Advanced Macroeconomics

Mathematical Foundations and Analytical Principles

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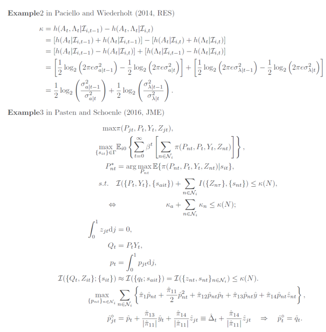

"A huge tree grows from a tiny sprout; A nine-story tower rises from piled earth; A thousand-mile journey begins beneath one's feet."—— Lao Tzu

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Chapter 1 Static Optimization Problem vs. Dynamic Optimization Problem

An optimization problem means that selecting one or more independent variables with or without constraints to make the objective function reach its extreme value, and the result is the value (set) of the choice variable(s) when the objective function reaches the extreme value.

1.1 Static Optimization Problem

Static optimization problems are divided into two common cases: unconstrained problems and constrained problems. Constrained problems are further divided into those with equality constraints and those with inequality constraints.

Unconstrained optimization is also called free optimization. Strictly speaking, it does not mean that there is no constraint on the choice variables, as the domain of the choice vairables is also a constraint if it exists. The word “unconstrained” here means that there is no constraint between the choice variables of the optimization problem. The choice variables are also called decision variables.

A single choice variable (hereinafter referred to as “single-variable”) does not have the chance to be tied to other selection variables, while it is possible that two selection variables (hereinafter referred to as “double-variable”) or more selection variables (hereinafter referred to as “multi-variable”) are mutually correlated and may increase or decrease at the expense of each other.

1.1.1 Unconstrained Optimization

We first discusses the necessary and sufficient conditions for the optimization problem when there are no constraints on the choice variables, including the solution of single-variable and multi-variable unconstrained optimization problems respectively.

1.1.1.1 Single Decision Variable

Using calculus, the optimization problem of a single choice variable directly takes the derivative of this independent variable and sets the derivative to 0, which is called the first-order condition. However, this requires that the function $O = f(x)$ is a continuously differentiable smooth curve (excluding constant functions and monotonic functions). The point where the derivative is 0 may be an “inflection point” (unstable point) or a “stationary point” (stable point). The stationary point may only be a local, relative extreme value (hereinafter referred to as ”extreme value” or ”local maximum/minimum”), rather than a global, absolute extreme value (hereinafter referred to as ”global maximum/minimum”). The discussion above involves at least six problems:

(1) Why exclude constant functions and monotonic functions?

(2) Why should the objective function be continuously differentiable?

(3) How to determine whether it is an inflection point or a stationary point?

(4) How to distinguish whether a stationary point is a local maximum or minimum?

(5) How to know whether it is an maximum or minimum?

(6) How to simply prove that the differentiation method is reasonable?

Problem 1: The value z of a constant function is fixed. No matter how the independent variable x is chosen, the target value can never be larger or smaller; The value at the endpoints of a monotonic function with a domain $[x_1, x_2]$ are easily judged as extreme values, which are of course the maximum/minimum. These can be expressed as, respectively:

$$f(x) = z; \quad \& \quad \begin{cases} \min f(x) = f(x_1), \\ \max f(x) = f(x_2). \end{cases}$$

Problem 2: The objective function may not be smooth and may have a sharp inflection point (corner point) where the derivative does not exist, so if the sharp inflection point is an extreme point, it cannot be judged by the derivative. This chapter focuses on calculus methods, so the basic premise is that the function can be differentiated at all orders; in other words, the first-order derivative, second-order derivative, third-order derivative, fourth-order derivative, and even any higher-order derivative are assumed to exist:

$$f'(x), f''(x), f^{(3)}(x), f^{(4)}(x), \dots$$

Promlem 3: For a continuously differentiable objective function, the first-order derivative is the slope. If the first-order derivative is greater than 0, the slope is positive, and if the first-order derivative is less than 0, the slope is negative. The first-order derivative at the extreme point must be 0, but a point where the first-order derivative is 0 is not necessarily an extreme point. The slopes on the left and right sides of the extreme point should have opposite signs. If the slopes on the left and right of the first-order derivative have the same sign when the first-order derivative is 0, it is an inflection point. Therefore, the first-order derivative of 0 is a necessary condition for the existence of an extreme value, but not a sufficient condition. This boils down to:

$$f'(x) = 0 \begin{cases} \text{infection point} \begin{cases} \text{the same sign} \begin{cases} f'(x) > 0 & \text{left} \\ f'(x) > 0 & \text{right} \end{cases} \\ \text{the same sign} \begin{cases} f'(x) < 0 & \text{left} \\ f'(x) < 0 & \text{right} \end{cases} \end{cases} \\ \text{stationary point} \begin{cases} \text{opposite signs} \begin{cases} f'(x) > 0 & \text{left} \\ f'(x) < 0 & \text{right} \end{cases} \\ \text{opposite signs} \begin{cases} f'(x) < 0 & \text{left} \\ f'(x) > 0 & \text{right} \end{cases} \end{cases} \end{cases}$$

Problem 4: The change in the sign of the first-order derivative in the neighborhood of the stationary point helps to determine the extreme value. The slope on the left side of a local maximum point is positive and the slope on the right side is negative. The slope on the left side of a local minimum point is negative and the slope on the right side is positive. The sign of the second-order derivative in the neighborhood of the stationary point can also helps. If the positive slope is getting slower and slower (the function value increases at a decreasing rate) and the negative slope is getting steeper and steeper (the function value decreases at an increasing rate), it is a local maximum (the second-order derivative is always less than 0). If the negative slope is getting slower and slower (the function value decreases at a decreasing rate) and the positive slope is getting steeper and steeper (the function value increases at an increasing rate), it is a minimum (the second-order derivative is always greater than 0). This comes down to:

$$\text{extreme values} \begin{cases} \text{local maximum} \begin{cases} f'(x) > 0, f''(x) < 0 & \text{left} \\ f'(x) < 0, f''(x) < 0 & \text{right} \end{cases} \\ \text{local minimum} \begin{cases} f'(x) < 0, f''(x) > 0 & \text{left} \\ f'(x) > 0, f''(x) > 0 & \text{right} \end{cases} \end{cases}$$

Problem 5: For a continuously differentiable objective function, if the extreme points can all be determined, we can simply compare the extreme points to determine the global maximum/minimum, but this is quite troublesome, and the extreme points may not be easy to exhaust. Another way is that if the concavity or convexity of the objective function are easy to determine, then the local maximum/minimum is also the global maximum/minimum, as objective functions with concave or convex characteristics are common in optimization problems regarding economics. Concave and convex functions can be determined according to the definition (in the following definition, $0 < \omega < 1$, and x_1, x_2 are any two points given on the curve), and the second-order derivative of a concave function is "everywhere" less than 0 and the second-order derivative of a convex function is "everywhere" greater than 0, so a concave objective function has a global maximum and a convex objective function has a global minimum. This boils down to:

$$\text{global maximum/minimum}\left\{\begin{array}{l} \text{global maximum}\left\{\begin{array}{ll} \omega f(x_1)+(1-\omega)f(x_2) \leq f[\omega x_1+(1-\omega)x_2] & \text{definition} \\ f''(x) < 0,\ \forall x & \text{property} \end{array}\right\}\text{convex;} \\ \text{global minimum}\left\{\begin{array}{ll} f''(x) > 0,\ \forall x & \text{property} \\ \omega f(x_1)+(1-\omega)f(x_2) \geq f[\omega x_1+(1-\omega)x_2] & \text{definition} \end{array}\right\}\text{concave.} \end{array}\right.$$

Only when the second-order derivative has a positive or negative sign can we judge whether it is a maximum or minimum from above. However, it does not rule out the possibility that the second-order derivative may be 0. In this case, it still fails to judge whether it is a maximum or a minimum. Therefore, $f''(x) \leq 0$ or $f''(x) \geq 0$ and $f'(x) = 0$ are necessary conditions for the existence of extreme values. Why? This is also the reason for the following question.

Problem 6: The derivation of the first-order necessary conditions uses the mean value theorem, and the derivation of the second-order (or higher) necessary or sufficient conditions uses the Taylor expansion.

i) If $O = f(x)$ is continuously differentiable in the domain, and O has an extreme value $f(x_o)$, then we must have

$$f'(x_o) = 0.$$

Suppose x is in the neighborhood of x_o, that is, $x \in x_o + \Delta x$, and suppose $f(x) \leq f(x_o)$, then we have $f(x_o + \Delta x) \leq f(x_o)$, then

$$\left.\begin{array}{l} \dfrac{f(x_o+\Delta x) - f(x_o) \leq 0}{\Delta x > 0} \leq 0; \\ \dfrac{f(x_o+\Delta x) - f(x_o) \leq 0}{\Delta x < 0} \geq 0. \end{array}\right\} \quad \Rightarrow \quad f'(x_o) = \left\{\begin{array}{l} f'_+(x_o) \equiv \lim\limits_{\Delta x \to 0^+} \dfrac{f(x_o+\Delta x) - f(x_o)}{\Delta x} \leq 0 \\ f'_-(x_o) \equiv \lim\limits_{\Delta x \to 0^-} \dfrac{f(x_o+\Delta x) - f(x_o)}{\Delta x} \geq 0 \end{array}\right\} = 0.$$

This is Fermat's lemma. [37] proved this first-order condition in the mathematical appendix directly using the mean value theorem. Let x_o be a point in the domain, and $x = x_o + \Delta x$ be another point ($\Delta x > 0$ or $\Delta x < 0$), then there is a point $\omega x + (1-\omega)x_o = x_o + \omega(x - x_o), \omega \in (0,1)$ between x and x_o, according to Lagrange's mean value theorem,

$$f'\left(x_o + \omega(x - x_o)\right) = \frac{f(x) - f(x_o)}{x - x_o} \quad \Leftrightarrow \quad f(x) - f(x_o) = f'\left(x_o + \omega(x - x_o)\right)(x - x_o).$$

If $f'(x_o) > 0$, according to the continuity assumption, there exists an interval $|x - x_o| < z$ such that $f'(x) > 0$ is true everywhere in the interval. Therefore, for any $x > x_0$, $f(x) - f(x_o) > 0$, which contradicts the assumption of $f(x) \leq f(x_o)$. Similarly, $f'(x_o) < 0$ will also lead to a contradiction. The first-order necessary conditions for the minimum can also be roughly proved in this way. Therefore, the necessary condition for the extreme value is $f'(x_o) = 0$. It can also be expressed by differential thinking, that is, $\mathrm{d}O = f'(x)\mathrm{d}x$. At the extreme point, any change in x ($\mathrm{d}x \neq 0$) will not cause a change in O ($\mathrm{d}O = 0$), then it can only be that $f'(x) = 0$. The solution of this equation is x_o, so the condition is equivalent to $f'(x_o) = 0$ [1]

We can also proof by contradiction. If at the extreme point, $f'(x_o) \neq 0$, then either $f'(x_o) > 0$, which means that as x_o increases, the target value will increase; or $f'(x_o) < 0$, which means that as x_o increases, the target value will decrease. In both cases, the target value will not be an extreme value, so the extreme point must have $f'(x_o) = 0$.

ii) Then let's see how to derive the necessary or sufficient conditions of the second or higher order. Let the function $O = f(x)$ be expanded in a Taylor series of order $n \in N^+$ around $x = x_o$ (the remainder is omitted and N^+ represents the

[1] See also Theorem 12.1 in [9, p.327] or the rigorous proof of the first-order condition in [**Sundaram2001**].

set of positive integers)

$$f(x) = \underbrace{f(x_o) + f'(x_o)(x - x_o)}_{\text{Linear approximation}} + \overbrace{\frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f^{(3)}(x_o)}{3!}(x - x_o)^3 + \frac{f^{(4)}(x_o)}{4!}(x - x_o)^4 + \cdots + \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n}^{\text{Nonlinear approximation}}.$$

(a) We focus on the function value at x ($x \neq x_o$) within the neighbourhood of x_o. If the first-order derivative $f'(x_o) \neq 0$, and the second-order derivative and higher-order derivatives are all 0, then:

$$f(x) - f(x_o) = f'(x_o)(x - x_o).$$

If we want to determine whether the function reaches an extreme value when $x = x_o$ (that is, when $f(x) < f(x_o)$ for any x is in the left or right neibourhood of x_o, x_o is a local maximum point; when $f(x) > f(x_o)$ for any x is in the left or right neibourhood of x_o, x_o is a local minimum point), it depends on the sign of $f'(x_o)(x - x_o)$, but it is the product of $f'(x_o)$ and $(x - x_o)$, so the sign cannot be determined.

But if $f(x_o)$ is an extreme value, take the maximum value as an example: On the left side of the maximum point, $x - x_o < 0$ and $f'(x_o) > 0$; on the right side of the maximum point, $x - x_o > 0$ and $f'(x_o) < 0$. It can be seen that there is always $f'(x_o)(x - x_o) < 0$ on both sides of the maximum point. Therefore, $f'(x_o) \neq 0$ means that $x = x_o$ is not an extreme point; in other words, the extreme point must have $f'(x_o) = 0$.

(b) Since the extreme point must have $f'(x_o) = 0$, assuming the second-order derivative is $f''(x_0) \neq 0$ and the third-order derivative and higher-order derivatives are 0. The Taylor expansion is:

$$f(x) - f(x_o) = 0 + \frac{f''(x_o)}{2}(x - x_o)^2.$$

Regardless of whether x is on the left or right side of x_o, $(x - x_o)^2 > 0$ always holds true. As long as the second-order derivative $f''(x_o) > 0$, then $f(x) > f(x_o)$, the value $f(x_o)$ is always smaller than that in the neiborhood so it is a minimum.

Regardless of whether x is on the left or right side of x_o, $(x - x_o)^2 > 0$ always holds true. As long as the second-order derivative $f''(x_o) < 0$, then $f(x) < f(x_o)$, the value $f(x_o)$ is always larger than that in the neiborhood so it is a maximum.

(c) But what if the first-order derivative $f'(x_o) = 0$ and the second-order derivative $f''(x_o) = 0$? Then the value $f(x_o)$ and the value $f(x)$ where x is in the neibourhood of x_0 cannot be compared by expanding to the second-order term. Assuming that the third-order derivative is not zero ($f^{(3)}(x_o) \neq 0$), and the fourth-order derivative and higher-order derivatives are 0, the Taylor expansion is:

$$f(x) - f(x_o) = 0 + 0 + \frac{f^{(3)}(x_o)}{3 \times 2 \times 1}(x - x_o)^3.$$

Since the sign of $(x - x_o)^3$ is not determined, the sign of $f^{(3)}$ cannot determine the sign of $f^{(3)}(x_o)(x - x_o)^3$, so the value $f(x_o)$ and the value $f(x)$ where x is in the neibourhood of x_0 cannot be compared uniformly So $x = x_o$ is not an extreme point. Since a point x_o such that $f'(x_o) = 0$ is not a stationary point, it must be an inflection point. The derivative of a derivative is 0, which shows that the derivative itself can be either a maximum or a minimum, which corresponds to a slightly different inflection point in form.

(d) Recursively, if the first-order derivative $f'(x_o) = 0$, the second-order derivative $f''(x_o) = 0$, the third-order derivative $f^{(3)}(x_o) = 0$, but assuming the fourth-order derivative is $f^{(4)}(x_o) \neq 0$, and the fifth-order derivative and higher-order derivatives are 0. The Taylor expansion is:

$$f(x) - f(x_o) = 0 + 0 + 0 + \frac{f^{(4)}(x_o)}{4!}(x - x_o)^4.$$

Regardless of whether x is on the left or right side of x_o, $(x - x_o)^4 > 0$ always holds true. As long as the fourth-order derivative $f^{(4)}(x_o) > 0$, then $f(x) > f(x_o)$, the value $f(x_o)$ is always smaller than that in the neiborhood, so it is a minimum.

Regardless of whether x is on the left or right side of x_o, $(x - x_o)^4 > 0$ always holds true. As long as the fourth-order derivative $f^{(4)}(x_o) < 0$, then $f(x) < f(x_o)$, the value $f(x_o)$ is always larger than that in the neiborhood, so it is a maximum.

By enumerating, the rule can be seen: if the first-order derivative at $x = x_o$, $f'(x_o) = 0$, then it is an inflection point or a stationary point. Let the first non-zero derivative value encountered in the second-order or higher-order derivatives be the jth-order derivative, that is, $f^{(j)}(x_o) \neq 0$. When j is an odd number, $(x_0, f(x_o))$ is an inflection point; when j is a evev number, $f(x_o)$ is a minimum when $f^{(j)}(x_o) > 0$, and $f(x_o)$ is a maximum when $f^{(j)}(x_o) < 0$.

The above six problems outline and successively introduce the first-order or higher-order conditions for unconstrained optimal solutions, but they are still confined to the scope of mathematics. In economics, what needs to be answered is how to select the choice variables that meet the corresponding economic conditions based on a specific economic environment to optimize the goals of economic entities. The following three examples respectively present the similarities and differences of "choice variables" in different structures of perfect competition and in the market environment of perfect competition and monopolistic competition.

Example 1. **Product supply in a perfectly competitive market**

In perfect competition, prices are determined by market supply and demand. Neither buyers nor sellers can control prices. Representative manufacturers choose product sales quantity Q_s (with subscript s to indicate supply) to optimize profits. Profits are defined as revenue after deducting costs. When both product and factor markets are perfectly competitive, revenue and costs are functions of sales, and the objective profit function is

$$\max_{Q_s} \Pi \equiv \mathcal{R}(Q_s) - C(Q_s).$$

Let's look at the first-order condition first:

$$\frac{\mathrm{d}\Pi}{\mathrm{d}Q_s} = \mathcal{R}'(Q_s) - C'(Q_s) = 0 \quad \Rightarrow \quad \underbrace{\overbrace{\mathcal{R}'(Q_s^\circ)}^{\text{marginal revenue}}}_{\text{MR}} = \underbrace{\overbrace{C'(Q_s^\circ)}^{\text{marginal cost}}}_{\text{MC}}.$$

This results in the equilibrium condition where marginal revenue equals marginal cost, from which the desired product supply (output level) Q_s° can be solved. However, the goal of maximizing profits may not be achieved at this point. At the point where profits are minimized or even continue to rise (an infection point), the revenue from an additional unit of sales is equal to the cost of an additional unit of sales. Therefore, this is only a necessary condition.

Let's look at the second-order condition:

$$\frac{\mathrm{d}^2\Pi}{\mathrm{d}Q_s^2} \equiv \frac{\mathrm{d}}{\mathrm{d}Q_s}\left(\frac{\mathrm{d}O}{\mathrm{d}Q_s}\right) = \mathcal{R}''(Q_s^\circ) - C''(Q_s^\circ) < 0 \quad \Rightarrow \quad \mathcal{R}''(Q_s^\circ) < C''(Q_s^\circ).$$

That is, when the rate of change of the marginal revenue is less than the rate of change of the marginal cost, it can ensure that the profit reaches its maximum value (a sufficient condition).

The explicit revenue curve, under perfect competition, is $\mathcal{R}(Q_s) = PQ_s$, with marginal revenue $\mathcal{R}'(Q_s^\circ) = P$, and the change rate of the marginal revenue $\mathcal{R}''(Q_s^\circ) = 0$. The first- and second-order conditions are:

$$\begin{cases} C'(Q_s^\circ) = P, \\ C''(Q_s^\circ) > 0. \end{cases}$$

This shows that the marginal cost at the optimal output level Q_s° is also the selling price, and the marginal cost at this point is increasing. Why? If the marginal cost is decreasing at Q_s, then producing one more unit at this time will still benefit P, while the marginal cost will be less than P, so Q_s will not be the optimal choice.

Example 2. **Labor demand in a perfectly competitive market**

Production activities depend on production factors, and common production factors include technology, capital, and labor. Omitting technology and exogenous capital, the demand equation for labor as a production factor of representative manufacturers is derived from profit optimization:

$$\max_{L_d \rightleftarrows Q_s} \Pi \equiv \overbrace{PQ_s}^{\text{revenue}} - \overbrace{WL_d}^{\text{variable cost}} - \overbrace{RPK}^{\text{fixed cost}}.$$

Both product and factor markets are perfectly competitive. The price level P, the wage level W, and the capital rental rate R (equals to the real interest rate r, which is the difference between the nominal interest rate i and the expected inflation rate π_e, plus the capital depreciation rate δ) are exogenous to the problem of optimizing corporate profits (exogeneity is indicated in gray). The demand for the production factor of physical or tangible capital K is relatively fixed in the short term, or is considered exogenous. The company only chooses the input factor of labor L_d (with subscript d indicating demand) to provide output supply Q_s (or the labor demand can be determined by choosing the output i.e., $L_d \rightleftarrows Q_s$), to achieve the optimal profit.

Back to the optimization problem above. Substitute the production function $Q_s = AF(K, L_d)$ under the technology level A into the defined profit identity:

$$\max_{L_d \rightleftarrows Q_s} \Pi \equiv PAF(K, L_d) - RPK - WL_d \qquad \Leftrightarrow \qquad \max_{L_d \rightleftarrows Q_s} \Pi \equiv PAF(K, L_d) - WL_d$$

Nominal capital PK and its price (nominal aggregate interest rate) R are both exogenous, so removing this term does not affect the first-order necessary conditions and the optimal choice variables determined by them:

$$\frac{d\Pi}{dL_d} = PAF_L(K, L_d^\circ) - W = 0 \xrightarrow{\text{labor demand curve}} \begin{aligned} \overbrace{W/P}^{\text{real wage}} &= \overbrace{AF_L(K, L_d^\circ)}^{\text{marginal product}}; \\ \underbrace{W/A}_{\text{marginal cost}} &= \underbrace{PF_L(K, L_d^\circ)}_{\text{marginal revenue}} \end{aligned} \xrightarrow{\text{desired labor demand}} L_d^\circ = f\left(\frac{W}{P}, A, K\right).$$

The optimal output is therefore:

$$Q_s^\circ = AF(K, L_d^\circ).$$

The functional relationship between the desired demand for labor as a production factor and the optimal output can be expressed by the inverse function F^{-1} as follows:

$$\Leftrightarrow \quad L_d^\circ = \frac{F^{-1}}{A} Q_s^\circ.$$

The above optimization problem can be expressed as a general function:

$$\max_{L_d \rightleftarrows Q_s} \Pi \equiv \underbrace{f\left(L_d; \frac{W}{P}, A, K\right)}_{\text{objective function}}.$$

First-order necessary conditions and the optimal choice variables determined by them:

$$f_{L_d} = 0 \qquad \Leftrightarrow \qquad L_d^\circ = f\left(\frac{W}{P}, A, K\right).$$

Substituting the above objective function into the value function, we can get the **value function**:

$$\Pi^\circ \equiv \overbrace{f\left(L_d^\circ\left(\frac{W}{P}, A, K\right); \frac{W}{P}, A, K\right)}^{\text{value function}}.$$

It has the following properties:

$$\frac{d\Pi^\circ}{dK} = \underbrace{\underbrace{\overbrace{f_{L_d^\circ}}^{\text{partial derivative}}}_{=0} \underbrace{\overbrace{\frac{dL_d^\circ}{dK}}^{\text{derivative}}}_{\neq 0}}_{\text{indirect effect}} + \underbrace{\overbrace{f_K}^{\text{partial derivative}}}_{\text{direct effect}},$$

$$= 0 + f_K.$$

It can be seen that the derivative value of the value function with respect to a certain exogenous variable is the partial derivative value of the objective function with respect to the exogenous variable, which is the **envelope theorem**. This is a reflection of the comparative analysis in Chapter 3, but it is also an extension of the optimization theme. The reason for this introduction is that, first, a similar function form is used in the dynamic programming solution method for the dynamic optimal problem presented later in this chapter; second, it reminds readers that although the value function can be regarded as a structural equation (there is correlation between the independent variables), it is not troubled by the endogeneity problem.

It is also necessary to use the second-order condition to determine whether the selected L_d° makes the objective function achieve a maximum or minimum value:

$$\frac{\mathrm{d}^2\Pi}{\mathrm{d}L_d^2} = PF_{LL} < 0.$$

The production function is assumed to have positive and decreasing marginal product ($F_L > 0, F_{LL} < 0$), so the second-order condition is less than 0, meaning that the above optimization problem produces a maximum value. Of course, according to the definition or the property, it is not difficult to see that the objective function of labor demand defined by the profit identity is a concave function; in other words, the cost in the profit function is linear, so the concave and convex characteristics of the profit function are determined by the production function that is closely related to the revenue; in other words, the concave and convex characteristics of the profit function in this example are consistent with the concave and convex characteristics of the production function. The concave production function is an important assumption in economics, so in the profit optimization problem, a local maximum is also a global maximum.

The production function is set to be $Q_s = AK^\alpha L_d^{1-\alpha}$, and the parameters $\alpha \in [0,1]$ and $(1-\alpha) \in [0,1]$ represent the output share of the corresponding factors (i.e., the percent change in output caused by a change of 1 percent of the factor). This is called the Cobb-Douglas function (the origin and properties will be systematically introduced in Chapter 3). If we do not consider the labor factor for the time being and instead examine oil resources as an input factor, assuming that the manufacturer maximize output eliminating the cost of oil , that is:

$$
\begin{aligned}
& Q_s = \max_{\text{Oil}}(\mathcal{A}K^\alpha \text{Oil}^{1-\alpha} - p_{\text{oil}}\text{Oil}),\\
\xRightarrow{\text{F.O.C.}}\quad & 0 = (1-\alpha)\mathcal{A}K^\alpha \text{Oil}^{-\alpha} - p_{\text{oil}},\\
\Rightarrow\quad & \text{Oil}^{-\alpha} = \frac{p_{\text{oil}}}{(1-\alpha)\mathcal{A}K^\alpha},\\
\Rightarrow\quad & \text{Oil} = \left[\frac{p_{\text{oil}}}{(1-\alpha)\mathcal{A}K^\alpha}\right]^{-\frac{1}{\alpha}},\\
\Rightarrow\quad & \text{Oil}^{1-\alpha} = \left[\frac{p_{\text{oil}}}{(1-\alpha)\mathcal{A}K^\alpha}\right]^{1-\frac{1}{\alpha}},\\
\Rightarrow\quad & Q_s = \mathcal{A}K^\alpha\left[\frac{P_{\text{Oil}}}{(1-\alpha)\mathcal{A}K^\alpha}\right]^{1-\frac{1}{\alpha}} - P_{\text{Oil}}\left[\frac{P_{\text{Oil}}}{(1-\alpha)\mathcal{A}K^\alpha}\right]^{-\frac{1}{\alpha}},\\
& = \mathcal{A}^{\frac{1}{\alpha}}P_{\text{oil}}^{1-\frac{1}{\alpha}}(1-\alpha)^{\frac{1}{\alpha}-1}K - \mathcal{A}^{\frac{1}{\alpha}}P_{\text{oil}}^{1-\frac{1}{\alpha}}K(1-\alpha)^{\frac{1}{\alpha}},\\
& = \mathcal{A}^{\frac{1}{\alpha}}P_{\text{oil}}^{1-\frac{1}{\alpha}}(1-\alpha)^{\frac{1}{\alpha}-1}K - \mathcal{A}^{\frac{1}{\alpha}}P_{\text{oil}}^{1-\frac{1}{\alpha}}K(1-\alpha)^{\frac{1}{\alpha}-1}(1-\alpha),\\
& = \underbrace{\alpha(1-\alpha)^{\frac{1}{\alpha}-1}\mathcal{A}^{\frac{1}{\alpha}}P_{\text{oil}}^{1-\frac{1}{\alpha}}}K,\\
& = AK.
\end{aligned}
$$

The last step is to set $A \equiv \alpha(1-\alpha)^{\frac{1}{\alpha}-1}\mathcal{A}^{\frac{1}{\alpha}}P_{\text{oil}}^{1-\frac{1}{\alpha}}$, where $\mathcal{A}$ is the technical level, and A is defined as the effective technical level, which is driven by the actual oil price p_{oil}. And because $1 - \frac{1}{\alpha} < 0$, an increase in the actual oil price will lower the effective technical level. [2]

Example 3. Optimal pricing of products in a monopolistic competition market

Given other input factors, the manufacturer chooses the desired labor to maximize profits (the concave production function ensures that the extreme value is the global maximum). Now it is still assumed that the labor factor market is perfectly competitive, but the manufacturer has a certain degree of monopoly power in the product sales market, which means:

(1) If we assume that one manufacturer corresponds to one product, it is not appropriate to use a representative manufacturer in a monopolistic competition environment, because monopolistic competition means that manufacturers have pricing power due to product differences. Assuming there are two manufacturers in the market, the products of manufacturer 1 and manufacturer 2 are not completely substitutable, and the elasticity of substitution is ϵ (the specific definition of this concept is detailed in Chapter 3).

(2) Capital and technology are still given (exogenous), and the capital is even normalized to 1. The production function is set to $Q_{is} = AL_{id}^{1-\alpha}$, where $\alpha = 0$ is set to represent constant returns to scale (see also Chapter 3), because the input factor increases n times, it is easy to prove that the output also increase n times, that is, $A(nL_{id}) = n(AL_{id}) = nQ_{is}$. The factor market is still perfectly competitive, and wages are determined by market supply and demand, set to W (exogenous for the manufacturer). However, because the manufacturer $i, i = \{1, 2\}$ has monopoly powers in the product market, the factor inputs are determined by the product demands, that is,

$$AL_{id} = \overbrace{Q_{is} = Q_{id}}^{\text{prodect supply determined by product demand}} \Rightarrow \overbrace{L_{id} = Q_{id}/A}^{\text{factor demand determined by product deand}}.$$

(3) The demand function for product i comes from the utility maximization of the household sector given a budget constraint. This is the solution to the optimization problem of equality constraints (strictly speaking, inequality constraints), which will be discussed later. Although the manufacturer has monopoly power for its products and can have pricing power for its products, the total price level also includes the prices determined by other manufacturers. Therefore, the total price index P is exogenous to any manufacturer; at the same time, the manufacturer's pricing will affect the demand for the priced product, but the total demand of the whole society Q_d is also exogenous to a single manufacturer (at equilibrium, total demand equals total supply equals Q). Based on these introductions, we will directly assume that the demand curve of product i is (Example 9 will explain its origin) without deduction:

$$Q_{id} = \left(\frac{P_i}{P}\right)^{-\epsilon} Q.$$

There are two similar approaches to solving the optimal price when the manufacturer's profit is optimized (that is, the desired selling price at the instantaneous state):

Method 1: The manufacturer first selects the product demand to maximize the profit function, and then uses the inverse demand function $P_i = \left(\frac{Q_{id}}{Q}\right)^{-\frac{1}{\epsilon}} P$ to solve the optimal price:

$$\begin{aligned}
\max_{P_i \rightleftarrows Q_{id} \leftrightarrows L_{id}} \Pi_i &\equiv P_i Q_{id} - W L_{id}, \\
&= P_i Q_{id} - W\frac{Q_{id}}{A} = \left(P_i - \frac{W}{A}\right) Q_{id}, \\
&= \left[\left(\frac{Q_{id}}{Q}\right)^{-\frac{1}{\epsilon}} P - \frac{W}{A}\right] Q_{id} = \frac{Q_{id}^{1-\frac{1}{\epsilon}}}{Q^{-\frac{1}{\epsilon}}} P - \frac{W}{A} Q_{id}.
\end{aligned}$$

The first-order necessary conditions are:

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}Q_{id}} = \left(1 - \frac{1}{\epsilon}\right) P Q^{\frac{1}{\epsilon}} Q_{id}^{-\frac{1}{\epsilon}} - \frac{W}{A} = 0.$$

This can be used to solve the desired product demand, and then the desired pricing level can be solved based on the

[2] The definition of effective technology in the production function uses Moll (2023) for reference.

inverse demand function. However, this process is slightly redundant, because by observing the first-order condition above, in which $PQ^{\frac{1}{\epsilon}}Q_{id}^{-\frac{1}{\epsilon}}$ is exactly P_i, this first-order condition is transformed into:

$$\left(1-\frac{1}{\epsilon}\right)P_i - \frac{W}{A} = 0.$$

This skips solving the optimal Q_{id}° and obtains the instantaneous optimal pricing when maximizing profits:

$$P_i^{\circ} = \underbrace{\overbrace{\frac{\epsilon}{\epsilon-1}}}_{\text{markup}} \cdot \underbrace{\overbrace{\frac{W}{A}}}_{\substack{\text{nominal}\\ \text{MC}}}.$$

The marginal output of labor is A, and the nominal wage divided by the marginal output of labor is the nominal marginal cost. $\frac{\epsilon}{\epsilon-1}$ is called the cost markup. Therefore, the above formula shows that the product price under monopolistic competition is equal to the cost markup multiplied by the marginal cost, which has an additional cost markup compared to the perfect competition environment.

Method 2: The manufacturer directly chooses the selling price of the product to maximize the profit function:

$$\begin{aligned}
\max_{L_{id}\rightleftarrows Q_{id}\rightleftarrows P_i} \Pi_i &\equiv P_i Q_{id} - W L_{id} = P_i Q_{id} - W\frac{Q_{id}}{A}, \\
&= \left(P_i - \frac{W}{A}\right)Q_{id} = \left(P_i - \frac{W}{A}\right)\left[\left(\frac{P_i}{P}\right)^{-\epsilon} Q\right], \\
&= \frac{P_i^{1-\epsilon}}{P^{-\epsilon}}Q - \frac{W}{A}\left(\frac{P_i}{P}\right)^{-\epsilon} Q.
\end{aligned}$$

The first-order necessary condition is:

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}P_i} = (1-\epsilon)\frac{P_i^{-\epsilon}}{P^{-\epsilon}}Q + \epsilon\frac{W}{A}\left(\frac{P_i}{P}\right)^{-\epsilon-1}\frac{1}{P}Q = 0.$$

From this we can get:

$$\frac{P_i^{\circ}}{P} = \overbrace{\underbrace{\frac{\epsilon}{\epsilon-1}}}^{\text{markup}} \cdot \overbrace{\underbrace{\frac{W/P}{A}}}^{\text{real MC}}.$$

The relative price of a product is defined as $\frac{P_i^{\circ}}{P}$, while $\frac{W}{P}$ represents the real wage level. The real wage divided by the marginal cost of labor is the real marginal cost. The above formula shows that the relative price of a product under monopolistic competition is equal to the cost markup multiplied by the real marginal cost. Eliminating the total price level P on both sides, we get the same derivation result as in method 1. An equilalent result can also be get by maximizing the real profit function $\frac{P_i}{P}Q_{id} - \frac{W}{P}L_{id}$.

Method 3: The manufacturer chooses a price higher than the marginal cost to maximize the profit function:

$$\begin{aligned}
\max_{Q_{id}\rightleftarrows P_i} \Pi_i &= (P_i - \mathrm{MC}_i)Q_{id}, \\
&= (P_i - \mathrm{MC}_i)\left[\left(\frac{P_i}{P}\right)^{-\epsilon} Q\right], \\
&= \frac{P_i^{1-\epsilon}}{P^{-\epsilon}}Q - \mathrm{MC}_i\left(\frac{P_i}{P}\right)^{-\epsilon} Q.
\end{aligned}$$

The first-order necessary conditions are:

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}P_i} = (1-\epsilon)\frac{P_i^{-\epsilon}}{P^{-\epsilon}}Q + \epsilon\mathrm{MC}_i\left(\frac{P_i}{P}\right)^{-\epsilon-1}\frac{1}{P}Q = 0.$$

From this we can get:

$$P_i^\circ = \frac{\epsilon}{\epsilon - 1}\mathrm{MC}_i.$$

The above solution for the optimal price or the subsequent solution for the optimal demand is based on a given marginal cost. The marginal cost can also be derived from the profit maximization of equality constraints or the cost minimization of equality constraints. Although this is an equality-constrained optimization problem, it can be converted into an unconstrained optimization problem for a single decision variable:

$$\begin{aligned}
&\min_{L_{id}} C_i \equiv W L_{id},\\
&\qquad\Updownarrow\\
&\min_{Q_{is}} C_i = W\frac{Q_{is}}{A},\\
\Rightarrow\quad &\mathrm{MC}_i \equiv \frac{\mathrm{d}C}{\mathrm{d}Q_{is}} = \frac{W}{A} = \mathrm{MC}.
\end{aligned}$$

It can be seen that under the corresponding assumptions, the marginal costs of each manufacturer are the same.

1.1.1.2 Multiple Decision Variables

I. **Two Decision Variables**

When explaining the first-order necessary conditions for single-variable optimization, we mentioned the idea of differentials, that is, $\mathrm{d}O = f'(x)\mathrm{d}x$. At the extreme point, the change of x ($\mathrm{d}x \neq 0$) will not cause the change of O ($\mathrm{d}O = 0$). So it can only be $f'(x) = 0$. This is very inspiring for obtaining the first-order necessary conditions for obtaining extreme values in the unconstrained optimization problem of two variables. Let the objective function of the two selected variables be $O = f(x, y)$. The total differential is $\mathrm{d}O = f_x\mathrm{d}x + f_y\mathrm{d}y$. At the extreme point, any changes in x and y ($\mathrm{d}x \neq 0 \neq \mathrm{d}y$) will not cause changes in O ($\mathrm{d}O = 0$), so it can only be

$$f_x = 0 = f_y.$$

This gives the first-order necessary condition for obtaining the extreme value of the two-variable unconstrained optimization problem.

The second-order and higher-order necessary or sufficient conditions for unconstrained optimization problems with two variables are similar to those for single variables, but usually the second-order conditions are sufficient (ie, the second-order conditions are not 0), so we will focus on them. By making the function $O = f(x, y)$ perform a second-order Taylor expansion around ($x = x_o, y = y_o$) and substituting the first-order necessary conditions into:

$$\begin{aligned}
f(x,y) - f(x_o,y_o) &= f_x^o(x-x_o) + f_y^o(y-y_o) + \frac{1}{2!}\left[f_{xx}^o(x-x_o)^2 + 2f_{xy}^o(x-x_o)(y-y_o) + f_{yy}^o(y-y_o)^2\right],\\
&= 0 + 0 + \frac{1}{2!}\left[f_{xx}^o(x-x_o)^2 + 2f_{xy}^o(x-x_o)(y-y_o) + f_{yy}^o(y-y_o)^2\right],\\
&= \frac{f_{xx}^o}{2}\left[(x-x_o)^2 + 2\frac{f_{xy}^o}{f_{xx}^o}(x-x_o)(y-y_o) + \frac{f_{yy}^o}{f_{xx}^o}(y-y_o)^2\right],\\
&= \frac{f_{xx}^o}{2}\left[(x-x_o)^2 + 2\frac{f_{xy}^o}{f_{xx}^o}(x-x_o)(y-y_o) + \frac{(f_{xy}^o)^2}{(f_{xx}^o)^2}(y-y_o)^2 - \frac{(f_{xy}^o)^2}{(f_{xx}^o)^2}(y-y_o)^2 + \frac{f_{yy}^o}{f_{xx}^o}(y-y_o)^2\right],\\
&= \frac{f_{xx}^o}{2}\left\{\left[(x-x_o)^2 + 2\frac{f_{xy}^o}{f_{xx}^o}(x-x_o)(y-y_o) + \frac{(f_{xy}^o)^2}{(f_{xx}^o)^2}(y-y_o)^2\right] + \left[\frac{f_{yy}^o}{f_{xx}^o}(y-y_o)^2 - \frac{(f_{xy}^o)^2}{(f_{xx}^o)^2}(y-y_o)^2\right]\right\},\\
&= \frac{f_{xx}^o}{2}\left\{\left[(x-x_o)^2 + 2\frac{f_{xy}^o}{f_{xx}^o}(x-x_o)(y-y_o) + \frac{(f_{xy}^o)^2}{(f_{xx}^o)^2}(y-y_o)^2\right] + \frac{f_{xx}^o f_{yy}^o - (f_{xy}^o)^2}{(f_{xx}^o)^2}(y-y_o)^2\right\},\\
&= \frac{f_{xx}^o}{2}\underbrace{\left\{\left[(x-x_o) + \frac{f_{xy}^o}{f_{xx}^o}(y-y_o)\right]^2 + \frac{f_{xx}^o f_{yy}^o - (f_{xy}^o)^2}{(f_{xx}^o)^2}(y-y_o)^2\right\}}_{\text{squaring}};
\end{aligned}$$

$$\overset{\text{or}}{=} \overbrace{\frac{1}{2}\begin{bmatrix} x-x_o & y-y_o \end{bmatrix} \underbrace{\begin{bmatrix} f_{xx}^o & f_{xy}^o \\ f_{yx}^o & f_{yy}^o \end{bmatrix}}_{\text{Hessian matrix}} \begin{bmatrix} x-x_o \\ y-y_o \end{bmatrix}}^{\text{quadratic form}} = \frac{1}{2!}\left[f_{xx}^o (x-x_o)^2 + 2 f_{xy}^o (x-x_o)(y-y_o) + f_{yy}^o (y-y_o)^2 \right].$$

Judging from the formula obtained by "squaring", to determine whether it is positive or negative, it depends on the positive or negative of f_{xx}^o and $f_{xx}^o f_{yy}^o - (f_{xy}^o)^2$, and first of all, $f_{xx}^o f_{yy}^o > (f_{xy}^o)^2$should be satisfied. Under this:

1) If $f_{xx}^o > 0$, $f(x, y) - f(x_o, y_o) > 0$, that is, $f(x, y) > f(x_o, y_o)$, so $f(x_o, y_o)$ is the minimum.

2) If $f_{xx}^o < 0$, $f(x, y) - f(x_o, y_o) < 0$, that is, $f(x, y) < f(x_o, y_o)$, so $f(x_o, y_o)$ is the maximum.

3) Since $f_{xx}^o f_{yy}^o > (f_{xy}^o)^2$ always holds, f_{xx}^o f_{yy}^o always have the same sign in the case above.

Judging from the formula obtained from the "quadratic form", the above conditions for determining positive and negative are easily reflected in the Hessian determinant:

4) When the "squared" formula is positive, the leading principal minors of the 2×2 -dimensional Hessian matrix are all positive, and the object function has a minimum, that is,

$$\text{minimum} \left\{ \begin{array}{lr} \text{first-order leading principal minor:} & \left| f_{xx}^o \right| > 0, \\ \qquad\qquad \text{and} & \\ \text{second-order leading principal minor:} & \begin{vmatrix} f_{xx}^o & f_{xy}^o \\ f_{yx}^o & f_{yy}^o \end{vmatrix} > 0. \end{array} \right\} \text{positive definite}$$

5) When the "squared" formula is negative, the leading principal minors of the 2×2 -dimensional Hessian matrix are first negative and then positive, and the object function has a maximum, that is,

$$\text{maximum} \left\{ \begin{array}{lr} \text{first-order leading principal minor:} & \left| f_{xx}^o \right| < 0, \\ \qquad\qquad \text{and} & \\ \text{second-order leading principal minor:} & \begin{vmatrix} f_{xx}^o & f_{xy}^o \\ f_{yx}^o & f_{yy}^o \end{vmatrix} > 0. \end{array} \right\} \text{negative definite}$$

If the second-order condition continues to be 0, then look at higher-order conditions until a higher-order sufficient condition that is not 0 appears. When the first-order necessary conditions are met and the positive or negative conditions of the second-order or higher-order conditions cannot be determined, it means that an "inflection point" appears in single variable optimization, but this is more complicated and may also be a "saddle point", that is, a maximum value on one cross section and a minimum value on another cross section.

Example 4. Capital and labor demand in a perfectly competitive market environment

Still taking the problem of maximizing the manufacturer's profit as an example, we add tangible capital as a choice variable. But note that $Q_s = F(K_d, L_d)$ is not a constraint, because the equation does not intend to describe the relationship between the two choice variables K_d and L_d. Therefore, the unconstrained optimization problem of the two choice variables is:

$$\max_{\{K_d, L_d\} \rightleftarrows Q_s} \Pi \equiv PF(K_d, L_d) - RPK_d - WL_d.$$

First-order necessary conditions:

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}\Pi}{\mathrm{d}K_d} = PF_K(K_d, L_d) - RP = 0, \\ \dfrac{\mathrm{d}\Pi}{\mathrm{d}L_d} = PF_L(K_d, L_d) - W \quad = 0. \end{array} \right.$$

The nominal wage of labor is W, the marginal output of labor (marginal material product) is $F_L(K_d, L_d)$; the rental rate of capital $R = r + \delta = i - \pi_e + \delta$ is called the marginal cost of capital use (where $r = i - \pi_e$ represents the real interest rate, π_e is the expected inflation rate, and δ is the capital depreciation rate. Example 12-1 and other places also explain the reason) $F_K(K_d, L_d)$ is the marginal product of capital.

The second-order condition can be given by the Hessian matrix:

$$\underbrace{\begin{bmatrix} \Pi_{KK} & \Pi_{KL} \\ \Pi_{LK} & \Pi_{LL} \end{bmatrix}}_{\text{Hessian matrix}} \Rightarrow \begin{cases} \text{first-order leading principal minor: } \left|\Pi_{KK}\right| = PF_{KK} < 0, \\ \text{second-order leading principal minor: } \begin{vmatrix} \Pi_{KK} & \Pi_{KL} \\ \Pi_{LK} & \Pi_{LL} \end{vmatrix} = P^2(F_{KK}F_{LL} - F_{KL}^2) > 0. \end{cases}$$

Although the orders of the cross partial derivatives of F_{KL} and F_{LK} are different, as long as they are continuous, according to Young's theorem, $F_{KL} = F_{LK}$.

When $F_{KK} < 0$, according to $F_{KK}F_{LL} > F_{KL}^2$, it should be that $F_{LL} < 0$. Therefore, the second-order sufficient condition for this optimization problem is:

$$\begin{cases} F_{KK} < 0, \\ F_{LL} < 0, \\ F_{KK}F_{LL} > F_{KL}^2. \end{cases}$$

It can be seen that the assumptions about some common properties of the production function are to make the profit own a maximum. If the production function is a concave curve (the second-order derivative is less than 0 everywhere), then the local maximum is also the global maximum. Similar to the second-order conditions for the maximum value of a single variable, the decreasing rate of change of the marginal output of the two factors of capital and labor in the case of two variables is also a part of sufficient conditions for the maximum value, but there is an additional condition for obtaining the extreme value that "the product of the rates of change of the same factors is greater than the product of the rates of change of the cross factors". The reason for adding this condition is that the change of one factor not only affects its own marginal output but also affects the marginal output of the other factor. Assume that $F_{KL} = F_{LK}$ is greater than F_{KK} and F_{LL}, or the impact of a factor change (such as capital) on the marginal output of the other factor (i.e., labor) is greater than the impact on the marginal output of the factor (ie, capital) itself. Although $F_{KK} < 0$, this will lead to a net increase in F_K rather than a decrease. At this time, increasing capital and reducing labor can still increase profits. Therefore, $F_{KK}F_{LL} > F_{KL}^2$ is another sufficient condition for achieving the extreme value [39, pp.76-77].

Example 5. **Optimal pricing of two products in a monopolistic competition market**

Assume that the prices of the two products of a manufacturer with a certain degree of monopoly power are P_1 and P_2 respectively, and the market demand functions for these two products are:

$$\left.\begin{aligned} Q_{1d} &= a - bP_1 + P_2, \\ Q_{2d} &= \alpha - \beta P_2 + P_1. \end{aligned}\right\} \quad \text{vs.} \quad \begin{cases} \log Q_{1d} = \log Q - \epsilon \log P_1 + \epsilon \log P, \\ \log Q_{2d} = \log Q - \epsilon \log P_2 + \epsilon \log P. \end{cases}$$

The two sets of demand curves can be compared: the left side is the ad-hoc form, and the right side is the demand curve of the two products picked out in Example 3 (it will be introduced later that it can be strictly derived based on maximizing the consumption of a basket of consumer goods under given household sector expenditure, so it has a micro foundation). Excepting the logarithm form of the two sides of the equations on the right, the two demand curves are very similar. The aggregate price level P includes the prices of products other than product 1, such as P_2. The different products on the right side correspond to different manufacturers, and what needs to be calculated is the profit maximization of a certain manufacturer, so only its price itself is the choice variable; while the two products on the left side belong to the same manufacturer, and what needs to be calculated is also the profit maximization of the manufacturer, so price 1 and price 2 are both choice variables. But in general, it can be seen from the demand function that the sales of the two commodities will affect each other. If the price of one commodity is high, the demand will be low, but if the price of the other commodity is high, the demand will also be high. This shows that both are normal commodities and are substitutable for each other. In particular, the right side gives a substitution elasticity that will not approach infinity as ϵ.

The profit of a monopolistically competitive firm is defined as total revenue minus total cost, that is,

$$\overbrace{\Pi}^{\text{total profit}} \equiv \overbrace{\mathcal{R}(P_1,P_2)}^{\text{total revenue}} - \overbrace{C(P_1,P_2)}^{\text{total cost}}.$$

The total revenue is

$$\begin{aligned}
\mathcal{R}(P_1,P_2) &= P_1Q_{1d} + P_2Q_{2d},\\
&= P_1(a - bP_1 + P_2) + P_2(\alpha - \beta P_2 + P_1),\\
&= aP_1 - bP_1^2 + P_1P_2 + \alpha P_2 - \beta P_2^2 + P_1P_2,\\
&= aP_1 - bP_1^2 + \alpha P_2 - \beta P_2^2 + 2P_1P_2.
\end{aligned}$$

The discussion of factor markets is ignored here, and a total cost function can be simply assumed:

$$\begin{aligned}
C(P_1,P_2) &= Q_{1d}^2 + Q_{1d}Q_{2d} + Q_{2d}^2,\\
&= (a - bP_1 + P_2)^2 + (a - bP_1 + P_2)(\alpha - \beta P_2 + P_1) + (\alpha - \beta P_2 + P_1)^2,\\
&= (a^2 + b^2P_1^2 + P_2^2 - 2abP_1 + 2aP_2 - 2bP_1P_2)\\
&\quad + (a\alpha - a\beta P_2 + aP_1 - b\alpha P_1 + b\beta P_1P_2 - bP_1^2 + \alpha P_2 - \beta P_2^2 + P_1P_2)\\
&\quad + (\alpha^2 + \beta^2P_2^2 + P_1^2 - 2\alpha\beta P_2 + 2\alpha P_1 - 2\beta P_1P_2),\\
&= (a^2 + a\alpha + \alpha^2) + (-2ab + a - b\alpha + 2\alpha)P_1 + (2a - a\beta + \alpha - 2\alpha\beta)P_2\\
&\quad + (b^2 - b + 1)P_1^2 + (1 - \beta + \beta^2)P_2^2 + (-2b + b\beta + 1 - 2\beta)P_1P_2.
\end{aligned}$$

So we get the objective function:

$$\max_{\{P_1,P_2\}} \Pi = -(a^2 + a\alpha + \alpha^2) + (2ab + b\alpha - 2\alpha)P_1 + (a\beta + 2\alpha\beta - 2a)P_2 - (1 + b^2)P_1^2 - (1 + \beta^2)P_2^2 + (1 + 2b - b\beta + 2\beta)P_1P_2.$$

The first-order necessary conditions are:

$$\begin{aligned}
\frac{\partial\Pi}{\partial P_1} &= (2ab + b\alpha - 2\alpha) - 2(1 + b^2)P_1 + (1 + 2b - b\beta + 2\beta)P_2 = 0,\\
\frac{\partial\Pi}{\partial P_2} &= (a\beta + 2\alpha\beta - 2a) - 2(1 + \beta^2)P_2 + (1 + 2b - b\beta + 2\beta)P_1 = 0.
\end{aligned}$$

Rearrange:

$$\left.\begin{aligned}
2(1 + b^2)P_1 - (1 + 2b - b\beta + 2\beta)P_2 &= b\alpha + 2ab - 2\alpha,\\
-(1 + 2b - b\beta + 2\beta)P_1 + 2(1 + \beta^2)P_2 &= a\beta + 2\alpha\beta - 2a.
\end{aligned}\right\} \quad \text{vs.} \quad \begin{cases}
F^1 \equiv 2(1 + b^2)P_1 - (1 + 2b - b\beta + 2\beta)P_2 - b\alpha - 2ab + 2\alpha = 0,\\
F^2 \equiv -(1 + 2b - b\beta + 2\beta)P_1 + 2(1 + \beta^2)P_2 - a\beta - 2\alpha\beta + 2a = 0.
\end{cases}$$

Written in matrix form:

$$\underbrace{\begin{bmatrix} 2(1 + b^2) & -(1 + 2b - b\beta + 2\beta)\\ -(1 + 2b - b\beta + 2\beta) & 2(1 + \beta^2)\end{bmatrix}}_{\mathbf{J}} \underbrace{\begin{bmatrix} P_1\\ P_2\end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} b\alpha + 2ab - 2\alpha\\ a\beta + 2\alpha\beta - 2a\end{bmatrix}}_{\mathbf{x}} \quad \text{vs.} \quad \underbrace{\begin{bmatrix} \frac{\partial F^1}{\partial P_1} & \frac{\partial F^1}{\partial P_2}\\ \frac{\partial F^2}{\partial P_1} & \frac{\partial F^2}{\partial P_2}\end{bmatrix}}_{\mathbf{J}} \underbrace{\begin{bmatrix} P_1\\ P_2\end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} b\alpha + 2ab - 2\alpha\\ a\beta + 2\alpha\beta - 2a\end{bmatrix}}_{\mathbf{x}}$$

If the Jacobi matrix is $\mathbf{J} \neq 0$, then either matrix inversion or Cramer's rule can be used to solve the endogenous variables P_1 and P_2 (goal equilibrium values), thereby obtaining the optimal selling price (desired price) of the two commodities:

$$\xRightarrow[\text{solve jointly}]{\text{matrix inversion}} \begin{bmatrix} P_1^\circ\\ P_2^\circ\end{bmatrix} = \begin{bmatrix} 2(1 + b^2) & -(1 + 2b - b\beta + 2\beta)\\ -(1 + 2b - b\beta + 2\beta) & 2(1 + \beta^2)\end{bmatrix}^{-1} \begin{bmatrix} b\alpha + 2ab - 2\alpha\\ a\beta + 2\alpha\beta - 2a\end{bmatrix};$$

or

$$\xrightarrow[\text{solve individually}]{\text{Cramer's rule}} \begin{cases} P_1^\circ = \dfrac{\begin{vmatrix} b\alpha + 2ab - 2\alpha & -(1+2b-b\beta+2\beta) \\ a\beta + 2\alpha\beta - 2a & 2(1+\beta^2) \end{vmatrix}}{\begin{vmatrix} 2(1+b^2) & -(1+2b-b\beta+2\beta) \\ -(1+2b-b\beta+2\beta) & 2(1+\beta^2) \end{vmatrix}}, \\ P_2^\circ = \dfrac{\begin{vmatrix} 2(1+b^2) & b\alpha + 2ab - 2\alpha \\ -(1+2b-b\beta+2\beta) & a\beta + 2\alpha\beta - 2a \end{vmatrix}}{\begin{vmatrix} 2(1+b^2) & -(1+2b-b\beta+2\beta) \\ -(1+2b-b\beta+2\beta) & 2(1+\beta^2) \end{vmatrix}}. \end{cases}$$

The next chapter will focus on equilibrium problems and will introduce the application of matrix algebra in solving the equilibrium of a linear system. The elimination method can also be used here. It is worth mentioning that the above solution process converts the objective function into a function containing only prices (choice variable), but it can also be solved like the method 1 of solving the previous Example 3, converting the demand function as an inverse demand function, writing the objective function as the function of product demands, and then using them as the choice variables. After solving the desired demand for the product, the desired price of each product is solved based on the inverse demand function. For this problem, this idea is actually simpler [8, pp.332-333].

The second-order sufficient condition is:

$$\underbrace{\begin{bmatrix} \Pi_{P_1P_1} & \Pi_{P_1P_2} \\ \Pi_{P_2P_1} & \Pi_{P_2P_2} \end{bmatrix}}_{\text{Hessian matrix}} \Rightarrow \begin{cases} \text{first-order leading principal minor: } \left|\Pi_{P_1P_1}\right|, \\ \text{second-order leading principal minor: } \begin{vmatrix} \Pi_{P_1P_1} & \Pi_{P_1P_2} \\ \Pi_{P_2P_1} & \Pi_{P_2P_2} \end{vmatrix}. \end{cases}$$

with

$$\left.\begin{aligned} \frac{\partial \Pi}{\partial P_1} &= (2ab + b\alpha - 2\alpha) - 2(1+b^2)P_1 + (1+2b-b\beta+2\beta)P_2 = 0, \\ \frac{\partial \Pi}{\partial P_2} &= (a\beta + 2\alpha\beta - 2a) - 2(1+\beta^2)P_2 + (1+2b-b\beta+2\beta)P_1 = 0. \end{aligned}\right\} \Rightarrow \begin{aligned} \Pi_{P_1P_1} &= -2(1+b^2), \\ \Pi_{P_1P_2} &= 1+2b-b\beta+2\beta, \\ \Pi_{P_2P_2} &= -2(1+\beta^2), \\ \Pi_{P_2P_1} &= 1+2b-b\beta+2\beta. \end{aligned}$$

The first-order leading principal minor is $-2(1+b^2) < 0$ and the second-order leading principal minor is the Hessian determinant itself: $4(1+b^2)(1+\beta^2) - (1+2b-b\beta+2\beta)^2$. To ensure that the value sought is a maximum (concave or quasi-concave function leads to a maximum value), the second-order leading principal minor should be positive, that is, $4(1+b^2)(1+\beta^2) > (1+2b-b\beta+2\beta)^2$.

II. **Multiple Decision Variables**

The following discusses the unconstrained optimization problem with $m \geq 3$ choice variables. Let the objective function be $O = f(x_1, x_2, x_3, \ldots, x_m)$, which is $\mathrm{d}O = f_1\mathrm{d}x_1 + f_2\mathrm{d}x_2 + f_3\mathrm{d}x_3 + \cdots + f_m\mathrm{d}x_m$ after total differentiation, where f_j represents the partial derivative of the function with respect to each choice variable x_j, and $j \in [1, m]$ is a positive integer. Similar to the discussion of the first-order necessary conditions in the double-variable section, at the extreme point, the change of $x_1, x_2, x_3, \ldots, x_m$ does not cause the change of $O(\mathrm{d}O = 0)$. Using ∇ to represent the set of partial derivatives of the function with respect to each element (gradient vector), the condition is,

$$\nabla f(x_1, x_2, x_3, \ldots, x_m) = \mathbf{0},$$

this gives the first-order necessary conditions for achieving extreme values in multi-variable unconstrained optimization problems.

Double-variable problems are already multi-variable in fact, so the second-order and higher-order necessary or sufficient conditions for unconstrained optimization problems with more variables are naturally still similar to the discussion steps for single-variable problems. For simplicity, only the second-order Taylor expansion of the function of 3 choice variables around $(x_1 = x_{1o}, x_2 = x_{2o}, x_3 = x_{3o})$ is listed, and the second-order partial derivatives at this point are marked as $f^o_{x_i x_j}$, $i = \{1, 2, 3\}$, $j = \{1, 2, 3\}$. Substituting the first-order necessary conditions into:

$$
\begin{aligned}
f(x_1, x_2, x_3) - f(x_{1o}, x_{2o}, x_{3o}) &= f^o_{x_1x_1}(x_1 - x_{1o})^2 + f^o_{x_1x_2}(x_1 - x_{1o})(x_2 - x_{2o}) + f^o_{x_1x_3}(x_1 - x_{1o})(x_3 - x_{3o}) \\
&\quad + f^o_{x_2x_1}(x_2 - x_{2o})(x_1 - x_{1o}) + f^o_{x_2x_2}(x_2 - x_{2o})^2 + f^o_{x_2x_3}(x_2 - x_{2o})(x_3 - x_{3o}) \\
&\quad + f^o_{x_3x_1}(x_3 - x_{3o})(x_1 - x_{1o}) + f^o_{x_3x_2}(x_3 - x_{3o})(x_2 - x_{2o}) + f^o_{x_3x_3}(x_3 - x_{3o})^2, \\
&= \underbrace{\text{Left to readers;}}_{\text{squaring}} \\
&\overset{\text{or}}{=} \overbrace{\begin{bmatrix} x_1 - x_{1o} & x_2 - x_{2o} & x_3 - x_{3o} \end{bmatrix} \underbrace{\begin{bmatrix} f^o_{x_1x_1} & f^o_{x_1x_2} & f^o_{x_1x_3} \\ f^o_{x_2x_1} & f^o_{x_2x_2} & f^o_{x_2x_3} \\ f^o_{x_3x_1} & f^o_{x_3x_2} & f^o_{x_3x_3} \end{bmatrix}}_{\text{Hessian Matrix}} \begin{bmatrix} x_1 - x_{1o} \\ x_2 - x_{2o} \\ x_3 - x_{3o} \end{bmatrix}}^{\text{quadratic form}}.
\end{aligned}
$$

For the Hessian determinant of 2×2 dimension, the concept of leading principal minor is not introduced. However, for the Hessian determinant of 3×3 and higher dimensions, the first one of each order principal minors is the leading principal minor, which is the object of our attention.

For the 3×3-dimensional Hessian determinant , if all leading principal minors are positive, the function has a minimum, that is,

$$
\text{minimum} \left\{ \begin{array}{lr} \text{first-order leading principal minor:} & \begin{vmatrix} f^o_{x_1x_1} \end{vmatrix} > 0, \\ \qquad\qquad \text{and} & \\ \text{second-order leading principal minor:} & \begin{vmatrix} f^o_{x_1x_1} & f^o_{x_1x_2} \\ f^o_{x_2x_1} & f^o_{x_2x_2} \end{vmatrix} > 0, \\ \qquad\qquad \text{and} & \\ \text{third-order leading principal minor:} & \begin{vmatrix} f^o_{x_1x_1} & f^o_{x_1x_2} & f^o_{x_1x_3} \\ f^o_{x_2x_1} & f^o_{x_2x_2} & f^o_{x_2x_3} \\ f^o_{x_3x_1} & f^o_{x_3x_2} & f^o_{x_3x_3} \end{vmatrix} > 0. \end{array} \right\} \text{positive definite}
$$

For the 3×3-dimensional Hessian determinant , if the leading principal minors are negative, positive and negetive in order, the function has a maximum, that is,

$$
\text{maximum} \left\{ \begin{array}{lr} \text{first-order leading principal minor:} & \begin{vmatrix} f^o_{x_1x_1} \end{vmatrix} < 0, \\ \qquad\qquad \text{and} & \\ \text{second-order leading principal minor:} & \begin{vmatrix} f^o_{x_1x_1} & f^o_{x_1x_2} \\ f^o_{x_2x_1} & f^o_{x_2x_2} \end{vmatrix} > 0, \\ \qquad\qquad \text{and} & \\ \text{third-order leading principal minor:} & \begin{vmatrix} f^o_{x_1x_1} & f^o_{x_1x_2} & f^o_{x_1x_3} \\ f^o_{x_2x_1} & f^o_{x_2x_2} & f^o_{x_2x_3} \\ f^o_{x_3x_1} & f^o_{x_3x_2} & f^o_{x_3x_3} \end{vmatrix} < 0. \end{array} \right\} \text{negative definite}
$$

Example 6. **Demand for physical capital, human capital, and labor in a perfectly competitive market**

In the previous **E**xample 4, it is implied that all workers have the same production technology or the same working hours. However, due to education, training, and learning by doing, workers will form an effective unit labor stock, which is called human capital and is represented by H. Therefore, the unconstrained optimization problem of the three choice variables is:

$$\max_{\{K_d,H_d,L_d\}} \Pi \equiv PF(K_d,H_d,L_d) - RPK_d - W_H H_d - W_L L_d.$$

First-order necessary conditions:

$$\begin{cases} \dfrac{d\Pi}{dK_d} = PF_K(K_d,H_d,L_d) - RP = 0, \\ \dfrac{d\Pi}{dH_d} = PF_H(K_d,H_d,L_d) - W_H = 0, \\ \dfrac{d\Pi}{dL_d} = PF_L(K_d,H_d,L_d) - W_L = 0. \end{cases}$$

The nominal wage of effective labor is W_H, and the marginal output (marginal material product) of effective labor is $F_H(K_d,H_d,L_d)$

The second-order conditions can be given by the Hessian matrix:

$$\underbrace{\begin{bmatrix} \Pi_{KK} & \Pi_{KH} & \Pi_{KL} \\ \Pi_{HK} & \Pi_{HH} & \Pi_{HL} \\ \Pi_{LK} & \Pi_{LH} & \Pi_{LL} \end{bmatrix}}_{\text{Hessian matrix}} \Rightarrow \begin{cases} \text{first-order leading principal minor: } \left|\Pi_{KK}\right| = PF_{KK} < 0, \\ \text{second-order leading principal minor: } \begin{vmatrix} \Pi_{KK} & \Pi_{KH} \\ \Pi_{HK} & \Pi_{HH} \end{vmatrix} = P^2(F_{KK}F_{HH} - F_{KH}^2) > 0, \\ \text{third-order leading principal minor: } \begin{vmatrix} \Pi_{KK} & \Pi_{KH} & \Pi_{KL} \\ \Pi_{HK} & \Pi_{HH} & \Pi_{HL} \\ \Pi_{LK} & \Pi_{LH} & \Pi_{LL} \end{vmatrix} \gtreqless 0? \end{cases}$$

From the analysis above, we can know that the second-order leading principal minor should be positive. On this basis, the odd-order leading principal minors have the same sign, that is,

$$\text{third-order: } \begin{vmatrix} \Pi_{KK} & \Pi_{KH} & \Pi_{KL} \\ \Pi_{HK} & \Pi_{HH} & \Pi_{HL} \\ \Pi_{LK} & \Pi_{LH} & \Pi_{LL} \end{vmatrix} = P^3(F_{KK}F_{HH}F_{LL} + 2F_{KH}F_{HL}F_{LK} - F_{KK}F_{HL}^2 - F_{LL}F_{KH}^2 - F_{HH}F_{KL}^2) < 0.$$

Under these conditions, the objective function has a maximum.

In summary, it is not difficult to find that:

(1) The first-order leading principal minor is the first element in the upper left corner of the Hessian matrix, the second-order leading principal minbor is the determinant of the four elements in the upper left corner of the Hessian matrix, and the third-order leading principal subformula is the Hessian determinant itself. These are the leading principal minors of the 3×3 Hessian determinant, and the the leading principal minors of $m \times m$ Hessian determinant are the same.

(2) For $m \times m$ Hessian determinant, when there is a minimum, all leading principal minors of each order are positive, and when there is a maximum, all leading principal minors of each order are alternately positive and negative; more precisely, the leading principal minors of even order are always positive, and if the leading principal minors of odd order are positive, there is a maximum; if the leading principal minors of odd order are negative, there is a minimum.

1.1.2 Constrained Optimization

In the previous section, the choice variables will not restrict each other, and the decision on a certain choice variable will not affect other choice variables, so it is also called free optimization. Starting from this section, we will consider optimization problems with various forms of constraints, including equality constraints, non-negative constraints, and inequality constraints. It is not difficult to imagine that constraints will reduce the domain of definition, and the range of the objective function will naturally become smaller, so the constrained extreme value will always be less than (when maximizing, or exactly equal to) or more than(when minimizing, or exactly equal to) the free extreme value. [3]

1.1.2.1 Equality Constraints Between Choice Variables

I. **Two-Variable Question with Single Equality Constraint**

Assume that the objective function and constraint conditions of the two-variable problem with single equality constraint are:

$$
\begin{aligned}
&\xleftrightarrow[\text{min}]{\text{max}} O = f(x, y), \\
&\text{s.t.} \quad g(x, y) = z,
\end{aligned}
$$

Among them, x and y are choice variables (independent variables), O is the extreme value (dependent variable), f and g are function symbols, and z is a constant (fixed number) or parameter (variable number). The constraints can also be written as $g(x, y, z) = 0$ or simplified to $g(x, y) = 0$ by letting $z = 0$.

i) **Elimination method**

One solution to the equality constrained optimization problem is to solve the function of one or more more variables through the constraints and substitute it into the objective function. In this way, the equality constrained optimization is transformed into a free optimization problem. After the dimension of the selected variables in the objective function is reduced, the necessary and sufficient conditions of unconstrained optimization can be directly applied.

ii) **Differentiation method**

From a geometric point of view, the optimal value is taken from the tangent point of the constraint line and the equal-value target line (indifference curve), that is, at this point, the slope of the constraint line is the same as the slope of the indifference curve. Its theoretical basis is:

For the unconstrained optimization problem of two variables $O = f(x, y)$, after total differentiation, we have $\mathrm{d}O = f_x \mathrm{d}x + f_y \mathrm{d}y$, and at the extreme point, we must have $f_x = 0 = f_y$, so $\mathrm{d}O = 0$. After adding the constraint $g(x, y) = z$, the objective function can still be fully differentiated, but $f_x = 0 = f_y$ does not need to be satisfied, as long as $\mathrm{d}O = 0$. $\mathrm{d}x$ and $\mathrm{d}y$ are no longer arbitrarily variable, and the range of variation is reduced to the straight line $g_x \mathrm{d}x + g_y \mathrm{d}y = 0$. To satisfy the above two conditions, we only need to have:

$$
\frac{f_x}{f_y} = \frac{g_x}{g_y} \quad \overset{\text{or}}{\Longleftrightarrow} \quad \frac{f_x}{g_x} = \frac{f_y}{g_y}.
$$

iii) **Multiplier method**

The derivation of the Lagrange multiplier method can also be transform a constrained optimization to an unconstrained optimization. Starting from the constraints, we have:

$$
\begin{aligned}
& g(x, y) = z, \\
\xRightarrow[g_y \neq 0]{g_y \text{ exists}} \quad & \mathrm{d}y = -\frac{g_x}{g_y}\mathrm{d}x, \\
\Rightarrow \quad & y = y(x), \\
\Rightarrow \quad & O = f[x, y(x)].
\end{aligned}
$$

This incorporates the constraints into the objective function. Then the second-order Taylor expansion is performed

[3] Taking bivariate equality-constrained optimization as an example, if there are two linear constraint equations intersecting at one point, these two constraints actually eliminate other possibilities for choice variables, rendering the constrained optimization problem meaningless. Therefore, the number of constraint equations should be less than the number of choice variables, so as to have a practical restrictive effect on the choice variables.

around point $x = x_o$:

$$f[x,y(x)]-f[x_o,y(x_o)] = \overbrace{\underbrace{\left\{ \underbrace{f_x^o[x,y(x)]}_{\substack{\text{first-order}\\ \text{partial deravative}}} + \underbrace{\overbrace{f_y^o[x,y(x)]}^{a}\,\overbrace{y_x^o(x)}^{b}}_{\substack{\text{composite first-order}\\ \text{partial deravative}}} \right\}(x-x_o)}_{\text{linear approximitation}} + \frac{(\overbrace{f_{xx}^o}^{\substack{\text{partial}\\ \text{deravative}}} + \overbrace{f_{xy}^o y_x^o}^{\substack{\text{composite}\\ \text{deravative}}}) + [\overbrace{(f_{yx}^o+f_{yy}^o y_x^o)}^{a'}\,\overbrace{y_x^o}^{b} + \overbrace{f_y^o}^{a}\,\overbrace{y_{xx}^o}^{b'}]}{2}(x-x_o)^2}^{\text{nonlinear approximitation}}.$$

The first-order necessary conditions satisfying the equality constraints are:

$$\underbrace{\underbrace{\overbrace{0 = f_x^o + f_y^o y_x^o}^{\text{F.O.C. after diemsion reduction}}}_{\frac{df[x,y(x)]}{dx}\big|_{x=x_0}} \xRightarrow[z=g(x_o,y_o)\Rightarrow y_x^o=-\frac{g_x^o}{g_y^o}]{y_x^o\equiv\frac{dy}{dx}|_{x=x_o}} \begin{Bmatrix} f_x^o - f_y^o\frac{g_x^o}{g_y^o}=0 \\ g(x_o,y_o)=z\end{Bmatrix} \xRightarrow{\frac{f_{y^o}}{g_{y^o}}\equiv-\lambda} \begin{Bmatrix} f_x^o+\lambda g_x^o=0\\ f_y^o+\lambda g_y^o=0 \\ g(x_o,y_o)=z\end{Bmatrix} \overset{x=x_o}{\underset{y=y_o}{\Longleftarrow}} \overbrace{\begin{Bmatrix}0=\frac{\partial\mathcal{L}}{\partial x}\\ 0=\frac{\partial\mathcal{L}}{\partial y}\\ 0=\frac{\partial\mathcal{L}}{\partial \lambda}\end{Bmatrix}}^{\text{F.O.C. by constructing Lagrangian function}} \Leftarrow \mathcal{L} \equiv f(x,y)+\lambda[g(x,y)-z]}_{\text{left and right ends are completely equivalent}}.$$

This constructs the Lagrangian function , with λ being the Lagrange multiplier.

Substituting the first-order necessary conditions into the above nonlinear expansion, we have

$$\begin{aligned} & f[x,y(x)] - f[x_o,y(x_o)] \\ &= 0 + (f_{xx}^o + f_{xy}^o y_x^o) + [(f_{yx}^o + f_{yy}^o y_x^o)y_x^o + f_y^o y_{xx}^o], \\ &= f_{xx}^o + 2f_{xy}^o y_x^o + f_{yy}^o(y_x^o)^2 + f_y^o y_{xx}^o \begin{cases} < 0 & \Leftrightarrow \quad \text{maximum;} \\ > 0 & \Leftrightarrow \quad \text{minimum.} \end{cases} \end{aligned}$$

This is equivalent to:

$$\begin{aligned} & f_{xx}^o + 2f_{xy}^o y_x^o + f_{yy}^o(y_x^o)^2 + f_y^o y_{xx}^o, \\ \xRightarrow[y_{xx}^o=?]{y_x^o=-\frac{g_x^o}{g_y^o}} &= f_{xx}^o - 2f_{xy}^o\frac{g_x^o}{g_y^o} + f_{yy}^o\left(\frac{g_x^o}{g_y^o}\right)^2 + f_y^o y_{xx}^o, \\ &= f_{xx}^o - 2f_{xy}^o\frac{g_x^o}{g_y^o} + f_{yy}^o\left(\frac{g_x^o}{g_y^o}\right)^2 + f_y^o\left[-\frac{g_{xx}^o}{g_y^o} + 2\frac{g_{xy}^o g_x^o}{(g_y^o)^2} - \frac{g_{yy}^o(g_x^o)^2}{(g_y^o)^3}\right], \\ &= f_{xx}^o - 2f_{xy}^o\frac{g_x^o}{g_y^o} + f_{yy}^o\left(\frac{g_x^o}{g_y^o}\right)^2 - \lambda\left[-g_{xx}^o + 2\frac{g_{xy}^o g_x^o}{g_y^o} - \frac{g_{yy}^o(g_x^o)^2}{(g_y^o)^2}\right], \\ &= [f_{xx}^o(g_y^o)^2 - 2f_{xy}^o g_x^o g_y^o + f_{yy}^o(g_x^o)^2 + \lambda g_{xx}^o(g_y^o)^2 - 2\lambda g_{xy}^o g_x^o g_y^o + \lambda g_{yy}^o(g_x^o)^2]\cancel{\frac{1}{(g_y^o)^2}}, \\ &= (f_{xx}^o + \lambda g_{xx}^o)(g_y^o)^2 - 2\left(f_{xy}^o + \lambda g_{xy}^o\right)g_x^o g_y^o + \left(f_{yy}^o + \lambda g_{yy}^o\right)(g_x^o)^2, \\ &= \mathcal{L}_{xx}^o(g_y^o)^2 - 2\mathcal{L}_{xy}^o g_x^o g_y^o + \mathcal{L}_{yy}^o(g_x^o)^2, \\ &\cancel{= [\mathcal{L}_{xx}^o(g_y^o)^2] \times 1^2 - 2(\mathcal{L}_{xy}^o g_x^o g_y^o) \times 1 \times 1 + [\mathcal{L}_{yy}^o(g_x^o)^2] \times 1^2,} \\ &\cancel{= \underbrace{\quad ? \quad}_{\text{no need of squaring}};} \end{aligned}$$

$$\overset{\text{or}}{\equiv} -\overbrace{\begin{bmatrix} ? & ? & ? \end{bmatrix} \underbrace{\begin{bmatrix} 0 & g_x^o & g_y^o \\ g_x^o & \mathcal{L}_{xx}^o & \mathcal{L}_{xy}^o \\ g_y^o & \mathcal{L}_{yx}^o & \mathcal{L}_{yy}^o \end{bmatrix}}_{\text{bordered Hessian matrix}} \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}}^{\text{no need of quadratic form}},$$

$$= -\overbrace{\begin{vmatrix} 0 & g_x^o & g_y^o \\ g_x^o & \mathcal{L}_{xx}^o & \mathcal{L}_{xy}^o \\ g_y^o & \mathcal{L}_{yx}^o & \mathcal{L}_{yy}^o \end{vmatrix}}^{\text{bordered Hessian matrix}} \gtrless 0 \quad \Leftrightarrow \quad f[x, y(x)] \gtrless f[x_o, y(x_o)] \quad \Leftrightarrow \quad \overbrace{\begin{vmatrix} 0 & g_x^o & g_y^o \\ g_x^o & \mathcal{L}_{xx}^o & \mathcal{L}_{xy}^o \\ g_y^o & \mathcal{L}_{yx}^o & \mathcal{L}_{yy}^o \end{vmatrix}}^{\text{bordered Hessian matrix}} \lessgtr 0.$$

It can be seen that, excluding the dimension of the added "edge", the objective function has a minimum value when the 2×2 bordered Hessian determinant is less than 0, and a maximum value when the bordered Hessian determinant is greater than 0. [4]

Example 7. Demand decisions for different consumer goods in a perfectly competitive market environment

Suppose there are two commodities C_1 and C_2 available for consumption, and the corresponding prices P_1, P_2 and nominal income PQ are exogenous to the household sector. The utility function and budget constraint are:

$$\begin{aligned} &\max_{C_1, C_2} U \equiv U(C_1, C_2), \\ \text{s.t.} \quad & P_1 C_1 + P_2 C_2 = PQ \equiv M. \end{aligned}$$

The problem can be solved by both elimination and construction of Lagrangian function. For this problem, the differential method is simple and intuitive. From the geometric plane of $C_1 - C_2$, it is easy to know that the optimal solution is the intersection of the indifference curve (the combination of C_1 and C_2 that can produce the same utility level) and the budget constraint line.

The slope of the budget constraint for a given nominal income PQ in monetary terms is:

$$\begin{aligned} & P_1 \mathrm{d}C_1 + P_2 \mathrm{d}C_2 = \mathrm{d}(PQ), \\ \Rightarrow \quad & P_1 \mathrm{d}C_1 + P_2 \mathrm{d}C_2 = 0, \end{aligned}$$

[4] This is a second-order sufficient condition for bivariate equality-constrained optimization problems. The derivation process is relatively cumbersome, and there are two minor additions to be made:

①

$$\begin{cases} \frac{f_{yo}}{g_{yo}} \equiv -\lambda \quad \Leftrightarrow \quad \mathcal{L} = f(x,y) + \lambda[g(x,y) - z] \quad \Leftrightarrow \quad z = g(x,y); \\ \frac{f_{yo}}{g_{yo}} \equiv +\lambda \quad \Leftrightarrow \quad \mathcal{L} = f(x,y) + \lambda[z - g(x,y)] \quad \Leftrightarrow \quad g(x,y) = z. \end{cases}$$

②

$$\begin{aligned} O &= f[x, y(x)], \\ f_x^o &= f_x^o[x, y(x)] + f_y^o[x, y(x)] y_x^o; \\ y_x^o &= -\frac{g_x^o[x, y(x)]}{g_y^o[x, y(x)]}, \\ \Rightarrow \quad y_{xx}^o &= -\frac{1}{g_y^o}(g_{xx}^o + g_{xy}^o y_x^o) + (-g_x^o)[-(g_y^o)^{-2}(g_{yx}^o + g_{yy}^o y_x^o)], \\ &= -\frac{1}{g_y^o}\left(g_{xx}^o - g_{xy}^o \frac{g_x^o}{g_y^o}\right) + (-g_x^o)\left[-(g_y^o)^{-2}\left(g_{yx}^o - g_{yy}^o \frac{g_x^o}{g_y^o}\right)\right], \\ &= -\frac{1}{g_y^o}\left(g_{xx}^o - g_{xy}^o \frac{g_x^o}{g_y^o}\right) + g_x^o (g_y^o)^{-2}\left(g_{yx}^o - g_{yy}^o \frac{g_x^o}{g_y^o}\right), \\ &= -\frac{g_{xx}^o}{g_y^o} + \frac{g_{xy}^o g_x^o}{(g_y^o)^2} + \frac{g_x^o g_{yx}^o}{(g_y^o)^2} - \frac{g_{yy}^o (g_x^o)^2}{(g_y^o)^3}, \\ &= -\frac{g_{xx}^o}{g_y^o} + 2\frac{g_{xy}^o g_x^o}{(g_y^o)^2} - \frac{g_{yy}^o (g_x^o)^2}{(g_y^o)^3}. \end{aligned}$$

$$\Rightarrow \qquad \frac{\mathrm{d}C_2}{\mathrm{d}C_1} = \underbrace{\overbrace{-\frac{P_1}{P_2}}}_{\substack{\text{slope of}\\ \text{budget line}}} .$$

The slope of the indifference curve for fixed immediate utility U is:

$$\begin{aligned} \mathrm{d}U &= U_1^o \mathrm{d}C_1 + U_2^o \mathrm{d}C_2, \\ \Rightarrow \qquad 0 &= U_1^o \mathrm{d}C_1 + U_2^o \mathrm{d}C_2, \\ \Rightarrow \qquad \frac{\mathrm{d}C_2}{\mathrm{d}C_1} &= \underbrace{\overbrace{-\frac{U_1^o}{U_2^o}}}_{\substack{\text{slope of}\\ \text{indifference curve}}} , \end{aligned}$$

where $U_i^o \equiv \frac{\partial U(\cdot)}{\partial C_i}$, $i = 1, 2$ is the first-order partial derivative of the utility function with respect to consumer good 1 and consumer good 2 at the optimal decision point.

Combine tangency condition and budget constraint:

$$\begin{cases} \overbrace{\overbrace{U_1^o / U_2^o}^{\text{marginal substitution rate}} = \overbrace{P_1/P_2}^{\text{realtive price}}}^{\text{Combining F.O.C. on } C_1 \text{ and } C_2} , \\ \underbrace{P_1 C_1 + P_2 C_2 = PQ}_{\text{F.O.C. on the Lagrange multiplier}} . \end{cases}$$

Since $U_1^o = U_1^o(C_1, C_2)$ and $U_2^o = U_2^o(C_1, C_2)$, if the specific form of the utility function is given, the desired consumption levels C_1° and C_2°can be solved. The following chapters will discuss how to conduct comparative static analysis of models with general functional forms, that is, the qualitative impact of changes in exogenous variables P_1, P_2 and PQ on the desired consumption choice bundle.

The second-order sufficient condition can still be written by constructing the Lagrangian function $\mathcal{L} \equiv U(C_1, C_2) + \lambda(PQ - P_1 C_1 - P_2 C_2)$ and rewriting the constraint condition as $g(C_1, C_2) \equiv P_1 C_1 + P_2 C_2 - PQ = 0$

$$\begin{aligned} \overbrace{\begin{vmatrix} 0 & g_1^o & g_2^o \\ g_1^o & \mathcal{L}_{11}^o & \mathcal{L}_{12}^o \\ g_2^o & \mathcal{L}_{21}^o & \mathcal{L}_{22}^o \end{vmatrix}}^{\text{bordered Hessian matrix}} &= \begin{vmatrix} 0 & P_1 & P_2 \\ P_1 & U_{11}^o & U_{12}^o \\ P_2 & U_{21}^o & U_{22}^o \end{vmatrix}, \\ &= 2P_1 P_2 U_{12}^o - P_1^2 U_{22}^o - P_2^2 U_{11}^o, \\ &\equiv |\bar{H}| \gtreqless 0? \end{aligned}$$

where $g_i^o \equiv \frac{\partial g(\cdot)}{\partial C_i}$, $i = 1, 2$ represent the partial derivatives of the budget constraint curve for consumer goods 1 and 2 at the optimal decision point; $\mathcal{L}_{ij}^o \equiv \frac{\partial \mathcal{L}(\cdot)}{\partial C_{ij}}$, $i, j = \{1, 2\}$ represents the second-order partial derivatives and cross-partial derivatives of the budget constraint curve function for consumer goods 1 and 2 at the optimal decision point; $U_{ij}^o \equiv \frac{\partial U(\cdot)}{\partial C_{ij}}$, $i, j = \{1, 2\}$ represents the second-order partial derivatives and cross-partial derivatives of the utility function for consumer goods 1 and 2 at the optimal decision point.

The above second-order condition is closely related to the curvature of the indifference curve (i.e., the slope of the slope), that is,

$$\begin{aligned} \frac{\mathrm{d}C_2}{\mathrm{d}C_1} &= -\frac{U_1^o(C_1, C_2)}{U_2^o(C_1, C_2)} = \frac{P_1}{P_2}, \\ \Rightarrow \qquad \frac{\mathrm{d}^2 C_2}{\mathrm{d}C_1^2} &= -\frac{\frac{\partial U_1^o(C_1, C_2(C_1))}{\partial C_1} U_2^o - U_1^o \frac{\partial U_2^o(C_1, C_2(C_1))}{\partial C_1}}{(U_2^o)^2}, \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\left(U_{11}^o + U_{12}\frac{\mathrm{d}C_2}{\mathrm{d}C_1}\right)U_2^o - U_1^o\left(U_{21}^o + U_{22}\frac{\mathrm{d}C_2}{\mathrm{d}C_1}\right)}{(U_2^o)^2},\\
&= -\frac{\left(U_{11}^o - U_{12}\frac{P_1}{P_2}\right)U_2^o - U_1^o\left(U_{21}^o - U_{22}\frac{P_1}{P_2}\right)}{(U_2^o)^2},\\
&= -\frac{\left(U_{11}^o - U_{12}\frac{P_1}{P_2}\right)U_2^o - \left(U_2^o\frac{P_1}{P_2}\right)\left(U_{21}^o - U_{22}\frac{P_1}{P_2}\right)}{(U_2^o)^2},\\
&= \frac{\frac{P_1}{P_2}\left(U_{21}^o - U_{22}\frac{P_1}{P_2}\right) - \left(U_{11}^o - U_{12}\frac{P_1}{P_2}\right)}{U_2^o},\\
&= \frac{\left[\frac{P_1}{P_2}\left(U_{21}^o - U_{22}\frac{P_1}{P_2}\right) - \left(U_{11}^o - U_{12}\frac{P_1}{P_2}\right)\right]P_2^2}{U_2^o P_2^2},\\
&= \frac{P_1P_2U_{21}^o - P_1^2U_{22} - P_2^2U_{11}^o + P_1P_2U_{12}^o}{U_2^o P_2^2},\\
&= \frac{2P_1P_2U_{12}^o - P_1^2U_{22} - P_2^2U_{11}^o}{U_2^o P_2^2},\\
&= \frac{|\bar{H}| \gtreqless 0?}{U_2^o P_2^2 > 0} \gtreqless 0?
\end{aligned}
$$

When $\frac{\mathrm{d}^2C_2}{\mathrm{d}C_1^2} > 0$, the indifference curve is strictly convex, and the second-order condition is also greater than 0, so the objective function has a maximum value. Therefore, objective function under equality constraints having a maximum value requires the indifference curve to be strictly convex, but this is a necessary condition, because when the second-order condition is equal to 0 (the indifference curve and the budget constraint line intersect on a line segment), there are multiple intersection points, but they are also extreme values.

It is important to note that if the prices of good 1 and good 2 both rise or fall by the same amount (and, accordingly, income rises or falls by the same amount), this will have no effect on the desired consumption outcome, because the budget constraint has not changed in any way:

$$
\begin{aligned}
&(\lambda P_1)C_1 + (\lambda P_2)C_2 = \lambda(PQ),\\
\Rightarrow\quad &\cancel{\lambda}(P_1C_1 + P_2C_2) = \cancel{\lambda}(PQ),\\
\Rightarrow\quad &P_1C_1 + P_2C_2 = PQ.
\end{aligned}
$$

This means:

$$
\begin{cases}
\lambda^0 C_1^* = C_1^*(P_1, P_2, M) = (\lambda P_1, \lambda P_2, \lambda PQ);\\
\lambda^0 C_2^* = C_2^*(P_1, P_2, M) = (\lambda P_1, \lambda P_2, \lambda PQ).
\end{cases}
$$

The above optimal consumer demand equation has the characteristic of zero-order homogeneity. We can equate monetary income $PQ$with money supply (the principle can be referred to the quantity theory of money: $MV = PQ$), which shows that the consumer demand equation points to the economic meaning of "monetary neutrality", that is, an increase in money supply only brings about an equal increase in prices, but does not have a real qualitative impact on actual economic activities.

If we slightly modify **Example** 1 in this chapter, we can obtain a similar double-variable equality-constrained optimization problem as this example:

$$
\begin{aligned}
\max_{Q_{s1}, Q_{s2}}\ &\Pi \equiv \mathcal{R}(Q_{s1}, Q_{s2}) - C(Q_{s1}, Q_{s2}),\\
\text{s.t.}\quad &Q_{s1} + Q_{s2} = \bar{Q}.
\end{aligned}
$$

Because of the constraint of production quota $\bar{Q}$, the selection variables Q_{s1} and Q_{s2} are no longer independent, and the free optimum turns to the constrained optimum.

Example 8. **Consumer demand and labor supply in a perfectly competitive market**

The utility function of the "representative" household sector is:

$$
\max_{C,L_s} \longleftrightarrow
\begin{cases}
\text{Type 1:} \quad U \equiv U(C, 1-L_s), & \begin{cases} U^o_C > 0,\ U^o_{1-L_s} > 0, \\ U^o_{CC} < 0,\ U^o_{1-L_s,1-L_s} < 0, \\ U^o_{C,1-L_s} > 0,\ U^o_{1-L_s,C} > 0. \end{cases} \\
\text{Type 2:} \quad U \equiv U(C, L_s), & \begin{cases} U^o_C > 0,\ U^o_{L_s} < 0, \\ U^o_{CC} < 0,\ U^o_{L_sL_s} < 0, \\ U^o_{CL_s} < 0,\ U^o_{L_sC} < 0. \end{cases}
\end{cases}
$$

$$
\text{s.t.} \qquad \underbrace{P_e C}_{\text{expected expenditure}} = \underbrace{W L_s}_{\text{wage income}},
$$

where C represents consumption demand. Note that it is different from the symbol C used to represent cost. In addition, it does not have a subscript d like other supply and demand variables, because there is no need to use symbols like C_s to distinguish consumption supply from output; L_s is for labor supply, or to allow leisure $1 - L_s$ to enter the utility function U, P_e is the instantaneous expectation or point expectation about prices, W is the nominal wage, and subscripts C and L represent the first-order, second-order, or cross-partial derivatives of the utility function with respect to the corresponding elements (the positive or negative assumptions are the "approximate" sufficient conditions for the objective function to achieve the maximum value). The objective functions of type 1 and type 2 are slightly different. The former enters the utility function with labor and the latter with leisure. After standardizing the labor time, $1 - L_s$ is leisure, which, like consumption, will bring positive utility.

To solve the optimization problem with two choice variables and an equality constraint, the elimination method or the Lagrange multiplier method can be used.

i) Since the objective function and constraints are very simple, the elimination method is more convenient. Taking the objective function of type 1 as an example, substituting the consumer demand variable $C = \frac{W}{P_e} L_s$ solved by the equality constraint, it is converted into an unconstrained single variable optimization problem:

$$
\max_{L_s} U \equiv U\left(\overbrace{\underbrace{\frac{W}{P_e} L_s}}^{C}, 1 - L_s \right).
$$

The first-order necessary condition is:

$$
\frac{\mathrm{d}U}{\mathrm{d}L_s} = U^o_C \frac{W}{P_e} - U^o_{1-L_s} = 0 \quad \Rightarrow \quad \underbrace{\overbrace{\frac{U^o_{1-L_s}(C, 1-L_s)}{U^o_C(C, 1-L_s)}}}_{\text{marginal rate of substitution}} = \underbrace{\overbrace{\frac{W}{P_e}}}_{\text{expected real wage}}.
$$

The second-order sufficient condition is:

$$
\begin{aligned}
\frac{\mathrm{d}(\mathrm{d}U/\mathrm{d}L_s)}{\mathrm{d}L_s} &= \frac{\mathrm{d}}{\mathrm{d}L_s}\left[\frac{W}{P_e} U_C \left(\overbrace{\underbrace{\frac{W}{P_e} L_s}}^{C}, 1 - L_s \right) - U_{1-L_s} \left(\overbrace{\underbrace{\frac{W}{P_e} L_s}}^{C}, 1 - L_s \right) \right], \\
&= \frac{W}{P_e} \left(\frac{W}{P_e} U^o_{CC} - U^o_{C,1-L_s} \right) - \left(\frac{W}{P_e} U^o_{1-L_s,C} - U^o_{1-L_s,1-L_s} \right), \\
&= \left(\frac{W}{P_e} \right)^2 U^o_{CC} + U^o_{1-L_s,1-L_s} - 2 \frac{W}{P_e} U^o_{C,1-L_s} > 0.
\end{aligned}
$$

Please pay attention and think for a moment: when converted into the optimization problem of an unconstrained single choice variable, according to the conclusion of the previous analysis, there is a minimum value when the second-order derivative is greater than 0. Why is the second-order derivative greater than 0 here still determined to be a maximum value?

ii) The Lagrange multiplier method will produce the same result. Introduce the multiplier λ to construct a new function:

$$\max_{C,L_s,\lambda} \mathcal{L} \equiv U(C, 1-L_s) + \lambda\left(\frac{W}{P_e}L_s - C\right).$$

The first-order necessary conditions are:

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial C} &= U_C^o - \lambda = 0, \\ \frac{\partial \mathcal{L}}{\partial L_s} &= -U_{1-L_s}^o + \lambda\frac{W}{P_e} = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \frac{W}{P_e}L_s - C = 0. \end{aligned}\right\} \Rightarrow \begin{cases} \dfrac{U_{1-L_s}^o(C, 1-L_s)}{U_C^o(C, 1-L_s)} = \dfrac{W}{P_e}, \\ \qquad\qquad\quad C = \dfrac{W}{P_e}L_s. \end{cases}$$

According to the constructed Lagrangian and the constraint $g(C, L_s) \equiv (W/P_e)L_s - C = 0$, the second-order sufficient condition is:

$$\begin{aligned} \overbrace{\begin{vmatrix} 0 & g_C^o & g_L^o \\ g_C^o & \mathcal{L}_{CC}^o & \mathcal{L}_{CL}^o \\ g_L^o & \mathcal{L}_{LC}^o & \mathcal{L}_{LL}^o \end{vmatrix}}^{\text{bordered Hessian determinant}} &= \begin{vmatrix} 0 & -1 & \frac{W}{P_e} \\ -1 & U_{CC}^o & -U_{C,1-L_s}^o \\ \frac{W}{P_e} & -U_{1-L_s,C}^o & U_{1-L_s,1-L_s}^o \end{vmatrix}, \\ &= 0 + \frac{W}{P_e}U_{C,1-L_s}^o + \frac{W}{P_e}U_{1-L_s,C}^o - 0 - U_{1-L_s,1-L_s}^o - \left(\frac{W}{P_e}\right)^2 U_{CC}^o, \\ &= 2\frac{W}{P_e}U_{C,1-L_s}^o - \left(\frac{W}{P_e}\right)^2 U_{CC}^o - U_{1-L_s,1-L_s}^o > 0. \end{aligned}$$

Notice:

(1) The utility function is concave so the extreme value is also the global maximum.

(2) The indifference curve is strictly convex so there is a maximum.

(3) The second-order conditions for single-variable optimization without constraints are opposite in sign to the second-order conditions for double-variable optimization under equality constraints.

To make it easier to see that the second-order conditions **E**xample 7 and **E**xample 8 are exactly the same in form, the subscripts in the former are reduced to the relevant variables, and both sides of the second-order conditions in the latter are multiplied by P_e^2, as shown below:

$$\begin{cases} \ \ 2P_1P_2U_{C1,C2}^o - P_1^2U_{C2,C2}^o - P_2^2U_{C1,C1}^o > 0; \\ 2WP_eU_{C,1-L}^o - W^2U_{C,C}^o - P_e^2U_{1-L_s,1-L_s}^o > 0. \end{cases}$$

Regardless of whether we choose two different consumer goods or choose consumption leisure, we follow the idea of [8, pp.376-377] to discuss this in detail. In a nutshell, it ensures that the indifference curve is strictly convex to the origin, so that it has a unique intersection with the budget constraint line, so that the objective function obtains a unique maximum value under the constraint conditions.

(4) Combining the first-order conditions and equality constraints allows solving for the optimal values of the choice variables (consumption demand and labor supply), explicit solution depends on the specific form of the utility function.

II. **Multi-Variable Questions with Single Equality Constraint**

Assume that the objective function and constraints for more than two variables but only a single equality constraint are:

$$\underset{\min}{\overset{\max}{\longleftrightarrow}} O = f(x_1, x_2, \ldots, x_m),$$

$$\text{s.t.} \quad g(x_1, x_2, \ldots, x_m) = 0.$$

When there are m choice variables and one equality constraint, the derivation process of the necessary and sufficient conditions for optimization is similar to that of two choice variables and one equality constraint. Now we can directly construct the Lagrangian function, that is,

$$\mathcal{L} \equiv f(x_1, x_2, \ldots, x_m) + \lambda[0 - g(x_1, x_2, \ldots, x_m)].$$

Not counting the dimension of the added "edge", when the leading principal minors of the $m \times m$ bordered Hessian determinant are all positive, there is a minimum, that is,

$$\text{minimum} \left\{ \begin{array}{lr}
\text{2nd-order leading principal minor:} & \begin{vmatrix} 0 & g^o_{x_1} & g^o_{x_2} \\ g^o_{x_1} & \mathcal{L}^o_{x_1x_1} & \mathcal{L}^o_{x_1x_2} \\ g^o_{x_2} & \mathcal{L}^o_{x_2x_1} & \mathcal{L}^o_{x_2x_2} \end{vmatrix} < 0, \\
\qquad\qquad \text{and} & \\
\text{3rd-order leading principal minor:} & \begin{vmatrix} 0 & g^o_{x_1} & g^o_{x_2} & g^o_{x_3} \\ g^o_{x_1} & \mathcal{L}^o_{x_1x_1} & \mathcal{L}^o_{x_1x_2} & \mathcal{L}^o_{x_1x_3} \\ g^o_{x_2} & \mathcal{L}^o_{x_2x_1} & \mathcal{L}^o_{x_2x_2} & \mathcal{L}^o_{x_2x_3} \\ g^o_{x_3} & \mathcal{L}^o_{x_3x_1} & \mathcal{L}^o_{x_3x_2} & \mathcal{L}^o_{x_3x_3} \end{vmatrix} < 0, \\
\qquad\qquad \text{and} & \\
\text{4th-order leading principal minor:} & \begin{vmatrix} 0 & g^o_{x_1} & g^o_{x_2} & g^o_{x_3} & g^o_{x_4} \\ g^o_{x_1} & \mathcal{L}^o_{x_1x_1} & \mathcal{L}^o_{x_1x_2} & \mathcal{L}^o_{x_1x_3} & \mathcal{L}^o_{x_1x_4} \\ g^o_{x_2} & \mathcal{L}^o_{x_2x_1} & \mathcal{L}^o_{x_2x_2} & \mathcal{L}^o_{x_2x_3} & \mathcal{L}^o_{x_2x_4} \\ g^o_{x_3} & \mathcal{L}^o_{x_3x_1} & \mathcal{L}^o_{x_3x_2} & \mathcal{L}^o_{x_3x_3} & \mathcal{L}^o_{x_3x_4} \\ g^o_{x_4} & \mathcal{L}^o_{x_4x_1} & \mathcal{L}^o_{x_4x_2} & \mathcal{L}^o_{x_4x_3} & \mathcal{L}^o_{x_4x_4} \end{vmatrix} < 0, \\
\qquad\qquad \text{and} & \\
\qquad\qquad \vdots &
\end{array} \right\} \text{positive definite}$$

Not counting the dimension of the added "edge", when the leading principal minors of the $m \times m$ bordered Hessian determinant are negative, positive, negative, ... in order, there is a maximum, that is,

$$\text{maximum} \left\{ \begin{array}{lr}
\text{2nd-order leading principal minor:} & \begin{vmatrix} 0 & g^o_{x_1} & g^o_{x_2} \\ g^o_{x_1} & \mathcal{L}^o_{x_1x_1} & \mathcal{L}^o_{x_1x_2} \\ g^o_{x_2} & \mathcal{L}^o_{x_2x_1} & \mathcal{L}^o_{x_2x_2} \end{vmatrix} > 0, \\
\qquad\qquad \text{and} & \\
\text{3rd-order leading principal minor:} & \begin{vmatrix} 0 & g^o_{x_1} & g^o_{x_2} & g^o_{x_3} \\ g^o_{x_1} & \mathcal{L}^o_{x_1x_1} & \mathcal{L}^o_{x_1x_2} & \mathcal{L}^o_{x_1x_3} \\ g^o_{x_2} & \mathcal{L}^o_{x_2x_1} & \mathcal{L}^o_{x_2x_2} & \mathcal{L}^o_{x_2x_3} \\ g^o_{x_3} & \mathcal{L}^o_{x_3x_1} & \mathcal{L}^o_{x_3x_2} & \mathcal{L}^o_{x_3x_3} \end{vmatrix} < 0, \\
\qquad\qquad \text{and} & \\
\text{4th-order leading principal minor:} & \begin{vmatrix} 0 & g^o_{x_1} & g^o_{x_2} & g^o_{x_3} & g^o_{x_4} \\ g^o_{x_1} & \mathcal{L}^o_{x_1x_1} & \mathcal{L}^o_{x_1x_2} & \mathcal{L}^o_{x_1x_3} & \mathcal{L}^o_{x_1x_4} \\ g^o_{x_2} & \mathcal{L}^o_{x_2x_1} & \mathcal{L}^o_{x_2x_2} & \mathcal{L}^o_{x_2x_3} & \mathcal{L}^o_{x_2x_4} \\ g^o_{x_3} & \mathcal{L}^o_{x_3x_1} & \mathcal{L}^o_{x_3x_2} & \mathcal{L}^o_{x_3x_3} & \mathcal{L}^o_{x_3x_4} \\ g^o_{x_4} & \mathcal{L}^o_{x_4x_1} & \mathcal{L}^o_{x_4x_2} & \mathcal{L}^o_{x_4x_3} & \mathcal{L}^o_{x_4x_4} \end{vmatrix} > 0, \\
\qquad\qquad \text{and} & \\
\qquad\qquad \vdots &
\end{array} \right\} \text{negative definite}$$

After observation, it is not difficult to find that:

(1) The bordered Hessian matrix of a multi-variable promblem with single equality constraint is symmetric along the main diagonal;

(2) Because the equality constraint is meaningful only for at least two choice variables, let's start with the second-order leading principal minor of the bordered Hessian determinant. If the leading principal minors of each order are negative, the matrix is positive definite and the objective function has a minimum value. (which is the opposite of the situation when there is no constraint, because the bordered Hessian determinant has a negative sign in addition to the added edge);

(3) If the leading principal minors of each order of the bordered Hessian determinant is alternately positive and negative. More precisely, the leading principal minors of the even-order are positive and those of the odd-order are negative, the matrix is negative definite and the objective function has a maximum value

Example 9. **A basket of consumer choices and the overall price level under monopolistic competition in product markets**

Note that it is the enterprise sector that has monopoly power in the product market, while the household sector can only choose consumption demand according to the budget constraint to optimize the goal of a basket of consumer goods. The objective function for the total consumption can be either continuous aggregation or discrete aggregation.

i) Continuous aggregation

Different consumer goods under monopolistic competition are not completely substitutable, and there is a certain elasticity of substitution (this term will be derived and further introduced in the examples of comparative static analysis), which is simply set to ϵ here. Consumers need to choose a basket of consumer goods. Since this basket of consumer goods is symmetrical, it is actually an optimization problem of a single variable under given budget constraint. For a better comparison, the dual property of the optimization problem is used here, that is, given a budget constraint to make a basket of goods as large as possible and given a basket of goods to make expenditures as small as possible will equally lead to the first-order conditions of the optimization. This set of dual optimization problems are:

$$
\begin{aligned}
&\max_{C_i} \quad \overbrace{C \equiv \left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}}^{\text{Dixit-Stiglitz aggregated CES function}}, &\overset{\text{dual}}{\Longleftrightarrow}\quad &\min_{C_i} \quad \overbrace{PQ = PC}^{Q=C+\cancel{I+G+\delta K}} = \textstyle\int_0^1 P_i C_i \mathrm{d}i,\\
&\text{s.t.} \quad \textstyle\int_0^1 P_i C_i \mathrm{d}i \le PQ = PC. & &\text{s.t.} \quad \left(\textstyle\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}} \ge C.
\end{aligned}
$$

CES refers to constant elasticity of substitution, that is, the elasticity of substitution between different consumer goods ϵ is a constant. A more detailed and comprehensive introduction to it will be left for Chapter 3. The equilibrium condition of the product market (that is, the national income identity) is $Q = C$, and the total income Q is the given resource endowment. Investment, government expenditure and depreciation are omitted, but will be restored to a certain extent when the classical model and Keynes model are introduced later. It seems that the inequality constraint problem is encountered here in advance. However, there is an Inada hypothesis in economics, which ensures $C_i > 0$, excluding corner point solutions, so the budget constraint takes an equal sign (the constraint is tight). The detailed discussion on the conversion of inequality constraints into equality constraints will be left later, and the reader only needs to have an impression here. Therefore, the Lagrange multiplier method is directly used to solve:

$$
\mathcal{L}^{\min} = \int_0^1 P_i C_i \mathrm{d}i + P\left[C - \left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}\right].
$$

$$
\begin{aligned}
\xRightarrow{\text{F.O.C.}} \quad & 0 = \frac{\mathrm{d}\mathcal{L}^{\min}}{\mathrm{d}C_i},\\
\Rightarrow \quad & P_i = P\frac{\epsilon}{\epsilon-1}\left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}-1}\frac{\epsilon-1}{\epsilon}C_i^{\frac{\epsilon-1}{\epsilon}-1},\\
\Rightarrow \quad & P_i = P\left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}-1} C_i^{\frac{\epsilon-1}{\epsilon}-1},
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad P_i &= P\left[\left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}\right]^{\frac{1}{\epsilon}} C_i^{\frac{-1}{\epsilon}},\\
\Rightarrow \quad \frac{P_i}{P} &= \left(\frac{C_i}{C}\right)^{\frac{-1}{\epsilon}},\\
\Rightarrow \quad C_i &= \left(\frac{P_i}{P}\right)^{-\epsilon} C.
\end{aligned}
$$

In this way, we can get the demand curve for consumer product i in a monopolistic competition market environment. Substituting it into the objective function, we can get the functional relationship between the total price level and the price of a single commodity i:

$$
\begin{aligned}
\int_0^1 P_i C_i \mathrm{d}i &= PC,\\
\Rightarrow \quad \int_0^1 P_i\left[\left(\frac{P_i}{P}\right)^{-\epsilon} C\right]\mathrm{d}i &= PC,\\
\Rightarrow \quad \int_0^1 P_i\left(\frac{P_i}{P}\right)^{-\epsilon}\mathrm{d}i &= P,\\
\Rightarrow \quad P &= \left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{1}{1-\epsilon}}.
\end{aligned}
$$

The above solution is the expenditure minimization problem. Based on the specific meaning of the Lagrange multiplier (the price of the consumption basket C), we can quickly get the demand curve for consumer good i and the price index. Assuming the budget is M, we can also get the the demand function for a certain heterogeneous product i in the basket from maximizing the basket.

$$
\mathcal{L}^{\max} = \left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}} + \lambda\left(M - \int_0^1 P_i C_i \mathrm{d}i\right).
$$

$$
\begin{aligned}
&\xRightarrow{\text{F.O.C.}} \quad \frac{\mathrm{d}\mathcal{L}^{\max}}{\mathrm{d}C_i} = 0,\\
\Rightarrow \quad & \frac{\epsilon}{\epsilon-1}\left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}-1}\frac{\epsilon-1}{\epsilon}C_i^{\frac{\epsilon-1}{\epsilon}-1} - \lambda P_i = 0,\\
\Rightarrow \quad & \left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}i\right)^{\frac{1}{\epsilon-1}} C_i^{-\frac{1}{\epsilon}} - \lambda P_i = 0,\\
\Rightarrow \quad & \left[\left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}\right]^{\frac{1}{\epsilon}} C_i^{-\frac{1}{\epsilon}} - \lambda P_i = 0,\\
\Rightarrow \quad & C^{\frac{1}{\epsilon}} C_i^{-\frac{1}{\epsilon}} - \lambda P_i = 0,\\
\Rightarrow \quad & C_i^{-\frac{1}{\epsilon}} = \lambda P_i C^{-\frac{1}{\epsilon}} \quad \xLeftrightarrow{\text{symmetry}} \quad C_j^{-\frac{1}{\epsilon}} = \lambda P_j C^{-\frac{1}{\epsilon}},\\
\Rightarrow \quad & \left(\frac{C_i}{C_j}\right)^{-\frac{1}{\epsilon}} = \frac{P_i}{P_j},\\
\Rightarrow \quad & C_i = \left(\frac{P_i}{P_j}\right)^{-\epsilon} C_j.
\end{aligned}
$$

This is the association between any two goods i and j. Under the tight constraint, the budget M is the total expenditure on each consumer good:

$$
\begin{aligned}
M &= \int_0^1 P_i C_i \mathrm{d}i,\\
&= \int_0^1 P_i\left(\frac{P_i}{P_j}\right)^{-\epsilon} C_{jt}\mathrm{d}i,
\end{aligned}
$$

$$
\begin{aligned}
&= CjP_j^{\epsilon}\int_0^1 P_i^{1-\epsilon}\mathrm{d}i,\\
\Rightarrow \quad C_j &= \frac{MP_j^{-\epsilon}}{\int_0^1 P_i^{1-\epsilon}\mathrm{d}i} \quad \overset{\text{symmetry}}{\Longleftrightarrow} \quad C_i = \frac{MP_i^{-\epsilon}}{\int_0^1 P_i^{1-\epsilon}\mathrm{d}i}.\\
C &= \left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}} = \left(\int_0^1 C_j^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}j\right)^{\frac{\epsilon}{\epsilon-1}},\\
&= \left[\int_0^1 \left(\frac{MP_j^{-\epsilon}}{\int_0^1 P_i^{1-\epsilon}\mathrm{d}i}\right)^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}j\right]^{\frac{\epsilon}{\epsilon-1}} = M\left[\int_0^1 \left(\frac{P_j^{-\epsilon}}{\int_0^1 P_i^{1-\epsilon}\mathrm{d}i}\right)^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}j\right]^{\frac{\epsilon}{\epsilon-1}} = M\left[\int_0^1 \frac{P_j^{1-\epsilon}}{\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{\epsilon-1}{\epsilon}}}\mathrm{d}j\right]^{\frac{\epsilon}{\epsilon-1}},\\
&= M\left[\frac{\left(\int_0^1 P_j^{1-\epsilon}\mathrm{d}j\right)}{\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{\epsilon-1}{\epsilon}}}\right]^{\frac{\epsilon}{\epsilon-1}} = M\left[\frac{\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)}{\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{\epsilon-1}{\epsilon}}}\right]^{\frac{\epsilon}{\epsilon-1}} = M\left[\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{1-\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}},\\
&= M\left[\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} = M\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{1}{\epsilon-1}}.
\end{aligned}
$$

Define the total expenditure of a basket of consumption as the total price index $P \equiv M|_{C=1}$, then:

$$
\begin{aligned}
C &= M\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{1}{\epsilon-1}},\\
&\Updownarrow\\
1 &= P\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{1}{\epsilon-1}},\\
\Rightarrow \quad P &= \left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{1}{1-\epsilon}}.
\end{aligned}
$$

$$
\begin{aligned}
M &= \int_0^1 P_i C_{id}\mathrm{d}i = \int_0^1 P_i\left(\frac{P_i}{P_j}\right)^{-\epsilon} C_j\mathrm{d}i,\\
&= C_jP_j^{\epsilon}\int_0^1 P_i^{1-\epsilon}\mathrm{d}i = C_jP_j^{\epsilon}\left[\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{1}{1-\epsilon}}\right]^{1-\epsilon},\\
&= C_jP_j^{\epsilon}P^{1-\epsilon},\\
\Rightarrow \quad C_j &= \frac{M}{P_j^{\epsilon}P^{1-\epsilon}} \quad \overset{\text{symmetry}}{\Longleftrightarrow} \quad C_i = \frac{M}{P_i^{\epsilon}P^{1-\epsilon}}.\\
C &= \left(\int_0^1 C_{id}^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}} = \left[\int_0^1\left(\frac{M}{P_i^{\epsilon}P^{1-\epsilon}}\right)^{\frac{\epsilon-1}{\epsilon}}\mathrm{d}i\right]^{\frac{\epsilon}{\epsilon-1}} = \frac{M}{P^{1-\epsilon}}\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}},\\
&= \frac{M}{P^{1-\epsilon}}\left[\left(\int_0^1 P_i^{1-\epsilon}\mathrm{d}i\right)^{\frac{1}{1-\epsilon}}\right]^{-\epsilon} = \frac{M}{P^{1-\epsilon}}P^{-\epsilon} = \frac{M}{P},\\
\Rightarrow \quad M &= PC,\\
&\Updownarrow\\
\Rightarrow \quad \int_0^1 P_iC_i\mathrm{d}i &= PC;\\
C_i &= \frac{M}{P_i^{\epsilon}P^{1-\epsilon}} = \frac{PC}{P_i^{\epsilon}P^{1-\epsilon}} = \frac{C}{P_i^{\epsilon}P^{-\epsilon}},
\end{aligned}
$$

$$\Rightarrow \qquad C_i = \left(\frac{P_i}{P}\right)^{-\epsilon} C.$$

This also derives the demand curve of consumer good i under a monoplistic competition environment. Another convenient approach comes from that the Lagrange multipliers of the dual problems are reciprocals of each other [8, p.437]. In this way, the Lagrangian function is:

$$\mathcal{L}^{\max} = \left(\int_0^1 C_i^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}} + \frac{1}{P}\left(M - \int_0^1 P_i C_i \mathrm{d}i\right).$$

ii) Discrete aggregation

A basket of goods or the total price level can be a continuous sum as shown above, or a discrete sum. It is also an optimization problem of converting inequality constraints into equality constraints based on economic assumptions:

$$\begin{aligned} &\max_{C_i} C \equiv \left(\tfrac{1}{I}\textstyle\sum_{i=1}^I C_i^{\frac{1}{1+\Lambda}}\right)^{1+\Lambda}, && \overset{\text{dual}}{\Longleftrightarrow} && \min_{C_i} PC \equiv \textstyle\sum_{i=1}^I P_i C_i, \\ &\text{s.t.} \quad \textstyle\sum_{i=1}^I P_i C_i \le PQ \equiv M. && && \text{s.t.} \quad \left(\tfrac{1}{I}\textstyle\sum_{i=1}^I C_i^{\frac{1}{1+\Lambda}}\right)^{1+\Lambda} \ge C. \end{aligned}$$

These are the dual problems about optimal consumption choice in the case of discrete aggregation. The Lagrangian functions are:

$$\begin{aligned} \mathcal{L}^{\max} &= \left(\frac{1}{I}\sum_{i=1}^I C_i^{\frac{1}{1+\Lambda}}\right)^{1+\Lambda} + \frac{1}{P}\left(M - \sum_{i=1}^I P_i C_i\right); \\ &\Updownarrow \\ \mathcal{L}^{\min} &= \sum_{i=1}^I P_i C_i + P\left[C - \left(\frac{1}{I}\sum_{i=1}^I C_i^{\frac{1}{1+\Lambda}}\right)^{1+\Lambda}\right], \end{aligned}$$

Note the reciprocal Lagrange multipliers in this dual problem. We can still use the latter to deduce:

$$\begin{aligned} &\Rightarrow \quad P_i = P(1+\Lambda)\left(\frac{1}{I}\sum_{i=1}^I C_i^{\frac{1}{1+\Lambda}}\right)^{1+\Lambda-1} \frac{1}{1+\Lambda} C_i^{\frac{1}{1+\Lambda}-1}\frac{1}{I}, \\ &\Rightarrow \quad I \times P_i = P\left[\left(\frac{1}{I}\sum_{i=1}^I C_i^{\frac{1}{1+\Lambda}}\right)^{1+\Lambda}\right]^{\frac{\Lambda}{1+\Lambda}} C_i^{\frac{-\Lambda}{1+\Lambda}}, \\ &\Rightarrow \quad I \times P_i = PC^{\frac{\Lambda}{1+\Lambda}} C_i^{\frac{-\Lambda}{1+\Lambda}}, \\ &\Rightarrow \quad I \times \frac{P_i}{P} = \left(\frac{C_i}{C}\right)^{\frac{-\Lambda}{1+\Lambda}}, \\ &\Rightarrow \quad C_i = \left(I \times \frac{P_i}{P}\right)^{\frac{1+\Lambda}{-\Lambda}} C; \end{aligned}$$

$$\begin{aligned} &\sum_{i=1}^I P_i C_i = PC, \\ \Rightarrow \quad &\sum_{i=1}^I P_i \left[\left(I \times \frac{P_i}{P}\right)^{\frac{1+\Lambda}{-\Lambda}} C\right] = PC, \\ \Rightarrow \quad &I^{\frac{1+\Lambda}{-\Lambda}} \sum_{i=1}^I P_i^{1-\frac{1+\Lambda}{\Lambda}} = P^{1-\frac{1+\Lambda}{\Lambda}}, \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \quad P &= I^{1+\Lambda}\left(\sum_{i=1}^{I} P_i^{\frac{1}{-\Lambda}}\right)^{-\Lambda}, \\
\Rightarrow \quad P &= I\left(\frac{1}{I}\sum_{i=1}^{I} P_i^{\frac{1}{-\Lambda}}\right)^{-\Lambda}.
\end{aligned}
$$

This results in essentially the same but slightly different consumer demand curve and overall price level function, where the elasticity of substitution between I different commodities is $1+\frac{1}{\Lambda}$, and Λ is the desired cost markup [34, pp.360-361]

Example 10. **Consumption demand, labor supply, and demand for real money balances in a centrally planned production environment**

Money has the functions of pricing, transaction and storage. When used for transactions, it saves labor time for searching for matches and increases leisure time, which makes it have positive utility. [13, ch.2.5.3] introduced the following optimization problem of money in the utility function (MIU) when discussing the optimal monetary policy of the classical model:

$$
\begin{aligned}
\max_{C, M_d/P, L_s} \quad & U \equiv U\left(C, \frac{M_d}{P}, L_s\right), \\
\text{s.t.} \quad & Q_s = AL_d^{1-\alpha}.
\end{aligned}
$$

The constraint condition is the total resource constraint; the choice variables include not only the consumer demand and the labor supply, but also the real money balance.

When the money market, product market, and labor market are cleared, the following equilibrium conditions exist:

$$
\begin{cases}
M_s = M_d = M, \\
Q_s = Q_d \ = C, \\
L_s = L_d \ = L.
\end{cases}
$$

Therefore, the three-variable optimization problem is:

$$
\begin{aligned}
\max_{C, M/P, L} \quad & U \equiv U\left(C, \frac{M}{P}, L\right), \\
\text{s.t.} \quad & C = AL^{1-\alpha}.
\end{aligned}
$$

Construct a Lagrangian function :

$$
\mathcal{L} \equiv U\left(C, \frac{M}{P}, L\right) + \lambda(AL^{1-\alpha} - C).
$$

By taking the derivatives of the variables in turn, we can obtain four first-order necessary conditions:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C} &= U_C - \lambda = 0, \\
\frac{\partial \mathcal{L}}{\partial L} &= U_L + (1-\alpha)\lambda AL^{-\alpha} = 0, \\
\frac{\partial \mathcal{L}}{\partial M/P} &= U_{M/P} = 0, \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= AL^{1-\alpha} - C = 0.
\end{aligned}
$$

The last first-order condition is the social budget constraint. Tiding-up:

$$
\overbrace{-U_L/U_C}^{\substack{\text{marginal rate} \\ \text{of substitution}}} = \overbrace{(1-\alpha)AL^{-\alpha}}^{\text{marginal product}};
$$

$$\underbrace{U_{M/P}}_{\text{marginal utility}} = \underbrace{0}_{\text{marginal cost}}.$$

The marginal rate of substitution between labor and consumption is equal to the marginal product of labor, which is essentially the same as the first-order condition of the decentralized competitive economy in **E**xample 8, that is, the decentralized competitive equilibrium configuration is Pareto optimal (the first welfare theorem); conversely, for different market arrangements, equilibrium prices that support Pareto optimality can also be found (the second welfare theorem). In addition, the above first-order condition shows that the marginal utility of real money balances is equal to the social marginal cost of money "production".

The specific form of the utility function is not given above. The most common ones are the separable and additive type and the split-and-combined type:

$$\left.\begin{aligned}
U &\equiv U\left(C, \frac{M}{P}, L\right),\\
&= f^1(C) + f^2\left(\frac{M}{P}\right) + f^3(L),\\
\xrightarrow[\nu\neq1]{\sigma\neq1}\quad &= \frac{C^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}} + \frac{(M/P)^{1-\frac{1}{\nu}}-1}{1-\frac{1}{\nu}} - \frac{L^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}},\\
\xRightarrow[\nu=1]{\sigma=1}\quad &\underbrace{= \log C + \log\frac{M}{P} - \frac{L^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}}_{\text{separable and additive}}.
\end{aligned}\right\} \quad \text{vs.} \quad \left\{\begin{aligned}
U &\equiv U\left(C, \frac{M}{P}, L\right),\\
&= f^1\left(C, \frac{M}{P}\right) + f^2(L),\\
&\overset{\sigma\neq1}{=} \frac{\left\{\left[(1-\alpha)C^{1-\frac{1}{\nu}} + \alpha\left(\frac{M}{P}\right)^{1-\frac{1}{\nu}}\right]^{\frac{1}{1-\frac{1}{\nu}}}\right\}^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} - \frac{L^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}},\\
&\underbrace{\overset{\sigma=1}{=} C^{1-\alpha}\left(\frac{M}{P}\right)^{\alpha} - \frac{L^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}.}_{\text{split-and-combined}}
\end{aligned}\right.$$

In the separable and additive function, $1/\sigma$ represents the constant relative risk aversion coefficient, $\sigma \equiv -\frac{U_C(\cdot)}{CU_{CC}(\cdot)}$ represents the intertemporal elasticity of substitution of consumption, ν represents the elasticity of monetary demand, φ represents the Frisch elasticity of labor supply; In the split-and-combined function, ν is the substitution elasticity between consumption and real money balance, α is the realtive weight on real money balance. These parameters mostly express the connotation of comparative analysis. Chapter 3 will further explore why the functions including the constant elasticity of substitution (CES) or the constant intertemporal elasticity of substitution (CIES) are transformed into logarithmic form or Cobb-Douglas form.

III. **Multi-Variable Questions with Multiple Equality Constraints**

Assume that there are m (more than two) variables and n (multiple) equality constraints. The objective function and the constraints are:

$$\begin{aligned}
\xleftrightarrow[\min]{\max}\ O &= f(x_1, x_2, \ldots, x_m),\\
\text{s.t.}\quad g^1(x_1, x_2, \ldots, x_m) &= 0,\\
g^2(x_1, x_2, \ldots, x_m) &= 0,\\
&\vdots\\
g^n(x_1, x_2, \ldots, x_m) &= 0.
\end{aligned}$$

For n constraints, we can introduce n Lagrange multipliers (note that $n \leq m-1$ to make the constraints meaningful):

$$\mathcal{L} \equiv f(x_1, x_2, \ldots, x_m) + \sum_{j=1}^{n} \lambda_j [0 - g^j(x_1, x_2, \ldots, x_m)].$$

Here, the bordered Hessian determinant is:

$$
\begin{vmatrix}
0 & 0 & \cdots & 0 & g^1_{x_1} & g^1_{x_2} & \cdots & g^1_{x_m} \\
0 & 0 & \cdots & 0 & g^2_{x_1} & g^2_{x_2} & \cdots & g^2_{x_m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & g^n_{x_1} & g^n_{x_2} & \cdots & g^n_{x_m} \\
g^1_{x_1} & g^2_{x_1} & \cdots & g^n_{x_1} & \mathcal{L}^o_{x_1x_1} & \mathcal{L}^o_{x_1x_2} & \cdots & \mathcal{L}^o_{x_1x_m} \\
g^1_{x_2} & g^2_{x_2} & \cdots & g^n_{x_2} & \mathcal{L}^o_{x_2x_1} & \mathcal{L}^o_{x_2x_2} & \cdots & \mathcal{L}^o_{x_2x_m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
g^1_{x_m} & g^2_{x_m} & \cdots & g^n_{x_m} & \mathcal{L}^o_{x_mx_1} & \mathcal{L}^o_{x_mx_2} & \cdots & \mathcal{L}^o_{x_mx_m}
\end{vmatrix}
$$

After observation, it is not difficult to find that:

(1) The bordered Hessian matrix of multi-variable optimization with multiple equality constraints is symmetric along the main diagonal;

(2) Because multiple equality constraints increase the dimension of the border of the Hessian determinant, judging whether it is positive definite, negative definite, or judging whether the function has a maximum, minimum is the same as the case of a single equality constraint with multiple variables. If the leading principle minors, starting from the second-order, of the Hessian determinant are all negative, the bordered Hessian matrix is positive definete, and the function has a minimum; If the odd-order leading principle minors of the Hessian determinant are all positive and the even-order leading principle minors of the Hessian determinant are all negative, the bordered Hessian matrix is negative definete, and the function has a maximum.

1.1.2.2 Inquality Constraints Between Choice Variables

In the optimization problem of m choice variable with n equality constraints, we have pointed out that in order to make the constraints effectively constraint the choice variables, $n < m$ should be hold. Taking$m = 2, n = 1$ as an example, the equality constraints and inequality constraints are:

$$
\left.\begin{aligned} \underset{\min}{\overset{\max}{\longleftrightarrow}} O &= f(x,y), \\ \text{s.t.} \quad g(x,y) &= 0. \end{aligned}\right\}
\overset{\text{vs.}}{\Longleftrightarrow}
\left\{\begin{aligned} \underset{\min}{\overset{\max}{\longleftrightarrow}} O &= f(x,y), \\ \text{s.t.} \quad g(x,y) &\le 0. \end{aligned}\right\}
\overset{\text{identical}}{\Longleftrightarrow}
\begin{cases} \underset{\min}{\overset{\max}{\longleftrightarrow}} O = f(x,y), \\ \text{s.t.} \quad g(x,y) + z = 0, \\ \qquad\qquad z \ge 0. \end{cases}
$$

The inequality constraint $g(x, y) \le 0$ can be transformed into an equality constraint by applying a non-negative parameter z. If it is not interfered by $z \ge 0$, it can be solved by the Lagrange multiplier method discussed in the optimization problem of equality constraints, that is, the first-order necessary condition is:

$$
\mathscr{L} = f(x,y) + \lambda[0 - z - g(x,y)] \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \mathscr{L}}{\partial x} = 0, \\ \dfrac{\partial \mathscr{L}}{\partial y} = 0, \\ \dfrac{\partial \mathscr{L}}{\partial z} = 0, \\ \dfrac{\partial \mathscr{L}}{\partial \lambda} = 0. \end{cases}
$$

But now that the inequality constraint $z \ge 0$ has been added, the first-order condition for z should change.

How to change?

Taking the maximization problem as an example, let's assume an objective function that only contains non-negative parameters z:

$$
\max_z O = f(z),
$$
$$
\text{s.t.} \quad z \ge 0.
$$

If there is not a non-negative constraint, $f(\cdot)$ should be a concave or quasi-concave function, in which case the maximum value should be located in the first quadrant of the O-z two-dimensional coordinate system or in the vertical direction. However, convex or quasi-convex functions may still obtain the maximum value on the ordinate axis under this non-negative constraint. This can be summarized by the following mathematical expression:

$$\left.\begin{aligned} \text{when } z>0,\quad f'(z)=0;\\ \text{when } z=0,\quad f'(z)=0;\\ \text{when } z=0,\quad f'(z)<0. \end{aligned}\right\} \Leftrightarrow \underbrace{z\ge 0}_{\text{non-negative}},\ \underbrace{f'(z)\le 0}_{\text{F.O.C.}},\ \text{and}\ \underbrace{zf'(z)=0}_{\text{complementary slackness}}.$$

Back to the optimization problem of 2 choice variables with 1 equality constraint and a non-negative constraint:

$$\left.\begin{aligned} \xleftrightarrow[\min]{\max} O=f(x,y),\\ \text{s.t.}\quad g(x,y)+z=0,\\ z\ge 0. \end{aligned}\right\} \xLeftrightarrow{\text{identical}} \left\{\begin{aligned} &\xleftrightarrow[\min]{\max} O=f(x,y),\\ &\text{s.t.}\quad g(x,y)\le 0. \end{aligned}\right.$$

Construct Lagrangian functions for optimization problems with and without auxiliary parameters z:

$$\begin{aligned} &\xleftrightarrow[\min]{\max} \mathscr{L}\equiv f(x,y)+\lambda[0-z-g(x,y)], &&\text{vs.} &&\xleftrightarrow[\min]{\max} \mathcal{L}\equiv f(x,y)+\lambda[0-g(x,y)],\\ &\text{s.t.}\quad z\ge 0. \end{aligned}$$

The conditions for obtaining extreme values are (the left and right ends are completely equivalent):

$$\underbrace{\overbrace{\left.\begin{aligned} \frac{\partial\mathscr{L}}{\partial x}&=0,\\ \frac{\partial\mathscr{L}}{\partial y}&=0,\\ z\ge 0,\ \frac{\partial\mathscr{L}}{\partial z}&\le 0,\ z\frac{\partial\mathscr{L}}{\partial z}=0,\\ \frac{\partial\mathscr{L}}{\partial\lambda}&=0. \end{aligned}\right\}}^{\text{F.O.C. of } \mathscr{L} \text{ when } z \text{ is nonnegative}} \Leftrightarrow \left\{\begin{aligned} \frac{\partial\mathscr{L}}{\partial x}&=0,\\ \frac{\partial\mathscr{L}}{\partial y}&=0,\\ z\ge 0,\ \frac{\partial\mathscr{L}}{\partial z}&=-\lambda\le 0,\ z(-\lambda)=0,\\ \frac{\partial\mathscr{L}}{\partial\lambda}&=0-z-g(x,y)=0. \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} &\frac{\partial\mathscr{L}}{\partial x}=0,\\ &\frac{\partial\mathscr{L}}{\partial y}=0,\\ &\left.\begin{aligned} z\ge 0,\ \lambda\ge 0,\ z\lambda=0,\\ z=0-g(x,y). \end{aligned}\right\} \end{aligned}\right. \Leftrightarrow \left\{\begin{aligned} &\frac{\partial\mathscr{L}}{\partial x}=0=\frac{\partial\mathcal{L}}{\partial x},\\ &\frac{\partial\mathscr{L}}{\partial y}=0=\frac{\partial\mathcal{L}}{\partial x},\\ &\left.\begin{aligned} 0-g(x,y)\ge 0,\ \lambda\ge 0,\ [0-g(x,y)]\lambda=0,\\ \overbrace{\cancel{\frac{\partial\mathscr{L}}{\partial\lambda}=0}}\ \text{vs.}\ \frac{\partial\mathcal{L}}{\partial\lambda}=0-g(x,y) \end{aligned}\right\} \end{aligned}\right. \xLeftrightarrow{\text{vs.}} \overbrace{\left\{\begin{aligned} \frac{\partial\mathcal{L}}{\partial x}&=0,\\ \frac{\partial\mathcal{L}}{\partial y}&=0,\\ \frac{\partial\mathcal{L}}{\partial\lambda}\ge 0,\ \lambda\ge 0,\ \lambda\frac{\partial\mathcal{L}}{\partial\lambda}&=0. \end{aligned}\right.}^{\text{F.O.C. of } \mathcal{L} \text{ without constraint}}}_{\text{the left and right ends are equivalent}}$$

For the two Lagrangian functions mentioned above, the results of the first-order partial derivatives are exactly the same for the unconstrained selection variables x and y. The third row on the left is the first-order condition for the optimization of function $\mathscr{L}(z)=f(x,y)+\lambda[0-z-g(x,y)]$ with respect to the non-negative constraint z Combining this condition with the equality constraint condition converted from the inequality, we can obtain the complementary slackness condition on the Lagrange multiplier λ on the third row on the right.

In the above optimization problem, if the two choice variables are also required to be non-negative, that is

$$\left.\begin{aligned} \xleftrightarrow[\min]{\max} O=f(x,y),\\ \text{s.t.}\quad g(x,y)+z=0,\\ x,y,z\ge 0. \end{aligned}\right\} \xLeftrightarrow{\text{identical}} \left\{\begin{aligned} &\xleftrightarrow[\min]{\max} O=f(x,y),\\ &\text{s.t.}\quad g(x,y)\le 0,\\ &\qquad\qquad x,y\ge 0. \end{aligned}\right.$$

Construct Lagrangian functions for optimization problems with and without auxiliary parameters z :

$$\begin{aligned} &\xleftrightarrow[\min]{\max} \mathscr{L}\equiv f(x,y)+\lambda[0-z-g(x,y)], &&\text{vs.} &&\xleftrightarrow[\min]{\max} \mathcal{L}\equiv f(x,y)+\lambda[0-g(x,y)],\\ &\text{s.t.}\quad x,y,z\ge 0. &&&&\text{s.t.}\quad x,y\ge 0. \end{aligned}$$

Therefore, the conditions an optimization problem where the variables are non-negative and there is an inequality constraint are:

$$\left.\begin{array}{l} x \geq 0,\ \dfrac{\partial \mathscr{L}}{\partial x} \leq 0,\ x\dfrac{\partial \mathscr{L}}{\partial x} = 0, \\ y \geq 0,\ \dfrac{\partial \mathscr{L}}{\partial y} \leq 0,\ y\dfrac{\partial \mathscr{L}}{\partial y} = 0, \\ z \geq 0,\ \dfrac{\partial \mathscr{L}}{\partial z} \leq 0,\ z\dfrac{\partial \mathscr{L}}{\partial z} = 0, \\ \dfrac{\partial \mathscr{L}}{\partial \lambda} = 0. \end{array}\right\} \overset{\text{vs.}}{\Longleftrightarrow} \left\{\begin{array}{l} x \geq 0,\ \dfrac{\partial \mathcal{L}}{\partial x} \leq 0,\ x\dfrac{\partial \mathcal{L}}{\partial x} = 0, \\ y \geq 0,\ \dfrac{\partial \mathcal{L}}{\partial y} \leq 0,\ y\dfrac{\partial \mathcal{L}}{\partial y} = 0, \\ \lambda \geq 0,\ \dfrac{\partial \mathcal{L}}{\partial \lambda} \geq 0,\ \lambda\dfrac{\partial \mathcal{L}}{\partial \lambda} = 0. \end{array}\right.$$

This is the Kuhn-Tucker condition similar to the first-order condition obtained in the nonlinear programming method for optimization problems with nonnegative and inequality constraints. The application analysis of inequality constraints in economics involves a considerable amount of trial and error, and the next edition may include additional discussions based on specific cases from cutting-edge papers.

Previously, unconstrained optimality and equality-constrained optimality were applied to the market environments of perfect competition and monopolistic competition respectively, which seemed to be separated from each other. However, in specific applications, the markets of perfect competition and monopolistic competition (or multiple monopolistic competitions) should be considered comprehensively. The following three examples are static examples of modern macroeconomics laying the foundation for microeconomics. They are appopriate to demonstrate the comprehensive application of static optimization.

Example 11-1. **The market for final product sales is of perfect competition, while the market for intermediate product sellers is of monopolistic competition.**

i) The optimization problem of the final product production department under perfect competition in the product buying and selling market.

The final product is indifferent, and the analytical framework for the representative production department to make a desired decision is:

$$\max_{Q_{id} \rightleftarrows Q_s} \Pi \equiv \overbrace{\underbrace{P\left(\left(\int_0^1 Q_{id}^{\frac{\epsilon_m-1}{\epsilon_m}} \mathrm{d}i\right)^{\frac{\epsilon_m}{\epsilon_m-1}}\right)}_{Q_s}}^{\text{revenue}\mathcal{R}[Q_s(Q_{id})]} - \overbrace{\int_0^1 P_i Q_{id} \mathrm{d}i}^{\text{cost } C[Q_s(Q_{id})]} \underset{\textbf{Example2}}{\overset{\text{vs.}}{\Longleftrightarrow}} \overbrace{\underbrace{\max_{L_d \rightleftarrows Q_s} \Pi \equiv \overbrace{PQ_s}^{\text{revenue}} - \overbrace{WL_d}^{\text{variable cost}}}_{\text{real wage=marginal product}}}^{\text{marginal revenue=marginal cost}}.$$

The choice of intermediate input factors Q_{id} is to choose the final product supply Q_s to maximize profits, where the final product is the Dixit-Stiglitz CES sum of the intermediate products and ϵ_m is the elasticity of substitution between the intermediate input basket.

The first-order necessary condition is:

$$\begin{aligned}
&\frac{\mathrm{d}\Pi}{\mathrm{d}Q_{id}} = 0 = P\frac{\epsilon_m}{\epsilon_m - 1}\left(\int_0^1 Q_{id}^{\frac{\epsilon_m-1}{\epsilon_m}} \mathrm{d}i\right)^{\frac{\epsilon_m}{\epsilon_m-1}-1} \frac{\epsilon_m - 1}{\epsilon_m} Q_{id}^{\frac{\epsilon_m-1}{\epsilon_m}-1} - P_i, \\
\Rightarrow \quad & P_i = P\left(\int_0^1 Q_{id}^{\frac{\epsilon_m-1}{\epsilon_m}} \mathrm{d}i\right)^{\frac{\epsilon_m}{\epsilon_m-1}-1} Q_{id}^{\frac{\epsilon_m-1}{\epsilon_m}-1}, \\
\Rightarrow \quad & P_i = P\left[\left(\int_0^1 Q_{id}^{\frac{\epsilon_m-1}{\epsilon_m}} \mathrm{d}i\right)^{\frac{\epsilon_m}{\epsilon_m-1}}\right]^{\frac{1}{\epsilon_m}} Q_{id}^{\frac{-1}{\epsilon_m}}, \\
\Rightarrow \quad & \frac{P_i}{P} = \left(\frac{Q_{id}}{Q_s}\right)^{\frac{-1}{\epsilon_m}}, \\
\Rightarrow \quad & Q_{id} = \left(\frac{P_i}{P}\right)^{-\epsilon_m} Q_s.
\end{aligned}$$

In this way, we can get the demand curve for intermediate product i in a perfectly competitive market environment.

Substituting it into the objective function, we can get the functional relationship between the total price level and the price of a single intermediate product i:

$$
\begin{aligned}
& PQ_s - \int_0^1 P_i Q_{id} \mathrm{d}i = \overbrace{\Pi = 0}^{\substack{\text{perfect}\\\text{competition}}} , \\
\Rightarrow \quad & \int_0^1 P_i Q_{id} \mathrm{d}i = PQ_s, \\
\Rightarrow \quad & \int_0^1 P_i \left[\left(\frac{P_i}{P} \right)^{-\epsilon_m} Q_s \right] \mathrm{d}i = PQ, \\
\Rightarrow \quad & \int_0^1 P_i \left(\frac{P_i}{P} \right)^{-\epsilon_m} \mathrm{d}i = P, \\
\Rightarrow \quad & P = \left(\int_0^1 P_i^{1-\epsilon_m} \mathrm{d}i \right)^{\frac{1}{1-\epsilon_m}} .
\end{aligned}
$$

It is not difficult to find that the required consumer demand curve and total price level index are consistent with **E**xample 9.

ii) Optimization problem of the intermediate product production department under monopolistic competition

The intermediate products are different. The basis for the heterogeneous intermediate product manufacturers i to make a desired decision is:

$$
\begin{aligned}
& \max_{P_i} \Pi_i \equiv P_i Q_{id} - W L_{id}, \\
\text{s.t.} \quad & Q_{id} = \left(\frac{P_i}{P} \right)^{-\epsilon_m} Q_s, \\
& Q_{is} = A L_{id}, \\
& Q_{id} = Q_{is} = Q_i .
\end{aligned}
$$

The simplified production function (capital input factor is exogenous and labor output share is 1) will be described in detail in Chapter 3. Substituting the constraints and equilibrium conditions into the objective function, we can transform the optimization problem about the demand for product i into an unconstrained optimization problem:

$$
\max_{Q_{id}} \Pi_i \equiv \left(P_i - \frac{W}{A} \right) Q_{id} \quad \xLeftrightarrow[\substack{\text{or} \min\limits_{Q_{is}} C_i = W L_{id} \\ \text{s.t. } Q_{is} = A L_{id}} \;\Rightarrow\; \mathrm{MC}_i \equiv \frac{\mathrm{d}C_i}{\mathrm{d}Q_{is}} = \frac{\mathrm{d}\left(W \frac{Q_{is}}{A} \right)}{\mathrm{d}Q_{is}} = \frac{W}{A} \equiv \mathrm{MC}]{\substack{\min\limits_{L_{id}} M_i = W L_{id} \\ \text{s.t. } A L_{id} \geq Q_{is}} \;\Rightarrow\; \min\limits_{L_{id}} \mathcal{L} \equiv W L_{id} + \mathrm{MC}(Q_{is} - A L_{id})} \quad \max_{Q_{id}} \Pi_i \equiv (P_i - \mathrm{MC}) Q_{id} .
$$

Then, the optimal price decision for product i is obtained through the inverse demand function $P_i = \left(\frac{Q_{id}}{Q_s} \right)^{-\frac{1}{\epsilon_m}} P$.

Or it can be directly transformed into the optimal price decision problem for product i:

$$
\max_{P_i} \Pi_i \equiv \left(P_i - \frac{W}{A} \right) \left[\left(\frac{P_i}{P} \right)^{-\epsilon_m} Q_s \right] \quad \xLeftrightarrow{\text{or}} \quad \max_{P_i} \Pi_i \equiv (P_i - \mathrm{MC}) \left[\left(\frac{P_i}{P} \right)^{-\epsilon_m} Q_s \right] .
$$

It is not difficult to find that this is a reappearance of **E**xample 3, and the demand curve directly given in **E**xample 3 comes from Example **E**xample 9. This example presents the close connection between **E**xample 9 and **E**xample 3 from a different perspective. Therefore, the solution is still marginal cost (a homogeneous production function makes the marginal cost of heterogeneous enterprises the same) with a cost markup to reflect the optimal pricing decision under monopolistic competition.

iii) The optimization problem of the household sector under perfect competition in the labor market

The decision analysis framework for the representative household sector not having monopoly power in both the final product buyer's market and the labor seller's market is:

$$
\begin{aligned}
&\max_{C,L_s} U \equiv U(C, L_s),\\
\text{s.t.}\quad &\underbrace{\int_0^1 P_i C_i \mathrm{d}i}_{PC} = \underbrace{\int_0^1 W_i L_{is} \mathrm{d}i}_{WL_s} + \underbrace{\int_0^1 \Pi_i \mathrm{d}i}_{\bar{\Pi}_i}.
\end{aligned}
$$

Note that the profit (sum) of the intermediate product manufacturer under monopolistic competition is $\bar{\Pi}_i \neq 0$, while the profit (sum) of the final product manufacturer under perfect competition is $\Pi = 0$. This comes back to **E**xample 8, except that the household's expected total price level is simplified to the instantaneous total price level and the profit and dividend income of the household as a shareholder of the enterprise is omitted. The solution is still the desired consumption demand and the optimal labor supply.

Based on the same market environment, the final product production sector and the household sector are both assumed to be representative, so the household sector's demand for product i in a basket of consumer goods corresponds exactly to the final product sector's demand for product i in a basket of intermediate goods when the market is cleared, which is what is solved in **E**xample 9. It can be seen that the hierarchical structure of the trinity of representative households, representative final product manufacturers, and heterogeneous intermediate product manufacturers has exactly the same modeling results as the hierarchical structure of the integration of representative households and heterogeneous manufacturers.

Therefore, we might as well return to the two-sector structure of households and firms, but in addition to considering that firms produce heterogeneous products, we can also consider that households have heterogeneous labor.

Example 11-2. **Monopolistic competition exists in both the product seller market and the labor seller market.**

This type of model comes from [11] and is explained in the famous works [13, ch.6] and [50, ch.7, pp.277-316], but all of them take into account both monopolistic competition and nominal rigidity to form a dynamic form.

i) Optimization Problems of Production Departments under Monopolistic Competition in Product Seller Market

The decision-making behavior of manufacturer i is divided into two steps.

The first step is to maximize a basket of labor demand given labor expenditure, or minimize labor expenditure given a basket of labor demand, which is back to a question similar to **E**xample 9:

$$
\left.
\begin{aligned}
&\min_{L_{ild}} \int_0^1 W_l L_{ild} \mathrm{d}l \equiv \underbrace{M_i}_{\text{given labor expenditure}},\\
\text{s.t.}\quad &\left(\int_0^1 L_{ild}^{\frac{\epsilon_h-1}{\epsilon_h}} \mathrm{d}l\right)^{\frac{\epsilon_h}{\epsilon_h-1}} \overbrace{\geq L_{id} = L_{is} = L_i}^{\text{a basket of labor}}.
\end{aligned}
\right\}
\Rightarrow
\left\{
\begin{aligned}
L_{ild} &= \left(\frac{W_l}{W}\right)^{-\epsilon_h} L_i,\\
W &= \left(\int_0^1 W_l^{1-\epsilon_h} \mathrm{d}l\right)^{\frac{1}{1-\epsilon_h}}.
\end{aligned}
\right\}
\Leftarrow
\quad WL_i = \int_0^1 W_l L_{ild} \mathrm{d}l.
$$

The production of firm i requires a basket of heterogeneous labor purchased from the representative household sector. Given the technical level, and assuming that the production share of labor is 1, the product i is the CES aggregate of heterogeneous labor, that is, $Q_{is} = A_i L_{id} = A_i \left(\int_0^1 L_{ild}^{\frac{\epsilon_h-1}{\epsilon_h}} \mathrm{d}l\right)^{\frac{\epsilon_h}{\epsilon_h-1}}$. Since the technical level of firm i is now more diverse, the marginal cost will also be different.

In the second step, given that its labor demand is determined by supply, the basis for heterogeneous firms i to make a desired decision is:

$$
\begin{aligned}
&\max_{P_i} \Pi_i \equiv (P_i - \mathrm{MC}_i) Q_{id},\\
\text{s.t.}\quad &Q_{id} = \left(\frac{P_i}{P}\right)^{-\epsilon_m} Q.
\end{aligned}
$$

The demand curve for product i results from minimizing consumption expenditure given a basket of heterogeneous consumer goods (combined with product market equilibrium conditions), which will be reviewed later when we introduce the decision-making behavior of the household sector.

The optimal price for manufacturer i is still:

$$P_i^\circ = \frac{\epsilon_m}{\epsilon_m - 1}\mathrm{MC}_i.$$

Determine the marginal cost of firm i by minimizing the cost:

$$\begin{aligned} &\min_{L_{ild}} M_i \equiv \overbrace{\int_0^1 W_l L_{ild} \mathrm{d}l}^{\text{cost of labor input}}, \\ \text{s.t.} \quad &\underbrace{A_i \left(\int_0^1 L_{ild}^{\frac{\epsilon_h - 1}{\epsilon_h}} \mathrm{d}l\right)^{\frac{\epsilon_h}{\epsilon_h - 1}}}_{\text{effective labor input}} \geq Q_{is}. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\min_{L_{is}} M_i \equiv \overbrace{W L_{id}}^{\text{cost of labor input}}, \\ \text{s.t.} \quad &\underbrace{A_i L_{id}}_{\text{effective labor input}} \geq Q_{is}. \end{aligned}$$

The Lagrange multiplier is the marginal cost:

$$\mathcal{L} \equiv W L_{id} + \mathrm{MC}_i (Q_{is} - A_i L_{id}).$$

The optimal price for firm i is then:

$$\begin{aligned} \underbrace{P_i^\circ}_{\substack{\text{nominal}\\ \text{price}}} &= \overbrace{\underbrace{\frac{\epsilon_m}{\epsilon_m - 1}}_{\text{markup}}}\overbrace{\underbrace{\frac{W}{A_i}}_{\substack{\text{nominal}\\ MC}}}, \\ \Rightarrow \quad \overbrace{\underbrace{\frac{P_i^\circ}{P}}_{\text{realtive price}}}^{\substack{\text{real}\\ \text{price}}} &= \underbrace{\overbrace{\frac{\epsilon_m}{\epsilon_m - 1}}^{\text{markup}}\overbrace{\frac{W/P}{A_i}}^{\substack{\text{real}\\ MC}}}_{\text{monoplistic competition}} \quad \text{vs.} \quad \underbrace{\frac{W}{P} = F_L^o}_{\text{perfect competition}}. \end{aligned}$$

If the technology level is $A_i = A$, then the marginal cost is $\mathrm{MC}_i = \mathrm{MC}$

ii) The optimization problem of the household sector under monopolistic competition in the labor seller market

The decision-making behavior of a representative household with heterogeneous labor force consists of two steps.

The first step is to maximize a basket of consumer goods given consumer expenditure, or minimize consumer expenditure given a basket of consumer goods, that is, return to **E**xample 9

$$\left.\begin{aligned} &\min_{C_i} \int_0^1 P_i C_i \mathrm{d}i \equiv M, \\ \text{s.t.} \quad &\left(\int_0^1 C_i^{\frac{\epsilon_m - 1}{\epsilon_m}} \mathrm{d}i\right)^{\frac{\epsilon_m}{\epsilon_m - 1}} \geq C. \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} C_i &= \left(\frac{P_i}{P}\right)^{-\epsilon_m} C, \\ P &= \left(\int_0^1 P_i^{1-\epsilon_m} \mathrm{d}i\right)^{\frac{1}{1-\epsilon_m}}. \end{aligned}\right\} \quad \Leftarrow \quad PC = \int_0^1 P_i C_i \mathrm{d}i.$$

In the second step, the household chooses wages to determine the supply of heterogeneous labor l, and then chooses the total consumption and total labor supply based on utility maximization:

$$
\begin{aligned}
&\max_{C,W_l} U \equiv U\left(C, \overbrace{\underbrace{\int_0^1 L_{ls}\mathrm{d}l}_{\text{the labor able to offer}}}^{\text{aggregate supply}}\right), && && \max_{C,W_l} U \equiv U\left(C, \overbrace{\underbrace{\int_0^1\int_0^1 L_{ild}\mathrm{d}l\mathrm{d}i}_{\text{the labor willing to offer}}}^{\text{aggregate demand}}\right),\\
\text{s.t.}\ &\underbrace{\int_0^1 P_iC_i\mathrm{d}i}_{=PC} = \underbrace{\int_0^1 W_lL_{ls}\mathrm{d}l}_{=\int_0^1 WL_i\mathrm{d}i} + \underbrace{\int_0^1 \Pi_i\mathrm{d}i}_{\bar{\Pi}_i}, && \xLeftrightarrow[d \text{ determines } s]{\text{identital}} && \text{s.t.}\ \underbrace{\int_0^1 P_iC_i\mathrm{d}i}_{=PC} = \underbrace{\int_0^1\int_0^1 W_lL_{ild}\mathrm{d}l\mathrm{d}i}_{=\int_0^1 WL_i\mathrm{d}i} + \underbrace{\int_0^1 \Pi_i\mathrm{d}i}_{\bar{\Pi}_i},\\
&L_{ls} = L_{ld} = \left(\frac{W_l}{W}\right)^{-\epsilon_h}\int_0^1 L_i\mathrm{d}i. && && \int_0^1\int_0^1 L_{ild}\mathrm{d}l\mathrm{d}i = \left(\frac{W_l}{W}\right)^{-\epsilon_h}\int_0^1 L_i\mathrm{d}i.
\end{aligned}
$$

The choice variable C can be solved from the constraint and substituted into the objective function together with the labor demand curve to transform it into an unconstrained optimization problem:

$$
\begin{aligned}
\max_{W_l} U \equiv\ & U\left[\frac{1}{P}\left(\int_0^1\int_0^1 W_lL_{ild}\mathrm{d}l\mathrm{d}i + \int_0^1 \Pi_i\mathrm{d}i\right), \int_0^1\int_0^1 L_{ild}\mathrm{d}l\mathrm{d}i\right],\\
=\ & U\left\{\frac{1}{P}\left[W_l\left(\frac{W_l}{W}\right)^{-\epsilon_h}\int_0^1 L_i\mathrm{d}i + \int_0^1 \Pi_i\mathrm{d}i\right], \left(\frac{W_l}{W}\right)^{-\epsilon_h}\int_0^1 L_i\mathrm{d}i\right\},\\
=\ & U\left\{\underbrace{\frac{1}{P}\left[W_l\left(\frac{W_l}{W}\right)^{-\epsilon_h}\bar{L}_i + \bar{\Pi}_i\right]}_{C}, \underbrace{\left(\frac{W_l}{W}\right)^{-\epsilon_h}\bar{L}_i}_{\{L_{ls}\}}\right\}.\\
\xRightarrow{\text{F.O.C.}}\quad 0 =\ & \frac{\mathrm{d}U}{\mathrm{d}W_l} = (1-\epsilon_h)U_C\frac{1}{P}\left(\frac{W_l}{W}\right)^{-\epsilon_h}\bar{L}_i - \epsilon_hU_L\left(\frac{W_l}{W}\right)^{-\epsilon_h}\bar{L}_iW_l^{-1},\\
\Rightarrow\quad & \overbrace{\underbrace{\frac{W_l^\circ}{P}}_{\text{realtive wage}}}^{\text{real wage}} = \underbrace{\overbrace{\frac{\epsilon_h}{\epsilon_h-1}}^{\text{markup}}\overbrace{\left(\frac{-U_L}{U_C}\right)}^{\text{MRS}}}_{\text{monopolistic competition}} \quad \text{vs.} \quad \frac{W_l}{P} = \underbrace{\frac{W}{P} = \frac{-U_L^o}{U_C^o}}_{\text{perfect competition}}.
\end{aligned}
$$

The focus of this example is to show how to obtain the optimal decision on prices and wages in an environment where both the product market and the labor market have monopolistic competitition.

The next example is still based on the trinity hierarchy and makes a similar setting. The household sector's labor market is still perfectly competitive, but in addition to the intermediate product production sector, the final product production sector is also in a monopolistic competition market environment. Considering that both the product market and the labor market are monopolistic competition is similar to considering that both the final product and intermediate product production stages are monopolistic competition.

Example 11-3. **The final product seller market and the intermediate product seller market are both monopolistic competition.**

This example is adapted from [17–19], and the specific application is expanded in [53, 54] The original papers are all time series analysis, that is, they consider the dynamic decision-making caused by nominal rigidity or information friction. As the first part of the advanced macroecnomics series, this book only considers the static decision-making problem of the final product seller market and the intermediate product seller market in a monopolistic competition market environment.

i) Optimization problem of representative household sector under perfect competition in labor market

The basis for the household sector to make desired decisions is:

$$
\begin{aligned}
&\max_{C,L_s} U \equiv U(C,L_s),\\
\text{s.t.}\quad &\underbrace{\int_0^1 P_j C_j \mathrm{d}j}_{P_f C} = \underbrace{\overbrace{\int_0^1 W L_{is} \mathrm{d}i}^{\substack{\text{wage income from}\\ \text{intermediate good production}}} + \overbrace{\int_0^1 W L_{js} \mathrm{d}j}^{\substack{\text{wage income from}\\ \text{final good production}}}}_{W L_s} + \underbrace{\overbrace{\int_0^1 \Pi_i \mathrm{d}i}^{\substack{\text{profit dividend from}\\ \text{intermediate good production}}} + \overbrace{\int_0^1 \Pi_j \mathrm{d}j}^{\substack{\text{profit dividend from}\\ \text{final good production}}}}_{\Pi}.
\end{aligned}
$$

From this, we can obtain the desired consumption demand and optimal labor supply as usual, by combining the labor supply curve and the budget constraint (depending on the specific utility function given):

$$
\begin{cases}
\dfrac{W}{P_f} = \dfrac{-U_{L_s}(C,L_s)}{U_C(C,L_s)},\\
P_f C = W L_s + \Pi.
\end{cases}
$$

The final goods sector is monopolistically competitive, and consumption demand for heterogeneous final goods j comes from the household sector maximizing a basket of final goods given its expenditure or minimizing its expenditure given a basket of final goods:

$$
\begin{aligned}
&\max_{C_j} \quad \overbrace{C \equiv \left(\int_0^1 C_j^{\frac{\epsilon_f-1}{\epsilon_f}} \mathrm{d}i\right)^{\frac{\epsilon_f}{\epsilon_f-1}}}^{\text{Dixit-Stiglitz aggregated CES functyion}}, &\overset{\text{vs.}}{\Longleftrightarrow}\quad &\min_{C_j} \quad \overbrace{P_f Q_f \equiv M_f \equiv P_f C}^{Q_f=Q_{fd}=Q_{fs}=C+\cancel{I+G+\delta K}} = \int_0^1 P_j C_j \mathrm{d}j,\\
&\text{s.t.} \quad \int_0^1 P_j C_j \mathrm{d}j \le P_f Q_f \equiv M_f. & &\text{s.t.} \quad \left(\int_0^1 C_j^{\frac{\epsilon_f-1}{\epsilon_f}} \mathrm{d}j\right)^{\frac{\epsilon_f}{\epsilon_f-1}} \ge C.
\end{aligned}
$$

From this, we can get the demand curve of heterogeneous final product j and the total price index of final product:

$$
C_j = \left(\frac{P_j}{P_f}\right)^{-\epsilon_f} C \quad \xLeftrightarrow[C_j=Q_{jd};\quad C=Q_f]{P_f=\left(\int_0^1 P_j^{1-\epsilon_f}\mathrm{d}j\right)^{\frac{1}{1-\epsilon_f}}} \quad Q_{jd} = \left(\frac{P_j}{P_f}\right)^{-\epsilon_f} Q_f.
$$

ii) Optimization problem of final product production department under monopolistic competition in product seller market

The final products are different, and the analytical framework for heterogeneous final product manufacturers j to make ideal decisions is:

$$
\begin{aligned}
&\max_{P_j} \Pi_j \equiv (P_j - \mathrm{MC}_j) \overbrace{Q_{jd}}^{\substack{\text{household}\\ \text{demand for } j}},\\
\text{s.t.}\quad & Q_{jd} = \left(\frac{P_j}{P_f}\right)^{-\epsilon_f} Q_f,\\
& Q_{js} = Q_{md}^{\alpha}[A_j L_{jd}]^{1-\alpha},\\
& Q_{ms} = \left(\int_0^1 Q_{jis}^{\frac{\epsilon_m-1}{\epsilon_m}} \mathrm{d}i\right)^{\frac{\epsilon_m}{\epsilon_m-1}},\\
& Q_{jid} = \left(\frac{P_i}{P_m}\right)^{-\epsilon_m} Q_m,\\
& Q_{jid} = Q_{jis}, \qquad \underbrace{Q_{md} = Q_{ms} = Q_m}_{\text{market clearing of intermediate goods}},\\
& Q_{jd} = Q_{js}, \qquad \overbrace{Q_{fd} = Q_{fs} = Q_f}^{\text{market clearing of final goods}}.
\end{aligned}
$$

Given the marginal cost, without considering the supply side, the optimal decision is simplified to:

$$\max_{P_j \rightleftarrows Q_{jd}} \Pi_j \equiv \left[\left(\frac{Q_{jd}}{Q_f}\right)^{-\frac{1}{\epsilon_f}} P_f - \mathrm{MC}_j\right] Q_{jd}.$$

This is the method 1 introduced in **E**xample 3. Following similar steps, we first solve the demand of the household sector for heterogeneous final products, and then using the inverse demand, we can know that the optimal pricing of the final product production department j is:

$$\underbrace{P_j^{\circ}}_{\text{nominal price}} = \overbrace{\underbrace{\frac{\epsilon_f}{\epsilon_f - 1}}_{\text{cost markup}}} \cdot \underbrace{\mathrm{MC}_j}_{\substack{\text{nominal}\\ \text{marginal cost}}}.$$

Next, we need to determine the marginal cost of final product manufacturer j by cost minimization:

$$\begin{aligned}
&\min_{Q_{jid}, L_{jd}} M_j \equiv \overbrace{\underbrace{\int_0^1 P_i Q_{jid} \mathrm{d}i}_{\substack{\text{expenditure of}\\ \text{intermidiate goods}}} + \underbrace{W L_{jd}}_{\substack{\text{expenditure of}\\ \text{labor input}}}}^{\text{total expenditure}}, \\
&\text{s.t.} \underbrace{\left[\left(\int_0^1 Q_{jid}^{\frac{\epsilon_m - 1}{\epsilon_m}} \mathrm{d}i\right)^{\frac{\epsilon_m}{\epsilon_m - 1}}\right]^{\alpha}}_{\substack{\text{a basket of}\\ \text{intermidiate goods input}}} \underbrace{(A_j L_{jd})^{1-\alpha}}_{\text{effective labor input}} \geq Q_{js}.
\end{aligned}
\quad \xLeftrightarrow{\text{equivelant}} \quad
\begin{aligned}
&\min_{Q_{md}, L_{jd}} M_j \equiv \overbrace{\underbrace{P_m Q_{md}}_{\substack{\text{expenditure of}\\ \text{intermidiate goods}}} + \underbrace{W L_{jd}}_{\substack{\text{expenditure of}\\ \text{labor input}}}}^{\text{total expenditure}}, \\
&\text{s.t.} \underbrace{Q_{md}^{\alpha}}_{\substack{\text{a basket of}\\ \text{intermidiate goods input}}} \underbrace{(A_j L_{jd})^{1-\alpha}}_{\text{effective labor input}} \geq Q_{js}.
\end{aligned}$$

The Lagrange multiplier is the marginal cost:

$$\mathcal{L}_j \equiv (P_m Q_{md} + W L_{jd}) + \mathrm{MC}_j \left[Q_{js} - Q_{md}^{\alpha}(A_j L_{jd})^{1-\alpha}\right].$$

The first order condition of the intermidiate good demand is:

$$\begin{aligned}
\frac{\partial \mathcal{L}_j}{\partial Q_{md}} &= 0, \\
\Rightarrow \quad \underbrace{P_m}_{\text{intermidiate good price index}} &= \alpha \mathrm{MC}_j Q_{md}^{\alpha-1}(A_j L_{jd})^{1-\alpha}, \\
&= \alpha \mathrm{MC}_j \underbrace{Q_{md}^{\alpha}(A_j L_{jd})^{1-\alpha}} Q_{md}^{-1}, \\
&= \alpha \mathrm{MC}_j \overbrace{Q_{js}} Q_{md}^{-1}, \\
\Rightarrow \quad \overbrace{Q_{md}}^{\substack{\text{a basket of intermidiate goods}\\ \text{used for the final good } j}} &= \alpha \frac{\mathrm{MC}_j}{P_m} Q_{js}, \\
\Rightarrow \quad Q_{jid}\left(\frac{P_i}{P_m}\right)^{\epsilon_m} &= \alpha \frac{\mathrm{MC}_j}{P_m} Q_{js}, \\
\Rightarrow \quad \underbrace{Q_{jid}}_{\substack{\text{manufacturer } j\text{'s demand}\\ \text{of intermidiate good } i}} &= \alpha \frac{\mathrm{MC}_j}{P_m}\left(\frac{P_i}{P_m}\right)^{-\epsilon_m} Q_{js}; \\
\Rightarrow \quad \overbrace{Q_{id}}^{\substack{\text{aggregate demand}\\ \text{of intermidiate good } i}} &\equiv \int_0^1 Q_{jid} \mathrm{d}j = \alpha \frac{\mathrm{MC}_j}{P_m}\left(\frac{P_i}{P_m}\right)^{-\epsilon_m} \int_0^1 Q_{js} \mathrm{d}j.
\end{aligned}$$

The first-order condition for the demand for labor is:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial L_{jd}} &= 0,\\
\Rightarrow\quad W &= (1-\alpha)\mathrm{MC}_j Q_{md}^{\alpha}(A_j L_{jd})^{-\alpha} A_j,\\
&= (1-\alpha)\mathrm{MC}_j Q_{md}^{\alpha}(A_j L_{jd})^{1-\alpha}\frac{1}{L_{jd}},\\
&= (1-\alpha)\mathrm{MC}_j \frac{Q_{js}}{L_{jd}},\\
\Rightarrow\quad \underbrace{L_{jd}}_{\substack{\text{manufacturer } j\text{'s}\\ \text{labor demand}}} &= (1-\alpha)\frac{\mathrm{MC}_j}{W} Q_{js},\\
\Rightarrow\quad \overbrace{\bar{L}_{jd}}^{\substack{\text{aggregate}\\ \text{labor demand}}} &\equiv \int_0^1 L_{jd}\mathrm{d}j = (1-\alpha)\frac{\mathrm{MC}_j}{W}\int_0^1 Q_{js}\mathrm{d}j.
\end{aligned}
$$

Combining the two equations in the above derivation process to further establish the marginal cost:

$$
\begin{aligned}
P_m &= \alpha \mathrm{MC}_j Q_{js} Q_{md}^{-1},\\
W &= (1-\alpha)\mathrm{MC}_j \frac{Q_{js}}{L_{jd}},\\
\Rightarrow\quad \frac{W}{P_m} &= \frac{1-\alpha}{\alpha}\frac{Q_{md}}{L_{jd}},\\
\Rightarrow\quad \frac{Q_{md}}{L_{jd}} &= \frac{\alpha}{1-\alpha}\frac{W}{P_m}.
\end{aligned}
$$

$$
\begin{aligned}
P_m &= \alpha \mathrm{MC}_j Q_{js} Q_{md}^{-1},\\
\Rightarrow\quad \mathrm{MC}_j &= \frac{1}{\alpha}\frac{P_m Q_{md}}{Q_{js}},\\
&= \frac{1}{\alpha}\frac{P_m Q_{md}}{Q_{md}^{\alpha}(A_j L_{jd})^{1-\alpha}},\\
&= \frac{1}{\alpha}\frac{P_m Q_{md}^{1-\alpha}}{(A_j L_{jd})^{1-\alpha}},\\
&= \frac{1}{\alpha}\frac{P_m}{A_j^{1-\alpha}}\left(\frac{Q_{md}}{L_{jd}}\right)^{1-\alpha},\\
&= \frac{1}{\alpha}\frac{P_m}{A_j^{1-\alpha}}\left(\frac{\alpha}{1-\alpha}\frac{W}{P_m}\right)^{1-\alpha},\\
&= \alpha^{-\alpha}(1-\alpha)^{\alpha-1}P_m^{\alpha}\left(\frac{W}{A_j}\right)^{1-\alpha},\\
\xRightarrow{\bar{\alpha}\equiv\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}}\quad \mathrm{MC}_j &= \bar{\alpha}P_m^{\alpha}\left(\frac{W}{A_j}\right)^{1-\alpha}.
\end{aligned}
$$

The optimal price of final product production department j is then

$$
P_j^{\circ} = \frac{\epsilon_f}{\epsilon_f - 1}\bar{\alpha}P_m^{\alpha}\left(\frac{W}{A_j}\right)^{1-\alpha}.
$$

iii) Optimization problem of intermediate product production department under monopolistic competition in product seller market.

Intermediate products are different. The desired decision of heterogeneous intermediate product manufacturer i is based on:

$$\max_{P_i} \Pi_i \equiv (P_i - MC_i)\overbrace{Q_{id}}^{\substack{\text{aggregate demand}\\ \text{of intermedaite good } i}},$$

$$\text{s.t.}\quad Q_{id} = \alpha\frac{\mathrm{MC}_j}{P_m}\left(\frac{P_i}{P_m}\right)^{-\epsilon_m}\underbrace{\int_0^1 Q_{js}\mathrm{d}j}_{\substack{\text{aggregate supply}\\ \text{of the final good}}},$$

$$\overbrace{Q_{is}}^{\substack{\text{aggregate supply}\\ \text{of intermedaite good } i}} = AL_{id},$$

$$Q_{id} = Q_{is}.$$

Given the marginal cost, $\mathrm{MC}_i = \frac{W}{A_i}$, without paying attention to supply, the equality-constrained optization is transformed into the unconstrained optimization

$$\max_{P_i} \Pi_i \equiv (P_i - MC_i)\alpha\frac{\mathrm{MC}_j}{P_m}\left(\frac{P_i}{P_m}\right)^{-\epsilon_m}\int_0^1 Q_{js}\mathrm{d}j.$$

$$\xRightarrow{\text{F.O.C.}}\quad 0 = \frac{\mathrm{d}\Pi_i}{\mathrm{d}P_i} = (1-\epsilon_m)\alpha\frac{\mathrm{MC}_j}{P_m}\left(\frac{P_i}{P_m}\right)^{-\epsilon_m}\int_0^1 Q_{js}\mathrm{d}j + \epsilon_m MC_i\alpha\frac{\mathrm{MC}_j}{P_m}\left(\frac{P_i}{P_m}\right)^{-\epsilon_m-1}\frac{1}{P_m}\int_0^1 Q_{js}\mathrm{d}j,$$

$$\Rightarrow\quad 0 = (\epsilon_m - 1)\alpha\frac{\mathrm{MC}_j}{P_m}\left(\frac{P_i}{P_m}\right)^{-\epsilon_m}\int_0^1 Q_{js}\mathrm{d}j - \epsilon_m MC_i\alpha\frac{\mathrm{MC}_j}{P_m}\left(\frac{P_i}{P_m}\right)^{-\epsilon_m-1}\frac{1}{P_m}\int_0^1 Q_{js}\mathrm{d}j,$$

$$\Rightarrow\quad P_i = \frac{\epsilon_m}{\epsilon_m - 1}MC_i = \frac{\epsilon_m}{\epsilon_m - 1}\frac{W}{A_i}.$$

In this way, we obtain the desired pricing equation for final product and intermediate product manufacturers both in a monopolistic competition market environment.

The above three examples connect the production sector and the household sector at different production stages, but they are still partial solutions. Combining the price index and the pricing equation, the aggregate supply curve or Phillips curve can be further derived; combining the aggregate supply and aggregate demand curves, monetary policy analysis can also be performed, which will be presented in the book [55]. In addition, if partial nominal rigidity is considered, the static optimization problem will be transformed into a dynamic optimization problem.

1.2 Dynamic Optimization Problem

The salient feature of a dynamic model is that it describes the relationship between variables at different times. Intuitively, all variables in a dynamic model have a variable t that represents time, but the reverse is not necessarily true. The key point is that there is a cross-period relationship between variables.

First, it should be noted that dynamic models can be divided into discrete time models and continuous time models.

(1) Discrete-time dynamics refers to the change of variables in a certain period of time, so $t = 0, 1, 2, 3, \ldots$, for example:

$$\begin{aligned}
\text{Savings growth rate: } S_{t+1} = (1+r)S_t &\quad\Leftrightarrow\quad S_{t+1} - S_t = rS_t &\quad\Leftrightarrow\quad \frac{\Delta S_t/\Delta t}{S_t} = r,\\
\text{Technology growth rate: } A_{t+1} = (1+g_a)A_t &\quad\Leftrightarrow\quad A_{t+1} - A_t = g_a A_t &\quad\Leftrightarrow\quad \frac{\Delta A_t/\Delta t}{A_t} = g_a,\\
\text{Labor growth rate: } L_{t+1} = (1+g_l)L_t &\quad\Leftrightarrow\quad L_{t+1} - L_t = g_l L_t &\quad\Leftrightarrow\quad \frac{\Delta L_t/\Delta t}{L_t} = g_l,\\
\text{Capital movement law: } K_{t+1} = I_t + (1-\delta)K_t &\quad\Leftrightarrow\quad K_{t+1} - K_t = I_t - \delta K_t &\quad\Leftrightarrow\quad \frac{\Delta K_t/\Delta t}{K_t} = \frac{I_t}{K_t} - \delta.
\end{aligned}$$

(2) Continuous-time dynamics means that the variable changes at every time point, $0, \delta, 2\delta, \ldots$, where $\delta > 0$. Δt is still used to represent any (small) time interval:

$$\left.\begin{array}{l} \text{Continuous compound growth} \begin{cases} S_{t+\Delta t} = e^{r\Delta t} S_t, \\ A_{t+\Delta t} = e^{g_a \Delta t} A_t, \\ L_{t+\Delta t} = e^{g_l \Delta t} L_t, \end{cases} \\ \text{Continuous investment growth}\ K_{t+\Delta t} - K_t = \Delta t I_t - \Delta t \delta K_t. \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dfrac{S_{t+\Delta t} - S_t}{\Delta t} = \dfrac{e^{r\Delta t} - 1}{\Delta t} S_t, \\ \dfrac{A_{t+\Delta t} - A_t}{\Delta t} = \dfrac{e^{g_a \Delta t} - 1}{\Delta t} A_t, \\ \dfrac{L_{t+\Delta t} - L_t}{\Delta t} = \dfrac{e^{g_l \Delta t} - 1}{\Delta t} L_t, \\ \dfrac{K_{t+\Delta t} - K_t}{\Delta t} = I_t - \delta K_t. \end{array}\right\} \xRightarrow{\Delta t \to 0} \begin{cases} \dot{S}(t) \equiv \dfrac{\mathrm{d}S(t)/\mathrm{d}t}{S(t)} = r, \\ \dot{A}(t) \equiv \dfrac{\mathrm{d}A(t)/\mathrm{d}t}{A(t)} = g_a, \\ \dot{L}(t) \equiv \dfrac{\mathrm{d}L(t)/\mathrm{d}t}{L(t)} = g_l, \\ \dot{K}(t) \equiv \dfrac{\mathrm{d}K(t)/\mathrm{d}t}{K(t)} = \dfrac{I(t)}{K(t)} - \delta. \end{cases}$$

The above derivation uses Taylor's first-order approximation or l'HÔpital rule, that is,

$$f(\Delta t) = e^{x\Delta t} \approx f(0) + f'(0)\frac{\mathrm{d}(x\Delta t)}{\mathrm{d}\Delta t}(\Delta t - 0) = 1 + x\Delta t \quad \text{or} \quad \lim_{\Delta t \to 0} \frac{e^{x\Delta t} - 1}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}\Delta t} e^{x\Delta t}}{(\Delta t)'} = \lim_{\Delta t \to 0} \frac{x}{1} = x.$$

In discrete time model, $\Delta X_t \equiv X_{t+1} - X_t$ $\Delta t \equiv (t+1) - t = 1$, where Δ represents the differential operator. Δt can also be used to represent any time interval, and when $\Delta t \longrightarrow 0$, the problem is converted to a continuous-time problem, that is, $\dot{X}(t) \equiv \frac{\mathrm{d}X(t)}{\mathrm{d}t} \equiv \lim_{\Delta t \to 0} \frac{\Delta X_t}{\Delta t}$ d represents the differential operator. t in discrete time problems often appears in the form of a right subscript, and t in continuous time problems is usually represented by brackets.

Furthermore, it is not difficult to imagine that the constraints of dynamic models are divided into endpoint constraints and process constraints.

(1) Endpoint constraints mainly refer to the initial and final conditions of the decision, which can be constants (fixed) or parameters (variable). The final conditions can be cross-sectional according to time or state. For simplicity, the initial and final conditions are given in the dynamic optimization problem introduced below. More specifically, the initial state and final state are often set to be 0.

(2) Process constraints may occur between decision variables and state variables, between decision variables, or between state variables. We will see later that from the perspective of establishing a close connection between multivariate static optimization and discrete dynamic optimization, the constraints between multiple static decision variables can be formally transformed into constraints between dynamic decision variables and state variables, so this type of constrained optimal problem can be directly solved by the Lagrangian method, but it is more convenient to use the optimal control method of the Hamiltonian function in continuous time. If there are still constraints between different decision variables in the dynamic optimization, the Lagrangian method is used on the basis of the Hamiltonian method. The following explains that the dynamic optimal solutions for discrete time and continuous time have different focuses. The former focuses on the decision variables from two periods to multiple periods, and the latter focuses on the state variables with no constraints and with constraints.

1.2.1 Discrete time

1.2.1.1 Two-period decision-making under perfect expectations

In the static optimization problem with two-variable equality constraints, x and y are used to represent the two selection variables.To better compare with the two-period discrete time problem, in the static optimization problem, x and y are changed to x_1 and x_2 to represent the two choice variables in the static state. In the dynamic problem, x_1 and x_2 are still used to represent the two selection variables, but the subscript has the meaning of "time", which means the selection variables at time $t = 1$ and $t = 2$.

$$\left.\begin{array}{ll} \xleftrightarrow[\min]{\max} & O = f(x_1, x_2), \\ \text{s.t.} & g(x_1, x_2) = z. \end{array}\right\} \text{one-period statics} \quad \overset{\text{vs.}}{\Longleftrightarrow} \quad \text{two-period discrete dynamics} \left\{\begin{array}{ll} \xleftrightarrow[\min]{\max} & O = f(x_1, x_2), \\ \text{s.t.} & g(x_1, x_2) = z. \end{array}\right.$$

After comparison, it is not difficult to find that although the meanings of the symbol subscripts are different, the two-period dynamic optimization problem in discrete time is completely consistent with the static optimization problem with two-variable equality constraints in form. Therefore, the elimination method, total differentiation method and Lagrange multiplier method introduced earlier are still effective.

Example 12-1. **Two-period consumption decision in an endowment economy**

$$\max_{C_1,C_2} U = u(C_1) + \beta u(C_2),$$

$$\text{s.t.} \quad \underbrace{\left.\begin{aligned} \overbrace{C_1 + S_1}^{\text{current-period expenditure}} &\le \overbrace{[r + (1-\delta)]S_0 + Q_1}^{\text{current-period income}}, \\ \underbrace{C_2 + S_2}_{\text{next-period expenditure}} &\le \underbrace{[r + (1-\delta)]S_1 + Q_2}_{\text{next-period income}}. \end{aligned}\right\}}_{\text{budget constrain of two periods}} \xrightarrow[C_1,C_2>0]{S_0=0=S_2} \underbrace{\overbrace{C_1 + \frac{C_2}{1+r}}^{\text{lifetime expenditure}} = \overbrace{Q_1 + \frac{Q_2}{1+r}}^{\text{lifetime income}}}_{\text{lifetime budget constraint}}.$$

Note:

(1) The utility function is assumed to be in a divisible and additive form, that is, $U(C_1, C_2) = u_1(C_1) + u_2(C_2)$;

(2) The utility at different times is different for different moments, and the utility of the second period is measured by the subjective discount factor $0 < \beta \equiv \frac{1}{1+\rho} < 1$ relative to the first period, where $\rho > 0$ is the subjective discount rate. The divisible and additive utility function considering the subjective discount rate is $U(C_1, C_2) = u(C_1) + \beta u(C_2)$;

(3) U represents the lifetime utility function based on the beginning of the first period; $u(\cdot)$ represents the immediate concave utility function, that is, $u'(\cdot) > 0$, $u''(\cdot) < 0$.

(4) An endowment economy means that the income of the two periods is given, so Q_1 and Q_2 are both exogenous and are marked in gray. From the perspective of the first period, Q_2 has not yet occurred. This implies the assumption of perfect expectations, i.e., $\mathbb{E}_1 Q_2 = Q_2$. $\mathbb{E}$ represents the expectation operator, and the subscript 1 represents the information set of the first period (if there are multiple periods, it represents the information set at the first period).

(5) Assuming that the Inada condition is satisfied, that is, there will always be consumption in each period and no resources will be wasted, this means that the inequality constraint of the budget constraint in each period are all tight, and the inequality constraints are transformed into equality constraints (this will be clearer in the inequality constraint optimization problem introduced later).

(6) Since only two periods are considered, there will be no savings before the first period and no savings in the second period, so $S_0 = 0 = S_2$, which is also the initial condition and the final condition. However, in addition to the endowment income in the second period, the principal and interest generated by the savings in the first period after depreciation in the second period will be consumed together, and the actual interest rate is r, which is determined by the market price. For the household sector, it is equivalent to an exogenous variable, also marked in gray. For simplicity, it is assumed that savings are not depreciated, that is, $\delta = 0$. If 1 unit of consumer goods in the first period is used as the measurement scale, the real price of consumer goods in the second period is $\frac{1}{1+r}$; if 1 unit of consumer goods in the second period is used as the measurement scale, the real price of the consumer goods in the first period is$1 + r$ To understand these two sets of relative real prices from another perspective, the value of consumer goods at different times should not be directly compared, so a certain time can be selected as the comparison benchmark. If the first period is selected as the benchmark, assuming that the real price of consumer goods in the first period is 1, then the real price of consumption in the second period is equivalent to $\frac{1}{1+r}$from the perspective of the first period, which is the present value in finance; if the second period is selected as the benchmark, assuming that the real price of consumer goods in the second period is 1, then the real price of consumer goods in the first period should be $(1 + r)$ in the second period. In the above model, the real prices of C_1 and Q_1 are set to 1, and the real prices C_2 and Q_2 are set to $\frac{1}{1+r}$.

(7) The flow budget constraints of period 1 and period 2 are combined through the intermediate link S_1 to obtain the inter-period lifetime budget constraint (right).

(8) In this two-period optimization problem, the choice variable can be C_1, that is, when C_1 is determined, S_1 can

be determined according to the budget constraint of the first period, and C_2 can be determined according to the budget constraint of the second period; the choice variable can be C_2, that is, when C_2 is determined, S_1 can be determined according to the budget constraint of the second period, and C_1 can be determined according to the budget constraint of the first period; the choice variable can also be S_1, that is, when S_1 is determined, C_1 can be determined according to the budget constraint of the first period, and C_2 can be determined according to the budget constraint of the second period. The choice variable is also called the control variable, and the control variable S_1 can also be called the endogenous state variable, and S_0 is called the initial state variable or the predetermined state variable.

i) **Elimination method**

Elimination method 1. Eliminate C_2 in the objective function according to the lifetime budget constraint, and only C_1 is left for the choice variable, which becomes an unconstrained optimization problem:

$$\max_{C_1} U = u(C_1) + \beta u \underbrace{[(1+r)(Q_1 - C_1) + Q_2]}_{C_2}.$$

The first-order necessary condition is:

$$\frac{\mathrm{d}U}{\mathrm{d}C_1} = 0 \quad \Rightarrow \quad \overbrace{\underbrace{\frac{\beta u'(C_2)}{u'(C_1)}}_{\text{MRS}}} = \overbrace{\underbrace{\frac{1}{1+r}}_{\text{relative price}}}.$$

The second-order sufficient condition is:

$$\frac{\mathrm{d}^2U}{\mathrm{d}C_1^2} = \frac{\mathrm{d}}{\mathrm{d}C_1}[u'(C_1) - \beta(1+r)u'(C_2)] = u''(C_1) + \beta^2(1+r)^2u''(C_2) < 0.$$

It can be seen that in order to ensure that the utility function reaches the maximum value, it should be hold that $u''(C_1) < 0$ and $u''(C_2) < 0$. The second-order derivative of the utility function is less than 0 everywhere, so the immediate utility function is a concave function, and the lifetime utility function is the linear addition of the immediate utility function, which is also a concave function. Therefore, the local maximum is also the global maximum.

Elimination method 2. Eliminate C_1 in the objective function according to the lifetime budget constraint, and only C_2 is left for the choice variable, which becomes an unconstrained optimization problem:

$$\max_{C_2} U = u \underbrace{\left(Q_1 + \frac{Q_2}{1+r} - \frac{C_2}{1+r}\right)}_{C_1} + \beta u(C_2).$$

The first-order necessary conditions are:

$$\frac{\mathrm{d}U}{\mathrm{d}C_2} = 0 \quad \Rightarrow \quad \overbrace{\underbrace{\frac{\beta u'(C_2)}{u'(C_1)}}_{\text{MRS}}} = \overbrace{\underbrace{\frac{1}{1+r}}_{\text{realtive price}}}.$$

Elimination method 3. Since there are two forms of flow budget constraints and lifetime budget constraints, in addition to solving it according to the elimination method of a lifetime budget constraint, it can also be solved according to two flow budget constraints:

$$\left.\begin{aligned} C_1 &= \cancel{[r + (1-\delta)]S_0} + Q_1 - S_1, \\ C_2 &= [r + (1-\delta)]S_1 + Q_2 - \cancel{S_2}. \end{aligned}\right\} \quad \Rightarrow \quad \max_{S_1} U = u \overbrace{(Q_1 - S_1)}^{C_1} + \beta u \overbrace{([r + (1-\cancel{\delta})]S_1 + Q_2)}^{C_2}.$$

The first-order necessary condition is:

$$\frac{\mathrm{d}U}{\mathrm{d}S_1} = 0 \quad \xRightarrow{-u'(C_1)+(1+r)\beta u'(C_2)=0} \quad \underbrace{\overbrace{\frac{\beta u'(C_2)}{u'(C_1)}}}_{\text{MRS}} = \underbrace{\overbrace{\frac{1}{1+r}}}_{\text{relative price}}.$$

The second-order conditions of elimination methods 2 and 3 can be obtained by taking the second-order derivatives of the corresponding choice variables.

ii) **Differentiation method**

On the C_1-C_2 plane, the slope of the indifference curve or isoutility curve ($\mathrm{d}U = 0$) is:

$$0 = u'(C_1)\mathrm{d}C_1 + \beta u'(C_2)\mathrm{d}C_2 \quad \Rightarrow \quad \frac{\mathrm{d}C_2}{\mathrm{d}C_1} = -\frac{u'(C_1)}{\beta u'(C_2)}.$$

On the C_1-C_2 plane, when the incomes of two periods ($\mathrm{d}Y_1 = 0 = \mathrm{d}Y_2$) are fixed, the slope of the dynamic budget constraint line is:

$$\mathrm{d}C_1 + \frac{1}{1+r}\mathrm{d}C_2 = 0 + 0 \quad \Rightarrow \quad \frac{\mathrm{d}C_2}{\mathrm{d}C_1} = -(1+r).$$

At the local or global maximum, the two are equal, resulting in the same first-order necessary condition:

$$\frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r}.$$

iii) **Multiplier method**

Constructing the Lagrangian function, the Lagrange multiplier λ is added into the choice variables in the optimization problem:

$$\max_{C_1,C_2,\lambda} \mathcal{L} \equiv [u(C_1) + \beta u(C_2)] + \lambda\left[\left(Q_1 + \frac{Q_2}{1+r}\right) - \left(C_1 + \frac{C_2}{1+r}\right)\right].$$

The first-order necessary conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_1} &= u'(C_1) - \lambda = 0,\\
\frac{\partial \mathcal{L}}{\partial C_2} &= \beta u'(C_2) - \lambda\frac{1}{1+r} = 0,\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= \left(Q_1 + \frac{Q_2}{1+r}\right) - \left(C_1 + \frac{C_2}{1+r}\right) = 0.
\end{aligned}$$

The third first-order condition is the dynamic budget constraint. Eliminating λ from the first two first-order conditions, we merge them to:

$$\frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r} \quad \Leftrightarrow \quad u'(C_1) = \beta(1+r)u'(C_2).$$

This is the well-known intertemporal Euler equation in dynamic macroeconomics. It states that consuming one unit less in the first period will reduce the marginal utility of the first period $u'(C_1)$, but the reduced consumption will increase income in the second period by $1+r$ after being saved, which will bring about an increase in marginal utility $\beta(1+r)u'(C_2)$ with the first period as the base period. At the optimal point, the two are equal (marginal cost equals marginal benefit).

We can also construct the Lagrangian function through the Lagrange multiplier μ according to the dual problem:

$$\min_{C_1,C_2} E = C_1 + \frac{C_2}{1+r},$$

$$\text{s.t.} \quad u(C_1) + \beta u(C_2) \geq U \quad \xRightarrow{\text{Inada conditions}} \quad u(C_1) + \beta u(C_2) = U.$$

$$\Rightarrow \quad \min_{C_1,C_2,\mu} \mathcal{L} \equiv \left(C_1 + \frac{C_2}{1+r}\right) + \mu\{U - [u(C_1) + \beta u(C_2)]\}.$$

The first-order necessary conditions are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_1} &= 1 - \mu u'(C_1) = 0, \\
\frac{\partial \mathcal{L}}{\partial C_2} &= \frac{1}{1+r} - \mu \beta u'(C_2) = 0, \\
\frac{\partial \mathcal{L}}{\partial \mu} &= U - [u(C_1) + \beta u(C_2)] = 0.
\end{aligned}
$$

The third first-order condition is to fix the utility level. The first two first-order conditions eliminate μ and merge into:

$$
\frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r}.
$$

The first-order conditions of the original problem and the dual problem are exactly the same. However, it is easy to find that the Lagange multipliers λ and μ are reciprocals, that is, $\lambda\mu = 1$.

Example 12-2. **Two-period consumption decision of a production economy** [5]

$$
\max_{C_1, C_2} U = u(C_1) + \beta u(C_2),
$$

$$
\text{s.t.} \quad \underbrace{\overbrace{\left.\begin{aligned} C_1 + K_2 &\le A_1 K_1, \\ C_2 + K_3 &\le A_2 K_2. \end{aligned}\right\}}^{\text{budget constraints of two periods}}}_{K_2 = I_1 + (1-\delta)K_1 \overset{\delta=1}{\Rightarrow} K_2 = I_1} \quad \xRightarrow[C_1, C_2 > 0]{K_1 \text{ given}, K_3 = 0} \quad \left\{\begin{aligned} C_1 + K_2 &= A_1 K_1, \\ C_2 + \ 0 \ &= A_2 K_2. \end{aligned}\right\} \Rightarrow \underbrace{\overbrace{C_1 + \frac{C_2}{A_2}}^{\text{lifetime expenditure}} = \overbrace{A_1 K_1}^{\text{lifetime income}}}_{\text{lifetime budget constraint}}.
$$

Constructing the Lagrangian function, the Lagrange multiplier λ is added into the choice variables in the optimization problem:

$$
\max_{C_1, C_2, \lambda} \mathcal{L} \equiv [u(C_1) + \beta u(C_2)] + \lambda \left(A_1 K_1 - C_1 - \frac{C_2}{A_2} \right).
$$

The first-order necessary conditions are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_1} &= u'(C_1) - \lambda = 0, \\
\frac{\partial \mathcal{L}}{\partial C_2} &= \beta u'(C_2) - \lambda \frac{1}{A_2} = 0, \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= A_1 K_1 - C_1 - \frac{C_2}{A_2} = 0.
\end{aligned}
$$

Slightly tidy up:

$$
\left.\begin{aligned} \frac{\beta u'(C_2^\circ)}{u'(C_1^\circ)} &= \frac{1}{A_2}, \\ C_1 + \frac{C_2}{A_2} &= A_1 K_1. \end{aligned}\right\} \xRightarrow[u(C_t) = \log C_t]{u(C_t) = \frac{C_t^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}}}
$$

$$
\text{CES} \left\{\begin{aligned} \left(\frac{C_2^\circ}{C_1^\circ}\right)^{-\frac{1}{\sigma}} &= \frac{1}{\beta A_2}, \\ A_2 C_1 + C_2 &= A_1 A_2 K_1. \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\left\{\begin{aligned} C_1^\otimes &= \frac{\left(\frac{1}{\beta A_2}\right)^\sigma A_2}{1+\left(\frac{1}{\beta A_2}\right)^\sigma A_2} A_1 K_1, \\ C_2^\otimes &= \frac{A_2}{1+\left(\frac{1}{\beta A_2}\right)^\sigma A_2} A_1 K_1. \end{aligned}\right. \\ &\xRightarrow{Y_1 = A_1 K_1} \left\{\begin{aligned} I_1^\otimes &= A_1 K_1 - C_1^\otimes = \frac{1}{1+\left(\frac{1}{\beta A_2}\right)^\sigma A_2} A_1 K_1, \\ K_2^\otimes &= I_1^\otimes, \\ Q_2^\otimes &= A_2 K_2^\otimes = \frac{A_2}{1+\left(\frac{1}{\beta A_2}\right)^\sigma A_2} A_1 K_1. \end{aligned}\right. \end{aligned}\right. .
$$

$$
\text{logarithm} \left\{\begin{aligned} C_2^\circ &= \beta A_2 C_1^\circ, \\ A_2 C_1 + C_2 &= A_1 A_2 K_1. \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\left\{\begin{aligned} C_1^\otimes &= \frac{1}{1+\beta} A_1 K_1, \\ C_2^\otimes &= \frac{\beta A_2}{1+\beta} A_1 K_1. \end{aligned}\right. \\ &\xRightarrow{Q_1 = A_1 K_1} \left\{\begin{aligned} I_1^\otimes &= A_1 K_1 - C_1^\otimes = \frac{\beta}{1+\beta} A_1 K_1, \\ K_2^\otimes &= I_1^\otimes, \\ Q_2^\otimes &= A_2 K_2^\otimes = \frac{\beta A_2}{1+\beta} A_1 K_1. \end{aligned}\right. \end{aligned}\right.
$$

[5] This example refers to Moll (2023) but adds the logarithm utility case to make it more clear.

The above lists the first-order conditions corresponding to the current-value utility function when the intertemporal substitution elasticity is not 1 (CES) and is 1 (logarithm). Combined with the constraints (which can also be said to be the components of the first-order conditions), the optimal consumption in the first period C_1, the optimal consumption in the second period C_2, the optimal investment in the first period I_1 , the optimal capital in the second period K_2 and the optimal output in the second period Y_2 can be solved. Since the capital in the first period K_1 is given, the output in the first period Y_1 is directly given. But in general, they are all functions of exogenous technical variables A_1 and (or) A_2.

Note that the optimal solution is a type of equilibrium, called the goal equilibrium. Therefore, the content of this chapter is a special case of the next chapter "Equilibrium Solution". In addition, these optimal solutions are explicit solutions (also called analytical solutions) , whose characteristics are that each endogenous variable is completely determined by the given initial state variables K_1 or exogenous variables (A_1, A_2) or parameters (σ, β). Initial state variables, exogenous variables or parameters can be collectively classified as exogenous variables, and this form of solution is also called a "simplified equation". If the goal equilibrium is a simplified equation and the corresponding function is continuously differentiable, the impact of changes in exogenous variables (also generally referred to as "shocks") on endogenous variables can be analyzed through calculus methods (that is, "comparative analysis". A systematic introduction to equilibrium solution and comparative analysis will be left after this chapter.

1.2.1.2 Multi-period decision-making under perfect expectations

In dynamics, we expand the scope from two-period questions to multi-period questions. Multi-period questions can still be divided into questions with finite periods and questions with infinite periods. Finite periods are represented by m or T, and $m \to \infty$ or $T \to \infty$ represents infinite periods.

I. **Finite periods**

In the static optimization problems with multiple variables and multiple equality constraints, $x_1, x_2, \dots, x_m$ is used to represent m choice variables. In the discrete time dynamic optimization problems with m periods, $x_1, x_2, \dots, x_m$ are used to represent m choice variables, but the subscripts represent the time $t = 1, t = 2, \dots, t = m$. It is not difficult to find that multi-variable static problems and multi-period discrete dynamic prbolems have great similarities. In order to avoid confusion, it is better to change the symbol m to T.

$$
\left.\begin{array}{l}
\left.\begin{aligned}
&\xleftrightarrow[\min]{\max} O = f(x_1, x_2, \dots, x_m),\\
&\text{s.t.}\quad g(x_1, x_2, \dots, x_m) = z.
\end{aligned}\right\}\\
\left.\begin{aligned}
&\xleftrightarrow[\min]{\max} O = f(x_1, x_2, \dots, x_m),\\
&\text{s.t.}\quad g^1(x_1, x_2, \dots, x_m) = z_1,\\
&\qquad\; g^2(x_1, x_2, \dots, x_m) = z_2,\\
&\qquad\qquad \vdots\\
&\qquad\; g^n(x_1, x_2, \dots, x_m) = z_n.
\end{aligned}\right\}
\end{array}\right\}
\begin{array}{c}\text{multi-variable}\\ \text{static}\end{array}
\quad \text{vs.} \quad
\begin{array}{c}\text{multi-period}\\ \text{discrete dynamic}\end{array}
\left\{\begin{array}{l}
\left\{\begin{aligned}
&\xleftrightarrow[\min]{\max} O = f(x_1, x_2, \dots, x_T),\\
&\text{s.t.}\quad g(x_1, x_2, \dots, x_T) = z.
\end{aligned}\right.\\
\left\{\begin{aligned}
&\xleftrightarrow[\min]{\max} O = f(x_1, x_2, \dots, x_T),\\
&\qquad \text{s.t.}\quad g^1(x_1, x_2) = z_1,\\
&\qquad\qquad\; g^2(x_2, x_3) = z_2,\\
&\qquad\qquad\qquad \vdots\\
&\qquad\quad g^{T-1}(x_{T-1}, x_T) = z.
\end{aligned}\right.
\end{array}\right.
$$

Although the meanings of the subscripts are different, the discrete-time T-period dynamic optimization problem and the m-variable static optimization problem with single or multiple equality constraints are still quite similar in form. The difference is that in the static model, problems with single or multiple equality constraints are two different types of optimization problems, while in the dynamic model, multiple equality constraints may be transformed to one constraint, as shown in the following example.

Example13-0. **Finite-period consumption decision of an endowment economy**[6]

$$\left.\begin{aligned} \xleftrightarrow[\min]{\max} U &= u(\mathbf{C}), \\ \text{s.t.} \quad \mathbf{P}\cdot\mathbf{C} &\leq \mathbf{P}\cdot\mathbf{Q}, \\ \mathbf{C} &\geq 0. \end{aligned}\right\} \begin{array}{c}\text{contemporaneous}\\ \text{discrete state}\end{array} \quad \overset{\text{vs.}}{\Longleftrightarrow} \quad \begin{array}{c}\text{multi-period}\\ \text{discrete dynamic}\end{array} \left\{\begin{aligned} \xleftrightarrow[\min]{\max} U &= u(\mathbf{C}), \\ \text{s.t.} \quad \mathbf{P}\cdot\mathbf{C} &\leq \mathbf{P}\cdot\mathbf{Q}, \\ \mathbf{C} &\geq 0. \end{aligned}\right.$$

where: $\mathbf{P}$ and $\mathbf{Q}$ are the exogenous price vector and exogenous income vector respectively. According to the above Inada condition, the consumption vector $\mathbf{C}$ is always positive, so the inequality constraint problem can be avoided first:

$$\left.\begin{aligned} \xleftrightarrow[\min]{\max} U &= u(\mathbf{C}), \\ \text{s.t.} \quad \mathbf{P}\cdot\mathbf{C} &= \mathbf{P}\cdot\mathbf{Q}. \end{aligned}\right\} \begin{array}{c}\text{contemporaneous}\\ \text{discrete state}\end{array} \quad \overset{\text{vs.}}{\Longleftrightarrow} \quad \begin{array}{c}\text{multi-period}\\ \text{discrete dynamic}\end{array} \left\{\begin{aligned} \xleftrightarrow[\min]{\max} U &= u(\mathbf{C}), \\ \text{s.t.} \quad \mathbf{P}\cdot\mathbf{C} &= \mathbf{P}\cdot\mathbf{Q}. \end{aligned}\right.$$

No matter it is a single-period static problem($\mathbf{X} = \{\mathbf{X}_i\}, \mathbf{X} = \{\mathbf{C}, \mathbf{P}, \mathbf{Q}\}, i = 1, 2, \ldots, m$) or a multi-period dynamic problem ($\mathbf{X} = \{\mathbf{X}_t\}, \mathbf{X} = \{\mathbf{C}, \mathbf{P}, \mathbf{Q}\}, t = 1, 2, \ldots, T$), this is a standard equality-constrained optimization problem, which can be solved by constructing a Lagrangian function:

$$\begin{aligned} & \max_{\mathbf{C}} \mathcal{L} \equiv u(\mathbf{C}) + \lambda \mathbf{P}\cdot(\mathbf{Q} - \mathbf{C}). \\ \xRightarrow{\text{F.O.C.}} \quad & \nabla u(\mathbf{C}) = \lambda \mathbf{P}, \\ \Rightarrow \quad & \mathbf{C} = \mathbf{C}(\lambda \mathbf{P}), \\ \xRightarrow{\text{substitute into the constraint}} \quad & \mathbf{P}\cdot\mathbf{C}(\lambda\mathbf{P}) = \mathbf{P}\cdot\mathbf{Q}, \\ \Rightarrow \quad & \lambda = \lambda(\mathbf{P}, \mathbf{Q}), \\ \Rightarrow \quad & \mathbf{C}^\circ = \mathbf{C}^\circ[\lambda(\mathbf{P}, \mathbf{Q})\mathbf{P}] = \mathbf{C}^\circ(\mathbf{P}, \mathbf{Q}). \end{aligned}$$

The price level has both a direct effect on consumption, namely the substitution effect, and an indirect effect, namely the income effect (and the wealth effect in the dynamics), which is generated through the Lagrange multiplier λ The substitution effect, income effect, and wealth effect of price changes are essentially the subject of comparative analysis, which will be discussed in Chapter 3.

For the discrete-time dynamic problem, the first-order condition can also be expressed as:

$$u'(C_t) = \lambda \mathbf{P}, \quad t = 1, 2, \ldots$$

The ratio of the marginal utilities of two consecutive periods represents the marginal rate of substitution. The optimal marginal rate of substitution is equal to the price ratio (relative price) of two consecutive periods:

$$\frac{u'(C_{t+1})}{u'(C_t)} = \frac{P_{t+1}}{P_t}.$$

It is a first-order difference equation about consumption (also known as the intertemporal Euler equation on consumption).

Using the divisible and additive lifetime utility function, the above formula can be rewritten as:

$$\begin{aligned} U_1(\mathbf{C}) &= \sum_{t=1}^{T} \beta^{t-1} u(C_t), \\ \Rightarrow \quad \frac{\beta u'(C_{t+1})}{u'(C_t)} &= \frac{P_{t+1}}{P_t}. \end{aligned}$$

After adding the subjective discount factor β , the total utility is the sum of future utilities discounted to the present (period 1) (i.e., U_1)

The inter-period consumption Euler equation is also obtained here, and its economic meaning is the same as that in the two-period consumption decision, except that the balance of consumption and savings between the first and second periods becomes the balance of consumption and savings between any two periods.

[6]This example is adapted fromEdmond (2019).

Example 13-1. **Finite-period consumption decision in an endowment economy**

The finite-period objective function is still assumed to be separable and additive, and the liquidity budget constraints of each period can be written as inter-period lifetime budget constraints. Therefore, the finite-period optimization problem is:

$$\max_{C_1, C_2, \ldots, C_T} U_1 = u(C_1) + \beta u(C_2) + \cdots + \beta^{T-1} u(C_T),$$

$$\text{s.t.} \quad \underbrace{\overbrace{\left.\begin{aligned} C_1 + S_1 &\le (1+r)S_0 + Q_1, \\ C_2 + S_2 &\le (1+r)S_1 + Q_2, \\ &\vdots \\ C_T + S_T &\le (1+r)S_{T-1} + Q_t, \\ 0 &\le C_1, C_2, \ldots, C_T. \end{aligned}\right\}}}_{\text{budget constraint of each period}} \xRightarrow[C_1, \ldots, C_T > 0]{S_0 = 0 = S_T} \underbrace{\overbrace{C_1 + \frac{C_2}{1+r} + \cdots + \frac{C_T}{(1+r)^{T-1}}}^{\text{lifetime expenditure}} = \overbrace{Q_1 + \frac{Q_2}{1+r} + \cdots + \frac{Q_t}{(1+r)^{T-1}}}^{\text{lifetime income}}}_{\text{finite-period lifetime budget constraint}}.$$

The above multiple budget constraints can be rewritten into a single inter-period lifetime budget constraint similar to the two-period case. Solving C_T and substituting it into the objective function, the finite-period dynamic optimization problem with equality constraints becomes an unconstrained optimization problem. Taking the partial derivative of $C_1, C_2, \ldots, C_{T-1}$, we can get $T-1$ first-order conditions. However, when there are many variables, it may be easier to construct a Lagrangian function to solve:

$$\max_{C_1, C_2, \ldots, C_T, \lambda} \mathcal{L} \equiv \overbrace{\left[u(C_1) + \beta u(C_2) + \cdots + \beta^{T-1} u(C_T)\right]}^{\text{lifetime utility}} + \lambda \overbrace{\left[\left(Q_1 + \frac{Q_2}{1+r} + \cdots + \frac{Q_t}{(1+r)^{T-1}}\right) - \left(C_1 + \frac{C_2}{1+r} + \cdots + \frac{C_T}{(1+r)^{T-1}}\right)\right]}^{\text{lifetime budget constraint}}.$$

The first-order necessary conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_1} &= u'(C_1) - \lambda = 0, \\
\frac{\partial \mathcal{L}}{\partial C_2} &= \beta u'(C_2) - \lambda \frac{1}{1+r} = 0, \\
\frac{\partial \mathcal{L}}{\partial C_3} &= \beta^2 u'(C_3) - \lambda \left(\frac{1}{1+r}\right)^2 = 0, \\
&\vdots \\
\frac{\partial \mathcal{L}}{\partial C_T} &= \beta^{\beta-1} u'(C_T) - \lambda \left(\frac{1}{1+r}\right)^{T-1} = 0, \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= \left(Q_1 + \frac{Q_2}{1+r} + \cdots + \frac{Q_m}{(1+r)^{T-1}}\right) - \left(C_1 + \frac{C_2}{1+r} + \cdots + \frac{C_T}{(1+r)^{T-1}}\right) = 0.
\end{aligned}$$

The last one is the repetition of lifetime expectation constraint. The first m first-order conditions are combined into the consumption Euler equation for any two consecutive periods:

$$\frac{\beta u'(C_{t+1})}{u'(t)} = \frac{1}{1+r}, \qquad t = 1, 2, \ldots, T.$$

Since t is used to represent a discrete moment, we can also replace the objective function and flow budget constraint with t:

$$\left.\begin{aligned}&\max_{\{C_t\}_{t=1}^{T}} U_1 \equiv \sum_{t=1}^{T} \beta^{t-1} u(C_t),\\ &\text{s.t.} \quad \left.\begin{aligned} C_t + S_t &\leq (1+r)S_{t-1} + Q_t,\\ 0 &\leq C_t. \end{aligned}\right\} t = 1, 2, \ldots, T.\\ &\qquad \text{given } S_0 = 0 = S_T. \end{aligned}\right\} \xrightarrow[\lim_{C_t \to \infty} \frac{\partial u(\cdot)}{\partial C_t} = 0 \Rightarrow C_t < \infty]{\lim_{C_t \to 0} \frac{\partial u(\cdot)}{\partial C_t} = \infty \Rightarrow C_t > 0} \left\{\begin{aligned}&\max_{\{C_t\}_{t=1}^{T}} U_1 \equiv \sum_{t=1}^{T} \beta^{t-1} u(C_t),\\ &\text{s.t.} \quad \begin{cases} C_t + S_t = (1+r)S_{t-1} + Q_t,\\ \qquad t = 1, 2, \ldots, T. \end{cases}\\ &\qquad \text{given } S_0 = 0 = S_T. \end{aligned}\right.$$

This is an optimization problem with superposition of non-negative constraints and inequality constraints, but based on the Inada condition, the inequality can be made a tight constraint to exclude corner point solutions. Therefore, the Lagrangian function can be directly reconstructed:

$$\max_{\{C_t, S_t, \lambda_t\}_{t=1}^{T}} \mathcal{L} \equiv \sum_{t=1}^{T} \left\{ \beta^{t-1} \overbrace{u(C_t)}^{\text{period utility function}} + \lambda_t \overbrace{[(1+r)S_{t-1} + Q_t - (C_t + S_t)]}^{\text{flow budget constraint}} \right\}.$$

The first-order necessary conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= \beta^{t-1}[u'(C_t) - \lambda_t] = 0,\\ \frac{\partial \mathcal{L}}{\partial S_t} &= -\lambda_t + \lambda_{t+1}(1+r) = 0,\\ \frac{\partial \mathcal{L}}{\partial \lambda_t} &= (1+r)S_{t-1} + Q_t - (C_t + S_t) = 0. \end{aligned}$$

The third condition is the budget constraint. Putting The first condition one period ahead:

$$u'(C_t) = \lambda_t \quad \Rightarrow \quad u'(C_{t+1}) = \lambda_{t+1}.$$

Combining two periods and using the second first-order condition gives us the familiar consumption Euler equation:

$$\frac{u'(C_{t+1})}{u'(C_t)} = \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{\beta(1+r)} \quad \Rightarrow \quad \frac{\beta u'(C_{t+1})}{u'(C_t)} = \frac{1}{1+r} \quad \text{vs.} \quad \frac{\beta u'(C_{t+1})}{u'(C_t)} = \frac{P_{t+1}}{P_t}.$$

Note that the liquidity budget constraint for period m is used here, so T Lagrange multipliers $\{\lambda_t\}_{t=1}^{T}$ are used. If combining the difference equation $S_t = (1+r)S_{t-1} + (Q_t - C_t)$ about savings with the initial conditions $S_0 = 0$ and the terminal conditions $S_T = 0$ according to the relevant assumptions, a single inter-period lifetime expectation constraint can still be obtained, then the problem can be solved by constructing a Lagrangian function with only one multiplier λ. In addition, in this example, the price of any period is 1, and the price of period $t+1$ discounted to period t is $\frac{1}{1+r}$, so the ratio of the price of period $t+1$ to that of period t (i.e., relative price) is $\frac{\frac{1}{1+r}}{1} = \frac{P_{t+1}}{P_t}$.

In other words:

$$\underbrace{\overbrace{u'(C_t)}^{\text{marginal cost in period } t}}_{\substack{\text{loss of marginal utility}\\ \text{when the consumption in period } t\\ \text{decreases by 1 unit}}} = \underbrace{\overbrace{\beta(1+r)u'(C_{t+1})}^{\text{marginal revenue discounted to period } t}}_{\substack{\text{gain of marginal utility}\\ \text{in period } t+1 \text{ for more consumption}\\ \text{brought by savings}}}.$$

II. **Infinite periods**

Example 13-2. **Infinite-period consumption decision in an endowment economy**

It is also more convenient to use the subscript t to represent infinite dynamic optimization problems, $t = 1 \to T \to \infty$

and thus $t = 1 \to \infty$ to represent the transition from finite to infinite:

$$\left.\begin{aligned} &\max_{\{C_t\}_{t=1}^{\infty}} U_1 \equiv \sum_{t=1}^{\infty} \beta^{t-1} u(C_t), \\ &\text{s.t.} \quad \left\{\begin{aligned} C_t + S_t &\leq (1+r)S_{t-1} + Q_t, \\ 0 &\leq C_t. \end{aligned}\right\} t = 1, 2, \ldots, \infty. \\ &\qquad \text{given } S_0 = 0 = \lim_{t\to\infty} S_t. \end{aligned}\right\} \xRightarrow[C_t<\infty]{C_t>0} \left\{\begin{aligned} &\max_{\{C_t\}_{t=1}^{\infty}} U_1 \equiv \sum_{t=1}^{\infty} \beta^{t-1} u(C_t), \\ &\text{s.t.} \qquad C_t + S_t = (1+r)S_{t-1} + Q_t, \\ &\qquad\qquad t = 1, 2, \ldots, \infty. \\ &\qquad \text{given } S_0 = 0 = \lim_{t\to\infty} S_t. \end{aligned}\right.$$

where $0 < C_t < \infty$ comes from the aforementioned Inada condition.

i) **Multiplier method**

In order to smoothly transition from the static model to the dynamic model, the variable subscripts have been mapped one by one. The above is given S_0 and the optimization problem starts from $t = 1$. Let's give $S_{-1}$and let the optimization problem start from $t = 0$ here, that is:

$$\begin{aligned} &\max_{\{C_t\}_{t=0}^{\infty}} U_0 \equiv \sum_{t=0}^{\infty} \beta^t u(C_t), \\ \text{s.t.} \quad &C_t + S_t = (1+r)S_{t-1} + Q_t, \\ &\text{given } S_{-1} = 0 = \lim_{t\to\infty} S_t. \end{aligned}$$

After fine-tuning, the constructed Lagrangian function is:

$$\max_{\{C_t, S_t, \lambda_t\}_{t=0}^{\infty}} \mathcal{L} \equiv \sum_{t=0}^{\infty} \left\{\beta^t u(C_t) + \lambda_t [(1+r)S_{t-1} + Q_t - (C_t + S_t)]\right\}.$$

Derivatives are taken for the choice variables C_t, S_t, λ_t respectively, and the results are the same as those in the finite-peroid dynamic optimization. Combining the consumption Euler equation with the flow budget constraint, the consumption demand equation for each period can be obtained. If an explicit expression of the consumption demand function is to be obtained, the immediate utility function needs to be defined.

ii) **Planning method**

The difference from static optimization problems is that in addition to common methods such as Lagrangian, there are also dynamic programming methods available.

Step 1) According to the budget constraint, we can solve:

$$C_t = (1+r)S_{t-1} + Q_t - S_t.$$

Substituting into the objective function, the dynamic optimization problem with equality constraints becomes unconstrained:

$$\max_{\{S_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u[(1+r)S_{t-1} + Q_t - S_t], \qquad \beta \in (0,1).$$

Step 2) Use $V(S_{-1}, Y_0)$ to represent the value function that optimizes the objective function given the initial state:

$$\begin{aligned} V(S_{-1}, Q_0) &\equiv \max_{\{S_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u[\overbrace{(1+r)S_{t-1} + Q_t - S_t}^{C_t}], \\ &= \max_{\{S_t\}_{t=0}^{\infty}} \left\{u[(1+r)S_{-1} + Q_0 - S_0] + \sum_{t=1}^{\infty} \beta^t u[(1+r)S_{t-1} + Q_t - S_t]\right\}, \\ &= \max_{\{S_t\}_{t=0}^{\infty}} \left\{u[(1+r)S_{-1} + Q_0 - S_0] + \beta \sum_{t=1}^{\infty} \beta^{t-1} u[(1+r)S_{t-1} + Q_t - S_t]\right\}, \\ &= \max_{S_0} \left\{u[(1+r)S_{-1} + Q_0 - S_0] + \beta \max_{\{S_t\}_{t=0}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u[(1+r)S_{t-1} + Q_t - S_t]\right\}, \end{aligned}$$

$$= \max_{S_0}\left\{u[\underbrace{(1+r)S_{-1}+Q_0-S_0}_{C_0}]+\beta V(S_0,Q_1)\right\}.$$

Without loss of generality, we can also use S_{t-1} as the initial state variable and S_t as the control variable to achieve inter-period lifetime optimization:

$$\underbrace{\overbrace{V(S_{t-1},Q_t)}^{\text{value function}} = \max_{S_t}\left\{\overbrace{u[\overbrace{(1+r)S_{t-1}+Q_t-S_t}^{C_t}]+\beta\,\overbrace{\underbrace{V(S_t,Q_{t+1})}_{\substack{S_t\text{ is chosen to achieve ovarall}\\ \text{(including the initial period) optimization}}}}^{\substack{S_{t+1}\text{ is chosen to achieve}\\ \text{the subsequent optimization}}}}^{\substack{S_t\text{ as state variable of}\\ \text{the subsequent optimization}}}\right\}.}_{S_{t-1}\text{ as state variable of the optimization}}$$

The dynamic optimization problem in the form of the recursive functional described above is called the Bellman equation. [7] Now what needs to be solved is not a variable that can be assigned a specific value, but a path function, that is, given the exogenous $u(C_t), Q_t, \beta$, we need to solve the endogenous function $V(S_{t-1})$.

Step 3) Observe the right side of the Bellman equation. If $V(S_t, Q_{t+1})$ is known, it is equivalent to a **two-period** dynamic optimization problem. It is easy to find the first-order condition and solve the control variable to obtain the policy function:

$$\begin{aligned}
&\max_{S_t}\left\{u[\overbrace{(1+r)S_{t-1}+Q_t-S_t}^{C_t}]+\beta V(S_t,Q_{t+1})\right\},\\
\xRightarrow{\text{F.O.C.}}\quad & u'(C_t)\frac{\partial C_t}{\partial S_t}+\beta\frac{\mathrm{d}V(S_t,Q_{t+1})}{\mathrm{d}S_t}=0,\\
\xRightarrow{\frac{\partial C_t}{\partial S_t}=-1}\quad & \beta\frac{\mathrm{d}V(S_t,Q_{t+1})}{\mathrm{d}S_t}=u'(C_t).
\end{aligned}$$

Note that $u'(C_t)$ is a function of S_{t-1}, S_t, Q_t , which is $u'(C_t) = u'(C_t)(S_{t-1}, S_t, Q_t)$.

The meaning of the first-order condition above is that, given that the objective function of the $t+1$-th period has been optimized, then choosing S_t makes it optimal from the t-th period onwards, thus forming a recursive selection structure. In other words, if $V(S_t, Q_{t+1})$ is known, its derivative with respect to S_t can be calculated, and thus the policy function or decision plan $S_t = g(S_{t-1}) = \arg\max_{S_t}[u(S_{t-1}, S_t) + \beta V(S_t)]$ can be solved. The question then becomes how to solve $\frac{\mathrm{d}V(S_t,Q_{t+1})}{\mathrm{d}S_t}$

[7] Named after the discoverer of this form, American mathematician Richard Bellman (1920-1984)

Step 4) Observe the left side of the Bellman equation. Put it one period ahead, we have:

$$\underbrace{\overbrace{V(S_t,Q_{t+1})}^{\text{value function}} = \max_{S_{t+1}} \left\{ \overbrace{\overbrace{u[(1+r)S_t+Q_{t+1}-S_{t+1}]}^{C_{t+1}} + \beta \overbrace{\underbrace{V(S_{t+1},Q_{t+2})}_{\substack{S_{t+1}\text{ is chosen to achieve ovarall}\\ \text{(including the initial period) optimization}}}}^{\substack{S_{t+2}\text{ is chosen to achieve}\\ \text{the subsequent optimization}}}}^{\substack{S_{t+1}\text{ as state variable of}\\ \text{the subsequent optimization}}} \right\}}_{S_t\text{ as state variable of the optimization}} \xLeftrightarrow[\text{different initial state}]{\text{identical structure}} V(S_{t-1},Q_t).$$

The condition obtained by choosing S_{t+1} after putting the equation one period ahead are the same as those in the previous step, but the problem to be solved in the previous step is the derivative of the value function of the previous period with respect to the state variable S_t. The policy function based on the value function is $S_{t+1} = g(S_t)$, so the Bellman equation of the period in advance can also be written as:

$$\overbrace{V(S_t,Q_{t+1})}^{\text{optimal value fuction}} = u[(1+r)S_t+Q_{t+1}-S_{t+1}(S_t)] + \beta \overbrace{V[S_{t+1}(S_t),Q_{t+1}]}^{\text{optimal value fuction}}.$$

According to the envelope theorem (comparative analysis of value functions with respect to state variables), it can be calculated: [8]

$$\begin{aligned}
\frac{\mathrm{d}V(S_t,Q_{t+1})}{\mathrm{d}S_t} &= \frac{\mathrm{d}u(\cdot)}{\mathrm{d}C_{t+1}}\frac{\partial C_{t+1}}{\partial S_t} + \frac{\mathrm{d}u(\cdot)}{\mathrm{d}C_{t+1}}\frac{\partial C_{t+1}}{\partial S_{t+1}}\frac{\mathrm{d}S_{t+1}}{\mathrm{d}S_t} + \beta\frac{\mathrm{d}V(\cdot)}{\mathrm{d}S_{t+1}}\frac{\mathrm{d}S_{t+1}}{\mathrm{d}S_t},\\
&= \frac{\mathrm{d}u(\cdot)}{\mathrm{d}C_{t+1}}\frac{\partial C_{t+1}}{\partial S_t} + \underbrace{\left[\frac{\mathrm{d}u(\cdot)}{\mathrm{d}C_{t+1}}\frac{\partial C_{t+1}}{\partial S_{t+1}} + \beta\frac{\mathrm{d}V(\cdot)}{\mathrm{d}S_{t+1}}\right]\frac{\mathrm{d}S_{t+1}}{\mathrm{d}S_t}}_{\text{derivative of choice variable is 0}},\\
&= u'(C_{t+1})\frac{\mathrm{d}C_{t+1}}{\mathrm{d}S_t} + 0\times g'(S_t),\\
&= u'(C_{t+1})(1+r).
\end{aligned}$$

Step 5) Substituting the above result of applying the envelope theorem into the first-order condition, we get the Euler equation which is the same as the Lagrange method:

$$\frac{\beta u'(C_{t+1})}{u'(C_t)} = \frac{1}{1+r}.$$

From the perspective of thought, dynamic programming seems to transform the dynamic optimal problem into a static optimal problem. The Bellman equation can be restated as:

$$V(x) = \max_y [u[f(x)-y] + \beta V(y)].$$

[8] The envelope theorem in dynamic settings operates in much the same way as in the static case. Let the state variable be x_t, the control variable be y_t, and suppose the state transition equation is given by $x_{t+1} = g(x_t, y_t)$. The objective function is $\max \sum_{t=0}^{\infty} \beta^t F(x_t, y_t)$, which leads to the corresponding Bellman equation: $V(x_t) = \max\{F(x_t,y_t) + \beta V(x_{t+1})\}$. Substituting the state transition equation, which serves as a constraint, into the Bellman equation yields $V(x_t) = \max\{F(x_t,y_t) + \beta V[g(x_t,y_t)]\}$. Given the optimal solution y_t°, the Bellman equation becomes $V(x_t) = F(x_t,y_t^\circ) + \beta V[g(x_t,y_t^\circ)]$. The envelope theorem implies $\underbrace{\frac{\mathrm{d}V(x_t)}{\mathrm{d}x_t}}_{\text{shadow price}} = \frac{\partial F(x_t,y_t^\circ)}{\partial x_t} + \frac{\partial F(x_t,y_t^\circ)}{\partial y_t^\circ}\frac{\mathrm{d}y_t^\circ}{\mathrm{d}x_t} + \beta\frac{\mathrm{d}V(x_{t+1})}{\mathrm{d}x_{t+1}}\left[\frac{\partial g(x_t,y_t^\circ)}{\partial x_t} + \frac{\partial g(x_t,y_t^\circ)}{\partial y_t^\circ}\frac{\mathrm{d}y_t^\circ}{\mathrm{d}x_t}\right] = \frac{\partial F(x_t,y_t^\circ)}{\partial x_t} + \underbrace{\left[\frac{\partial F(x_t,y_t^\circ)}{\partial y_t^\circ} + \beta\frac{\mathrm{d}V(x_{t+1})}{\mathrm{d}x_{t+1}}\frac{\partial g(x_t,y_t^\circ)}{\partial y_t^\circ}\right]}_{\text{first-order condition=0}}\frac{\mathrm{d}y_t^\circ}{\mathrm{d}x_t} + \beta\frac{\mathrm{d}V(x_{t+1})}{\mathrm{d}x_{t+1}}\frac{\partial g(x_t,y_t^\circ)}{\partial x_t} = \frac{\partial F(x_t,y_t^\circ)}{\partial x_t} + \beta\frac{\mathrm{d}V(x_{t+1})}{\mathrm{d}x_{t+1}}\frac{\partial g(x_t,y_t^\circ)}{\partial x_t}$. The envelope theorem indicates that, under the optimal policy, the effect of a small change in a parameter (such as the initial state) on the value function depends only on its direct effect, while the indirect effect via the control variable can be ignored. This is because, at the optimum, the control variable has already been adjusted to equate marginal benefit and marginal cost, making the net effect of its variation zero.

In the above formula, x is the state variable (i.e., S_{t-1}), y is the endogenous state variable as the choice variable (i.e., S_t, and according to the budget constraint, the other choice variable (control variable) can be expressed as $f(x)-y$ (equivalent to $(1+r)S_{t-1}+Q_t-S_t$) in the above. According to the envelope theorem in the value function:

$$
\begin{aligned}
&\frac{\mathrm{d}V(x)}{\mathrm{d}x} = u'[f(x)-y]f'(x),\\
\xRightarrow[y=g(x)]{\text{policy function}}\quad & V'(x) = u'[f(x)-g(x)]f'(x),\\
\xRightarrow[\text{endogenous state variable becomes state variable}]{\text{one period ahead}}\quad & V'(y) = u'[f(y)-g(y)]f'(y).
\end{aligned}
$$

The first-order necessary condition for optimizing the Bellman equation is:

$$
\begin{aligned}
&\frac{\mathrm{d}V(x)}{\mathrm{d}y} = 0 = -u'[f(x)-y]+\beta V'(y),\\
\Rightarrow\quad & u'[f(x)-y] = \beta V'(y),\\
\Rightarrow\quad & u'[f(x)-y] = \beta \overbrace{u'[f(y)-g(y)]f'(y)}^{V'(y)},\\
\xRightarrow[g(x)\equiv\arg\max_y u(f(x)-y)+\beta V(y)]{\text{substitute into the policy function}y=g(x)}\quad & u'[f(x)-g(x)] = \beta u'[f[g(x)]-g[g(x)]f'[g(x)],\\
\xRightarrow{\text{i.e.,}}\quad & u'[f(x_{t-1})-g(x_{t-1})] = \beta u'[f[g(x_{t-1})]-g[g(x_{t-1})]f'[g(x_{t-1})],\\
\xRightarrow[x_{t+1}=g(x_t)=g(g(x_{t-1}))]{x_t=g(x_{t-1})}\quad & u'[f(x_{t-1})-x_t] = \beta u'[f(x_t)-x_{t+1}]f'(x_t),\\
\Rightarrow\quad & \frac{\beta u'[f(x_t)-x_{t+1}]}{u'[f(x_{t-1})-x_t]} = \frac{1}{f'(x_t)}.
\end{aligned}
$$

Omitting endowment income, the constraint can also be expressed as:

$$C_t + \frac{1}{1+r}S_{t+1} = S_t,$$

That is, the savings in the next period are discounted to the current period.

Given the same initial and final conditions, the optimization problem is also:

$$
\begin{aligned}
\max_{\{C_t\}_{t=0}^{\infty}} \ U_0 &\equiv \sum_{t=0}^{\infty}\beta^t u(C_t),\\
\text{s.t.}\quad S_{t+1} &= (1+r)(S_t-C_t).
\end{aligned}
$$

Substituting the budget constraint (state transition equation) into the Bellman equation:

$$V(S_t) = \max_{C_t}\{u(C_t)+\beta V[\overbrace{(1+r)(S_t-C_t)}^{S_{t+1}}]\}.$$

Given the optimal solution (policy function) $C_t^\circ = C_t^\circ(S_t)$, the Bellman equation is:

$$V(S_t) = u[C_t^\circ(S_t)] + \beta V\{(1+r)[S_t - C_t^\circ(S_t)]\}.$$

Applying the first-order condition to the envelope theorem also yields the consumption Euler equation:

$$
\begin{aligned}
V'(S_t) &= u'(C_t^\circ)\frac{\mathrm{d}C_t^\circ}{\mathrm{d}S_t} + \beta V'(S_{t+1})(1+r)\left(1-\frac{\mathrm{d}C_t^\circ}{\mathrm{d}S_t}\right),\\
&= \underbrace{\left[u'(C_t^\circ)-\beta(1+r)V'(S_{t+1})\right]}_{\text{F.O.C.}=0}\frac{\mathrm{d}C_t^\circ}{\mathrm{d}S_t} + \beta(1+r)V'(S_{t+1}),\\
\Rightarrow\quad V'(S_t) &= \beta(1+r)V'(S_{t+1}),\\
\Rightarrow\quad u'(C_t) &= \beta(1+r)u'(C_{t+1}).
\end{aligned}
$$

Example14. **Infinite-horizon consumption decisions in a production economy**

A) **Economy with discentralized competition**

a. Representative manufacturers

Given the real wage and the real interest rate, the firm chooses capital and labor to optimize profit:

$$\max_{K_t} F(K_t, L_t, A_t) - (r+\delta)K_t - WL_t.$$

This is essentially a single-variable unconstrained static optimization problem, and the first-order necessary conditions are:

$$F_K(K_t, L_t, A_t) = r + \delta \equiv R.$$

Depreciation rate $\delta \in [0,1]$: $\delta = 0$ means no depreciation at all, $\delta = 1$ means full depreciation.

b. Representative households

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t),$$

$$\text{s.t.} \quad \underbrace{C_t + K_{t+1} + (1-\delta)K_t}_{I_t = S_t} = \underbrace{WL_t + RK_t}_{Y_t} + \underbrace{D_t}_{0},$$

$$\text{given } K_0 = 0 = \lim_{t\to\infty} K_{t+1}.$$

D_t can be regarded as the asset income of the household, which comes from the economic profit of the enterprise in which the household has equity. Under perfect competition, the economic profit of the enterprise is 0, so $D_t = 0$. If necessary, government transfer payments or lump-sum taxes can also be added to balance the budget constraint.

B) **Economy with centralized arrangement**

In an endowment economy, it is assumed that there is an income that "falls from the sky" in each period, and the interest earned on savings is used as the income of inter-period allocation of consumption. This example considers the conversion of savings into investment. If capital is fully depreciated ($\delta = 1$) , investment is all the newly added capital, which is used for production, and the output is also income. In addition to capital input, output should also have labor input and technology input. For simplicity, labor and technology are assumed to be exogenous and constant. Therefore, in this example, the main markets here are a capital market and a product market, and supply and demand are equal when the market is cleared. Therefore:

$$\left.\begin{aligned} C_t + S_t &= Q_{dt}, \\ Q_{st} &= F(K_t^d, L_t, A_t), \\ K_{s,t+1} &= I_t, \end{aligned}\right\} \text{behavioral equations} \qquad \text{equilibrium conditions} \begin{cases} I_t = S_t, \\ K_{dt} = K_{st} = K_t, \\ Q_{dt} = Q_{st} = Q_t. \end{cases}$$

Combining the above behavioral equations and equilibrium conditions, we can obtain:

$$C_t + K_{t+1} = F(K_t, L_t, A_t) = (1+r)K_t + WL_t.$$

The second equation above is from a decentralized economy. Compare the budget constraint in the endowment economy above with the budget constraint in the production economy here:

$$\begin{cases} C_t + S_t \quad = (1+r)S_{t-1} + Q_t; \\ C_t + K_{t+1} = (1+r)K_t \quad + WL_t. \end{cases}$$

In the endowment economy mentioned above, the assumption $\delta = 1$ is also implied; the two adjacent periods K_{t+1} and K_t can also be written as K_t and K_{t-1} , so the production function is $Y = F(K_{t-1}, L_t)$; the endowment income Q_t and labor WL_t of the two models are both exogenous. In contrast, the two budget constraints can be highly consistent in form and are basically the same in essence.

This leads to the following utility optimization problem with budget constraints:

$$\max_{\{C_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t),$$
$$\text{s.t.} \quad C_t + K_{t+1} = F(K_t, L_t, A_t),$$
$$\text{given } K_0 = 0 = \lim_{t\to\infty} K_{t+1}.$$

For simplicity, the exogenous variables production technology and labor are set as constants and further standardized to 1, that is, $A_t = A = 1 = L = L_t$. Solving C_t from the budget constraint and substituting it into the objective function, we can write the Bellman equation:

$$\begin{aligned}
& V(K_0) \equiv \max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u[F(K_t) - K_{t+1}], \\
\Rightarrow \quad & V(K_0) = \max_{K_1}\{u[f(K_0) - K_1] + \beta V(K_1)\}, \\
\Rightarrow \quad & V(K_t) = \max_{K_{t+1}}\{u[F(K_t) - K_{t+1}] + \beta V(K_{t+1})\}, \\
\Rightarrow \quad & V(K) = \max_{X}\{u[F(K) - X] + \beta V(X)\},
\end{aligned}$$

The last but one Bellman equation mentioned above is an extension from the beginning of period 0 to the beginning of any period t ; the last Bellman equation expresses that this recursive structure is independent of the specific period, and it is always given the previous material capital K (state variable) to select the optimal material capital X for the next period, so X is the control variable.

The first-order condition for optimization is:

$$\begin{aligned}
& \frac{\mathrm{d}u[F(K) - X]}{\mathrm{d}X} + \beta\frac{\mathrm{d}V(X)}{\mathrm{d}X} = 0, \\
\Rightarrow \quad & \frac{\mathrm{d}u[F(K) - X]}{\mathrm{d}[F(K) - X]}\frac{\mathrm{d}[F(K) - X]}{\mathrm{d}X} + \beta\frac{\mathrm{d}V(X)}{\mathrm{d}X} = 0, \\
\Rightarrow \quad & -u'[F(K) - X] + \beta\frac{\mathrm{d}V(X)}{\mathrm{d}X} = 0, \\
\Rightarrow \quad & u'[F(K) - X] = \beta\frac{\mathrm{d}V(X)}{\mathrm{d}X} \qquad \text{vs.} \qquad u'(C_t) = \lambda_t.
\end{aligned}$$

If $V(X)$ is known, then $\mathrm{d}V(X)/\mathrm{d}X$ can be known, and from this we can get the policy function (or decision rule, feedback rule) $X = f(K)$. In addition, after comparison, it is easy to know the Lagrange multiplier $\lambda_t \equiv \beta(\mathrm{d}V(K_{t+1})/\mathrm{d}K_{t+1})$, which is the shadow price of newly produced tangible capital (or investment expenditure), that is, the present value of the lifetime utility increased by an additional unit of tangible capital.

Substituting it into the above formula:

$$u'[F(K) - f(K)] = \beta\frac{\mathrm{d}V[f(K)]}{\mathrm{d}f(K)}.$$

Substituting it into the Bellman equation, we can deduce according to the envelope theorem:

$$\begin{aligned}
& V(K) = u[F(K) - f(K)] + \beta V[f(K)], \\
\Rightarrow \quad & \frac{\mathrm{d}V(K)}{\mathrm{d}K} = \frac{\partial}{\partial K}\{u[F(K) - f(K)] + \beta V[f(K)]\}, \\
& = u'[F(K) - f(K)][F'(K) - f'(K)] + \beta\frac{\mathrm{d}V[f(K)]}{\mathrm{d}f(K)}f'(K), \\
& = u'[F(K) - f(K)]F'(K) - u'[F(K) - f(K)]f'(K) + \beta\frac{\mathrm{d}V[f(K)]}{\mathrm{d}f(K)}f'(K), \\
& = u'[F(K) - f(K)]F'(K) - \beta\frac{\mathrm{d}V[f(K)]}{\mathrm{d}f(K)}f'(K) + \beta\frac{\mathrm{d}V[f(K)]}{\mathrm{d}f(K)}f'(K), \\
& = u'[F(K) - f(K)]F'(K),
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow && \frac{\mathrm{d}V[f(K)]}{\mathrm{d}f(K)} = u'\left\{F[f(K)] - f[f(K)]\right\} F'[f(K)],\\
&\Rightarrow && u'[F(K) - f(K)] = \beta u'\left\{F[f(K)] - f[f(K)]\right\} F'[f(K)],\\
&\xrightarrow[K_{t+2}=f(f(K_t))]{K_{t+1}=f(K_t)} && u'[\underbrace{F(K_t) - K_{t+1}}_{C_t}] = \beta u'[\underbrace{F(K_{t+1}) - K_{t+2}}_{C_{t+1}}]F'(K_{t+1}).
\end{aligned}
$$

This is also the Euler equation after substituting the budget constraint. It is a second-order difference equation for tangible capital; the initial condition is a given non-negative capital stock (i.e., $K_0 \geq 0$), and the cross-sectional condition is $\lim_{t\to\infty} \beta^t u'(C_t)K_{t+1} = 0$.

It can be seen that whether solving the Lagrangian function to obtain the Euler equation or solving the value function through the Bellman equation, the policy function can be obtained. If the objective function is not too complicated, the former is relatively convenient and fast to solve; especially when the objective function has complete information rational expectations or incomplete information rational expectations (see Chapter 5), the latter is more robust and reliable.

Linear quadratic dynamic programming is another form of dynamic programming, where the objective function is quadratic and the constraints are linear.

Question 1: How to get the quadratic objective function? [9]

Starting from the two-vector Taylor expansion,

$$
\begin{aligned}
F(\mathbf{x}_t,\mathbf{y}_t) &\approx F(\mathbf{x}^{\odot},\mathbf{y}^{\odot}) + \overbrace{\begin{bmatrix} F_{\mathbf{x}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})' & F_{\mathbf{y}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})' \end{bmatrix}}^{\text{gradient vector}} \begin{bmatrix} \mathbf{x}_t - \mathbf{x}^{\odot} \\ \mathbf{y}_t - \mathbf{y}^{\odot} \end{bmatrix} + \begin{bmatrix} (\mathbf{x}_t - \mathbf{x}^{\odot})' & (\mathbf{y}_t - \mathbf{y}^{\odot})' \end{bmatrix} \overbrace{\begin{bmatrix} \frac{F_{\mathbf{xx}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})}{2} & \frac{F_{\mathbf{xy}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})}{2} \\ \frac{F_{\mathbf{yx}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})}{2} & \frac{F_{\mathbf{yy}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})}{2} \end{bmatrix}}^{\frac{1}{2}\text{Hessian matrix}} \begin{bmatrix} \mathbf{x}_t - \mathbf{x}^{\odot} \\ \mathbf{y}_t - \mathbf{y}^{\odot} \end{bmatrix},\\
&= F(x_1^{\odot}, x_2^{\odot}, \cdots, x_k^{\odot}, y_1^{\odot}, y_2^{\odot}, \cdots, y_l^{\odot})\\
&\quad + F_{x1}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(x_{1t} - x_1^{\odot}) + F_{x2}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(x_{2t} - x_2^{\odot}) + \cdots + F_{y1}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(y_{1t} - y_1^{\odot}) + F_{y2}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(y_{2t} - y_2^{\odot}) + \cdots,\\
&\quad + \frac{1}{2}\Big[F_{x1x1}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(x_{1t} - x_1^{\odot})^2 + F_{x1x2}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(x_{1t} - x_1^{\odot})(x_{2t} - x_2^{\odot}) + \cdots + F_{x1y1}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(x_{1t} - x_1^{\odot})(y_{1t} - y_1^{\odot}) + \cdots\\
&\quad + \quad F_{y1y1}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(y_{1t} - y_1^{\odot})^2 + F_{y1y2}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(y_{1t} - y_1^{\odot})(y_{2t} - y_2^{\odot}) + \cdots + F_{y1x1}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})(y_{1t} - y_2^{\odot})(x_{1t} - x^{\odot}) + \cdots\Big],\\
&= F(\mathbf{x}^{\odot},\mathbf{y}^{\odot}) - F_{\mathbf{x}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})'\mathbf{x}^{\odot} - F_{\mathbf{y}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})'\mathbf{y}^{\odot}\\
&\quad + \frac{1}{2}\mathbf{x}^{\odot\prime}F_{\mathbf{xx}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{x}^{\odot} + \frac{1}{2}\mathbf{x}^{\odot\prime}F_{\mathbf{xy}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{y}^{\odot} + \frac{1}{2}\mathbf{y}^{\odot\prime}F_{\mathbf{yx}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{x}^{\odot} + \frac{1}{2}\mathbf{y}^{\odot\prime}F_{\mathbf{yy}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{y}^{\odot}\\
&\quad + (\cdot)\mathbf{x}_t + (\cdot)\mathbf{y}_t + \mathbf{x}_t'(\cdot)\mathbf{x}_t + \mathbf{x}_t'(\cdot)\mathbf{y}_t + \mathbf{y}_t'(\cdot)\mathbf{y}_t,\\
&= F(\mathbf{x}^{\odot},\mathbf{y}^{\odot}) - F_{\mathbf{x}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})'\mathbf{x}^{\odot} - F_{\mathbf{y}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})'\mathbf{y}^{\odot} + \frac{1}{2}\mathbf{x}^{\odot\prime}F_{\mathbf{xx}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{x}^{\odot} + \mathbf{x}^{\odot\prime}F_{\mathbf{xy}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{y}^{\odot} + \frac{1}{2}\mathbf{y}^{\odot\prime}F_{\mathbf{yy}}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{y}^{\odot}\\
&\quad + (\cdot)\mathbf{x}_t + (\cdot)\mathbf{y}_t + \mathbf{x}_t'(\cdot)\mathbf{x}_t + \mathbf{x}_t'(\cdot)\mathbf{y}_t + \mathbf{y}_t'(\cdot)\mathbf{y}_t.
\end{aligned}
$$

The quadratic term of Taylor expansion is a quadratic form. To include the steady-state point and the linear term in the quadratic form, we need to redefine the vector $\mathbf{z}_t = [1 \ \ \mathbf{x}_t \ \ \mathbf{y}_t]'$ and its steady state $\mathbf{z}^{\odot} = [1 \ \ \mathbf{x}^{\odot} \ \ \mathbf{y}^{\odot}]'$, the equivalence of the two is:

$$
F(\underbrace{\mathbf{x}_t}_{k\times k}, \underbrace{\mathbf{y}_t}_{l\times l}) \approx \underbrace{\begin{bmatrix} 1 & \mathbf{x}_t & \mathbf{y}_t \end{bmatrix}}_{\mathbf{z}_t'} \underbrace{\begin{bmatrix} d_{11} & \mathbf{d}_{12} & \mathbf{d}_{13} \\ \mathbf{d}_{21} & \mathbf{d}_{22} & \mathbf{d}_{23} \\ \mathbf{d}_{31} & \mathbf{d}_{32} & \mathbf{d}_{33} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 \\ \mathbf{x}_t \\ \mathbf{y}_t \end{bmatrix}}_{\mathbf{z}_t} = \underbrace{d_{11}}_{1\times 1} + (\mathbf{d}_{12} + \mathbf{d}_{21}')\mathbf{x}_t + (\mathbf{d}_{13} + \mathbf{d}_{31}')\mathbf{y}_t + \mathbf{x}_t' \underbrace{\mathbf{d}_{22}}_{k\times k} \mathbf{x}_t + \mathbf{x}_t'(\mathbf{d}_{23} + \mathbf{d}_{32}')\mathbf{y}_t + \mathbf{y}_t' \underbrace{\mathbf{d}_{33}}_{l\times l} \mathbf{y}_t.
$$

The coefficient matrix is defined as:

$$
\begin{aligned}
d_{11} &= F(\mathbf{x}^{\odot},\mathbf{y}^{\odot}) - \mathbf{x}^{\odot\prime}F_x(\mathbf{x}^{\odot},\mathbf{y}^{\odot}) - \mathbf{y}^{\odot\prime}F_y(\mathbf{x}^{\odot},\mathbf{y}^{\odot}) + \frac{\mathbf{x}^{\odot\prime}F_{xx}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{x}^{\odot}}{2} + \mathbf{x}^{\odot\prime}F_{xy}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{y}^{\odot} + \frac{\mathbf{y}^{\odot\prime}F_{yy}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})\mathbf{y}^{\odot}}{2},\\
d_{22} &= \frac{F_{xx}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})}{2},
\end{aligned}
$$

[9][21] expressed the objective function in a quadratic form,[25] provided a brief introduction in the book, and this section draws inspiration from it.

$$
\begin{aligned}
d_{33} &= \frac{F_{yy}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})}{2},\\
d_{12} = d_{21}' &= \frac{F_x(\mathbf{x}^{\odot},\mathbf{y}^{\odot})' - \mathbf{x}^{\odot\prime}F_{xx}(\mathbf{x}^{\odot},\mathbf{y}^{\odot}) - \mathbf{y}^{\odot\prime}F_{yx}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})}{2},\\
d_{13} = d_{31}' &= \frac{F_y(\mathbf{x}^{\odot},\mathbf{y}^{\odot})' - \mathbf{x}^{\odot\prime}F_{xy}(\mathbf{x}^{\odot},\mathbf{y}^{\odot}) - \mathbf{y}^{\odot\prime}F_{yy}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})}{2},\\
d_{23} = d_{32}' &= \frac{F_{xy}(\mathbf{x}^{\odot},\mathbf{y}^{\odot})}{2}.
\end{aligned}
$$

The objective function of infinite-period optimization can therefore be expressed in quadratic form.

Question 2: How to obtain the optimal constraints of linear quadratic form?

Let $\mathbf{y}_t$ be the choice variable (including control variables and endogenous state variables), incorporate 1 into the state vector $\mathbf{x}_t$ to form an augmented state vector $\mathbf{x}_t^1$ (one dimension added), redefine $\mathbf{z}_t \equiv [\mathbf{x}_t^1 \quad \mathbf{y}_t]'$, and combine the linear constraints to obtain the linear quadratic form:

$$
\begin{aligned}
&\max_{\mathbf{y}_t}\sum_{t=0}^{\infty}\beta^t\{F(\mathbf{x}_t,\mathbf{y}_t)=\mathbf{z}_t'\mathbf{D}\mathbf{z}_t\} \Leftrightarrow \max_{\mathbf{y}_t}\sum_{t=0}^{\infty}\beta^t\left\{\mathbf{z}_t'\mathbf{D}\mathbf{z}_t=\begin{bmatrix}\mathbf{x}_t' & \mathbf{y}_t'\end{bmatrix}\begin{bmatrix}\mathbf{R} & \mathbf{W}'\\ \mathbf{W} & \mathbf{Q}\end{bmatrix}\begin{bmatrix}\mathbf{x}_t'\\ \mathbf{y}_t'\end{bmatrix}\right\} \Leftrightarrow \max_{\mathbf{y}_t}\sum_{t=0}^{\infty}\beta^t[\mathbf{x}_t^{1\prime}\mathcal{R}\mathbf{x}_t^1+\mathbf{y}_t'\mathcal{Q}\mathbf{y}_t+2\mathbf{y}_t'\mathcal{W}\mathbf{x}_t^1],\\
&\text{s.t. } \mathbf{x}_{t+1}=G(\mathbf{x}_t,\mathbf{y}_t)=\mathbf{A}\mathbf{x}_t+\mathbf{B}\mathbf{y}_t \Leftrightarrow \underbrace{\begin{bmatrix}1\\ \mathbf{x}_{t+1}\end{bmatrix}}_{\mathbf{x}_{t+1}^1}=\underbrace{\begin{bmatrix}1 & \mathbf{0}\\ \mathbf{0} & \mathbf{A}\end{bmatrix}}_{\mathcal{A}}\begin{bmatrix}1\\ \mathbf{x}_t\end{bmatrix}+\underbrace{\begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{bmatrix}}_{\mathcal{B}}\mathbf{y}_t \Leftrightarrow \mathbf{x}_{t+1}^1=\mathcal{A}\mathbf{x}_t^1+\mathcal{B}\mathbf{y}_t.
\end{aligned}
$$

For the sake of convenience, we will also write $\mathbf{x}_t^1$ as $\mathbf{x}_t$, $\mathcal{A}$ as $\mathbf{A}$, and $\mathcal{B}$ as $\mathbf{B}$, but the reader should now know that if $\mathbf{x}_t$ is an augmented vector containing element 1, then all elements in the first row of matrix $\mathbf{A}$ are 0 except for the first column, and all elements in the first row of $\mathbf{B}$ are 0, and the Hessian matrix is adjusted accordingly. The linear quadratic form is restated as:

$$
\begin{aligned}
&\max_{\mathbf{y}_t}\sum_{t=0}^{\infty}\beta^t[\mathbf{x}_t'\mathbf{R}\mathbf{x}_t+\mathbf{y}_t'\mathbf{Q}\mathbf{y}_t+2\mathbf{y}_t'\mathbf{W}\mathbf{x}_t],\\
&\text{s.t.}\quad \underbrace{\mathbf{x}_{t+1}}_{(k+1)\times 1}=\underbrace{\mathbf{A}}_{(k+1)\times(k+1)}\underbrace{\mathbf{x}_t}_{(k+1)\times 1}+\underbrace{\mathbf{B}}_{(k+1)\times l}\underbrace{\mathbf{y}_t}_{l\times 1}
\end{aligned}
$$

Bellman equation:

$$
\begin{aligned}
V(\mathbf{x}_t) &= \max_{\mathbf{y}_t}[\mathbf{x}_t'\mathbf{R}\mathbf{x}_t+\mathbf{y}_t'\mathbf{Q}\mathbf{y}_t+2\mathbf{y}_t'\mathbf{W}\mathbf{x}_t+\beta V(\mathbf{x}_{t+1})],\\
&= \max_{\mathbf{y}_t}[\mathbf{x}_t'\mathbf{R}\mathbf{x}_t+\mathbf{y}_t'\mathbf{Q}\mathbf{y}_t+2\mathbf{y}_t'\mathbf{W}\mathbf{x}_t+\beta V(\mathbf{A}\mathbf{x}_t+\mathbf{B}\mathbf{y}_t)].
\end{aligned}
$$

Question 3: How to get the solution of linear quadratic optimization?

Make the following conjecture:

$$
\begin{cases}
\overbrace{V(\mathbf{x}_t)=\mathbf{x}_t'\mathbf{P}\mathbf{x}_t}^{\text{value function}};\\
\Rightarrow \underbrace{\mathbf{y}_t=-\mathbf{F}\mathbf{x}_t}_{\text{policy function}}.
\end{cases}
$$

Why do we have this conjecture about the value function? Recall that the value function in the static model is always a function of exogenous variables (including parameters), and the value function in the dynamic model is always a function of exogenous variables (including state variables).

Substitute the conjectured value function into the Bellman equation:

$$
\begin{aligned}
V(\mathbf{x}_t) &= \max_{\mathbf{y}_t}\{\mathbf{x}_t'\mathbf{R}\mathbf{x}_t+\mathbf{y}_t'\mathbf{Q}\mathbf{y}_t+2\mathbf{y}_t'\mathbf{W}\mathbf{x}_t+\beta[(\mathbf{A}\mathbf{x}_t+\mathbf{B}\mathbf{y}_t)'\mathbf{P}(\mathbf{A}\mathbf{x}_t+\mathbf{B}\mathbf{y}_t)]\},\\
&= \max_{\mathbf{y}_t}\{\mathbf{x}_t'\mathbf{R}\mathbf{x}_t+\mathbf{y}_t'\mathbf{Q}\mathbf{y}_t+2\mathbf{y}_t'\mathbf{W}\mathbf{x}_t+\beta\{[(\mathbf{A}\mathbf{x}_t)'+(\mathbf{B}\mathbf{y}_t)']\mathbf{P}(\mathbf{A}\mathbf{x}_t+\mathbf{B}\mathbf{y}_t)\},
\end{aligned}
$$

$$
\begin{aligned}
&= \max_{y_t}\{x_t'Rx_t + y_t'Qy_t + 2y'Wx_t + \beta[(x_t'A' + y_t'B')P(Ax_t + By_t)]\},\\
&= \max_{y_t}\{x_t'Rx + y'Qy_t + 2y_t'Wx_t + \beta[(x_t'A'P + y_t'B'P)(Ax_t + By_t)]\},\\
&= \max_{y_t}\{x_t'Rx_t + y_t'Qy_t + 2y_t'Wx_t + \beta[x_t'A'PAx_t + x_t'A'PBy_t + y_t'B'PAx_t + y_t'B'PBy_t]\},\\
&= \max_{y_t}\{x_t'Rx_t + y_t'Qy_t + 2y_t'Hx_t + \beta[x_t'A'PAx_t + x_t'A'PBy_t + (y_t'B'PAx_t)' + y_t'B'PBy_t]\},\\
&= \max_{y_t}\{x_t'Rx_t + y_t'Qy_t + 2y_t'Wx_t + \beta[x_t'A'PAx_t + x_t'A'PBy_t + x'A'PBy_t + y_t'B'PBy_t]\},
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & V(x_t) = \max_{y_t}\{x_t'Rx_t + y_t'Qy_t + 2y_t'Wx_t + \beta[x_t'A'PAx_t + 2x_t'A'PBy_t + y_t'B'PBy_t]\}.\\
\Rightarrow\quad & \frac{\partial V(x_t)}{\partial y_t} = (Q + Q')y_t + 2Wx_t + \beta\{2(x'A'PB)' + [(B'PB) + (B'PB)']y_t\},\\
& = (Q + Q)y_t + 2Wx_t + \beta[2B'P'Ax_t + (B'PB + B'PB)y_t], \qquad \leftarrow Q \text{ is symmetric}\\
& = 2Qy_t + 2Wx_t + 2\beta(B'P'Ax_t + B'PBy_t),\\
\Rightarrow\quad & 0 = 2Qy + 2Wx_t + 2\beta(B'PAx_t + B'PBy_t), \qquad \leftarrow P \text{ is symmetric}\\
\Rightarrow\quad & 0 = Qy_t + Wx_t + \beta B'PAx_t + \beta B'PBy_t,\\
\Rightarrow\quad & 0 = (Q + \beta B'PB)y_t + (W + \beta B'PA)x_t,\\
\Rightarrow\quad & y_t = -\underbrace{(Q + \beta B'PB)^{-1}(W + \beta B'PA)}_{F}\, x_t.
\end{aligned}
$$

Substitute the policy function derived above into the Bellman equation to determine the coefficient matrix P:

$$
\begin{aligned}
V(x_t) &= x_t'Rx + y_t'Qy_t + 2y_t'Wx_t + \beta\left(x_t'A'PAx_t + 2x_t'A'PBy_t + y_t'B'PBy_t\right),\\
&= x_t'Rx_t + (-Fx_t)'\,Q\,(-Fx_t) + 2\,(-Fx_t)'\,Wx_t + \beta\left[x_t'A'PAx_t + 2x_t'A'PB\,(-Fx_t) + (-Fx_t)'\,B'PB\,(-Fx_t)\right],\\
&= x_t'Rx_t + x_t'F'QFx_t - 2x'F'Wx_t + \beta\left(x_t'A'PAx_t - 2x'A'PBFx_t + x_t'F'B'PBFx_t\right),\\
&= x_t'Rx_t + x_t'\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]'Q\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\\
&\quad - 2x_t'\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]'Wx_t\\
&\quad + \beta\Big(x_t'A'PAx_t - 2x_t'A'PB\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\\
&\quad + x_t'\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]'B'PB\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\Big),\\
&= x_t'Rx_t + x_t'\left\{(W + \beta B'PA)'\left[(Q + \beta B'PB)^{-1}\right]'\right\}Q\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\\
&\quad - 2x_t'\left\{(W + \beta B'PA)'\left[(Q + \beta B'PB)^{-1}\right]'\right\}Wx_t\\
&\quad + \beta\Big(x_t'A'PAx_t - 2x_t'A'PB\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\\
&\quad + x_t'\left\{(W + \beta B'PA)'\left[(Q + \beta B'PB)^{-1}\right]'\right\}B'PB\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\Big),\\
&= x_t'Rx_t + x_t'\left[(W + \beta B'PA)'(Q + \beta B'PB)^{-1}\right]Q\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\\
&\quad - 2x_t'\left[(W + \beta B'PA)'(Q + \beta B'PB)^{-1}\right]Wx_t\\
&\quad + \beta\Big(x_t'A'PAx_t - 2x_t'A'PB\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\\
&\quad + x_t'\left[(W + \beta B'PA)'(Q + \beta B'PB)^{-1}\right]B'PB\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\Big),\\
&= x_t'Rx_t + x_t'\left[(W' + \beta A'PB)(Q + \beta B'PB)^{-1}\right]Q\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t\\
&\quad - 2x_t'\left[(W' + \beta A'PB)(Q + \beta B'PB)^{-1}\right]Wx_t\\
&\quad + \beta\Big(x_t'A'PAx_t - 2x_t'A'PB\left[(Q + \beta B'PB)^{-1}(W + \beta B'PA)\right]x_t
\end{aligned}
$$

$$
\begin{aligned}
& + \mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{B}'\mathbf{PB} \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t \Big), \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{Q} \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t \\
& - 2\mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{W}\mathbf{x}_t \\
& + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t - 2\beta \mathbf{x}_t'\mathbf{A}'\mathbf{PB} \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t \\
& + \beta \mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{B}'\mathbf{PB} \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{Q} \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t \\
& + \beta \mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] (\mathbf{B}'\mathbf{PB}) \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t, \\
& - 2\mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{W}\mathbf{x}_t \\
& + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t - 2\beta \mathbf{x}_t' (\mathbf{A}'\mathbf{PB}) \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{Q} \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t \\
& + \mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] (\beta \mathbf{B}'\mathbf{PB}) \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t, \\
& - 2\mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{W}\mathbf{x}_t \\
& + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t - 2\beta \mathbf{x}_t' (\mathbf{A}'\mathbf{PB}) \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] (\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB}) \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t \\
& - 2\mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{W}\mathbf{x}_t \\
& + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t - 2\beta \mathbf{x}_t' (\mathbf{A}'\mathbf{PB}) \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] (\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t \\
& - 2\mathbf{x}_t' \left[(\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1}\right] \mathbf{W}\mathbf{x}_t \\
& + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t - 2\beta \mathbf{x}_t' (\mathbf{A}'\mathbf{PB}) \left[(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA})\right] \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t \\
& - 2\mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} \mathbf{W}\mathbf{x}_t \\
& + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t - 2\beta \mathbf{x}_t' (\mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t \\
& - 2\mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} \mathbf{W}\mathbf{x}_t - 2\beta \mathbf{x}_t' (\mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t, \\
& + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t \\
& - 2\left[\mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} \mathbf{W}\mathbf{x}_t\right]' - 2\beta \mathbf{x}_t' (\mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t \\
& - 2\mathbf{x}_t'\mathbf{W}' (\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t - 2\beta \mathbf{x}_t' (\mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t \\
& - 2\mathbf{x}_t'\mathbf{W}' (\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t - 2\mathbf{x}_t' (\beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t + \mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t \\
& - 2\mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t, \\
= {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t - \mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t, \\
\Rightarrow \quad \mathbf{x}_t'\mathbf{P}\mathbf{x}_t = {} & \mathbf{x}_t'\mathbf{R}\mathbf{x}_t - \mathbf{x}_t' (\mathbf{W}' + \beta \mathbf{A}'\mathbf{PB})(\mathbf{Q} + \beta \mathbf{B}'\mathbf{PB})^{-1} (\mathbf{W} + \beta \mathbf{B}'\mathbf{PA}) \mathbf{x}_t + \beta \mathbf{x}_t'\mathbf{A}'\mathbf{PA}\mathbf{x}_t,
\end{aligned}
$$

$$\Rightarrow \qquad \mathbf{P} = \mathbf{R} - (\mathbf{W}' + \beta\mathbf{A}'\mathbf{P}\mathbf{B})\,(\mathbf{Q} + \beta\mathbf{B}'\mathbf{P}\mathbf{B})^{-1}\,(\mathbf{W} + \beta\mathbf{B}'\mathbf{P}\mathbf{A}) + \beta\mathbf{A}'\mathbf{P}\mathbf{A}.$$

This gives us the Riccati equations, which can be solved numerically by iteration for the matrix **P** (a function of the matrix**R**, **Q**, **A**, and **B**) [10]This understanding of linear quadratic deterministic dynamic programming is helpful in understanding linear quadratic stochastic dynamic programming, which is further discussed in Chapter 5 of this book.

1.2.2 Continuous time

The static part of the previous section focused on the optimization problems of equality constraints and inequality constraints **between different choice variables**. The discrete-time part of the previous subsection basically dealt with the constraints **on the same chioce variable** at different times, but the focus was on the dynamic optimization problem from two periods to multiple periods. In continuous time problems, there will naturally be constraints **on the same choice variable at different times** and **between different choice variables**, but the continuous time part will also briefly introduce the dynamic optimization with constraints **between different state variables** that have not yet been mentioned.

This leads to a conceptual question: what are state variables? [25, p.51] defined that at a certain time t(decision period), state variables refer to those variables whose values have been determined by past decision actions or by natural motion processes. [49, p.55] defined it more broadly, arguing that all variables that can determine control variables other than parameters are state variables. In other words, control variables should be expressible as functions of state variables.

1.2.2.1 No constraints between state variables

I. **Single state variable**

When there is a single state variable, there may be a single control variable or multiple control variables.

i. **Single control variable**

A single state variable and a single control variable form a constraint on the time axis.

Example 15. **Continuous-time consumption decisions in an endowment economy**

This example is the continuous time version of Example 12. The objective function and constraints of the optimization problem are rewritten into continuous time versions:

$$\left.\begin{aligned} &\max_{\{C_t\}_{t=0}^T} U_0 \equiv \sum_{t=0}^{T} \beta^t u(C_t), \\ &\text{s.t.} \quad C_t + S_t - S_{t-1} = rS_{t-1} + Q_t, \\ &\qquad \text{given } S_{-1} = 0 = S_T. \end{aligned}\right\}\text{discrete time} \quad \text{vs.} \quad \text{continuous time}\left\{\begin{aligned} &\max_{C(t)} U \equiv U(0) = \int_{t=0}^{T} e^{-\rho t} u[C(t)]\mathrm{d}t, \\ &\text{s.t.} \quad C(t) + \dot{S}(t) = rS(t) + Q(t), \\ &\qquad \text{given } S(0) = 0 = S(T). \end{aligned}\right.$$

A few notes:

(1) In discrete time, the summation operator $\sum$ is used for summation, while in continuous time, the integral symbol $\int$ (and the differential symbol dt) are used for summation.

(2) For discrete multi-period dynamic optimization problems, the discount factor $\beta \equiv 1/(1+\rho)$ is used, where ρ is the subjective discount rate, which has the same meaning as ρ used in continuous time problems.

(3) Why is the discount factor $e^{-\rho t}$used in continuous time? The definition of e is related to the calculation of continuous compound interest.

[10] The above derivation uses high-dimensional differential rules $\frac{\partial \mathbf{A}'\mathbf{x}_t}{\partial \mathbf{x}_t} = \mathbf{A}$; $\frac{\partial \mathbf{x}_t'\mathbf{A}\mathbf{x}_t}{\partial \mathbf{x}_t} = (\mathbf{A}+\mathbf{A}')\mathbf{x}_t$, $\frac{\partial^2 \mathbf{x}_t'\mathbf{A}\mathbf{x}_t}{\partial \mathbf{x}_t \partial \mathbf{x}_t'} = (\mathbf{A}+\mathbf{A}')$, $\frac{\partial \mathbf{x}_t'\mathbf{A}\mathbf{x}_t}{\partial \mathbf{A}} = \mathbf{x}_t'\mathbf{x}_t$; $\frac{\partial \mathbf{y}_t'\mathbf{B}\mathbf{x}_t}{\partial \mathbf{x}_t} = \mathbf{B}'\mathbf{y}_t$, $\frac{\partial \mathbf{y}_t'\mathbf{B}\mathbf{x}_t}{\partial \mathbf{y}_t} = \mathbf{B}\mathbf{x}_t$, $\frac{\partial \mathbf{y}_t'\mathbf{B}\mathbf{x}_t}{\partial \mathbf{B}} = \mathbf{y}_t'\mathbf{x}_t$ [22, 1st ed., solution to exercise 4.1] provided Matlab code olrp.m for solving Riccati equation [25, p.155, eq.7.5] also presented the use of numerical methods to approximate and solve the undetermined coefficient matrix through the following construction $\mathbf{P}_{k+1} = \mathbf{R} - (\mathbf{W}' + \beta\mathbf{A}'\mathbf{P}_k\mathbf{B})\,(\mathbf{Q} + \beta\mathbf{B}'\mathbf{P}_k\mathbf{B})^{-1}\,(\mathbf{W} + \beta\mathbf{B}'\mathbf{P}_k\mathbf{A}) + \beta\mathbf{A}'\mathbf{P}_k\mathbf{A}$, where $\lim_{k\to\infty} \mathbf{P}_k = \mathbf{P}$, $\mathbf{P}_0$ is given.

Step 1, assuming that the initial principal is 1 unit of currency, the interest rate is 100% per period, and the interest is calculated once per period. After one period, the sum of principal and interest is: $v = 1 + 100\% = 1 + 1/1 = 2$ units; if the interest is changed to half-period, the half-period interest rate is 50% , and the sum of principal and interest after half a period is used as the principal for the second half of the period, and the interest is calculated twice in one period, then the sum of principal and interest is: $v = (1 + 50\%)(1 + 50\%) = (1 + 1/2)^2$. Similarly, if the interest is calculated m in one period, the interest rate is $1/m$, and the sum of principal and interest after one period is: $v = (1 + 1/m)^m$. When m approaches infinity, the sum of principal and interest after one period is: $v = \lim_{m\to\infty} [1 + 1/m]^m \equiv e = e^1 = e^x|_{x=1} = [1 + x + (1/2!)x^2 + (1/3!)x^3 + \cdots]_{x=1} = 1 + 1 + 1/(2!) + 1/(3!) + \cdots \approx 2.71828$.

Step 2: Continuously calculate compound interest. After the first period, 1 unit of principal is e units of currency. Let it be the new principal for the next period. After the second period, each unit of the e units will become e units of currency again, so the total is $v = e \times e = e^2$. By analogy, the sum of principal and interest at the end of the t-th period is $v = e^t$ units.

Step 3: If the period interest rate per period is not 100% but an arbitrary nominal interest rate i, then the sum of principal and interest after the first period is $v = \lim_{m\to\infty} (1 + i/m)^m = \lim_{m\to\infty} [(1 + i/m)^{m/i}]^i = e^i$, and the sum of principal and interest calculated by compound interest after the t-th period is$v = e^{it}$.

Step 4: If the initial principal is not 1 but an arbitrary d, then the sum of principal and interest calculated by compound interest after the t-th period is $v = de^{it}$.

Step 5: If it is known that the sum of principal and interest calculated based on continuous compound interest after the t-th period is $v = de^{it}$, then the initial principal is $d = ve^{-it}$.

Step 6: The nominal interest rate i is replaced by the subjective discount rate ρ and the current utility of future consumption is discounted according to $e^{-\rho t}$.

Step 7: for discrete time, the interest rate per period is i, the sum of principal and interest after the first period is $v = d(1 + i)$, after the second period is $v = d(1 + i)^2$ and after the t-th period is $v = d(1 + i)^t$. If it is known that the sum of principal and interest calculated according to discrete compound interest after the t-th period is $v = d(1 + i)^t$, then the initial principal is $d = v(1 + i)^{-t}$. When the nominal interest rate is replaced by the subjective discount rate ρ, the present-value utility of future consumption is discounted according to $(1 + \rho)^{-t}$, that is, β^t.

Step 8: The discount factor of discrete time and the discount factor of continuous time can establish an equivalent relationship:

$$\underbrace{\beta^t = \left(\frac{1}{1+\rho}\right)^t}_{\text{discrete time}} \quad \text{vs.} \quad \underbrace{e^{-\rho t} = (e^\rho)^{-t} = [(e^\rho)^{-1}]^t = [(1+\rho)^{-1}]^t = \left(\frac{1}{1+\rho}\right)^t}_{\text{continuous time}}.$$

i) **From Lagrangian function to Hamiltonian function**

If $S(t)$ is continuously differentiable, the variational method can be used; if $S(t)$ has a sharp inflection point, optimal control can be used.

(1) Predecessor: classical variation

By solving the budget constraint, we choose the variable C_t and substitute it into the objective function to obtain a most basic continuous-time dynamic optimization problem that only contains state variables and their changes:

$$\left.\begin{aligned} &\max_{C(t)} U = \int_{t=0}^{T} e^{-\rho t} u[C(t)]\mathrm{d}t, \\ \text{s.t.} \quad & C(t) + \dot{S}(t) = rS(t) + Q(t), \\ & \text{given} \ \ S(0) = 0 = S(T). \end{aligned}\right\} \xrightarrow[\text{converted to no constraint}]{\text{equality constraint}} \left\{\begin{aligned} &\max_{S(t)} U = \int_{t=0}^{T} e^{-\rho t} u[rS(t) + Q(t) - \dot{S}(t)]\mathrm{d}t, \\ &\text{given } S(0) = 0 = S(T). \end{aligned}\right.$$

In addition to the exogenous variables r and $Q(t)$, the main elements of this basic problem of continuous-time dynamic optimization are time t, state variables $S(t)$ and their time derivatives $\dot{S}(t)$. The objective functional in general form is:

$$\left.\begin{aligned}&\max_{S(t)} U[S(t)] = \int_{t=0}^{T} e^{-\rho t} U[t, S(t), \dot{S}(t)]\mathrm{d}t,\\ &\text{given } S(0) = 0 = S(T).\end{aligned}\right\} \xRightarrow{\text{omit discounting for now}} \left\{\begin{aligned}&\max_{S(t)} U[S(t)] = \int_{t=0}^{T} \cancel{e^{-\rho t}} U[t, S(t), \dot{S}(t)]\mathrm{d}t,\\ &\text{given } S(0) = 0 = S(T).\end{aligned}\right.$$

It is called a functional because U is not a composite function of t , but a function of the entire path $S(t)$; in other words, U is a function of S, not t. Also, note that the time and state of the start and end conditions are fixed, eliminating variability. [7, ch1, pp.8-12] introduces several cases of variable endpoints: fixed time (vertical endpoint), fixed state (horizontal endpoint), and fixed time-state relationship (curve or surface endpoint); variable start points are also briefly mentioned.

Since S is not a number but a function of time t, and U is not a function of t, the optimization problem of $U(S)$ cannot be differentiated with respect to S or t to find the optimal path. Based on the idea of calculus, a feasible idea is to convert the objective function into a value-to-value problem instead of a function-to-value problem. In this way, the continuous-time dynamic optimization problem can be solved directly using calculus methods. How to convert it?

Assuming that the optimal curve $S^\circ(t)$ is known, and the adjacent perturbation path to the optimal curve is known to be $\xi(t)$, then any state path and its changes over time can be expressed as:

$$\begin{aligned}S(t) &= S^\circ(t) + a\xi(t),\\ \dot{S}(t) &= \dot{S}^\circ(t) + a\dot{\xi}(t).\end{aligned}$$

Therefore, the objective functional of U with respect to S is transformed into the objective function of U with respect to a:

$$\begin{aligned}&\max_{a} U(a) = \int_{t=0}^{T} u[t, \overbrace{S^\circ(t) + a\xi(t)}^{S(t)}, \overbrace{\dot{S}^\circ(t) + a\dot{\xi}(t)}^{\dot{S}(t)}]\mathrm{d}t,\\ &\text{given } S(0) = 0 = S(T),\\ &\quad\text{and } \xi(0) = 0 = \xi(T).\end{aligned}$$

This has become a single variable unconstrained static optimization problem. The first-order necessary condition is naturally (for simplicity, the lower limit of the integral is directly written as the starting point 0):

$$\begin{aligned}\left.\frac{\mathrm{d}U}{\mathrm{d}a}\right|_{a=0} &= \frac{\mathrm{d}}{\mathrm{d}a}\left\{\int_0^T u[t, S^\circ(t) + a\xi(t), \dot{S}^\circ(t) + a\dot{\xi}(t)]\mathrm{d}t\right\} = 0,\\ \xRightarrow{\text{Leibniz rule}} &= \int_0^T \left(\frac{\partial u}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}a} + \frac{\partial u}{\partial \dot{S}}\frac{\mathrm{d}\dot{S}}{\mathrm{d}a}\right)\mathrm{d}t = 0,\\ &= \int_0^T u_S\xi(t)\mathrm{d}t + \underbrace{\int_0^T u_{\dot{S}}\dot{\xi}_t\mathrm{d}t}_{} = 0,\\ &= \int_0^T u_S\xi(t)\mathrm{d}t + \overbrace{\underbrace{\left[u_{\dot{S}}\xi(t)\right]_0^T}_{=0-0} - \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}u_{\dot{S}}\right)\xi(t)\mathrm{d}t}^{ab=a'b+ab' \xRightarrow{\text{integrate by part}} ab'=(ab)'-a'b} = 0,\\ &= \int_0^T \xi(t)\left(u_S - \frac{\mathrm{d}}{\mathrm{d}t}u_{\dot{S}}\right)\mathrm{d}t = 0,\\ &= \overbrace{u_S - \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}u_{\dot{S}}[t, S(t), \dot{S}(t)]}_{} = 0}^{\text{Euler equation}} \quad \text{or} \quad \overbrace{\int u_S\mathrm{d}t - u_{\dot{S}} = 0}^{\text{Euler euqtion}},\end{aligned}$$

$$
\begin{aligned}
&= u_S - \overbrace{\left(\frac{\partial u_{\dot S}}{\partial t} + \frac{\partial u_{\dot S}}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\partial u_{\dot S}}{\partial \dot S}\frac{\mathrm{d}\dot S}{\mathrm{d}t}\right)},\\
&= u_S - \left[u_{t\dot S} + u_{S\dot S}\dot S(t) + u_{\dot S\dot S}\ddot S(t)\right] = 0,\\
&= \underbrace{u_{\dot S\dot S}\ddot S(t) + u_{S\dot S}\dot S(t) + u_{t\dot S} - u_S = 0}_{\text{(second-order nonlinear differentiation) Euler euqation}} \quad .
\end{aligned}
$$

The original objective functional contains a discount factor, and the integrand should be $e^{-\rho t}u$, so the above Euler equation can be further expressed as:

$$
e^{-\rho t}u_{\dot S\dot S}\ddot S(t) + e^{-\rho t}u_{S\dot S}\dot S(t) + (-\rho)e^{-\rho t}u_{t\dot S} - e^{-\rho t}u_S = 0.
$$

The second-order condition does not change substantially due to the discount factor and can be simplified to:

$$
\begin{aligned}
\frac{\mathrm{d}^2 u}{\mathrm{d}a^2} &= \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\mathrm{d}u}{\mathrm{d}a}\right),\\
&= \frac{\mathrm{d}}{\mathrm{d}a}\int_0^T \left[u_S\xi(t) + u_{\dot S}\dot\xi(t)\right]\mathrm{d}t,\\
\xRightarrow{\text{Leibniz rule}} &= \int_0^T \underbrace{\left(\frac{\mathrm{d}}{\mathrm{d}a}u_S\right)}\xi(t) + \underbrace{\left(\frac{\mathrm{d}}{\mathrm{d}a}u_{\dot S}\right)}\dot\xi(t)\mathrm{d}t,\\
&= \int_0^T \left\{\left[\overbrace{\left(u_{SS}\frac{\mathrm{d}u}{\mathrm{d}a} + u_{\dot S S}\frac{\mathrm{d}\dot S}{\mathrm{d}a}\right)}\xi(t)\right] + \left[\overbrace{\left(\frac{\mathrm{d}}{\mathrm{d}a}u_{\dot S}\right)\dot\xi(t)}\right]\right\}\mathrm{d}t,\\
&= \int_0^T \left\{\left[u_{SS}\xi(t) + u_{\dot S S}\dot\xi(t)\right]\xi(t) + \left[u_{S\dot S}\xi(t) + u_{\dot S\dot S}\dot\xi(t)\right]\dot\xi(t)\right\}\mathrm{d}t,\\
&= \int_0^T \left[u_{SS}\xi^2(t) + 2u_{S\dot S}\xi(t)\dot\xi(t) + u_{\dot S\dot S}\dot\xi^2(t)\right]\mathrm{d}t,\\
&= \int_0^T \begin{bmatrix}\xi(t) & \dot\xi(t)\end{bmatrix}\begin{bmatrix}u_{SS} & u_{S\dot S}\\ u_{\dot S S} & u_{S\dot S}\end{bmatrix}\begin{bmatrix}\xi(t)\\ \dot\xi(t)\end{bmatrix}\mathrm{d}t;\\
&\overset{\text{or}}{=} \int_0^T \left[u_{\dot S\dot S}\dot\xi^2(t) + 2u_{\dot S S}\dot\xi(t)\xi(t) + u_{SS}\xi^2(t)\right]\mathrm{d}t,\\
&= \int_0^T \begin{bmatrix}\dot\xi(t) & \xi(t)\end{bmatrix}\begin{bmatrix}u_{\dot S\dot S} & u_{\dot S S}\\ u_{S\dot S} & u_{SS}\end{bmatrix}\begin{bmatrix}\dot\xi(t)\\ \xi(t)\end{bmatrix}\mathrm{d}t.
\end{aligned}
$$

(2) Extension: Optimal Control

The variational method is to eliminate the choice variables of optimization problems without differential equation form and transform it into an optimization problem with only state variables and their differential equations. The solution idea is to transform the state path into a function of the optimal path and the adjacent perturbation path, and the dynamic optimal problem in functional form is transformed into a static optimal problem similar to an ordinary function. The state differential equation as part of the constraint condition will enter the objective function through variable substitution, and the original choice variable (i.e. the control variable in the optimal control method) will be eliminated. Since it is converted into a static optimization problem similar to that solved by calculus, it is naturally required that the differential equation of state transfer is continuously differentiable.

The control variables that were eliminated in the calculus of variations (ie, the choice variables in static problems, a type of choice variables in dynamic problems) become the protagonists. It is the the control variable that is in the objective function, and the constraints are the state differential equations containing the control variables. Like static optimization or discrete dynamic optimization problems, Lagrangian functions are constructed, and the first-order necessary conditions from the perspective of Hamiltonian functions are derived (maximum principle). The control variables can be continuous or discontinuous; the state variables can only be continuous (allowing sharp inflection points).

Back to the original question:

$$\left.\begin{aligned} &\max_{C(t)} U = \int_0^T e^{-\rho t} u[C(t)]\mathrm{d}t, \\ \text{s.t.}\quad &C(t) + \dot{S}(t) = rS(t) + Q(t), \\ &\text{given } S(0) = 0 = S(T). \end{aligned}\right\} \xrightarrow{\text{omit discounting for now}} \left\{\begin{aligned} &\max_{C(t)} U = \int_0^T \cancel{e^{-\rho t}} u[C(t)]\mathrm{d}t, \\ \text{s.t.}\quad &C(t) + \dot{S}(t) = rS(t) + Q(t), \\ &\text{given } S(0) = 0 = S(T). \end{aligned}\right.$$

The Lagrangian is constructed just as in discrete time, the only difference being that instead of a summation operator it is now an integration operator:

$$\begin{aligned}
\mathcal{L}(t) &\equiv \int_0^T \left\{ \underbrace{u[C(t)] + \lambda(t)\left[rS(t) + Q(t) - C(t) - \dot{S}(t)\right]}_{\text{current-value Lagrangian}} \right\} \mathrm{d}t, \\
&= \int_0^T \left\{ \overbrace{u[C(t)] + \lambda(t)\left[rS(t) + Q(t) - C(t)\right]}^{\text{current-value Hamiltonian, abbreviated as } \mathcal{H}(t)} - \lambda(t)\dot{S}(t) \right\} \mathrm{d}t, \\
&= \int_0^T \left\{ \mathcal{H}[t, S(t), C(t), \lambda(t)] - \lambda(t)\dot{S}(t) \right\} \mathrm{d}t, \\
&\qquad \frac{\partial \mathcal{L}(t)}{\partial \lambda(t)} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{H}(t)}{\partial \lambda(t)} - \dot{S}(t) = 0 \\
&= \int_0^T \mathcal{H}[t, S(t), C(t), \lambda(t)]\mathrm{d}t - \int_0^T \underbrace{\lambda(t)\dot{S}(t)}_{\text{integrate by part}} \mathrm{d}t, \\
&= \int_0^T \mathcal{H}[t, S(t), C(t), \lambda(t)]\mathrm{d}t - \left\{ \overbrace{[\lambda(t)S(t)]|_0^T - \int_0^T \dot{\lambda}(t)S(t)\mathrm{d}t}^{ab' = (ab)' - a'b} \right\}, \\
&= \int_0^T \mathcal{H}[t, S(t), C(t), \lambda(t)] + \dot{\lambda}(t)S(t)\mathrm{d}t - [\lambda(t)S(t)]|_0^T, \\
&= \int_0^T \mathcal{H}[t, S(t), C(t), \lambda(t)] + \dot{\lambda}(t)S(t)\mathrm{d}t - [\lambda(T)S(T) - \lambda(0)S(0)], \\
&= \int_0^T \mathcal{H}[t, S(t), C(t), \lambda(t)] + \dot{\lambda}(t)S(t)\mathrm{d}t - (0 - 0). \\
&\qquad \frac{\partial \mathcal{L}(t)}{\partial S(t)} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{H}(t)}{\partial S(t)} + \dot{\lambda}(t) = 0
\end{aligned}$$

The Lagrange multiplier is also called the co-state variable of the Hamiltonian function. The first-order necessary condition is:

$$\left.\begin{aligned} \frac{\partial \mathcal{L}(t)}{\partial C(t)} &= 0, \\ \frac{\partial \mathcal{L}(t)}{\partial S(t)} &= 0, \\ \frac{\partial \mathcal{L}(t)}{\partial \lambda(t)} &= 0. \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} \frac{\partial \mathcal{H}(t)}{\partial C(t)} &= 0, \\ \frac{\partial \mathcal{H}(t)}{\partial S(t)} &= -\dot{\lambda}(t), \\ \frac{\partial \mathcal{H}(t)}{\partial \lambda(t)} &= \dot{S}(t). \end{aligned}\right.$$

It can be seen that, with $C(t)$ as the control variable, constructing the Hamiltonian only requires using it as the choice variable:

$$\max_{C(t)} \mathcal{H}(t) \equiv u[C(t)] + \lambda(t)[rS(t) + Q(t) - C(t)],$$

$$
\begin{aligned}
\text{F.O.C. of control variable:} \quad 0 &= \frac{\partial \mathcal{H}(t)}{\partial C(t)}, \\
\text{motion of co-state variable:}\ \dot{\lambda}(t) &= -\frac{\partial \mathcal{H}(t)}{\partial S(t)}, \\
\text{motion of state variable:}\ \dot{S}(t) &= \frac{\partial \mathcal{H}(t)}{\partial \lambda(t)}.
\end{aligned}
$$

The Lagrangian function considering the subjective discount factor is:

$$
\begin{aligned}
\mathcal{L}(0) &\equiv \int_0^T \left\{ e^{-\rho t} u[C(t)] + \lambda(t) \left[r S(t) + Q(t) - C(t) - \dot{S}(t) \right] \right\} \mathrm{d}t, \\
&= \int_0^T \left\{ \underbrace{e^{-\rho t} u[C(t)] + \lambda(t) \left[r S(t) + Q(t) - C(t) \right]}_{\text{present-value Hamiltonian, abbrivates as} \mathcal{H}(0)} - \lambda(t) \dot{S}(t) \right\} \mathrm{d}t.
\end{aligned}
$$

The present-value Hamiltonian function including the subjective discount factor is:

$$
\begin{aligned}
& \mathcal{H}(0) \equiv e^{-\rho t} u[C(t)] + \lambda(t) [r S(t) + Q(t) - C(t)], \\
\Rightarrow \quad & \underbrace{e^{\rho t} \mathcal{H}(0)}_{\text{current-value}} = u[C(t)] + \underbrace{e^{\rho t} \lambda(t)}_{\text{current-value}} [r S(t) + Q(t) - C(t)], \\
\Rightarrow \quad & \overbrace{\mathcal{H}(t)}^{\substack{\text{current-value} \\ \text{Hamiltonian}}} = u[C(t)] + \overbrace{\eta(t)}^{\substack{\text{current-value} \\ \text{Lagrange} \\ \text{multiplier}}} [r S(t) + Q(t) - C(t)].
\end{aligned}
$$

The relevant conditions for optimization are:

$$
\left.
\begin{aligned}
& \max_{C(t)} \mathcal{H}(0) \equiv e^{-\rho t} u[C(t)] + \lambda(t) [r S(t) + Q(t) - C(t)], \\
& \text{F.O.C. of control variable:} \quad 0 = \frac{\partial \mathcal{H}(0)}{\partial C(t)}, \\
& \text{motion of co-state variable:}\ \dot{\lambda}(t) = -\frac{\partial \mathcal{H}(0)}{\partial S(t)}, \\
& \text{motion of state variable:}\ \dot{S}(t) = \frac{\partial \mathcal{H}(0)}{\partial \lambda(t)}.
\end{aligned}
\right\}
\xrightarrow[\lambda(t) = e^{-\rho t} \eta(t)]{\mathcal{H}(0) = e^{-\rho t} \mathcal{H}(t)}
\left\{
\begin{aligned}
& \max_{C(t)} \mathcal{H}(t) \equiv u[C(t)] + \eta(t) [r S(t) + Q(t) - C(t)], \\
& \text{F.O.C. of control variable:} \quad 0 = \frac{\partial \mathcal{H}(t)}{\partial C(t)}, \\
& \text{motion of co-state variable:}\ \dot{\eta}(t) = \rho \eta(t) - \frac{\partial \mathcal{H}(t)}{\partial S(t)}, \\
& \text{motion of state variable:}\ \dot{S}(t) = \frac{\partial \mathcal{H}(t)}{\partial \eta(t)}.
\end{aligned}
\right.
$$

The first set of equations above are equivalent, because the time t is fixed when the extreme value is obtained, that is,

$$
\begin{aligned}
& \frac{\partial \mathcal{H}(0)}{\partial C(t)} = 0, \\
\Rightarrow \quad & e^{\rho t} \frac{\partial \mathcal{H}(0)}{\partial C(t)} = 0, \\
\Rightarrow \quad & \frac{\partial [e^{\rho t} \mathcal{H}(0)]}{\partial C(t)} = 0, \\
\Rightarrow \quad & \frac{\partial \mathcal{H}(t)}{\partial C(t)} = 0.
\end{aligned}
$$

The second set of equations above are equivalent because:

$$
\begin{aligned}
& \eta(t) = e^{\rho t} \lambda(t), \\
\Rightarrow \quad & \frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = \frac{\mathrm{d}[e^{\rho t} \lambda(t)]}{\mathrm{d}t}, \\
\Rightarrow \quad & \dot{\eta}(t) = \rho e^{\rho t} \lambda(t) + e^{\rho t} \dot{\lambda}(t), \\
& \qquad = \rho \eta(t) - e^{\rho t} \frac{\partial \mathcal{H}(0)}{\partial S(t)}, \\
& \qquad = \rho \eta(t) - \frac{\partial [e^{\rho t} \mathcal{H}(0)]}{\partial S(t)},
\end{aligned}
$$

$$\Rightarrow \qquad \dot{\eta}(t) = \rho\eta(t) - \frac{\partial\mathcal{H}(t)}{\partial S(t)}.$$

The third set of equations above are equivalent because:

$$\begin{aligned} \dot{S}(t) &= \frac{\partial\mathcal{H}(0)}{\partial\lambda(t)}, \\ &= \frac{\partial[e^{\rho t}\mathcal{H}(0)]}{\partial[e^{\rho t}\lambda(t)]}, \\ \Rightarrow \qquad \dot{S}(t) &= \frac{\partial\mathcal{H}(t)}{\partial\eta(t)}. \end{aligned}$$

ii) **From Bellman equation to Hamilton-Jacobi-Bellman equation**

[**Bellman1957**] creatively proposed the idea of dynamic programming for dynamic optimization problems in discrete time and continuous time. To commemorate Richard Bell-man's contribution in this area, the value function of the recursive functional form he discovered was called the Bellman equation; the Bellman equation in continuous time is an extension of the Hamilton-Jacobi equation in classical mechanics, so it is also called the Hamilton-Jacobi-Bellman (abbreviated as HJB) equation. [11]

From **Example** 13-1, we can see that the discrete-time Bellman equation is:

$$\left.\begin{aligned} &\max_{\{C_t\}_{t=0}^{T}} U_0 \equiv \sum_{t=0}^{T} \beta^t u(C_t), \\ &\text{s.t.} \quad S_{t+1} - S_t = rS_t + Q_{t+1} - C_{t+1}, \\ &\quad \text{given } S(0) = 0 = S(T); \\ &\uparrow \text{start from } 1 \qquad \downarrow \text{start from } t+1 \\ &\max_{\{C_{t+h+1}\}_{h=1}^{T}} U_t \equiv \sum_{h=1}^{T} \beta^h u(C_{t+h+1}), \\ &\text{s.t.} \quad S_{t+h+1} - S_{t+h} = rS_{t+h} + Q_{t+h+1} - C_{t+h+1}, \\ &\xRightarrow[\left\{\begin{smallmatrix} \lim_{\Delta\to\infty} \beta(\Delta) = 0 \\ \lim_{\Delta\to 0} \beta(\Delta) = 1 \\ \Delta = 1,\ \beta(\Delta) = \beta \end{smallmatrix}\right.]{\text{Bellman equation}} V(S_t) = \max_{C_t}[\Delta u(C_t) + \beta(\Delta)V(S_{t+\Delta})], \\ &\text{s.t.} \quad S_{t+\Delta} - S_t = rS_t + \Delta Q_t - \Delta C_t, \\ &\quad \text{given } S_t = 0 = S_{t+T}. \end{aligned}\right\} \text{vs.} \left\{\begin{aligned} &\max_{\{C(t)\}_{t=0}^{T}} U(0) \equiv \int_{t=0}^{T} e^{-\rho t} u[C(t)]\mathrm{d}t, \\ &\text{s.t.} \quad \dot{S}(t) = rS(t) + Q(t) - C(t), \\ &\text{given } S(0) = 0 = S(T); \\ &\uparrow \text{start from } 0 \qquad \downarrow \text{start from } t \\ &\max_{\{C(h)\}_{h=t}^{T}} U(t) \equiv \int_{h=t}^{T} e^{-\rho(h-t)} u[C(h)]\mathrm{d}h, \\ &\text{s.t.} \quad \dot{S}(h) = rS(h) + Q(h) - C(h), \\ &\xRightarrow[\left\{\begin{smallmatrix} \beta(\Delta) = e^{-\rho\Delta} \\ \lim_{\Delta\to 0} e^{-\rho\Delta} = 1 - \rho\Delta \\ \lim_{\Delta\to 0} (e^{\rho\Delta})^{-1} = (1+\rho\Delta)^{-1} \end{smallmatrix}\right.]{\text{HJB equation}} V[S(t)] = ? \\ &\text{s.t.} \quad \dot{S}(t) = rS(t) + Q(t) - C(t), \\ &\text{given } S(t) = 0 = S(T). \end{aligned}\right.$$

In discrete time dynamic programming, the whole time chain appears to be divided into periods t and $t+1$, but there is no discrete period $t+1$ in continuous time, so it is represented by the time span Δ, and the next period is $t+\Delta$. In discrete time, $\Delta = 1$ is a special case. After sorting out the symbols, the value function of the continuous-time dynamic optimaization can be written as:

[11] Previously, envelope theorems in static optimization and discrete-time dynamic optimization were introduced. Envelope theorems also exist in continuous-time dynamic optimization. Assume that the state variable is $x(t)$, the control variable is $y(t)$, the state transition equation is$\dot{x}(t) = g[x(t), y(t), t]$, the objective function is$\max_{y(\tau)} \int_t^T F[x(\tau), y(\tau), \tau]\mathrm{d}\tau$, and the HJB equation is $-\frac{\partial V(x,t)}{\partial t} = \max_y\{F(x,y,t) + \frac{\partial V(x,t)}{\partial x} g(x,y,t)\}$. given the optimal solution y_t°, the HJB equation becomes $-\frac{\partial V(x,t)}{\partial t} = F(x,y^\circ,t) + \frac{\partial V(x,t)}{\partial x} g(x,y^\circ,t)$, and the envelope theorem means that $-\frac{\partial^2 V(x,t)}{\partial t\partial x} = \frac{\partial F(x,y^\circ,t)}{\partial x} + \frac{\partial F(x,y^\circ,t)}{\partial y^\circ}\frac{\partial y^\circ}{\partial x} + \frac{\partial^2 V(x,t)}{\partial x^2} g(x,y^\circ,t) + \frac{\partial V}{\partial x}\left[\frac{\partial g(x,y^\circ,t)}{\partial x} + \frac{\partial g(x,y^\circ,t)}{\partial y^\circ}\frac{\partial y^\circ}{\partial x}\right] = \frac{\partial F(x,y^\circ,t)}{\partial x} + \underbrace{\left[\frac{\partial F(x,y^\circ,t)}{\partial y^\circ} + \frac{\partial V}{\partial x}\frac{\partial g(x,y^\circ,t)}{\partial y^\circ}\right]}_{\text{F.O.C.}=0}\frac{\partial y^\circ}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial g(x,y^\circ,t)}{\partial x} + \frac{\partial^2 V(x,t)}{\partial x^2} g(x,y^\circ,t) = \frac{\partial F(x,y^\circ,t)}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial g(x,y^\circ,t)}{\partial x} + \frac{\partial^2 V(x,t)}{\partial x^2} g(x,y^\circ,t)$. This further reveals the sensitivity of the optimal value function to the state, including two aspects: the direct effect (the change of state directly affects the current income and future state) and the indirect effect (when the control variable is optimized, the marginal income has been equal to the marginal cost, so the indirect effect generated by the adjustment of the optimal control variable is 0).

$$
\begin{aligned}
V(t) &\equiv \max_{\{C(h)\}_{h=t}^{T}} \int_{h=t}^{T} e^{-\rho(h-t)} u[C(h)]\mathrm{d}h, \quad \text{s.t.} \quad \dot{S}(h) = rS(h) + Q(h) - C(h), \\
&= \max_{\{C(h)\}_{h=t}^{T}} \left\{ \int_{h=t}^{t+\Delta} e^{-\rho(h-t)} u[C(h)]\mathrm{d}h + \int_{h=t+\Delta}^{T} e^{-\rho(h-t-\Delta)} u[C(h)]\mathrm{d}h \right\}, \\
\Rightarrow \quad V[S(t)] &= \max_{C(t)} \left\{ \Delta u[C(t)] + \frac{1}{1+\rho\Delta} V[S(t+\Delta)] \right\}, \\
\Rightarrow \quad (1+\rho\Delta)V[S(t)] &= \max_{C(t)} \{ (1+\rho\Delta)\Delta u[C(t)] + V[S(t+\Delta)] \}, \\
\Rightarrow \quad \rho\Delta V[S(t)] &= \max_{C(t)} \{ (1+\rho\Delta)\Delta u[C(t)] + V[S(t+\Delta)] - V[S(t)] \}, \\
\Rightarrow \quad \rho V[S(t)] &= \max_{C(t)} \left\{ (1+\rho\Delta) u[C(t)] + \frac{V[S(t+\Delta)] - V[S(t)]}{\Delta} \right\}, \\
\xRightarrow[\lim_{\Delta\to 0}\equiv \mathrm{d}t]{\Delta\to 0} \quad \rho V[S(t)] &= \max_{C(t)} \left\{ u[C(t)] + \frac{\mathrm{d}V[S(t)]}{\mathrm{d}t} \right\}, \\
&= \max_{C(t)} \left\{ u[C(t)] + \frac{V[S(t)]}{\mathrm{d}S(t)} \frac{\mathrm{d}S(t)}{\mathrm{d}t} \right\}, \\
&= \max_{C(t)} \left\{ u[C(t)] + V'[S(t)]\dot{S}(t) \right\}, \\
\Rightarrow \quad \rho V[S(t)] &= \max_{C(t)} \{ u[C(t)] + V'[S(t)][rS(t) + Q(t) - C(t)] \}.
\end{aligned}
$$

[49, pp.142-] pointed out that the HJB equation can be derived from another perspective:

$$
\begin{aligned}
\max_{\{C(h)\}_{h=t}^{T}} V(t) &\equiv \int_{h=t}^{T} e^{-\rho(h-t)} u[C(h)]\mathrm{d}h, \\
\Rightarrow \quad \dot{V}(t) &= -e^{-\rho(t-t)} u[C(t)] + \int_{h=t}^{T} \frac{\mathrm{d}}{\mathrm{d}t} e^{-\rho(h-t)} u[C(h)]\mathrm{d}h, \\
\Rightarrow \quad \dot{V}(t) &= -u[C(t)] + \rho u(t), \\
\Rightarrow \quad \rho u(t) &= u[C(t)] + \dot{V}(t), \\
\Rightarrow \quad \rho V[S(t)] &= u[C(t)] + \dot{V}[S(t)], \\
\Rightarrow \quad \rho V[S(t)] &= u[C(t)] + \underbrace{V'[S(t)]}[rS(t) + Q(t) - C(t)]; \\
\text{vs.} \quad \mathcal{H}(t) &= u[C(t)] + \overbrace{\eta(t)}\,[rS(t) + Q(t) - C(t)].
\end{aligned}
$$

Moll (2012) compared the Hamiltonian of optimal control and the HJB equation of continuous dynamic programming, and pointed out that $\eta(t) = V'[S(t)]$ is the link between the two, that is, the covariate in the Hamiltonian are the shadow prices in the HJB equation.

The first-order condition is:

$$
\underbrace{u'[C(t)]}_{\text{marginal revenue}} = \underbrace{V'[S(t)]}_{\text{marginal coat}} = \underbrace{\eta(t)}_{\text{shadow price}}.
$$

Derivation of the state variable based on the HJB equation:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}S(t)}\{\rho V[S(t)]\} &= \frac{\mathrm{d}}{\mathrm{d}S(t)}\{V'[S(t)]\}[rS(t) + Q(t) - C(t)] + V'[S(t)]\frac{\mathrm{d}}{\mathrm{d}S(t)}\{rS(t) + Q(t) - C(t)\}, \\
\Rightarrow \quad \rho V'[S(t)] &= V''[S(t)][rS(t) + Q(t) - C(t)] + rV'[S(t)], \\
\Rightarrow \quad (\rho - r)V'[S(t)] &= V''[S(t)][rS(t) + Q(t) - C(t)], \\
&= \frac{\mathrm{d}V'[S(t)]}{\mathrm{d}t} \equiv \dot{V}[S(t)],
\end{aligned}
$$

$$\Rightarrow \qquad \rho - r = \frac{\dot{V}[S(t)]}{V'[S(t)]} = \frac{\dot{\eta}(t)}{\eta(t)}.$$

This is the equation of motion for the marginal utility (shadow price) of savings, which is also the co-state variable: (Hamilton multiplier).

The first-order condition is derived with respect to time t:

$$\begin{aligned}
\frac{\mathrm{d}u'[C(t)]}{\mathrm{d}t} &= \frac{\mathrm{d}V'[S(t)]}{\mathrm{d}t}, \\
&= \dot{V}[S(t)], \\
&= (\rho - r)V'[S(t)], \\
&= (\rho - r)u'[C(t)], \\
\Rightarrow \qquad \frac{\frac{\mathrm{d}u'[C(t)]}{\mathrm{d}t}}{u'[C(t)]} &= \rho - r = \frac{\dot{\eta}(t)}{\eta(t)}, \\
\Rightarrow \qquad \frac{u''[C(t)]\dot{C}(t)}{u'[C(t)]} &= \rho - r, \\
\Rightarrow \qquad \overbrace{\underbrace{-\frac{C(t)u''[C(t)]}{u'[C(t)]}}_{1/\sigma}}^{\text{intertemporal elasticity of substitution}} \frac{\dot{C}(t)}{C(t)} &= r - \rho, \\
\Rightarrow \qquad \frac{1}{\sigma}\frac{\dot{C}(t)}{C(t)} &= r - \rho, \\
\Rightarrow \qquad \frac{\dot{C}(t)}{C(t)} &= \sigma(r - \rho).
\end{aligned}$$

This is the famous Keynes-Ramsey rule, or the intertemporal Euler equation for consumption or the Euler equation for the consumption growth rate. σ is the intertemporal elasticity of substitution that appeared in **E**xample 10. This parameter will be introduced in more detail in Chapter 3.

ii. **Multiple Control Variables** [12]

State variables and control variables may form constraints on the time axis, or different control variables may also form constraints in time and space. Back to the original question, add control variables and make them constrain each other:

$$\left.\begin{array}{l} \max\limits_{C(t)} U_0 = \int_0^T e^{-\rho t} u[C(t)]\mathrm{d}t, \\ \text{s.t.} \quad C(t) + \dot{S}(t) = r(t)S(t) + Q(t), \\ \quad \text{given } S(0) = 0 = S(T). \end{array}\right\} \xRightarrow[\text{to two control variable}]{\text{from one control variable}} \left\{\begin{array}{l} \max\limits_{C(t),L(t)} U_0 = \int_0^T e^{-\rho t} u[C(t), L(t)]\mathrm{d}t, \\ \text{s.t.} \quad C(t) + \dot{S}(t) = r(t)S(t) + w(t)L(t), \\ \quad \text{given } S(0) = 0 = S(T). \end{array}\right\} \xRightarrow[\text{to nominal variables}]{\text{from real varaibles}} \left\{\begin{array}{l} \max\limits_{C(t),L(t)} U_0 = \int_0^T e^{-\rho t} u[C(t), L(t)]\mathrm{d}t, \\ \text{s.t.} \quad P(t)C(t) + \dot{B}(t) = i(t)B(t) + W(t)L(t), \\ \quad \text{given } B(0) = 0 = B(T). \end{array}\right.$$

The increase from a single control variable to two control variables is labor supply $L(t)$ (real wages are $w(t)$), and the increase from real variables to nominal variables is the aggregate price level $P(t)$ and the aggregate wage level $W(t)$. Savings can be expressed as real variables or nominal variables, and are often converted into bonds $B(t)$.

The Hamiltonian and optimality conditions are:

$$\begin{aligned}
\max_{C(t),L(t),B(t),\eta(t)} \quad & \mathcal{H}(t) \equiv u[C(t), L(t)] + \eta(t)[i(t)B(t) + W(t)L(t) - P(t)C(t)], \\
\text{F.O.C. of control variable:} \quad 0 = \frac{\partial \mathcal{H}(t)}{\partial C(t)} \quad & \Rightarrow \quad U_{Ct} = \eta(t)P(t), \\
\text{F.O.C. of control variable:} \quad 0 = \frac{\partial \mathcal{H}(t)}{\partial L(t)} \quad & \Rightarrow \quad U_{Lt} = \eta(t)W(t), \\
\text{motion of co-state variable:} \quad \dot{\eta}(t) = \rho\eta(t) - \frac{\partial \mathcal{H}(t)}{\partial B(t)} \quad & \Rightarrow \quad \dot{\eta}(t) = \rho\eta(t) - \eta(t)i(t), \\
\text{motion of state variable:} \quad \dot{B}(t) = \frac{\partial \mathcal{H}(t)}{\partial \eta(t)} \quad & \Rightarrow \quad \dot{B}(t) = i(t)B(t) + W(t)L(t) - P(t)C(t).
\end{aligned}$$

[12] [7, pp.276-277] and Moll (2016) presented such dynamic optimal problems.

Define the current-value utility function as in **E**xample 10:

$$u[C(t), L(t)] = \log C(t) - \frac{L(t)^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}.$$

From this we can obtain the intertemporal consumption Euler equation and labor supply equation:

$$\left.\begin{aligned} \frac{1}{C(t)} &= \eta(t)P(t) \\ L(t)^{\frac{1}{\varphi}} &= \eta(t)W(t) \end{aligned}\right\} \xRightarrow{\Pi(t)=\frac{\dot{P}(t)}{P(t)}} \left\{\begin{aligned} \frac{\dot{C}(t)}{C(t)} &= r(t) - \rho, \\ C(t)L(t)^{\frac{1}{\varphi}} &= \frac{W(t)}{P(t)}. \end{aligned}\right.$$

The second equation on the left is divided by the first equation on the left to obtain the second equation on the right; the first equation on the right can be obtained by first taking the logarithm of the first equation on the left, that is, $-\log C(t) = \log \eta(t) + \log P(t)$, calculating its derivative with respect to time $-\frac{\dot{C}(t)}{C(t)} = \frac{\dot{\eta}(t)}{\eta(t)} + \frac{\dot{P}(t)}{P(t)}$and then combining it with the motion equation of the co-state variable $\frac{\dot{\eta}(t)}{\eta(t)} = \rho - i(t)$, and defining the real interest rate as $r(t) \equiv i(t) - \Pi(t)$.

Dynamic optimal problem with special equality constraints between control variables (subjective discounting is not considered for simplicity):

$$\begin{aligned} \xleftrightarrow[\min]{\max} U &= \int_0^T u(t, B, C, L)\mathrm{d}t, \\ \text{s.t.} \quad \dot{B} &= f(t, B, C, L), \\ z &= g(t, B, C, L), \\ B(0) &= 0 = B(T). \end{aligned}$$

Where t is the time variable, B is the state variable, C and L are control variables, f and g are two function expressions, and z is a constant.

If there is no equality constraint between the two control variables, it is the most basic optimal control problem. Let's put it aside and construct the Hamiltonian:

$$\max_{B(t),C(t),L(t),\eta(t)} \mathcal{H}(t) \equiv u(t, B, C, L) + \eta(t)f(t, B, C, L).$$

The initial problem becomes optimizing the Hamiltonian given the equality constraints:

$$\begin{aligned} \max_{B(t),C(t),L(t),\eta(t)} \mathcal{H}(t) &\equiv u(t, B, C, L) + \eta(t)f(t, B, C, L), \\ \text{s.t.} \quad z &= g(t, B, C, L). \end{aligned}$$

Then construct the Lagrangian function to find the optimal solution:

$$\begin{aligned} \max_{B(t),C(t),L(t),\eta(t)} \mathcal{L}(t) &\equiv \mathcal{H}(t) + \lambda(t)[z - g(t, B, C, L)], \\ &= [u(t, B, C, L) + \eta(t)f(t, B, C, L)] + \lambda(t)[z - g(t, B, C, L)]. \end{aligned}$$

$$\Rightarrow \left\{\begin{aligned} &\text{F.O.C. of Lagrangian} \left\{\begin{aligned} \frac{\partial \mathcal{L}(t)}{\partial C(t)} &= 0 = \frac{\partial u(\cdot)}{\partial C(t)} + \eta(t)\frac{\partial f(\cdot)}{\partial C(t)} - \lambda(t)\frac{\partial g(\cdot)}{\partial C(t)}, \\ \frac{\partial \mathcal{L}(t)}{\partial L(t)} &= 0 = \frac{\partial u(\cdot)}{\partial L(t)} + \eta(t)\frac{\partial f(\cdot)}{\partial L(t)} - \lambda(t)\frac{\partial g(\cdot)}{\partial L(t)}, \\ \frac{\partial \mathcal{L}(t)}{\partial \lambda(t)} &= 0 = z - g(t, B, C, L). \end{aligned}\right. \\ &\text{maximum principle of Hamiltonian} \left\{\begin{aligned} \dot{\eta}(t) &= -\frac{\partial \mathcal{L}(t)}{\partial B(t)} = \lambda(t)\frac{\partial g(\cdot)}{\partial B(t)} - \frac{\partial \mathcal{H}(t)}{\partial B(t)}, \\ \dot{B}(t) &= \frac{\partial \mathcal{L}(t)}{\partial \eta(t)} = \frac{\partial \mathcal{H}(t)}{\partial \eta(t)}. \end{aligned}\right. \end{aligned}\right.$$

If there is an inequality constraint, it is similar to a static optimization problem, which is broadened to the Kuhn-Tucker condition based on the first-order condition of the equality constraint.

II. **Multiple state variables**

Equality or inequality constraints in dynamic problems may otherwise exist in integral form, but the integral constraints can be replaced by new state variables, which in turn increase the number of state variables (ignoring multiple control variables and their constraints and subjective discounting):

$$\left.\begin{aligned} \overset{\max}{\underset{\min}{\longleftrightarrow}}\ U &= \int_0^T u(t,B,C,\cancel{L})\mathrm{d}t,\\ \text{s.t.}\quad \dot{B}(t) &= f(t,B,C,\cancel{L}),\\ \cancel{z} &\cancel{= g(t,B,C,L)},\\ z &= \int_0^T g(t,B,C,\cancel{L})\mathrm{d}t,\\ B(0) &= 0 = B(T). \end{aligned}\right\} \xRightarrow[\substack{\Gamma(t)\equiv-\int_0^t g(t,B,C)\mathrm{d}t\\ \Gamma(0)=-\int_0^0 g(t,B,C)\mathrm{d}t=0\\ \Gamma(T)=-\int_0^T g(t,B,C)\mathrm{d}t=-z}]{\substack{\text{define the integral constraint}\\ \text{as a new state variable}\Gamma(t)}} \left\{\begin{aligned} \overset{\max}{\underset{\min}{\longleftrightarrow}}\ U &= \int_0^T u(t,B,C)\mathrm{d}t,\\ \text{s.t.}\quad \dot{B}(t) &= f(t,B,C),\\ \\ \dot{\Gamma}(t) &= -g(t,B,C),\\ \Gamma(0) &= 0,\quad \Gamma(T)=-z,\\ B(0) &= 0,\quad B(T)=0. \end{aligned}\right.$$

Now it is transformed into an unconstrained dynamic optimal problem with two state variables. The Hamiltonian function and optimality conditions are:

$$\begin{aligned} \max_{C(t),B(t),\Gamma(t),\eta(t),\mu(t)} \quad & \mathcal{H}(t) \equiv u[C(t)] + \eta(t)f(t,B,C) - \mu(t)g(t,B,C),\\ \text{F.O.C. of control variable:}\quad & 0 = \frac{\partial\mathcal{H}(t)}{\partial C(t)},\\ \text{motion of co-state variable:}\quad & \dot{\eta}(t) = -\frac{\partial\mathcal{H}(t)}{\partial B(t)},\\ \text{motion of co-state variable:}\quad & \dot{\mu}(t) = -\frac{\partial\mathcal{H}(t)}{\partial \Gamma(t)},\\ \text{motion of state variable:}\quad & \dot{B}(t) = \frac{\partial\mathcal{H}(t)}{\partial \eta(t)},\\ \text{motion of state variable:}\quad & \dot{\Gamma}(t) = \frac{\partial\mathcal{H}(t)}{\partial \mu(t)}. \end{aligned}$$

Just to explain, since $\dot{\mu}(t) = -\frac{\partial\mathcal{H}(t)}{\partial\Gamma(t)} = 0$, it can be seen that $\mu(t)$ is a constant μ.

If subjective discounting is retained, the present-value Hamiltonian is constructed; if the integral equality constraint is relaxed to an integral inequality constraint, the optimal condition is more complicated, and the reader is left to further explore it when he has the energy; the solution to the unconstrained dynamic optimal solution between more than two state variables is the same and will not be repeated here.

1.2.2.2 Equality constraints between state variables

Before introducing the optimal problem of equality constraints between state variables in continuous time, let's review **E**xample 12-1, the two-period consumption decision problem of an endowment economy. If, in addition to consumption, the real money balance also enters the utility function (real savings become nominal savings, so the real interest rate r becomes the nominal interest rate i):

$$\max_{C_1,C_2,M_1/P_1,M_2/P_2} U = u\left(C_1,\frac{M_1}{P_1}\right) + \beta u\left(C_2,\frac{M_2}{P_2}\right),$$

$$\text{s.t.}\ \underbrace{\left.\begin{aligned} \overbrace{P_1C_1+M_1+S_1}^{\substack{\text{expenditure and}\\ \text{money balance}\\ \text{in this period}}} &\le \overbrace{[i+(1-\cancel{\delta})]\cancel{S_0}+\cancel{M_0}+P_1Q_1}^{\text{income in this period}},\\ \underbrace{P_2C_2+M_2+\cancel{S_2}}_{\substack{\text{expenditure and}\\ \text{money balance}\\ \text{in the next period}}} &\le \underbrace{[i+(1-\cancel{\delta})]S_1+M_1+P_2Q_2}_{\text{income in next period}}. \end{aligned}\right\}}_{\text{per-period (dynamic) budget constraint}} \xRightarrow[C_1,C_2,M_1,M_2>0]{M_0=S_0=0=S_2} \underbrace{\overbrace{P_1C_1+\frac{P_2}{1+i}C_2+\frac{i}{1+i}M_1+\frac{M_2}{1+i}}^{\text{lifetime expenditure \& money balance}} = \overbrace{P_1Q_1+\frac{P_2}{1+i}Q_2}^{\text{lifetime income}}}_{\text{lifetime (intertemporal) budget constraint}}.$$

Still assume that saving in the last period is meaningless, so the terminal condition is $S_2 = 0$, but the money balance M_2 in the last period has the same utility increasing at a decreasing rate like consumption, so it does not have to be 0. Combine the two budget constraints:

$$
\begin{aligned}
&M_1 + S_1 = P_1 Q_1 - P_1 C_1, \\
\xRightarrow{A_1 \equiv M_1 + S_1} \quad &A_1 = P_1 Q_1 - P_1 C_1. \\
&P_2 C_2 + M_2 = (1+i)\,(\overbrace{\underbrace{A_1}\; - M_1}^{S_1}) + M_1 + P_2 Q_2, \\
\Rightarrow \quad &P_2 C_2 + M_2 = (1+i)[\overbrace{(P_1 Q_1 - P_1 C_1) - M_1}] + M_1 + P_2 Q_2, \\
\Rightarrow \quad &P_2 C_2 + M_2 = (1+i)(P_1 Q_1 - P_1 C_1) - (1+i) M_1 + M_1 + P_2 Q_2, \\
\Rightarrow \quad &P_2 C_2 + M_2 = (1+i) P_1 (Q_1 - C_1) - i M_1 + P_2 Q_2, \\
\Rightarrow \quad &(1+i) P_1 C_1 + P_2 C_2 + i M_1 + M_2 = (1+i) P_1 Q_1 + P_2 Q_2, \\
\Rightarrow \quad &P_1 C_1 + \frac{P_2}{1+i} C_2 + \frac{i}{1+i} M_1 + \frac{M_2}{1+i} = P_1 Q_1 + \frac{P_2}{1+i} Q_2, \\
\Rightarrow \quad &C_1 + \frac{P_2/P_1}{1+i} C_2 + \frac{i}{1+i}\frac{M_1}{P_1} + \frac{1}{1+i}\frac{M_2}{P_1} = Q_1 + \frac{P_2/P_1}{1+i} Q_2, \\
\Rightarrow \quad &C_1 + \frac{P_2/P_1}{1+i} C_2 + \frac{i}{1+i}\frac{M_1}{P_1} + \frac{P_2/P_1}{1+i}\frac{M_2}{P_2} = Q_1 + \frac{P_2/P_1}{1+i} Q_2.
\end{aligned}
$$

This is how the lifetime budget constraint is obtained. Construct the Lagrangian function:

$$
\max \mathcal{L} \equiv \left[u\left(C_1, \frac{M_1}{P_1}\right) + \beta u\left(C_2, \frac{M_2}{P_2}\right)\right] + \lambda \left[\left(Q_1 + \frac{P_2/P_1}{1+i} Q_2\right) - \left(C_1 + \frac{P_2/P_1}{1+i} C_2 + \frac{i}{1+i}\frac{M_1}{P_1} + \frac{P_2/P_1}{1+i}\frac{M_2}{P_2}\right)\right].
$$

The first-order necessary conditions are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_1} &= u_{C1} - \lambda = 0, \\
\frac{\partial \mathcal{L}}{\partial C_2} &= \beta u_{C2} - \lambda \frac{P_2/P_1}{1+i} = 0, \\
\frac{\partial \mathcal{L}}{\partial \frac{M_1}{P_1}} &= u_{M1/P1} - \lambda \frac{i}{1+i} = 0.
\end{aligned}
$$

The first and the second first-order conditions above are combined into the intertemporal consumption Euler equation, and the first and the third first-order conditions are combined into the demand equation for the real money balance in the first period:

$$
\begin{aligned}
\frac{\beta u_{C2}}{u_{C1}} &= \frac{P_2/P_1}{1+i}, \\
\frac{u_{M1/P1}}{u_{C1}} &= \frac{i}{1+i},
\end{aligned}
$$

Given the current-value utility function, after logarithmic linearization, we can obtain the linearized dynamic IS curve (the core of which is the inverse relationship between output and real interest rate) and the money demand curve (the key is to give the positive relationship between real money balance and output and the inverse relationship with nominal interest rate), which will be described in detail in [55].

It should be noted that the state variables in the first period are only S_0 and M_0. For the sake of simplicity, they are all assumed to be 0. In other words, the equality constraint $S_0 + M_0 = 0$ between the state variables in the first period greatly simplifies the calculation, which leaves only the exogenous state variable (current endowment income Q_1) in the first period; the equality constraint between the endogenous state variables in the second period is $S_1 + M_1 = A_1$, and there is also an exogenous state variable (current endowment income Q_2).

The appearance of the above two endogenous state variables does not bring additional trouble to the solution. Through variable substitution, it is also the dynamic optimal problem of a single endogenous state variable. However, the **inequality** constraint of a single endogenous state variable will make the solution more complicated, which has been explained in [7, pp.298-313] and [20, pp.230-239], so this book will not repeat it.