

Forecasts Based on Conditional Expectation

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1 Questions

- 1) Which series can be forecast?
- 2) How to forecast time series?
- 3) Unconditional moments vs. Conditional moments
- 4) Variance vs. Forecast error variance
- 5) State variables vs. Control variables (rf. McCandless 2008, p.51)

2 MSE

A quadratic loss function is choosed to

$$\min \text{MSE}(y_{t+1}^*|t) \equiv \mathbb{E}(y_t - y_{t+1}^*)^2, \quad \text{where } y_{t+1}^*|t = \mathbb{E}(y_{t+1}|y_t, y_{t-1}, \dots) \equiv \mathbb{E}_t y_{t+1}.$$

The MSE of this optimal forecast is

$$\mathbb{E}(y_{t+1} - \mathbb{E}_t y_{t+1})^2.$$

See Hamilon (1994, pp.72-73)

3 Forecasts of ARMA Models

- 1) MA(∞)

$$y_t = \sum_{i=0}^{\infty} \theta_i \epsilon_{t-i} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots, \quad \theta_0 \equiv 1.$$

$$\begin{aligned}
&\xrightarrow{\text{move forward}} \begin{cases} y_{t+1} = \sum_{i=0}^{\infty} \theta_i \epsilon_{t+1-i} = \epsilon_{t+1} + \theta_1 \epsilon_t + \theta_2 \epsilon_{t-1} + \theta_3 \epsilon_{t-2} + \dots, \\ y_{t+2} = \sum_{i=0}^{\infty} \theta_i \epsilon_{t+2-i} = \epsilon_{t+2} + \theta_1 \epsilon_{t+1} + \theta_2 \epsilon_t + \theta_3 \epsilon_{t-1} + \dots, \\ \vdots \\ y_{t+h} = \sum_{i=0}^{\infty} \theta_i \epsilon_{t+h-i} = \epsilon_{t+h} + \theta_1 \epsilon_{t+h-1} + \theta_2 \epsilon_{t+h-2} + \dots + \theta_h \epsilon_t + \dots + \theta_{h+1} \epsilon_{t-1} + \dots; \end{cases} \\
&\xrightarrow{\text{forecast ahead}} \begin{cases} \mathbb{E} y_t = 0, \\ \mathbb{E}_{\textcolor{red}{t}} y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots, \\ \mathbb{E}_t y_{t+1} = \mathbb{E}_t(\epsilon_{t+1} + \theta_1 \epsilon_t + \theta_2 \epsilon_{t-1} + \theta_3 \epsilon_{t-2} + \dots) = 0 + \theta_1 \epsilon_t + \theta_2 \epsilon_{t-1} + \theta_3 \epsilon_{t-2} + \dots, \\ \mathbb{E}_t y_{t+2} = \mathbb{E}_t(\epsilon_{t+2} + \theta_1 \epsilon_{t+1} + \theta_2 \epsilon_t + \theta_3 \epsilon_{t-1} + \dots) = 0 + 0 + \theta_2 \epsilon_t + \theta_3 \epsilon_{t-1} + \dots, \\ \vdots \\ \mathbb{E}_t y_{t+h} = \mathbb{E}_t(\epsilon_{t+h} + \theta_1 \epsilon_{t+h-1} + \dots + \theta_h \epsilon_t + \theta_{h+1} \epsilon_{t-1} + \dots) = \theta_h \epsilon_t + \theta_{h+1} \epsilon_{t-1} + \dots; \end{cases} \\
&\xrightarrow{\text{variance}} \begin{cases} \text{var}_t y_t = \mathbb{E}[(y_t - \mathbb{E} y_t)^2] = \mathbb{E}(y_t^2) = (1 + \theta_1^2 + \theta_2^2 + \dots) \sigma_{\epsilon}^2, \\ \text{var}_t y_t = \mathbb{E}_t[(y_t - \mathbb{E}_t y_t)^2] = 0, \\ \text{var}_t y_{t+1} = \mathbb{E}_t[(y_{t+1} - \mathbb{E}_t y_{t+1})^2] = \mathbb{E}_t \epsilon_{t+1}^2 = \sigma_{\epsilon}^2, \\ \text{var}_t y_{t+2} = \mathbb{E}_t[(y_{t+2} - \mathbb{E}_t y_{t+2})^2] = \mathbb{E}_t(\epsilon_{t+2} + \theta_1 \epsilon_{t+1})^2 = (1 + \theta_1^2) \sigma_{\epsilon}^2, \\ \vdots \\ \text{var}_t y_{t+h} = \mathbb{E}_t[(y_{t+h} - \mathbb{E}_t y_{t+h})^2] = (1 + \theta_1^2 + \theta_2^2 + \dots + \theta_{h-1}^2) \sigma_{\epsilon}^2. \end{cases}
\end{aligned}$$

$$\begin{cases} \mathbb{E}_t y_{t+h} \equiv \mathbb{E}(y_{t+h} | y_t, y_{t-1}, y_{t-2}, \dots, \epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \dots) \leftarrow \text{form predictions of } y \text{ given its past} \\ \text{var}_t y_{t+h} \equiv \text{var}(y_{t+h} | y_t, y_{t-1}, y_{t-2}, \dots, \epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \dots) \leftarrow \text{how certain about the prediction} \end{cases}$$

2) AR(1)

$$y_t = \phi y_{t-1} + \epsilon_t = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}.$$

$$\begin{aligned}
& \xrightarrow{\text{move forward}} \begin{cases} y_{t+1} = \phi y_t + \epsilon_{t+1}, \\ y_{t+2} = \phi y_{t+1} + \epsilon_{t+2} = \phi(\phi y_t + \epsilon_{t+1}) + \epsilon_{t+2} = \phi^2 y_t + \phi \epsilon_{t+1} + \epsilon_{t+2}, \\ \vdots \\ y_{t+h} = \phi y_{t+h-1} + \epsilon_{t+h}; \end{cases} \\
& \xrightarrow{\text{unconditional expectation}} \mathbb{E} y_t = 0, \\
& \xrightarrow{\text{conditional expectation}} \mathbb{E}_t y_t = \phi y_{t-1}, \\
& \xrightarrow{\text{1 step ahead forecast}} \mathbb{E}_t y_{t+1} = \mathbb{E}_t(\phi y_t) + \mathbb{E}_t \epsilon_{t+1} = \phi \mathbb{E}_t y_t + 0 = \phi y_t, \\
& \xrightarrow{\text{2 steps ahead forecasts}} \mathbb{E}_t y_{t+2} = \mathbb{E}_t(\phi y_{t+1}) = \phi \mathbb{E}_t y_{t+1} = \phi(\phi y_t) = \phi^2 y_t, \\
& \quad \vdots \\
& \xrightarrow{\text{h steps ahead forecasts}} \mathbb{E}_t y_{t+h} = \mathbb{E}_t(\phi y_{t+h-1}) = \phi \mathbb{E}_t y_{t+h-1} = \phi^h y_t; \\
& \xrightarrow{\text{forecast error variances}} \begin{cases} \text{var}_t y_t = \mathbb{E}[(y_t - \mathbb{E} y_t)^2] = \mathbb{E} y_t^2 = \mathbb{E}[(\epsilon_t + \phi \epsilon_{t-1} + \phi^2 \epsilon_{t-2} + \dots)^2] = \frac{\sigma_\epsilon^2}{1 - \phi^2}, \\ \text{var}_t y_t = \mathbb{E}_t[(y_t - \mathbb{E}_t y_t)^2] = \mathbb{E}_t[(y_t - \phi y_{t-1})^2] = \mathbb{E}_t \epsilon_t^2 = \sigma_\epsilon^2, \\ \text{var}_t y_{t+1} = \mathbb{E}_t[(y_{t+1} - \mathbb{E}_t y_{t+1})^2] = \mathbb{E}_t[(y_{t+1} - \phi y_t)^2] = \sigma_\epsilon^2, \\ \text{var}_t y_{t+2} = \mathbb{E}_t[(y_{t+2} - \phi^2 y_t)^2] = \mathbb{E}_t[(\phi \epsilon_{t+1} + \epsilon_{t+2})^2] = (1 + \phi^2) \sigma_\epsilon^2, \\ \vdots \\ \text{var}_t y_{t+h} = \mathbb{E}_t[(y_{t+h} - \phi^h y_t)^2] = (1 + \phi^2 + \phi^4 + \dots + \phi^{2(h-1)}) \sigma_\epsilon^2. \end{cases}
\end{aligned}$$

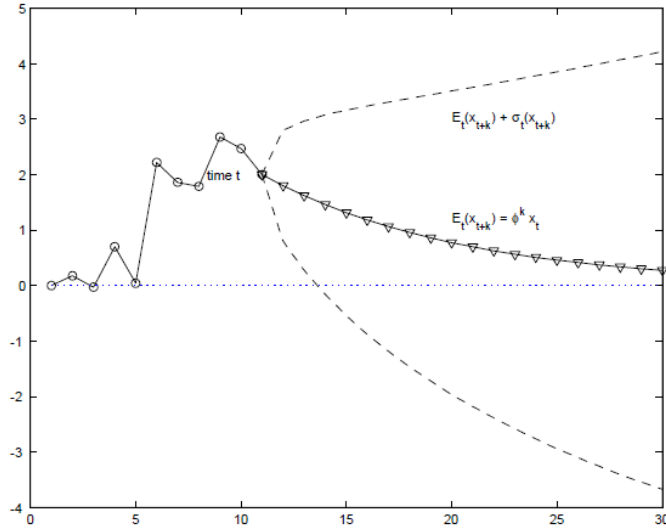


Figure 1: AR(1) forecast and standard deviation ¹

4 Forecasts of VAR Processes

1) Vector MA(∞)

$$y_t = \sum_{i=0}^{\infty} \theta_i \epsilon_{t-i} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots, \quad \theta_0 \equiv \mathbf{I}.$$

¹Source: Cochrane (2005, p.33)

$$\begin{aligned}\mathbb{E}_t \mathbf{y}_{t+h} &= \boldsymbol{\theta}_h \boldsymbol{\epsilon}_t + \boldsymbol{\theta}_{h+1} \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\theta}_{h+2} \boldsymbol{\epsilon}_{t-2} + \cdots; \\ \text{var}_t \mathbf{y}_{t+h} &= \boldsymbol{\sigma}^2 + \boldsymbol{\theta}_1 \boldsymbol{\sigma}^2 \boldsymbol{\theta}'_1 + \cdots + \boldsymbol{\theta}_{h-1} \boldsymbol{\sigma}^2 \boldsymbol{\theta}'_{h-1}.\end{aligned}$$

2) VAR(1)

$$\begin{aligned}\mathbf{y}_t &= \mathbf{F} \mathbf{y}_{t-1} + \mathbf{G} \boldsymbol{\epsilon}_t \quad \Leftrightarrow \quad \mathbf{y}_t = \mathbf{F} \mathbf{y}_{t-1} + \mathbf{G} \boldsymbol{\xi}_t. \\ \mathbf{G} &\equiv \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \sigma^2 \\ 0 \\ \sigma^2 \end{bmatrix} \quad \mathbb{E}(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}'_t) = \boldsymbol{\sigma}^2 \rightarrow \mathbb{E} \boldsymbol{\xi}_t \boldsymbol{\xi}'_t = \mathbf{I}.\end{aligned}$$

$$\begin{aligned}\mathbf{y}_t &= \mathbf{F} \mathbf{y}_{t-1} + \mathbf{G} \boldsymbol{\xi}_t = \sum_{i=0}^{\infty} \mathbf{F}^i \mathbf{G} \boldsymbol{\xi}_{t-i}. \\ \mathbf{y}_{t+1} &= \mathbf{F} \mathbf{y}_t + \mathbf{G} \boldsymbol{\xi}_{t+1} \\ \mathbf{y}_{t+2} &= \mathbf{F} \mathbf{y}_{t+1} + \mathbf{G} \boldsymbol{\xi}_{t+2} = \mathbf{F}^2 \mathbf{y}_t + \mathbf{F} \mathbf{G} \boldsymbol{\xi}_{t+1} + \mathbf{G} \boldsymbol{\xi}_{t+2} \\ &\vdots \\ \mathbf{y}_{t+h} &= \mathbf{F} \mathbf{y}_{t+h-1} + \mathbf{G} \boldsymbol{\xi}_{t+h}. \\ \mathbb{E}_t \mathbf{y}_{t+1} &= \mathbb{E}_t(\mathbf{F} \mathbf{y}_t) = \mathbf{F} \mathbb{E}_t \mathbf{y}_t = \mathbf{F} \mathbf{y}_t; \\ \mathbb{E}_t \mathbf{y}_{t+2} &= \mathbb{E}_t(\mathbf{F} \mathbf{y}_{t+1}) = \mathbf{F} \mathbb{E}_t \mathbf{y}_{t+1} = \mathbf{F}^2 \mathbf{y}_t; \\ &\vdots \\ \mathbb{E}_t \mathbf{y}_{t+h} &= \mathbb{E}_t(\mathbf{F} \mathbf{y}_{t+h-1}) = \mathbf{F} \mathbb{E}_t \mathbf{y}_{t+h-1} = \mathbf{F}^h \mathbf{y}_t.\end{aligned}$$

$$\begin{aligned}\text{var}_t \mathbf{y}_{t+1} &= \mathbb{E}_t[(\mathbf{y}_{t+1} - \mathbb{E}_t \mathbf{y}_{t+1})(\mathbf{y}_{t+1} - \mathbb{E}_t \mathbf{y}_{t+1})'] = \mathbb{E}_t[(\mathbf{G} \boldsymbol{\epsilon}_{t+1})(\mathbf{G} \boldsymbol{\epsilon}_{t+1})'] = \mathbf{G} \mathbf{G}', \\ \text{var}_t \mathbf{y}_{t+2} &= \mathbb{E}_t[(\mathbf{y}_{t+2} - \mathbb{E}_t \mathbf{y}_{t+2})(\mathbf{y}_{t+2} - \mathbb{E}_t \mathbf{y}_{t+2})'] = \mathbb{E}_t[(\mathbf{F} \mathbf{G} \boldsymbol{\epsilon}_{t+1} + \mathbf{G} \boldsymbol{\epsilon}_{t+2})(\mathbf{F} \mathbf{G} \boldsymbol{\epsilon}_{t+1} + \mathbf{G} \boldsymbol{\epsilon}_{t+2})'] = \mathbf{F} \mathbf{G} \mathbf{G}' \mathbf{F} + \mathbf{G} \mathbf{G}', \\ &\vdots \\ \text{var}_t \mathbf{y}_{t+h} &= \mathbb{E}_t[(\mathbf{y}_{t+h} - \mathbb{E}_t \mathbf{y}_{t+h})(\mathbf{y}_{t+h} - \mathbb{E}_t \mathbf{y}_{t+h})'] = \sum_{i=0}^{h-1} \mathbf{F}^i \mathbf{G} \mathbf{G}' \mathbf{F}^i.\end{aligned}$$

$$\xrightarrow{\text{programming in a simple loop}} \begin{cases} \mathbb{E}_t \mathbf{y}_{t+h} = \mathbf{F} \mathbb{E}_t \mathbf{y}_{t+h-1}, \\ \text{var}_t \mathbf{y}_{t+h} = \mathbf{F} \text{var}_t \mathbf{y}_{t+h-1} \mathbf{F}' + \mathbf{G} \mathbf{G}'. \end{cases}$$

5 The State-Space Representation

Any process can be mapped into a VAR(1), which leads to easy programming of forecasts.

1) MA(1) → VAR(1)

$$y_t = c + \epsilon_t + \theta \epsilon_{t-1} \xleftrightarrow{\mu \equiv \mathbb{E} y_t = c} y_t = \mu + \epsilon_t + \theta \epsilon_{t-1}.$$

$$\begin{aligned}\begin{bmatrix} \epsilon_{t+1} \\ \epsilon_t \end{bmatrix} &\stackrel{\text{state equation}}{=} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \epsilon_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{t+1} \\ 0 \end{bmatrix}, \\ y_t &\stackrel{\text{observation eq.}}{=} \mu + [1 \quad \theta] \begin{bmatrix} \epsilon_t \\ \epsilon_{t-1} \end{bmatrix}.\end{aligned}$$

2) AR(p)→VAR(1)

$$\begin{cases} y_t - \mu = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \epsilon_t, \\ y_{t+1} - \mu = \phi_1(y_t - \mu) + \phi_2(y_{t-1} - \mu) + \cdots + \phi_p(y_{t+1-p} - \mu) + \epsilon_{t+1}. \end{cases}$$

$$\begin{bmatrix} y_{t+1} - \mu \\ y_t - \mu \\ \vdots \\ y_{t+1-(p-2)} - \mu \\ y_{t+1-(p-1)} - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t+1-(p-1)} - \mu \\ y_{t+1-p} - \mu \end{bmatrix} + \begin{bmatrix} \epsilon_{t+1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$y_t = \mu + [1 \ 0 \ \cdots \ 0] \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t+1-(p-1)} - \mu \\ y_{t+1-p} - \mu \end{bmatrix}$$

3) VAR(∞)→VAR(1)

$$\begin{cases} y_t = \phi_{yy}^{(1)} y_{t-1} + \phi_{yy}^{(2)} y_{t-2} + \cdots + \phi_{yz}^{(1)} z_{t-1} + \phi_{yz}^{(2)} z_{t-2} + \cdots + \epsilon_{yt}, \\ z_t = \phi_{zy}^{(1)} y_{t-1} + \phi_{zy}^{(2)} y_{t-2} + \cdots + \phi_{zz}^{(1)} z_{t-1} + \phi_{zz}^{(2)} z_{t-2} + \cdots + \epsilon_{zt} \end{cases}$$

$$\begin{bmatrix} y_t \\ z_t \\ y_{t-1} \\ z_{t-1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \phi_{yy}^{(1)} & \phi_{yz}^{(1)} & \phi_{yy}^{(2)} & \phi_{yz}^{(2)} & \cdots \\ \phi_{zy}^{(1)} & \phi_{zz}^{(1)} & \phi_{zy}^{(2)} & \phi_{zz}^{(2)} & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \\ y_{t-2} \\ z_{t-2} \\ \vdots \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{zt} \end{bmatrix}$$

$$\mathbf{x}_t = \mathbf{F} \mathbf{x}_{t-1} + \mathbf{G} \boldsymbol{\epsilon}_t \quad \begin{cases} \mathbb{E} \boldsymbol{\epsilon}_t = \mathbf{0} \\ \mathbb{E}(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t') = \boldsymbol{\sigma}^2 \end{cases}$$

4) ARMA(2, 1)→VAR(1)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1}.$$

$$\begin{bmatrix} y_t \\ y_{t-1} \\ \epsilon_t \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \theta_1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \epsilon_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \epsilon_t.$$

$$\mathbf{y}_t = \mathbf{F} \mathbf{y}_{t-1} + \mathbf{G} \epsilon_t \quad \Leftrightarrow \quad \mathbf{y}_t = \mathbf{F} \mathbf{y}_{t-1} + \mathbf{G} \xi_t.$$

$$\mathbf{G} \equiv \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_\epsilon^2 \\ 0 \\ \sigma_\epsilon^2 \end{bmatrix} \quad \mathbb{E} \epsilon_t^2 = \sigma_\epsilon^2 \rightarrow \mathbb{E} \xi_t^2 = 1.$$

5) ARMA(2, 2)→VAR(1)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}.$$

$$\begin{bmatrix} y_t \\ y_{t-1} \\ \epsilon_t \\ \epsilon_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \theta_1 & \theta_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \epsilon_{t-1} \\ \epsilon_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \epsilon_t.$$

6) ARMA(p, q)→VAR(1)

$$\begin{cases} y_t - \mu = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \dots + \phi_p(y_{t-p} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \dots + \theta_q\epsilon_{t-q}, \\ y_{t+1} - \mu = \phi_1(y_t - \mu) + \phi_2(y_{t-1} - \mu) + \dots + \phi_p(y_{t+1-p} - \mu) + \epsilon_{t+1} + \theta_1\epsilon_t + \theta_2\epsilon_{t-1} + \dots + \theta_q\epsilon_{t+1-q}. \end{cases}$$

$$\begin{bmatrix} y_{t+1} - \mu \\ y_t - \mu \\ \vdots \\ y_{t+1-(p-2)} - \mu \\ y_{t+1-(p-1)} - \mu \\ \epsilon_{t+1} \\ \epsilon_t \\ \vdots \\ \epsilon_{t+1-(q-2)} \\ \epsilon_{t+1-(q-1)} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_{p-1} & \phi_p & \theta_1 & \theta_2 & \dots & \theta_{q-1} & \theta_q \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t+1-(p-1)} - \mu \\ y_{t+1-p} - \mu \\ \epsilon_t \\ \epsilon_{t-1} \\ \vdots \\ \epsilon_{t+1-(q-1)} \\ \epsilon_{t+1-q} \end{bmatrix} + \begin{bmatrix} \epsilon_{t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_t = \mu + [1 \ 0 \ \dots \ 0 \ 1 \ \theta_1 \ \dots \ \theta_{q-1}] \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t+1-(p-1)} - \mu \\ y_{t+1-p} - \mu \\ \epsilon_t \\ \epsilon_{t-1} \\ \vdots \\ \epsilon_{t+1-(q-1)} \\ \epsilon_{t+1-q} \end{bmatrix}$$

6) ARMA(2, 2) with two variables y_t and z_t →VAR(1)

$$\begin{cases} y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}, \\ z_t = \rho z_{t-1} + \epsilon_t. \end{cases}$$

$$\begin{bmatrix} y_t \\ y_{t-1} \\ z_t \\ \epsilon_t \\ \epsilon_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & 0 & \theta_1 & \theta_2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ z_{t-1} \\ \epsilon_{t-1} \\ \epsilon_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \epsilon_t.$$

6 How to Calculate the Impulse-Response

The impulse response function is the path that y follows if it is kicked by a single unit shock. It allows us to start thinking about “causes” and “effects”.

1) MA(∞) or AR(1)

$$\begin{cases} y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots = \sum_{i=0}^{\infty} \theta_i \epsilon_{t-i}, & \theta_0 \equiv 1; \\ y_t = \phi y_{t-1} + \epsilon_t & \text{the closed form solution} \quad \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}. \end{cases}$$

Table 1: IRF of two simple processes							
MA(∞)	ϵ_t :	0	0	1	0	0	...
	y_t :	0	0	1	θ_0	θ_1	θ_2 ...
AR(1)	ϵ_t :	0	0	1	0	0	...
	y_t :	0	0	1	ϕ	ϕ^2	ϕ^3 ...

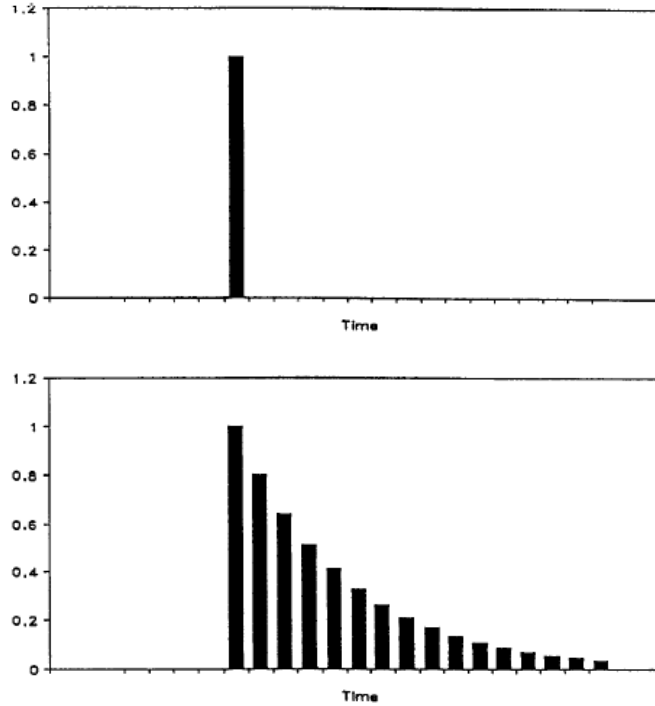


Figure 2: Value of y when it's kicked by a single unit shock ϵ_t (i.e., $\epsilon_{t-h} = \epsilon_{t+h} = 0$ but $\epsilon_t = 1$)¹

2) Vector MA(∞) or VAR(1)

$$\mathbf{x}_t = (\mathbf{B}_0 + \mathbf{B}_1 L + \mathbf{B}_2 L^2 + \dots) \epsilon_t \equiv \mathbf{B}(L) \epsilon_t \quad \Leftrightarrow \quad \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \theta_{yy}(L) & \theta_{yz}(L) \\ \theta_{zy}(L) & \theta_{zz}(L) \end{bmatrix} \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{zt} \end{bmatrix},$$

so $\theta_{yy}(L)$ gives the response of y_{t+h} to a unit shock ϵ_{yt} , and $\theta_{yz}(L)$ gives the response of y_{t+h} to a unit shock ϵ_{zt} .

$$\mathbf{x}_t = \mathbf{F} \mathbf{x}_{t-1} + \mathbf{G} \epsilon_t = \sum_{i=0}^{\infty} \mathbf{F}^i \mathbf{G} \epsilon_{t-i} \xrightarrow{\sum_{i=0}^{\infty} \beta^i \frac{\partial \mathbf{x}_t}{\partial \epsilon_0}} \mathbf{G}, \mathbf{F} \mathbf{G}, \mathbf{F}^2 \mathbf{G}, \dots, \mathbf{F}^i \mathbf{G}, \dots \leftarrow \epsilon_0 = 1, \beta = 1.$$

7 Numerical Solution Using Dynare

see Miao(2014, pp.32,69)

¹Source: Hamilton (1994, p.5)