



# Advanced Macroeconomics

## Mathematical Foundations and Analytical Principles

Author: DENG Yanfei, SHEN Ji, WU Jieran, ZHANG Jun

Institute:

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Example2 in Paelelli and Wöhrle (2014, EES)

$$\begin{aligned} n &= h(A_t, A_t)I_{t,t-1} - h(A_t, A_t)I_{t,t-1} \\ &= [h(A_t)I_{t,t-1} - h(A_t)I_{t,t-1}] - [h(A_t)I_{t,t-1} - h(A_t)I_{t,t-1}] \\ &= [h(A_t)I_{t,t-1} - h(A_t)I_{t,t-1}] - [h(A_t)I_{t,t-1} - h(A_t)I_{t,t-1}] \\ &= \left[ \frac{1}{2} \log_2 \left( \frac{A_t}{A_{t-1}} \right) - \frac{1}{2} \log_2 \left( \frac{A_t}{A_{t-1}} \right) \right] + \left[ \frac{1}{2} \log_2 \left( \frac{A_t}{A_{t-1}} \right) - \frac{1}{2} \log_2 \left( \frac{A_t}{A_{t-1}} \right) \right] \\ &= \frac{1}{2} \log_2 \left( \frac{A_t}{A_{t-1}} \right) + \frac{1}{2} \log_2 \left( \frac{A_t}{A_{t-1}} \right) \end{aligned}$$

Example3 in Paelelli and Wöhrle (2014, EES)

$$\begin{aligned} \max_{\{P_t, Y_t, Z_t\}} & \left\{ \sum_{t=0}^{\infty} \beta^t \left[ \sum_{i=0}^{\infty} \pi_i P_{t+i} Y_{t+i} Z_{t+i} \right] \right\} \\ \text{s.t. } & P_t = \max_{i=0,1,2} \{ \pi_i P_{t+i} Y_{t+i} Z_{t+i} \} \\ \text{s.t. } & \mathbb{I}(\{P_t, Y_t, Z_t\}) + \sum_{i=0}^{\infty} \pi_i \mathbb{I}(Z_{t+i}) \leq n(N) \\ & n_t + \sum_{i=0}^{\infty} n_i \leq n(N) \\ & \int_0^1 \gamma_i d\gamma = 0 \\ & Q_t = P_t Y_t \\ & n = \int_0^1 \gamma_i d\gamma \\ & \mathbb{I}(\{Q_t, Z_t\}; \{n_t\}) \leq \mathbb{I}(\{Q_t, n_t\}) - \mathbb{I}(\{n_t, n_t\}) \leq n(N) \\ & \max_{\{n_t\}} \sum_{i=0}^{\infty} \left\{ \pi_i P_{t+i} + \frac{\pi_{i+1}}{2} P_{t+i+1} + \pi_{i+2} P_{t+i+2} + \pi_{i+3} P_{t+i+3} \right\} \\ & \beta_{t+1} = \beta_t + \frac{\pi_{t+1}}{\pi_{t+1}} \beta_t + \frac{\pi_{t+2}}{\pi_{t+1}} \beta_t \Rightarrow \beta_t = \beta_t \end{aligned}$$

"A huge tree grows from a tiny sprout; A nine-story tower rises from piled earth; A thousand-mile journey begins beneath one's feet."—— Lao Tzu

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