classmate

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The solution to the regularized LS problem is given by $\hat{\beta}_\tau = (X^T X + \tau \mathbb{I})^{-1} X^T Y$

We can write $Y = X\beta + \epsilon$, giving us,

$$\hat{\beta}_\tau = (X^T X + \tau \mathbb{I})^{-1} X^T (X\beta + \epsilon)$$
$$= (X^T X + \tau \mathbb{I})^{-1} X^T (X\beta) + (X^T X + \tau \mathbb{I})^{-1} X^T \epsilon$$

Now,

$$E[\hat{\beta}_\tau | X] = E[(X^T X + \tau \mathbb{I})^{-1} X^T (X\beta + \epsilon) \,|\, X]$$
$$= E[(X^T X + \tau \mathbb{I})^{-1} X^T \epsilon \,|\, X] + E[(X^T X + \tau \mathbb{I})^{-1} X^T X \beta | X]$$
$$= (X^T X + \tau \mathbb{I})^{-1} X^T \underbrace{E[\epsilon | X]}_{\hookrightarrow = 0} + (X^T X + \tau \mathbb{I})^{-1} X^T X \beta$$
$$= (X^T X + \tau \mathbb{I})^{-1} X^T X \beta = \underline{\underline{S_\tau^{-1} S_\tau \beta}}$$

where $S_\tau = (X^T X + \tau \mathbb{I})$, $S = X^T X$

$$Cov(\hat{\beta}_\epsilon) = Cov((X^T X + \tau \mathbb{I})^{-1} X^T (X\beta + \epsilon))$$
$$= Cov((X^T X + \tau \mathbb{I})^{-1} X^T \epsilon)$$
$$= (X^T X + \tau \mathbb{I})^{-1} X^T Cov(\epsilon) X (X^T X + \tau \mathbb{I})^{-1}$$

$Cov(\epsilon) = \sigma^2$, therefore

$$Cov(\hat{\beta}_\tau) = \sigma^2 (X^T X + \tau \mathbb{I})^{-1} X^T X (X^T X + \tau \mathbb{I})^{-1}$$
$$= \underline{\underline{\sigma^2 S_\tau^{-1} S S_\tau^{-1}}}$$