Lag Operators and Matrix Operation

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December 30, 2019

1 Exercises and Questions

1. Exercises:

Enders (2015, ch.1: E1, E2, E3, E4)

2. Qusetions:

(1) What is a time series? Why firstly use a linear DE to describe a time series?

(2) Why we need a solution to DEs and why the solution be requested to converge?

(3) Why we have developed at least 4 methods (backward iteration et al.) to solve DEs?

(4) Why not make much use of forward iteration?

(5) Why p_t^h represents the deviation from the intertemporal equilibrium and p_t^p means the intertemporal equilibrum?

2 Outline of 2 Groups of DEs and 4 Solution Methods

1. Linear DEs with constant coefficients and constant terms (1st- vs. 2nd- vs. pth- order);

2. Linear DEs with constant coefficients and variable terms (deterministic vs. stochastic terms; 1st- vs. 2nd- vs. pth- order).

(1) Iteration (with- vs. without- an initial condition)

(2) $Y_t = Y_t^h + Y_t^p$, where $y_t^h = A\lambda^t$ and $y_t^p = k, kt, kt^2, \dots$

(3) Lag Operators

(4) Matrix Operation

3 Lag- & Forward- Operators

3.1 The properties of lag operators

$$
\begin{aligned}
L^i y_t &\equiv y_{t-i},\\
L^{-i} y_t &= y_{t+i} \equiv F^i y_t,\\
Lc &= c,\\
(L^i + L^j) y_t &= \cdots \text{the distributive algebraic laws for “+” and “}\times\text{”}\\
L^i L^j y_t &= L^i (L^j y_t) = \cdots \text{the associative algebraic laws}\\
L^i L^j y_t &= L^j (L^i y_t) = \cdots \text{the commutative algebraic laws}\\
L^i L^j y_t &= L^{i+j} y_t = \cdots\\
L^0 y_t &= \cdots\\
(1 + \phi L + \phi^2 L^2 + \cdots) y_t &= \sum_{i=0}^{\infty} (\phi L)^i y_t = \frac{y_t}{1 - \phi L}, \quad \text{for } |\phi| < 1;\\
[1 + (\phi L)^{-1} + (\phi L)^{-2} + \cdots] y_t &= \frac{1}{1 - (\phi L)^{-1}} y_t = \frac{-\phi L}{1 - \phi L} y_t, \quad \text{for } |\phi| > 1,\\
\Rightarrow \frac{y_t}{1 - \phi L} &= -(\phi L)^{-1} \sum_{i=0}^{\infty} (\phi L)^{-i} y_t, \quad \text{for } |\phi| > 1.
\end{aligned}
$$

The 1st- & 2nd- & pth- & (p, q)- order equation

$$
\begin{aligned}
y_t = c + \phi y_{t-1} + \epsilon_t \Rightarrow y_t = c + \phi L y_t + \epsilon_t \Rightarrow (1 - \phi L) y_t = c + \epsilon_t &\xrightarrow{\phi(L) \equiv 1 - \phi L} \phi(L) y_t = c + \epsilon_t;\\
y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t \ \Rightarrow \ (1 - \phi_1 L - \phi_2 L^2) y_t = c + \epsilon_t &\xRightarrow{\phi(L) \equiv 1 - \phi_1 L - \phi_2 L^2} \phi(L) y_t = c + \epsilon_t;\\
y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \epsilon_t &\xRightarrow{\phi(L) \equiv 1 - \phi_1 L - \cdots - \phi_p L^p} \phi(L) y_t = c + \epsilon_t;\\
y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q} &\xRightarrow{\theta(L) \equiv 1 + \theta_1 L + \cdots + \theta_q L^q} \phi(L) y_t = c + \theta(L) \epsilon_t.
\end{aligned}
$$

3.2 Using lag operators to solve DEs

1) AR(1) $(|\phi| \lesseqgtr 1)$ note that $|\phi| \neq 1$

$$
\begin{aligned}
y_t &= c + \phi y_{t-1} + \epsilon_t, \quad \text{where } |\phi| < 1,\\
\Rightarrow (1 - \phi L) y_t &= c + \epsilon_t\\
&= \frac{c}{1 - \phi L} + \frac{\epsilon_t}{1 - \phi L}\\
&= (1 + \phi L + \phi^2 L^2 + \cdots) c + (1 + \phi L + \phi^2 L^2 + \cdots) \epsilon_t\\
&= (1 + \phi + \phi^2 + \cdots) c + (\epsilon_t + \phi \epsilon_{t-1} + \epsilon_{t-2} + \cdots)\\
&= \frac{1}{1 - \phi} c + \epsilon_t + \phi \epsilon_{t-1} + \epsilon_{t-2} + \cdots\\
\Rightarrow y_t^p &= \frac{c}{1 - \phi} + \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}.
\end{aligned}
$$

$$
\begin{aligned}
y_t &= c + \phi y_{t-1} + \epsilon_t, \quad \text{where } |\phi| > 1, \\
\Rightarrow (1-\phi L) y_t &= c + \epsilon_t \\
\Rightarrow y_t &= \frac{c}{1-\phi L} + \frac{\epsilon_t}{1-\phi L} \quad \overset{\text{or}}{=} \frac{c}{1-\phi L} + \frac{\epsilon_t(\phi L)^{-1}}{(1-\phi L)(\phi L)^{-1}} = \frac{c}{1-\phi} - \frac{(\phi L)^{-1}\epsilon_t}{1-(\phi L)^{-1}} \\
\Rightarrow &= \frac{c}{1-\phi} - (\phi L)^{-1} \sum_{i=0}^{\infty} (\phi L)^{-i} \epsilon_t \\
&= \frac{c}{1-\phi} - \phi^{-1} \sum_{i=0}^{\infty} (\phi L)^{-i} (L^{-1}\epsilon_t) \\
&= \frac{c}{1-\phi} - \phi^{-1} \sum_{i=0}^{\infty} \phi^{-i} [(L^{-i}L^{-1})\epsilon_t] \\
\Rightarrow y_t^p &= \frac{c}{1-\phi} - \phi^{-1} \sum_{i=0}^{\infty} \phi^{-i} \epsilon_{t+i+1}.
\end{aligned}
$$

2) ARMA(1, 1)

$$
\begin{aligned}
y_t &= c + \phi y_{t-1} + \epsilon_t + \theta \epsilon_{t-1}, \quad \text{where } |\phi| < 1 \\
\Rightarrow y_t &= \frac{c}{1-\phi L} + \frac{(1+\theta L)\epsilon_t}{1-\phi L} \\
&= \frac{c}{1-\phi} + \frac{\epsilon_t}{1-\phi L} + \frac{\theta \epsilon_{t-1}}{1-\phi L}, \quad \text{note that } \phi \neq 1. \\
&= \frac{c}{1-\phi} + (1 + \phi L + \phi^2 L^2 + \cdots)\epsilon_t + \theta(1 + \phi L + \phi^2 L^2 + \cdots)\epsilon_{t-1} \\
&= \frac{c}{1-\phi} + (\epsilon_t + \phi \epsilon_{t-1} + \phi \epsilon_{t-2} + \cdots) + (\theta \epsilon_{t-1} + \theta\phi \epsilon_{t-2} + \theta \phi^2 \epsilon_{t-3} + \cdots) \\
\Rightarrow y_t^p &= \frac{c}{1-\phi} + \epsilon_t + (\phi + \theta)\epsilon_{t-1} + (\phi + \theta\phi)\epsilon_{t-2} + (\phi + \theta\phi^2)\epsilon_{t-3} + \cdots
\end{aligned}
$$

3) AR(2)

$$
\begin{aligned}
& y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \\
\Rightarrow\ & y_t = \frac{c+\epsilon_t}{\underbrace{1-\phi_1 L - \phi_2 L^2}_{}\leftarrow \text{inverse characteristic equation}}, \\
\Rightarrow\ & y_t = \frac{c+\epsilon_t}{(1-\lambda_1 L)(1-\lambda_2 L)} \\
&\quad = (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}(c+\epsilon_t) \\
&\quad \frac{x_1}{1-\lambda_1 L} + \frac{x_2}{1-\lambda_2 L} \ \&\ x_1 + x_2 = 1 \ \Rightarrow x_1, x_2 = f(\lambda_1, \lambda_2) \equiv c_1, c_2 \\
&\quad = (\lambda_1 - \lambda_2)^{-1}\left(\frac{\lambda_1}{1-\lambda_1 L} - \frac{\lambda_2}{1-\lambda_2 L}\right)(c+\epsilon_t) \\
&\quad = \left[\frac{\lambda_1}{\lambda_1-\lambda_2}(1-\lambda_1)^{-1} - \frac{\lambda_2}{\lambda_1-\lambda_2}(1-\lambda_2)^{-1}\right]c + \left[\frac{\lambda_1}{\lambda_1-\lambda_2}(1-\lambda_1 L)^{-1} - \frac{\lambda_2}{\lambda_1-\lambda_2}(1-\lambda_2 L)^{-1}\right]\epsilon_t \\
&\quad = [\cdots]c + \left[\frac{\lambda_1}{\lambda_1-\lambda_2}(1+\lambda_1 L + \lambda_1^2 L^2 + \cdots) - \frac{\lambda_2}{\lambda_1-\lambda_2}(1+\lambda_2 L + \lambda_2^2 L^2 + \cdots)\right]\epsilon_t, \quad \text{where } |\lambda_1|, |\lambda_2| < 1 \\
\Rightarrow\ & y_t^p = [\cdots]c + \epsilon_t + (\lambda_1+\lambda_2)\epsilon_{t-1} + \left(\frac{\lambda_1^3-\lambda_2^3}{\lambda_1-\lambda_2}\right)\epsilon_{t-2} + \left(\frac{\lambda_1^4-\lambda_2^4}{\lambda_1-\lambda_2}\right)\epsilon_{t-3} + \cdots \\
&\quad \overset{\text{or}}{=} [\cdots]c + (c_1+c_2)\epsilon_t + (c_1\lambda_1 + c_2\lambda_2)\epsilon_{t-1} + (c_1\lambda_1^2 + c_2\lambda_2^2)\epsilon_{t-2} + \cdots \ \text{where } c_1 = \frac{\lambda_1}{\lambda_1-\lambda_2}, c_2 = -\frac{\lambda_2}{\lambda_1-\lambda_2}.
\end{aligned}
$$

Recall that $y_t^h = A\lambda^t$. Substituting it into the homogeneous equation yields the following equation and makes a comparison

$$
\left.\begin{aligned} \lambda^2 - \phi_1\lambda - \phi_2 = 0 \\ 1 - \phi_1 L - \phi_2 L^2 = 0 \end{aligned}\right\} \Rightarrow \begin{cases} \lambda_1, \lambda_2 = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} \qquad \text{where } \sqrt{\phi^2 + 4\phi_2} \gtreqless 0 & \Rightarrow \lambda_1 + \lambda_2 = \phi_1,\ \lambda_1\lambda_2 = -\phi_2; \\ (1-\lambda_1 L)(1-\lambda_2 L) = 1 - (\lambda_1+\lambda_2)L + \lambda_1\lambda_2 L^2 = 0 & \Rightarrow \lambda_1 + \lambda_2 = \phi_1,\ \lambda_1\lambda_2 = -\phi_2. \end{cases}
$$

$$
\Rightarrow\ \lambda_1 = \frac{1}{L_1},\ \lambda_2 = \frac{1}{L_2} \xrightarrow{|\lambda_1|, |\lambda_2| < (1,1) \text{ (stability condition)}} |L_1|, |L_2| > (1,1) \text{ (stability condition)}
$$

4) AR(p)

$$
\begin{aligned}
& y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t, \\
\Rightarrow\ & y_t = \overline{\frac{\epsilon_t}{1-\phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p} \leftarrow \text{inverse characteristic equation}} \\
\Rightarrow \quad & y_t = [(1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}\cdots(1-\lambda_p L)^{-1}]\epsilon_t, \\
& \quad = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)\cdots(1-\lambda_p L)}\epsilon_t, \\
& \quad \frac{x_1}{1-\lambda_1 L} + \frac{x_2}{1-\lambda_2 L} + \cdots + \frac{x_p}{1-\lambda_p L} \ \& \ x_1 + x_2 + \cdots + x_p = 1 \ \Rightarrow x_i = f(\lambda_1, \lambda_2, \ldots, \lambda_p) \equiv c_i \\
& \quad = [c_1(1-\lambda_1 L)^{-1} + c_2(1-\lambda_2 L)^{-1} + \cdots + c_p(1-\lambda_p L)^{-1}]\epsilon_t, \\
& \quad = [c_1(1+\lambda_1 L + \lambda_1^2 L^2 + \cdots) + c_2(1+\lambda_2 L + \lambda_2^2 L^2 + \cdots) + \cdots + c_p(1+\lambda_p L + \lambda_p^2 L^2 + \cdots)]\epsilon_t, \\
& \quad = (c_1 + c_2 + \cdots + c_p)\epsilon_t + (c_1\lambda_1 + c_2\lambda_2 + \cdots + c_p\lambda_p)\epsilon_{t-1} + (c_1\lambda_1^2 + c_2\lambda_2^2 + \cdots + c_p\lambda_p^2)\epsilon_{t-2}\cdots \\
& \quad \equiv \psi_0\epsilon_t + \psi_1\epsilon_{t-1} + \psi_2\epsilon_{t-2} + \cdots = (\psi_0 + \psi_1 L + \psi_2 L^2 + \cdots)\epsilon_t \equiv \psi(L)\epsilon_t. \\
& \quad \Rightarrow \frac{\partial y_{t+j}}{\partial \epsilon_t} = c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j = \psi_j. \\
& \quad \xRightarrow{\text{the effects of } \epsilon_t \text{ on the present value of } y} \frac{\partial \sum_{j=0}^{\infty}\beta^j y_{t+j}}{\partial \epsilon_t} = \sum_{j=0}^{\infty}\beta^j \frac{\partial y_{t+j}}{\partial \epsilon_t} = \sum_{j=0}^{\infty}\beta^j\psi_j. \\
& \quad \xRightarrow{\text{the long-run multiplier with } \beta=1} \lim_{j\to\infty}\left(\frac{\partial y_{t+j}}{\partial \epsilon_t} + \frac{\partial y_{t+j}}{\partial \epsilon_{t+1}} + \cdots + \frac{\partial y_{t+j}}{\partial \epsilon_{t+j}}\right) = \frac{1}{1-\phi_1-\phi_2-\cdots-\phi_p}.
\end{aligned}
$$

$$
\begin{aligned}
1-\phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p &= (1-\lambda_1 L)(1-\lambda_2 L)\cdots(1-\lambda_p L), \\
\Leftrightarrow \quad \underbrace{1-\phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p}_{\text{lie outside the unit circle}} &= (1-\lambda_1 z)(1-\lambda_2 z)\cdots(1-\lambda_p z), \\
\Leftrightarrow \quad (1-\phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p)z^{-p} &= (1-\lambda_1 z)(1-\lambda_2 z)\cdots(1-\lambda_p z)z^{-p}, \\
\Leftrightarrow \quad \frac{(1-\phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p)}{z^p} &= \frac{(1-\lambda_1 z)}{z}\frac{(1-\lambda_2 z)}{z}\cdots\frac{(1-\lambda_p z)}{z}, \\
\xRightarrow{z^{-1}\equiv\lambda} \underbrace{\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p}_{\text{lie inside the unit circle}} &= (z^{-1}-\lambda_1)(z^{-1}-\lambda_2)\cdots(z^{-1}-\lambda_p) = 0, \\
\to \quad (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_p) &= (z^{-1}-\lambda_1)(z^{-1}-\lambda_2)\cdots(z^{-1}-\lambda_p) = 0.
\end{aligned}
$$

3.3 Using forward operators to solve DEs with rational expectations

Given an initial condition, a stochastic DE will have a backward- ($|\phi| < 1$) and a forward- ($|\phi| > 1$) looking solution.

Knowing how to obtain forward-looking solutions is useful for solving rational expectations models although future realizations of stochastic variables are not directly observable.

1) Using forward iterations to solve the 1st-order equation

$$
\begin{aligned}
y_t &= c + \phi y_{t-1} + \epsilon_t \\
\Rightarrow y_{t-1} &= \frac{y_t - c - \epsilon_t}{\phi} \\
\xRightarrow{\text{updating 1 period}} y_t &= \frac{y_{t+1} - c - \epsilon_{t+1}}{\phi} \\
\xRightarrow{y_{t+1}=\frac{y_{t+2}-c-\epsilon_{t+2}}{\phi}} y_t &= \frac{\left(\frac{y_{t+2}-c-\epsilon_{t+2}}{\phi}\right) - c - \epsilon_{t+1}}{\phi} \\
&= \frac{y_{t+2}}{\phi^2} - \frac{c}{\phi^2} - \frac{c}{\phi} - \frac{\epsilon_{t+2}}{\phi^2} - \frac{\epsilon_{t+1}}{\phi} \\
&\vdots \\
&= \frac{y_{t+p}}{\phi^p} - c\sum_{i=1}^{p}\phi^{-i} - \sum_{i=1}^{p}\phi^{-i}\epsilon_{t+i} \\
&\overset{p\to\infty}{=} 0 - \frac{\phi^{-1}}{1-\phi^{-1}}c - \sum_{i=1}^{\infty}\phi^{-i}\epsilon_{t+i} \quad \text{when } |\phi| > 1 \\
&= \frac{c}{1-\phi} - \sum_{i=1}^{\infty}\phi^{-i}\epsilon_{t+i} \\
&= \frac{c}{1-\phi} - \phi^{-1}\sum_{i=0}^{\infty}\phi^{-i}\epsilon_{t+i+1}.
\end{aligned}
$$

This forward-looking solution will (as p gets infinitely large)

$$
\left.\begin{matrix}\text{converge} \\ \text{diverge}\end{matrix}\right\} \quad \text{if} \quad \left\{\begin{matrix}|\phi| > 1; \\ |\phi| < 1.\end{matrix}\right.
$$

Notice that the key point is that the future values of the disturbances affect the present.

2) Using lag operators to solve the previous problem:

$$
\begin{aligned}
y_t &= c + \phi y_{t-1} + \epsilon_t, \\
\Rightarrow y_t &= \frac{c + \epsilon_t}{1 - \phi L} \\
&= \frac{c}{1-\phi} + \frac{\epsilon_t \phi^{-1} L^{-1}}{(1-\phi L)\phi^{-1}L^{-1}} \quad \text{when } |\phi| > 1 \\
&= \frac{c}{1-\phi} - \frac{\epsilon_t \phi^{-1} L^{-1}}{(1-\phi^{-1}L^{-1})} \\
&= \frac{c}{1-\phi} - \frac{\phi^{-1}\epsilon_{t+1}}{(1-\phi^{-1}L^{-1})} \\
&= \frac{c}{1-\phi} - (1 + \phi^{-1}L^{-1} + \phi^{-2}L^{-2} + \cdots)\phi^{-1}\epsilon_{t+1} \\
&= \frac{c}{1-\phi} - \sum_{i=1}^{\infty}\phi_i^{-i}\epsilon_{t+i} \\
&= \frac{c}{1-\phi} - \phi^{-1}\sum_{i=0}^{\infty}\phi^{-i}\epsilon_{t+i+1}.
\end{aligned}
$$

3.4 3 models (cf. Mankiw and Reis, 2002, QJE)

1) Adaptive expectations (backward-looking)

$$\begin{cases} \pi_t = \kappa y_t + \pi_{t-1}, \quad \kappa > 0 \\ y_t = m_t - p_t. \end{cases}$$

Sloving the system yields

$$\begin{aligned} p_t - p_{t-1} &= \kappa(m_t - p_t) + (p_{t-1} - p_{t-2}) \\ \Rightarrow (1+\kappa)p_t &= 2p_{t-1} - p_{t-2} + \kappa m_t \\ \Rightarrow (1+\kappa)p_t - 2p_{t-1} + p_{t-2} &= 0 \leftarrow \text{the homogeneous equation} \\ \Rightarrow \lambda_1, \lambda_2 &= \frac{1 \pm \sqrt{-\kappa}}{1+\kappa} = \frac{1 \pm \sqrt{\kappa} i}{1+\kappa} = a \pm bi = R^t(\cos\theta t \pm i \sin\theta t) \\ \Rightarrow p_1^h, p_2^h &= A_1\lambda_1^t + A_2\lambda_2^t = R^t[(A_1 + A_2)\cos\theta t + (A_1 - A_2) i \sin\theta t], \end{aligned}$$

where $R = \sqrt{a^2 + b^2}$ and $|R| < 1$, the solutions will converge with oscillatory behavior.

2) Rational expectations (forward-looking)

$$\begin{cases} \pi_t = \kappa y_t + \mathbb{E}_t \pi_{t+1}, \quad \kappa > 0 \\ y_t = m_t - p_t. \end{cases}$$

Sloving the system yields

$$\begin{aligned} p_t - p_{t-1} &= \kappa(m_t - p_t) + (\mathbb{E}_t p_{t+1} - p_t) \\ \Rightarrow \mathbb{E}_t p_{t+1} &= (2+\kappa)p_t - p_{t-1} - \kappa m_t \\ \Rightarrow \mathbb{E}_t p_{t+1} - (2+\kappa)p_t + p_{t-1} &= -\kappa m_t \leftarrow \text{an expectational DE} \\ \Rightarrow [F^2 - (2+\kappa)F + 1]L p_t^e &= -\kappa m_t^e \leftarrow F = L^{-1} \\ \Rightarrow (1-\lambda_1 F)(1-\lambda_2 F)L p_t^e &= -\kappa m_t^e \Rightarrow \lambda_1 + \lambda_2 = 2+\kappa,\ \lambda_1\lambda_2 = 1 \\ \Rightarrow (1-\lambda_1 F)(L-\lambda_2)p_t^e &= -\kappa m_t^e \Leftarrow (1-\lambda_1 F)[(1-\lambda_2 F)L]p_t^e = -\kappa m_t^e \\ \Rightarrow (1-\theta F)(1-\theta L)p_t^e &= (-\theta)(-\kappa)m_t^e \leftarrow \lambda_1 = \theta,\ \lambda_2 = \frac{1}{\theta},\ \kappa = \frac{(\theta-1)^2}{\theta} \\ \Rightarrow (1-\theta L)p_t^e &= (1-\theta)^2(1-\theta F)^{-1}m_t^e \\ \Rightarrow (1-\theta L)p_t^e &= (1-\theta)^2(1 + \theta F + \theta^2 F^2 + \cdots)m_t^e \\ \Rightarrow p_t &= \theta p_{t-1} + (1-\theta)^2 \sum_{i=0}^{\infty} \theta^i \mathbb{E}_t m_{t+i}. \end{aligned}$$

3) Rtional expectations (lagged-looking)
TBA

4 Matrix Operation

cf. Chiang (2005, ch.4-5, pp.48-) and Hamilton (1994, ch. 1 & A.4, p.721)
0) The 1st-order DE

$$y_t = c + \phi y_{t-1} + \epsilon_t \xrightarrow{\text{dynamic multiplier}} \begin{cases} \frac{\partial y_t}{\partial \epsilon_{t-i}} = \phi^i, \quad |\phi| < 1 \Leftarrow \text{back iteration;} \\ \frac{\partial y_t}{\partial \epsilon_{t+i}} = \phi^{-i}, |\phi| > 1 \Leftarrow \text{forward iteration.} \end{cases}$$

1) The characteristic roots of a 2nd-order DE

$$
\begin{aligned}
y_t &= c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \\
\mathbb{E}y_t \equiv \mu &= c + \phi_1\mu + \phi_2\mu \quad \Leftrightarrow \quad \mu = (1-\phi_1-\phi_2)^{-1}c \overset{c=0}{=\!=} 0 \\
y_t - \mu &= \phi_1(y_{t-1}-\mu) + \phi_2(y_{t-2}-\mu) + \epsilon_t \quad \overset{\mu=0}{\Longleftrightarrow} \quad y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t.
\end{aligned}
$$

Let $\mathbf{y}_t = [y_t, y_{t-1}]'$, then

$$\mathbf{y}_t = \mathbf{F}\mathbf{y}_{t-1} + \boldsymbol{\nu}_t.$$

That is

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}$$

Recall that the eigenvalues of a maxtrix $\mathbf{F}$ are those numbers λ for which

$$|\mathbf{F} - \lambda \mathbf{I}_2| = 0.$$

Substitute and yield

$$\left| \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0 \quad \Rightarrow \quad \begin{vmatrix} \phi_1 - \lambda & \phi_2 \\ 1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - \phi_1\lambda - \phi_2 = 0.$$

The two eigenvalues of $\mathbf{F}$ for a 2nd-order DE are thus given by

$$\lambda_1, \lambda_2 = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

2) The dynamic multipliers of a 2nd-order DE

Blackboard-Writing

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \\
\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} &= \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix} \\
\mathbf{y}_t &= \mathbf{F}\mathbf{y}_{t-1} + \boldsymbol{\nu}_t. \\
\mathbf{y}_1 &= \mathbf{F}\mathbf{y}_0 + \boldsymbol{\nu}_1, \\
\mathbf{y}_2 &= \mathbf{F}^2\mathbf{y}_0 + \mathbf{F}\nu_1 + \boldsymbol{\nu}_2, \\
&\ \ \vdots \\
\mathbf{y}_t &= \mathbf{F}^t\mathbf{y}_0 + \mathbf{F}^{t-1}\boldsymbol{\nu}_1 + \mathbf{F}^{t-2}\boldsymbol{\nu}_2 + \cdots + \mathbf{F}^2\boldsymbol{\nu}_{t-2} + \mathbf{F}\boldsymbol{\nu}_{t-1} + \boldsymbol{\nu}_t, \\
\mathbf{y}_t &= \mathbf{F}^t\mathbf{y}_0 + \sum_{i=0}^{t-1} \mathbf{F}^i \boldsymbol{\nu}_{t-i} \overset{\text{or}}{=} \mathbf{F}^{t+1}\mathbf{y}_{-1} + \sum_{i=0}^{t} \mathbf{F}^i \boldsymbol{\nu}_{t-i}. \\
\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} &= \mathbf{F}^t \begin{bmatrix} y_0 \\ y_{-1} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{t-1} \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{t-2} \\ 0 \end{bmatrix} + \cdots \\
&= \qquad \cdots \qquad + \begin{bmatrix} \phi_1^2 + \phi_2 & \phi_1\phi_2 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \epsilon_{t-2} \\ 0 \end{bmatrix} + \cdots \\
y_t &= \mathbf{F}_{11}^t y_0 + \mathbf{F}_{12}^t y_{-1} + \epsilon_t + \phi_1\epsilon_{t-1} + (\phi_1^2 + \phi_2)\epsilon_{t-2} + \cdots \\
&= \mathbf{F}_{11}^t y_0 + \mathbf{F}_{12}^t y_{-1} + \epsilon_t + \mathbf{F}_{11}\epsilon_{t-1} + \mathbf{F}_{11}^2\epsilon_{t-2} + \cdots
\end{aligned}
$$

$$|\mathbf{F} - \lambda\mathbf{I}_2| = 0 \ \Rightarrow \lambda_1, \lambda_2 \quad \& \quad \mathbf{F}^2 = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1} \ \Rightarrow \mathbf{F}_{11}^2.$$

3) The pth-order DE

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_{p-1} y_{t-(p-1)} + \phi_p y_{t-p} + \epsilon_t.$$

Let $\mathbf{y}_t = [y_t, y_{t-1}, y_{t-2}, \cdot, y_{t-(p-1)}]'$, then

$$\mathbf{y}_t = \mathbf{F}\mathbf{y}_{t-1} + \boldsymbol{\nu}_t.$$

That is

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The eigenvalues of the matrix $\mathbf{F}$ are the value of λ that satisfy

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p = 0.$$

Once we know the eigenvalues, it's straightforward to characterize the dynamic behavior of the system.

Case 1. distinct and real roots (eigenvalues)

Recall that if the eigenvalues of a (p, p) matrix $\mathbf{F}$ are distinct, there exists a nonsingular (p, p) matrix $\mathbf{T}$ (eigenvector) such that

$$\underbrace{\mathbf{F}}_{(p\times p)} = \underbrace{\mathbf{T}}_{(p\times p)} \underbrace{\boldsymbol{\Lambda}}_{(p\times p)} \mathbf{T}^{-1},$$

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p \end{bmatrix}$$

This enables us to characterize the dynamic multiplier ($\phi^j \to \mathbf{F}^j$) very easily.

For example

$$\begin{aligned} \mathbf{F}^2 &= \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1} \times \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1} \\ &= \mathbf{T} \times \boldsymbol{\Lambda} \times (\mathbf{T}^{-1}\mathbf{T}) \times \boldsymbol{\Lambda} \times \mathbf{T}^{-1} \\ &= \mathbf{T} \times \boldsymbol{\Lambda} \times \mathbf{I}_p \times \boldsymbol{\Lambda} \times \mathbf{T}^{-1} \\ &= \mathbf{T}\boldsymbol{\Lambda}^2\mathbf{T}^{-1}. \end{aligned}$$

where

$$\boldsymbol{\Lambda}^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p^2 \end{bmatrix}$$

More generally,

$$\mathbf{F}^j = \mathbf{T}\boldsymbol{\Lambda}^j\mathbf{T}^{-1},$$

where

$$\mathbf{\Lambda}^j = \begin{bmatrix} \lambda_1^j & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p^j \end{bmatrix}$$

Let T_{ij} (T^{ij}) denote the row i column j element of $\mathbf{T}$ ($\mathbf{T}^{-1}$), thus

$$\begin{aligned} \mathbf{F}^j = \mathbf{T}\mathbf{\Lambda}^j\mathbf{T}^{-1} &= \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1p} \\ T_{21} & T_{22} & \cdots & T_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ T_{p1} & T_{p2} & \cdots & T_{pp} \end{bmatrix} \begin{bmatrix} \lambda_1^j & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p^j \end{bmatrix} \begin{bmatrix} T^{11} & T^{12} & \cdots & T^{1p} \\ T^{21} & T^{22} & \cdots & T^{2p} \\ \vdots & \vdots & \cdots & \vdots \\ T^{p1} & T^{p2} & \cdots & T^{pp} \end{bmatrix} \\ &= \begin{bmatrix} T_{11}\lambda_1^j & T_{12}\lambda_2^j & \cdots & T_{1p}\lambda_p^j \\ T_{21}\lambda_1^j & T_{22}\lambda_2^j & \cdots & T_{2p}\lambda_p^j \\ \vdots & \vdots & \cdots & \vdots \\ T_{p1}\lambda_1^j & T_{p2}\lambda_2^j & \cdots & T_{pp}\lambda_p^j \end{bmatrix} \begin{bmatrix} T^{11} & T^{12} & \cdots & T^{1p} \\ T^{21} & T^{22} & \cdots & T^{2p} \\ \vdots & \vdots & \cdots & \vdots \\ T^{p1} & T^{p2} & \cdots & T^{pp} \end{bmatrix} \\ &= \begin{bmatrix} (T_{11}T^{11})\lambda_1^j + (T_{12}T^{21})\lambda_2^j + \cdots + (T_{1p}T^{p1})\lambda_p^j & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{aligned}$$

from which the (1, 1) element of $\mathbf{F}^j$ (the dynamic multiplier) is given by

$$\frac{\partial y_{t+j}}{\partial \epsilon_t} = \psi_j = F_{11}^{(j)} = A_1\lambda_1^j + A_2\lambda_2^j + \cdots + A_p\lambda_p^j, \quad \text{where } A_i = (T_{1i}T^{i1}) = \frac{\lambda_i^{p-1}}{\prod_{k=1,k\neq i}^{p}(\lambda_i - \lambda_k)} \text{ and } \sum_{i=1}^{p} A_i = 1.$$

Case 2. repeated real roots (eigenvalues)

Assume that $\mathbf{F}$ has p repeated eigenvalues and q linearly independent eigenvecotr. Using the Jordan decomposition (note that $q < p$)

$$\underbrace{\mathbf{F}}_{(p\times p)} = \underbrace{\mathbf{M}}_{(p\times p)} \underbrace{\mathbf{J}}_{(q\times q)} \mathbf{M}^{-1},$$

where

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_q \end{bmatrix} \quad \text{with} \quad \underbrace{\mathbf{J}_i}_{(n\times n)} = \begin{bmatrix} 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

More generally,

$$\mathbf{F}^j = \mathbf{M}\mathbf{J}^j\mathbf{M}^{-1},$$

where

$$\mathbf{J}^j = \begin{bmatrix} \mathbf{J}_1^j & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2^j & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_q^j \end{bmatrix} \quad \text{with} \quad \mathbf{J}_i^j = \begin{bmatrix} \binom{j}{0}\lambda_i^j & \binom{j}{1}\lambda_i^{j-1} & \binom{j}{2}\lambda_i^{j-2} & \cdots & \binom{j}{n-1}\lambda_i^{j-(n-1)} \\ 0 & \lambda_i^j & \binom{j}{1}\lambda_i^{j-1} & \cdots & \binom{j}{n-2}\lambda_i^{j-(n-2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i^j \end{bmatrix}$$

where

$$\binom{j}{n} \equiv \begin{cases} \frac{j(j-1)(j-2)\cdots(j-n+1)}{n(n-1)\cdots 3\cdot 2\cdot 1} & \text{for } j \geq n; \\ 0 & \text{otherwise.} \end{cases} \qquad \binom{j}{n} + \binom{j}{n-1} = \binom{j+1}{n}.$$

Let M_{ij} (M^{ij}) denote the row i column j element of $\mathbf{M}$ ($\mathbf{M}^{-1}$), thus

$$\mathbf{F}^j = \mathbf{M}\mathbf{J}^j\mathbf{M}^{-1}$$

$$= \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1p} \\ M_{21} & M_{22} & \cdots & M_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ M_{p1} & M_{p2} & \cdots & M_{pp} \end{bmatrix} \begin{bmatrix} \binom{j}{0}\lambda_i^j & \binom{j}{1}\lambda_i^{j-1} & \binom{j}{2}\lambda_i^{j-2} & \cdots & \binom{j}{n-1}\lambda_i^{j-(n-1)} \\ 0 & \lambda_i^j & \binom{j}{1}\lambda_i^{j-1} & \cdots & \binom{j}{n-2}\lambda_i^{j-(n-2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i^j \end{bmatrix} \begin{bmatrix} M^{11} & M^{12} & \cdots & M^{1p} \\ M^{21} & M^{22} & \cdots & M^{2p} \\ \vdots & \vdots & \cdots & \vdots \\ M^{p1} & M^{p2} & \cdots & M^{pp} \end{bmatrix}$$

$$= \begin{bmatrix} M_{11}\binom{j}{0}\lambda_i^j & M_{11}\binom{j}{1}\lambda_i^j + M_{12}\lambda_i^j & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} M^{11} & M^{12} & \cdots & M^{1p} \\ M^{21} & M^{22} & \cdots & M^{2p} \\ \vdots & \vdots & \cdots & \vdots \\ M^{p1} & M^{p2} & \cdots & M^{pp} \end{bmatrix}$$

$$= \begin{bmatrix} ? & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Take the 2nd-order DE as an example

$$\begin{aligned} \mathbf{F}^j &= \mathbf{M}\mathbf{J}^j\mathbf{M}^{-1} \\ &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \lambda^j & j\lambda^{j-1} \\ 0 & \lambda^j \end{bmatrix} \begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix} \\ &= \begin{bmatrix} M_{11}\lambda^j & M_{11}j\lambda^{j-1} + M_{12}\lambda^j \\ M_{21}\lambda^j & M_{21}j\lambda^{j-1} + M_{22}\lambda^j \end{bmatrix} \begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix} \\ &= \begin{bmatrix} (M_{11}M^{11} + M_{12}M^{21})\lambda^j + (M_{11}M^{21})j\lambda^{j-1} & \cdots \\ \cdots & \cdots \end{bmatrix} \end{aligned}$$

so that the dynamic multiplier takes the form

$$\frac{\partial y_{t+j}}{\partial \epsilon_t} = F_{11}^j = A_1\lambda^j + A_2 j\lambda^{j-1}.$$

Case 3. complex roots
TBA