lun}$. Hence $F^*(\alpha)$ automatically commutes with actions.

We thus obtain a functor

$$\begin{aligned} E : \mathbb{V}^{\mathbb{D}} &\to A - \mathrm{Mod}_{\mathbb{V}^{\mathbb{C}}} \\ M &\mapsto (F^*(M), \lambda_M) \\ \alpha &\mapsto F^*(\alpha) \end{aligned} \tag{1.12}$$

By construction, we obtain the first half of the comparison of the adjunctions:

$$U \circ E = F^*\,. \tag{1.13}$$

Let's study the monoidality of the functor E. In general, we have:

Lemma 1.4.2. *The lax structure of $F^* : \mathbb{V}^{\mathbb{D}} \to \mathbb{V}^{\mathbb{C}}$ lifts along the forgetful functor U to define a lax structure on $E : \mathbb{V}^{\mathbb{D}} \to A - \mathrm{Mod}_{\mathbb{V}^{\mathbb{C}}}$.*

Proof. We know from corollary 1.3.5 that the functor F^* is lax monoidal, and from the defining coequalizer (1.1) of proposition 1.1.16 that the monoidal structure $\otimes_A$ of the category of A-modules is a quotient of that of $\mathbb{V}^{\mathbb{C}}$. We claim that the lax structure of F^* factors through that quotient; that is, for $M, M' \in \mathbb{V}^{\mathbb{D}}$, there is a morphism making the following diagram commutative:

$$\begin{tikzcd} F^*(M) \otimes_A F^*(M') \arrow[rr, dashed, "\mathrm{lax}^E_{M,M'}"] & & F^*(M \boxtimes M') \\ F^*(M) \boxtimes_{\mathbb{C}} F^*(M') \arrow[u, two heads] \arrow[urr, "\mathrm{lax}^{F^*}_{M,M'}"'] & & \\ F^*(M) \boxtimes_{\mathbb{C}} F^*(1) \boxtimes_{\mathbb{C}} F^*(M') \arrow[u, shift left] \arrow[u, shift right] & & \end{tikzcd} \tag{1.14}$$

There ban be but one such dashed arrow, and it will give the lax structure of E, hence we denote it $\mathrm{lax}^E_{M,M'}$.

Note that it is really enough to show that $\mathrm{lax}^{F^*}_{M,M'}$ factors through the coequalizer $F^*(M) \otimes_A F^*(M')$, because the factored map is automaticaly $A-$ equivariant (as we know from 1.3.12), and the coherence axioms hold because they do for lax^{F^*} and because the coequalizer's map is an epimorphism.

Note that the lax structure map lax^E_1 at the unit is also the unique A-equivariant factorization through the unit u_A of A, that is, the identity:

$$\begin{tikzcd} A = F^*(1) \arrow[r, dashed, "\mathrm{lax}^E_1"] & F^*(1) = U \circ E(1) \\ 1_{\mathbb{V}^{\mathbb{C}}} \arrow[u, "u_A"] \arrow[ur, "\mathrm{lax}^{F^*}_1"'] & \end{tikzcd} \tag{1.15}$$

Indeed, by 1.1.12 and 1.1.13, $u_A := F^*(\mathrm{Id}_{1_{\mathbb{V}^{\mathbb{D}}}}) \circ \mathrm{lax}^{F^*}_1$.

Let us show that $\mathrm{lax}^{F^*}_{M,M'}$ factors as claimed. For this, we need to make the coequalizer (1.1) explicit in our situation. First, recall from point 3 of proposition 1.2.16 the associativity of $\boxtimes_{\mathbb{C}}$ is explicitly given, for N, N', N'' in $\mathbb{V}^{\mathbb{C}}$ and c an object in $\mathbb{C}$, by

$$\mathfrak{a}_{\mathbb{C},c} : [u \otimes t \xrightarrow{f} c, r \otimes s \xrightarrow{g} u, x \in N(r), y \in N'(s), z \in N''(t)] \mapsto [f \circ (g \otimes \mathrm{Id}_t), \mathrm{Id}_{s \otimes t}, x, y, z]$$

The mindful reader will have noted that the coequaliazer diagram (1.14) uses a notation that is not defined yet, since the Day convolution product is not strictly associative. Note that both $- \boxtimes (- \boxtimes -)$ and $(- \boxtimes -) \boxtimes -$ are equivalent to a more symmetrical form, written as follows for our three functors N, N' and N'', at c:

$$(N \boxtimes N' \boxtimes N'')(c) := \int^{a,a',a'' \in \mathbb{C}} \mathbb{C}(a \otimes a' \otimes a'', c) \otimes_{\mathbb{k}} N(a) \otimes_{\mathbb{k}} N'(a') \otimes_{\mathbb{k}} N''(a'')$$

The isomorphisms are given by the maps below:

$$(N \boxtimes N' \boxtimes N'')(c) \to ((N \boxtimes N') \boxtimes N'')(c)$$

$$[f, x, x', x''] \mapsto [f, \mathrm{Id}_u, x, x', x'']$$
$$(N \boxtimes N' \boxtimes N'')(c) \to (N \boxtimes (N' \boxtimes N''))(c)$$
$$[f, x, x', x''] \mapsto [f, \mathrm{Id}_u, x, x', x'']$$

The well-defineness and the fact that these are isomorphisms are straightforward if the reader refers to the coend formula in the point 3 of 1.2.16. We can now explicitly write the coequalizer (1.14) we are interested in at an object c in $\mathbb{C}$:

$$\begin{tikzcd}
 & (F^*(M) \boxtimes (F^*(1) \boxtimes F^*(M')))(c) \arrow[dr, "\mathrm{Id}\boxtimes\lambda_{F^*(M')}"] & \\
(F^*(M) \boxtimes F^*(1) \boxtimes F^*(M'))(c) \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & & (F^*(M) \boxtimes F^*(M'))(c) \\
 & ((F^*(M) \boxtimes F^*(1)) \boxtimes F^*(M'))(c) \arrow[ur, "(\lambda_{F^*(M)}\circ\mathfrak{s})\boxtimes\mathrm{Id}"'] &
\end{tikzcd}$$

Using the coend formula, we get this:

$$\begin{tikzcd}
 & \int^{a,v}\mathbb{C}(a \otimes v, c) \otimes_{\Bbbk} \int^{b,a'} \mathbb{C}(b \otimes a', v) \otimes_{\Bbbk} M(Fa) \otimes_{\Bbbk} \mathbb{D}(1, Fb) \otimes_{\Bbbk} M'(Fa') \arrow[dr, "\mathrm{Id}\boxtimes\lambda_{F^*(M')}"] & \\
\int^{a,b,a'} \mathbb{C}(a \otimes b \otimes a', c) \otimes_{\Bbbk} M(Fa) \otimes_{\Bbbk} \mathbb{D}(1, Fb) \otimes_{\Bbbk} M'(Fa') \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & & (\int^{w,w'\in\mathbb{C}} \mathbb{C}(w \otimes w', c) \otimes_{\Bbbk} M(Fw) \otimes_{\Bbbk} M'(Fw') \\
 & \int^{u,a'} \mathbb{C}(u \otimes a', c) \otimes_{\Bbbk} \int^{a,b} \mathbb{C}(a \otimes b, u) \otimes_{\Bbbk} M(Fa) \otimes_{\Bbbk} \mathbb{D}(1, Fb) \otimes_{\Bbbk} M'(Fa') \arrow[ur, "(\lambda_{F^*(M)}\circ\mathfrak{s})\boxtimes\mathrm{Id}"'] &
\end{tikzcd}$$

Then, once we recall the definition of $\lambda_{F^*(M)} = F^*(\mathfrak{lun}_M) \circ \mathrm{lax}^{F^*}_{1,M}$ from 1.1.12 and 1.1.13 and we refer ourselves to both the point 4 of 1.2.16 and the corollary 1.3.6, the maps are fairly obvious. We set $k := \mathrm{strg}^F_{a,b} \circ (\mathrm{Id}_{Fa} \otimes g) \circ \mathrm{run}^{-1}_{Fa}$ and $\ell := \mathrm{strg}^F_{b,a'} \circ (g \otimes \mathrm{Id}_{Fa'}) \circ \mathrm{lun}^{-1}_{Fa'}$, and then for every

$$[f, x, g, x'] \in (F^*M \boxtimes F^*M' \boxtimes F^*M'')(c)$$

we obtain

$$[f, x, g, x'] \mapsto \zeta := [f, x, M'(\ell))(x')]$$

upstairs and

$$[f, x, g, x'] \mapsto \xi := [f, M(k)(x), x']$$

downstairs. By the definition of coequalizers, the images of ζ and ξ are the same in the quotient object $\big(F^*(M) \otimes_A F^*(M')\big)(c)$, which is precisely the quotient of $\big(F^*(M) \boxtimes F^*(M')\big)(c)$ by the $\Bbbk$-linear hull of the relations $\zeta = \xi$ for all $[f, x, g, x']$.

Thus, to prove that $\mathrm{lax}^{F^*}_{M,M'}$ factors through the coequalizer, it suffices to verify that it identifies the class of ξ to that of ζ. Indeed,

$$\mathrm{lax}^{F^*}_{M,M'}(\zeta) = [Ff \circ \big(\mathrm{strg}^F\big)^{-1}, M(k)(x), x']$$

and

$$\mathrm{lax}^{F^*}_{M,M'}(\xi) = [Ff \circ \big(\mathrm{strg}^F\big)^{-1}, x, M'(\ell)(x')],$$

and if we consider the following diagram,

$$\begin{tikzcd}
F(a \otimes b) \otimes F(a') \arrow[r, "Ff \circ \mathrm{strg}_{a\otimes b, a'}"] & Fc \arrow[d, equal] \\
F(a) \otimes F(a') \arrow[u, "k \otimes \mathrm{Id}_{F(a')}"] \arrow[d, "\mathrm{Id}_{F(a)} \otimes \ell"'] \arrow[r, dashed, "?"] & F(c) \arrow[d, equal] \\
F(a) \otimes F(b \otimes a') \arrow[r, "Ff \circ \mathrm{strg}_{a, b\otimes a'}"'] & F(c)
\end{tikzcd}$$

we can see we have a zig-zag of basic relations in $F^*(M \boxtimes M')(c)$, given by k and ℓ, on the condition that there exists a map in the place of the dashed arrow that makes the graph commutative, in other words, on the condition that

$$Ff \circ \mathrm{strg}_{a\otimes b, a'} \circ (k \otimes \mathrm{Id}_{F(a')}) = Ff \circ \mathrm{strg}_{a, b\otimes a'} \circ (\mathrm{Id}_{F(a)} \otimes \ell).$$

But this is easily obtained via the hexagon axiom for the tensor functor F (see 1.1.2), thus ending the proof of lemma 1.4.2. □

From now on, we assume that F is full and essentially surjective and that the tensor category $\mathbb{C}$ is rigid. We note at this point that since F is an essentially surjective strong tensor functor, it follows by 1.1.10 that $\mathbb{D}$ is also rigid. We need this fact in order to use the special formulas 1.2.18 for the Day convolutions in $\mathbb{V}^{\mathbb{C}}$ and $\mathbb{V}^{\mathbb{D}}$.

Lemma 1.4.3. *The morphism* $\mathrm{lax}^{F^*}_{M,N}$ *is an isomorphism for any* $M, N \in \mathbb{V}^{\mathbb{D}}$.

Proof. For $M, N \in \mathbb{V}^{\mathbb{D}}$, and for an object c of $\mathbb{C}$, we recall the coend formulas:

$$(F^*(M) \boxtimes F^*(N))(c) = \int^{u,v\in\mathbb{C}} \mathbb{C}(u \otimes v, c) \otimes_{\Bbbk} M(Fu) \otimes_{\Bbbk} N(Fv)$$

$$F^*(M \boxtimes N)(c) = \int^{r,s\in\mathbb{D}} \mathbb{D}(r \otimes s, Fc) \otimes_{\Bbbk} M(r) \otimes_{\Bbbk} N(s)$$

Lemma 1.2.18 gives the following isomorphism

$$(F^*(M) \boxtimes F^*(N))(c) \simeq \int^{u\in\mathbb{C}} M(Fu) \otimes_{\Bbbk} N(F(u^{\vee} \otimes c)) \tag{1.16}$$

$$[f, x, y] \mapsto [x, N(\overline{Ff})(y)]$$

where we use the notation

$$\overline{Ff} := \mathrm{strg}^F_{u^\vee,c} \circ (\mathrm{Id}_{Fu^\vee} \otimes (Ff \circ \mathrm{strg}^F_{u,v})) \circ (\sigma_{\mathbb{D}} \otimes \mathrm{Id}_{Fv}) \circ (\eta_{Fu} \otimes \mathrm{Id}_{Fv}) \circ \mathrm{lun}^{-1}_{Fv}$$

and the isomorphism

$$\begin{aligned} F^*(M \boxtimes N)(c) &\simeq \int^{d\in\mathbb{D}} M(d) \otimes_{\Bbbk} N(d^\vee \otimes Fc) \\ [g, x', y'] &\mapsto [x', N(\bar{g})(y')] \end{aligned} \tag{1.17}$$

with $d = r$ and

$$\bar{g} := (\mathrm{Id}_{r^\vee} \otimes g) \circ (\sigma_{r,r^\vee} \otimes \mathrm{Id}_s) \circ (\eta_r \otimes \mathrm{Id}_s) \circ \mathrm{lun}^{-1}_{\mathbb{D},s}\,.$$

Now, we claim that under the identifications (1.16) and (1.17), the map $\mathrm{lax}^{F^*}_{M,N,c}$ becomes as follows:

$$\begin{aligned} \int^{u\in\mathbb{C}} M(Fu) \otimes_{\Bbbk} N(F(u^\vee \otimes c)) &\longrightarrow \int^{d\in\mathbb{D}} M(d) \otimes_{\Bbbk} N(d^\vee \otimes Fc) \\ [x, y] &\mapsto [x, N(\mathrm{strg}^F_{u^\vee,c})^{-1}(y)] \end{aligned}$$

It is straightforward to check that this is indeed the case, and also – by using that F is essentially surjective and full – that the assignment $[x, z] \mapsto [x, N(\mathrm{strg}^F)(z)]$ is well-defined and thus yields an inverse map. Therefore $\mathrm{lax}^{F^*}_{M,N,c}$ is also an isomorphism, as wished. □

Remark 1.4.4. Note however that lemma 1.4.3 does not make F^* a *strong* tensor functor, because in general the structure map $\mathrm{lax}^{F^*}_1 : \mathbb{C}(1,-) \to \mathbb{D}(1, F-)$ is not invertible.

Corollary 1.4.5. *The functor E is strong monoidal.*

Proof. We have the following commutative triangle:

$$\begin{tikzcd} F^*(M) \otimes_A F^*(N) \arrow[r, "\mathrm{lax}^E_{M,N}"] & F^*(M \boxtimes N) \\ F^*(M) \boxtimes F^*(N) \arrow[u, two heads, "\mathrm{coeq}"] \arrow[ur, "\mathrm{lax}^{F^*}_{M,N}"'] & \end{tikzcd}$$

Since, by lemma 1.4.3, $\mathrm{lax}^{F^*}_{M,N}$ is invertible, and since the coequalizer's map is an epimorphism, both the coequalizer's map and $\mathrm{lax}^E_{M,N}$ are isomorphisms. Indeed, we have

$$\big((\mathrm{lax}^{F^*}_{M,N})^{-1} \circ \mathrm{lax}^E_{M,N}\big) \circ \mathrm{coeq} = \mathrm{Id}_{F^*(M)\boxtimes F^*(N)}$$

of course, and since coeq is an epimorphism,

$$\mathrm{coeq}\circ\big((\mathrm{lax}^{F^*}_{M,N})^{-1}\circ\mathrm{lax}^E_{M,N}\big)\circ\mathrm{coeq} = \mathrm{coeq}\circ\mathrm{Id}_{F^*(M)\boxtimes F^*(N)} = \mathrm{Id}_{F^*(M)\otimes_A F^*(N)}\circ\mathrm{coeq}$$

implies that

$$\mathrm{coeq} \circ \left((\mathrm{lax}^{F^*}_{M,N})^{-1} \circ \mathrm{lax}^{E}_{M,N} \right) = \mathrm{Id}_{F^*(M) \otimes_A F^*(N)}$$

and coeq is an isomorphism. Hence $\mathrm{lax}^E_{M,N}$ is an isomorphism.

Furthermore $\mathrm{lax}^E_1 : A \to E(1)$ is the identity by (1.15). Hence, E is strong monoidal. $\square$

Proposition 1.4.6. *Let $(M, \lambda) \in A - \mathrm{Mod}_{\mathbb{V}^{\mathbb{C}}}$ be any A-module. Then the underlying functor $M : \mathbb{C} \to \mathbb{V}$ factors through $F : \mathbb{C} \to \mathbb{D}$.*

Proof. Let (M, λ) be an object in $A - \mathrm{Mod}_{\mathbb{V}^{\mathbb{C}}}$. We want to show that, up to a canonical isomorphism, the underlying functor M factors through F: given such an A-module, we need a way to construct a functor $\overline{M} : \mathbb{D} \to \mathbb{V}$ and a natural isomorphism $M \cong \overline{M} \circ F$.

For any object d in $\mathbb{D}$, we choose an object c_d in $\mathbb{C}$ and an isomorphism $\varphi_d \in \mathbb{D}(d, F(c_d))$. We can do so because F is essentially surjective. Moreover, if d is already of the form Fc we can choose φ_d to be an identity map for simplicity, namely $\varphi_c := \mathrm{Id}_{Fc} = \mathrm{Id}_d$ for such a $c \in \mathbb{C}$. We set $\overline{M}(d) := M(c_d)$ on objects.

For $\psi \in \mathbb{D}(d, d')$, the following graph defines $\overline{M}(\psi)$:

$$\begin{tikzcd}
\overline{M}(d) \arrow[r, equal] \arrow[d, "\overline{M}(\psi)"'] & M(c_d) \arrow[d, "M(\tilde{\psi})"] \\
\overline{M}(d') \arrow[r, equal] & M(c_{d'})
\end{tikzcd}$$

where $\tilde{\psi} : c_d \to c_{d'}$ is chosen such that

$$F(\tilde{\psi}) = \varphi_{d'} \circ \psi \circ \varphi_d^{-1} \in \mathbb{D}(F(c_d), F(c_{d'}))$$

and exists by the fullness of F.

We need to check that this is well defined, i.e. that for any choice of such a $\tilde{\psi}$, the definition of $\overline{M}$ is unchanged. Let $\tilde{\psi}_1, \tilde{\psi}_2 \in \mathbb{C}(c_d, c_{d'})$ be such that $F(\tilde{\psi}_1) = F(\tilde{\psi}_2) = \varphi_{d'} \circ \psi \circ \varphi_d^{-1}$. Consider the diagram

$$\begin{tikzcd}
 & (A \boxtimes M)(c_d) \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, shift right, "(A\boxtimes M)(\tilde{\psi}_1)"'] \arrow[d, shift left, "(A\boxtimes M)(\tilde{\psi}_2)"] & \\
M(c_d) \arrow[d, "M(\tilde{\psi}_1)"'] & (A \boxtimes M)(c_{d'}) \arrow[dl, two heads] \arrow[dr, two heads] & M(c_d) \arrow[d, "M(\tilde{\psi}_2)"] \\
M(c_{d'}) & & M(c_{d'})
\end{tikzcd}$$

where the diagonal arrows are A-action components at c_d and $c_{d'}$ respectively. Recall that these maps are coequalizer maps, see lemma 1.1.17. Now let us compute the central downward arrows, using the isomorphisms of lemma 1.2.18:

$$\begin{aligned} \int^w A(w^\vee \otimes c_d) \otimes_{\Bbbk} M(w) &\simeq \int^w \mathbb{D}(F(w), F(c_d)) \otimes_{\Bbbk} M(w) \\ &\to \int^u \mathbb{D}(F(u), F(c_{d'})) \otimes_{\Bbbk} M(u) \\ &\simeq \int^u A(u^\vee \otimes c_d) \otimes_{\Bbbk} M(u) \end{aligned} \tag{1.18}$$

These two maps (1.18) are in fact the same: for $\beta \in \mathbb{D}(F(w), F(c_d))$ and $x \in M(w)$, they become respectively the maps

$$[\beta, x] \mapsto [F(\tilde{\psi}_1) \circ \beta, x] \quad \text{and} \quad [\beta, x] \mapsto [F(\tilde{\psi}_2) \circ \beta, x]$$

at $u = w$. By hypothesis on the $\tilde{\psi}_i$, we have $F(\tilde{\psi}_1) = F(\tilde{\psi}_2)$ and therefore the two maps are equal. Thus

$$(A \boxtimes M)(\tilde{\psi}_1) = (A \boxtimes M)(\tilde{\psi}_2),$$

from which it follows that $M(\tilde{\psi}_1) = M(\tilde{\psi}_2)$ on the quotients, as claimed. This shows that $\overline{M}$ is well-defined on morphisms.

It is obvious from the definition that $\overline{M}(\mathrm{Id}) = \mathrm{Id}$, and the fact that it commutes with composition follows from the well defineness on morphisms: since by definition we get $F(\widetilde{\psi' \circ \psi}) = F(\tilde{\psi}') \circ F(\tilde{\psi})$, by the same reasoning as before we have

$$\overline{M}(\psi' \circ \psi) := M(\widetilde{\psi' \circ \psi}) = M(\tilde{\psi}' \circ \tilde{\psi}) =: \overline{M}(\psi') \circ \overline{M}(\psi)\,.$$

Hence $\overline{M}$ is a functor $\mathbb{D} \to \mathbb{V}$.

Through the choices of pairs (c_d, φ_d), with the simplification $\varphi_{Fc} = \mathrm{Id}_{Fc}$ as above, we obtain the identity $\overline{M} \circ F = M$. □

Proof of theorem 1.3.13. We can now assemble all the pieces of our proof. By (1.13), we have for every pair of objects $M, N \in \mathbb{V}^{\mathbb{D}}$ the following commutative triangle:

$$\begin{tikzcd} & \mathbb{V}^{\mathbb{C}}(F^*M, F^*N) & \\ \mathbb{V}^{\mathbb{D}}(M, N) \arrow[ur, "F^*_{M,N}"] \arrow[rr, "E_{M,N}"'] & & A-\mathrm{Mod}(EM, EN) \arrow[ul, "U_{M,N}"'] \end{tikzcd}$$

The diagonal maps are bijections since F^* and U are fully faithful (see 1.3.2 and 1.3.12), hence so is the horizontal one, that is, E is fully faithful too. By proposition 1.4.6, we see that E is also essentially surjective, because for every A-module (M, λ) its underlying functor is of the form $M = \overline{M} \circ F = F^*(\overline{M})$, hence $(M, \lambda) = E(\overline{M})$. Therefore E is an equivalence of categories. Moreover, it is a strong tensor functor by 1.4.5.

It remains to find a natural isomorphism $E \circ \mathrm{Lan}_F \simeq \mathrm{Free}$. Let E^{-1} be an adjoint quasi-inverse of E. Then we obtain an isomorphism $F^* \circ E^{-1} \simeq U$ from $U \circ E = F^*$, and both Free and $E \circ \mathrm{Lan}_F$ (as a composite of two left adjoints) are left adjoints of U. Hence by uniqueness of left adjoints they must be canonically isomorphic, as wished. $\square$

Chapter 2

Cohomological Mackey functors and precomposition

In this chapter, we apply our first Main Theorem in order to reprove a well-known result relating Mackey functors and cohomological Mackey functors for a finite group G, essentially due to Yoshida [28]. Thanks to our proof, we will gain insight into the tensor structure of cohomological Mackey functors which as far as we know was not yet available in the literature. A similar (but somewhat different) functorial approach to ours on Yoshida's results can be found in section 10 of [22].

Notation 2.0.1. We fix G a finite group and $\Bbbk$ a commutative ring throughout this chapter. We denote by Set the category of finite sets and maps, by G-set the category of finite G-sets and G-equivariant maps, and by $\mathbb{V}$ the category of $\Bbbk$-modules, as before.

2.1 From the Lindner category to permutation modules

In this section, we define and describe the category $\Bbbk\mathrm{Sp}(G)$ of spans over G-sets and the category $\mathrm{perm}_{\Bbbk}$ of permutation $\Bbbk$-modules. We recall the universal property of span categories from [18], and define a $\mathbb{V}$-functor $\Bbbk\mathrm{Sp}(G) \to \mathrm{perm}_{\Bbbk}$ that we prove to be strong monoidal, full and essentially surjective.

We denote by

- $1 := G/G$ "the" final object in G-set, and the exclamation mark denotes the only morphism to it,
- $0 := \emptyset$ the initial object in G-set, and the exclamation mark denotes the only morphism from it,

- $\Delta = \Delta_X$ the diagonal application $X \to X \times X$, $x \mapsto (x, x)$,
- $[G/H]$ a set of representatives of the equivalence classes of the quotient of G by one of its subgroup H,
- $\Bbbk G$ the group algebra of G, that is the free $\Bbbk$-module on the set G equipped with the multiplication extended $\Bbbk$-bilinearly from the product of G,
- $\Bbbk G-\mathrm{mod}$ the category of finitely generated left modules over this algebra.

Definition 2.1.1. Let $\mathrm{Sp}(G)$ be the category with objects finite G-sets, and morphisms between X and Y are given by equivalence classes of diagrams

$$Y \xleftarrow{g} V \xrightarrow{f} X$$

in G-set, for the following equivalence relation: two diagrams are equivalent if there exists a G-equivariant isomorphism h such that the following diagram commutes

$$\begin{tikzcd}
& V \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[dd, "h"] & \\
Y & & X \\
& V' \arrow[ul, "g'"] \arrow[ur, "f'"'] &
\end{tikzcd}.$$

Composition is obtained by pullbacks, and the identity of a G-set X is the equivalence class of the span

$$X \xleftarrow{\mathrm{Id}_X} X \xrightarrow{\mathrm{Id}_X} X\,.$$

A span will either be written as a flat diagram, as for the identity span above, or as a triangle (as in the next diagram below), or as a tuple (Y, g, V, f, X). By default, a span will be read from the right hand side to the left hand side, but sometimes it is convenient to consider both reading directions. When there will be a risk of confusion, a doted arrow will denote the reading direction, as in:

$$\begin{tikzcd}
& V \arrow[dl, "g"'] \arrow[dr, "f"] & \\
Y & & X. \arrow[ll, dotted]
\end{tikzcd}$$

Sometimes, the abuse of calling “span” its equivalence class will be made, but all constructions will be well defined regarding the equivalence relation, hence robust to such abuse.

Proposition 2.1.2. $\mathrm{Sp}(G)$ *is a semi-additive rigid tensor category. That is, it has the structure of a semi-additive category (see definition 1.2.4) and the structure of a rigid tensor category, such that the tensor product functor preserves the additive structure on the Hom sets (and therefore also preserves biproducts of objects).*

Proof. We only give elements of the proof, enough to describe the different structures and give an idea of how things can be computed. A more detailled proof can be found in Serge Bouc's book [7]. A more general approach is done by Panchadcharam and Street in [22], where they study the category of spans over any *lextensive* category.

Any binary pullback in Set can be described explicitly as follows: let X, Y, and Z be sets, and $f : X \to Z$, $g : Y \to Z$ be maps. Then the set

$$P = \{(x, y) \in X \times Y | f(x) = g(y) \in Z\} \ ,$$

together with canonical projections, gives the pullback square

$$\begin{tikzcd} P \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

It is straightforward to check this is, when sets are G-sets, a pullback of G-sets. Thus, one can define the composition of two composable morphisms in $\mathrm{Sp}(G)$ as follows

$$\begin{tikzcd}[column sep=small] & Y \arrow[dl, "g'"'] \arrow[dr, "g"] & & ; & & X \arrow[dl, "f"'] \arrow[dr, "f'"] & & \mapsto & & P \arrow[dl, "g' \circ p_Y"'] \arrow[dr, "f' \circ p_X"] & \\ T & & Z & & Z & & U & & T & & U. \end{tikzcd}$$

Now, suppose that the spans below are equivalent via respectively $k : Y \to Y_1$ and $\ell : X \to X_1$

$$\left(T \xleftarrow{g_1'} Y_1 \xrightarrow{g_1} Z\right) \sim \left(T \xleftarrow{g'} Y \xrightarrow{g} Z\right)$$

$$\left(Z \xleftarrow{f_1'} X_1 \xrightarrow{f_1} U\right) \sim \left(Z \xleftarrow{f'} X \xrightarrow{f} U\right).$$

Then, the span obtained by pullback on the left hand side

$$T \xleftarrow{g_1' \circ p_{Y_1}} P_1 \xrightarrow{f_1' \circ p_{X_1}} U$$

is in the same class as

$$T \xleftarrow{g' \circ p_Y} P \xrightarrow{f' \circ p_X} U$$

via $(x, y) \mapsto (\ell(x), k(y))$, and composition is well defined. One can easily check that composition is associative, and that the identity span is neutral for relevant compositions.

It follows from the fact that $0 := \emptyset$ is initial in $G-\mathrm{set}$ that it is a *zero object* in $\mathrm{Sp}(G)$, with , for any object X, exactly one span from X to 0 and thus exactly one span from 0 to X:

$$\begin{tikzcd} & 0 \arrow[dl, equal, "!"'] \arrow[dr, "!"] & & & & 0 \arrow[dl, "!"'] \arrow[dr, equal, "!"] & \\ 0 & & X \arrow[ll, dotted, "!"] & & X & & 0. \arrow[ll, dotted, "!"] \end{tikzcd}$$

More generally, for any two objects X and Y, there exists a zero span from one to the other, which is the only span in $\mathrm{Sp}(G)(X,Y)$ that factors through 0:

$$\begin{tikzcd} & 0 \arrow[dl, "!"'] \arrow[dr, "!"] & \\ Y & & X. \arrow[ll, dotted, "0_{XY}"] \end{tikzcd}$$

Now, we have all the ingredients to show that the disjoint union of G-sets gives a biproduct in $\mathrm{Sp}(G)$. Indeed, considering the class of the four following spans:

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, hook] & & Y \arrow[dl, hook'] \arrow[dr, equal] & \\ X \arrow[rr, dotted, bend right, "i_X"'] & & X \coprod Y \arrow[ll, dotted, bend right, "p_X"'] \arrow[rr, dotted, bend left, "p_Y"] & & Y, \arrow[ll, dotted, bend left, "i_Y"] \end{tikzcd}$$

one can easily check that

$$p_X \circ i_Y = 0_{YX}\ ,\ p_Y \circ i_X = 0_{XY}\,,$$

$$p_X \circ i_X = \mathrm{Id}_X \text{ , and } p_Y \circ i_Y = \mathrm{Id}_Y\,.$$

Furthermore, one can define the sum of two morphisms in $\mathrm{Sp}(G)(Z,T)$ as follows:

$$\begin{tikzcd}[column sep=small] & V \arrow[dl, "g'"'] \arrow[dr, "g"] & & & & U \arrow[dl, "f'"'] \arrow[dr, "f"] & & & & & V \coprod U \arrow[dll, "g' \cup f'"'] \arrow[drr, "g \cup f"] & & \\ T & & Z & + & T & & Z & = & T & & & & Z. \end{tikzcd}$$

Checking this is well defined on the *class of spans* is straightforward, and then the final relation for biproducts $i_X \circ p_X + i_Y \circ p_Y = \mathrm{Id}_{X \coprod Y}$ follows from this definition of the sum in $\mathrm{Sp}(G)(X \coprod Y, X \coprod Y)$. As expected, one can check that for any two objects X and Y, the hom-set $\mathrm{Sp}(G)(X,Y)$ equipped with the sum is a commutative monoid with 0_{XY} as neutral element. It is straightforward to check that composition is bilinear.

The cartesian product of G-sets induces a symmetric monoidal structure on $\mathrm{Sp}(G)$:

1. It induces a functor $\mathrm{Sp}(G) \times \mathrm{Sp}(G) \to \mathrm{Sp}(G)$, simply by sending a pair of spans $f : X \xleftarrow{\alpha} V \xrightarrow{\beta} Y$ and $f' : X' \xleftarrow{\alpha'} V' \xrightarrow{\beta'} Y'$ to the product span $f \times f' : X \times X' \xleftarrow{(\alpha,\alpha')} V \times V' \xrightarrow{(\beta,\beta')} Y \times Y'$.
2. The G-set 1 is the unit object.

3. The right unitor at X is given by the class of the span

$$\begin{tikzcd}
& X \times 1 \arrow[dl, "\mathrm{run}_X"'] \arrow[dr, equal] & \\
X & & X \times 1 \arrow[ll, dotted, "\mathrm{run}_X"'] ,
\end{tikzcd}$$

containing the right unitor of X in $(G-\mathrm{set}, \times, 1)$, and similarly for the left unitor.

4. The associator and the symmetry are constructed similarly. For instance, the latter is given by the class of the span

$$\begin{tikzcd}
& X \times Y \arrow[dl, "(x,y)\mapsto(y,x)"'] \arrow[dr, equal] & \\
Y \times X & & X \times Y \arrow[ll, dotted, "\sigma_{X,Y}"'] .
\end{tikzcd}$$

Checking the coherence conditions is straightforward.

For the monoidal structure described above, $\Bbbk\mathrm{Sp}(G)$ is rigid if one sets that any G-set is its own monoidal dual, and that the unit and counit are as follows:

$$\begin{tikzcd}
& X \arrow[dl, "\Delta"'] \arrow[dr, "!"] & \\
X \times X^\vee & & 1 \arrow[ll, dotted, "\eta_X"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, "!"'] \arrow[dr, "\Delta"] & \\
1 & & X^\vee \times X. \arrow[ll, dotted, "\epsilon_X"]
\end{tikzcd}$$

(Duality is denoted by $X^\vee = X$, the $^\vee$ being used to keep track of the variance (co- or contra-) of our constructions.)

We check that the first relation holds

$$\begin{tikzcd}
& & & W \arrow[dll, "p_1"'] \arrow[drr, "p_2"] & & & \\
& X \times X \arrow[dl, "\mathrm{Id}\times !"'] \arrow[drr, "\mathrm{Id}\times\Delta"] & & & & X \times X \arrow[dll, "\Delta\times\mathrm{Id}"'] \arrow[dr, "!\times\mathrm{Id}"] & \\
X \times 1 & & & X \times X^\vee \times X \arrow[lll, dotted, "\mathrm{Id}\times\epsilon_X"'] & & & 1 \times X \arrow[lll, dotted, "\eta_X\times\mathrm{Id}"']
\end{tikzcd}$$

W is the set of quadruplets $(x,y,z,t) \in X \times X \times X \times X$ such that $(x,y,y) = (z,z,t)$, hence it is isomorphic to $1 \times X$ via $(x,x,x,x) \mapsto (1,x)$. p_1 is the projection along the first two coordinates, p_2 the projection along the third and fourth. We obtain the following commuting diagram, and the result follows by

$\mathrm{run}_X \circ \sigma_{1,X} \circ \mathrm{lun}_X^{-1} = \mathrm{Id}_X$.

$$\begin{tikzcd}
 & W \arrow[dl, "(\mathrm{Id}\times !)\circ p_1"'] \arrow[dr, "(!\times \mathrm{Id})\circ p_2"] \arrow[dd, "\cong"] & \\
X \times 1 & & 1 \times X \\
 & 1 \times X \arrow[ul, "(1,x)\mapsto(x,1)"] \arrow[ur, equal] &
\end{tikzcd}$$

The second condition is checked in a similar way. □

Definition 2.1.3. We define the $\mathbb{V}$-category $\Bbbk\mathrm{Sp}(G)$ to be the $\Bbbk$-linearization of $\mathrm{Sp}(G)$ as in proposition 1.2.5. Concretely, given two G-sets X and Y, consider the monoid $\mathrm{Sp}(G)(X,Y)$ and make it a group by Grothendieck group completion, that is by giving any morphism a formal inverse. This group completion can be seen as a change of scalars:

$$\mathrm{Sp}(G)(X,Y) \to \mathbb{Z} \otimes_{\mathbb{N}} \mathrm{Sp}(X,Y).$$

This map is injective because, as one can easily check, the monoid is *cancellative*: $a+b = c+b$ in $\mathrm{Sp}(G)(X,Y)$ implies that $a = c$. We obtain the category $\Bbbk\mathrm{Sp}(G)$ by going one step further, and changing the base ring to $\Bbbk$:

$$\mathbb{Z} \otimes_{\mathbb{N}} \mathrm{Sp}(G)(X,Y) \to \Bbbk \otimes_{\mathbb{Z}} \mathbb{Z} \otimes_{\mathbb{N}} \mathrm{Sp}(X,Y) =: \Bbbk\mathrm{Sp}(G)(X,Y).$$

The composition maps in $\Bbbk\mathrm{Sp}(G)$ are simply obtained by extending $\Bbbk$-bilinearly the composition maps of $\mathrm{Sp}(G)$.

Proposition 2.1.4. $\Bbbk\mathrm{Sp}(G)$ *is an additive and rigid tensor* $\mathbb{V}$*-category.*

Proof. This is straightforward from the definition. □

Definition 2.1.5. We denote by $\mathrm{perm}_{\Bbbk}(G)$ the full subcategory of $\Bbbk G-\mathrm{mod}$ with objects the *permutation modules* over $\Bbbk G$, which are the $\Bbbk G$-modules which admit a finite G-invariant basis X (in particular they are finitely generated free over $\Bbbk$). Given a permutation module M, each such basis X is a G-set and yields an isomorphism $\Bbbk X \simeq M$ with the free $\Bbbk$-module $\Bbbk X$ over X equipped with the linearly extended $\Bbbk G$-action.

Remark 2.1.6. As a full subcategory, $\mathrm{perm}_{\Bbbk}(G)$ inherits the $\Bbbk$-linear structure of $\Bbbk G-\mathrm{mod}$, as well as its tensor-structure because the tensor product $M \otimes_{\Bbbk} N$ of two permutation modules is again a permuation module. Indeed, given two permutation $\Bbbk G$-modules M, N with invariant bases X, Y respectively, the set $\{x \otimes y \mid x \in X, y \in Y\}$ provides an invariant basis of $M \otimes_{\Bbbk} N$.

Definition 2.1.7. We define a functor $F : \Bbbk\mathrm{Sp}(G) \to \mathrm{perm}_{\Bbbk}(G)$ by the following assignments:

1. A finite G-set X is mapped to the permutation module $\Bbbk X$.

2. The class of any span $(Y \xleftarrow{f} V \xrightarrow{g} X)$ is mapped to the module homomorphism defined on the $\Bbbk$-basis X of $\Bbbk X$ by $x \mapsto \sum_{v \in g^{-1}(x)} f(v)$.

In order to study F, in particular to see that it is a functor, we need the two following lemmas, which we will refer to as Lindner's lemma and Yoshida's lemma, because they are the key to bridge Lindner's work on Mackey functors to Yoshida's work on cohomological Mackey functors.

Lemma 2.1.8 (Lindner's lemma)**.** *Let $\mathbb{C}$ be a category with every pullback. The category of spans on $\mathbb{C}$ is constructed analogously to* $\mathrm{Sp}(G)$*, and we denote it by* $\mathrm{Sp}(\mathbb{C})$*. One can embed $\mathbb{C}$ and $\mathbb{C}^{op}$ into* $\mathrm{Sp}(\mathbb{C})$ *via the faithful functors:*

$$X \mapsto X\ ,\ \ (f : X \to Y) \mapsto (Y, f, X, \mathrm{Id}_X, X)$$

and

$$X \mapsto X\ ,\ \ (f : X \to Y) \mapsto (X, \mathrm{Id}_X, X, f, Y).$$

Now, if $F_1 : \mathbb{C} \to \mathbb{D}$ and $F_2 : \mathbb{C}^{op} \to \mathbb{D}$ are two functors such that:

- *for any object $X \in \mathrm{Ob}(\mathbb{C})$, we have $F_1(X) = F_2(X)$*
- *for any pullback $d \circ c = b \circ a$ in $\mathbb{C}$, we have $\big(F_1(a) \circ F_2(c)\big) = \big(F_2(b) \circ F_1(d)\big)$,*

then there exists a unique functor F making the following diagram commutative.

$$\begin{tikzcd}
\mathbb{C} \arrow[d, hook] \arrow[dr, "F_1"] & \\
\mathrm{Sp}(\mathbb{C}) \arrow[r, dotted, "F"] & \mathbb{D} \\
\mathbb{C}^{op} \arrow[u, hook] \arrow[ur, "F_2"'] &
\end{tikzcd}$$

It is given by $X \mapsto F_1(X) = F_2(X) =: F(X)$ on objects and $(Y, f, Z, g, X) \mapsto \big(F_1(f) \circ F_2(g)\big)$ on morphisms.

Proof. This is an easy exercise. □

Lemma 2.1.9 (Yoshida's lemma)**.** *Let H and K be subgroups of G, then the following is an isomorphism:*

$$\Phi : \Bbbk\, H \backslash G / K \to \mathrm{Hom}_{\Bbbk G}(\Bbbk\, G/H, \Bbbk\, G/K)$$

$$HxK \mapsto (gH \mapsto \sum_u guxK),$$

where u runs over a complete set of representatives of $H/(H \cap xKx^{-1})$.

Proof. This is half of lemma 3.1 of [28]. The proof is a direct verification, and the author cites 3.4 of [25]. □

Proposition 2.1.10. *As defined in 2.1.7, F is a strong tensor $\mathbb{V}$-functor which is essentially surjective on objects and full.*

Proof. 1. Considering F_1, that maps any finite G-set to the module $\Bbbk X$ and any G-equivariant map $f : X \to Y$ to its $\Bbbk$-linear extension on $\Bbbk X$, it is straightforward to check that it is a functor $G - \text{set} \to \text{perm}_{\Bbbk}(G)$.

Considering F_2, that maps any finite G-set to the module $\Bbbk X$ and any G-equivariant map $f : X \to Y$ to the $\Bbbk G$-module homomorphism

$$\Bbbk Y \to \Bbbk X,\ y \mapsto \sum_{x \in f^{-1}(y)} x\ ,$$

where $f^{-1}(y) := \{x \in X | f(x) = y\}$, it is straightforward to check that it is a functor $G - \text{set}^{op} \to \text{perm}_{\Bbbk}(G)$.

Those two functors meet the requirements of Lindner's lemma 2.1.8: given any pullback in G-set $d \circ c = b \circ a$, the induced square is commutative.

$$\begin{tikzcd}
\Bbbk X \arrow[r, "F_1(a)"] & \Bbbk Y \\
\Bbbk Z \arrow[u, "F_2(c)"] \arrow[r, "F_1(d)"'] & \Bbbk T \arrow[u, "F_2(b)"']
\end{tikzcd}$$

Indeed, given any $z \in Z$,

$$(F_1(a) \circ F_2(c))(z) = \sum_{x \in c^{-1}(z)} a(x),$$

and

$$(F_2(b) \circ F_1(d))(z) = \sum_{y \in b^{-1}(d(z))} y.$$

Furthermore, we can assume that $X = \{(z, y) \in Z \times Y | d(z) = b(y)\}$ and that a and c are the two projections, and from this assumption it follows that $a(c^{-1}(z)) = \{y \in Y | b(y) = d(z)\}$, hence $a(c^{-1}(z)) = b^{-1}(d(z))$ and the square is commutative.

By lemma 2.1.8, there exists a unique functor $\tilde{F}$ extending F_1 and F_2 on $\text{Sp}(G)$. Since $\text{perm}_{\Bbbk}(G)$ is a $\mathbb{V}$-category, $\tilde{F}$ extends uniquely to a $\mathbb{V}$-functor on $\Bbbk\text{Sp}(G)$, which is given by the formula of definition 2.1.7, as one check immediately. This show that F as in definition 2.1.7 is indeed a functor, in fact a $\mathbb{V}$-fuctor, and is the unique one making the diagram below commutative. It is essentially surjective by construction.

$$\begin{tikzcd}
G - \text{set} \arrow[d, hook] \arrow[dr, "F_1"] & \\
\Bbbk\text{Sp}(G) \arrow[r, "F"] & \text{perm}_{\Bbbk}(G) \\
G - \text{set}^{op} \arrow[u, hook'] \arrow[ur, "F_2"'] &
\end{tikzcd}$$

2. Let us check that F is full. Let H and K be subgroups of G, and fix $x \in G$. We set
$$L_x = H \cap xKx^{-1}$$
and
$$\psi_x : \Bbbk G/H \to \Bbbk G/K\ ,\ gH \mapsto \sum_{u \in [H/L_x]} guxK.$$
Set $p_x : G/L_x \to G/H$ to be the canonical projection, and
$$f : G/L_x \to G/K\ ,\ gL_x \mapsto gxK.$$
The map f is well defined, because $L_x x \subseteq xK$, hence
$$\forall \ell \in L_x, \exists k \in K$$
such that $\ell x = xk$, and so
$$f(g\ell L_x) = g\ell xK = gxkK = gxK = f(gL_x).$$
We can construct a span from those maps, and compute its image by our functor F:
$$F\big(G/K, f, G/L_x, p_x, G/H\big) : \Bbbk G/H \to \Bbbk G/K$$
is an homomorphism of $\Bbbk G$-modules, defined on the basis by
$$gH \quad \mapsto \sum_{z \in p_x^{-1}(gH)} f(z).$$
Since $p_x^{-1}(gH) = \{ghL_x | h \in H\}$, we get
$$\sum_{z \in p_x^{-1}(gH)} f(z) = \sum_{vL_x \in \{ghL_x | h \in H\}} vxK.$$
If we change the variable by $u := g^{-1}v$, we obtain
$$\sum_{uL_x \in \{hL_x | h \in H\}} guxK.$$
And since $\{hL_x | h \in H\} = H/L_x$, we have
$$F(G/K, f, G/L_x, p_x, G/H) = \psi_x.$$
It follows from Yoshida's lemma 2.1.9 that every element of the Hom set $\mathrm{Hom}_{\Bbbk G}(\Bbbk G/H, \Bbbk G/K)$ is a $\Bbbk$-linear combinations of the maps ψ_x (for $x \in [H\backslash G/K]$), and every morphism in the category $\mathrm{perm}_{\Bbbk}(G)$ can be obtained by additivity from those, since any G-set can be written as a disjoint union of quotients of G. Hence, F is full.

3. Consider the morphism in $\mathrm{perm}_{\Bbbk}(G)(F(X) \otimes_{\Bbbk} F(Y), F(X \times Y))$:
$$\mathrm{strg}^F_{X,Y} : F(X) \otimes_{\Bbbk} F(Y) \to F(X \times Y) = \Bbbk(X \times Y)$$
given on the basis by
$$x \otimes y \mapsto (x, y)$$
Being a bijection on G-invariant bases, this is an isomorphism in $\mathrm{perm}_{\Bbbk}(G)$. Now, consider the morphism $\mathrm{strg}^F_1 \in \mathrm{perm}_{\Bbbk}(G)(\Bbbk, F(1))$ to be the identity of $\Bbbk$. Checking that these two morphisms give a strong tensor structure to F is straightforward.

This ends the proof. □

2.2 Mackey functors

There are, to the author's knowledge, at least four equivalent definitions of Mackey functors. They were first introduced by Green in 1971 in [11], as an axiomatic approach to group representation. We recall this definition for context, but the definition we use in our work is historically the third one, and due to Harald Lindner. For a more complete approach of Mackey functors, one could read Bouc's book [7], Webb's paper [27], and the paper he wrote with Thevenaz [26].

Definition 2.2.1 (Green's classical definition)**.** A *Mackey functor* for G over $\Bbbk$ is a function
$$M : \{\text{subgroups of } G\} \to \Bbbk - \mathrm{Mod}$$
together with $\Bbbk$-linear morphisms
$$\begin{aligned} I^H_K &: M(K) \to M(H), \\ R^H_K &: M(H) \to M(K), \\ c_g &: M(H) \to M({}^gH), \end{aligned}$$
for all subgroups $K \leq H \leq G$ and all $g \in G$, such that

- I^H_H, R^H_H, $c_H : M(H) \to M(H)$ are the identity morphisms for all surbgroups H and all $h \in H$.
- $R^K_J \circ R^H_K = R^H_J$, for all subgroups $J \leq K \leq H$.
- $I^H_K \circ I^K_J = I^H_J$, for all subgroups $J \leq K \leq H$.
- $c_g \circ c_h = c_{gh}$, for all $g, h \in G$.
- $R^{{}^gH}_{{}^gK} \circ c_g = c_g \circ R^H_K$, for all subgroups $K \leq H$ and $g \in G$.
- $I^{{}^gH}_{{}^gK} \circ c_g = c_g \circ I^H_K$, for all subgroups $K \leq H$ and $g \in G$.
- $R^H_J \circ I^H_K = \sum_x I^J_{J \cap {}^xK} \circ c_x \circ R^K_{J^x \cap K}$, where x runs through a set of representatives of $J \backslash H / K$ for all subgroups $J, K \leq H$.

Definition 2.2.2 (Dress-Lindner's definition)**.** A *Mackey functor* is a $\Bbbk$-linear functor $\Bbbk\mathrm{Sp}(G) \to \Bbbk-\mathrm{Mod}$. We denote $\mathrm{Mack}_{\Bbbk}(G)$ the category of these functors with morphisms the natural transformations between them.

Remark 2.2.3. It is one of the main results of [18] to show that this definition is equivalent to that of Green.

Proposition 2.2.4. *The category* $\mathrm{Mack}_{\Bbbk}(G)$ *inherits a symmetric monoidal structure via Day convolution.*

Proof. This follows directly from what we know of the structure of $\Bbbk\mathrm{Sp}(G)$, see 2.1.4, and the proposition of our first chapter on Day convolution 1.2.16. An explicit description of this structure can be found in [7], at 1.6.2. It can also be obtained by application of the general description we give in 1.2.16. □

Definition 2.2.5 (Classical definition of Green functors)**.** A Mackey functor M (for G over $\Bbbk$) in the sense of Green is a *Green functor* if:

- for any subgroup H of G, $M(H)$ has a $\Bbbk$-algebra structure,
- for any subgroups $K \leq H$ of G and any $g \in G$, both R^H_K and c_g are $\Bbbk$-algebra homomorphisms
- and the two *Frobenius formulas*

$$I^H_K(x \cdot R^H_K(y)) = I^H_K(x) \cdot y \quad \text{and} \quad I^H_K(R^H_K(y) \cdot x) = y \cdot I^H_K(x)$$

  are satisfied for all $K \leq H \leq G$.

And here is the alternative definition that we will use:

Definition 2.2.6. A Mackey functor $R : \Bbbk\mathrm{Sp}(G) \to \Bbbk - \mathrm{Mod}$ is a *Green functor* if it is a monoid object of the category $\mathrm{Mack}_{\Bbbk}(G)$ equiped with the Day convolution product.

Proposition 2.2.7. *The last two definitions coincide.*

Proof. See the beginning of [7] for the equivalence of the different definitions of Mackey and Green functors. Let us just say here that $M(H)$ corresponds to $M(G/H)$, and that the restriction, conjugation and induction maps of the classical definitions are obtained, respectively, from the spans

$$\begin{tikzcd} & G/H \arrow[dl, "p"'] \arrow[dr, equal] & \\ G/K & & G/H \arrow[ll, dotted] \end{tikzcd} \qquad \begin{tikzcd} & G/H \arrow[dl, "xH\mapsto gxg^{-1}gH"'] \arrow[dr, equal] & \\ G/{}^gH & & G/H \arrow[ll, dotted] \end{tikzcd}$$

$$\begin{tikzcd} & G/H \arrow[dl, equal] \arrow[dr, "p"] & \\ G/H & & G/K \arrow[ll, dotted] \end{tikzcd}$$

by applying the functor $M : \Bbbk\mathrm{Sp}(G) \to \Bbbk - \mathrm{Mod}$ to them. □

Definition 2.2.8. It follows that a *module over a Green functor* R is simply an object of the category $R - \mathrm{Mod}_{\mathrm{Mack}_{\Bbbk}(G)}$, as in definition 1.1.15.

Definition 2.2.9 (Cohomological Mackey functors: classical definition)**.** A Mackey functor is said to be *cohomological* if for every pair of subgroups $H \leq K$ of G the map $I_H^K \circ R_H^K$ is multiplication by $|K : H|$.

Let us give an well-known example of a Green functor, which is also a cohomological Mackey functor:

Example 2.2.10 (The fixed point Green functor)**.** In terms of the classical definitions, the *fixed point Green functor* $FP_{\Bbbk}$ is defined as follows. Its value at $H \leq G$ is simply the $\Bbbk$-algebra $\Bbbk$ for every subgroup. This forces the restrictions and conjugations to be all identities. The induction map $I_K^H : \Bbbk \to \Bbbk$ is given by multiplication by the index $|H : K|$.

The name 'fixed point' comes from the fact that this Mackey functor is the special case with $V = \Bbbk$ of a family of Mackey functors FP_V, for V any given $\Bbbk G$-module, which are such that $FP_V(H) = V^H$ is the submodule of the H-fixed points in V.

Definition 2.2.11 (Yoshida's Cohomological Mackey functors)**.** *Cohomological Mackey functors* are $\Bbbk$-linear functors $\mathrm{perm}_{\Bbbk}(G) \to \Bbbk - \mathrm{Mod}$. We denote $\mathrm{CoMack}_{\Bbbk}(G)$ the category of these functors with morphisms the natural transformations between them.

Remark 2.2.12. It actually is the main theorem of Yoshida's paper [28] that the classically defined category of cohomological Mackey functors is isomorphic to this one.

Proposition 2.2.13. *This category* $\mathrm{CoMack}_{\Bbbk}(G)$ *of definition 2.2.11 inherits a tensor structure by Day convolution.*

Proof. As before, this is a direct consequence of 1.2.16 and of what we know of $\mathrm{perm}_{\Bbbk}(G)$, that is, it is a symmetric tensor $\mathbb{V}$-category. □

Definition 2.2.14. Consider the tensor functor $F\colon \Bbbk\mathrm{Sp}(G) \to \Bbbk - \mathrm{Mod} = \mathbb{V}$ of proposition 2.1.10. As in section 1.3, restriction along F is a lax monoidal functor $F^* : \mathrm{CoMack}_{\Bbbk}(G) \to \mathrm{Mack}_{\Bbbk}(G)$ with respect to the Day convolution tensor products, and therefore we obtain a commutative monoid $A := F^*(1)$ in $\mathrm{Mack}_{\Bbbk}(G)$, that is, a commutative Green functor.

Lemma 2.2.15. *The commutative Green functor of definition 2.2.14 is isomorphic to the fixed point Green functor recalled in example 2.2.10.*

Proof. Note that for every G-set X we have an isomorphism

$$A(X) = \mathrm{perm}_{\Bbbk}(G)(\Bbbk, FX) \cong (\Bbbk X)^G \cong \Bbbk\, G\backslash X \tag{2.1}$$

where the first isomorphism identifies a $\Bbbk$-linear map $f\colon \Bbbk \to FX = \Bbbk X$ with the image of $1 \in \Bbbk$, which is a G-fixed point of $\Bbbk X$. The second isomorphism is because we have a $\Bbbk$-linear inclusion $\Bbbk\, G\backslash X \to (\Bbbk X)^G$ sending the orbit Gx to $\sum_{y\in Gx} y$, and if $\sum_x \lambda_x x \in \Bbbk X$ is a G-invariant element then the coefficients map $x \mapsto \lambda_x$ is constant on the G-orbits of X, so the inclusion is actually an isomorphism (as it gives a bijection on bases).

We can now see explicitly what is the classical description of this Green functor $A = \mathrm{perm}_{\Bbbk}(\Bbbk, F-)$. For a subgroup $H \leq G$, it gives

$$A(H) = A(G/H) \cong \Bbbk\, G\backslash G/H = \Bbbk \cdot \{GeH\}$$

as in the above isomorphism (2.1). Therefore this $\Bbbk$-module is isomorphic to $\Bbbk$, by mapping $1 \in \Bbbk$ to its unique basis element GeH.

Therefore, it follows that the algebra structure is simply $\Bbbk$ for each H, and the restriction and conjugation maps are all the identity. It only remains to check that for all subgroups $K \leq H \leq G$ the induction maps I_K^H are given by multiplication by $|H : K|$. As recalled in the proof of proposition 2.2.7, the map $I_K^H : A(K) \to A(H)$ is the image under the functor A of the span $G/H = G/H \to G/K$ from G/K to G/H. By the definition of F, this maps sends $gK \in G/K$ to $gH \in G/H$. Since there are $|H : K|$ different K-cosets mapping to the same H-coset, by following the isomorphisms (2.1) we see that this map is indeed multiplication by $|H : K|$ as wished. □

Remark 2.2.16. The monoidal unit of $\mathrm{Mack}_{\Bbbk}(G)$, given by the functor

$$1_{\mathrm{Mack}_{\Bbbk}(G)} := \mathrm{Sp}_{\Bbbk}(G)(1, -),$$

is known as the *Burnside functor*. Since F is full, and since $F(1) \cong \Bbbk$, the following map gives an epimorphism from the Burnside functor to the the functor of fixed points:

$$\begin{array}{ccc} \mathrm{Sp}_{\Bbbk}(G)(1, X) & \to & \mathrm{perm}_{\Bbbk}(G)(\Bbbk, F(X)) \\ f & \mapsto & F(f) \circ \mathrm{strg}_1^F \end{array}$$

This is the unit map of the monoid A.

Corollary 2.2.17 (Cohomological vs ordinary Mackey functors)**.** *There is an equivalence of tensor categories between:*

- *The category of modules over the fixed-point Green functor $FP_{\Bbbk}$, equipped with its tensor product of modules;*
- *Representations of the category of permutation $\Bbbk G$-modules, equipped with the Day convolution product.*

Moreover, both categories identify canonically with the full subcategory of ordinary Mackey functors for G over $\Bbbk$ satisfying the 'cohomological axiom' ($L \leq H \leq G$):

$$\mathrm{ind}_L^H \circ \mathrm{res}_L^H = [H : L] \cdot \mathrm{Id}.$$

Proof. This is a corollary of theorem 1.3.13 and proposition 2.1.10, that is, this result is obtained by considering the rigid tensor $\mathbb{V}$-categories $\mathbb{C} := \Bbbk\mathrm{Sp}(G)$ and $\mathbb{D} := \mathrm{perm}_{\Bbbk}(G)$, via precomposition by our tensor $\mathbb{V}$-functor $F : \Bbbk\mathrm{Sp}(G) \to \mathrm{perm}_{\Bbbk}(G)$, giving

$$F^* : \mathrm{CoMack}_{\Bbbk}(G) \to \mathrm{Mack}_{\Bbbk}(G),$$

from which the situation studied in chapter 1 follows. The identification of our monoid A with the classical fixed point functor $FP_{\Bbbk}$ is in lemma 2.2.14

The moreover part follows now from the fact, proved in proposition 16.3 of [26] by very easy explicit calculations, that a Mackey module over the Green functor $FP_{\Bbbk}$ is the same thing as a Mackey functor satisfying the above-mentioned extra relations. $\square$

Chapter 3

Bicategorical notions

This chapter is a short introduction to the bicategorical notions needed for chapter four. It is based on the works of Benabou [3] and Leinster [17].

We denote by 1 “the” category with exactly one object and one morphism.

Definition 3.0.1. A *bicategory* $\mathcal{C}$ consists of the following data subject to the following axioms:

1. A collection of objects $\mathrm{Ob}(\mathcal{C})$, with elements denoted by capital roman letters: X, Y, Z,...
2. A category $\mathcal{C}(X, Y)$ for any ordered pair of given objects X and Y. The category's objects are denoted by small roman letters and its morphisms are usually denoted by small greek letters, with the exception of the structural maps described below. They are occasionally called *1-cells* and *2-cells* respectively.
3. A “composition” functor for any ordered triple of objects (X, Y, Z):
$$\begin{aligned} \circ_{\mathcal{C},XYZ} : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) &\to \mathcal{C}(X,Z) \\ (g, f) &\mapsto g \circ_{\mathcal{C}} f \\ (\beta, \alpha) &\mapsto \beta *_{\mathcal{C}} \alpha \end{aligned}$$
4. An “identity” functor for any $X \in \mathrm{Ob}(\mathcal{C})$:
$$I_{\mathcal{C},X} : 1 \to \mathcal{C}(X, X)$$
5. A natural isomorphism called *associator* for any ordered quadruple (X, Y, Z, T) of objects:
$$\begin{tikzcd} & \mathcal{C}(Z,T) \times \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \arrow[dl, "\circ_{\mathcal{C}} \times \mathrm{Id}"'] \arrow[dr, "\mathrm{Id} \times \circ_{\mathcal{C}}"] \arrow[d, phantom, "\overset{\mathfrak{a}_{XYZT}}{\Rightarrow}"] & \\ \mathcal{C}(Y,T) \times \mathcal{C}(X,Y) \arrow[dr, "\circ_{\mathcal{C}}"'] & \, & \mathcal{C}(Z,T) \times \mathcal{C}(X,Z) \arrow[dl, "\circ_{\mathcal{C}}"] \\ & \mathcal{C}(X,T) & \end{tikzcd}$$

6. Natural isomorphims called *right (or left) unitors* for any pair of objects X and Y

$$\begin{tikzcd} & 1 \times \mathcal{C}(X,Y) \arrow[dl, "I_{\mathcal{C},Y}\times \mathrm{Id}"'] \arrow[dr, "\cong"] \arrow[d, phantom, "\mathfrak{lun}_{X,Y} \Rightarrow"] & \\ \mathcal{C}(Y,Y)\times\mathcal{C}(X,Y) \arrow[rr, "\circ_{\mathcal{C}}"'] & & \mathcal{C}(X,Y) \end{tikzcd} \quad \begin{tikzcd} & \mathcal{C}(X,Y)\times 1 \arrow[dl, "\mathrm{Id}\times I_{\mathcal{C},X}"'] \arrow[dr, "\cong"] \arrow[d, phantom, "\overset{\Rightarrow}{\mathfrak{run}}_{X,Y}"] & \\ \mathcal{C}(X,Y)\times\mathcal{C}(X,X) \arrow[rr, "\circ_{\mathcal{C}}"'] & & \mathcal{C}(X,Y) \end{tikzcd}$$

7. Associators are subject to the pentagon axiom:

$$\begin{tikzcd} & ((k\circ_{\mathcal{C}} h)\circ_{\mathcal{C}} g)\circ_{\mathcal{C}} f \arrow[r, Rightarrow, "\mathfrak{a}*\mathrm{Id}_g"] \arrow[dl, Rightarrow, "\mathfrak{a}"'] & (k\circ_{\mathcal{C}} (h\circ_{\mathcal{C}} g))\circ_{\mathcal{C}} f \arrow[dr, Rightarrow, "\mathfrak{a}"] & \\ (k\circ_{\mathcal{C}} h)\circ_{\mathcal{C}} (g\circ_{\mathcal{C}} f) \arrow[drr, Rightarrow, "\mathfrak{a}"'] & & & k\circ_{\mathcal{C}} ((h\circ_{\mathcal{C}} g)\circ_{\mathcal{C}} f) \arrow[dl, Rightarrow, "\mathrm{Id}*\mathfrak{a}"] \\ & & k\circ_{\mathcal{C}} (h\circ_{\mathcal{C}} (g\circ_{\mathcal{C}} f)) & \end{tikzcd}$$

8. Associators are compatible with left and right unitors in the following way:

$$\begin{tikzcd} (g\circ_{\mathcal{C}} \mathrm{Id})\circ_{\mathcal{C}} f \arrow[rr, Rightarrow, "\mathfrak{a}"] \arrow[dr, Rightarrow, "\mathfrak{run}*\mathrm{Id}"'] & & g\circ_{\mathcal{C}} (\mathrm{Id}\circ_{\mathcal{C}} f) \arrow[dl, Rightarrow, "\mathrm{Id}*\mathfrak{lun}"] \\ & g\circ_{\mathcal{C}} f & \end{tikzcd}$$

Remark 3.0.2.

- The subscripts are often omitted when context makes confusion unlikely. We also write Id_X instead of $I_{\mathcal{C},X}(1)$.
- If the associators and unitors are all identities, then $\mathcal{C}$ is 2-category, and the axioms hold automatically.
- Every ordinary category, or *1-category*, can be seen as a 2-category (and thus as a bicategory) having only identity 2-cells.
- The axioms and the vocabulary resemble those for monoidal categories, and indeed monoidal categories can be identified with bicategories having a unique (unnamed) object. In particular, the coherence theorem for monoidal categories generalizes to bicategories.
- Given a bicategory $\mathcal{C}$, one can define the opposite bicategory $\mathcal{C}^{op}$ where the 1-cells are formally reversed. The bicategory $\mathcal{C}^{co}$ is obtained by formally reversing the 2-cells instead of the 1-cells, and one could do both things and obtain a bicategory $\mathcal{C}^{op\,co}$.

Definition 3.0.3. A *pseudofunctor* F from a bicategory $\mathcal{C}$ to a bicategory $\mathcal{D}$ consists of the following data subject to the following axioms:

1. A function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
2. For any pair of objects X and Y, a functor
$$F_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY).$$
3. For any triple of object X, Y, and Z, an invertible natural transformation
$$\phi_{F,\,(X,Y,Z)} : \circ_{\mathcal{D}}(F_{Y,Z} \times F_{X,Y}) \Rightarrow F_{X,Z}(\circ_{\mathcal{C}}).$$
4. For any object X, an invertible natural transformation
$$\phi_{F,X} : I_{\mathcal{D},F(X)} \Rightarrow F_{X,X} \circ I_{\mathcal{C},X}.$$
5. For $(f,g,h) \in \mathrm{Ob}\big(\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \times \mathcal{C}(Z,T)\big)$, the following "hexagon axiom" holds:
$$\begin{tikzcd}
(Fh \circ_{\mathcal{C}} Fg) \circ_{\mathcal{D}} Ff \arrow[r, Rightarrow, "\phi * \mathrm{Id}"] \arrow[d, Rightarrow, "\mathfrak{a}_{\mathcal{D}}"'] & F(h \circ_{\mathcal{C}} g) \circ_{\mathcal{D}} Ff \arrow[r, Rightarrow, "\phi"] & F((h \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} f) \arrow[d, Rightarrow, "F(\mathfrak{a}_{\mathcal{C}})"] \\
Fh \circ_{\mathcal{D}} (Fg \circ_{\mathcal{D}} Ff) \arrow[r, Rightarrow, "\mathrm{Id} * \phi"'] & Fh \circ_{\mathcal{D}} F(g \circ_{\mathcal{C}} f) \arrow[r, Rightarrow, "\phi"'] & F(h \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} f)).
\end{tikzcd}$$
6. For any $f \in \mathrm{Ob}(\mathcal{C}(X,Y))$, the following holds:
$$\begin{tikzcd}
Ff \circ \mathrm{Id}_{F(X)} \arrow[r, Rightarrow, "\mathrm{Id} * \phi_X"] \arrow[d, Rightarrow, "\mathfrak{run}_{\mathcal{D}}"'] & Ff \circ F(\mathrm{Id}_X) \arrow[r, Rightarrow, "\phi"] & F(f \circ \mathrm{Id}_X) \arrow[d, Rightarrow, "F(\mathfrak{run}_{\mathcal{C}})"] \\
Ff \arrow[rr, equal] & & Ff \\
\mathrm{Id}_{F(Y)} \circ Ff \arrow[r, Rightarrow, "\phi_Y * \mathrm{Id}"'] \arrow[u, Rightarrow, "\mathfrak{lun}_{\mathcal{D}}"] & F(\mathrm{Id}_Y) \circ Ff \arrow[r, Rightarrow, "\phi"'] & F(\mathrm{Id}_Y \circ f) \arrow[u, Rightarrow, "F(\mathfrak{lun}_{\mathcal{C}})"']
\end{tikzcd}$$

Definition 3.0.4. An adjunction in a bicategory $\mathcal{C}$, often written $f \dashv g$, consists of 1-cells $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,X)$, together with 2-cells (the *unit and counit of the adjunction*), $\eta : \mathrm{Id}_X \Rightarrow g \circ_{\mathcal{C}} f$, and $\epsilon : f \circ_{\mathcal{C}} g \Rightarrow \mathrm{Id}_Y$, satisfying triangle identities that generalize those the usual adjunctions in Cat satisfy:

$$\begin{tikzcd}
f \arrow[r, Rightarrow, "\mathfrak{run}_{\mathcal{C}\,f}"] \arrow[drr, equal] & f \circ_{\mathcal{C}} \mathrm{Id}_X \arrow[r, Rightarrow, "\mathrm{Id}_f *_{\mathcal{C}} \eta"] & f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} f) \arrow[d, Rightarrow, "(\epsilon *_{\mathcal{C}} \mathrm{Id}_f) \circ_{\mathcal{C}} \mathfrak{a}_{\mathcal{C}}^{-1}"] \\
 & & f
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, Rightarrow, "\mathfrak{lun}_{\mathcal{C}\,g}"] \arrow[drr, equal] & \mathrm{Id}_X \circ_{\mathcal{C}} g \arrow[r, Rightarrow, "\eta *_{\mathcal{C}} \mathrm{Id}_g"] & (g \circ_{\mathcal{C}} f) \circ_{\mathcal{C}} g \arrow[d, Rightarrow, "(\mathrm{Id}_g *_{\mathcal{C}} \epsilon) \circ_{\mathcal{C}} \mathfrak{a}_{\mathcal{C}}"] \\
 & & g
\end{tikzcd}$$

Definition 3.0.5. Let $\mathcal{C}$ be a bicategory. We call *1-truncation of* $\mathcal{C}$, and denote $\tau_1\mathcal{C}$, the category with objects $\mathrm{Ob}(\mathcal{C})$ and with morphisms between two objects X and Y the isomorphism classes of 1-cells $X \to Y$, that is, equivalence classes of the 1-cells in $\mathcal{C}(X,Y)$ for the equivalence relation $f \sim f'$ if and only if there is an invertible 2-cell $f \Rightarrow f'$ in $\mathcal{C}$. It is immediate to see that this is a category.

We denote by $[f]$ the class in $\tau_1\mathcal{C}$ of a 1-cell $f \in \mathcal{C}$.

Proposition 3.0.6. *There is an obvious (pseudo)functor* $\mathcal{C} \to \tau_1\mathcal{C}$ *which is the identity on objects and maps* $f \mapsto [f]$ *on 1-cells and* $\alpha \mapsto \mathrm{Id}$ *on 2-cells, which is initial among functors from* $\mathcal{C}$ *to 1-categories.*

Definition 3.0.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor. We denote $\tau_1 F$, and call *1-truncation of* F, the functor $\tau_1\mathcal{C} \to \tau_1\mathcal{D}$ that maps every object $X \in \mathrm{Ob}(\mathcal{C})$ to its image $F(X)$ by F, and every class of 1-cell f in $\tau_1\mathcal{C}$ to the class of $F(f)$ in $\tau_1\mathcal{D}$. It is immediate to see that this is a functor.

Chapter 4

Global Mackey functors and precomposition

In this chapter, we describe a situation between the 2-category of finite groupoids and the bicategory of bimodules that evokes the 1-categorical Universal Property of span categories, see 2.1.8, which we used in chapter 2. Using a result of Balmer and Dell'Ambrogio in [1], we construct a pseudofunctor $\mathcal{R}$ between the bicategory of spans of groupoids and the bicategory of bimodules over groupoids, and show that its 1-truncation $\tau_1\mathcal{R}$ is a functor F satisfying the hypothesis of our first theorem 1.3.13. This is the content of our second Main Theorem of the introduction, that is theorem 4.2.12. A corollary of this theorem is the embedding of biset functors inside global Mackey functor. Both biset functors, introduced by Bouc (see [6]), and global Mackey functors, introduced by Webb (see [27]), can be seen as ways to extend the axiomatic study of finite groups representations from the subgroups of a given 'ambient' group G (as with usual Mackey functors) to *all* finite groups simultaneously. In [27], Webb already states the above-mentioned inclusion. It is later proven by Ganter in [10] in the same terminology, and independently by Nakaoka in [20], in a different setting that is equivalent to ours. Our proof enhances theirs by also considering the tensor structures. Moreover, it is simplified by using general arguments, in parallel with those used for cohomological Mackey functor in chapter 2.

Notation 4.0.1. Throughout, we make the hypothesis that groupoids are finite and the value sets of bimodules are finite, but most of our constructions hold under the weaker hypothesis that they are small. The finiteness is assumed in order to fit the finite groups/ finite bisets context of the last section of the chapter. Throughout, we will be using the basic notations and terminology of bicategory theory as recalled in chapter 3.

4.1 Three bicategories of groupoids

In this section we introduce three bicategories, $\mathcal{G}$, $\mathcal{S}$ and $\mathcal{B}$, whose objects are all finite groupoids. Here $\mathcal{G}$ is the usual 2-category of finite groupoids, $\mathcal{S}$ is a bicategory of spans in it, and $\mathcal{B}$ a bicategory of bimodules (a.k.a. profunctors or distributors).

Definition 4.1.1 (The 2-category of groupoids)**.** By $\mathcal{G}$, we denote the 2-category with objects finite groupoids, 1-cells functors between them, and 2-cells the natural transformations.

Remark 4.1.2. Since the composition of functors and both the horizontal and vertical composition of natural transformations in $\mathcal{G}$ are the classical ones, we omit the subscripts.

Proposition 4.1.3. *Equipped with usual compositions and identities for functors and natural transformations, $\mathcal{G}$ is indeed a 2-category.*

Proof. This is well known, so we only give an outline of the proof.

1. Let X and Y be two small groupoids, then $\mathcal{G}(X,Y)$ is the category of functors $X \to Y$ and natural transformations between them.

2. Let Z be a third small groupoid, then standard composition and the horizontal composition of natural transformations give the composition functor $\mathcal{G}(Y,Z) \times \mathcal{G}(X,Y) \to \mathcal{G}(X,Z)$, whose functoriality is given by the Interchange Law $(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha)$

3. Composition is associative on the nose, and the unitors are also identities.

Thus $\mathcal{G}$ is a 2-category. $\square$

Remark 4.1.4. Since a natural transformation is given by a collection of morphisms in a groupoid, and since these are all invertible, every natural transformation is a natural isomorphism.

Definition 4.1.5. A 2-cell in $\mathcal{G}$

$$\begin{tikzcd}
& b/a \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, phantom, "\overset{\Leftarrow}{\Lambda}"] & \\
B \arrow[dr, "b"'] & & A, \arrow[dl, "a"] \\
& C &
\end{tikzcd}$$

is an *iso-comma square* over the cospan (b, C, a) if it has the following universal property:

- For every 2-cell $\delta : a \circ f \Rightarrow b \circ g$ as below, there is exactly one functor h such that the following equality holds.

$$
\begin{tikzcd}
 & T \arrow[ddl, bend right, "g"'] \arrow[ddr, bend left, "f"] & \\
 & & \\
B \arrow[dr, "b"'] \arrow[rr, phantom, "\overset{\delta}{\Leftarrow}"] & & A \arrow[dl, "a"] \\
 & C &
\end{tikzcd}
=
\begin{tikzcd}
 & T \arrow[d, "h"'] \arrow[ddl, bend right, "g"'] \arrow[ddr, bend left, "f"] & \\
 & b/a \arrow[dl, "q"'] \arrow[dr, "p"] & \\
B \arrow[dr, "b"'] \arrow[rr, phantom, "\underset{\Lambda}{\Leftarrow}"] & & A \arrow[dl, "a"] \\
 & C &
\end{tikzcd}
$$

- For every pair of 2-cells $\tau_B : q \circ h' \Rightarrow q \circ h$ and $\tau_A : p \circ h' \Rightarrow p \circ h$ such that:

$$
\begin{tikzcd}
 & T \arrow[d, bend left, "h'"] \arrow[ddl, bend right, "q \circ h"'] & \\
\Leftarrow & b/a \arrow[dl, "q"'] \arrow[dr, "p"] & \\
B \arrow[dr, "b"'] \arrow[rr, phantom, "\Leftarrow"] & & A \arrow[dl, "a"] \\
 & C &
\end{tikzcd}
=
\begin{tikzcd}
 & T \arrow[d, bend right, "h"'] \arrow[ddr, bend left, "p \circ h'"] & \\
 & b/a \arrow[dl, "q"'] \arrow[dr, "p"] & \Leftarrow \\
B \arrow[dr, "b"'] \arrow[rr, phantom, "\Leftarrow"] & & A \arrow[dl, "a"] \\
 & C &
\end{tikzcd}
$$

  there exists a unique 2-cell $\tau : h' \Rightarrow h$ such that $q * \tau = \tau_B$ and $p * \tau = \tau_A$.

Proposition 4.1.6. *The 2-category $\mathcal{G}$ has all iso-comma squares: given a cospan $B \xrightarrow{b} C \xleftarrow{a} A$, let b/a be the category with objects (x, y, λ) where $y \in \mathrm{Ob}(B)$, $x \in \mathrm{Ob}(A)$ and $\lambda \in C(a(x), b(y))$. Given two objects, the Hom set $b/a((x, y, \lambda), (x', y', \lambda'))$ is the set of pairs $(u, v) \in A(x, x') \times B(y, y')$ such that the obvious square commutes in C: $\lambda' \circ a(u) = b(v) \circ \lambda$. Composition and identities are induced by those in A and B. Then, we define p, q and Λ by*

$$p(x, y, \lambda) = x \quad p(u, v) = u,$$

$$q(x, y, \lambda) = y \quad q(u, v) = v,$$

$$\Lambda_{(x,y,\lambda)} = \lambda.$$

Proof. One can easily check that

1. $h : T \to b/a$ is given by $h(t) = (g(t), f(t), \delta_t)$ and is the unique functor yielding the correct diagram, and

2. given 1-cells h and h' and 2-cells τ_A and τ_B as above, the natural transformation $\tau : h' \Rightarrow h$ with component at $t \in \mathrm{Ob}(T)$ given by

$$\tau_t = (\tau_{A,t}, \tau_{B,t}) \in b/a(h'(t), h(t))$$

is the only one doing the job.

This shows that b/a with Λ, p and q as defined give an iso-comma square. $\square$

Definition 4.1.7 (The bicategory of spans)**.** We denote by $\mathcal{S}$ the following bicategory.

1. The objects of $\mathcal{S}$ are finite groupoids: $\mathrm{Ob}(\mathcal{S}) = \mathrm{Ob}(\mathcal{G})$.

2. For any ordered pair of objects, the category $\mathcal{S}(X, Y)$ whose objects are spans of 1-cells $(Y \xleftarrow{f} Z \xrightarrow{g} X)$ in $\mathcal{G}$, which will sometimes be written as quintuplets (Y, f, Z, g, X), and whose morphisms are equivalence classes of triples

$$[\alpha, k, \beta] : (Y \xleftarrow{f} Z \xrightarrow{g} X) \to (Y \xleftarrow{f'} Z' \xrightarrow{g'} X),$$

where $k : Z \to Z'$ is a functor and both $\alpha : f \Rightarrow (f' \circ k)$ and $\beta : g \Rightarrow (g' \circ k)$ are natural transformations. Given two such triples (α, k, β) and (α_1, k_1, β_1), we have

$$(\alpha, k, \beta) \sim (\alpha_1, k_1, \beta_1)$$

if and only if there exists a natural isomorphism $\varphi : k \Rightarrow k_1$ such that the following equalities hold in $\mathcal{G}$:

$$(\mathrm{Id}_{f'} * \varphi) \circ \alpha = \alpha_1 \quad ; \quad (\mathrm{Id}_{g'} * \varphi) \circ \beta = \beta_1$$

It is often more practical, even though it is somewhat missleading, to deliver these data in the form of diagrams in $\mathcal{G}$:

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "k"] \arrow[ddl, phantom, "\Downarrow\alpha"] \arrow[ddr, phantom, "\Downarrow\beta"] & \\
Y & & X \\
 & Z \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

The identity morphism in $\mathcal{S}(X,Y)\big((Y \xleftarrow{f} Z \xrightarrow{g} X), (Y \xleftarrow{f} Z \xrightarrow{g} X)\big)$ is the class of the triple $(\mathrm{Id}_f, \mathrm{Id}_Z, \mathrm{Id}_g)$, and composition of morphisms is

$$[\alpha', k', \beta'] \circ_{\mathcal{S}} [\alpha, k, \beta] = [(\alpha' * k) \circ \alpha, k' \circ k, (\beta' * k) \circ \beta],$$

that is the following pasting:

$$
\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, "k"] \arrow[dr, "g"] \arrow[dl, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[dr, phantom, shift right=3ex, "\Downarrow\beta"] & \\
Y & Z' \arrow[l] \arrow[r] \arrow[d, "k'"] \arrow[dl, phantom, shift right=3ex, "\Downarrow\alpha'"] \arrow[dr, phantom, shift left=3ex, "\Downarrow\beta'"] & X \\
 & Z'' \arrow[ul, "f''"] \arrow[ur, "g''"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dd] \arrow[dr, "g"] & \\
Y \arrow[r, phantom, "\Downarrow(\alpha'*k)\circ\alpha"] & {} \arrow[r, phantom, "\Downarrow(\beta'*k)\circ\beta"] & X. \\
 & Z'' \arrow[ul, "f''"] \arrow[ur, "g''"'] &
\end{tikzcd}
$$

Since, by using the Interchange Law, one gets

$$
\begin{aligned}
((\alpha'' * (k' \circ k)) \circ (\alpha' * k)) \circ \alpha &= (((\alpha'' * k') * k) \circ (\alpha' * k)) \circ \alpha \\
&= (((\alpha'' * k') \circ \alpha') * k) \circ \alpha,
\end{aligned}
$$

vertical composition is associative.

3. For any ordered triple of objects X, Y, Z, the composition functor

$$\mathcal{S}(Y,Z) \times \mathcal{S}(X,Y) \to \mathcal{S}(X,Z)$$

is induced an iso-comma square as follows: it sends a pair of morphisms of spans as on the left to the morphism of spans as on the right:

$$
\begin{tikzcd}
\big((Z,b,U,g,Y),(Y,f,T,a,X)\big) \arrow[rr, mapsto] \arrow[d, "\big((\alpha_2,\ell,\beta_2),(\alpha_1,k,\beta_1)\big)"'] & & (Z, b\circ p_U, g/f, a\circ p_T, X) \arrow[d, "(\alpha_2 * p_U, c, \beta_1 * p_T)"] \\
\big((Z,b',U',g',Y),(Y,f',T',a',X)\big) & \text{|->} & (Z, b'\circ p_{U'}, g'/f', a'\circ p_{T'}, X),
\end{tikzcd}
$$

where c is the vertical functor given by the equality below,

$$
\begin{tikzcd}
 & g/f \arrow[dl, bend right, "\ell\circ p_U"'] \arrow[dr, bend left, "k\circ p_T"] & \\
U' \arrow[dr, "g'"'] & \Leftarrow & T' \arrow[dl, "f'"] \\
 & Y &
\end{tikzcd}
=
\begin{tikzcd}
 & g/f \arrow[d, "c"] \arrow[ddl, bend right, "\ell\circ p_U"'] \arrow[ddr, bend left, "k\circ p_T"] & \\
 & g'/f' \arrow[dl, "p_{T'}"'] \arrow[dr, "p_{U'}"] & \\
U' \arrow[dr] & \overset{\Lambda'}{\Leftarrow} & T', \arrow[dl] \\
 & Y &
\end{tikzcd}
$$

where the unnamed 2-cell is $(\beta_2 * p_U) \circ \Lambda \circ (\alpha_1^{-1} * p_T)$.

4. Given any object X, the identity functor I_X has $(X, \mathrm{Id}_X, X, \mathrm{Id}_X, X)$ for image in $\mathcal{S}(X, X)$.

5. For any object of $\mathcal{S}(T,U) \times \mathcal{S}(U,V) \times \mathcal{S}(W,V)$, that is a triple of composable spans, the associativity isomorphism is given by the Universal Property of iso-comma squares.

$$\begin{tikzcd}
 & & a/b \arrow[dl] \arrow[dr] & & b'/c \arrow[dl] \arrow[dr] & & \\
 & A \arrow[dl, "t"'] \arrow[dr, "a"] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & B \arrow[dl, "b"'] \arrow[dr, "b'"] \arrow[rr, phantom, "\overset{\beta}{\Leftarrow}"] & & C \arrow[dl, "c"'] \arrow[dr, "w"] & \\
T & & U & & V & & W
\end{tikzcd}$$

If one composes from left to right, i.e. by constructing a/b and the corresponding span first, one gets the following iso-comma square ρ, and a unique functor r is defined by the equality below and the universal property of iso-comma squares 4.1.5:

$$\begin{tikzcd}
 & (a/b)/c \arrow[dl] \arrow[dr] & \\
a/b \arrow[d] \arrow[dr] \arrow[rr, phantom, "\overset{\Leftarrow}{\rho}"] & & C \arrow[dl] \\
B \arrow[r] & V &
\end{tikzcd}
=
\begin{tikzcd}
 & (a/b)/c \arrow[dl] \arrow[d, "r"'] \arrow[dr] & \\
a/b \arrow[d] & b'/c \arrow[r] \arrow[dl] \arrow[d, phantom, "\overset{\beta}{\Leftarrow}"] & C \arrow[dl] \\
B \arrow[r] & V &
\end{tikzcd}$$

If one composes from right to left, a unique functor ℓ emerges:

$$\begin{tikzcd}
 & a/(b'/c) \arrow[dl] \arrow[dr] & \\
A \arrow[dr] \arrow[rr, phantom, "\overset{\lambda}{\Leftarrow}"] & & b'/c \arrow[dl] \arrow[d] \\
 & U & B \arrow[l]
\end{tikzcd}
=
\begin{tikzcd}
 & a/(b'/c) \arrow[r] \arrow[dl] \arrow[d, "\ell"] & b'/c \arrow[d] \\
A \arrow[dr] & a/b \arrow[l] \arrow[r] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] & B \arrow[dl] \\
 & U &
\end{tikzcd}$$

considering the following diagrams, a functor $h : a/(b'/c) \to (a/b)/c$ emerges:

$$\begin{tikzcd}
a/b \arrow[d] & a/(b'/c) \arrow[l, "\ell"'] \arrow[dl] \arrow[d, "q"'] \arrow[dr] & \\
B \arrow[dr] & b'/c \arrow[l] \arrow[r] \arrow[d, phantom, "\overset{\beta}{\Leftarrow}"] & C \arrow[dl] \\
 & V &
\end{tikzcd}
=
\begin{tikzcd}
 & a/(b'/c) \arrow[r] \arrow[dl, "\ell"'] \arrow[dr] & b'/c \arrow[d] \\
a/b \arrow[d] \arrow[dr] \arrow[rr, phantom, "\overset{\beta * q}{\Leftarrow}"] & & C \arrow[dl] \\
B \arrow[r] & V &
\end{tikzcd}
=
\begin{tikzcd}
 & a/(b'/c) \arrow[r] \arrow[dl] \arrow[d, "h"'] \arrow[dr] & b'/c \arrow[d] \\
a/b \arrow[d] \arrow[dr] & (a/b)/c \arrow[l] \arrow[r] \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & C \arrow[dl] \\
B \arrow[r] & V &
\end{tikzcd}$$

The unlabelled 2-arrow is $\beta * q$.

The same idea yields a functor $h' : (a/b)/c \to a/(b'/c)$:

$$
\begin{tikzcd}
& (a/b)/c \arrow[dl] \arrow[d, "p"'] \arrow[dr] \arrow[r, "r"'] & b'/c \arrow[d] \\
A \arrow[dr] & a/b \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow\alpha"] & B \arrow[dl] \\
& U &
\end{tikzcd}
=
\begin{tikzcd}
& (a/b)/c) \arrow[dl] \arrow[dr, "r"] & \\
A \arrow[dr] & \arrow[phantom, "\overset{\alpha * p}{\Leftarrow}"] & b'/c \arrow[dl] \arrow[d] \\
& U & B \arrow[l]
\end{tikzcd}
=
\begin{tikzcd}
a/b \arrow[d] & (a/b)/c) \arrow[l] \arrow[d, "h'"'] \arrow[dr, "r"] & \\
A \arrow[dr] & a/(b'/c) \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow\lambda"] & b'/c \arrow[dl] \arrow[d] \\
& U & B \arrow[l]
\end{tikzcd}
$$

The unlabelled 2-arrow is $\alpha * p$.

Now, let's check that h and h' are strictly mutually invertible: by the first property of iso-comma squares, the following equalities hold and yield $h \circ h' = \mathrm{Id}_{(a/b)/c}$ and $h' \circ h = \mathrm{Id}_{a/(b'/c)}$.

$$
\begin{tikzcd}
& (a/b)/c \arrow[ddl, bend right] \arrow[d, "h'"'] \arrow[dr, "r"] & \\
& a/(b'/c) \arrow[dl, "l"'] \arrow[d, "h"'] \arrow[dr] \arrow[r] & b'/c \arrow[d] \\
a/b \arrow[dr] & (a/b)/c \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow\ell"] & C \arrow[dl] \\
& V &
\end{tikzcd}
=
\begin{tikzcd}
& (a/b)/c \arrow[dl, bend right] \arrow[d, equal] \arrow[dr, bend left] & \\
a/b \arrow[dr] & (a/b)/c \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow\ell"] & C \arrow[dl] \\
& V &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& a/(b'/c) \arrow[dl, "l"'] \arrow[d, "h"'] \arrow[ddr, bend left] & \\
a/b \arrow[d] & (a/b)/c \arrow[l] \arrow[dl] \arrow[d, "h'"'] \arrow[dr, "r"] & \\
A \arrow[dr] & a/(b'/c) \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow\lambda"] & b'/c \arrow[dl] \\
& U &
\end{tikzcd}
=
\begin{tikzcd}
& a/(b'/c) \arrow[dl, bend right] \arrow[d, equal] \arrow[dr, bend left] & \\
A \arrow[dr] & a/(b'/c) \arrow[l] \arrow[r] \arrow[d, phantom, "\Leftarrow\lambda"] & b'/c \arrow[dl] \\
& U &
\end{tikzcd}
$$

Remark 4.1.8. • A proof that this is indeed a bicategory can be extracted from the proof of theorem 3.0.3 of [12]. As the title of the paper suggests, Hoffnung actually treats a more general case and proves there is more structure than what we have found useful for this work. A number of details about this bicategory can also be found in [1].

- The necessity of taking classes of triples (α, k, β) as 2-morphisms, instead of the triples themselves, appears when one wants to prove that $(X = X = X)$ is a unit for horizontal composition.
- One can check that a morphism of spans $[\alpha, k, \beta]$ is invertible if and only if k is an equivalence of categories. This is the content of lemma 5.1.12 of [1].

Definition 4.1.9 (The bicategory of bimodules)**.** We denote by $\mathcal{B}$ the bicategory defined by the following data.

1. The same class of objects as before, $\mathrm{Ob}(\mathcal{B}) = \mathrm{Ob}(\mathcal{G})$.
2. For any ordered pair of objects, a category $\mathcal{B}(X,Y)$ with objects functors $X^{op} \times Y \to \mathrm{Set}$, where Set is the category of finite sets, and morphisms natural transformations between them, with usual identities and composition.
3. For any three objects X, Y and Z, we get a composition functor

$$\circ_{\mathcal{B}} : \mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \to \mathcal{B}(X,Z)$$

by mapping

- any pair of bimodules $(\psi, \phi) \in \mathcal{B}(Y,Z) \times \mathcal{B}(X,Y)$ to the functor

$$\psi \circ_{\mathcal{B}} \varphi : X^{op} \times Z \longrightarrow \mathrm{Set}$$

defined on objects by

$$(x,z) \mapsto (\psi \circ_{\mathcal{B}} \varphi)(x,z) := \Big(\sqcup_{y \in \mathrm{Ob}(Y)} \psi(y,z) \times \varphi(x,y) \Big) / \approx$$

where the equivalence relation $\approx$ is that generated by

$$(v, \varphi(\mathrm{Id}, b)(u)) \approx (\psi(b, \mathrm{Id})(v), u),$$

with $x \in X$, $y, y' \in Y$, $z \in Z$, $u \in \varphi(x,y)$, $v \in \psi(y',z)$, and $b \in Y(y',y)$. We denote by $[v,u]$ the equivalence class of (v,u). On morphisms, $\psi \circ_{\mathcal{B}} \varphi$ is defined by

$$(a : x' \to x, c : z \to z') \mapsto ([v,u] \mapsto [\psi(\mathrm{Id}, c)(v), \varphi(a, \mathrm{Id})(u)]) \,.$$

- any pair of natural transformations $(\beta : \psi \Rightarrow \psi', \alpha : \varphi \Rightarrow \varphi')$ to the natural transformation $\beta *_{\mathcal{B}} \alpha : \psi \circ_{\mathcal{B}} \varphi \Rightarrow \psi' \circ_{\mathcal{B}} \varphi'$ with component at $(x,z) \in \mathrm{Ob}(X^{op} \times Z)$ given by

$$[v,u] \mapsto [\beta_{y,z}(v), \alpha_{x,y}(u)]$$

where $v \in \psi(y,z)$ and $u \in \varphi(x,y)$.

4. Associators are given by the correspondence

$$[[w,v],u] \mapsto [w,[v,u]]$$

5. The identity bimodule on X is its Hom functor $X(-,-)$, and the right and left unitors are given respectively by

$$\mathfrak{run}_{\mathcal{B}}(x,y)\ : [v,u] \in (\psi \circ X(-,-))(x,y) \mapsto \psi(u,\mathrm{Id})(v) \in \psi(x,y)$$

and

$$\mathfrak{lun}_{\mathcal{B}}(x,y)\ : [v,u] \in (Y(-,-) \circ \psi)(x,y) \mapsto \psi(\mathrm{Id},v)(u) \in \psi(x,y)\,.$$

Note that this are invertible, by definition of the quotient: for instance $\mathfrak{run}_{\mathcal{B}}(x,y)^{-1}\ : u \mapsto [u,\mathrm{Id}]$.

Remark 4.1.10. This is indeed a bicategory, which actually is the opposite bicategory of the one Borceux describes in ([5], 7.8.2). We chose here to call bimodule what Benabou and Borceux call distributor, and what others have called a profunctor. Note that the composite bimodule $\psi \circ_{\mathcal{B}} \phi$ is given by a coend in sets; compare the coends in $\Bbbk$-modules which were used in chapter 1.

4.2 The pseudofunctor of span realization

In this section we define a pseudofunctor $\mathcal{R} : \mathcal{S} \to \mathcal{B}$ which "realizes" every span of groupoid as a bimodule. Although not immediately obvious, our construction is also related to Nakaoka's "range" of a span, see [20].

Remark 4.2.1. In bicategory theory, most proofs are lengthy and require a lot of notations. Since there are only so many letters we want to use, we will often "reset" the notations from one proof to the other. We will try to make this clear when it is not self explanatory.

In a way very similar to what we did when considering spans of G-sets, we describe two ways of including the bicategory $\mathcal{G}$ in the bicategory $\mathcal{S}$ via pseudofunctors.

Definition 4.2.2. Let $\iota_* : \mathcal{G} \to \mathcal{S}$ be the following pseudofunctor.

1. To any groupoid X, it associates itself $\iota_*(X) = X$.
2. To a functor $f \in \mathcal{G}(X,Y)$, it associates the span $\iota_{*X,Y}(f) = (Y, f, X, \mathrm{Id}_X, X)$.
3. To a natural transformation $\alpha : f \Rightarrow g \in \mathcal{G}(X,Y)$, it associates the morphism of spans $\iota_{*X,Y}(\alpha) = (\alpha, \mathrm{Id}_X, \mathrm{Id}_{\mathrm{Id}_X})$.
4. For any groupoid X, the structure isomorphism $\phi_X : (X = X = X) \xrightarrow{\sim} \iota_{*X,X}(\mathrm{Id}_X)$ is simply the identity.

5. For any triple of groupoids (X, Y, Z) and any functors $f \in \mathcal{G}(X, Y)$ and $g \in \mathcal{G}(Y, Z)$, the structure isomorphism of spans $\phi_{f,g} : \iota_{*Y,Z}(g) \circ \iota_{*X,Y}(f) \xrightarrow{\sim} \iota_{*X,Z}(g \circ f)$ is given by $(g * \Lambda^{-1}, p_X, \mathrm{Id}_{p_X})$, where $\Lambda : f \circ p_X \Rightarrow p_Y$ is given by the iso-comma square from the composition:

$$\begin{tikzcd}
 & \mathrm{Id}_Y/f \arrow[dl, "g\circ p_Y"'] \arrow[dr, "p_X"] \arrow[dd, "p_X"] & \\
Z \arrow[r, phantom, "\overset{g*\Lambda^{-1}}{\Downarrow}"] & {} \arrow[r, phantom, "/\!/"] & X \\
 & X \arrow[ul, "g\circ f"] \arrow[ur, equal] &
\end{tikzcd}$$

Proposition 4.2.3. *ι_* is indeed a pseudofunctor.*

Proof. 1. For any groupoids X and Y, $\iota_{*X,Y}$ is a functor: one can easily check that for any $f, g, h : X \to Y$ and any $\alpha : f \Rightarrow g$, $\beta : g \Rightarrow h$, the following hold

$$\iota_{*X,Y}(\beta \circ \alpha) = \iota_{*X,Y}(\beta) \circ \iota_{*X,Y}(\alpha),$$
$$\iota_{*X,Y}(\mathrm{Id}_f) = \mathrm{Id}_{\iota_{*X,Y}(f)}.$$

2. For any groupoid X, $\iota_{*X,X}(\mathrm{Id}_X) = I_{\mathcal{S},X}(1) = (X, \mathrm{Id}_X, X, \mathrm{Id}_X, X)$.

3. The collection $\phi_{X,Y,Z}$ of all relevant morphisms of spans $\phi_{f,g}$ is a natural transformation $\circ_{\mathcal{S}}(\iota_{*Y,Z} \times \iota_{*X,Y}) \Rightarrow \iota_{*X,Z}(\circ_{\mathcal{G}})$. Indeed, the following holds for any pair of two cells $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$, with $X \xrightarrow{f} Y \xrightarrow{g} Z$:

$$\begin{aligned}
&\phi_{f',g'} \circ \big(\circ_{\mathcal{S}} (\iota_{*Y,Z} \times \iota_{*X,Y})(\beta, \alpha)\big) \\
&\qquad = (g' * \Lambda'^{-1}, p'_X, \mathrm{Id}_{p'_X}) \circ \big(\beta * p_Y, c, \mathrm{Id}_{p_X}\big) \\
&\qquad = \big((g' * \Lambda'^{-1} * c) \circ (\beta * p_Y), p'_X \circ c, \mathrm{Id}_{p_X}\big) \\
&\big(\iota_{*X,Z}\big(\circ_{\mathcal{G}} (\beta, \alpha)\big)\big) \circ \phi_{f,g} \\
&\qquad = (\beta * \alpha, \mathrm{Id}_X, \mathrm{Id}) \circ (g * \Lambda^{-1}, p_X, \mathrm{Id}_{p_X}) \\
&\qquad = \big((\beta * \alpha * p_X) \circ (g * \Lambda^{-1}), p_X, \mathrm{Id}_{p_X}\big),
\end{aligned}$$

where c is the unique relevant functor from Id_Y/f to Id_Y/f', obtained from the first property of iso-comma squares, see 4.1.5:

$$\begin{tikzcd}
 & Id_Y/f \arrow[dl, bend right, "p_Y"'] \arrow[dr, bend left, "p_X"] & \\
Y \arrow[rr, phantom, "\overset{\Lambda\circ(\alpha^{-1}*p_X)}{\Leftarrow}"] \arrow[dr, equal] & & X \arrow[dl, "f'"] \\
 & Y &
\end{tikzcd}
=
\begin{tikzcd}
 & Id_Y/f \arrow[d, "c"] \arrow[ddl, bend right, "p_Y"'] \arrow[ddr, bend left, "p_X"] & \\
 & Id_Y/f' \arrow[dl, "p'_Y"'] \arrow[dr, "p'_X"] & \\
Y \arrow[rr, phantom, "\underset{\Lambda'}{\Leftarrow}"] \arrow[dr, equal] & & X \arrow[dl, "f'"] \\
 & Y &
\end{tikzcd}$$

The functors $p'_X \circ c$ and p_X are equal by definition of c, and the left-hand side natural transformations are equal because, using the Interchange Law and the definition of c, we obtain respectively $\beta * (\Lambda'^{-1} * c)$ and $\beta * ((\alpha * p_X) \circ \Lambda^{-1})$, which are equal, again by definition of c.

4. We now need to check that the hexagon axiom holds. Be aware that we "reset" the notations for this part of the proof.

   Given three functors $(f, g, h) \in \mathcal{G}(X, Y) \times \mathcal{G}(Y, Z) \times \mathcal{G}(Z, T)$, and since functor composition is associative on the nose, we have to check that the following diagram is commutative

$$\begin{tikzcd} \Big((T \xleftarrow{h} Z = Z) \circ_{\mathcal{S}} (Z \xleftarrow{g} Y = Y)\Big) \circ_{\mathcal{S}} (Y \xleftarrow{f} X = X) \arrow[d, shift right, Rightarrow, "\phi_{g\circ f,h} \circ_{\mathcal{S}} \left(\iota_{*Z,T}(h) *_{\mathcal{S}} \phi_{f,g}\right) \circ_{\mathcal{S}} \mathfrak{a}_{\mathcal{S}}"'] \arrow[d, shift left, Rightarrow, "\phi_{f,h\circ g} \circ_{\mathcal{S}} \left(\phi_{g,h} *_{\mathcal{S}} \left(\iota_{*X,Y}(f)\right)\right)"] \\ (T \xleftarrow{h\circ g\circ f} X = X). \end{tikzcd}$$

   In order to do so, we need to explicit the different 2-cells involved. We begin by the right-hand side. First, consider the three following iso-comma squares:

$$\begin{tikzcd}[column sep=small] & \mathrm{Id}_Z/g \arrow[dl, "p_Z"'] \arrow[dr, "p_Y"] & \\ Z \arrow[dr, equal] \arrow[rr, phantom, "\overset{\Leftarrow}{\Lambda}"] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \quad ; \quad \begin{tikzcd}[column sep=small] & p_Y/f \arrow[dl, "q"'] \arrow[dr, "q_X"] & \\ \mathrm{Id}_Z/g \arrow[dr, "p_Y"'] \arrow[rr, phantom, "\overset{\Leftarrow}{\Delta}"] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd} \quad ; \quad \begin{tikzcd}[column sep=small] & \mathrm{Id}_Y/f \arrow[dl, "p'_Y"'] \arrow[dr, "p'_X"] & \\ Y \arrow[dr, equal] \arrow[rr, phantom, "\overset{\Leftarrow}{\Lambda'}"] & & X. \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

   Then, the source of $\Big(\phi_{g,h} *_{\mathcal{S}} \left(\iota_{*X,Y}(f)\right)\Big)$ is the span

$$(T \xleftarrow{h \circ p_Z \circ q} p_Y/f \xrightarrow{q_X} X),$$

   and the target is the span

$$(T \xleftarrow{h \circ g \circ p'_Y} \mathrm{Id}_Y/f \xrightarrow{p'_X} X).$$

   Hence, by definition of $\phi_{g,h}$, the 2-cell corresponding to $\Big(\phi_{g,h} *_{\mathcal{S}} \left(\iota_{*X,Y}(f)\right)\Big)$ is

$$\begin{tikzcd} & p_Y/f \arrow[dl, "h\circ p_Z\circ q"'] \arrow[dr, "q_X"] \arrow[dd, "c"] & \\ T \arrow[r, phantom, "\overset{h*\Lambda^{-1}*q}{\Downarrow}"] & \arrow[r, phantom, "/\!/"] & X \\ & \mathrm{Id}_Y/f \arrow[ul, "h\circ g\circ p'_Y"] \arrow[ur, "p'_X"'] & \end{tikzcd}$$

where c is the functor defined by the equality:

$$
\begin{tikzcd}
 & p_Y/f \arrow[dl, "p_Y\circ q"'] \arrow[dr, "q_X"] & \\
Y \arrow[dr, "\mathrm{Id}_Y"'] \arrow[rr, phantom, "\overset{\Delta}{\Leftarrow}"] & & X \arrow[dl, "f"] \\
 & Y &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & p_Y/f \arrow[d, "c"] \arrow[ddl, bend right, "p_Y\circ q"'] \arrow[ddr, bend left, "q_X"] & \\
 & \mathrm{Id}_Y/f \arrow[dl, "p'_Y"'] \arrow[dr, "p'_X"] & \\
Y \arrow[dr, "\mathrm{Id}_Y"'] \arrow[rr, phantom, "\overset{\Lambda'}{\Leftarrow}"] & & X. \arrow[dl, "f"] \\
 & Y &
\end{tikzcd}
$$

The 2-cell corresponding to $\phi_{f,h\circ g}$ is

$$
\begin{tikzcd}
 & \mathrm{Id}_Y/f \arrow[dl, "h\circ g\circ p'_Y"'] \arrow[dr, "p'_X"] \arrow[dd, "p'_X"] & \\
T \arrow[r, phantom, "\overset{h*g*\Lambda'^{-1}}{\Rightarrow}"] & \arrow[r, phantom, "//"] & X \arrow[dl, equal] \\
 & X \arrow[ul, "h\circ g\circ f"] &
\end{tikzcd}
$$

by defintion, and we can now compute the right hand side:

$$
\begin{aligned}
& (h * g * \Lambda'^{-1},\, p'_X,\, \mathrm{Id}_{p'_X}) \circ_{\mathcal{S}} (h * \Lambda^{-1} * q,\, c,\, \mathrm{Id}_{q_X}) \\
= & \Big((h * g * \Lambda'^{-1} * c) \circ (h * \Lambda^{-1} * q),\, p'_X \circ c,\, \mathrm{Id}_{q_X}\Big) \\
= & \Big(h * \big((g * \Lambda'^{-1} * c) \circ (\Lambda^{-1} * q)\big),\, q_x,\, \mathrm{Id}_{q_X}\Big).
\end{aligned}
$$

Now, on the left hand side, $\mathfrak{a}_{\mathcal{S},(f,g,h)}$ has

$$(T \xleftarrow{h} Z = Z) \circ_{\mathcal{S}} \Big((Z \xleftarrow{g} Y = Y) \circ_{\mathcal{S}} (Y \xleftarrow{f} X = X)\Big)$$

for target. By considering the following iso-comma square

$$
\begin{tikzcd}
 & \mathrm{Id}_Z/(g\circ p'_Y) \arrow[dl, "q_Z"'] \arrow[dr, "q_1"] & \\
Z \arrow[dr, equal] \arrow[rr, phantom, "\underset{\Delta_1}{\Leftarrow}"] & & \mathrm{Id}_Y/f, \arrow[dl, "g\circ p'_Y"] \\
 & Z &
\end{tikzcd}
$$

we can see that the span in question is $(T \xleftarrow{h\circ q_Z} \mathrm{Id}_Z/(g\circ p'_Y) \xrightarrow{p'_X\circ q_1} X)$.

We define a functor a via the universal property of iso-comma squares:

$$
\begin{tikzcd}
 & p_Y/f \arrow[dl, "p_Z\circ q"'] \arrow[dr, "c"] & \\
Z \arrow[rr, phantom, "\overset{\Leftarrow}{\Lambda * q}"] \arrow[dr, equal] & & \mathrm{Id}_Y/f \arrow[dl, "g\circ p'_Y"] \\
 & Z &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & p_Y/f \arrow[dll, "q"'] \arrow[d, "a"] \arrow[ddr, bend left, "c"] & \\
\mathrm{Id}_Z/g \arrow[d, "p_Z"'] & & \mathrm{Id}_Z/(g\circ p'_Y) \arrow[dll, "q_Z"'] \arrow[dr, "q_1"] & \\
Z \arrow[rrr, phantom, "\overset{\Leftarrow}{\Delta_1}"] \arrow[drr, equal] & & & \mathrm{Id}_Y/f, \arrow[dl, "g\circ p'_Y"] \\
 & & Z &
\end{tikzcd}
$$

and we obtain the 2-cell corresponding to $\mathfrak{a}_{\mathcal{S},(f,g,h)}$, that is:

$$
\begin{tikzcd}
 & & p_Y/f \arrow[dll, "h\circ p_Z\circ q"'] \arrow[dd, "a"] \arrow[drr, "q_X"] & & \\
T \arrow[rr, phantom, "\Vert"] & & & & X. \arrow[ll, phantom, "\Vert"'] \\
 & & \mathrm{Id}_Z/(g\circ p'_Y) \arrow[ull, "h\circ q_Z"] \arrow[urr, "p'_X\circ q_1"'] & &
\end{tikzcd}
$$

The 2-cell $\iota_{*_{Z,T}}(h) *_{\mathcal{S}} \phi_{f,g}$ is by definition $(\mathrm{Id}_h,\ \mathrm{Id}_Z,\ \mathrm{Id}_{\mathrm{Id}_Z}) *_{\mathcal{S}} (g*\Lambda'^{-1},\ p'_X,\ \mathrm{Id}_{p'_X})$. We use the universal property of the following iso-comma square in order to define the relevant functor $\mathrm{Id}_Z/(g\circ p'_Y) \xrightarrow{d} \mathrm{Id}_Z/(g\circ f)$:

$$
\begin{tikzcd}
 & \mathrm{Id}_Z/(g\circ f) \arrow[dl, "r_Z"'] \arrow[dr, "r_X"] & \\
Z \arrow[rr, phantom, "\overset{\Leftarrow}{\Delta'_1}"] \arrow[dr, equal] & & X, \arrow[dl, "g\circ f"] \\
 & Z &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathrm{Id}_Z/(g\circ p'_Y) \arrow[dl, "q_Z"'] \arrow[dr, "p'_X\circ q_1"] & \\
Z \arrow[rr, phantom, "\underset{\Leftarrow}{\Delta_1\circ(g*\Lambda'*q_1)}"] \arrow[dr, equal] & & X \arrow[dl, "g\circ f"] \\
 & Z &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathrm{Id}_Z/(g\circ p'_Y) \arrow[ddl, bend right, "q_Z"'] \arrow[d, "d"] \arrow[ddr, bend left, "p'_X\circ q_1"] & \\
 & \mathrm{Id}_Z/(g\circ f) \arrow[dl, "r_Z"'] \arrow[dr, "r_X"] & \\
Z \arrow[rr, phantom, "\underset{\Leftarrow}{\Delta'_1}"] \arrow[dr, equal] & & X. \arrow[dl, "g\circ f"] \\
 & Z &
\end{tikzcd}
$$

Now, we can give the 2-cell corresponding to $\iota_{*Z,T}(h) *_{\mathcal{S}} \phi_{f,g}$, that is:

$$\begin{tikzcd}
& \mathrm{Id}_Z/(g\circ p'_Y) \arrow[dl, "h\circ q_Z"'] \arrow[dr, "p'_X\circ q_1"] \arrow[dd, "d"] & \\
T \arrow[r, phantom, "\mathbin{\backslash\!\backslash}"] & {} \arrow[r, phantom, "/\!/"] & X. \\
& \mathrm{Id}_Z/(g\circ f) \arrow[ul, "h\circ r_Z"] \arrow[ur, "r_X"'] &
\end{tikzcd}$$

We now have all the elements for computing the left hand side:

$$\begin{aligned}
&\phi_{g\circ f,h} \circ_{\mathcal{S}} \left(\iota_{*Z,T}(h) *_{\mathcal{S}} \ \phi_{f,g}\right) \circ_{\mathcal{S}} \ \mathfrak{a}_{\mathcal{S},(f,g,h)} \\
&= ((h * \Delta_1')^{-1},\ r_X,\ \mathrm{Id}_{r_X}) \circ_{\mathcal{S}} (\mathrm{Id}_{h\circ q_Z},\ d,\ \mathrm{Id}_{r_X\circ d}) \circ_{\mathcal{S}} (\mathrm{Id}_{h\circ p_Z\circ q},\ a,\ \mathrm{Id}_{q_X}) \\
&= \Big((h * \Delta_1')^{-1} * d,\ r_X \circ d,\ \mathrm{Id}_{r_X\circ d}\Big) \circ_{\mathcal{S}} (\mathrm{Id}_{h\circ p_Z\circ q},\ a,\ \mathrm{Id}_{q_X}) \\
&= \Big((h * \Delta_1')^{-1} * d * a,\ q_X,\ \mathrm{Id}_{q_X}\Big).
\end{aligned}$$

In order to check that the left hand side of our diagram is equal to the right hand side, we now only need to check that $h*((g*\Lambda'^{-1}*c)\circ(\Lambda^{-1}*q))$ and $(h*\Delta_1')^{-1}*d*a$ are the same natural isomorphisms. We do so, using the definition of d for the first step, and the definition of a for the last one:

$$\begin{aligned}
h * (\Delta_1' * d)^{-1} * a &= h * \left(\Delta_1 \circ (g * \Lambda' * q_1)\right)^{-1} * a \\
&= h * \left((g * \Lambda'^{-1} * q_1) \circ \Delta_1^{-1}\right) * a \\
&= h * \left((g * \Lambda'^{-1} * q_1 * a) \circ (\Delta_1^{-1} * a)\right) \\
&= h * \left((g * \Lambda'^{-1} * c) \circ (\Lambda^{-1} * q)\right).
\end{aligned}$$

5. The two remaining axioms hold, since for $f \in \mathcal{G}(X,Y)$, $\mathfrak{run}_{\mathcal{S},f} = \ \phi_{\mathrm{Id}_X,f}$ and $\mathfrak{lun}_{\mathcal{S},f} = \phi_{f,\mathrm{Id}_Y}$, and all other arrows are identities.

This ends the proof of the pseudo functoriality of ι_*. □

Definition 4.2.4. Let $\iota^* : \mathcal{G}^{op} \to \mathcal{S}$ be the pseudofunctor defined as follows.

1. To any groupoid X, it associates a groupoid $\iota^*(X) = X$.

2. To a functor $f \in \mathcal{G}(X,Y)$, it associates a span $\iota^*_{X,Y}(f) = (X, \mathrm{Id}_X, X, f, Y)$.

3. To a natural transformation $\alpha : f \Rightarrow g \in \mathcal{G}(X,Y)$, it associates a morphism of spans $\iota^*_{X,Y}(\alpha) = (\mathrm{Id}_{\mathrm{Id}_X}, \mathrm{Id}_X, \alpha)$.

4. For any groupoid X, let $\phi^*_X = \mathrm{Id}_{\iota^*_{X,X}(\mathrm{Id}_X)}$.

5. For any triple of groupoids (X, Y, Z) and any functors $f \in \mathcal{G}(X, Y)$ and $g \in \mathcal{G}(Y, Z)$, a morphism of spans $\phi^*_{f,g} : \iota^*_{Y,X}(f) \circ \iota^*_{Z,Y}(g) \to \iota^*_{Z,X}(g \circ f)$ given by $(\mathrm{Id}_{p_X}, p_X, g*\Lambda)$, where $\Lambda : p_Y \Rightarrow f \circ p_X$ is given by the iso-comma square from the composition:

$$\begin{tikzcd}
& f/\mathrm{Id}_Y \arrow[dl, "p_X"'] \arrow[dr, "g \circ p_Y"] \arrow[dd, "p_X"] & \\
X \arrow[dr, equal] \arrow[r, phantom, "\mathbin{\rotatebox{45}{$=$}}"] & \arrow[r, phantom, "\Swarrow g*\Lambda"] & Z \\
& X \arrow[ur, "g \circ f"'] &
\end{tikzcd}$$

Proposition 4.2.5. *ι^* is indeed a pseudofunctor.*

Proof. One can check directly that this is the case, as we did in the proof of 4.2.3. Another method is to consider the following evident pseudofunctor:

$$(-)^\vee : \mathcal{S}^{op} \to \mathcal{S}$$

$$X^\vee = X$$

$$(Y \xleftarrow{g} U \xrightarrow{f} X)^\vee = (X \xleftarrow{f} U \xrightarrow{g} Y)$$

$$(\beta, k, \alpha)^\vee = (\alpha, k, \beta)$$

Given two composable spans $(Y \xleftarrow{g} U \xrightarrow{f} X)$ and $(Z \xleftarrow{k} V \xrightarrow{h} Y)$, the structural morphism $\phi_{(-)^\vee\,(X,Y,Z)}$ sends $(Y \xleftarrow{g} U \xrightarrow{f} X)^\vee \circ_{\mathcal{S}} (Z \xleftarrow{k} V \xrightarrow{h} Y)^\vee$ to the span $(X \xleftarrow{f \circ p_U} h/g \xrightarrow{k \circ p_V} Z)$ via $g/h \to h/g$, $(v, u, \lambda) \mapsto (u, v, \lambda)$.

Then, one can check that $(-)^\vee \circ \iota_* = \iota^*$, which ends the proof. $\square$

Now, we want to do something similar with the bicategory $\mathcal{B}$ of bimodules: we are going to send $\mathcal{G}$ to $\mathcal{B}$ in two ways, one covariant and one contravariant. The two first families of examples that come to one's mind when reading the definition of bimodules may be, given a functor $X \xrightarrow{f} Y$:

$$Y(f-, -) \colon X^{op} \times Y \to \mathrm{Set} \quad \text{and} \quad Y(-, f-) \colon Y^{op} \times X \to \mathrm{Set}\,.$$

We show that these families induce two pseudofunctors $\mathcal{R}_* : \mathcal{G} \to \mathcal{B}$ and $\mathcal{R}^* : \mathcal{G}^{op} \to \mathcal{B}$. It is also natural to expect an adjunction relation inside the bicategory $\mathcal{B}$ of bimodules between these bimodules, and we prove it is as expected.

Definition 4.2.6. Let $\mathcal{R}_* : \mathcal{G} \to \mathcal{B}$ be the following pseudofunctor:

1. To any groupoid X, associate $\mathcal{R}_*(X) = X$.
2. To any functor $f \in \mathcal{G}(X, Y)$, associate $\mathcal{R}_*(f) := Y(f-, -)$.

3. To any natural transformation $\alpha : f \Rightarrow f' \in \mathcal{G}(X,Y)$, associate the natural transformation $\mathcal{R}_*(\alpha) : \mathcal{R}_*(f) \Rightarrow \mathcal{R}_*(f')$, given at (x,y) by
$$u \in \ Y(fx,y) \mapsto u \circ \alpha_x^{-1} \in Y(f'x,y).$$

4. Given three groupoids X, Y, and Z, the structure isomorphism $\phi_{\mathcal{R}_*\,(X,Y,Z)}$ for two functors $f \in \mathcal{G}(X,Y)$ and $g \in \mathcal{G}(Y,Z)$ is the natural transformation $\mathcal{R}_*(g) \circ_{\mathcal{B}} \mathcal{R}_*(f) \Rightarrow \mathcal{R}_*(g \circ f)$ with component at $(x,z) \in X^{op} \times Z$ given by
$$\begin{aligned} \phi_{\mathcal{R}_*\,(g,f)}(x,z) : (\mathcal{R}_*(g) \circ_{\mathcal{B}} \mathcal{R}_*(f))(x,z) &\to \mathcal{R}_*(g \circ f)(x,z) \\ [v,u] &\mapsto v \circ g(u). \end{aligned}$$

5. Given any groupoid X, the structure isomorphism
$$\phi_{\mathcal{R}_*\,X} : X(\mathrm{Id}_X -, -) \to X(-,-)$$
is given by the identity.

Proposition 4.2.7. *$\mathcal{R}_* : \mathcal{G} \to \mathcal{B}$ is indeed a pseudofunctor.*

Proof.

1. It is fairly obvious that $Y(f-,-)$ is a bimodule $X^{op} \times Y \to \mathrm{Set}$ and that $- \circ \alpha_x^{-1}$ gives a natural transformation
$$\mathcal{R}_*(\alpha) : Y(f-,-) \Rightarrow Y(f'-,-).$$

2. We prove that for any two groupoids X and Y, we have a functor:
$$\mathcal{R}_{*\,X,Y} \ : \ \mathcal{G}(X,Y) \ \to \ \mathcal{B}(X,Y).$$
   - Let $\alpha : f \Rightarrow f'$ and $\alpha' : f' \Rightarrow f''$ be in $\mathcal{G}(X,Y)$. Since, for any $x \in \mathrm{Ob}(X)$, $y \in \mathrm{Ob}(Y)$ and $u \in Y(fx,y)$, we have
   $$u \circ (\alpha' \circ \alpha)_x^{-1} = u \circ \alpha_x^{-1} \circ {\alpha'_x}^{-1},$$
   it is clear that $\mathcal{R}_{*\,X,Y}(\alpha' \circ \alpha) = \mathcal{R}_{*\,X,Y}(\alpha') \circ \mathcal{R}_{*\,X,Y}(\alpha)$.
   - It is clear that $\mathcal{R}_{*\,X,Y}(\mathrm{Id}_f) = \mathrm{Id}_{\,Y(f-,-)}$.

3. We verify that $\phi_{\mathcal{R}_*\ (X,Y,Z)}$ is well-defined. Consider functors $f : X \to Y$ and $g : Y \to Z$, four objects $x \in \mathrm{Ob}(X)$, $z \in \mathrm{Ob}(Z)$ and $y, y' \in \mathrm{Ob}(Y)$, and three morphisms $u \in Y(fx,y')$, $b \in Y(y',y)$ and $v \in Z(gy,z)$. We recall that according to the definition of $\circ_{\mathcal{B}}$, we have equalities
$$[v, b \circ u] = [v \circ g(b), u] \in \big(\mathcal{R}_*(g) \circ_{\mathcal{B}} \mathcal{R}_*(f)\big)(x,z),$$
and these relations define the quotient set on the right-hand side. Hence, to prove that $\phi_{\mathcal{R}_*\,(f,g)}(x,z)$ is well defined, it suffices to check that
$$\phi_{\mathcal{R}_*\,(f,g)}(x,z)([v, b \circ u]) = \phi_{\mathcal{R}_*\,(f,g)}(x,z)([v \circ g(b), u]),$$

which is immediate. Furthermore, we can see that $\phi_{\mathcal{R}_*\,(f,g)}(x,z)$ is always an isomorphism: indeed, one can check that $(g \circ f(x) \xrightarrow{a} z) \mapsto [a, \mathrm{Id}_{fx}]$ is an inverse isomorphism, since we always have $[v,u] = [v \circ g(u), \mathrm{Id}_{fx}]$.

We now check the naturality of $\phi_{\mathcal{R}_*\,(X,Y,Z)}$, that is, for any pair of 2-cells $\alpha : f \Rightarrow f' \in \mathcal{G}(X,Y)$ and $\beta : g \Rightarrow g' \in \mathcal{G}(Y,Z)$, the commutativity of the square

$$\begin{tikzcd}
\mathcal{R}_*(g) \circ_{\mathcal{B}} \mathcal{R}_*(f) \arrow[r, "\phi_{\mathcal{R}_*\,(f,g)}"] \arrow[d, "\mathcal{R}_*(\beta) *_{\mathcal{B}} \mathcal{R}_*(\alpha)"'] & \mathcal{R}_*(g \circ f) \arrow[d, "\mathcal{R}_*(\beta * \alpha)"] \\
\mathcal{R}_*(g') \circ_{\mathcal{B}} \mathcal{R}_*(f') \arrow[r, "\phi_{\mathcal{R}_*\,(f',g')}"'] & \mathcal{R}_*(g' \circ f').
\end{tikzcd}$$

Let $v \in Z(gy,z)$ and $u \in Y(fx,y)$. One path sends $[v,u]$ to $v \circ g(u) \circ (\beta * \alpha)_x^{-1}$ and the other sends $[v,u]$ to $v \circ \beta_y^{-1} \circ g'(u \circ \alpha_x^{-1})$: it remains to check that those two morphisms in $Z(g \circ f(x), z)$ are equal:

$$\begin{aligned} v \circ \beta_y^{-1} \circ g'(u) \circ g'(\alpha_x^{-1}) &= v \circ g(u) \circ \beta_{fx}^{-1} \circ g'(\alpha_x^{-1}) \\ &= v \circ g(u) \circ (\beta * \alpha)_x^{-1}. \end{aligned}$$

4. The "hexagon axiom" holds: indeed, given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T \in \mathcal{G}$, the commutativity of

$$\begin{tikzcd}
\Big(T(h-,-) \circ_{\mathcal{B}} Z(g-,-)\Big) \circ_{\mathcal{B}} Y(f-,-) \arrow[d, shift right, "\phi_{\mathcal{R}_*\,g\circ f,h} \circ_{\mathcal{B}} \left(\mathrm{Id}_{\mathcal{R}_*(h)} *_{\mathcal{B}} \phi_{\mathcal{R}_*\,f,g}\right) \circ_{\mathcal{B}} \mathfrak{a}_{\mathcal{B}}"'] \arrow[d, shift left, "\phi_{\mathcal{R}_*\,f,h\circ g} \circ_{\mathcal{B}} \left(\phi_{\mathcal{R}_*\,g,h} *_{\mathcal{B}} \mathrm{Id}_{\mathcal{R}_*(f)}\right)"] \\
T(h \circ g \circ f-,-).
\end{tikzcd}$$

boils down to checking that $w \circ h(v) \circ (h \circ g)(u) = w \circ h(v \circ g(u))$, which is immediate by functoriality of h.

5. It is immediate from the definitions that the last two axioms hold.

This ends the proof that $\mathcal{R}_*$ is a pseudofunctor. ☐

Proposition 4.2.8. *For every functor $f \in \mathcal{G}(X,Y)$, there is an adjunction $\mathcal{R}_*(f) \dashv \mathcal{R}^*(f)$, where $\mathcal{R}^*(f) := Y(-, f-) : Y^{op} \times X \to \mathrm{Set}$, inside the bicategory $\mathcal{B}$ in the sense of 3.0.4.*

Proof. Let $f \in \mathcal{G}(X,Y)$ be a functor.

- Let $(x,x') \in \mathrm{Ob}(X^{op} \times X)$. We define a map $\eta_f(x,x')$:

$$X(x,x') \to \big(\mathcal{R}^*(f) \circ_{\mathcal{B}} \mathcal{R}_*(f)\big)(x,x')$$

$$a \mapsto [\mathrm{Id}_{fx}, f(a)^{-1}].$$

We set $\eta_f := \{\eta_f(x,x')|(x,x') \in X^{op} \times X\}$, and check that it is a 2-cell of $\mathcal{B}(X,X)$: for any two morphisms $b \in X(x_1,x)$ and $b' \in X(x',x'_1)$, the following square is commutative

$$\begin{tikzcd} X(x,x') \arrow[r, "\eta_f(x,x')"] \arrow[d, "X(b,b')"'] & \big(\mathcal{R}^*(f) \circ_{\mathcal{B}} \mathcal{R}_*(f)\big)(x,x') \arrow[d, "\big(\mathcal{R}^*(f)\circ_{\mathcal{B}}\mathcal{R}_*(f)\big)(b,b')"] \\ X(x_1,x'_1) \arrow[r, "\eta_f(x_1,x'_1)"'] & \big(\mathcal{R}^*(f) \circ_{\mathcal{B}} \mathcal{R}_*(f)\big)(x_1,x'_1), \end{tikzcd}$$

since $[\mathrm{Id}_{f(x_1)},\, f(b' \circ a \circ b)^{-1}] = [f(b)^{-1},\, f(a)^{-1} \circ f(b')^{-1}]$ in the quotient $\big(\mathcal{R}^*(f) \circ_{\mathcal{B}} \mathcal{R}_*(f)\big)(x_1,x'_1)$, which is immediate.

- Let $y, y' \in \mathrm{Ob}(Y)$, we define a map $\epsilon_f(y,y')$:

$$\big(\mathcal{R}_*(f) \circ_{\mathcal{B}} \mathcal{R}^*(f)\big)(y,y') \to Y(y,y')$$
$$[v,\, u] \mapsto (v \circ u)^{-1}$$

We set $\epsilon_f := \{\epsilon_f(y,y')|(y,y') \in Y^{op} \times Y\}$, and check that it is a 2-cell of $\mathcal{B}(Y,Y)$: for any two morphism $c \in Y(y_1,y)$ and $c' \in Y(y',y'_1)$, the following square is commutative

$$\begin{tikzcd} \big(\mathcal{R}_*(f) \circ_{\mathcal{B}} \mathcal{R}^*(f)\big)(y,y') \arrow[r, "\epsilon_f(y,y')"] \arrow[d, "\big(\mathcal{R}_*(f)\circ_{\mathcal{B}}\mathcal{R}^*(f)\big)(c,c')"'] & Y(y,y') \arrow[d, "Y(c,c')"] \\ \big(\mathcal{R}_*(f) \circ_{\mathcal{B}} \mathcal{R}^*(f)\big)(y_1,y'_1) \arrow[r, "\epsilon_f(y,y')"'] & Y(y_1,y'_1), \end{tikzcd}$$

since $\big((c^{-1} \circ v) \circ (u \circ c'^{-1})\big)^{-1} = c' \circ (v \circ u)^{-1} \circ c$.

- Now let us check that η_f and ϵ_f satisfy the triangular equalities for an adjunction. For any $(x,y) \in \mathrm{Ob}(X^{op} \times Y)$, consider the following composition of maps:

$$\begin{array}{llll} \mathcal{R}_*(f)(x,y) & \xrightarrow{\mathfrak{run}^{-1}} (\mathcal{R}_*(f) \circ_{\mathcal{B}} X(-,-))(x,y) & a \mapsto [a, \mathrm{Id}_x] \\ & \xrightarrow{\mathrm{Id}*_{\mathcal{B}}\eta_f} (\mathcal{R}_*(f) \circ_{\mathcal{B}} (\mathcal{R}^*(f) \circ_{\mathcal{B}} \mathcal{R}_*(f)))(x,y) & \mapsto [a, [\mathrm{Id}_{fx}, \mathrm{Id}_{fx}]] \\ & \xrightarrow{\mathfrak{a}} ((\mathcal{R}_*(f) \circ_{\mathcal{B}} \mathcal{R}^*(f)) \circ_{\mathcal{B}} \mathcal{R}_*(f))(x,y) & \mapsto [[a, \mathrm{Id}_{fx}], \mathrm{Id}_{fx}] \\ & \xrightarrow{\epsilon_f*_{\mathcal{B}}\mathrm{Id}} ((-,-) \circ_{\mathcal{B}} \mathcal{R}_*(f))(x,y) & \mapsto [a, \mathrm{Id}_{fx}] \\ & \xrightarrow{\mathfrak{lun}} \mathcal{R}_*(f)(x,y) & \mapsto a \end{array}$$

The second condition is equally straightforward to check, and ends the proof. □

Definition 4.2.9. Let $\mathcal{R}^* : \mathcal{G}^{op} \to \mathcal{B}$ be the following pseudofunctor:

1. To any groupoid X, associate itself.
2. To any functor $f \in \mathcal{G}(X, Y)$, associate $\mathcal{R}^*(f) := Y(-, f-)$.
3. To any natural transformation $\alpha : f \Rightarrow f' \in \mathcal{G}(X, Y)$, associate the natural transformation $\mathcal{R}^*(\alpha) : \mathcal{R}^*(f) \Rightarrow \mathcal{R}^*(f')$, given at (y, x) by
$$u \in Y(y, fx) \mapsto \alpha_x \circ u.$$
4. Given any three groupoids X, Y, and Z, a natural transformation $\phi_{\mathcal{R}^*\,(Z,Y,X)}$, such that given two functors $f \in \mathcal{G}(X, Y)$ and $g \in \mathcal{G}(Y, Z)$ and objects $z \in Z$ and $x \in X$, we get a map
$$\phi_{\mathcal{R}^*\,(g,f)}(z, x) : (R^*(f) \circ_{\mathcal{B}} R^*(g))(z, x) \to R^*(g \circ f)(z, x),$$
$$[u, v] \mapsto g(u) \circ v.$$
5. Given a groupoid X, the functor $\phi_{\mathcal{R}^*\,X} : X(-, \mathrm{Id}_X-) \to X(-, -)$ is the identity.

Proposition 4.2.10. *$\mathcal{R}^* : \mathcal{G}^{op} \to \mathcal{B}$ is a pseudofunctor.*

Proof. The proof is similar as for $\mathcal{R}_*$. Alternatively, this can be deduced from the fact that $\mathcal{R}_* : \mathcal{G} \to \mathcal{B}$ is a pseudofunctor and that for each f we have an adjunction $\mathcal{R}_*(f) \dashv \mathcal{R}^*(f)$: by taking mates under these adjunctions of the structure isomorphisms of the pseudo-functor $\mathcal{R}_*$, we obtain structure isomorphisms for a pseudo-functor $\mathcal{R}^* : \mathcal{G}^{op\,co} \to \mathcal{B}$ sending the 1-cell f to the right adjoint $\mathcal{R}^*(f)$ (see for example [1, Remark A.2.10]). Then, after precomposing with $\alpha \mapsto \alpha^{-1}$ (inversion of 2-cells) one can check that we obtain precisely the pseudofunctor $\mathcal{R}^* : \mathcal{G}^{op} \to \mathcal{B}$ described above in 4.2.9. $\square$

We are going to use the following general construction, which is treated in details and greater generality in [1, Ch. 5].

Proposition 4.2.11. *Let $\mathcal{C}$ be any bicategory, and suppose we have two pseudo-functors $\mathcal{F}_*$ and $\mathcal{F}^*$ on groupoids as follows*

$$\begin{tikzcd}
\mathcal{G} \arrow[d, "\iota_*"'] \arrow[dr, bend left, "\mathcal{F}_*"] & \\
\mathcal{S} \arrow[r, dotted, "\mathcal{F}"] & \mathcal{C} \\
\mathcal{G}^{op} \arrow[u, "\iota^*"] \arrow[ur, bend right, "\mathcal{F}^*"'] &
\end{tikzcd} \tag{4.1}$$

such that:

- *$\mathcal{F}_*$ and $\mathcal{F}^*$ coincide on objects: $\mathcal{F}_*(X) = \mathcal{F}^*(X)$ for all $X \in \mathrm{Ob}(\mathcal{G})$.*

- *For every $f : X \to Y$ in $\mathcal{G}$ we have an adjunction $(\mathcal{F}_*(f) \dashv \mathcal{F}^*(f), \eta_f, \varepsilon_f)$ in $\mathcal{C}$. Assume moreover that all this data satisfies the Beck-Chevalley condition, namely that every iso-comma square in $\mathcal{G}$*

$$\begin{tikzcd} & g/f \arrow[dl, "p_Y"'] \arrow[dr, "p_X"] & \\ Y \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\Leftarrow}{\Lambda}"] & & X \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

induces an invertible 2-cell $\tilde{\Lambda}$ in $\mathcal{B}$ as follows:

$$\begin{tikzcd} & g/f \arrow[dr, "\mathcal{F}_*(p_X)"] & \\ Y \arrow[ur, "\mathcal{F}^*(p_Y)"] \arrow[dr, "\mathcal{F}_*(g)"'] \arrow[rr, phantom, "\tilde{\Lambda}\ \Downarrow"] & & X \\ & Z \arrow[ur, "\mathcal{F}^*(f)"'] & \end{tikzcd} \quad := \quad \begin{tikzcd} & & X \arrow[dd, equal] \\ & g/f \arrow[ur, "\mathcal{F}_*(p_X)"] \arrow[r, phantom, "\overset{\Rightarrow}{\epsilon_{p_X}}"] & \\ Y \arrow[ur, "\mathcal{F}^*(p_Y)"] \arrow[dd, equal] \arrow[rr, phantom, "\overset{\Rightarrow}{\mathcal{F}^*(\Lambda^{-1})}"] & & X \arrow[ul] \\ & Z \arrow[ul] \arrow[ur, "\mathcal{F}^*(f)"'] \arrow[l, phantom, "\overset{\Rightarrow}{\eta_g}"] & \\ Y \arrow[ur, "\mathcal{F}_*(g)"'] & & \end{tikzcd}$$

- *For simplicity, we also assume that the structural isomorphisms $\phi_{\mathcal{F}_*\,X}$ and $\phi_{\mathcal{F}^*\,X}$ are the identity:* $\mathrm{Id}_{\mathcal{F}_*X} = \mathcal{F}_*(\mathrm{Id}_X)$ *and* $\mathrm{Id}_{\mathcal{F}^*X} = \mathcal{F}^*(\mathrm{Id}_X)$.

Then, there exists a unique pseudo-functor $\mathcal{F}$ coinciding with $\mathcal{F}_*$ and $\mathcal{F}^*$ on objects, mapping spans as follows

$$(\, Y \xleftarrow{g} U \xrightarrow{f} X \,) \qquad \mapsto \qquad \mathcal{F}_*(g) \circ \mathcal{F}^*(f)$$

and such that: $\mathcal{F} \circ \iota_* = \mathcal{F}_*$ and $\mathcal{F} \circ \iota^* = \mathcal{F}^*$.

Proof. This follows as a special case of [1, Theorem 5.2.1]. Let us give some explanations. Beware that conventions here and in loc. cit. are a little different: in particular, the direction of the 2-cell in iso-comma squares is opposite.

Here we apply the theorem with a few adjustments. First, we must choose (in the notations of loc. cit.) $\mathbb{J} = \mathbb{G} = \mathcal{G}$, which is possible by [1, Remark 5.1.2]. Second, in loc. cit. the target $\mathcal{C}$ is assumed to be a *strict* 2-category, but this can be easily extended to a bicategory by the strictification theorem (remark 3.0.2); see [1, § 5.3] for details.

In loc. cit., the pseudofunctor $\mathcal{F}$ is constructed from the pseudofunctor $\mathcal{F}^*$ together with a choice of adjunctions $(\mathcal{F}_*(f) \dashv \mathcal{F}^*(f), \eta_f, \varepsilon_f)$ (there noted $\mathcal{F}_!(f)$ instead of $\mathcal{F}_*(f)$), and the uniqueness of $\mathcal{F}$ is only up to the choice of these adjunctions. Since here we give the adjunctions as part of the data, the resulting pseudo-functor $\mathcal{F}$ is unique. Moreover, the two triangles in (4.1) commute strictly, rather than just up to an isomorphism, because we have assumed that the unit structure isomorphisms of $\mathcal{F}_*$ and $\mathcal{F}^*$ are identities. $\square$

We are now ready to give the main construction of this chapter:

Theorem 4.2.12. *There exists a unique pseudofunctor $\mathcal{R} : \mathcal{S} \to \mathcal{B}$ from the span bicategory (definition 4.1.7) to the bimodule bicategory (definition 4.1.9) of finite groupoids, such that*

- *it sends every finite groupoid X to itself,*
- *it sends a span $Y \xleftarrow{g} U \xrightarrow{f} X$ to the bimodule $\mathcal{R}_*(g) \circ_{\mathcal{B}} \mathcal{R}^*(f)$,*
- *and $\mathcal{R} \circ \iota_* = \mathcal{R}_*$ and $\mathcal{R} \circ \iota^* = \mathcal{R}^*$, where $\mathcal{R}_*$ and $\mathcal{R}^*$ are the pseudofunctors of 4.2.6 and 4.2.9.*

Remark 4.2.13. Note that in order to horizontally compose spans of groupoids we use *iso-comma squares*, not pullbacks of groupoids. This is the correct composition that must be used in order to obtain the pseudo-functoriality of $\mathcal{R}$. Note that Hoffnung already foresaw this, although he provided no proof yet (see Claim 13 in [13]).

Proof. It suffices to specialize proposition 4.2.11 to the pseudofunctors $\mathcal{R}_*$ and $\mathcal{R}^*$ and the adjunctions described in the proof of proposition 4.2.8, once we have verified the Beck-Chevalley condition for this data.

Let us do the latter. Consider the following iso-comma square in $\mathcal{G}$:

$$\begin{tikzcd}[column sep=small, row sep=small]
& g/f \arrow[dl, "p_Y"'] \arrow[dr, "p_X"] \arrow[dd, phantom, "\overset{\Leftarrow}{\Lambda}"] & \\
Y \arrow[dr, "g"'] & & X. \arrow[dl, "f"] \\
& Z &
\end{tikzcd}$$

It induces the following diagrams in $\mathcal{B}$, where the left square is obtained by applying $\mathcal{R}^*$ to the iso-comma square:

$$\begin{tikzcd}[column sep=small, row sep=small]
& & X \arrow[dd, equal] \\
& g/f \arrow[ur] \arrow[r, phantom, "\overset{\Rightarrow}{\epsilon_{p_X}}"] & \\
Y \arrow[ur] \arrow[dd, equal] \arrow[rr, phantom, "\overset{\Rightarrow}{\mathcal{R}^*(\Lambda^{-1})}"] & & X \arrow[ul] \\
\arrow[r, phantom, "\overset{\Rightarrow}{\eta_g}"] & Z \arrow[ul] \arrow[ur] & \\
Y \arrow[ur] & &
\end{tikzcd}$$

and, via composition,

$$\begin{tikzcd}[column sep=small, row sep=small]
& g/f \arrow[dr, "\mathcal{R}_*(p_X)"] & \\
Y \arrow[ur, "\mathcal{R}^*(p_Y)"] \arrow[dr, "\mathcal{R}_*(g)"'] \arrow[rr, phantom, "\tilde{\Lambda}\ \Downarrow"] & & X. \\
& Z \arrow[ur, "\mathcal{R}^*(f)"'] &
\end{tikzcd}$$

Concretely and very explicitly, the component at $(y, x) \in \mathrm{Ob}(Y^{op} \times X)$ of this natural transformation

$$\tilde{\Lambda} : \big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} \mathcal{R}^*(p_Y)\big) \Rightarrow \big(\mathcal{R}^*(f) \circ_{\mathcal{B}} \mathcal{R}_*(g)\big) : Y^{op} \times X \longrightarrow \mathrm{Set}$$

is given step by step below. Let $t_1 = (x_1, y_1, \lambda_1) \in \ \mathrm{Ob}(g/f)$:

$$\begin{array}{rl}
\big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} \mathcal{R}^*(p_Y)\big)(y,x) & \to \big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} (\mathcal{R}^*(p_Y) \circ_{\mathcal{B}} Y(-,-))\big)(y,x) \\
{[v \in X(x_1, x),\ u \in Y(y, y_1)]} & \mapsto [v, [u, \mathrm{Id}_y]] \\
 & \to \big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} (\mathcal{R}^*(p_Y) \circ_{\mathcal{B}} (\mathcal{R}^*(g) \circ_{\mathcal{B}} \mathcal{R}_*(g)))\big)(y,x) \\
\text{compose with the unit } \eta : & \mapsto [v, [u, [\mathrm{Id}_{gy}, \mathrm{Id}_{gy}]]] \\
 & \to \big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} ((\mathcal{R}^*(p_Y) \circ_{\mathcal{B}} \mathcal{R}^*(g)) \circ_{\mathcal{B}} \mathcal{R}_*(g))\big)(y,x) \\
 & \mapsto [v, [[u, \mathrm{Id}_{gy}], \mathrm{Id}_{gy}]] \\
 & \to \big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} (\mathcal{R}^*(g \circ p_Y) \circ_{\mathcal{B}} \mathcal{R}_*(g))\big)(y,x) \\
 & \mapsto [v, [g(u), \mathrm{Id}_{gy}]] \\
 & \to \big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} (\mathcal{R}^*(f \circ p_X) \circ_{\mathcal{B}} \mathcal{R}_*(g))\big)(y,x) \\
\text{compose with } \mathcal{R}^*(\Lambda^{-1}) : & \mapsto [v, [\lambda_1^{-1} \circ g(u), \mathrm{Id}_{gy}]] \\
 & \to \big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} ((\mathcal{R}^*(p_X) \circ_{\mathcal{B}} \mathcal{R}^*(f)) \circ_{\mathcal{B}} \mathcal{R}_*(g))\big)(y,x) \\
 & \mapsto [v, [[\mathrm{Id}_{x_1}, \lambda_1^{-1} \circ g(u)], \mathrm{Id}_{gy}]] \\
 & \to \big(\,(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} (\mathcal{R}^*(p_X) \circ_{\mathcal{B}} \mathcal{R}^*(f))) \circ_{\mathcal{B}} \mathcal{R}_*(g)\big)(y,x) \\
 & \mapsto [[v, [\mathrm{Id}_{x_1}, \lambda_1^{-1} \circ g(u)]], \mathrm{Id}_{gy}] \\
 & \to \big(\,(\,(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} \mathcal{R}^*(p_X)) \circ_{\mathcal{B}} \mathcal{R}^*(f)) \circ_{\mathcal{B}} \mathcal{R}_*(g)\big)(y,x) \\
 & \mapsto [[[v, \mathrm{Id}_{x_1}], \lambda_1^{-1} \circ g(u)], \mathrm{Id}_{gy}] \\
 & \to \big(\,(\,X(-,-) \circ_{\mathcal{B}} \mathcal{R}^*(f)) \circ_{\mathcal{B}} \mathcal{R}_*(g)\big)(y,x) \\
\text{compose with the counit } \epsilon : & \mapsto [[v^{-1}, \lambda_1^{-1} \circ g(u)], \mathrm{Id}_{gy}] \\
 & \to \big(\mathcal{R}^*(f) \circ_{\mathcal{B}} \mathcal{R}_*(g)\big)(y,x) \\
 & \mapsto [f(v) \circ \lambda_1^{-1} \circ g(u), \mathrm{Id}_{gy}]
\end{array}$$

where the unnamed maps are the evident associators and unitors of $\mathcal{B}$ and pseudofunctoriality isomorphisms of $\mathcal{R}^*$, as necessary.

We need to prove this map is an isomorphism. We recall that

$$\Lambda : f \circ p_X \Rightarrow g \circ p_Y$$

is an iso-comma square, with g/f the category with objects triples

$$(x, y, \lambda) \quad \text{with} \quad x \in \mathrm{Ob}(X), y \in \mathrm{Ob}(Y), \lambda \in Z(f(x), g(y))$$

and given two objects $t_1 = (x_1, y_1, \lambda_1)$ and $t_2 = (x_2, y_2, \lambda_2)$, the hom-set $(g/f)(t_1, t_2)$ is the set of pairs $(a, b) \in X(x_1, x_2) \times Y(y_1, y_2)$ such that

$$\lambda_2 \circ f(a) = g(b) \circ \lambda_1.$$

Now, let $[x_1 \xrightarrow{v} x, y \xrightarrow{u} y_1]$ and $[x_2 \xrightarrow{v'} x, y \xrightarrow{u'} y_2]$ be elements of $\big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} \mathcal{R}^*(p_Y)\big)(y,x)$, and assume that

$$[f(v) \circ \lambda_1^{-1} \circ g(u), \mathrm{Id}_{gy}] = [f(v') \circ \lambda_2^{-1} \circ g(u'), \mathrm{Id}_{gy}] \in \big(\mathcal{R}^*(f) \circ_{\mathcal{B}} \mathcal{R}_*(g)\big)(y,x),$$

which is equivalent to

$$f(v) \circ \lambda_1^{-1} \circ g(u) = f(v') \circ \lambda_2^{-1} \circ g(u'),$$

since any morphism in Z giving a relation between the two pairs has to be Id_{gy}. Let's show that this implies that $[v,u] = [v',u'] \in \big(\mathcal{R}_*(p_X) \circ_{\mathcal{B}} \mathcal{R}^*(p_Y)\big)(y,x)$. Indeed, the latter is equivalent to the existence of a morphism $(a,b) \in (g/f)(t_1,t_2)$ such that:

$$\begin{tikzcd} x_1 \arrow[r, "v"] \arrow[d, "a"'] & x \\ x_2 \arrow[ur, "v'"'] & \end{tikzcd} \quad ; \quad \begin{tikzcd} y \arrow[r, "u"] \arrow[dr, "u'"'] & y_1 \arrow[d, "b"] \\ & y_2 \end{tikzcd}$$

Since X and Y are groupoids, there is no choice but $a := v'^{-1} \circ v$ and $b := u' \circ u^{-1}$, and it only remains to check that $(v'^{-1} \circ v,\ u' \circ u^{-1})$ is a morphism in g/f, i.e. that

$$\lambda_2 \circ f(v'^{-1} \circ v) = g(u' \circ u^{-1}) \circ \lambda_1,$$

which by functoriality of f and g is exactly the condition above.

Hence, $\tilde{\Lambda}(y,x)$ is injective.

For any $z \in \mathrm{Ob}(Z)$ and $[z \xrightarrow{c} f(x), g(y) \xrightarrow{d} z] \in \big(\mathcal{R}^*(f) \circ_{\mathcal{B}} \mathcal{R}_*(g)\big)(y,x)$, note that $(c \circ d)^{-1}$ is a morphism in $Z(f(x), g(y))$, and as a consequence $t := (x, y, (c \circ d)^{-1}) \in \mathrm{Ob}(g/f)$, and

$$\tilde{\Lambda}(y,x)\big([p_X(t) \xrightarrow{\mathrm{Id}_x} x, y \xrightarrow{\mathrm{Id}_y} p_Y(t)]\big) = [c,d].$$

Hence, $\tilde{\Lambda}(y,x)$ is surjective, $\tilde{\Lambda}$ is an invertible 2-cell, and the Beck-Chevalley condition holds. □

4.3 The 1-truncated picture

We remind the reader that our goal is to apply our first theorem 1.3.13 to this situation. In order to do so, we have to consider the 1-truncation (see 3.0.5) of the bicategories $\mathcal{S}$ and $\mathcal{B}$, and check that those categories and the functor $\tau_1\mathcal{R}$ carry enough structure so that the theorem applies. We recall that an isomorphism of 1-cells in $\mathcal{S}$ is the class of a relevant triple (α, k, β) where k is an equivalence of category, hence $\tau_1\mathcal{S}$ is the category where objects are finite morphisms and the isomorphism classes of spans for this definition of isomorphisms.

Lemma 4.3.1. *The functor $\tau_1\mathcal{R} : \tau_1\mathcal{S} \to \tau_1\mathcal{B}$ is full.*

Proof. Let X and Y be finite groupoids, and $\psi : X^{op} \times Y \to \mathrm{Set}$ be an object of $\mathcal{B}(X,Y)$. We define a groupoid Q_ψ as follows:

$$\mathrm{Ob}(Q_\psi) = \bigsqcup_{(x,y)\in X^{op}\times Y} \{(x,y,u) | u \in \psi(x,y)\},$$

$$Q_\psi\big((x,y,u),(x',y',u')\big) = \{(a,b) \in X(x',x) \times Y(y,y') | \psi(a,b)(u) = u'\}.$$

Composition and identities of Q_ψ come from that of X and Y, and this is clearly a groupoid. Now, we define a span $\mathcal{Q}(\psi) := (Y \xleftarrow{q_Y} Q_\psi \xrightarrow{q_X} X)$ by defining the two functors $q_Y((x,y,u)) = y$, $q_Y((a,b)) = b$ and $q_X(x,y,u)) = x$, $q_X((a,b)) = a^{-1}$ (beware of the inverse!).

The image of this span by our pseudofunctor $\mathcal{R}$, defined at 2.1.10, is isomorphic to ψ in $\mathcal{B}$: indeed, given any $x \in \mathrm{Ob}(X)$ and $y \in \mathrm{Ob}(Y)$, we consider the following map

$$\big(\mathcal{R}_*(q_Y) \circ_{\mathcal{B}} \mathcal{R}^*(q_X)\big)(x,y) \to \psi(x,y) \tag{4.2}$$
$$[q_Y((x_1,y_1,u_1)) \xrightarrow{g} y,\, x \xrightarrow{f} q_X((x_1,y_1,u_1))] \mapsto \psi(f,g)(u_1)$$

Surjectivity is obvious, since $\forall u \in \psi(x,y)$, $u = \psi(\mathrm{Id}_x, \mathrm{Id}_y)(u)$. We now prove the injectivity.

Given two elements of $\mathcal{R}(\mathcal{Q}(\psi))(x,y)$

$$[q_Y((x_1,y_1,u_1)) \xrightarrow{g} y,\, x \xrightarrow{f} q_X((x_1,y_1,u_1))]$$

and

$$[q_Y((x_2,y_2,u_2)) \xrightarrow{g'} y,\, x \xrightarrow{f'} q_X((x_2,y_2,u_2))],$$

assume $\psi(f,g)(u_1) = \psi(f',g')(u_2)$. This implies that

$$\psi(f'^{-1}, g'^{-1})\big(\psi(f,g)(u_1)\big) = u_2,$$

meaning that $(f \circ f'^{-1}, g'^{-1} \circ g)$ is a morphism in $Q_\psi\big((x_1,y_1,u_1),(x_2,y_2,u_2)\big)$, and via this morphism $[g,f] = [g',f'] \in \mathcal{R}(\mathcal{Q}(\psi))(x,y)$, hence the map is injective.

Finally, given morphisms $x' \xrightarrow{a} x \in X$ and $y \xrightarrow{b} y' \in Y$, we need to check that the following square is commutative:

$$\begin{tikzcd}
\mathcal{R}(\mathcal{Q}(\psi))(x,y) \arrow[r] \arrow[d, "\mathcal{R}(\mathcal{Q}(\psi))(a,b)"'] & \psi(x,y) \arrow[d, "\psi(a,b)"] \\
\mathcal{R}(\mathcal{Q}(\psi))(x',y') \arrow[r] & \psi(x',y')
\end{tikzcd}$$

For relevant (x_1,y_1,u_1) and $[g,f]$, the equality $\psi(b\circ g, f\circ a)(u_1) = \psi(a,b)\big(\psi(f,g)(u_1)\big)$ holds, and this shows naturality in $(x,y) \in X^{op} \times Y$ of the isomorphism (4.2).

This proves that, for any bimodule ψ, there exists a span $\mathcal{Q}(\psi)$ such that

$$\tau_1\mathcal{R}([\mathcal{Q}(\psi)]) = \psi \quad \text{in } \tau_1\mathcal{B}.$$

Hence, $\tau_1\mathcal{R}$ is full. $\square$

Remark 4.3.2. The construction of Q_ψ and $\mathcal{Q}_\psi$ is similar to what Benabou wrote in [4], using *categories of elements*. Benabou has considered the similarities between spans and bimodules from the moment he defined bicategories, but since back then he used pullbacks for horizontal composition of spans, the comparison he made could not be functorial, as we already remarked before.

Definition 4.3.3 (Elements of the additive structures)**.** Let X_1 and X_2 be two objects in $\mathcal{G}$. We define $X_1 \sqcup X_2$ to be the groupoid with objects $\mathrm{Ob}(X_1) \sqcup \mathrm{Ob}(X_2)$, with obvious morphisms, identities and composition. It follows from this definition that for $j \in \{1,2\}$, a functor $i_{\mathcal{G}\,X_j} : X_j \to X_1 \sqcup X_2$ is given by the obvious embedding. If notation is somewhat ambiguous, we trust context will always be explanatory. Binary coproducts in $\mathcal{G}$ are given by the triples $(X_1 \sqcup X_2, i_{\mathcal{G}\,X_1}, i_{\mathcal{G}\,X_2})$.

For each $j \in \{1,2\}$, we now define two 1-cells in $\mathcal{S}$,

$$i_{\mathcal{S}\,X_j} := \iota_*(i_{\mathcal{G}\,X_j}) = (X_1 \sqcup X_2 \xleftarrow{i_{\mathcal{G}\,X_j}} X_j = X_j)$$

and

$$p_{\mathcal{S}\,X_j} := \iota^*(i_{\mathcal{G}\,X_j}) = (X_j = X_j \xrightarrow{i_{\mathcal{G}\,X_j}} X_1 \sqcup X_2),$$

and two 1-cells in $\mathcal{B}$,

$$i_{\mathcal{B}\,X_j} := \mathcal{R}_*(i_{\mathcal{G}\,X_j}) = (X_1 \sqcup X_2)(i_{\mathcal{G}\,X_j}-,-)$$

and

$$p_{\mathcal{B}\,X_j} := \mathcal{R}^*(i_{\mathcal{G}\,X_j}) = (X_1 \sqcup X_2)(-,i_{\mathcal{G}\,X_j}-).$$

Proposition 4.3.4. *The category $\tau_1\mathcal{S}$ is semi-additive.*

Proof. The empty groupoid, that we denote 0, is the zero object of $\tau_1\mathcal{S}$: indeed, a span from X to 0 has to be $(0 = 0 \xrightarrow{!} X)$, and a span from 0 to X has to be $(X \xleftarrow{!} 0 = 0)$. For any two groupoids X and Y, write $0_{XY} := (Y \xleftarrow{!} 0 \xrightarrow{!} X)$. There is a commutative monoid structure on every hom-set of $\tau_1\mathcal{S}$ that is induced by coproducts in $\mathcal{G}$: indeed, given two spans, we can define a third span via

$$(Y, g_1, U_1, f_1, X) + (Y, g_2, U_2, f_2, X) := (Y, g_1 \sqcup g_2, U_1 \sqcup U_2, f_1 \sqcup f_2, X),$$

and 0_{XY} is the neutral element of $(\tau_1\mathcal{S}(X,Y), +)$. Commutativity is obvious, and it is easy to check the well defineness of +, as well as the other axioms. Furthermore, we claim that the 1-cells in $\mathcal{S}$ described in 4.3.3 give biproducts in $\tau_1\mathcal{S}$.

The iso-comma category induced by the composition $p_{\mathcal{S}\,X_1} \circ_{\mathcal{S}} i_{\mathcal{S}\,X_2}$ is the empty groupoid. From this, it is obvious that

$$p_{\mathcal{S}\,X_1} \circ_{\mathcal{S}} i_{\mathcal{S}\,X_2} = 0_{X_2X_1}$$

$$p_{\mathcal{S}\,X_2} \circ_{\mathcal{S}} i_{\mathcal{S}\,X_1} = 0_{X_1X_2}$$

The composed span $p_{\mathcal{S}\,X_1} \circ_{\mathcal{S}} i_{\mathcal{S}\,X_1} = (X_1 \leftarrow (i_{\mathcal{G}\,X_1}/i_{\mathcal{G}\,X_1}) \rightarrow X_1)$ is isormorphic in $\mathcal{B}$ to the identity span for X_1 via the morphism of spans

$$(\mathrm{Id}, (x, x', \lambda) \mapsto x, \Lambda).$$

Indeed, it is straightforward to check that this functor is an equivalence of category, with quasi-inverse induced by the map $x \mapsto (x, x, \mathrm{Id}_x)$, and it follows that for $j \in \{1, 2\}$, we have

$$p_{\mathcal{S}\,X_j} \circ_{\mathcal{S}} i_{\mathcal{S}\,X_j} = [X_j = X_j = X_j] \in \tau_1\mathcal{S}.$$

It is straightforward to check that the sum in $\tau_1\mathcal{S}(X_1 \sqcup X_2, X_1 \sqcup X_2)$ of the compositions of $p_{\mathcal{S}X_j}$ with $i_{\mathcal{S}X_j}$ gives the span $\big((X_1 \sqcup X_2 \xleftarrow{f} U \xrightarrow{f} (X_1 \sqcup X_2)\big)$, where U is the category with set of objects

$$\{(x, x, \mathrm{Id}_x) | x \in \mathrm{Ob}((X_1 \sqcup X_2))\},$$

morphisms $U\big((x, x, \mathrm{Id}_x), (x', x', \mathrm{Id}_{x'})\big) = \{(a, a) | a \in (X_1 \sqcup X_2)(x, x')\}$, and obivous composition and identities. The above f is the functor given by

$$f(x, x, \mathrm{Id}_x) = x \quad ; \quad f(a, a) = a.$$

The functor f is an equivalence, and gives an isomorphism of spans in $\mathcal{S}$:

$$\big((X_1 \sqcup X_2, f, U, f, (X_1 \sqcup X_2)\big) \xrightarrow{(\mathrm{Id}, f, \mathrm{Id})} \mathrm{Id}_{(X_1 \sqcup X_2)},$$

hence we have the fifth equality for biproducts in $\tau_1\mathcal{S}$, namely:

$$[i_{\mathcal{S}X_1} \circ_{\mathcal{S}} p_{\mathcal{S}X_1}] + [i_{\mathcal{S}X_2} \circ_{\mathcal{S}} p_{\mathcal{S}X_2}] = \mathrm{Id}_{(X_1 \sqcup X_2)}.$$

Proving that composition in $\tau_1\mathcal{S}$ is bilinear with respect to $+$ ends the proof. □

Proposition 4.3.5. *The category $\tau_1\mathcal{B}$ is semi-additive.*

Proof. We define a monoid structure on every hom-set $\tau_1\mathcal{B}(X, Y)$: given ψ_1, $\psi_2 \in \mathcal{B}(X, Y)$, let

$$\psi_1 \sqcup \psi_2 : X^{op} \times Y \to \mathrm{Set}$$
$$(x, y) \mapsto \psi_1(x, y) \sqcup \psi_2(x, y)$$
$$(x' \xrightarrow{a} x, y \xrightarrow{b} y') \mapsto \psi_1(a, b) \sqcup \psi_2(a, b)$$

Composition in $\tau_1\mathcal{B}$ is bilinear with respect to that structure. Indeed, given a bimodule $\varphi : Y^{op} \times Z \to \mathrm{Set}$, it is straightforward to check that the map

$$\big((\psi_1 \sqcup \psi_2) \circ_{\mathcal{B}} \varphi\big)(x, z) \to \big((\psi_1 \circ_{\mathcal{B}} \varphi) \sqcup (\psi_2 \circ_{\mathcal{B}} \varphi)\big)(x, z)$$
$$[v_i \in \psi_i(x, y),\, u \in \varphi(y, z)] \mapsto [v_i, u]$$

induces a natural isomorphism $(\psi_1 \sqcup \psi_2) \circ_{\mathcal{B}} \varphi\big) \simeq \big((\psi_1 \circ_{\mathcal{B}} \varphi) \sqcup (\psi_2 \circ_{\mathcal{B}} \varphi)\big)$.

Furthermore, we claim that the 1-cells in $\mathcal{B}$ described in 4.3.3 give biproducts in $\tau_1\mathcal{B}$. Now, we prove this claim.

Since for any objects x_1 of X_1 and x_2 of X_2, and any object $x \in X_1 \sqcup X_2$, either the set $X_1 \sqcup X_2(x_1, x)$ or the set $X_1 \sqcup X_2(x, x_2)$ is empty, hence both $p_{\mathcal{B}\, X_1} \circ_{\mathcal{B}} i_{\mathcal{B}\, X_2}$ and $p_{\mathcal{B}\, X_2} \circ_{\mathcal{B}} i_{\mathcal{B}\, X_1}$ are constant functors evaluating in the empty set.

On the other hand, for any morphisms $x \xrightarrow{a} x_1'$ and $x_1 \xrightarrow{b} x$ of X_1, we have the equality $[a, b] = [a', b'] \in \big(p_{\mathcal{B}\, X_1} \circ_{\mathcal{B}} i_{\mathcal{B}\, X_1}\big)(x_1, x_1')$, which means the map $\big(p_{\mathcal{B}\, X_1} \circ_{\mathcal{B}} i_{\mathcal{B}\, X_1}\big)(x_1, x_1') \to X_1(x_1, x_1')$, $[a, b] \mapsto a \circ b$ is injective.

It is obviously surjective, hence $[p_{\mathcal{B}\, X_1} \circ_{\mathcal{B}} i_{\mathcal{B}\, X_1}] = [X_1(-, -)] \in \tau_1\mathcal{B}$. The fifth equation is obtained easily by considering the monoid structure of $\big(\tau_1\mathcal{B}(X_1 \sqcup X_2, X_1 \sqcup X_2), \sqcup\big)$. $\square$

Lemma 4.3.6. *The functor $\tau_1\mathcal{R}$ sends biproducts in $\tau_1\mathcal{S}$ on biproducts in $\tau_1\mathcal{B}$.*

Proof. This is rather obvious from the definitions of the structures and from the fact that $\mathcal{R} \circ \iota_* = \mathcal{R}_*$ by theorem 4.2.12. $\square$

Remark 4.3.7. It is well known that the category of groupoids and functors is a symmetric monoidal category, with the cartesian monoidal product. It is straightforward to check that the category $\tau_1\mathcal{G}$ is also symmetric monoidal via the cartesian product, i.e. that the functor sending a groupoid to itself and a functor to its equivalence class in the truncated 2-category is strong symmetric monoidal.

Notation 4.3.8 (Elements of the monoidal structures)**.** We fix the notations for the tensor structure of remark 4.3.7:

- the functor $\tau_1\mathcal{G} \times \tau_1\mathcal{G} \xrightarrow{- \times -} \tau_1\mathcal{G}$ induced by the cartesian product,
- the tensor unit 1, which denotes from now on a fixed groupoid with one object and one morphism,
- the associator $a : (- \times -) \times - \Rightarrow - \times (- \times -)$,
- the left unitor $\mathrm{lun}_X : 1 \times X \to X$, and the right unitor run,
- the symmetry $\sigma_{XY} : X \times Y \to Y \times X$.

Proposition 4.3.9. *The tensor structure on $\tau_1\mathcal{G}$ induces a tensor structure on $\tau_1\mathcal{S}$ via*

1. *a functor $\tau_1\mathcal{S} \times \tau_1\mathcal{S} \to \tau_1\mathcal{S}$: for any groupoids X, X', Y, Y' and for any spans $Y \xleftarrow{g} V \xrightarrow{f} X$ and $Y' \xleftarrow{g'} V' \xrightarrow{f'} X'$, the monoidal product of the classes of our two spans is given by the class of the product span $[Y \times Y' \xleftarrow{(g,g')} V \times V' \xrightarrow{(f,f')} X \times X']$,*
2. *the monoidal unit 1,*

3. *the associators* $a_{\tau_1\mathcal{S}}$*, the left/right unitors* $\mathrm{lun}_{\tau_1\mathcal{S}}/\mathrm{run}_{\tau_1\mathcal{S}}$ *and the symmetry isomorphisms* $\sigma_{\tau_1\mathcal{S}}$ *are all given by the images of those of* $\tau_1\mathcal{G}$ *by the functor* $\tau_1\iota_*$*.*

Furthermore, the category $\tau_1\mathcal{S}$ *is rigid with respect to this monoidal structure.*

Proof. The fact that this gives $\tau_1\mathcal{S}$ a monoidal product follows from the fourth part of [12], via truncation. Since Hoffnung does not say anything about symmetry, probably because there is no suitable definition of a monoidal symmetric tricategory, this is what remains to be studied.

This is pretty straightforward, since both

$$\begin{tikzcd}
(X\times Y)\times Z \arrow[r, "\tau_1\iota_*(a)"] \arrow[d, "\tau_1\iota_*(\sigma)\times\mathrm{Id}"'] & X\times(Y\times Z) \arrow[r, "\tau_1\iota_*(\sigma)"] & (Y\times Z)\times X \arrow[d, "\tau_1\iota_*(a)"] \\
(Y\times X)\times Z \arrow[r, "\tau_1\iota_*(a)"] & Y\times(X\times Z) \arrow[r, "\mathrm{Id}\times\tau_1\iota_*(\sigma)"] & Y\times(Z\times X)
\end{tikzcd}$$

and

$$\tau_1\iota_*(\sigma_{X\,Y})\circ_{\tau_1\mathcal{S}}\tau_1\iota_*(\sigma_{Y\,X})=\mathrm{Id}_{X\times Y}\in\tau_1\mathcal{S}$$

always hold by functoriality of $\tau_1\iota_*$. Hence, we only need to show that $\sigma_{\tau_1\mathcal{S}}$ is a natural transformation, i.e. that given any four groupoids X, X', Y, Y' and for any two spans $Y'\xleftarrow{g'}V\xrightarrow{g}Y$ and $X'\xleftarrow{f'}U\xrightarrow{f}X$, the following square commutes in $\tau_1\mathcal{S}$

$$\begin{tikzcd}
X\times Y \arrow[r, "\sigma_{\tau_1\mathcal{S}}"] \arrow[d, "\big((f',g'),U\times V,(f,g)\big)"'] & Y\times X \arrow[d, "\big((g',f'),U\times V,(g,f)\big)"] \\
X'\times Y' \arrow[r, "\sigma_{\tau_1\mathcal{S}}"'] & Y'\times X'
\end{tikzcd}$$

that is, the classes of the spans

$$Y'\times X'\xleftarrow{(g',f')\circ q}(g,f)/\sigma_{X\,Y}\xrightarrow{p}X\times Y$$

and

$$Y'\times X'\xleftarrow{\sigma_{X'\,Y'}\circ q_1}\mathrm{Id}/(f',g')\xrightarrow{(f,g)\circ p_1}X\times Y$$

are the same, with data coming from the iso-comma squares below:

$$\begin{tikzcd}
& (g,f)/\sigma_{X\,Y} \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, phantom, "\overset{\Leftarrow}{\Lambda}"] & \\
V\times U \arrow[dr, "(g,f)"'] & & X\times Y, \arrow[dl, "\sigma"] \\
& Y\times X &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{Id}/(f',g') \arrow[dl, "q_1"'] \arrow[dr, "p_1"] & \\
X'\times Y' \arrow[dr, equal] \arrow[rr, phantom, "\overset{\Leftarrow}{\Lambda_1}"] & & U\times V. \arrow[dl, "{(f',g')}"] \\
& X'\times Y' &
\end{tikzcd}$$

One can check that the mapping $c : \big((v,u);(x,y);\lambda\big) \mapsto \big((f'(u), g'(v));(u,v);\mathrm{Id}\big)$ induces an isomorphism of spans

$$\begin{tikzcd}
& (g,f)/\sigma_{X\,Y} \arrow[dl, "{(g',f')\circ q}"'] \arrow[dr, "p"] \arrow[dd, "c"] & \\
Y'\times X' \arrow[r, phantom, "="] & {} \arrow[r, phantom, "\Swarrow \Lambda*\sigma_{Y\,X}"] & X\times Y \\
& \mathrm{Id}/(f',g') \arrow[ul, "\sigma\circ q_1"] \arrow[ur, "{(f,g)\circ p_1}"'] &
\end{tikzcd}$$

in $\mathcal{S}$, and this ends the proof for the tensor structure.

Finally, rigidity of $\tau_1\mathcal{S}$ follows from the fact that there is a pseudofunctor $(-)^\vee : \mathcal{S}^{op} \to \mathcal{S}$, as stated in the proof of proposition 4.2.4, which moreover is clearly inverse of itself. More precisely, we set:

- $X^\vee := X$ as the dual of any object,
- $\eta_X : 1 \to X\times X^\vee \in \tau_1\mathcal{S}$ is given by the class of the span $X\times X \xleftarrow{\Delta} X \xrightarrow{!} 1$
- $\varepsilon_X : X^\vee \times X \to 1$ is given by the class of the span $1 \xleftarrow{!} X \xrightarrow{\Delta} X\times X$, where ! denotes the only functor to 1 and Δ the diagonal functor.

The first axiom is equivalent to the following equality in $\tau_1\mathcal{B}$:

$$[X\times 1 \leftarrow (\mathrm{Id}\times\Delta)/(\Delta\times\mathrm{Id}) \to 1\times X] = [X\times 1 \xleftarrow{\mathrm{run}^{-1}} X \xrightarrow{\mathrm{lun}^{-1}} 1\times X],$$

which holds because the map $X \to (\mathrm{Id}\times\Delta)/(\Delta\times\mathrm{Id})$, $x \mapsto ((x,x),(x,x),(\mathrm{Id}_x,\mathrm{Id}_x,\mathrm{Id}_x))$ is an equivalence of category. The latter is true, indeed:

- every object in the iso-comma category is isomorphic to the image of an object of X, as the data below shows.

$$\begin{tikzcd}
x, & x', & x_1, & x'_1 \\
x \arrow[u, equal] & x, \arrow[u, "\lambda_3^{-1}\circ\lambda_2"'] & x, \arrow[u, "\lambda_1"'] & x, \arrow[u, "\lambda_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, "\lambda_1"] & x_1, \\
x \arrow[u, equal] \arrow[r, equal] & x, \arrow[u, "\lambda_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "\lambda_2"] & x'_1, \\
x \arrow[u, equal] \arrow[r, equal] & x, \arrow[u, "\lambda_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x' \arrow[r, "\lambda_3"] & x'_1 \\
x \arrow[u, "\lambda_3^{-1}\circ\lambda_2"'] \arrow[r, equal] & x \arrow[u, "\lambda_2"']
\end{tikzcd}$$

- the functor is full and faithful since every morphism (a,b,c,d) in the comma category between the images of two objects $x, x' \in X$

$$\begin{tikzcd} x', & x', & x', & x' \\ x, \arrow[u, "a"] & x, \arrow[u, "b"] & x, \arrow[u, "c"] & x, \arrow[u, "d"] \end{tikzcd}$$

$$\begin{tikzcd} x' \arrow[r, equal] & x', & x' \arrow[r, equal] & x', & x' \arrow[r, equal] & x' \\ x \arrow[u, "a"] \arrow[r, equal] & x, \arrow[u, "c"'] & x \arrow[u] \arrow[r, "a"] & x, \arrow[u, "d"'] & x \arrow[u, "b"] \arrow[r, equal] & x \arrow[u, "d"'] \end{tikzcd}$$

  has to be of the form (a,a,a,a).

This proves the first axiom, and the second one holds by a similar proof. □

Proposition 4.3.10. *The tensor structure on $\tau_1\mathcal{G}$ induces a tensor structure on $\tau_1\mathcal{B}$ via*

1. *a functor $\tau_1\mathcal{B} \times \tau_1\mathcal{B} \to \tau_1\mathcal{B}$: for any groupoids X, X', Y, Y' and for any bimodules $\varphi : X^{op} \times Y \to Set$ and $\varphi' : X'^{op} \times Y' \to Set$, $\varphi \times_{\tau_1\mathcal{B}} \varphi' : (X \times X')^{op} \times (Y \times Y') \to \mathrm{Set}$ is given by $(\varphi \times \varphi')$, precomposed by the relevant shuffling of the information,*

$$\begin{aligned}(X \times X')^{op} \times (Y \times Y') &\xrightarrow{\simeq} (X'^{op} \times X^{op}) \times (Y \times Y') \\ &\xrightarrow{a} ((X'^{op} \times X^{op}) \times Y) \times Y' \\ &\xrightarrow{a \times \mathrm{Id}} (X'^{op} \times (X^{op} \times Y)) \times Y' \\ &\xrightarrow{\sigma \times \mathrm{Id}} ((X^{op} \times Y) \times X'^{op}) \times Y' \\ &\xrightarrow{a} (X^{op} \times Y) \times (X'^{op} \times Y')\end{aligned}$$

2. *the monoidal unit* 1,

3. *the associators $a_{\tau_1\mathcal{B}}$, the left/right unitors $\mathrm{lun}_{\tau_1\mathcal{B}}/\mathrm{run}_{\tau_1\mathcal{B}}$ and the symmetry isomorphisms $\sigma_{\tau_1\mathcal{B}}$ are all given by the images of those of $\tau_1\mathcal{G}$ by the functor $\tau_1\mathcal{R}_*$.*

Proof. The verifications that the above data defines a tensor structure are easy consequences of the analogous fact for the product of sets, and the uniqueness of the shuffling identification in 1. We leave the remaining details to the reader. □

Lemma 4.3.11. *The functor $\tau_1\mathcal{R}$ is a strong tensor functor.*

Proof. We consider the following diagram

$$\begin{tikzcd}
\tau_1\mathcal{S} \times \tau_1\mathcal{S} \arrow[r, "\times_{\tau_1\mathcal{S}}"] \arrow[d, "\tau_1\mathcal{R}\times\tau_1\mathcal{R}"'] & \tau_1\mathcal{S} \arrow[d, "\tau_1\mathcal{R}"] \\
\tau_1\mathcal{B} \times \tau_1\mathcal{B} \arrow[r, "\times_{\tau_1\mathcal{B}}"'] & \tau_1\mathcal{B}
\end{tikzcd}$$

Fix four groupoids X, X', Y, Y' and two spans $Y \xleftarrow{g} V \xrightarrow{f} X$ and $Y' \xleftarrow{g'} V' \xrightarrow{f'} X'$, and define

$$\times_{\tau_1\mathcal{B}} \circ \big(\tau_1\mathcal{R} \times \tau_1\mathcal{R}\big)\Big([Y', g', V', f', X'], [Y, g, V, f, X]\Big) =$$

$$\times_{\tau_1\mathcal{B}}\big(\mathcal{R}_*(g') \circ_{\mathcal{B}} \mathcal{R}^*(f'), \mathcal{R}_*(g) \circ_{\mathcal{B}} \mathcal{R}^*(f)\big) =: \psi$$

and

$$\tau_1\mathcal{R} \circ (\times_{\tau_1\mathcal{S}})\big([Y', g', V', f', X'], [Y, g, V, f, X]\big) =$$
$$\mathcal{R}_*(g' \times g) \circ_{\mathcal{B}} \mathcal{R}^*(f' \times f) =: \psi_1$$

Given any object $\big((x', x), (y', y)\big) \in (X' \times X)^{op} \times (Y' \times Y)$, we consider an element

$$\big([y' \xrightarrow{b'} g'(v'), x' \xrightarrow{a'} f'(v')], [y \xrightarrow{b} g(v), x \xrightarrow{a} f(v)]\big) \in \psi_1\big((x', x), (y', y)\big)$$

and we map it to

$$[(b', b), (a', a)] \in \psi\big((x', x), (y', y)\big).$$

Checking this induces a natural isomorphism $\mathrm{strg}^{\mathcal{R}}$ is straightforward, and one can note that the mapping above is only a shuffling of the information.

It is rather obvious, from the definitions of the structures (both inherited from the *cartesian* tensor structure on $\tau_1\mathcal{G}$) and from the fact that $\mathcal{R} \circ \iota_* = \mathcal{R}_*$ by theorem 4.2.12, that the axiom of a strong tensor functor follow. ☐

Definition 4.3.12. From now on, we define

- $\mathbb{C}$ to be the category $\Bbbk\tau_1\mathcal{S}$,
- $\mathbb{D}$ to be the category $\Bbbk\tau_1\mathcal{B}$, with the prefix $\Bbbk$ meaning we $\Bbbk$-linearized the category as in 1.2.5 both times
- $F : \mathbb{C} \to \mathbb{D}$ to be the $\Bbbk$-linear extension of $\tau_1\mathcal{R}$ to $\mathbb{C}$ (see 1.2.6).

Lemma 4.3.13. *The $\mathbb{V}$-functor $F : \mathbb{C} \to \mathbb{D}$ is full and strong monoidal.*

Proof. This follows from lemma 4.3.1, 4.3.6 and 4.3.11.

Note that the monoidal structures survive the $\Bbbk$-linearization, and are compatible with it. Furthermore, F is *defined* as the $\mathbb{V}$-functor extending the semi-additive functor $\tau_1\mathcal{R}$ to $\mathbb{C}$. It follows that the fullness of $\tau_1\mathcal{R}$ implies that F is full. ☐

4.4 Biset functors and global Mackey functors

Now, we establish the link with Serge Bouc's theory of biset functors, and the comparison studied by Ganter and Nakaoka in [10], [20] and [21].

Definition 4.4.1. Following Bouc [6], we define the *biset category* $\mathbb{B}$ to be the category with

- objects the finite groups;
- morphisms $\mathbb{B}(H, G) := \Bbbk B(G, H)$, that is the $\Bbbk$-linearization of the Grothendieck group associated to the monoid of isomorphism classes of (G, H)-bisets together with the disjoint union (recall that a (G, H)-*biset* is the same as a left $H^{op} \times G$-set, seen as a set equipped with commuting left G-action and right H-action);
- composition induced by the following construction: given three finite groups G, H and K, and given U a (G, H)-biset and V a (H, K)-biset, we denote $V \times_H U$ the (G, K)-biset with underlying set the quotient of the set $V \times U$ by the action of H given by $(v, u) \cdot h = (v \cdot h, h^{-1} \cdot u)$, and with (G, K)-action induced by $k \cdot (v, u) \cdot g = (k \cdot v, u \cdot g)$;
- identities given by Id_G the biset with underlying set G and actions of G given by multiplication on both sides.

Proposition 4.4.2. *This definition is indeed that of a category.*

Proof. See Bouc's book [6] for a proof and a treatment of this subject. □

Proposition 4.4.3. *There is an equivalence of categories between Bouc's category of* biset functors $\mathbb{V}^{\mathbb{B}}$ *(denoted by $\mathcal{F}$ in [6]) and $\mathbb{V}^{\mathbb{D}}$.*

Proof. This is essentially equivalent to theorem 4.3 of [14], as we can see $\mathbb{D}$ as the *additive completion* of $\mathbb{B}$ (that is the smallest additive category containing $\mathcal{B}$ as a full subcategory), noting that

- every finite groupoid is equivalent to the disjoint union of a finite family of groupoids with one object, the latter being essentially finite groups, as we will see below;
- the "fractions" X/G defined in [14] for G a finite groupe and X finite G-set are essentially *transport groupoids*;

see Nakaoka's paper [21] for details.

Indeed, we can define a functor $I : \mathbb{B} \to \mathbb{D}$, precomposition by which will give the equivalence with the arguments of [14]. The functor is as follows:

- a finite group G is mapped to the groupoid $\underline{G}$ with a single object $*$, such that $\underline{G}(*, *) = G$;

- the class of a (G,H)-biset U is mapped to the class of the following bimodule

$$\underline{U} : \underline{G}^{op} \times \underline{H} \to \mathrm{Set}$$

$$\underline{U}(*,*) = U$$

$$\underline{U}(g,h) : u \mapsto h \cdot u \cdot g$$

Functoriality holds if $I(V \times_H U) \simeq \underline{V} \circ_{\mathcal{B}} \underline{U} \in \mathcal{B}$, which follows from the definitions of both quotients.

The functor I is full, since given any two finite groups G and H, any bimodule $\varphi : \underline{G}^{op} \times \underline{H} \to \mathrm{Set}$ gives a finite set $U_\varphi := \varphi(*,*)$, with actions given by $\forall u \in U_\varphi$, $\forall g \in G$, and $\forall h \in H$

$$u \cdot g = \varphi(g, \mathrm{Id})(u) \quad ; \quad h \cdot u = \varphi(\mathrm{Id}, h)(u)$$

The fact that the actions commute is obvious enough, and so is the fact that $I(U_\varphi) = \varphi \in \tau_1\mathcal{B}$. Furthermore, it is clear that this describes an isomorphism between $\mathbb{B}(G,H)$ and $\mathbb{D}(\underline{G},\underline{H})$, and that I is actually *fully faithful*.

Every object in $\mathbb{D}$ is a direct sum of objects in the image of I, hence $\mathbb{D}$ is (equivalent to) the additive completion of $\mathbb{B}$. As a consequence, the restriction functor along I

$$I^* : \mathbb{V}^{\mathbb{D}} \longrightarrow \mathbb{V}^{\mathbb{B}}$$

is an equivalence of $\mathbb{V}$-categories. □

Definition 4.4.4. Following Ganter [10] and Nakaoka [20], we define *global Mackey functors* to be the $\mathbb{V}$-functors from $\mathbb{C} = \Bbbk\tau_1\mathcal{S}$ to the category of $\Bbbk$-modules.

We can now directly apply the theorem 1.3.13 of chapter 1 to the functor F of definition 4.3.12 and we obtain the following result, most of which (safe the monoidal part) has already appeared in work of Nakaoka [20] [21]:

Corollary 4.4.5 (Biset functors vs global Mackey functors)**.** *There is an equivalence of tensor categories between:*

- *The category of modules over a certain global Green functor (see [21]);*
- *The category of biset functors $\mathbb{V}^{\mathbb{B}}$ of Bouc [6].*

Moreover, both categories identify canonically with the reflexive full subcategory of global Mackey functors satisfying the 'deflation axiom' ($N \trianglelefteq G$):

$$\mathrm{def}^G_{G/N} \circ \mathrm{inf}^G_{G/N} = \mathrm{Id}.$$

Proof. By lemma 4.3.13, the functor $F : \mathbb{C} \to \mathbb{D}$ meets the requirements of 1.3.13, and by 4.3.9 the category $\mathbb{C}$ is rigid, hence the results follows. The mentioned global Green functor is, of course, the commutative monoid $A = F^*(1_{\mathbb{V}^{\mathbb{D}}})$ in the tensor category $\mathbb{V}^{\mathbb{C}}$ of global Mackey functors.

The ‘moreover’ part is proved in [20]. Note however that it is not immediately evident why the definition of global Mackey functor given there agrees with ours; one must first notice that Nakaoka’s bicategory $\mathbb{S}$ is biequivalent to that of groupoids (see [21] for details). $\square$

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