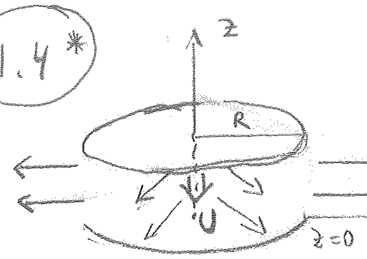
1.4*



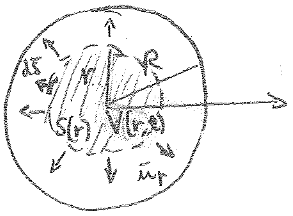
A plate moves with U

$\bar{u} = ?$ (assume that u_r independent of z)

flow incompressible $\triangleq$ $\rho = \text{const.}$

Solution:
$$\frac{dV}{dt} = -(V\text{-flux out})$$

Looking from above:
$$V(r,t) = \pi r^2 (H_0 - Ut)$$



$$\Rightarrow \frac{dV}{dt} = -\pi r^2 U \quad (1)$$

Total V-flux through $S(r)$
$$Q = \iint_S \bar{u} \cdot d\bar{S} = u_r \cdot 2\pi r (H_0 - Ut) \quad (2)$$

Now (1) & (2) $\Rightarrow$
$$\cancel{-}\cancel{\pi} r^{\cancel{2}} U = \cancel{-} u_r \cdot 2 \cancel{\pi} \cancel{r} (H_0 - Ut)$$
$$\Leftrightarrow \quad u_r = \frac{Ur}{2(H_0 - Ut)}$$

u_z is obtained from $\nabla \cdot \bar{u} = 0$
$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0$$

$$\Rightarrow \frac{\partial u_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{Ur^2}{2(H_0 - Ut)}\right] = -\frac{U}{H_0 - Ut}$$

$$\overset{\int}{\Rightarrow} u_z = -\frac{Uz}{H_0 - Ut} + \text{const.}$$

Boundary conditions: $\begin{cases} u_z(0,t) = 0 \\ u_z(H_0 - Ut, t) = -U \end{cases}$ $\Rightarrow$ const. $= 0$

$$\Rightarrow u_z = -\frac{Uz}{H_0 - Ut}$$

1.5 A model of 2-D "stationary" turbulent flow, $\bar{u} = (u_x, u_y)$

$$u_x = \sum_i A_i \cos(k_i x) \sin(k_i y)$$

a) $u_y = ?$

b) P (for one mode)

Solution:

a) If only one mode $\Rightarrow$

$$u_x = A \cos(kx) \sin(ky)$$

incomp. ($\rho = \text{const.}$) $\Rightarrow \nabla \cdot \bar{u} = 0 \iff \dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} = 0$

$$\Rightarrow \frac{\partial u_y}{\partial y} = + kA \sin(kx) \sin(ky)$$

$$\overset{\int}{\Rightarrow} \quad u_y = - A \sin(kx) \cos(ky) \; (\underbrace{+ C(x)}_{=0\text{, elementary mode}})$$

All modes

$$u_y = - \sum_i A_i \sin(k_i x) \cos(k_i y)$$

$\leftarrow$ we sum up all elementary modes

b) Pressure for 1 mode

Euler equation $\Rightarrow (\bar{u} \cdot \nabla)\bar{u} = -\dfrac{1}{\rho} \nabla P$

$$\Rightarrow \begin{cases} u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} = -\dfrac{1}{\rho} \dfrac{\partial P}{\partial x} & (1) \\ u_x \dfrac{\partial u_y}{\partial x} + u_y \dfrac{\partial u_y}{\partial y} = -\dfrac{1}{\rho} \dfrac{\partial P}{\partial y} & (2) \end{cases}$$

$$\Rightarrow \begin{cases} -kA^2 \cos(kx)\sin(kx)\sin^2(ky) - kA^2 \cos(kx)\sin(kx)\cos^2(ky) = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x} \\ -kA^2 \cos^2(kx)\cos(ky)\sin(ky) - kA^2 \sin^2(kx)\cos(ky)\sin(ky) = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial y} \end{cases}$$

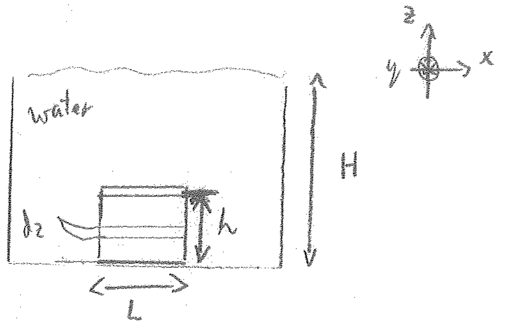
Since $\sin(kx)\cos(kx) = \frac{1}{2}\sin(2kx)$

$$\Rightarrow \begin{cases} \dfrac{\partial P}{\partial x} = \frac{1}{2}\rho A^2 k \sin(2kx) \\ \dfrac{\partial P}{\partial y} = \frac{1}{2}\rho A^2 k \sin(2ky) \end{cases} \Rightarrow \begin{cases} P = -\frac{1}{4}\rho A^2 \cos(2kx) + f_1(y) \\ P = -\frac{1}{4}\rho A^2 \cos(2ky) + f_2(x) \end{cases}$$

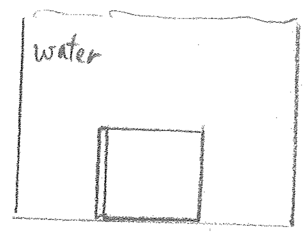
$$\Rightarrow \quad P = -\frac{1}{4}\rho A^2 \left[\cos(2kx) + \cos(2ky)\right]$$

1.8

a)



b)



Find F and T acting on the gate. P_{atm} is neglected.

Solution:

Force on the gate caused by water pressure in both of the cases a and b.

$$d\bar{F} = -P\,d\bar{S} \Rightarrow \bar{F} = -\iint_{S_0} P\,d\bar{S}, \quad \frac{dP}{dz} = -\rho g \overset{\int}{\Rightarrow} P = -\rho g z\,(+P_0) \leftarrow P_{atm}$$

$$d\bar{S} = +L\,dz\,\hat{e}_y \leftarrow \text{surface of the container} \Rightarrow \text{gate}$$

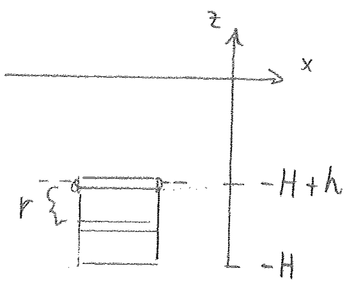
$$\Rightarrow d\bar{F} = (-1)\cdot(-\rho g z)\cdot(+L\,dz)\,\hat{e}_y \overset{z<0}{\Rightarrow} d\bar{F} \uparrow\uparrow -\hat{e}_y$$

$$\Rightarrow F = -\int_{-H}^{-H+h} \rho g L z\,dz = -\rho g L\left[\frac{z^2}{2}\right]\Bigg|_{-H}^{-H+h}$$

$$= \frac{1}{2}\rho g L\underbrace{\left[H^2-(H-h)^2\right]}_{=\cancel{H^2}-\cancel{H^2}+2Hh-h^2}$$

$$= \frac{1}{2}\rho g L h(2H-h), \quad \bar{F} \uparrow\uparrow -\hat{e}_y$$

Torque



a) $\bar{T} = \bar{r} \times \bar{F}$

$$\overset{\bar{r} \perp d\bar{F}}{\Rightarrow} dT = r\,dF, \quad d\bar{T} \uparrow\uparrow -\hat{e}_x$$

$$\begin{cases} r = -H+h-z & (,\ z<0) \\ dF = P\,dS = -\rho g L z\,dz \end{cases}$$

$$\Rightarrow T = -\rho g L\int_{-H}^{-H+h} z(-H+h-z)\,dz$$

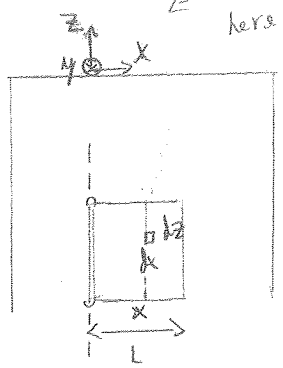
$$= -\rho g L\left[-(H-h)\frac{z^2}{2} - \frac{z^3}{3}\right]\Bigg|_{-H}^{-(H-h)}$$

$$= \rho g L\left[\frac{H^3}{3} - \frac{(H-h)^3}{3} + \frac{(H-h)^3}{2} - (H-h)\frac{H^2}{2}\right] = \frac{1}{6}\rho g L h^2(3H-h)$$

continues...

1.8 ... continues

b)



$$dT = x\,dF \quad , \quad d\bar{T} \upuparrows -\hat{e}_z$$

$$\Rightarrow \quad dT = x\,P\,dx\,dz$$

$$\Rightarrow \quad T = \int_0^L \int_{-H}^{-H+h} x(-\rho g z)\,dx\,dz$$

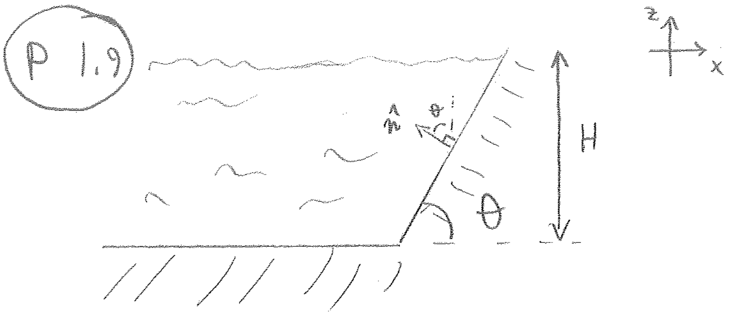
$$= \underbrace{\int_0^L x\,dx}_{=\frac{L^2}{2}} \int_{-H}^{-H+h} (-\rho g z)\,dz$$

$$= \frac{L}{2} \underbrace{\int_{-H}^{-H+h} (-\rho g L z)\,dz}_{=F}$$

$$= \frac{L}{2} \cdot F$$

$$= \underbrace{\frac{1}{4} \rho g L^2 h (2H - h)}$$

P 1.9



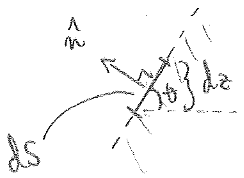
$\rho = \rho_0 - \alpha z$

Force on the wall $F_{total} = ?$

Solution : $d\bar{F} = -P\, dS\, \hat{n}$, $\hat{n} = -\hat{e}_x \sin\theta + \hat{e}_z \cos\theta$

$$\frac{dP}{dz} = -\rho g = -\rho_0 g + g\alpha z$$

$$\Rightarrow \quad P = -\rho_0 g z + g\alpha \frac{z^2}{2} \qquad \text{(neglecting } P_{atm}\text{)}$$



$$dS = L_y \cdot dl = L_y \cdot \frac{dz}{\sin\theta}$$

$$\frac{dz}{dl} = \sin\theta \quad \Rightarrow dl = \ldots$$

$$\Rightarrow \quad \bar{F} = -\hat{n}\, F$$

$$\Rightarrow \quad F = \int_{-H}^{0} \left(-\rho_0 g z + g\alpha \frac{z^2}{2}\right) L_y \frac{dz}{\sin\theta}$$

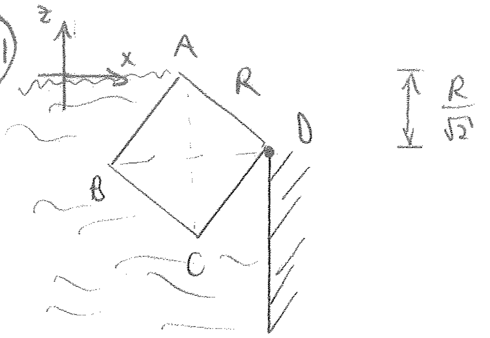
$$= \left[-\rho_0 g \frac{z^2}{2} + g\alpha \frac{z^3}{6}\right]\Bigg|_{-H}^{0} \frac{L_y}{\sin\theta}$$

$$= \frac{L_y}{\sin\theta}\left(\rho_0 g \frac{H^2}{2} + g\alpha \frac{H^3}{6}\right)$$

$$\Rightarrow \quad \bar{F} = -\hat{n}\, \frac{g L_y}{2\sin\theta} H^2 \left(\rho_0 + \frac{\alpha H}{3}\right)$$

$$\Rightarrow \quad \bar{F}/\text{unit length} = \frac{\bar{F}}{L_y} = -\hat{n}\, \frac{g}{2\sin\theta} H^2 \left(\rho_0 + \frac{\alpha H}{3}\right)$$

P. 1.11



Horizontal force

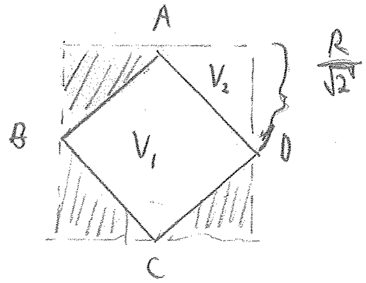
$$F_x = F_{AB} + \cancel{F_{BC}} \underbrace{- \cancel{F_{CD}}}_{= -F_{BC}} = F_{AB}$$

$$p = -\rho g z$$

$$dF_x = \int_{-\frac{R}{\sqrt{2}}}^{0} (-\rho g z) \cdot \overbrace{L_y\, dz}^{= ds} = -\rho g L_y \left[\frac{z^2}{2} \right] \Big|_{-\frac{R}{\sqrt{2}}}^{0} = \rho g L_y \frac{R^2}{4}$$

$$\Rightarrow \frac{dF_x}{\text{unit length}} = \frac{dF_x}{L_y} = \underline{\rho g \frac{R^2}{4}}$$

Vertical force



$$F_z = F_{BCD} - F_{AB}$$

Archimedes

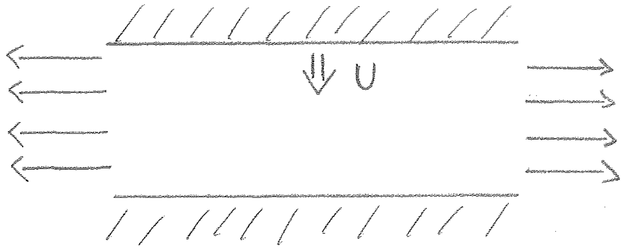
$$= \rho_{fluid}\, g\, V_{body}$$

$$= \rho g (V_1 + V_2)$$

$$= \rho g L_y \left(R^2 + \frac{1}{2} \frac{R^2}{2} \right) = \frac{5}{4} \rho g L_y R^2$$

$$\Rightarrow \frac{dF_z}{\text{unit length}} = \frac{dF_z}{L_y} = \underline{\frac{5 \rho g R^2}{4}}$$

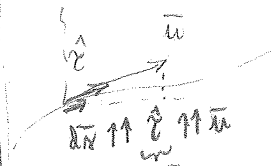
Example:



$$u_x = \frac{U}{H_0 - Ut}\, x$$

$$u_y = -\frac{U}{H_0 - Ut}\, y$$

Streamlines ?



$d\bar{r} \uparrow\uparrow \underbrace{\hat{\tau}}_{=\frac{\bar{u}}{|\bar{u}|}} \uparrow\uparrow \bar{u}$ $\qquad \dfrac{dy}{dx} = \dfrac{u_y}{u_x}$

$d\bar{r} = dx\,\hat{e}_x + dy\,\hat{e}_y$

Solution:

$$\frac{dx}{u_x} = \frac{dy}{u_y}$$

$$\Rightarrow \quad \cancel{(H_0 - Ut)}\frac{dx}{Ux} = -\frac{dy}{Uy}\cancel{(H_0 - Ut)}$$

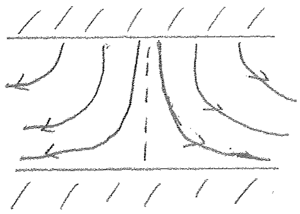
$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0$$

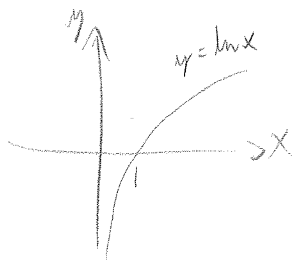
$$\overset{\int}{\Rightarrow} \quad \ln x + \ln y = C'$$

$$\Rightarrow \quad xy = C \quad , \; C = \text{const.}$$

$\ln(xy) = C'$
$xy = e^{C'} = C$

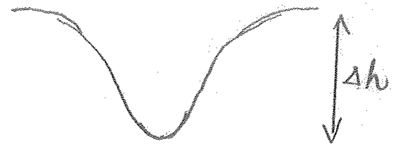
$$\Rightarrow \quad y = \frac{C}{x}$$





2.2 The Rankine vortex

$$\begin{cases} u_\theta = \Omega r, & r < R \\ u_\theta = \Omega R^2/r, & r > R \end{cases}$$



Shape of the free surface $z_f = f(r) = ?$

Euler $\Rightarrow$ $\Delta h = ?$

Solution: r: $\quad -\dfrac{u_\theta^2}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} \qquad (1)$

z: $\quad 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g \qquad (2)$

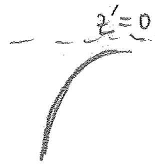
a) $r < R$

$$\frac{\partial p}{\partial r} = \rho\Omega^2 r \quad \overset{\int(1)\ \&\ \int(2)}{\Longrightarrow} \quad p = \tfrac{1}{2}\rho\Omega^2 r^2 - \rho g z + p_0$$

$$p_0 = p_{atm} \Rightarrow \text{at } z_f: \ p_{z=z_f} = p_{atm}$$

$$\Rightarrow z_f = \frac{\Omega^2 r^2}{2g} \qquad z=0=z_f \text{ at } r=0 \Leftarrow \text{our choice}$$

b) $r > R$



$$\frac{\partial p}{\partial r} = \frac{\rho\Omega^2 R^4}{r^3} \quad (3) \quad \overset{\int(3)\ \&\ \int(2)}{\Longrightarrow} \quad p = -\frac{1}{2}\frac{\rho\Omega^2 R^4}{r^2} - \rho g z' + \underbrace{p_\infty}_{=p_{atm}}$$

Now $z' = z - \Delta h$

$$\Rightarrow p = -\frac{1}{2}\rho\frac{\Omega^2 R^4}{r^2} - \rho g z + \rho g\Delta h + p_{atm}$$

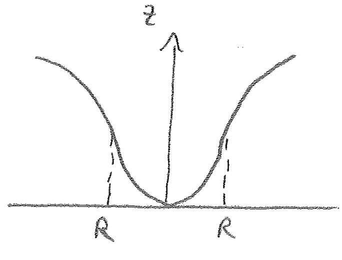
$$p_{z=z_f} = p_{atm}$$

$$\Rightarrow \cancel{p_{atm}} = -\frac{1}{2}\rho\frac{\Omega^2 R^4}{r^2} - \rho g z_f + \rho g\Delta h + \cancel{p_{atm}}$$

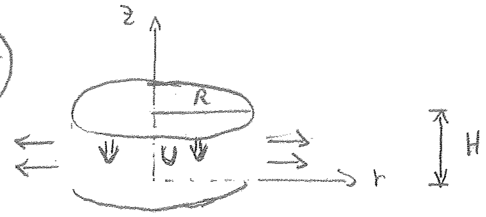
$$\Leftrightarrow z_f = -\frac{1}{2}\frac{\Omega^2 R^4}{g r^2} + \Delta h$$

Now $z_f\big|_{R_-} = z_f\big|_{R_+} \Leftrightarrow \dfrac{\Omega^2 R^2}{2g} = -\dfrac{\Omega^2 R^{\cancel{4}\,2}}{2g\cancel{R^2}} + \Delta h \Rightarrow \Delta h = \dfrac{\Omega^2 R^2}{g}$

$$\Rightarrow z_f = \begin{cases} \dfrac{\Omega^2 r^2}{2g}, & r < R \\ -\dfrac{\Omega^2 R^4}{2g r^2} + \dfrac{\Omega^2 R^2}{g}, & r > R \end{cases}$$



1.4**



$P(r) = ?$, $P(r=0) = ?$, $F_{p,flow} = ?$ at the bottom plate $z=0$

Solution: From 1.4* $\Rightarrow \begin{cases} u_r = \dfrac{Ur}{2(H_0 - Ut)} \\ u_z = -\dfrac{Uz}{H_0 - Ut} \end{cases}$

At the bottom plate $u_z = 0$ $(z=0)$

A quasi-stationary flow $\Rightarrow H_0 - Ut \approx H_0$

Bernoulli's eq. $P + \frac{1}{2}\rho u^2 = \text{const.}$

$(u_z=0) \Rightarrow \quad P + \frac{1}{2}\rho u_r^2 = P_{atm} + \frac{1}{2}\rho u_r^2\Big|_{r=R}$

$$\Rightarrow P = P_{atm} + \frac{1}{2}\rho U^2\left(\frac{R^2}{4H_0^2} - \frac{r^2}{4H_0^2}\right) = P_{atm} + \frac{\rho U^2}{8H_0^2}(R^2 - r^2)$$

$$\Rightarrow P(r=0) = P_{atm} + \frac{\rho U^2 R^2}{8H_0^2}$$

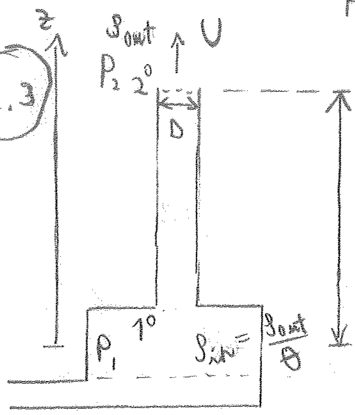
Total force due to the flow:

$$F_{p,flow} = \iint_S (P - P_{atm})\,ds = \int_0^R \frac{\rho U^2}{8H_0^2}(R^2 - r^2)\underbrace{2\pi r\,dr}_{=ds}$$

$$= \frac{\rho U^2}{8H_0^2}\cdot 2\pi\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]\Bigg|_0^R$$

$$= \frac{\rho U^2 \pi R^4}{16 H_0^2}$$

2.3



At the exit of the stack $P = P_2$

$$Q_{out} = ? = US$$

Solution: (flow inside)

Bernoulli's integral $\dfrac{P}{\rho_{in}} + \dfrac{u^2}{2} + gz = \text{const.}$

$\left(P + \rho_{in} \dfrac{u^2}{2} + \rho_{in} g z = \text{const.} \right)$

Now at 1° (inside the stove, burning chamber):
$$\begin{cases} P_1 = P_2 + \rho_{out} g H \\ U_1 \approx 0 \\ z = 0 \end{cases}$$

and at 2° (exit of the stack):
$$\begin{cases} P_2 \\ U_2 = U \\ z = H \\ \rho_{in} = \rho_{out}/\theta \end{cases}$$

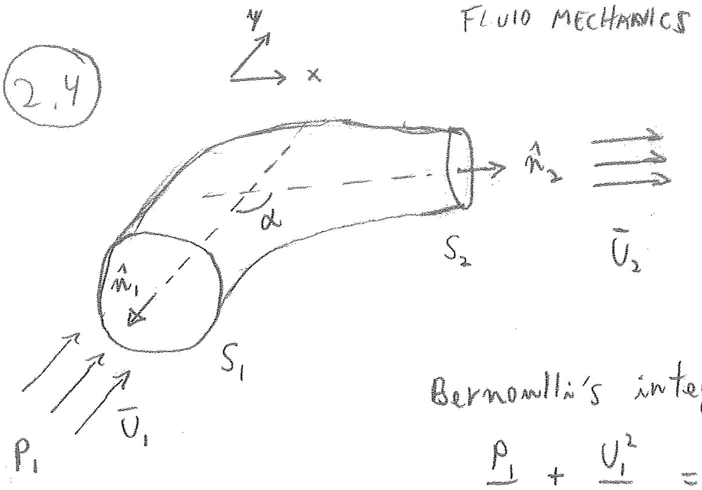
Bernoulli $\overset{1° = 2°}{\Rightarrow}$
$$\underbrace{\frac{\cancel{P_2} + \rho_{out} g H}{\rho_{in}} + 0 + 0}_{1°,\ \text{stove}} = \underbrace{\cancel{\frac{P_2}{\rho_{in}}} + \frac{U^2}{2} + gH}_{2°,\ \text{exit of the stack}}$$

$$\Rightarrow \quad \theta g H = \frac{U^2}{2} + gH$$

$$\Leftrightarrow \quad U^2 = 2(\theta - 1) g H$$

$$\Rightarrow \quad Q = US = \frac{\pi}{4} D^2 \sqrt{2(\theta - 1) g H}$$

(2.4)



Horizontal tube, $\bar{F}_{\text{on the wall}} = ? = \bar{F}$

Solution:

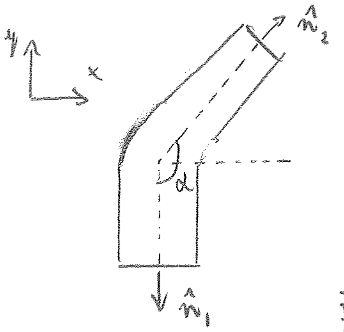
$$S_1 U_1 = S_2 U_2 \Rightarrow U_2 = \frac{S_1}{S_2} U_1 \quad (1)$$

Bernoulli's integral

$$\frac{P_1}{\rho} + \frac{U_1^2}{2} = \frac{P_2}{\rho} + \frac{U_2^2}{2} \quad (2)$$

(1) & (2) $\Rightarrow$

$$P_2 = P_1 + \frac{\rho U_1^2}{2}\left(1 - \frac{S_1^2}{S_2^2}\right)$$

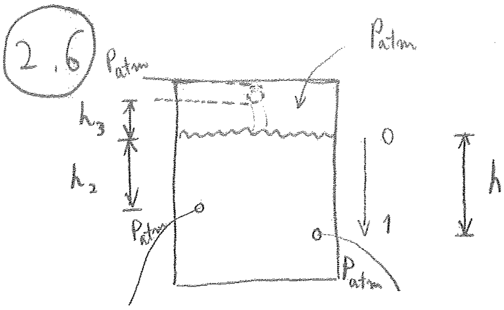


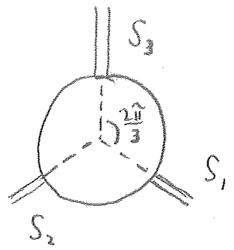
$$\begin{cases} \bar{F} \overset{(1.22)}{=} -\sum_i (\rho U_i^2 + P_i) S_i \hat{n}_i \quad \left(= -\int_{S_1, S_2} [(\rho \bar{U}) \bar{U} \cdot d\bar{S} + P d\bar{S}] \right) \\ \hat{n}_1 = -\hat{e}_y \\ \hat{n}_2 = -\hat{e}_y \cos\alpha + \hat{e}_x \sin\alpha \end{cases}$$

$$\bar{F} = -(\rho U_1^2 + P_1) S_1 (-\hat{e}_y) - \left[\left(\rho \frac{S_1^2}{S_2^2} U_1^2 \right) + \left(P_1 + \frac{\rho U_1^2}{2}\left(1 - \frac{S_1^2}{S_2^2}\right) \right) \right] S_2 \overbrace{(-\hat{e}_y \cos\alpha + \hat{e}_x \sin\alpha)}^{\hat{n}_2}$$

$$= (P_1 + \rho U_1^2) S_1 \hat{e}_y + \left[P_1 + \tfrac{1}{2}\rho U_1^2 \left(1 + \frac{S_1^2}{S_2^2}\right) \right] S_2 (\hat{e}_y \cos\alpha - \hat{e}_x \sin\alpha)$$

2.6





$\bar{F}_{\text{on the tank}} = ? = \bar{F}$

Formal equation $\bar{F} \overset{(1.22)}{=} -\sum_i (\rho U_i^2 + \cancel{P_i}) S_i \hat{n}_i$, $P_i = P_{atm}$ for all jets and P_{atm} generally surrounding the tank, $\Rightarrow P_i$'s may be neglected

$\Rightarrow$ more convenient eq. $\bar{F} = -\sum_i Q_i \rho \bar{U}_i$, $Q_i = \bar{U}_i \cdot S_i \hat{n}_i$

Let's find first U_1 and U_2 :

Bernoulli's integral $\Rightarrow$ velocity at the surface << U_1

$0 \to 1 :\quad \cancel{\frac{P_{atm}}{\rho}} + 0 + 0 = \cancel{\frac{P_{atm}}{\rho}} + \frac{U_1^2}{2} - g h_1 \quad \Rightarrow U_1 \approx \sqrt{2 g h_1}$

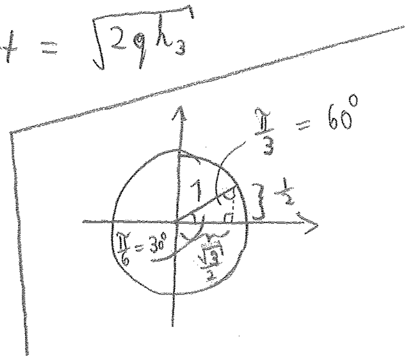
similarly $U_2 \approx \sqrt{2 g h_2}$

Total discharge : $Q_{out} = S_1 \sqrt{2gh_1} + S_2 \sqrt{2gh_2}$

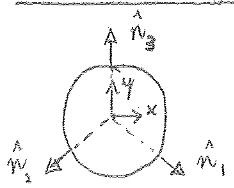
Discharge of the coming water : $Q_{in} = Q_{out}$

$\Rightarrow \quad U_3 = \frac{S_1}{S_3}\sqrt{2gh_1} + \frac{S_2}{S_3}\sqrt{2gh_2}$

Vertical velocity component of the coming fluid $\underbrace{U_{vert}}_{\sim \text{Free fall velocity}} = \sqrt{2gh_3}$



Horizontal forces :



$\hat{n}_3 = \hat{e}_y$

$\hat{n}_1 = \frac{\sqrt{3}}{2}\hat{e}_x - \frac{1}{2}\hat{e}_y$

$\hat{n}_2 = -\frac{\sqrt{3}}{2}\hat{e}_x - \frac{1}{2}\hat{e}_y$

$$\bar{F}_{horiz} = -\hat{e}_y \rho U_3^2 S_3 - \left(\frac{\sqrt{3}}{2}\hat{e}_x - \frac{1}{2}\hat{e}_y\right) S_1 \rho \cdot \overbrace{2gh_1}^{=U_1^2} - \left(-\frac{\sqrt{3}}{2}\hat{e}_x - \frac{1}{2}\hat{e}_y\right) S_2 \rho \overbrace{2gh_2}^{=U_2^2}$$

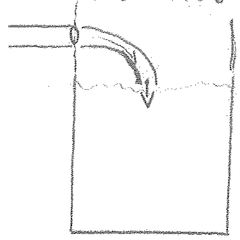
$$= -\hat{e}_y \frac{\rho}{S_3}\left(S_1\sqrt{2gh_1} + S_2\sqrt{2gh_2}\right)^2 + \frac{1}{\cancel{2}}\hat{e}_y \rho \cancel{2}g(h_1 S_1 + h_2 S_2)$$

$$- \frac{\sqrt{3}}{\cancel{2}}\hat{e}_x \rho \cancel{2}g(S_1 h_1 - S_2 h_2)$$

$$= \hat{e}_x \sqrt{3}\rho g (S_2 h_2 - S_1 h_1) + \hat{e}_y \rho g \left[h_1 S_1 + h_2 S_2 - \frac{2}{S_3}\left(S_1\sqrt{h_1} + S_2\sqrt{h_2}\right)^2\right]$$

continues ...

2.6 ... continues

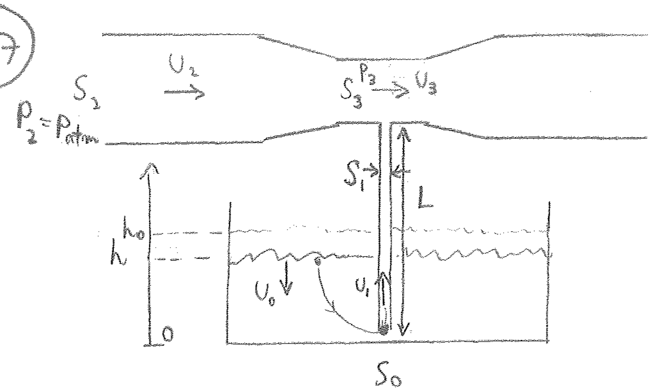


$$\bar{F}_{vert} = -\hat{e}_z \rho Q_{in} U_{vert}$$

$$= -\hat{e}_z \rho \sqrt{2gh_3}\left(S_1\sqrt{2gh_1} + S_2\sqrt{2gh_2}\right)$$

$$= -\hat{e}_z 2\rho g \sqrt{h_3}\left(S_1\sqrt{h_1} + S_2\sqrt{h_2}\right)$$

2.7



a) $U_{2,\,empt.} = ?$

b) $h(t) = ?$

solution:

$$\begin{cases} P_{atm} + \rho \frac{U_2^2}{2} = P_3 + \rho \frac{U_3^2}{2} \\ U_2 S_2 = U_3 S_3 \end{cases} \Rightarrow P_3 = P_{atm} - \rho \frac{U_2^2}{2}\left(\frac{S_2^2}{S_3^2} - 1\right)$$

Bernoulli's integral in the reservoir (assuming $U_0 \ll U_1$)

$$\cancel{P_{atm}} + \rho g h = \overbrace{P_3 + \rho g L}^{P_{at\ bottom}} + \rho \frac{U_1^2}{2}$$

$$= \cancel{P_{atm}} + \rho g L + \rho \frac{U_1^2}{2} - \frac{U_2^2}{2}\left(\frac{S_2^2}{S_3^2} - 1\right)$$

$$\Rightarrow U_1^2 = U_2^2\left(\frac{S_2^2}{S_3^2} - 1\right) - 2g(L-h)$$

In addition $U_1 = \frac{S_0}{S_1} U_0 = -\frac{S_0}{S_1}\frac{dh}{dt}$

$$\Rightarrow \left(\frac{dh}{dt}\right)^2 = \frac{S_1^2}{S_0^2}\left(2g(h-L) + U_2^2\left(\frac{S_2^2}{S_3^2} - 1\right)\right)$$

The flow in pipe stops $\Leftrightarrow U_1 = 0$ at

$$h_\infty = L - \frac{U_2^2}{2g}\left(\frac{S_2^2}{S_3^2} - 1\right)$$

$\Rightarrow$ reservoir empty $\Leftrightarrow h_\infty = 0$

$$\Rightarrow 0 = L - \frac{U_{2,empt}^2}{2g}\left(\frac{S_2^2}{S_3^2} - 1\right)$$

$$\Rightarrow U_{2,empt} = \sqrt{\frac{2gL}{S_2^2 - S_3^2}}\, S_3 \qquad a)$$

$\Downarrow$

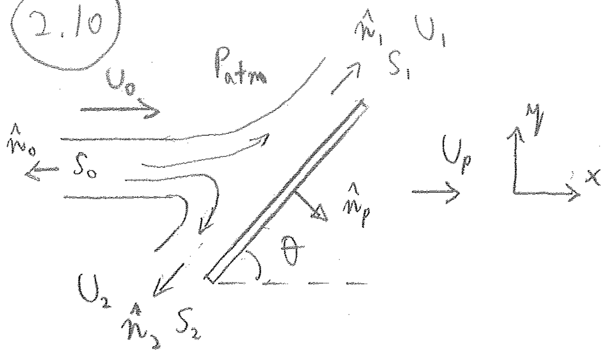
$$-\left(\frac{dh}{dt}\right) = \frac{S_1}{S_2}\sqrt{2g}\sqrt{h - h_\infty}$$

$$\Rightarrow -\int_{h_0}^{h} \frac{dh}{\sqrt{h - h_\infty}} = \frac{S_1}{S_2}\sqrt{2g}\, t$$

$$\Rightarrow -2\sqrt{h - h_\infty} + 2\sqrt{h_0 - h_\infty} = \frac{S_1}{S_2}\sqrt{2g}\, t$$

$$\Rightarrow h = h_\infty + (h_0 - h_\infty)\left(1 - \frac{S_1}{S_2}\sqrt{\frac{g t^2}{2(h_0 - h_\infty)}}\right)^2 \qquad b)$$

2.10



a) $S_1 = ?$, $S_2 = ?$

b) $\bar{F}_p = ?$

c) maximal work/unit time $W_{max} = ?$

assuming no losses, no gravity

Solution: In the rest frame of the plate: $U_0' = U_0 - U_p$

Bernoulli's integral $\Rightarrow$ $\frac{(U_0')^2}{2} + \cancel{\frac{P_{atm}}{\rho}} = \frac{U_1^2}{2} + \cancel{\frac{P_{atm}}{\rho}} = \frac{U_2^2}{2} + \cancel{\frac{P_{atm}}{\rho}}$

$\Leftrightarrow$ $U_0' = U_1 = U_2$

Mass conservation $\Rightarrow$ $S_0 U_0' = S_1 U_1 + S_2 U_2 \Rightarrow S_0 = S_1 + S_2$

The net force on the fluid

$$0 = -\int_S \rho \bar{u} (\bar{u} \cdot d\bar{s}) + \bar{F}_{ext}$$

$\frac{dp}{dt} = -(\bar{p}\text{ - flux out}) + \underbrace{\bar{F}_{\text{on the fluid}}}_{= F_{ext}}$

$$\Rightarrow \bar{F}_{ext} = \rho S_0 (U_0')^2 \hat{n}_0 + \rho S_1 (U_0')^2 \hat{n}_1 + \rho S_2 (U_0')^2 \hat{n}_2$$

$\hat{n}_2 = -\hat{n}_1$

$F_{ext} \uparrow\uparrow -\hat{n}_p$, $\hat{n}_p = \hat{e}_x \sin\theta - \hat{e}_y \cos\theta$

$\hat{=}$ the external force along the plate is zero! (the fluid is ideal)

$$\Rightarrow \hat{n}_1 \cdot \bar{F}_{ext} = 0 = S_0 \underbrace{(\hat{n}_0 \cdot \hat{n}_1)}_{= -\cos\theta} + S_1 - S_2$$

Now $\begin{cases} S_1 - S_2 = S_0 \cos\theta \\ S_1 + S_2 = S_0 \end{cases} \Rightarrow \begin{cases} S_1 = \frac{S_0}{2}(1 + \cos\theta) \\ S_2 = \frac{S_0}{2}(1 - \cos\theta) \end{cases}$ a)

Checking: $\cos\theta = 0$, $\theta = \frac{\pi}{2}$ $\Rightarrow S_1 = S_2$ Ok

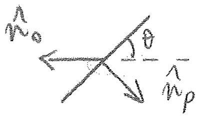
$\theta = 0$, $\cos\theta = 1$ $\Rightarrow S_1 = S_0$, $S_2 = 0$ Ok

continues...

2.10 ... continues

b) Looking for the force $\Rightarrow$

$$\hat{n}_p \cdot \bar{F}_{ext} = \rho S_0 (U_0')^2 \underbrace{\hat{n}_0 \cdot \hat{n}_p}_{} + \rho S_1 (U_0')^2 \underbrace{\hat{n}_1 \cdot \hat{n}_p}_{=0} + \rho S_2 (U_0')^2 \underbrace{\hat{n}_2 \cdot \hat{n}_p}_{=0}$$



$$= -\cos\left(\tfrac{\pi}{2} - \theta\right)$$
$$= -\sin\theta$$

$$\Rightarrow F_p = -F_{ext} = -\rho S_0 (U_0')^2 (\hat{n}_0 \cdot \hat{n}_p) = \rho S_0 (U_0')^2 \sin\theta$$

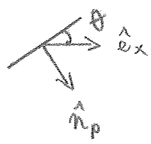
Going to the laboratory frame

$$U_0' = U_0 - U_p$$

$$\Rightarrow \bar{F}_p = \hat{n}_p \rho S_0 (U_0 - U_p)^2 \sin\theta$$

c) $$\frac{\text{Work}}{\text{unit time}} = \text{power} = W = \bar{F}_p \cdot \bar{U}_p$$

$$= \rho S_0 (U_0 - U_p)^2 U_p \sin\theta \underbrace{\hat{e}_x \cdot \hat{n}_p}_{=\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta}$$



$$W = \rho S_0 (U_0 - U_p)^2 U_p \sin^2\theta$$

$W_{max} = ?$

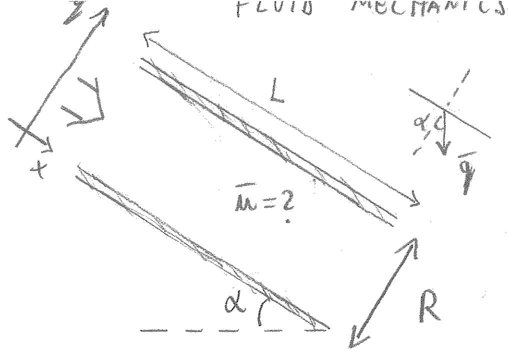
$$\frac{dW}{dU_p} = 0 \Rightarrow \rho S_0 \sin^2\theta \left[(U_0 - U_p)^2 - 2U_p(U_0 - U_p)\right] = 0$$

Maximum corresponds to

$$U_p = \frac{U_0}{3}$$

$$\Rightarrow W_{max} = \frac{4}{27} \rho S_0 U_0^3 \sin^2\theta$$

(5.10)



velocity $\bar{u} = ?$
discharge $Q = ?$

$$\rho\left(\cancel{\frac{\partial \bar{u}}{\partial t}} + \cancel{(\bar{u}\cdot\nabla)\bar{u}}\right) = -\cancel{\nabla P} + \rho\bar{g} + \mu\nabla^2\bar{u}$$

x-direction : the channel open to atmosphere $\Rightarrow \frac{\partial P}{\partial x} = 0$

$u_x = u_x(y) \Rightarrow (\bar{u}\cdot\nabla)\bar{u} = 0$

$$\frac{\partial P}{\partial y} = -\rho g\cos\alpha$$

$$\Rightarrow P = -\rho g\cos\alpha\, y + P_x(x)$$

bound. cond. $P_x(x=0) = P_{atm}$

$P_x(x=L) = P_{atm}$

$$\Rightarrow \frac{\partial P}{\partial x} = 0$$

$$\Rightarrow \mu\frac{\partial^2 u_x}{\partial y^2} + \rho g\sin\alpha = 0$$

$$\Rightarrow \frac{\partial^2 u_x}{\partial y^2} = -\frac{\rho g}{\mu}\sin\alpha$$

$$\Rightarrow u_x = -\frac{\rho g}{\mu}\sin\alpha\frac{y^2}{2} + By + C$$

Boundary conditions : $u_x = 0$ at $y = 0 \Rightarrow C = 0$

$u_x = 0$ at $y = R \Rightarrow -\frac{\rho g}{\mu}\sin\alpha\frac{R^2}{2} + BR = 0$

$$\Rightarrow B = \frac{\rho g}{\mu}\sin\alpha\frac{R}{2}$$

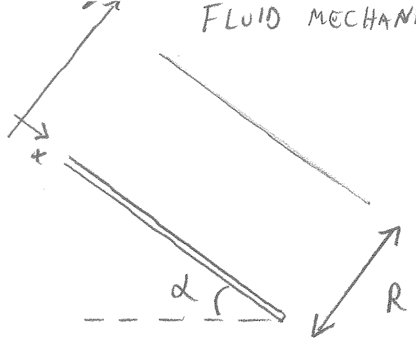
$$\Rightarrow u_x = \frac{\rho g}{2\mu}\sin\alpha\,(R-y)\,y$$

The discharge per L_z

$$\frac{Q}{L_z} = \int_0^R u_x\,dy = \frac{\rho g}{2\mu}\sin\alpha\int_0^R (Ry - y^2)\,dy = \frac{\rho g}{2\mu}\sin\alpha\left(\frac{R^3}{2} - \frac{R^3}{3}\right)$$

$$= \frac{\rho g}{12\mu}\sin\alpha\, R^3$$

5.11



$\downarrow \bar{g}$

$u_x = ?$

$u_{x,max} = ?$

$\tau = \tau_{xy} = \left.\frac{F}{S}\right|_{y=0} = ?$ Force on the bottom plate

Navier - Stokes $\quad \mu \frac{\partial^2 u_x}{\partial y^2} + \rho g \sin\alpha = 0$

$$\Rightarrow \quad u_x = -\frac{\rho g \sin\alpha}{\mu}\frac{y^2}{2} + By + C$$

Boundary conditions:

$u_x = 0$ at $y = 0 \quad \Rightarrow \quad C = 0$

$\frac{\partial u_x}{\partial y} = 0$ at $y = R \quad \hat{=}$ zero stress $\quad \left(\tau = \mu \frac{\partial u_x}{\partial y}\right)$

$$\Leftrightarrow \quad \left.\frac{\partial u_x}{\partial y}\right|_{y=R} = \left.\left(-\frac{\rho g \sin\alpha}{\mu} y + B\right)\right|_{y=R} = 0 \quad \Rightarrow \quad B = \frac{\rho g \sin\alpha}{\mu} R$$

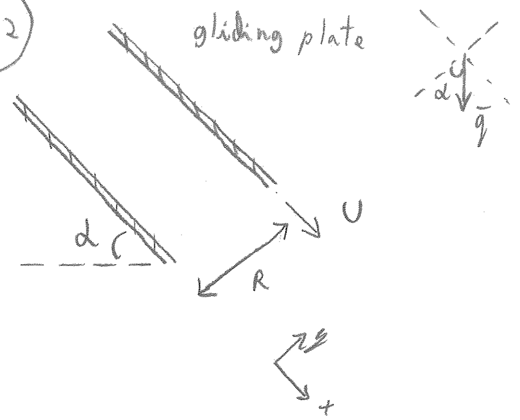
$$\Rightarrow \quad u_x = \frac{\rho g \sin\alpha}{2\mu}(2R - y)\, y$$

$$u_{x,max} = \left.u_x\right|_{y=R} = \frac{\rho g \sin\alpha}{2\mu} R^2$$

Viscous stress (force) acting on the lower plate

$$\frac{F}{S} = \mu \left.\frac{\partial u_x}{\partial y}\right|_{y=0} = \cancel{\mu}\,\frac{\rho g \sin\alpha}{\cancel{\mu}} R = \rho g R \sin\alpha$$

5.12



$U = ?$, $\dfrac{\text{plate mass}}{\text{unit surface area}} = \dfrac{M}{S}$

Solution: Balance of forces on the (upper) plate $\Leftrightarrow$ $Mg \sin\alpha = \xi S$ $(*)$

Navier-Stokes along x

$$\mu \frac{\partial^2 u_x}{\partial y^2} = -\rho g \sin\alpha \overset{\int}{\Longrightarrow} u_x = -\frac{\rho g \sin\alpha}{2\mu} y^2 + Ay + B$$

Boundary conditions: $u_x = 0$ at $y = 0$ $\Rightarrow$ $B = 0$

$u_x = U$ at $y = R$ $\Rightarrow$ $A = \dfrac{U}{R} + \rho g \dfrac{\sin\alpha}{2\mu} R$

Then $$u_x = \frac{\rho g \sin\alpha}{2\mu} y(R-y) + \frac{Uy}{R}$$

$\Rightarrow$ Viscous stress ξ at $y = R$

$$\xi = \mu \frac{\partial u_x}{\partial y}\bigg|_{y=R} = \mu\left[\frac{\rho g \sin\alpha}{2\mu} R - \frac{\rho g \sin\alpha}{2\mu} \cdot 2R + \frac{U}{R}\right]$$

$$= \mu\frac{U}{R} - \frac{\rho g \sin\alpha}{2} R$$

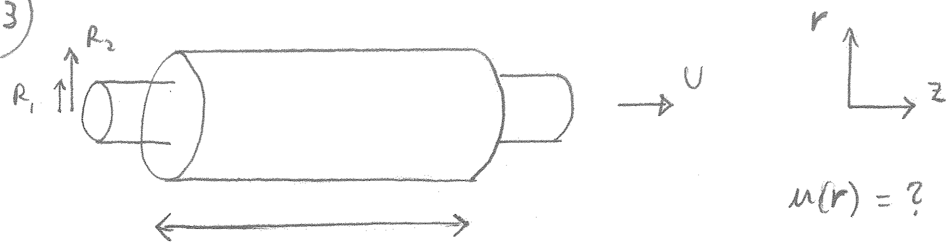
Now $(*)$ $\Rightarrow$ $$\xi = \mu\frac{U}{R} - \frac{\rho g \sin\alpha}{2} R = \frac{Mg\sin\alpha}{S}$$

$$\Rightarrow \left(M + \tfrac{1}{2}\rho R S\right) g \sin\alpha = \mu \frac{U}{R} S$$

$$\Rightarrow U = \frac{Rg\sin\alpha}{\mu S}\left(M + \tfrac{1}{2}\rho R S\right)$$

or $$U = \frac{MgR\sin\alpha}{\mu S} + \frac{\rho R^2 g \sin\alpha}{2\mu}$$

5.13



$u(r) = ?$

$\Delta P > 0$

Solution:

$$-\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) = 0$$

$$\frac{\partial P}{\partial r} = 0 \quad \Rightarrow \quad \frac{\partial P}{\partial z} = \frac{\Delta P}{L} = \text{const.}$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) = \frac{\Delta P}{\mu L} \quad \Big| \int$$

$$\Rightarrow \quad u_z = \frac{\Delta P}{\mu L}\frac{r^2}{4} + A \ln r + B$$

Boundary conditions: $u_z = U$ at $r = R_1$

$u_z = 0$ at $r = R_2$

Then

$$\frac{\Delta P}{4\mu L} R_1^2 + A \ln R_1 + B = U$$

$$-\quad \frac{\Delta P}{4\mu L} R_2^2 + A \ln R_2 + B = 0$$

$$\frac{\Delta P}{4\mu L}\left(R_1^2 - R_2^2\right) + A \ln \frac{R_1}{R_2} = U \quad \Rightarrow \quad A = \ln \frac{R_2}{R_1}\left[U - \frac{\Delta P}{4\mu L}\left(R_1^2 - R_2^2\right)\right]$$

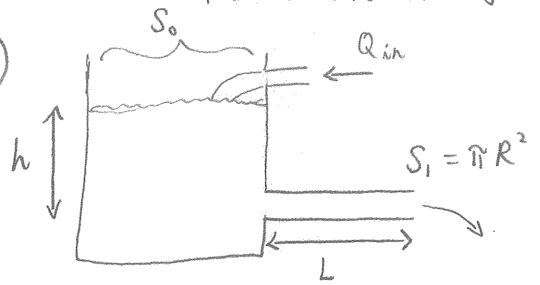
$$\Rightarrow \quad B = -\ln\frac{R_2}{R_1}\left[\quad\right] \ln R_2 - \frac{\Delta P}{4\mu L} R_2^2$$

$$\Rightarrow \quad u_z = \frac{\Delta P}{4\mu L} r^2 + \left[U + \frac{\Delta P}{4\mu L}\left(R_2^2 - R_1^2\right)\right] \ln \frac{R_2}{R_1} \cdot \ln r$$

$$- \left[U + \frac{\Delta P}{4\mu L}\left(R_2^2 - R_1^2\right)\right] \ln \frac{R_2}{R_1} \ln R_2 - \frac{\Delta P}{4\mu L} R_2^2$$

$$= \frac{\Delta P}{4\mu L}\left(r^2 - R_2^2\right) + \left[U + \frac{\Delta P}{4\mu L}\left(R_2^2 - R_1^2\right)\right] \frac{\ln (r/R_2)}{\ln (R_1/R_2)}$$

5.14



$h(t) = ?$

Pressure variations for the tube $\Delta P = \rho g h$

Navier - Stokes in the tube

$$-\frac{\partial P}{\partial x} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_x}{\partial r}\right) = 0$$

$$\frac{\partial P}{\partial x} = -\frac{\Delta P}{L} = -\frac{\rho g h}{L}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) = -\frac{\rho g h}{\mu L}$$

$$\overset{\int}{\Rightarrow} \quad u_x = \frac{\rho g h}{4\mu L}(R^2 - r^2)$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) = -\frac{\rho g h}{\mu L} r$$

$$\overset{\int}{\Rightarrow} \quad r\frac{\partial u_x}{\partial r} = -\frac{\rho g h}{\mu L}\frac{r^2}{2} + A$$

$$\Rightarrow \quad \frac{\partial u_x}{\partial r} = -\frac{\rho g h}{\mu L}\frac{r}{2} + \frac{A}{r}$$

$$\overset{\int}{\Rightarrow} \quad u_x = -\frac{\rho g h}{\mu L}\frac{r^2}{4} + A\ln r + B$$

Boundary cond.

$u_x > 0$ at $r = 0 \Rightarrow A = 0$

$u_x = 0$ at $r = R$

$$\Rightarrow B = \frac{\rho g h}{4\mu L} R^2$$

$$Q_{out} = \int_0^R u_x 2\pi r\, dr = \frac{\pi}{8}\frac{\rho g h}{\mu L} R^4$$

$$Q_{out} - Q_{in} = -S_0 \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{Q_{in}}{S_0} - \frac{\pi \rho g R^4}{8\mu L S_0} h$$

Water level $h = ?$

When $\frac{dh}{dt} = 0 \Leftrightarrow h_\infty = \frac{8\mu L Q_{in}}{\pi \rho g R^4}$

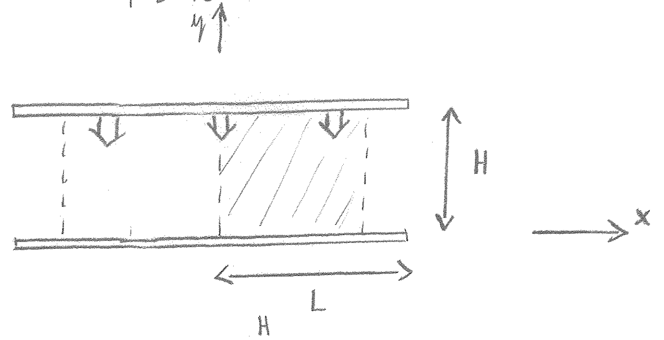
Typical time scale $\tau \left(= \frac{h_\infty}{U_0}\right) = \frac{8\mu L S_0}{\pi \rho g R^4}$

Then $\frac{dh}{dt} = -\frac{h - h_\infty}{\tau}$

$$\Rightarrow \quad \ln(h - h_\infty) - \ln(h_0 - h_\infty) = -\frac{t}{\tau}$$

$$\Rightarrow \quad h = h_\infty + (h_0 - h_\infty)\exp(-t/\tau)$$

5.15



Viscous walls,
far from centre $u_x(y) \sim$ Poiseuille
$U_{max,x} = U_{max} = ?$

Solution: $V = x\,(\overbrace{H_0 - Ut}^{H})\,L_z$

$$\frac{dV}{dt} = -xUL_z = -Q_{out}$$

Quasi-Poiseuille flow:

$$u_x \approx A\,y\,(H-y)$$

Poiseuille flow
$$u_x = \frac{\Delta P}{2\mu L}\,y\,(H-y)$$

$$U_{max} \Rightarrow \frac{du_x}{dy} = 0 \Leftrightarrow A(H-y) - Ay = 0$$
$$\Rightarrow y = \frac{H}{2}$$
$$\Rightarrow U_{max} = A\frac{H}{2}H - A\frac{H^2}{4} \Rightarrow U_{max} = A\frac{H^2}{4}$$
$$\Leftrightarrow A = \frac{4U_{max}}{H^2}$$

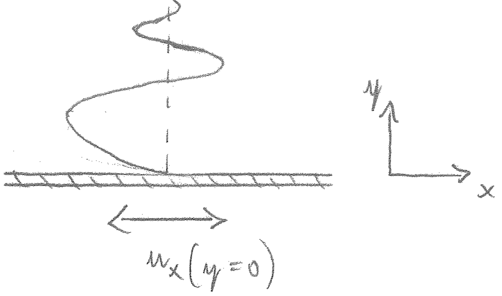
$$\Rightarrow u_x = 4U_{max}\frac{y}{H}\left(1-\frac{y}{H}\right)$$

$$Q_{out} = \int_0^H u_x L_z\,dy$$
$$= L_z\,4U_{max}\underbrace{\int_0^H \frac{y}{H}\left(1-\frac{y}{H}\right)dy}$$
$$= \int_0^H \left(\frac{y}{H} - \frac{y^2}{H^2}\right)dy = \left[\frac{y^2}{2H} - \frac{y^3}{3H^2}\right]\Big|_0^H = \frac{H}{6}$$

$$= L_z\,4U_{max}\frac{H}{6}$$

Now $\cancel{L_z}\frac{2U_{max}H}{3} = xU\cancel{L_z} \Rightarrow U_{max} = \frac{3}{2}U\frac{x}{H}$
$$\begin{cases} x = L \\ U_{max} = \frac{3}{2}U\frac{L}{H} \end{cases}$$

5.24



$$u_x(y,t) = ?$$

$$u_x(y=0) = U\sin(\omega t) = \mathrm{Im}\left[U\exp(i\omega t)\right]$$

Navier-Stokes $\qquad \dfrac{\partial u_x}{\partial t} = \nu \dfrac{\partial^2 u_x}{\partial y^2}$

$$u_x = W(y)\exp(i\omega t)$$

k can be complex

$$\Rightarrow \quad i\omega W \cancel{\exp(i\omega t)} = \nu \frac{d^2 W}{dy^2}\cancel{\exp(i\omega t)} \quad \Rightarrow \quad W \propto \exp(ky)$$

$$\Rightarrow \quad k^2 = i\frac{\omega}{\nu} \quad \Rightarrow \quad k = (\pm)\frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}$$

$$\text{or} \quad k = -(1+i)\sqrt{\frac{\omega}{2\nu}}$$

"minus" $\hat{=}$ damping solution

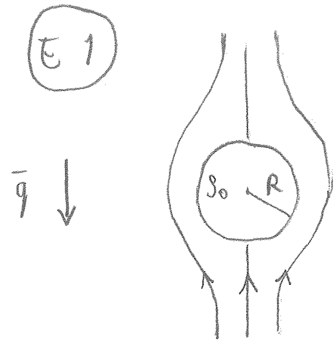
$$(1+i)^2 = 1 + 2i - 1 = 2i \Rightarrow i = \frac{(1+i)^2}{2}$$

Then $\qquad u_x = A\exp\left(i\omega t - i\sqrt{\frac{\omega}{2\nu}}\,y - \sqrt{\frac{\omega}{2\nu}}\,y\right)$

Boundary conditions: $\quad u_x = \mathrm{Im}\left[U\exp(i\omega t)\right]$ at $y=0 \Rightarrow A = U$

$$\Rightarrow \quad u_x = U\exp\left(-\sqrt{\frac{\omega}{2\nu}}\,y\right)\sin\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\,y\right)$$

E 1



$\rho_f = \alpha \rho_0$, $\alpha < 1$, $Re \ll 1$ (slow flow)
quasi-stationary

$U(t) = ?$

Solution:

$$M \frac{dU}{dt} = Mg - F_{Archim.} - F_{Drag}$$

$$M = \frac{4\pi}{3} R^3 \rho_0 \quad , \quad F_{Arch.} = \alpha \underbrace{\rho_0 \frac{4\pi}{3} R^3}_{=M} g = \alpha M g$$

$$F_{Drag} = 6\pi \mu U R \quad \text{(Stokes formula)}$$

$$\Rightarrow \frac{dU}{dt} = (1-\alpha) g - \frac{6\pi\mu R}{M} U$$

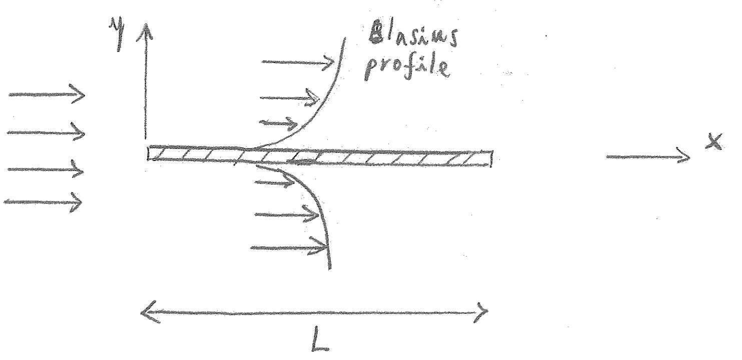
Introducing $\tau = \dfrac{M}{6\pi\mu R} = \dfrac{4\cancel{\pi}}{3} \rho_0 R^{\cancel{3}\,2} \cdot \dfrac{1}{6\cancel{\pi}\mu \cancel{R}} = \dfrac{2\rho_0 R}{9\mu}$

$$\Rightarrow \frac{dU}{dt} = (1-\alpha) g - \frac{U}{\tau} \quad \Big| \int$$

$$\Rightarrow \ln\Big[(1-\alpha) g \tau - U\Big]_{U=0}^{U} = -\frac{t}{\tau}$$

$$\Rightarrow \frac{(1-\alpha) g \tau - U}{(1-\alpha) g \tau} = \exp\left(-\frac{t}{\tau}\right) \quad \Rightarrow \quad U = (1-\alpha) g \tau \left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$$

6.1

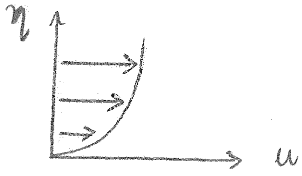


$\mu = \rho \nu$

Blasius profile $u = U f(\eta)$

$\eta = y/\delta(x)$

$\frac{df}{d\eta} \approx 0.33$ at $\eta = 0$



$F_{drag}/L_z = ?$

Solution :

$$dF_{drag}/L_z = \underbrace{2}_{\text{top + bottom}} \tau \, dx = 2\mu \left.\frac{\partial u}{\partial y}\right|_{y=0} dx$$

$$= 2\mu U \underbrace{\left(\frac{df}{d\eta}\right)\Big|_{\eta=0}}_{= 0.33} \underbrace{\frac{\partial \eta}{\partial y}}_{= \frac{1}{\delta(x)}} dx$$

$$\Rightarrow \quad dF_{drag}/L_z = 0.66 \frac{\mu U}{\delta(x)} dx \qquad , \qquad \delta(x) = \sqrt{\frac{\nu x}{U}}$$

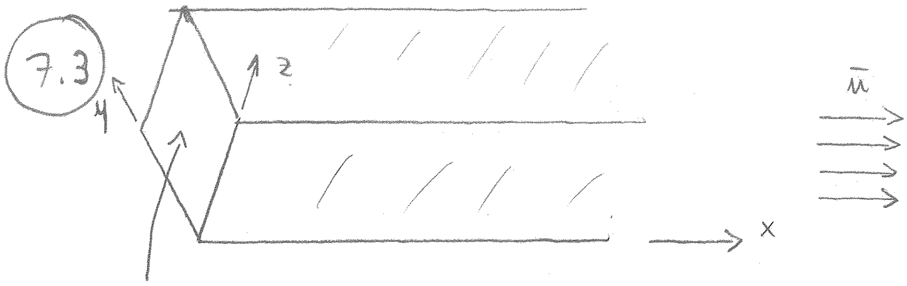
$$\mu = \rho \nu$$

$$= 0.66 \rho \nu U \sqrt{\frac{U}{\nu x}} dx$$

Now $F_{drag}/L_z = 0.66 \rho \nu U \sqrt{\frac{U}{\nu}} \int_0^L \frac{dx}{\sqrt{x}} = 2 \cdot 0.66 \rho \nu U \sqrt{\frac{U}{\nu}} \sqrt{L}$

$$= 1.32 \rho \nu U \sqrt{\frac{UL}{\nu}}$$

7.3



$T_0 + \Theta \sin\left(\frac{\pi y}{R}\right) \sin\left(\frac{\pi z}{R}\right)$, $\bar{u} = U \hat{e}_x$, $T(x,y,z) = ?$

Solution:

temperature drift

$$\cancel{\frac{\partial T}{\partial t}} + (\bar{u} \cdot \nabla) T = \chi \nabla^2 T$$

$$\Rightarrow \quad \frac{U}{\chi} \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$T = T_0 + \vartheta(x) \sin\left(\frac{\pi y}{R}\right) \sin\left(\frac{\pi z}{R}\right)$$

$$\cancel{\sin \sin} \frac{U}{\chi} \frac{d\vartheta}{dx} = \frac{d^2 \vartheta}{dx^2} \cancel{\sin \sin} - 2 \frac{\pi^2}{R^2} \vartheta \cancel{\sin \sin}$$

$$\vartheta = A \exp(-\alpha x)$$

$$\Rightarrow \quad \alpha^2 + \frac{U}{\chi} \alpha - 2 \frac{\pi^2}{R^2} = 0$$

$$\Rightarrow \quad \alpha = -\frac{U}{2\chi} \underset{(-)}{+} \sqrt{\frac{U^2}{4\chi^2} + 2 \frac{\pi^2}{R^2}}$$

$$\Rightarrow \quad T = T_0 + \Theta \exp(-\alpha x) \sin\left(\frac{\pi y}{R}\right) \sin\left(\frac{\pi z}{R}\right)$$

7.4 Boussinesq convection

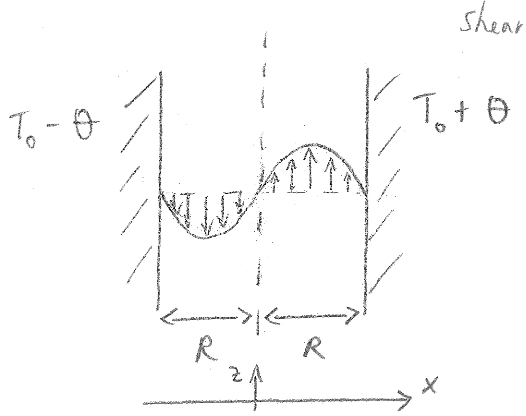
$\rho \approx$ const in all terms except for $\rho \bar{g}$, where $\rho = \rho_0 + \Delta\rho$

$$\Delta\rho \approx \underbrace{\left(\frac{\partial \rho}{\partial T}\right)_p}_{\text{constant}} \Delta T = -\alpha \rho_0 \Delta T \quad , \text{ designation } \Delta T = \vartheta$$

Equations:
$$\nabla \cdot \bar{u} = 0$$
$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} = -\frac{1}{\rho_0}\nabla p + \frac{1}{\rho_0}(\rho_0 + \underbrace{\Delta\rho}_{=-\alpha\rho_0\vartheta})\bar{g} + \nu \nabla^2 \bar{u}$$
$$\underbrace{\cancel{\frac{\partial \vartheta}{\partial t}}}_{\text{stationary}} + (\bar{u}\cdot\nabla)\vartheta = \chi \nabla^2 \vartheta$$

Navier - Stokes:
$$\underbrace{\cancel{\frac{\partial \bar{u}}{\partial t}}}_{\text{stationary}} + \cancel{(\bar{u}\cdot\nabla)\bar{u}} = \underbrace{-\frac{1}{\rho_0}\nabla p + \bar{g}}_{\text{pressure coincides with the hydrostatic one}} - \alpha\vartheta\bar{g} + \nu\nabla^2\bar{u}$$

$\bar{u} = u_z(x)\hat{e}_z$ shear



$$\bar{u} = u_z(x)\hat{e}_z \quad , \quad T(x) = ? \quad u_z(x) = ?$$

$$\bar{g} = -g\hat{e}_z$$
$$T = T_0 + \vartheta \quad , \quad \vartheta = \vartheta(x)$$

$$\cancel{u_z \frac{\partial}{\partial z} u_z} = g\alpha\vartheta + \nu\frac{\partial^2 u_z}{\partial x^2}$$
$$\cancel{u_z\frac{\partial}{\partial z}\vartheta} = \chi\nabla^2\vartheta$$

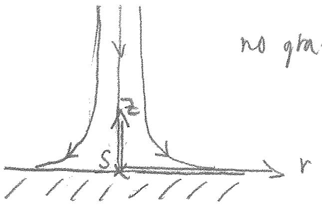
Then $\dfrac{\partial^2\vartheta}{\partial x^2} = 0 \Rightarrow \vartheta = Ax + B = \theta\dfrac{x}{R}$, Boundary conditions
$\vartheta(x=0) = 0 \Rightarrow B = 0$
$\vartheta(x=R) = \theta \Rightarrow A = \dfrac{\theta}{R}$

$$\Rightarrow T = T_0 + \theta\frac{x}{R}$$

And $g\alpha\vartheta + \nu\dfrac{\partial^2 u_z}{\partial x^2} = 0 \iff \dfrac{\partial^2 u_z}{\partial x^2} = -\dfrac{g\alpha\theta}{\nu R}x$

$\Rightarrow u_z = -\dfrac{g\alpha\theta}{6R\nu}x^3 + Ax + B$, Boundary conditions $u_z = 0$ at $x = \pm R$ $\Rightarrow u_z = \dfrac{g\alpha\theta}{6R\nu}(R^2 - x^2)x$

(2.13)



no gravity $\phi(r,z) = ?$, $\begin{cases} u_r = ? \\ u_z = ? \end{cases}$, Streamlines $= ?$, $P(r,z) = ?$

Let's look for $\phi(r,z) = \phi_1(r) + \phi_2(z)$

$$\nabla^2\phi = 0 \iff \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Then $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_1}{\partial r}\right) + \frac{\partial^2 \phi_2}{\partial z^2} = 0 \Rightarrow \begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_1}{\partial r}\right) = A & (1) \\ \frac{\partial^2 \phi_2}{\partial z^2} = -A & (2) \end{cases}$

Now (2) $\Rightarrow$ $\phi_2 = -\frac{Az^2}{2} + B_2 z + \underbrace{C_2}_{=0}$, Choosing $\phi_2(z=0) = 0$ $\Rightarrow C_2 = 0$

$$\Rightarrow u_z = \frac{\partial \phi}{\partial z} = \frac{\partial \phi_2}{\partial z} = -Az + B_2$$

Bound. cond. $u_z = 0$ at $z = 0$ $\xRightarrow{B_2=0}$ $u_z = -Az$ and $\phi_2 = -\frac{Az^2}{2}$

(1) $\Rightarrow$ $r\frac{\partial \phi_1}{\partial r} = \frac{Ar^2}{2} + B_1$

$\Rightarrow$ $\phi_1 = \frac{Ar^2}{4} + \underbrace{B_1}_{=0} \ln r + \underbrace{C_1}_{=0}$, Choosing $\phi_1(r=0) = 0$ $\Rightarrow B_1 = 0,\ C_1 = 0$

$\Rightarrow$ $\phi = \phi_1 + \phi_2 = \frac{Ar^2}{4} - \frac{Az^2}{2}$ and $u_r = \frac{\partial \phi}{\partial r} = \frac{\partial \phi_1}{\partial r} = \frac{Ar}{2}$

Streamlines : $\frac{dz}{u_z} = \frac{dr}{u_r} \iff -\frac{dz}{z} = 2\frac{dr}{r}$

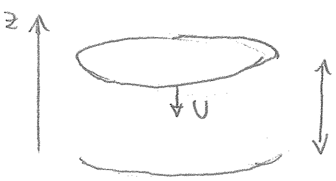
$\xRightarrow{\int}$ $-\ln z + \text{const}' = 2\ln r \Rightarrow \ln r^2 + \ln z = \text{const}'$

$\Rightarrow$ $zr^2 = \text{const.}$

$P(r,z) = ?$ Bernoulli $\Rightarrow$ $P + \frac{1}{2}\rho u^2 = P_s + 0$

$$\Rightarrow P = P_s - \frac{1}{2}\rho A^2\left(z^2 + \frac{r^2}{4}\right)$$

Example



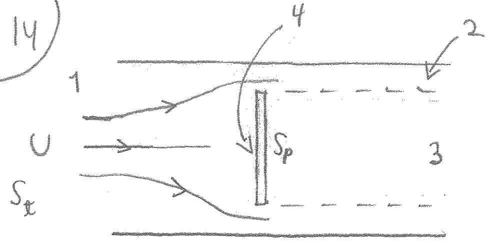
$H = H_0 - Ut$

$u_z = -U\big|_{z = H_0 - Ut}$

On the other hand $u_z = -Az\big|_{z = H_0 - Ut}$ $\Rightarrow U = A\cdot(H_0 - Ut) \iff A = \frac{U}{H_0 - Ut}$

Then $\phi = \phi(t)$ and Bernoulli : $\rho\frac{\partial \phi}{\partial t} + \rho\frac{u^2}{2} + P = f(t)$

2.14



a) $\Delta P = P_4 - P_3 = ?$

$$P_4 = P_1 + \tfrac{1}{2}\rho U^2$$

$$P_2 + \tfrac{1}{2}\rho U_2^2 = P_1 + \tfrac{1}{2}\rho U^2$$

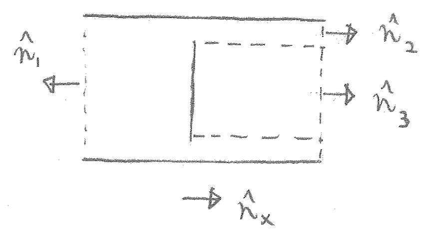
$$\Rightarrow P_3 = P_2 = \underbrace{P_1 + \tfrac{1}{2}\rho U^2}_{P_4} - \tfrac{1}{2}\rho U_2^2 \quad \Rightarrow \Delta P = \tfrac{1}{2}\rho U_2^2$$

From continuity eq. $U S_t = U_2 (S_t - S_p)$

$$\Rightarrow \Delta P = \frac{1}{2}\rho U^2 \frac{S_t^2}{(S_t - S_p)^2}$$

b) $F_{drag} = ?$, by using momentum conservation

$$\bar{F}_{drag} = \bar{F} = -\sum (P_i + \rho U_i^2) S_i \hat{n}_i$$



$$\begin{cases} \hat{n}_1 = -\hat{n}_x \\ \hat{n}_2 = \hat{n}_x \\ \hat{n}_3 = \hat{n}_x \end{cases}$$

$P_2 = P_3$

$$\begin{cases} F = P_1 S_t + \rho U^2 S_t - P_2 (S_t - \cancel{S_p}) - \rho U_2^2 (S_t - S_p) - \cancel{P_3 S_p} \\ P_2 = P_1 + \tfrac{1}{2}\rho U^2 - \tfrac{1}{2}\rho U_2^2 \end{cases}$$

$$\Rightarrow F = \cancel{P_1 S_t} + \rho U^2 S_t - \cancel{P_1 S_t} - \tfrac{1}{2}\rho U^2 S_t + \tfrac{1}{2}\rho U_2^2 S_t - \rho U_2^2 (S_t - S_p)$$

$U_2 = \dfrac{S_t}{S_t - S_p} U$

$$= \tfrac{1}{2}\rho U^2 \left[S_t + \frac{S_t^3}{(S_t - S_p)^2} - 2\frac{S_t^2 (S_t - S_p)}{(S_t - S_p)^2} \right]$$

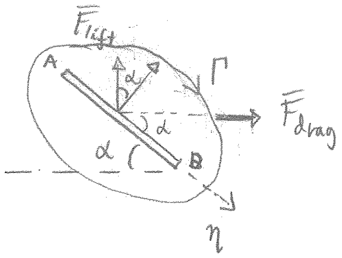
$$= \tfrac{1}{2}\rho U^2 \left[\frac{\cancel{S_t^3} - \cancel{2 S_t^2 S_p} + S_t S_p^2 + \cancel{S_t^3} - \cancel{2 S_t^3} + \cancel{2 S_t^2 S_p}}{(S_t - S_p)^2} \right]$$

$$= \tfrac{1}{2}\rho U^2 \frac{S_t S_p^2}{(S_t - S_p)^2}$$

$$F = \frac{C_{drag}}{2} \rho U^2 S_p$$

$$\Rightarrow C_{drag} = \frac{S_t S_p}{(S_t - S_p)^2}$$

2.15



Jukowsky – Kutta $\Rightarrow$ $$\frac{F_{lift}}{L_z} = -\rho U \Gamma$$

On the other hand

$$\frac{F_{lift}}{L_z} = \overbrace{\int_A^B (P_{bot} - P_{top})\, d\eta}^{\text{elementary pressure force}} \cos\alpha$$

$$\frac{F_{drag}}{L_z} = \int_A^B (P_{bot} - P_{top})\, d\eta \sin\alpha$$

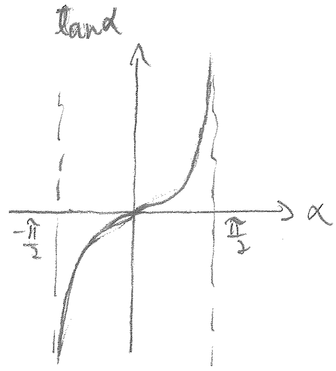
$$\Rightarrow \quad \frac{F_{drag}}{F_{lift}} = \tan\alpha$$

Then $$\frac{F_{drag}}{L_z} = -\rho U \Gamma \tan\alpha$$

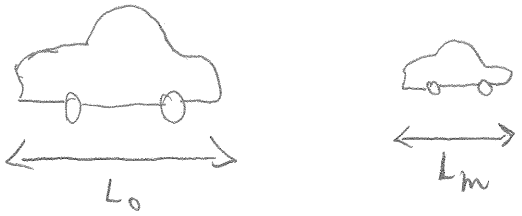
Checking : $\alpha = 0 \Rightarrow F_{drag} = 0$

$\alpha \to \frac{\pi}{2} \Rightarrow F_{drag} \to \infty$

$\hat{=}$ the model does not work



5.1



$$\frac{L_0}{L_m} = 10$$

$$\nu_0 = \text{air} = 1.5 \cdot 10^{-5} \ \frac{m^2}{s}$$

$$\nu_m = \text{water} = 10^{-6} \ \frac{m^2}{s}$$

a) $U_m = ?$

$$Re = \frac{\rho U L}{\mu} \qquad \nu = \frac{\mu}{\rho}$$

$$Re_0 = Re_m$$

$$\Leftrightarrow \quad \frac{U_0 L_0}{\nu_0} = \frac{U_m L_m}{\nu_m}$$

$$\Rightarrow \quad U_m = \frac{L_0 \nu_m}{L_m \nu_0} U_0 = 10 \cdot \frac{10^{-6} \frac{m^2}{s}}{1.5 \cdot 10^{-5} \frac{m^2}{s}} U_0$$

$$= \frac{2}{3} U_0$$

b) $F_0 = ?$

$$[F] = [\rho U^2 S]$$

$$F' = \frac{F_0}{\rho_0 U_0^2 L_0^2} = \frac{F_m}{\rho_m U_m^2 L_m^2}$$

$$\Rightarrow \quad F_0 = \frac{\rho_0 U_0^2 L_0^2}{\rho_m U_m^2 L_m^2} F_m$$

$\rho_0 = 1.2 \ kg/m^3$

$\rho_m = 10^3 \ kg/m^3$

$$a) \Rightarrow \left(\frac{U_0 L_0}{U_m L_m}\right)^2 = \left(\frac{\nu_0}{\nu_m}\right)^2$$

$$\Rightarrow \quad F_0 = \frac{\rho_0 \nu_0^2}{\rho_1 \nu_1^2} F_m = 0.27 \ F_m$$

Dimensional analysis:

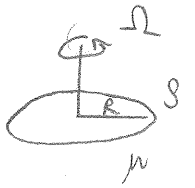
Buckingham Pi Theorem

$\underbrace{\text{A dimensionally homogeneous}}_{\text{units are in balance (the same in the both sides)}}$ equation involving n variables can be reduced to a relationship among $n - m$ dimensionless products, where m is the minimum number of reference dimensions (= number of basic dimensions)

Eight steps:

1. List all the variables that are involved in the problem
2. Express each of the variables in terms of basic dimensions
3. Determine the required number of pi terms
4. Select a number of repeating variables (no two repeating variables can have the same dimension), where the number required is equal to the number of reference (=basic) dimensions.
5. Form a pi term by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.
6. Repeat Step 5 for each of the remaining nonrepeating variables.
7. Check all the resulting pi terms to make sure they are dimensionless.
8. Express the final form as a relationship among the pi terms, and think about what it means!

(5.3)



$T \propto \Omega$, T on R ?

$[T] = [FR] = [\rho U^2 R^2 R]$

1. R, Ω, ρ, μ, T
2. $[R] = [L]$, $[\Omega] = [t^{-1}]$, $[\rho] = \left[\frac{M}{L^3}\right]$, $[\mu] = \left[\frac{M}{Lt}\right]$, $[T] = \left[\frac{M}{\cancel{L^3}}\frac{L^2}{t^2}\cancel{L^3}\right]$
3. $\begin{cases} n = 5 \\ m = 3 \end{cases}$ $([L], [t], [M])$ $\Rightarrow$ $n - m = 2$ pi terms

$Re = \dfrac{\rho(\Omega R)R}{\mu} = \dfrac{\rho\Omega R^2}{\mu}$

4. 3 repeating variables (R, Ω, ρ)
5. $[\Pi_1] = \left[\dfrac{\rho R^2 \Omega}{\mu}\right] = \left[\dfrac{\cancel{M}}{\cancel{L^3}} \cdot \dfrac{\cancel{L}\cancel{t}}{\cancel{M}} \cdot \cancel{L^2} \cdot \dfrac{1}{\cancel{t}}\right]$ dimensionless $\rightarrow Re$

continues...

5.3 ... continues

6. $$[\Pi_2] = \left[\frac{T}{\rho\Omega^2 R^2 R^3}\right] = \left[\frac{\cancel{M}\cancel{L^2}}{\cancel{t^2}}\frac{\cancel{L^3}}{\cancel{M}}\cancel{t^2}\cdot\frac{1}{\cancel{L^5}}\right]$$ dimensionless

8. $$\Rightarrow \Pi_2 = f(\Pi_1)$$

$$\Leftrightarrow \frac{T}{\rho\Omega^2 R^5} = f(Re) = \frac{C}{Re} \quad \Rightarrow \quad T \stackrel{T\propto\Omega}{=} \rho\Omega^2 R^{\cancel{5}\,3}\cdot C\frac{\mu}{\cancel{\rho}\cancel{\Omega}\cancel{R^2}}$$

$$= C\mu\Omega R^3$$

Vitaly's way: R, Ω, ρ, μ , $T \propto \Omega$

T on R ?

$$Re = \frac{\rho(\Omega R)\cdot R}{\mu} = \frac{\rho\Omega R^2}{\mu}$$

Dimension of torque

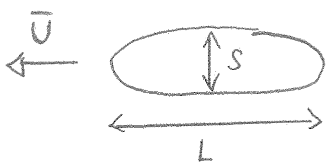
$$[T] = [FR] = [\rho U^2 R^2 R] = [\rho\Omega^2 R^2 R^3] = [\rho\Omega^2 R^5]$$

$$T' = \frac{T}{T_{scaling}} = \frac{T}{\rho\Omega^2 R^5} = f(Re)$$

$$\Rightarrow T = \rho\Omega^2 R^5 f\left(\frac{\rho\Omega R^2}{\mu}\right)$$

$$\stackrel{T\propto\Omega}{=} C\cancel{\rho}\Omega^{\cancel{2}}R^{\cancel{5}\,3}\cdot\frac{\mu}{\cancel{\rho}\cancel{\Omega}\cancel{R^2}} = C\mu\Omega R^3$$

5.4



ρ

μ

$F \propto \frac{1}{\sqrt{\rho}}$ $\quad F(L) = ?$

$F \propto \sqrt{S}$

Solution: Dimensionless parameters

$$Re = \frac{\rho U L}{\mu} \quad , \quad \frac{S}{L^2}$$

$$[F] = [\rho U^2 S]$$

$$F' = \frac{F}{\rho U^2 S} = f\left(\frac{\rho U L}{\mu}, \frac{S}{L^2}\right)$$

Assuming power laws $\Rightarrow$

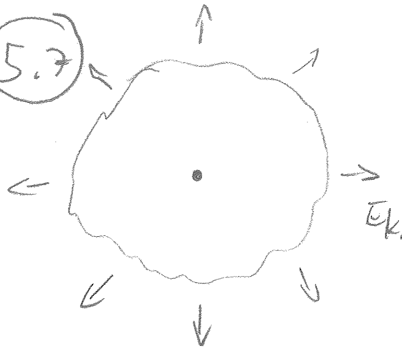
$$F \propto \rho U^2 S \left(\frac{\rho U L}{\mu}\right)^n \left(\frac{S}{L^2}\right)^m = \underline{\rho^{n+1}}\, U^{n+2}\, \underline{S^{m+1}}\, L^{n-2m}\, \mu^{-n}$$

Now $n+1 = -\frac{1}{2} \Rightarrow n = -\frac{3}{2}$

$m+1 = \frac{1}{2} \Rightarrow m = -\frac{1}{2}$

$$\Rightarrow \quad F \propto \sqrt{\frac{U S \mu^3}{\rho L}}$$

5.7



$R = R(t) = ?$

Sedov - Taylor strong shock

$E_{kin} \sim \frac{1}{2} M v^2 \longrightarrow \underbrace{E, \rho, R, t}_{4}$ → only 1 dimensionless parameter

$[E] = [M v^2] = \left[M \frac{L^2}{t^2}\right] = \left[\rho \frac{L^5}{t^2}\right]$

$[\rho] = \left[\frac{M}{L^3}\right]$

$\left[\frac{E}{\rho}\right] = \left[\frac{L^5}{t^2}\right]$

Only one dimensionless parameter $\frac{E t^2}{\rho R^5} = \text{const}$

$\Rightarrow R = \text{Const} \left(\frac{E t^2}{\rho}\right)^{1/5}$

$\Rightarrow R \propto t^{2/5}$