Solutions Manual for Macroeconomic Theory

DENG Yan-fei*

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Contents

1 Benjamin Keen (2022)

1.1 Exercises in Lec1

1. Prove that the s-s growth rate of output in the Solow growth model is $g_A + g_L + g_A g_L$ where g_A is the growth rate of labor-augmenting technology and g_L is the population growth rate. Assume the production function is CRS.

$$
\begin{aligned}
&Y_t = F(K_t, A_t L_t), \\
\xRightarrow{\text{CRS}}\ &y_t = \frac{Y_t}{A_t L_t},
\end{aligned}
$$

*https://idengyf.github.io/

$$
\begin{aligned}
\Rightarrow \quad Y_t &= y_t A_t L_t, \\
\Rightarrow \quad 1 + g_Y \equiv \frac{Y_t}{Y_{t-1}} &= \frac{y_t A_t L_t}{y_{t-1} A_{t-1} L_{t-1}}, \\
&= \frac{y_t A_t L_t}{y_{t-1} A_{t-1} L_{t-1}}, \\
&= \frac{y_t}{y_{t-1}} \frac{A_t}{A_{t-1}} \frac{L_t}{L_{t-1}}, \\
&= (1 + g_y)(1 + g_A)(1 + g_L), \\
g_k \overset{\text{s-s}}{=} 0 \;\Rightarrow\; g_y &\overset{\text{s-s}}{=} 0 \\
\Rightarrow \quad 1 + g_Y &= (1 + g_A)(1 + g_L), \\
\Rightarrow \quad g_Y &= g_A + g_L + g_A g_L.
\end{aligned}
$$

2. How does the depreciation rate δ affect the balanced growth level of capital? That is, mathematically show that what happens to the balanced growth level of capital when the depreciation rate rises? Repeat this analysis for an increase in both the growth rates of labor-augmenting technological progress g_A and the population growth rate g_L.

$$
\begin{aligned}
K_{t+1} &= I_t + (1 - \delta) K_t, \\
\Rightarrow \quad K_{t+1} - K_t &= I_t - \delta K_t, \\
&= S_t - \delta K_t, \\
&= s Y_t - \delta K_t, \\
\Rightarrow \quad g_K \equiv \frac{K_{t+1} - K_t}{K_t} &= s \frac{Y_t}{K_t} - \delta, \\
K_t = k_t L_t \;\Rightarrow\; g_K &= g_k + g_L = 0 + 0 \\
Y_t = y_t L_t \;\Rightarrow\; g_Y &= g_y + g_L = 0 + 0 \\
\frac{Y_t}{K_t} = \frac{y_t}{k_t} &= \frac{\delta}{s} \;\Leftarrow\; s y_t = \delta k_t \;\Leftarrow\; k_{t+1} - k_t = s y_t - \delta k_t \;\Leftarrow\; A_t = 1, L_t = L \\
\Rightarrow \quad g_K &= 0; \\
K_t = k_t L_t \;\Rightarrow\; g_K &= g_k + g_L = 0 + g_L \\
Y_t = y_t L_t \;\Rightarrow\; g_Y &= g_y + g_L = 0 + g_L \\
\frac{Y_t}{K_t} = \frac{y_t}{k_t} &= \frac{g_L + \delta}{s} \;\Leftarrow\; A_t = 1, L_t = (1 + g_L) L_{t-1} \\
\Rightarrow \quad g_K &= g_L; \\
K_t = k_t A_t L_t \;\Rightarrow\; g_K &= g_k + g_A + g_L = 0 + g_A + g_L \\
Y_t = y_t A_t L_t \;\Rightarrow\; g_Y &= g_y + g_A + g_L = 0 + g_A + g_L \\
\frac{Y_t}{K_t} = \frac{y_t}{k_t} &= \frac{g_A + g_L + g_A g_L + \delta}{s} \;\Leftarrow\; A_t = (1 + g_A) A_{t-1}, L_t = (1 + g_L) L_{t-1} \\
\Rightarrow \quad g_K &= g_A + g_L + g_A g_L.
\end{aligned}
$$

Notice that all of the above results are nothing to do with δ, and note that

$$
K_{t+1} = s Y_t + (1 - \delta) K_t,
$$

$$
\begin{aligned}
\xrightarrow{A_t=1,\, L_t=L} \quad \frac{K_{t+1}}{L} &= s\frac{Y_t}{L} + (1-\delta)\frac{K_t}{L}, \\
\Rightarrow \quad k_{t+1} &= sy_t + (1-\delta)k_t, \\
\Rightarrow \quad k_{t+1} - k_t &= sy_t - \delta k_t, \\
\Rightarrow \quad \frac{y_t}{k_t} &= \frac{\delta}{s};
\end{aligned}
$$

$$
\begin{aligned}
K_{t+1} &= sY_t + (1-\delta)K_t, \\
\xrightarrow{A_t=1,\, L_t=(1+g_L)L_{t-1}} \quad \frac{K_{t+1}}{L_t} &= s\frac{Y_t}{L_t} + (1-\delta)\frac{K_t}{L_t}, \\
\Rightarrow \quad \frac{K_{t+1}}{L_{t+1}}\frac{L_{t+1}}{L_t} &= s\frac{Y_t}{L_t} + (1-\delta)\frac{K_t}{L_t}, \\
\Rightarrow \quad (1+g_L)k_{t+1} &= sy_t + (1-\delta)k_t, \\
\Rightarrow \quad k_{t+1} &= \frac{s}{1+g_L}y_t + \frac{1-\delta}{1+g_L}k_t, \\
\Rightarrow \quad k_{t+1} - k_t &= \frac{s}{1+g_L}y_t - \left(1 - \frac{1-\delta}{1+g_L}\right)k_t, \\
\Rightarrow \quad \frac{y_t}{k_t} &= \frac{1-\frac{1-\delta}{1+g_L}}{\frac{s}{1+g_L}}, \\
&= \frac{g_L+\delta}{s};
\end{aligned}
$$

$$
\begin{aligned}
K_{t+1} &= sY_t + (1-\delta)K_t, \\
\xrightarrow{A_t=(1+g_A)A_{t-1},\, L_t=(1+g_L)L_{t-1}} \quad \frac{K_{t+1}}{A_tL_t} &= s\frac{Y_t}{A_tL_t} + (1-\delta)\frac{K_t}{A_tL_t}, \\
\Rightarrow \quad \frac{K_{t+1}}{A_{t+1}L_{t+1}}\frac{A_{t+1}L_{t+1}}{A_tL_t} &= s\frac{Y_t}{A_tL_t} + (1-\delta)\frac{K_t}{A_tL_t}, \\
\Rightarrow \quad (1+g_A)(1+g_L)k_{t+1} &= sy_t + (1-\delta)k_t, \\
\Rightarrow \quad k_{t+1} &= \frac{s}{(1+g_A)(1+g_L)}y_t + \frac{1-\delta}{(1+g_A)(1+g_L)}k_t, \\
\Rightarrow \quad k_{t+1} - k_t &= \frac{s}{(1+g_A)(1+g_L)}y_t - \left[1 - \frac{1-\delta}{(1+g_A)(1+g_L)}\right]k_t, \\
\Rightarrow \quad \frac{y_t}{k_t} &= \frac{1-\frac{1-\delta}{(1+g_A)(1+g_L)}}{\frac{s}{(1+g_A)(1+g_L)}}, \\
&= \frac{g_A+g_L+g_Ag_L+\delta}{s}.
\end{aligned}
$$

The critical point is that how to interpret the balanced growth level of capital and the balanced-growth-path level of capital per unit of effective labor. For the latter, I refer the reader to the solution:

3. Suppose that the production function is CD.

Problem 1.3

(a) The slope of the break-even investment line is given by (n + g + δ) and thus a fall in the rate of depreciation, δ, decreases the slope of the break-even investment line.

The actual investment curve, sf(k) is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k* to k^*_{NEW} .

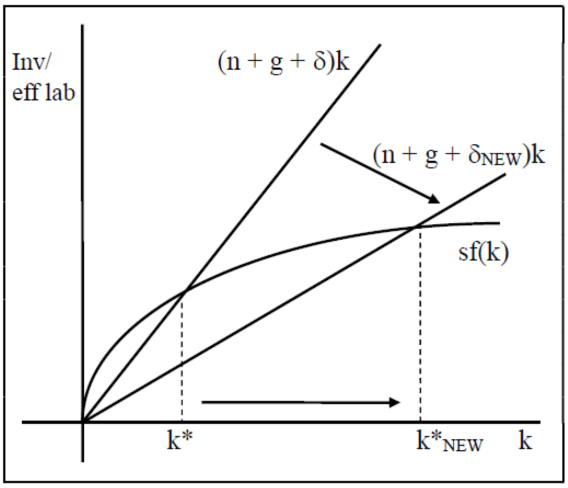

Figure 1: Solutions to ch1 of Romer (2018, 5th)

A. Express the production function $F(K_t, A_tL_t)$ in terms of output per unit of effective labor. That is, find $f(k_t)$.

$$
\begin{aligned}
& Y_t = F(K_t, A_tL_t) = K_t^{\alpha}(A_tL_t)^{1-\alpha}, \\
\Rightarrow \quad & \lambda^m Y_t = F(\lambda^m K_t, \lambda^m A_tL_t), \\
\xrightarrow{m=1,\, \lambda\equiv\frac{1}{A_tL_t}} \quad & \frac{Y_t}{A_tL_t} = F\left(\frac{K_t}{A_tL_t}, 1\right), \\
\Rightarrow \quad & y_t = f(k_t), \\
& \quad = k_t^{\alpha}.
\end{aligned}
$$

B. Find expression for k^*, y^* and c^* as functions of the parameters of the model g_A, g_L, s, δ, and α.

$$
\begin{aligned}
\frac{y^*}{k^*} &= \frac{1 - \frac{1-\delta}{(1+g_A)(1+g_L)}}{\frac{s}{(1+g_A)(1+g_L)}}, \\
&= \frac{g_A + g_L + g_Ag_L + \delta}{s}, \\
\Rightarrow \quad sy^* &= (g_A + g_L + g_Ag_L + \delta)k^*, \\
\Rightarrow \quad k^* &= \left(\frac{g_A + g_L + g_Ag_L + \delta}{s}\right)^{\frac{1}{\alpha-1}}, \\
&\overset{\text{or}}{\equiv} \left(\frac{s}{g_A + g_L + g_Ag_L + \delta}\right)^{\frac{1}{1-\alpha}}; \\
\Rightarrow \quad y^* &= (k^*)^{\alpha}, \\
&= \left(\frac{s}{g_A + g_L + g_Ag_L + \delta}\right)^{\frac{\alpha}{1-\alpha}}; \\
\Rightarrow \quad c* &= (1-s)y^*, \\
&= (1-s)\left(\frac{s}{g_A + g_L + g_Ag_L + \delta}\right)^{\frac{\alpha}{1-\alpha}}.
\end{aligned}
$$

C. What is the golden-rule value of k^*? (let it k^{**})

$$
\begin{aligned}
c^* &= (1-s)y^*, \\
&= y^* - \underbrace{sy^*}, \\
&= f(k^*(s)) - \overbrace{(g_A + g_L + g_A g_L + \delta)k^*(s)}, \\
\Rightarrow \quad \frac{\partial c^*}{\partial s} = 0 &= f'(k^*)k_s^* - (g_A + g_L + g_A g_L + \delta)k_s^*, \\
&= [f'(k^*) - (g_A + g_L + g_A g_L + \delta)]k_s^*, \\
\Rightarrow \quad f'(k^{**}) &= g_A + g_L + g_A g_L + \delta, \\
\Rightarrow \quad \alpha(k^{**})^{\alpha-1} &= g_A + g_L + g_A g_L + \delta, \\
\Rightarrow \quad k^{**} &= \left(\frac{g_A + g_L + g_A g_L + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}}, \\
&\overset{\text{or}}{=} \left(\frac{\alpha}{g_A + g_L + g_A g_L + \delta}\right)^{\frac{1}{1-\alpha}}.
\end{aligned}
$$

D. What savings rate is needed to yield the golden-rule value of k^*?

$$
\begin{aligned}
sf(k^*) &= (g_A + g_L + g_A g_L + \delta)k^*, \\
\Rightarrow \quad s &= \frac{(g_A + g_L + g_A g_L + \delta)k^*}{f(k^*)}, \\
&= \frac{f'(k^{**})k^{**}}{f(k^{**})}, \\
&= \alpha \in (0,1).
\end{aligned}
$$

4. Assume labor and capital are paid their marginal products in perfectly competitive markets. That is $R_t = F_K(K_t, A_tL_t)\cancel{-\delta}$ and $W_t = F_L(K_t, A_tL_t)$.

$$
\max_{K_t, L_t} \Pi_t = F(K_t, A_tL_t) - W_tL_t - R_tK_t \cancel{-\delta K_t}.
$$

A. Show that the marginal product of labor $F_L(K_t, A_tL_t)$ is equal to $A_t[f(k_t) - k_tf'(k_t)]$.

$$
\begin{aligned}
Y_t &= F(K_t, A_tL_t), \\
\Rightarrow \quad \frac{Y_t}{A_tL_t} &= f\left(\frac{K_t}{A_tL_t}\right) = f(k_t), \\
\Rightarrow \quad Y_t &= A_tL_tf\left(\frac{K_t}{A_tL_t}\right), \\
\Rightarrow \quad \frac{\partial Y_t}{\partial K_t} &\equiv F_K = f'(k_t). \\
Y_t &\overset{\text{CRS}}{=} F_KK_t + F_LL_t, \\
\Rightarrow \quad \frac{Y_t}{L_t} &= F_K\frac{K_t}{L_t} + F_L, \\
\Rightarrow \quad A_t\frac{Y_t}{A_tL_t} &= A_tF_K\frac{K_t}{A_tL_t} + F_L,
\end{aligned}
$$

$$\Rightarrow \quad F_L = A_t[f(k_t) - k_t f'(k_t)].$$

Another method:

$$\begin{aligned}
F(K_t, A_t L_t) = Y_t &= A_t L_t f\left(\frac{K_t}{A_t L_t}\right), \\
\Rightarrow \quad W_t &\equiv \frac{\partial F(K_t, A_t L_t)}{\partial L_t}, \\
&= A_t f(k_t) + A_t L_t f'(K_t)\left(-\frac{K_t}{A_t L_t^2}\right), \\
&= A_t[f(k_t) - k_t f'(k_t)].
\end{aligned}$$

B. Show that under a CRS production function $F(K_t, A_t L_t) = R_t K_t + W_t L_t$.

$$\left.\begin{aligned}
R_t &= F_K(K_t, A_t L_t) \\
W_t &= F_L(K_t, A_t L_t) \\
Y_t &= F(K_t, A_t L_t) \\
Y_t &\overset{\text{CRS}}{=} F_K K_t + F_L L_t
\end{aligned}\right\} \Rightarrow \quad F(K_t, A_t L_t) = R_t K_t + W_t L_t.$$

C. Calculate the growth rates of W_t and R_t on a balanced growth path.

$$\begin{aligned}
W_t &= A_t[f(k_t) - k_t f'(k_t)], \\
\Rightarrow \quad \log W_t &= \log A_t + \log[f(k_t) - k_t f'(k_t)], \\
\Rightarrow \quad \frac{\mathrm{d}\log W_t}{\mathrm{d}t} &= \frac{\mathrm{d}\log A_t}{\mathrm{d}t} + \frac{\mathrm{d}\log[f(k_t) - k_t f'(k_t)]}{\mathrm{d}t}, \\
\Rightarrow \quad \frac{\mathrm{d}W_t}{W_t} &= \frac{\mathrm{d}A_t}{A_t} + \frac{f'(k_t)\frac{\mathrm{d}k_t}{\mathrm{d}t} - \frac{\mathrm{d}k_t}{\mathrm{d}t} f'(k_t) - k_t f''(k_t)\frac{\mathrm{d}k_t}{\mathrm{d}t}}{f(k_t) - k_t f'(k_t)}, \\
&= g_A - \frac{k_t f''(k_t)\frac{\mathrm{d}k_t}{\mathrm{d}t}}{f(k_t) - k_t f'(k_t)}, \\
&\overset{\frac{\mathrm{d}k_t}{\mathrm{d}t} \overset{\text{BGP}}{=} 0}{=} g_A.
\end{aligned}$$

$$\begin{aligned}
R_t &= f'(k_t), \\
\Rightarrow \quad \log R_t &= \log f'(k_t), \\
\Rightarrow \quad \frac{\mathrm{d}\log R_t}{\mathrm{d}t} &= \frac{\mathrm{d}\log f'(k_t)}{\mathrm{d}t}, \\
\Rightarrow \quad \frac{\mathrm{d}R_t}{R_t} &= \frac{f''(k_t)\frac{\mathrm{d}k_t}{\mathrm{d}t}}{f'(k_t)}, \\
&\overset{\frac{\mathrm{d}k_t}{\mathrm{d}t} \overset{\text{BGP}}{=} 0}{=} 0.
\end{aligned}$$

D. Suppose the economy begins with a level of $k_t < k^*$. As moves toward k^*, is W_t growing at a greater than, less than, or equal to its growth rate on the balanced growth path? What about R_t?

I refer the reader to the solution to the problem 1.9 of Romer (2018, 5th).

1.2 Exercises in Lec2

1.3 Exercises in Lec3