A motivic approach to twisted Mackey functors

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In short

This is a PhD project in higher algebra and stable homotopy, whose goal is to unify two successful formalisms: Grothendieck–Neeman duality on one side, and (higher) Mackey functors on the other. The first captures important aspects of various duality theories throughout mathematics; the second is a powerful approach to equivariant mathematics which is mostly useful when the groups are finite (or under similar strong finiteness hypothesis). Typically, the restriction of group actions from a group G to a finite-index subgroup H provides a functor which fits in both theories. In this case, however, the resulting Grothendieck–Neeman duality is a straightforward ambidexterity (an isomorphism between induction and coinduction), whereas there are many examples in geometry, topology and representation theory, of restriction functors which only enjoy *twisted* ambidexterity, giving rise to more interesting dualities. Current Mackey functors formalisms do not handle this, and the idea is to find a way to do so which is compatible with the methods of Grothendieck–Neeman duality. The answer will probably require higher algebraic tools, such as Lurie's ∞-categories and possibly $(\infty, 2)$-categories.

A successful unifying framework should provide, among other things, a good way of keeping track of non-trivial dualizing objects and their various properties, which has potential applications in several areas of pure mathematics. For example, algebraic K-theory of equivariant exact or Waldhausen categories provides a source of examples of spectral Mackey functors that encode very interesting invariants.

Grothendieck–Neeman duality

This theory was developed in [BDS16] at the level of tensor-triangulated categories (without recurring to higher categories or other enhancements) by integrating many previous results, most notably from Grothendieck duality theory in algebraic geometry as approached by Amnon Neeman. The idea is to study abstract tensor-exact functors $f^*\colon \mathcal{D} \to \mathcal{C}$ between 'nice' stable homotopy theories (technically: rigidly-compactly generated tensor triangulated categories), by looking at the existence and properties of the functors adjoint to f^*, *their* adjoints, and so on. In the examples f^* is often some kind of derived pullback or restriction functor, whence the notation.

The theory says that there is always a triple of adjoint functors, $f^* \dashv f_* \dashv f^!$, and that the special object $\omega_f := f^!(\mathbf{1}_{\mathcal{D}})$ of $\mathcal{C}$ is a well-behaved dualizing objects in many situations. In good cases, f^* also has a left adjoint $f_!$ which is necessarily subject to the relation $f_!(x) \simeq f_*(\omega_f \otimes x)$, which is a form of twisted ambidexterity. A prominent example is when f^* is the restriction functor $\mathrm{Ho}(\mathrm{Sp}^G) \to \mathrm{Ho}(\mathrm{Sp}^H)$ from (genuine) G-spectra to H-spectra, where $H \leq G$ is a closed subgroup of a compact Lie group, in which case the above relation is known as the Wirthmüller isomorphism between induction and coinduction; unless G/H is finite, the twist is non-trivial: $\omega_f \neq \mathbf{1}_{\mathcal{C}}$.

Mackey functors and Burnside categories

Mackey functors, as well as Green functors (the corresponding ring objects) [Gre71], have been used in representation theory since the early 70s to encode the induction, restriction and conjugation maps inherited by the additive invariants of finite groups. Working within Lurie's ∞-categorical setting [Lur09] [Lur17], Barwick [Bar17] has initiated a theory of Mackey functors taking values in a stable ∞-category rather than an abelian category. For example, spectral Mackey functors (those taking values in the stable ∞-category of spectra) for a finite group G can be identified with genuine G-spectra, Sp^G. In a similar vein, the Mackey and Green 2-functors introduced in [BD20] [Del22] model (algebraic) 2-categorical aspects of inductions and restrictions functors, such as their adjunctions. What is common to all such Mackey functor formalisms is the possibility of a 'motivic' formulation, whereby there is a universal target for Mackey functors, called *Burnside category*, through which all others must factor. The Burnside category can be constructed as a certain (1- or 2-) category of spans, and its symmetry and structures inform the theory. Constructions with spans only work when ambidexterity holds directly (induction and coinduction coincide), but appear to fail when there is only a *twisted* ambidexterity, e.g. as in the general case of the Wirthmüller isomorphism in topology.

A common framework

The goal is twofold. First, to find a suitable axiomatics for Mackey functors with twisted ambidexterity. Second, to construct the corresponding Burnside category, that is the universal target of such a Mackey functor, and reveal its structure.

For the first goal, we could follow the lead of Cnossen [Cno23] who uses parametrized homotopy theory to introduce a twisted version of Hopkins–Lurie ambidexterity, motivated precisely by the example of the Wirthmüller isomorphism for compact Lie groups. Similar ideas appear when formalizing categories of six-functor formalisms à la Mann–Scholze [Sch22], although of course the latter involve functors on schemes, whereas we should rather have some kind of topological groups or groupoids (or a formal analogue).

For the second goal, the successful identification of a universal six-functor formalism in some contexts [DG22] points the way towards a suitable construction of a twisted Burnside category. Once again the example of G-spectra for compact Lie groups, where transfers shift degrees, suggests that this construction may not yield a 1- or $(2,1)$-category anymore, as with finite groups, but a more general object, and it may be necessary to use techniques from [GR17] on $(\infty, 2)$-categories of correspondences.

Applications to equivariant algebraic K-theory

The development of higher algebra allowed for new foundations of equivariant algebraic K-theory [Bar17], [BGS20], providing a rich source of spectral Mackey functors. Fundamental examples include the algebraic K-theory of G-Galois extensions of rings in algebra or stable homotopy theory [Rog08], [Aus10], [CMNN20], or the A-theory of smooth manifolds with G-action [MM20]. These have been studied mainly in the case of finite groups G. The extension of the theory in this project will provide a useful formalism for the study of the spectral Mackey functors for compact Lie groups G provided by equivariant algebraic K-theory, or related functors like equivariant topological Hochschild homology.

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