Advanced Macroeconomics

Mathematical Foundations and Analytical Principles

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Example2 in Pacielo and Wiederholt (2014, RES)

$$\begin{aligned}
\kappa &= h(A_t, A_t|\mathcal{I}_{t,t-1}) - h(A_t, A_t|\mathcal{I}_{t,t}) \\
&= [h(A_t|\mathcal{I}_{t,t-1}) + h(A_t|\mathcal{I}_{t,t-1})] - [h(A_t|\mathcal{I}_{t,t}) + h(A_t|\mathcal{I}_{t,t})] \\
&= [h(A_t|\mathcal{I}_{t,t-1}) - h(A_t|\mathcal{I}_{t,t})] + [h(A_t|\mathcal{I}_{t,t-1}) - h(A_t|\mathcal{I}_{t,t})] \\
&= \left[\frac{1}{2}\log_2\left(2\pi e\sigma^2_{\eta|t-1}\right) - \frac{1}{2}\log_2\left(2\pi e\sigma^2_{\eta|t}\right)\right] + \left[\frac{1}{2}\log_2\left(2\pi e\sigma^2_{\lambda|t-1}\right) - \frac{1}{2}\log_2\left(2\pi e\sigma^2_{\lambda|t}\right)\right] \\
&= \frac{1}{2}\log_2\left(\frac{\sigma^2_{\eta|t-1}}{\sigma^2_{\eta|t}}\right) + \frac{1}{2}\log_2\left(\frac{\sigma^2_{\lambda|t-1}}{\sigma^2_{\lambda|t}}\right).
\end{aligned}$$

Example3 in Pasten and Schoenle (2016, JME)

$$\max \pi(P_{jt}, P_t, Y_t, Z_{jt}),$$

$$\max_{\{s_{it}\}\in\Omega} \mathbb{E}_{\mathcal{B}_t}\left\{\sum_{t=0}^{\infty} \beta^t \left[\sum_{\kappa\in\mathcal{N}_t} \pi(P_{\kappa t}, P_t, Y_t, Z_{\kappa t})\right]\right\},$$

$$P^*_{\kappa t} = \arg\max_{P_{\kappa t}} \mathbb{E}\{\pi(P_{\kappa t}, P_t, Y_t, Z_{\kappa t})|s_{\kappa t}\},$$

$$s.t. \quad \mathcal{I}(\{P_t, Y_t\}, \{s_{\kappa it}\}) + \sum_{n\in\mathcal{N}_t} I(\{Z_{\kappa t}\}, \{s_{\kappa t}\}) \le \kappa(N),$$

$$\Leftrightarrow \qquad \kappa_{\kappa} + \sum_{n\in\mathcal{N}_t} \kappa_{\kappa} \le \kappa(N);$$

$$\int_0^1 \hat{z}_{jt} dj = 0,$$

$$Q_t = P_t Y_t,$$

$$p_t = \int_0^1 p_{jt} dj,$$

$$\mathcal{I}(\{Q_t, Z_{it}\}, \{s_{it}\}) \approx \mathcal{I}(\{q_t; s_{\kappa it}\}) = \mathcal{I}(\{s_{it}, s_{\kappa t}\}_{\kappa\in\mathcal{N}_t}) \le \kappa(N).$$

$$\max_{\{s_{\kappa t}\}_{\kappa\in\mathcal{N}_t}} \sum_{\kappa\in\mathcal{N}_t} \left\{\hat{\pi}_1 \hat{p}_{\kappa t} + \frac{\hat{\pi}_{11}}{2}\hat{p}_{\kappa t} + \hat{\pi}_{12}\hat{p}_{\kappa t}\hat{p}_t + \hat{\pi}_{13}\hat{p}_{\kappa t}\hat{q}_t + \hat{\pi}_{14}\hat{p}_{\kappa t}\hat{z}_{\kappa t}\right\},$$

$$\hat{p}^*_{jt} = \hat{p}_t + \frac{\hat{\pi}_{13}}{|\hat{\pi}_{11}|}\hat{q}_t + \frac{\hat{\pi}_{14}}{|\hat{\pi}_{11}|}\hat{z}_{jt} \equiv \hat{\Delta}_t + \frac{\hat{\pi}_{14}}{|\hat{\pi}_{11}|}\hat{z}_{jt} \quad \Rightarrow \quad \hat{p}^*_t = \hat{q}_t.$$

"A huge tree grows from a tiny sprout; A nine-story tower rises from piled earth; A thousand-mile journey begins beneath one's feet."—— Lao Tzu

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